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Network Reliability Analysis and Intractability of Counting Diameter-Critical Graphs

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Abstract. Consider a stochastic network, where nodes are perfect but links fail independently, ruled by failure probabilities. Additionally, we are given distinguished nodes, called terminals, and a positive integer, called diameter. The event under study is to connect terminals by paths not longer than the given diameter. The probability of this event is called diameter-constrained reliability (DCR, for short).

Since the DCR subsumes connectedness probability of random graphs, its computation belongs to the class of $\mathcal{NP}$-Hard problems. The computational complexity for DCR is known for fixed values of the number of terminals $k \leq n$ and diameter d, being n the number of nodes in the network.

The contributions of this article are two-fold. First, we extend the computational complexity of the DCR when the terminal size is a function of the number of nodes, this is, when $k = k(n)$. Second, we state counting diameter-critical graphs belongs to the class of $\mathcal{NP}$-Hard problems.

Keywords: Computational Complexity, Network reliability, Diameter-Critical Graphs

1 Motivation

The object under study is the probability of a desired property on network connectivity, when this network is exposed to failures, inspired in real engineering problems. In delay-sensitive applications, distinguished nodes, called terminals, should deliver their messages timely and efficiently. For this purpose, the property we wish to preserve is a "hop-limit" between terminals, called diameter. We warn the reader that this diameter is not necessarily the diameter of the network. The mentioned diameter is a practical constraint selected by the network operator.

The diameter constrained reliability was introduced in 2001 by Héctor Cancela and Luis Petingi, inspired by delay sensitive applications [4]. The scientific literature in this field has been increasing in recent years. We believe that the main reasons of this emergence of network analysis under diameter constraint are its applications, ranging from flooding to multi-conferencing, peer-to-peer live video streaming, wavelength division multiplexing and degraded analog systems, not being exhaustive [6]. Indeed, the IETF Network Working Group designed the IPv6 protocol with a "hop-limit" field, reinforcing engineering applications with a prescribed number of forward messages [9].

This article is organized in the following manner. Section 2 presents the problem, and the terminology used throughout the document. Section 3 summarizes its computational complexity, when the number of terminals and diameter are fixed. The main contributions of this paper are summarized in Section 4. There, we extend the complexity analysis when the number of terminals is a function of the number of nodes. Additionally, the complexity of counting diameter-critical subgraphs is characterized. The proof is inspired in network reliability analysis, and the result is another example of the strong link between network reliability and graph theory. Finally, Section 5 contains concluding remarks and open problems.

2 Problem and Terminology

We are given a simple graph $G = (V, E)$, a terminal-set $K \subseteq V$, a positive integer d and the elementary link reliabilities $p_e \in [0, 1]$ for all $e \in E$. With graph G we associate a random partial graph (same nodes, subset of edges) where edge e is removed from G with probability p_e, and where the event "link e has been removed" is independent of the corresponding events associated with any other subset of edges not including e. In this random graph, the event "all pair of terminals have distance d or less" is denoted by $A^d_{K,G}$. The diameter-constrained reliability (DCR) is the probability of $A^d_{K,G}$, and it is denoted $R^d_{K,G}$:

$$R^d_{K,G} = P(A^d_{K,G}) = P(d(u, v) \leq d, \forall \{u, v\} \in K). \tag{1}$$

The object of this paper is to study the number $R^d_{K,G}$, for an arbitrary graph G, terminal set K and diameter d. ¿From now on, we will use the symbols $m = |E|$, $n = |V|$, $k = |K|$.

A *cutset* is link-subset $E' \subseteq E$ such that if we remove from G all links in E', there are at least a couple of nodes in K whose distance is strictly greater than d. In the same way, a *pathset* is a link-subset $E' \subseteq E$ such that if none of its edges is removed from G, then the distance between any pair of terminals is less than or equal to d. We will denote by c the minimum cardinality cutset, and by f the minimum cardinality pathset. Additionally, n_c will denote the number of such cutsets, while n_f will denote the number of minimum cardinality pathsets.

The following definitions from graph theory and computational complexity will be used throughout the document. Graph G is said to be *diameter-critical* if its diameter is increased under the remotion of an arbitrary link. More specifically, we will say G is d-critical when it is diameter-critical and its diameter is d. A *vertex cover* for $G = (V, E)$ is a subset $V' \subseteq V$ such that for each $e = \{x, y\} \in E$, either $x \in V'$ or $y \in V'$ (i.e., V' meets all links). A simple graph $G = (V, E)$ is *bipartite* if there is a bipartition $V = V_1 \cup V_2$ such that $E \subseteq \{\{x, y\} : x \in V_1, y \in V_2\}$. A *cycle with n nodes*, denoted by C_n, is a graph with node-set $\{x_1, \ldots, x_n\}$ and link-set $E = \{\{x_i, x_{i+1}\}, i = 1, \ldots, n-1\} \cup \{x_n, x_1\}$.

The class $\mathcal{NP}$ is the set of problems solvable by a non-deterministic Turing machine in polynomial time [8]. A problem is $\mathcal{NP}$-Hard if it is at least as hard as every problem in the set $\mathcal{NP}$ (formally, if every problem in $\mathcal{NP}$ has a polynomial reduction to the former). It is widely believed that $\mathcal{NP}$-Hard problems are intractable (i.e. there is no polynomial-time algorithm to solve them). An $\mathcal{NP}$-Hard problem is $\mathcal{NP}$-Complete if it is inside the class $\mathcal{NP}$. Stephen Cook proved that the joint satisfiability of an input set of clauses in disjunctive form is an $\mathcal{NP}$-Complete decision problem; in fact, it is the first known problem of this class [7]. In this way, he provided a systematic procedure to prove that a certain problem is $\mathcal{NP}$-Complete. Specifically, it suffices to prove that the problem is inside the class $\mathcal{NP}$, and that it is at least as hard as an $\mathcal{NP}$-Complete problem. Richard Karp followed this hint, and presented the first 21 combinatorial problems inside this class [10]. Leslie Valiant defines the class $\#\mathcal{P}$ of counting problems, such that testing whether an element should be counted or not can be accomplished in polynomial time [14]. A problem is $\#\mathcal{P}$-Complete if it is in the set $\#\mathcal{P}$ and it is at least as hard as any problem of that class.

Finding the diameter-constrained reliability is at least as hard as recogniting and counting minimum cardinality mincuts/minpaths. In our terms, finding the reliability is at least as hard as the computation of the numbers c, f, n_c and n_f. We invite the reader to consult [1] for a more general approach of reliability, in the context of stochastic binary systems.

3 Computational Complexity

If $d \geq n-1$, the diameter is not a constraint, and $R^d_{K,G} = R_{K,G}$ is the K-terminal or classical reliability. In 1977, Arnon Rosenthal observed that finding f in the classical K-terminal reliability is at least as hard as the Steiner Tree Problem in graphs [13]. Since the Steiner Tree Problem is inside Karp's list, classical K-terminal reliability computation belongs to the class of $\mathcal{NP}$-Hard problems [10]. By inclusion, the exact DCR computation is $\mathcal{NP}$-Hard as well [8]. In this section, we outline the computational complexity for fixed pairs (k, d). Sketch-of-proofs are here provided, since the main contributions are based on these proofs.

For pedagogical and aesthetic reasons, we start our study with diameter $d = 1$. We invite the reader to convince himself that all pair of terminals must be directly connected, and then, that

$$R^1_{K,G} = \prod_{\{u,v\} \in K} p_{u,v}. \tag{2}$$

A similar situation occurs in diameter-critical graphs when $K = V$ for the computation of $R^d_{K,G}$ where d is the diameter of G. As a consequence, the exact DCR in diameter-critical graphs is found directly (it is just the product of all elementary link reliabilities). The determination whether a given graph $G = (V, E)$ is diameter-critical or not is straightforward; just find the diameter of $G - e$ for every link e. However, we will see that counting diameter-critical subgraphs is a hard task.

The following natural step is to study the simplest case with diameter $d = 2$, which is the case of $K = \{s, t\}$ (i.e., $k = 2$). In this case, nodes s and t must be either directly connected or both linked with a third node:

$$R^2_{\{s,t\},G} = 1 - (1 - p_{s,t}) \prod_{v \in V - \{s,t\}} (1 - p_{s,v} p_{v,t}). \tag{3}$$

A more sophisticated argument shows that an efficient DCR computation is possible for any fixed k when $d = 2$. A proof is included in [3]. The key point is to present a family of exhaustive and disjoint events for all pathsets.

Curiously, if we consider fixed pairs (k, d) with $d \geq 3$ and $k \geq 2$, the exact DCR computation belongs to the class of $\mathcal{NP}$-Hard computational problems. This result was formally proved by Héctor Cancela and Louis Petingi in 2004 [5]. There, they introduced a family of networks that are essential for this paper. Here, we call them Cancela-Petingi networks:

Definition 1 (Cancela-Petingi Network). *Let $G = (A \cup B, E)$ be a bipartite graph, and $d \geq 3$ a positive integer. Consider a path P with node set $V(P) = \{s, s_1, \ldots, s_{d-3}\}$.* Cancela-Petingi Network *is defined as the graph $G' = (V', E')$ with node-set $V' = A \cup B \cup V(P) \cup \{t\}$ and edge-set $E' = E \cup E(P) \cup I$, being $I = \{\{s_{d'}, a\}, a \in A\} \cup \{\{b, t\}, b \in B\}$. In G', all links are perfect but the ones from I, which fail independently with identical probabilities $p = 1/2$.*

Figure 1 illustrates Cancela-Petingi network for bipartite graph C_6, when $d = 6$. The main idea of the proof can be summarized in two items:

- Counting vertex-covers in bipartite graphs is a $\#\mathcal{P}$-Complete problem [11].
- Finding the DCR of Cancela-Petingi networks where $K = \{s, t\}$ is at least as hard as counting vertex-covers in bipartite graphs.

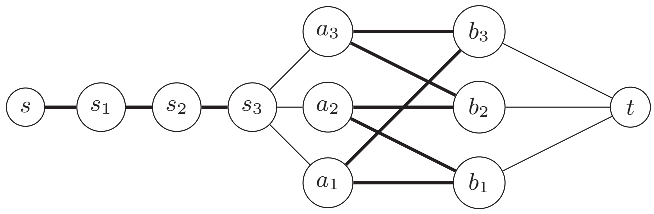

Fig. 1. Cancela-Petingi Network for $G = C_6$ and $d = 6$. Thick lines are perfect (thin lines fail with $p = 1/2$).

Here a sketch of the proof is provided.

Proposition 1. *[5] Consider fixed pairs (k, d) with $d \geq 3$ and $k \geq 2$. The general DCR computation for k terminals and diameter d belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. First, consider $k = 2$. The DCR of Cancela-Petingi network for fixed values (k, d) is related with the number of vertex-covers $\mathcal{B}$ in a bipartite graph $G = (A \cup B, E)$. Indeed, if $I' \subseteq I$ is a cutset, then the set of all nodes from $A \cup B$ incident to some link in I' is a vertex cover for G. We invite the reader to check that the converse is true. As a consequence, there is a bijective correspondence between cutsets and vertex-covers for G. Since all cutsets are equally likely (non-perfect links reliabilities assume the value $p = 1/2$), we get that

$$1 - R^d_{\{s,t\},G'} = \frac{|\mathcal{B}|}{2^{|A|+|B|}} \tag{4}$$

Therefore, finding $R^d_{\{s,t\},G'}$ is at least as hard as counting $|\mathcal{B}|$. If $k > 2$, just connect new terminals in a trivial fashion (i.e. perfect direct links) with the first two terminals s and t. Again, the problem is at least as hard as the source-terminal case. ♠

The following corollary is first proved in [12]

Corollary 1. *The DCR in the all-terminal case belongs to the class of $\mathcal{NP}$-Hard problems for any fixed $d \geq 3$.*

Proof. For any bipartite graph $G = (A \cup B, E)$, extend Cancela-Petingi network, adding perfect links between all pairs of the component A, and all pairs of the component B. The pathsets in the source-terminal Cancela-Petingi network are precisely the pathsets in the extended network for the all-terminal scenario. Then, the all-terminal DCR computation is at least as hard as finding the reliability of Cancela-Petingi networks. ♠

For completeness, it suffices to study the all-terminal scenario with diameter $d = 2$. Recall that counting vertex-covering in graphs is a $\#\mathcal{P}$-Complete problem. The main idea is to capture the complexity of this task. The following result was first proved in [2].

Proposition 2. *The DCR in the all-terminal case belongs to the class of $\mathcal{NP}$-Hard problems when $d = 2$.*

Proof. Consider $G' = (V \cup \{a, b\}, E \cup \{\{x, a\}, \{x, b\}, \forall x \in V\})$ for any given graph $G = (V, E)$. Clearly, G' has diameter 2. Define all perfect links but the ones connected to node a, with independent elementary reliabilities $p(ax) = 1/2$.

A minimum cardinality pathset H in G' has all perfect links, and some links $\{a, x_1\}, \ldots, \{a, x_r\}$ for certain nodes $x_i \in V$. The diameter of $G_H = (V, H)$ is two, and it is increased under an arbitrary link deletion. Observe that node b reaches every other node in one or two steps (a is reached in two steps), and every pair of nodes $x, y \in V$ are reachable in two steps, using path xby. It suffices to study reachability between node a and every node from the set V. Consider the neighbor-set for node a, $N_a = \{x : \{a, x\} \in H\}$. Then, *$a$ reaches every node in two steps if and only if N_a is a vertex cover of G*. Indeed, suppose a reaches every node in two steps. Then, for any $x \in V \setminus N_a$ there exists a path xya, so $y \in N_a$ and thus N_a is a vertex cover. Conversely, if N_a covers V, let $x \in V$. Then, either $x \in N_a$ and x is adjacent with a, or $x \in V \setminus N_a$ and there exists $y \in N_a \cap N_x$, so xya is a path of two hops between x and a. The minimality of N_a as a cover follows from the minimality of H as a pathset. Then, minimum cardinality pathset counting (i.e., counting n_f) is at least as hard as counting minimum vertex-covering in graphs. ♠

Observe that there is a strict connection between graph theory and network reliability. In all previous propositions, the bridge is complexity theory. The full picture of complexity for fixed pairs (k, d) is given in Figure 2. The main contributions of this paper are summarized in Section 4. There, we further explore the connection between network reliability and graph theory. Indeed, Theorem 2 states a new result from graph theory, inspired by the DCR complexity in the all-terminal case with diameter two.

4 Main Theorems

Theorem 1 further generalizes the complexity analysis resumed in Section 2, considering $k = k(n)$. The result is inspired by Corollary 1. Theorem 2 states the hardness of counting diameter-critical subgraphs, based on Proposition 2.

Theorem 1. *The exact DCR computation for $d \geq 3$ (and an arbitrary terminal-set) belongs to the class of $\mathcal{NP}$-Hard problems.*

Proof. The statement makes sense whenever $2 \leq k(n) \leq n$. Recall that the network from Corollary 1 adds perfect links to the components of a bipartite graph in the Cancela-Petingi Network. Let us call G' the Cancela-Petingi network coming from the arbitrary bipartite graph $G = (A \cup B, E)$, and G'' the extended network. Minpaths in G'' in the all-terminal scenario are precisely the minpaths of G' when $K = \{s, t\}$. Therefore, $R^d_{V,G''} = R^d_{\{s,t\},G'}$. For every number of terminals $k = k(n)$ choose a terminal-set $K(n)$ such that $\{s, t\} \subseteq K(n)$ and

	k (fixed)		$k = n$ or free
	2	$3 \ldots$	
2	$O(n)$ [5]	$O(n)$ [3]	$\mathcal{NP}$-Hard [2]
d: $3 \ldots n-2$	$\mathcal{NP}$-Hard [5]		$\mathcal{NP}$-Hard [12]
$n-1 \ldots$	$\mathcal{NP}$-Hard [13]		$\mathcal{NP}$-Hard [11]

Fig. 2. DCR Complexity in terms of the diameter d and number of terminals $k = |K|$

$k(n) = |K(n)|$. Again, the pathsets in this scenario is precisely the ones in the all-terminal case. Therefore, $R^d_{K(n),G''} = R^d_{\{s,t\},G'}$, and the exact DCR computation with terminal-set $K(n)$ is at least as hard as in the source-terminal case. ♠

Theorem 2. *Counting diameter-critical graphs is $\mathcal{NP}$-Hard, for any fixed diameter $d \geq 2$.*

Proof. The case $d = 2$ is just a corollary of Proposition 2. There, a correspondence between minimum cardinality pathsets and vertex coverings is presented. But minimum cardinality pathsets are precisely 2-critical graphs. If $d > 2$, extend the graph from Proposition 2, adding a simple perfect path of length $d - 2$ pending from node b. There is a coincidence between d-critical graphs in the extended graph, and 2-critical subgraphs in the original graph. ♠

Observe that all subgraphs of $G = (V, E)$ with diameter d are an extension of d-critical graphs. Inspired in Theorem 2, we propose the following

Conjecture 1. Counting all subgraphs with a fixed diameter belongs to the class of $\mathcal{NP}$-Hard problems.

5 Conclusions

In network reliability, the goal is to preserve a specific property in a network, subject to stochastic failures. Here, we studied the diameter-constrained reliability, which joints connectivity with a hop-constraint. Its applications are diverse, ranging, for instance, from video streaming to flooding-based systems.

The exact DCR computation belongs to the class of $\mathcal{NP}$-Hard problems, since it subsumes connectedness probability in random graphs. Furthermore, from prior results it is known that the DCR computation belongs to this class even when $k \geq 2$ and $d \geq 3$ are fixed. In this article, we proved that the exact DCR belongs to the class of $\mathcal{NP}$-Hard problems for an arbitrary function $k = k(n)$ on the number of nodes n within the network. Finally, the hardness of counting 2-critical graphs is proved, inspired in the complexity theory of network reliability.

There are several open problems in this area, from both practical and theoretical aspects. Here, we consider two of them. The DCR computation for the all-terminal case with diameter two is $\mathcal{NP}$-Hard. However, the complexity under homogeneous link failures is still an open problem. What is more, all graphs are equally likely when $p = 1/2$, and $2^m R^2_{V,G}(1/2)$ is the number of subgraphs with diameter two. To the best of our knowledge, this counting problem is not solved yet, and it remains at the heart of graph theory. Inspired in the complexity of counting diameter-critical graphs, we conjecture that counting partial graphs with a fixed diameter is hard as well.

In practical networks, it happens that nodes mail fail as well, and that links fail in clusters. It is a major concern to extend the type of analysis of this paper to the case of dependent-failures.

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