Skeleton of locally most separable 3-terminal graphs

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Abstract

A 3-terminal graph is a graph with 3 distinguished vertices, called terminals. Let $T_{n,m}$ be the set of all connected 3-terminal graphs on n vertices and m edges. Let G be in $T_{n,m}$. A spanning subgraph H of G is 3-separable when H has precisely 3 components, each containing one terminal. For each p in $[0,1]$, the separability of G at p, denoted $S_G(p)$, is the probability that the resulting subgraph of G is 3-separable after each edge in G is removed independently with probability $1-p$. The graph G is a locally most separable 3-terminal graph (LMS3TG) if for each H in $T_{n,m}$ there exists $\delta > 0$ such that $S_G(p) \geq S_H(p)$ whenever $p \in (1-\delta, 1)$.

Let $\mathcal{C}_{n,m}$ be the set of all connected simple graphs on n vertices and m edges. For each G in $\mathcal{C}_{n,m}$, the skeleton G' of G is the graph arising from G by the contraction of each of its bridges. An almost-complete graph is a graph G in $\mathcal{C}_{n,m}$ such that $m \geq \binom{n-1}{2} + 2$. In this note, we show that the skeleton of each LMS3TG in $T_{n,m}$ is an almost-complete graph.

1. Introduction

The reliability polynomial of a graph has been a central topic since the seminal work of Moore and Shannon [5], who showed that it is possible to construct highly reliable circuits from imperfect components. A deep conjecture posed by Brown and Colbourn [3] asserts that the roots of the reliability polynomial of all graphs lie in the closed unit disc of the complex plane. Counterexamples were built by Brown and Mol [2], using the *separability of a graph*. Recently, Brown and McMullin [1] introduced the concept of a *uniformly most separable graph* for two terminals, and characterized these graphs in nonempty classes of two-terminal multigraphs. Later, Romero [6] proved the nonexistence of uniformly most separable simple graph in infinitely many classes of two-terminal graphs. As a byproduct, the author succeeds in characterizing locally most separable graphs in each nonempty class of two-terminal graphs.

In this note, the concept of separability is extended to three terminals. This note is organized as follows. Section 2 presents the concept of locally most separable 3-terminal graph (LMS3TG). Related work is revisited in Section 3. The main result of this note is given in Section 4, yielding a necessary condition for a 3-terminal graph to be LMS3TG, namely, that every LMS3TG has an almost-complete skeleton.

2. Concepts

Let $\mathcal{C}_{n,m}$ be the class of all connected simple graphs on n vertices and m edges, and let G be any graph in $\mathcal{C}_{n,m}$. The distance between two vertices u and v in G is denoted $d_G(u, v)$. Let $b(G)$ be the number of bridges of G, and let $b(n, m)$ be the maximum value of $b(G)$ among all graphs G in $\mathcal{C}_{n,m}$. The *skeleton* of G, here denoted G', is the simple graph that arises from G by the contraction of each of its bridges.

A 3-terminal graph is a simple connected graph equipped with 3 distinguished vertices, called terminals. Let $T_{n,m}$ be the set consisting of all 3-terminal graphs on n vertices and m edges. We will use the symbols t_1, t_2 and t_3 for the terminal vertices. Let G be any 3-terminal graph in $T_{n,m}$. A *3-separable subgraph of G* is a spanning subgraph of G having precisely 3 components, each one including one terminal. For each p in $[0, 1]$, the *separability of G evaluated at p*, denoted $S_G(p)$, is the probability of having a 3-separable subgraph of G after each of its edges is independently deleted with probability $1 - p$. For each $i \in \{1, 2, \ldots, m\}$, we let $F_i(G)$ be the number of 3-separable subgraphs of G obtained by removing i edges. By additivity, $S_G(p)$ can be computed as follows,

$$S_G(p) = \sum_{i=1}^{m} F_i(G)(1 - p)^i p^{m-i}.$$

In this note we study the concept of locally most separable 3-terminal graphs (LMS3TG).

Definition 1. *We say G in $T_{n,m}$ is a* locally most separable 3-terminal graph *(LMS3TG) if for each H in $T_{n,m}$ there exists $\delta > 0$ such that if $p \in (1 - \delta, 1)$ then $S_G(p) \geq S_H(p)$.*

3. Background

Our goal is to find necessary properties satisfied by all LMS3TGs. In this section, we list previous results which will be useful for our purpose. The following result can be proved using elementary calculus.

Lemma 2 (Brown and McMullin [1]). *Let G and H be in $T_{n,m}$. If there exists $j \in \{0, 1, \ldots, m\}$ such that $F_k(G) = F_k(H)$ for all $k \in \{0, 1, \ldots, j - 1\}$ and $F_j(G) > F_j(H)$, then there exists $\delta > 0$ such that $S_G(p) > S_H(p)$ for all $p \in (1 - \delta, 1)$.*

For each G in $T_{n,m}$ we define its F-tuple by $F(G) = (F_1(G), F_2(G), \ldots, F_m(G))$. A consequence of Lemma 2 is that a 3-terminal graph G in $T_{n,m}$ belongs to LMS3TG if and only if its F-tuple is lexicographically maximum among all tuples $F(G)$ such that $G \in T_{n,m}$. In fact, define $T_{n,m}(0) = T_{n,m}$ and, for each $i \in \{0, 1, \ldots, m-1\}$, define the set $T_{n,m}(i+1)$ as $T_{n,m}(i + 1) = \{G : G \in T_{n,m}(i)\ ,\ F_{i+1}(G) \geq F_{i+1}(H)$ for each H in $T_{n,m}(i)\}$. Lemma 2 gives a characterization of LMS3TG.

Lemma 3 (Romero [6]). *The set $\mathcal{G}_{n,m}$ of all LMS3TG in $T_{n,m}$ equals $T_{n,m}(m)$.*

By construction, for each i in $\{1, 2, \dots, m-1\}$ it holds that $T_{n,m}(i+1) \subseteq T_{n,m}(i)$, and the intersection of all sets of this nested sequence gives, by Lemma 3, the set of all LMS3TGs in $T_{n,m}$. The set $T_{n,m}(i+1)$ is obtained by solving a combinatorial optimization problem, consisting of the maximization of $F_{i+1}(G)$ among all 3-terminal graphs in $T_{n,m}(i)$. The following concepts give tools to solve the previous combinatorial optimization problems.

Definition 4. *A tuple $(x_1, x_2, ..., x_t) \in \mathbb{Z}_+^t$ is fair if $|x_i - x_j| \leq 1$ for all $i, j \in \{1, 2, ..., t\}$*

A known result on the maximization of elementary symmetric polynomials is given in Lemma 5.

Lemma 5 (Marshall et al. [4])**.** *Let k and t be integers such that $2 \leq k \leq t$ and define*

$$\phi_t^{(k)}(\ell_1, \ell_2, \dots, \ell_t) = \sum_{J \in \binom{[t]}{k}} \prod_{i \in J} \ell_i$$

for each $(\ell_1, \ell_2, \dots, \ell_t) \in \mathbb{Z}_+^t$. Let m be any integer such that $m \geq t$ and set

$$L_{t,m} = \{\, (\ell_1, \ell_2, \dots, \ell_t) \in \mathbb{Z}_+^t : \ell_1 + \ell_2 + \cdots + \ell_t = m \,\}.$$

Then the maximum of $\phi_t^{(k)}(\ell_1, \ell_2, \dots, \ell_t)$ as $(\ell_1, \ell_2, \dots, \ell_t)$ ranges over $L_{t,m}$, denoted $\overline{\phi}_t^{(k)}(m)$, is attained precisely at those tuples that are fair*.*

Remark 6. *In the conditions of Lemma 5 it holds that the maximum value $\overline{\phi}_t^{(k)}(m)$ among all tuples in $L_{t,m}$ is an increasing function with respect to both integers t as well as m.*

The combinatorial optimization problem whose goal is to maximize the number of bridges among all graphs in $\mathcal{C}_{n,m}$ is completely solved. For convenience, we say that an *almost-complete* graph G is a graph in $\mathcal{C}_{n,m}$ such that $m \geq \binom{n-1}{2} + 2$.

Proposition 7 (Romero [6])**.** *A graph G in $\mathcal{C}_{n,m}$ has $b(n,m)$ bridges if and only if its skeleton G' has at least $\binom{n-1}{2} + 2$ edges.*

4. Analysis

The main result in this note is Proposition 17, which asserts that the skeleton of each LMS3TG is almost-complete. As $T_{n,m}(2) \subseteq T_{n,m}(m)$, by Lemma 3 it suffices to prove that the skeleton of each G in $T_{n,m}(2)$ is almost-complete. By Proposition 7, we will equivalently show that each G in $T_{n,m}(2)$ has precisely $b(n,m)$ bridges.

In Remark 8, we see that $T_{n,m}(1) = T_{n,m}$. Then, we will find $T_{n,n-1}(2)$, i.e., all 3-terminal tree-graphs G in $T_{n,n-1}(1)$ maximizing $F_2(G)$. Finally, we will find $T_{n,m}(2)$ when $m \geq n$. The maximization of $F_2(G)$ when G is a 3-terminal graphs on more edges than vertices will be reduced to the maximization of $F_2(T(G))$ for some tree-graph $T(G)$ corresponding to G. Such 3-terminal graphs have the maximum number of bridges yielding the main result of this note.

The number of components of a graph is increased by a unit when one edge is removed if and only if a bridge is deleted. Therefore, Remark 8 holds.

Remark 8. *For each G in $T_{n,m}$ it holds that $G-e$ has at most 2 components thus $F_1(G)=0$. Each 3-separable edge set of G that arises by removing only two edges consists of 2 bridges.*

Remark 8 gives that $T_{n,m}(1)=T_{n,m}$. Now, let us find the set $T_{n,n-1}(2)$. Each tree-graph in $T_{n,n-1}$ has a special vertex given in Definition 9.

Definition 9. *Let T be a 3-terminal tree-graph in $T_{n,n-1}$. The* median *of $\{t_1,t_2,t_3\}$ is the unique vertex belonging to the three paths joining the terminals.*

Let G be in $T_{n,n-1}$ with terminals t_1, t_2, t_3 and median $m(G)$. Either one of its terminals lie on the path joining the other two (in which case the median coincides with a terminal), or the three terminal paths intersect at an internal vertex. In the first case, Lemma 10 yields an upper bound for $F_2(G)$.

Lemma 10. *If G is in $T_{n,n-1}$ and one of its terminals belongs to the path joining the other two terminal, then $F_2(G)\leq \overline{\phi}_2^{(2)}(n-1)$*

Proof. Let G be any 3-terminal graph satisfying the conditions of the statement. Assume without loss of generality that the only path joining t_1 and t_3 traverses t_2. Define $\ell_1=d_G(t_1,t_2)$, $\ell_2=d_G(t_2,t_3)$. The only way to separate the three terminals is to pick 1 out of the ℓ_1 edges joining t_1 and t_2, plus 1 out of the ℓ_2 edges joining t_2 and t_3, thus $F_2(G)=\ell_1\ell_2$. The result follows applying Lemma 5 with $t=2$, $k=2$ and $m=n-1$. □

In the second case Lemma 11 gives an upper bound for $F_2(G)$.

Lemma 11. *If G is in $T_{n,n-1}$ and none of its terminals belongs to the path joining the other two, then $F_2(G)\leq \overline{\phi}_3^{(2)}(n-1)$.*

Proof. Let G be any 3-terminal graph in $T_{n,n-1}$ satisfying the conditions of the statement. Its median, $m(G)$, is included in each of the three paths $P[t_1,t_2]$, $P[t_1,t_3]$, and $P[t_2,t_3]$. Define $\ell_1=d_G(t_1,m(G))$, $\ell_2=d_G(t_2,m(G))$, and $\ell_3=d_G(t_3,m(G))$. Every 3-separable subgraph obtained by deleting two edges of G is uniquely determined by choosing two of the three paths and deleting one edge from each, thus $F_2(G)=\ell_1\ell_2+\ell_1\ell_3+\ell_2\ell_3$. Three paths $P[t_1,m(G)]$, $P[t_2,m(G)]$, $P[t_3,m(G)]$ are edge-disjoint thus $\ell_1+\ell_2+\ell_3\leq n-1$. Applying Lemma 5 with $k=2$, $t=3$ and $m=n-1$, the maximum of $F_2(G)$ is precisely $\overline{\phi}_3^{(2)}(n-1)$. □

A fair subdivision of $K_{1,3}$ arises by replacing each of its edges e_1, e_2 and e_3 by paths of respective lengths ℓ_1, ℓ_2 and ℓ_3 such that (ℓ_1,ℓ_2,ℓ_3) is a fair tuple. We are in position to determine all 3-terminal tree-graphs G in $T_{n,n-1}$ maximizing $F_2(G)$.

Lemma 12. *For each 3-terminal tree-graph G in $T_{n,n-1}$ it holds that $F_2(G)\leq \overline{\phi}_3^{(2)}(n-1)$, and the equality holds if and only if G is a fair subdivision of $K_{1,3}$.*

Proof. Lemmas 10 and 11 give that $F_2(G)\leq \max\left\{\overline{\phi}_2^{(2)}(n-1),\overline{\phi}_3^{(2)}(n-1)\right\}$. By Remark 6, it follows that $\overline{\phi}_2^{(2)}(n-1)<\overline{\phi}_3^{(2)}(n-1)$. Therefore, $F_2(G)\leq \overline{\phi}_3^{(2)}(n-1)$, and the equality holds precisely when the tuple with the distances between the median and the terminals is fair and its sum equals $n-1$. This condition occurs precisely when G is a fair subdivision of $K_{1,3}$. □

Now we will maximize $F_2(G)$ among all 3-terminal graphs G in $T_{n,m}$ when $m \geq n$. We will reduce the counting to tree-graphs using Definition 13.

Definition 13. *Let G be a 3-terminal graph in $T_{n,m}$ such that every block of G contains at most one terminal. The* tree *of G, denoted by $T(G)$, is the 3-terminal graph obtained from G by contracting every block into a single vertex, where the resulting vertex is declared to be a terminal if and only if the contracted block contains a terminal.*

Observe that $T(G)$ is indeed a tree.

Lemma 14. *For each 3-terminal graph G in $T_{n,m}$ whose blocks include 0 or 1 terminal it holds that $F_2(G) = F_2(T(G))$.*

Proof. Let G be as in Definition 13. Let e_1 and e_2 be a pair of edges of G. Then, $G - e_1 - e_2$ is a 3-separable subgraph of G if and only if e_1 and e_2 are bridges and further, there is no pair of terminals of G connected in $G - e_1 - e_2$. This occurs precisely when $T(G) - e_1 - e_2$ is a 3-separable subgraph of $T(G)$. Consequently there exists a bijective mapping between 3-separable subgraphs arising by removing 2 edges in G and $T(G)$, and the lemma follows. $\square$

By Lemma 14, the maximization of $F_2(G)$ only depends on its tree-graph $T(G)$.

Lemma 15. *Let n and m be integers such that $b(n,m) \geq 2$. If G is in $T_{n,m}(2)$ then $b(G) = b(n,m)$ and $T(G)$ is a fair subdivision of $K_{1,3}$.*

Proof. Let n and m be integers such that $b(n,m) \geq 2$. Let G be in $T_{n,m}$. By Remark 8, each 3-terminal subgraph of G that arises by removing 2 edges includes 2 bridges. Therefore, if G has either 0 or 1 bridges then $F_2(G) = 0$. Additionally, if G has some of its blocks with two or more terminals, then these terminals cannot be separated by removing only bridges thus $F_2(G) = 0$. Otherwise, $T(G)$ is well-defined and Lemma 14 asserts that $F_2(G) = F_2(T(G))$. Now, by Lemma 12, $F_2(G) = F_2(T(G)) \leq \overline{\phi}_3^{(2)}(b(G))$, with equality if and only if $T(G)$ is a fair subdivision of $K_{1,3}$. Finally, Remark 6 gives that $\overline{\phi}_3^{(2)}(b(G)) \leq \overline{\phi}_3^{(2)}(b(n,m))$, with equality precisely when $b(G) = b(n,m)$. Consequently, G maximizes $F_2(G)$ precisely when $b(G) = b(n,m)$ and $T(G)$ is a fair subdivision of $K_{1,3}$. $\square$

Lemma 15 determines the bridge structure of 3-terminal graphs in $T^3_{n,m}(2)$ when $b(n,m) \geq 2$. The remaining cases are covered in Remark 16.

Remark 16. *Let $T_{n,m}$ be a nonempty set such that $b(n,m) \in \{0,1\}$. On the one hand, if $b(n,m) = 0$ then by Proposition 7 we know that each G in $T_{n,m}$ is almost-complete and its skeleton is precisely G. On the other hand, if $b(n,m) = 1$ then $F_2(G) = 0$ for each G in $T_{n,m}$. Then, the optimization starts with the subsequent entries of the F-tuple. A straightforward inspection of this case shows that every LMS3TG must have its unique bridge incident with a terminal. Hence its skeleton is almost-complete by Proposition 7.*

We are in position to prove the main result of this note.

Proposition 17. *The skeleton of each LMS3TG is almost-complete.*

Proof. Let $T_{n,m}$ be any nonempty set. If $b(n,m) \geq 2$ then $T_{n,m}(2)$ consists of 3-terminal graphs G such that $b(G) = b(n,m)$ in which $T(G)$ is a fair subdivision of $K_{1,3}$. As $b(G) = b(n,m)$, by Proposition 7 we know that its skeleton is almost-complete. The remaining cases are covered in Remark 16. The proposition follows. $\square$

Further analysis should be conducted to characterize all LMS3TGs in each nonempty set $T_{n,m}$. The general determination of locally most separable k-terminal graph when $k \geq 3$ is an open problem worth exploring.

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