Radiación gravitacional y teoremas soft logarítmicos

Gianni Boschetti

Programa de Posgrado en Física
Pedeciba, Facultad de Ciencias, Facultad de Ingenieria
Universidad de la República

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Directores:
Dr. Miguel Campiglia
Dr. Guzmán Hernández

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RESUMEN

La radiación gravitacional emitida en un proceso genérico está gobernada, en el límite de baja frecuencia, por los denominados *teoremas soft*. Estos son expresiones universales que dependen únicamente de los momentos de los objetos involucrados, y se aplican tanto a cuerpos celestes en procesos astrofísicos como a partículas elementales.

En este trabajo nos centramos en el segundo término en la expansión mencionada, el cual depende logarítmicamente de la frecuencia y, por tanto, se denomina *teorema soft logarítmico*. En particular, presentamos una prueba de dicho teorema que depende únicamente de las propiedades de la relatividad general acoplada a materia. Para ello se utiliza el formalismo asintótico, en el cual se resuelven las ecuaciones de Einstein en las distintas regiones asintóticas del espacio-tiempo.

Este enfoque permite además responder algunas interrogantes presentes en la literatura acerca de esta expresión. En particular, el hecho de que, tras ciertas cancelaciones no triviales, el resultado final no depende de las partículas sin masa salientes.

Finalmente, se extiende la expresión de este teorema al caso en que se considera radiación gravitacional incidente en el proceso de *scattering*. Esto permite resolver otra tensión en la literatura, ya que las expresiones propuestas hasta ahora no respetaban la invarianza bajo inversiones en el tiempo.

Palabras claves:
Ondas gravitacionales, Scattering, Teoremas soft, Simetrías asintóticas, Relatividad general.

ABSTRACT

The gravitational radiation emitted in a generic process is governed, in the low-frequency limit, by the so-called *soft theorems*. These are universal expressions that depend only on the momenta of the objects involved, and apply both to celestial bodies in astrophysical processes and to elementary particles.

In this work we focus on the second of the terms in the aforementioned expansion, which depends logarithmically on the frequency and is therefore referred to as the *logarithmic soft theorem*. In particular, we present a proof of this theorem that relies only on the general properties of general relativity coupled to matter. To this end, we employ the asymptotic formalism, in which Einstein's equations are solved in the different asymptotic regions of spacetime.

This approach also allows us to address certain questions raised in the literature regarding this expression. In particular, the fact that, after certain non-trivial cancellations, the final result does not depend on the outgoing massless particles.

Finally, the expression of this theorem is extended to the case in which incoming gravitational radiation is included in the scattering process. This allows us to resolve another tension present in the literature, the fact that the previously proposed expressions did not respect invariance under time reversal.

Keywords:
Gravitational waves, Scattering, Soft theorems, Asymptotic symmetries, General relativity.

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Capítulo 1

Introducción

A grandes distancias, la física de nuestro universo está dominada por la interacción gravitacional. Esta interacción describe fenómenos conocidos desde la antigüedad, como el movimiento de los planetas, así como otros cuya existencia fue teorizada mucho más recientemente y confirmada sólo en las últimas décadas, entre ellos los agujeros negros y —particularmente relevante para este trabajo— las ondas gravitacionales.

De manera análoga a como una carga acelerada emite radiación electromagnética, una masa acelerada —y, en general, cualquier distribución de energía en movimiento no uniforme— produce perturbaciones que se propagan en el campo gravitatorio en forma de ondas gravitacionales. Así como la luz emitida por un objeto contiene información sobre sus propiedades físicas, como su composición química o su temperatura, las ondas gravitacionales transportan información acerca de los sistemas astrofísicos que las generan.

Uno de los objetivos centrales de la física de ondas gravitacionales es precisamente establecer la relación entre la radiación observada y las fuentes que la producen. De este modo, las ondas gravitacionales constituyen una nueva ventana a través de la cual observar el universo.

1.1. Scattering Gravitacional

El desarrollo de esta tesis se enmarca en un programa más amplio que busca entender la emisión de ondas gravitacionales producidas en un proceso de *scattering*, con particular énfasis en la universalidad que se observa en el límite de baja frecuencia.

Entendemos por scattering gravitacional a un proceso en el cual un conjunto de objetos que interactúan únicamente gravitacionalmente se dispersa, emitiendo en el proceso ondas gravitacionales.

A primera vista, puede resultar difícil encontrar sistemas físicos reales que satisfagan estrictamente esta definición, ya que en general pueden intervenir otros tipos de interacciones —como interacciones nucleares o fuerzas de contacto— además de la gravitacional. Sin embargo, es posible hacer la siguiente observación. Supongamos que existe una región espacio-temporal de tamaño característico R fuera de la cual las interacciones entre los objetos son exclusivamente gravitacionales. En ese caso, las ondas gravitacionales emitidas con longitud de onda mucho mayor que R estarán determinadas únicamente por la interacción gravitacional. En otras palabras, la radiación de muy baja frecuencia se vuelve insensible a la física de cortas distancias.

Esta propiedad permite extender la noción de scattering gravitacional a procesos más generales, a costa de perder información sobre la radiación de alta frecuencia, que sí depende de los detalles microscópicos de las interacciones involucradas. Otras generalizaciones que se considerarán son la posibilidad de que la cantidad de objetos involucrados cambie -por ejemplo debido a colisiones o explosiones-, y que los objetos no tengan masa. Esto último se utiliza para modelar radiación, por ejemplo electromagnética.

Algunas motivaciones para estudiar estos procesos son las siguientes.

- **Scattering astrofísicos:** Si bien no son la mayoría, una importante cantidad de sistemas astrofísicos se encuentran realizados trayectorias no ligadas, de manera que su dinámica se puede modelar como un scattering
- **Sistemas ligados:** Aunque no es usual describir sistemas binarios en estos términos, también es posible incorporarlos dentro de este marco conceptual. En particular, un sistema binario que colisiona puede interpretarse como un proceso efectivo de scattering $2 \to 1 +$ radiación. En este caso el tamaño característico R del sistema es del orden del radio orbital —y por lo tanto una escala macroscópica—, pero aun así es posible extraer información en el régimen de muy baja frecuencia. En dichos casos este enfoque proporciona un chequeo de consistencia para cálculos desarrollados específicamente para este tipo de sistemas.
- **Cantidades conservadas y Teoremas Soft:** El hecho de que las formulas de la radiación emitida presenten una estructura universal -esto

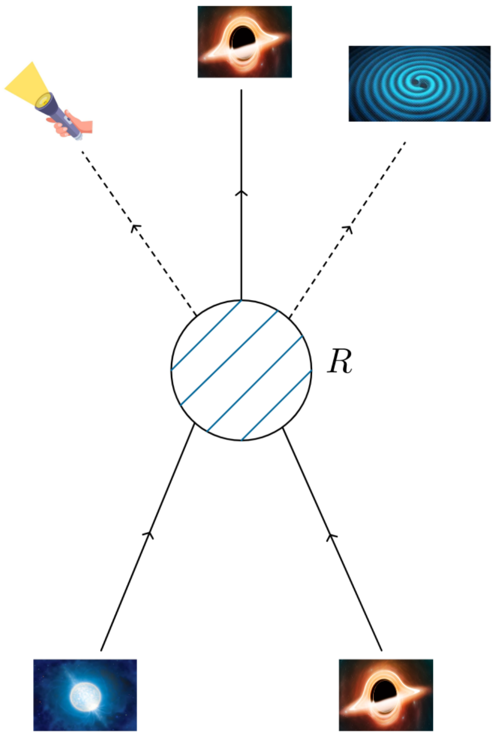

Figura 1.1: Ilustración de un proceso de scattering. Un agujero negro y una estrella de neutrones colisionan dando lugar a otro agujero negro, radiación electromagnética y radiación gravitacional.

es, expresiones concretas que dependen únicamente de los momentos de los objetos involucrados -, ha motivado recientemente la formulación de nuevas cantidades conservadas en el scattering gravitacional, cuya ley de conservación resulta equivalente a dichas fórmulas [34]. Curiosamente, la primera de estas fórmulas ya habían sido estudiadas en el contexto de la física cuántica, donde recibió el nombre de *teorema soft* [35], al ser un resultado universal que involucra radiación de baja energía o "suave".

- **Otras consideraciones formales:** Estos desarrollos también han motivado una reevaluación de trabajos históricos sobre la estructura del espacio de fases de la gravedad. En particular ha llevado a nuevas perspectivas sobre qué es una simetría, en el contexto de las teorías de gauge [7].
- **Gravedad cuántica:** Las fórmulas que describen la radiación emitida en un proceso de scattering están íntimamente relacionadas con las amplitudes de scattering, observables paradigmáticos en las teorías cuánticas de campos. Si bien aún no se dispone de una teoría completa de la gravedad cuántica, cualquier candidato consistente se debe reducir a la Relatividad General de Einstein en el límite de largas distancias. En este contexto, los teoremas soft gravitacionales proporcionan tests de consistencia no triviales para diferentes propuestas de cuantización, y han motivado nuevos enfoques conceptuales, como el programa conocido como *Celestial Holography* [29].

1.2. Relatividad general y ondas gravitacionales

La relatividad general describe la gravedad como una manifestación de la geometría del espacio-tiempo. En esta teoría, la interacción gravitatoria no se representa mediante una fuerza en el sentido tradicional, sino mediante cantidades geométricas. En particular la métrica del espacio-tiempo, que determina cómo se miden distancias y tiempos entre eventos

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{1.1}$$

donde ds^2 es la distancia infinitesimal entre dos eventos separados por dx^μ.

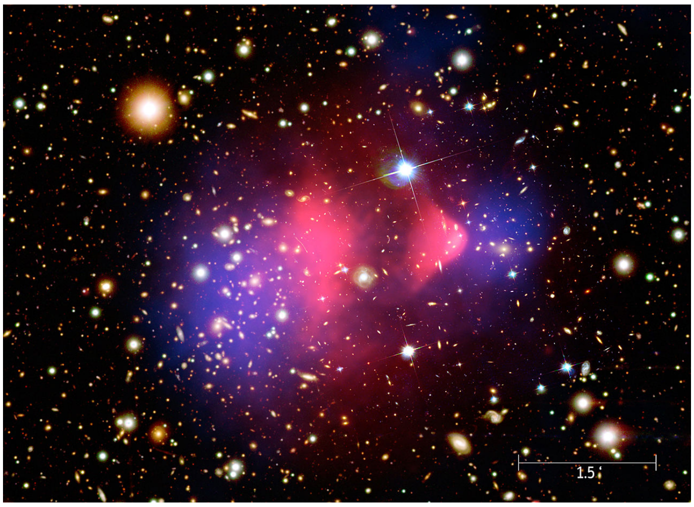

Figura 1.2: El *Bullet Cluster* como un ejemplo de scatering astrofísico. Imagen del observatorio de rayos X Chandra (NASA).

La relación entre la geometría del espacio-tiempo y la materia está dada por las ecuaciones de campo de Einstein

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.2}$$

donde $G_{\mu\nu}$ es el tensor de Einstein construido a partir de la métrica y hasta derivadas segundas, Λ es la constante cosmológica que será fijada a cero en el resto del trabajo, G la constante de Newton, y $T_{\mu\nu}$ es el tensor estrés-energía, que describe de la materia y energía presente en el espacio-tiempo. Un ejemplo reelevante es el caso de una partícula puntual, donde este tensor es una distribución con soporte en la trayectoria de dicha partículas.

$$T^{\mu\nu}(x) = \int \delta^{(4)}\Big(x - X(s)\Big) \frac{dX^\mu}{ds} \frac{dX^\nu}{ds} ds, \tag{1.3}$$

donde X^μ es la trayectoria parametrizada por s. A su vez, el movimiento de la partícula sobre el campo gravitatorio -en ausencia de otras fuerzas -está dado por la ecuación de geodésicas

$$\frac{d^2X^\mu}{ds^2} = -\Gamma^\mu_{\nu\sigma} \frac{dX^\nu}{ds} \frac{dX^\sigma}{ds}, \tag{1.4}$$

donde Γ son los símbolos de Christoffel, construidos a partir de derivadas primeras de la métrica, y su inversa. Las geodésicas representan las trayectorias "más rectas posibles" en un espacio-tiempo curvo y generalizan el concepto de movimiento inercial de la física newtoniana.

Estas ecuaciones (1.2,1.4) expresan la idea central de la teoría: la materia curva el espacio-tiempo y esta curvatura determina el movimiento de la materia.

Una propiedad importante de la teoría es la invarianza bajo transformaciones generales de coordenadas o *difeomorfismos*. Esto significa que las leyes físicas descritas por la teoría no dependen de la elección particular de coordenadas utilizada para describir el espacio-tiempo.

En la práctica, esta libertad permite imponer ciertas condiciones sobre las coordenadas —o equivalentemente sobre la métrica— con el fin de simplificar las ecuaciones. Estas elecciones suelen estar guiadas por las simetrías o características físicas del sistema que se desea estudiar. Por ejemplo, al buscar soluciones que describan configuraciones estáticas, es natural utilizar coordenadas en las cuales la métrica no dependa del tiempo $\partial_t g_{\mu\nu} = 0$.

En un espacio-tiempo de cuatro dimensiones es posible imponer al menos cuatro condiciones independientes, una asociada a cada coordenada. Estas restricciones se conocen habitualmente como condiciones de gauge. Aunque diferentes elecciones de gauge pueden simplificar los cálculos de maneras distintas, las cantidades físicas observables deben ser independientes de dicha elección.

Poco después de su presentación, fue observado que las ecuaciones de campo pueden presentar soluciones ondulatorias. Estas son soluciones dinámicas en las que perturbaciones de la métrica se propagan a la velocidad de la luz. Dichas perturbaciones se producen debido a la aceleración de distribuciones de masa y energía, y en particular, la propia gravedad puede radiar.

Durante décadas estas ondas constituyeron una predicción puramente teórica de la relatividad general. Sin embargo, evidencias indirectas de su existencia fueron obtenidas en el estudio de sistemas binarios compactos, en los cuales la pérdida de energía orbital es consistente con la emisión de radiación gravitacional [22]. La confirmación directa llegó en 2015 con la primera detección realizada por el observatorio LIGO (Laser Interferometer Gravitational-Wave Observatory). En ese experimento se observó la señal producida por la fusión de dos agujeros negros [1]. Desde entonces, se han incorporado nuevos observatorios y se han reportado cientos de detecciones adicionales [2].

1.3. Teoremas soft clásicos

La radiación gravitacional se codifica en el *campo de radiación* h, que se relaciona con la métrica de la siguiente manera:

$$g_{\mu\nu}(u,r,\hat{n}) = \eta_{\mu\nu} + \frac{1}{r}h_{\mu\nu}(u,\hat{n}) + \cdots\,, \tag{1.5}$$

donde r es una coordenada radial apropiadamente definida, $u = t - r$ es el tiempo retardado, $\hat{n}(\theta,\phi)$ la dirección de observación, η es la métrica de Minkowski, y $\cdots$ representa términos de mayor orden en $1/r$. Intuitivamente, estos decaen demasiado rápido y su flujo de energía cuando $r \to \infty$ es cero.

Como se mencionó anteriormente, nos interesa el límite de baja frecuencia de la radiación, por lo que podemos centrarnos en la transformada de Fourier temporal de h en dicho límite. Como se discutirá en esta sección, para un proceso de scattering este toma la forma

$$\lim_{\omega\to 0} \tilde{h}_{\mu\nu}(\omega,\hat{n}) = \frac{1}{\omega}\tilde{h}^{(0)}_{\mu\nu}(\hat{n}) + \log\omega\,\tilde{h}^{(\log)}_{\mu\nu}(\hat{n}) + \omega^0\tilde{h}^{(1)}_{\mu\nu}(\hat{n}) + \cdots\,, \tag{1.6}$$

donde ahora los puntos suspensivos representan términos de orden superior en ω.

Históricamente, el término dominante $\tilde{h}^{(0)}$ fue calculado en dos contextos diferentes. Por un lado, en 1965, Weinberg considera un scattering de partículas elementales en gravedad cuántica perturbativa [35]. Su resultado, reinterpretado en términos del campo de radiación la toma la forma

$$\tilde{h}^{(0)}_{\mu\nu} = -4Gi\sum_i \frac{p^i_\mu p^i_\nu}{p_i\cdot n}, \tag{1.7}$$

donde G es la constante de Newton, p^i_μ los cuadrimomentos de las particulas entrantes y salientes, y $n^\mu = (1,\hat{n})$ el cuadrivector nulo contruido a partir de $\hat{n}$. Notamos que la suma involucra a todas las partículas participantes en el scattering, y en particular a la radiación emitida, que en el cálculo perturbativo corresponde a *gravitones* salientes.

Por otro lado en 1971 Zeldovich y Polnarev consideran un scattering clásico de objetos astrofísicos, en la aproximación linealizada de la Relatividad General [37]. En un trabajo posterior, publicado en 1991, Christodoulou logra incorporar los efectos no lineales de la gravedad al resultado de Zeldovich y

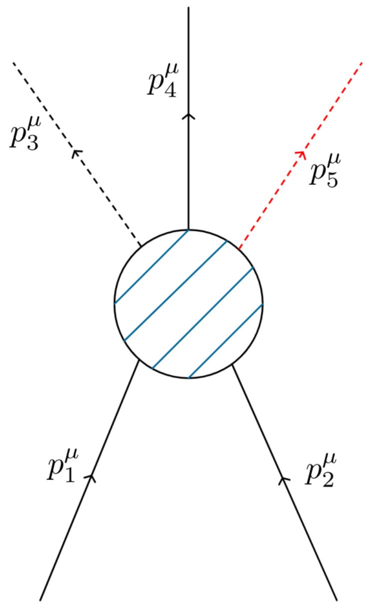

Figura 1.3: Representación diagramática del mismo proceso de scattering que en 1.1. Utilizamos líneas puntadas para denotar a los objetos sin masa y denotamos en rojo a la radiación gravitacional emitida.

Polnarev, obteniendo una expresión que coincide con la de Weinberg; en particular, recupera la contribución de la radiación saliente [15].

Resulta notable que ambos enfoques conduzcan al mismo resultado y que, además, este presente una estructura universal: depende únicamente de los cuadrimomentos de los objetos involucrados y no de otras propiedades, como el espín (clásico o cuántico). Por estos motivos a este resultado se le da el nombre de *teorema soft dominante*. Notamos también que el resultado análogo en electromagnetismo es conocido desde finales del siglo XIX, como radiación de frenado o *bremsstrahlung* [23]; siendo la diferencia principal que la teoría de Maxwell, el campo electromagnético no tiene autointeracciones, por lo que no hay una contribución de la propia radiación.

Recientemente, este tipo de análisis ha recobrado interés debido al descubrimiento de su conexión con una propiedad de la Relatividad General estudiada por Bondi, van der Burg, Metzner y Sachs en la década de 1960 [10, 30]. Estos autores analizaron las ecuaciones de Einstein en espacio-tiempos asintóticamente planos, es decir, espacios que, en un sentido apropiado, se aproximan al espacio de Minkowski en el límite $r \to \infty$. Encontraron que estos espacio-tiempos poseen, además de las simetrías usuales del espacio plano (traslaciones y transformaciones de Lorentz), un nuevo conjunto de simetrías. Estas se denominan supertraslaciones y, esquemáticamente, actúan de la forma

$$u \to u + f(\theta, \phi), \tag{1.8}$$

donde f es una función arbitraria sobre la esfera. Es decir, estas transformaciones desplazan el tiempo retardado de manera dependiente de la dirección angular.

De forma análoga a como las traslaciones poseen cantidades conservadas asociadas mediante el teorema de Noether [28] —el momento lineal-, estas transformaciones también tienen cargas asociadas, cuya conservación resulta equivalente al teorema soft dominante [34].

Esto motivó diversos avances, entre ellos extensiones de estos análisis a órdenes superiores en la expansión en frecuencia (1.6). En particular, Cachazo y Strominger [13] consideraron un análisis similar al de Weinberg, pero a un orden mayor en ω, correspondiente al orden ω^0. De esta forma obtienen evidencia de un denominado *teorema soft subdominante*

$$\tilde{h}^{(1)}_{\mu\nu} = 4G \sum_i \frac{p^i_{(\mu} J^i_{\nu)\rho} n^\rho}{p_i \cdot n}, \tag{1.9}$$

donde ahora $J^i_{\mu\nu}$ corresponde al momento angular total (orbital y de espín) de las partículas involucradas y los paréntesis indican simetrización de los índices. A diferencia del término dominante, este término es sensible a más detalles del proceso de scattering, como el espín y los parámetros de impacto, aunque la expresión resultante sigue siendo universal. Es natural esperar que, al considerar órdenes aún mayores en la expansión en frecuencia, comiencen a aparecer cada vez más detalles del proceso de dispersión.

Un aspecto clave del análisis de Cachazo y Strominger es que consideran gravedad perturbativa a orden G (aproximación *tree level*). Por lo tanto, si existieran términos con potencias mayores de G, pero a un orden menor que ω^0 en la expansión en frecuencia, estos podrían volverse dominantes en el límite de baja frecuencia. Esto es precisamente lo que ocurre: es necesario considerar correcciones logarítmicas que se vuelven dominantes respecto de $\tilde{h}^{(1)}$.

Para desarrollar una intuición sobre dichas correcciones, podemos observar el siguiente cálculo. Consideremos partículas salientes que siguen trayectorias casi libres cuando su tiempo propio $s_i \to \infty$.

$$X_i^\mu(s_i) = V_i^\mu s_i + b_i^\mu + \delta X_i^\mu(s_i), \tag{1.10}$$

donde $\{V_i^\mu\}$ son las velocidades asintóticas, $\{b_i^\mu\}$ los parámetros de impacto, y $\{\delta X_i^\mu(s_i)\}$ correcciones a las trayectorias libres. En el límite mencionado la interacción gravitacional se puede aproximar mediante una fuerza inversamente proporcional al cuadradro de la distancia (una versión relativista de la gravedad newtoniana),

$$\frac{d^2 X_i^\mu}{ds_i^2} = \sum_{i \neq j} \frac{c_{ij}^\mu}{|X_i - X_j|^2} \sim \frac{c_i^\mu}{s_i^2}, \tag{1.11}$$

donde $\{c_{ij}^\mu\}$ y $\{c_i^\mu\}$ son coeficientes que dependen de las velocidades. La solución de esta ecuación diferencial es

$$\delta X_i^\mu = c_i^\mu \log s_i, \tag{1.12}$$

por lo que las trayectorias sufren desviaciones logarítmicas. Podemos ahora calcular el momento angular orbital

$$J_i^{\mu\nu} = m_i(X_i^\mu V_i^\nu - X_i^\nu V_i^\mu) = \log s_i (c_i^\mu p_i^\nu - c_i^\nu p_i^\mu) + O(s_i^0), \tag{1.13}$$

donde vemos que este adquiere un término logarítmico, que además, es divergente para $s_i \to \infty$. Reemplazando esta expresión en el teorema soft subdominante (1.9), e identificando heurísticamente $s_i \to \omega^{-1}$, obtenemos una expresión para la corrección logarítmica

$$\tilde{h}_{\mu\nu}^{(\log)} = -4G \sum_i \frac{p_{(\mu}^i J_{\nu)\rho}^{i(div)} n^\rho}{p_i \cdot n}, \tag{1.14}$$

donde introducimos el *momento angular divergente*

$$J_i^{\mu\nu(div)} := c_i^\mu p_i^\nu - c_i^\nu p_i^\mu. \tag{1.15}$$

Finalmente, notamos que la expresión (1.14) es de orden $\log \omega$ por lo que resulta subdominante respecto al término ω^{-1}, pero dominante respecto al ω^0. Si bien no es evidente a primera vista, esta expresión es de órden G^2, puesto que los coeficientes c^μ tienen una potencia de G al provenir de la interacción gravitatoria.

En una serie de artículos [26, 27, 32, 25, 31] Laddha, Saha, Sahoo, y Sen mostraron mediante un análisis en gravedad perturbativa a là Weinberg, pero ahora a orden G^2 (*1-loop*), que esta es efectivamente la estructura y calcularon explícitamente este término, proporcionando evidencia de un *teorema soft logarítmico*. Presentaremos a continuación la expresión.

$$\begin{aligned}\tilde{h}^{(\log)}_{\mu\nu} =& 4G^2 \sum_i \frac{p^i_\mu n^\rho}{p_i \cdot n} \sum_{\substack{i,j\in \\ in/out \\ i\neq j}} \frac{\Big(2(p_i\cdot p_j)^3 - 3(p_i\cdot p_j) m_i^2 m_j^2\Big)}{\Big((p_i\cdot p_j)^2 - m_i^2 m_j^2\Big)^{3/2}} (p^i_\rho p^j_\nu - p^j_\rho p^i_\nu) \\ &+ 8G^2 \sum_i p_i \cdot n \sum_{j\in in} \frac{p^j_\mu p^j_\nu}{p_j \cdot n} + \tilde{h}^{(cuant)}_{\mu\nu},\end{aligned} \tag{1.16}$$

donde $m_i^2 = -p_i \cdot p_i$. El primer término contiene una suma sobre pares de partículas diferentes que pueden ser ambas entrantes o ambas salientes, mientras que el segundo término contiene una suma sobre partículas entrantes únicamente. Notamos que al igual que $\tilde{h}^{(0)}$ la expresión depende únicamente de los momentos de las partículas involucradas. $\tilde{h}^{(cuant)}_{\mu\nu}$ corresponde a términos presentes el cálculo cuántico, que se anulan en el límite clásico, y no serán considerados en el resto del trabajo.

Una observación que puede realizarse sobre esta expresión es la siguiente: a diferencia del teorema soft dominante (1.7), en este término las partículas entrantes y salientes no aparecen en pie de igualdad, debido a la estructura de la segunda suma. Esto sugiere que la expresión no es invariante bajo el intercambio de los estados inicial y final, o de forma equivalente, que no respeta la simetría bajo inversiones temporales $t \to -t$. Este hecho resulta sorprendente, puesto que dicha transformación es una invarianza de la relatividad general.

Otra observación fue realizada en [33]. Los autores notan que, si se escoge que algunas partículas salientes no tengan masa, se producen cancelaciones tales que la expresión del teorema soft logarítmico no depende de ellas. Esto podría resultar útil desde el punto de vista observacional, ya que evita la necesidad de modelar explícitamente la radiación emitida en términos de partículas sin masa. Sin embargo, no se da un principio o mecanismo que explique dicha cancelación.

El objetivo central de la tesis es adquirir una comprensión más profunda del teorema soft logarítmico. En particular, se busca entender cómo surge esta expresión desde el punto de vista de la relatividad general clásica, así como profundizar en las interrogantes mencionadas anteriormente.

Implicaciones observacionales

Las implicaciones observacionales de los teoremas soft se relacionan con la determinación de las componentes de baja frecuencia de la onda gravitacional emitida. Sin embargo, en la actualidad esta idea enfrenta ciertos obstáculos. Por un lado, los detectores de ondas gravitacionales actuales no son sensibles a frecuencias muy bajas, debido tanto a limitaciones de diseño como al ruido sísmico presente en la superficie terrestre. Por otro lado, los procesos de scattering, en los cuales estos términos son más relevantes, son en general menos energéticos que los procesos ligados, lo que dificulta aún más su detección.

El primer problema podría mitigarse mediante la incorporación de detectores espaciales, como LISA [21], los cuales al no estar ligados a la superficie terrestre evitarían la problemática mencionada. El segundo requiere identificar procesos astrofísicos particulares en los cuales la señal sea lo suficientemente intensa como para ingresar en el rango de sensibilidad de los detectores actuales o futuros.

En cuanto al término logarítmico, en [25] los autores estiman su orden de magnitud para distintos procesos e identifican al menos dos posibles candidatos. Uno de ellos son las *hypervelocity stars*, que corresponden a sistemas binarios en los cuales una de las estrellas es capturada por un agujero negro supermasivo, mientras que la otra es expulsada a gran velocidad. Otro candidato son las *core-collapse supernovae*, en las que la estrella de neutrones resultante del colapso puede, en algunos casos, adquirir velocidades muy elevadas. En ambos escenarios, para que la señal sea detectable sería necesario que el evento ocurra dentro de nuestra galaxia, lo cual limita considerablemente el número de eventos potencialmente observables.

1.4. Organización de le tesis

La organización de la tesis es la siguiente. En el capítulo 2 se expone, de manera no técnica, el formalismo asintótico de la gravedad, que luego es utilizado durante el resto del trabajo. Los capítulos 3 y 4 conforman el cuerpo de la tesis y corresponden a dos artículos, [12] y [11], junto con su respectiva descripción. En el capítulo 5 se resumen los resultados obtenidos y se discuten posibles direcciones futuras de investigación. Finalmente se presenta la declaración de autoría correspondiente a los artículos.

Capítulo 2

Gravedad a largas distancias

Para poder analizar de manera no perturbativa los teoremas soft, haremos uso del formalismo asintótico de la gravedad. Este formalismo fue ampliamente desarrollado en la década de 1960, ya que permite estudiar las ondas gravitacionales sin necesidad de considerar pequeñas perturbaciones alrededor del espacio-tiempo plano. Sin embargo, gran parte de su éxito corresponde a cuestiones formales de la teoría. El reciente interés en los teoremas soft, por su parte renovó el interés en el formalismo asintótico, ahora no solo como una herramienta conceptual, sino también como una herramienta de cálculo, complementaria a otros enfoques.

2.1. Estructura astintótica del espacio de Minkowski

Consideremos el diagrama de Minkowski en una dimensión espacial y una temporal. Si analizamos los distintos límites hacia el infinito, encontramos varias posibilidades.

Podemos tomar $t \to \infty$ permaneciendo dentro del cono de luz futuro ($t > x$); esto corresponde al infinito temporal futuro ($\mathcal{H}^+$), que es la región donde terminan las trayectorias de partículas con masa (geodésicas de tipo temporal). De manera análoga se define el infinito temporal pasado ($\mathcal{H}^-$).

Un rayo de luz (o una partícula sin masa) se mueve a lo largo de trayectorias $t = \pm x$ (geodésicas nulas), y cuando $t \to \infty$ alcanza la región denominada infinito nulo futuro ($\mathcal{I}^+$). De forma análoga se define el infinito nulo pasado ($\mathcal{I}^-$), como la región donde comienzan las geodésicas nulas.

Finalmente, otra posibilidad es tomar el límite $x \to \infty$ fuera del cono de luz ($|x| > t$); esta región se denomina infinito espacial ($\mathcal{H}^0$). Si bien en esta región no existen partículas u objetos físicos, puede interpretarse como el lugar donde comienzan o terminan geodésicas de tipo espacial. Su análisis es relevante, por ejemplo, en la definición de superficies de tiempo constante (*superficies de Cauchy*), las cuales pueden utilizarse para formular leyes de conservación.

Una coordinatización útil para estudiar $\mathcal{H}^+$ consiste en considerar la coordenada temporal $\tau = \sqrt{t^2 - r^2}$, la cual folia el cono de luz futuro en hiperboloides (figura 2.1). Identificamos $\mathcal{H}^+$ con el límite $\tau \to \infty$. En este hiperboloide podemos escoger tres coordenadas espaciales x^a que representarán el espacio de velocidades finales de las geodésicas masivas. Por ejemplo, una posible elección son los ángulos esféricos $(\theta,\, \phi)$ y una coordenada radial

$$\rho = \frac{r}{\sqrt{t^2 - r^2}}. \tag{2.1}$$

Esta última se puede interpretar como el módulo de la tres-velocidad de la partícula, por lo que la velocidad de una partícula de masa despreciable corresponderá a $\rho \to \infty$. Para $\mathcal{H}^-$ es posible hacer una descripción análoga, pero con τ tomando ahora valores negativos, y foliando el cono de luz pasado.

En cuanto al infinito nulo, tomaremos el límite $r \to \infty$ manteniendo el tiempo retardado u constante. Rayos de luz emitidos en el pasado lejano corresponden a $u \to -\infty$, y los rayos de luz emitidos en el futuro lejano corresponden a $u \to \infty$ (figura 2.2). A esta coordenada se le añaden nuevamente los ángulos esféricos $(\theta,\, \phi)$ que indican la dirección de dichos rayos. Es posible pensar en esta región como el cilindro $\mathbb{R} \times S^2$, donde el primer factor corresponde a u y el segundo es la esfera bidimensional. Nuevamente, $\mathcal{I}^-$ se puede entender de la misma forma, con v tomando el papel de u.

Finalmente, $\mathcal{H}^0$ se puede pensar como el hiperboloide que resulta de tomar la coordenada $\tilde{\rho} = \sqrt{r^2 - t^2}$ a infinito. Se pueden utilizar nuevamente coordenadas angulares, así como la coordenada temporal

$$\tilde{\tau} = \frac{t}{\sqrt{r^2 - t^2}}. \tag{2.2}$$

Si bien esto resulta similar a $\mathcal{H}^+$, una diferencia clave es que esta variedad posee una dirección temporal y dos espaciales [1]. Estas regiones deben tratarse

[1] Otra diferencia relevante es que $\mathcal{H}^0$ posee topología no trivial, como sugiere la figura 2.1

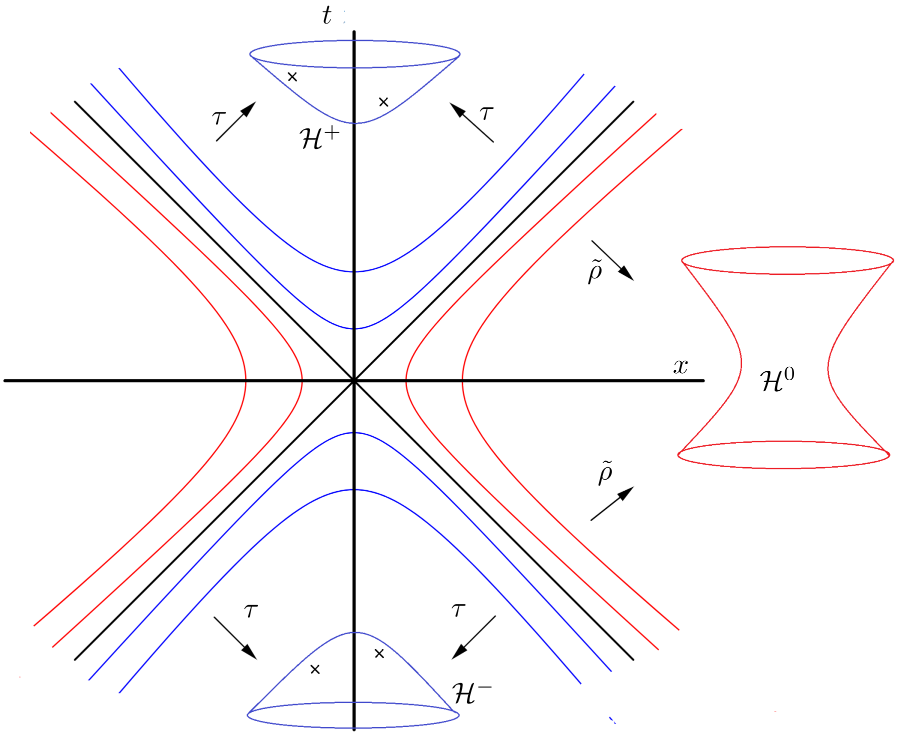

Figura 2.1: Diagrama de Minkowski 1+1 dimensional. En azul se muestra la foliación hiperbólica utilizada dentro del cono de luz futuro/pasado. $\mathcal{H}^{\pm}$ corresponde al hiperboloide euclídeo que se obtiene en el límite $\tau \to \pm\infty$, donde las partículas salientes/entrantes corresponden a puntos determinados. En rojo se muestra la foliación hiperbólica utilizada fuera de los conos luz. $\mathcal{H}^0$ corresponde al hiperboloide lorentziano que se obtiene al tomar $\tilde{\rho} \to \infty$. Estos espacios se muestran de manera pictórica en su versión en una dimensión más.

como parte de un mismo espacio-tiempo, y por ende se deben imponer condiciones de consistencia en las regiones en las que se unen. Esto da lugar a un esquema de la forma de la figura 2.3.

En este trabajo consideraremos espacios asintóticamente planos, es decir, espacios que no son planos debido a la gravedad, pero que a grandes distancias se comportan como el espacio de Minkowski. De esta manera, la noción de infinito introducida resulta relevante incluso en situaciones en las que la curvatura es muy grande en el interior.

Los espacios asintóticamente planos constituyen una muy buena aproximación para describir sistemas astrofísicos aislados. Sin embargo, hasta hace relativamente poco tiempo no estaba claro que permitieran describir procesos de scattering. Su incorporación implicó una extensión de los formalismos in-

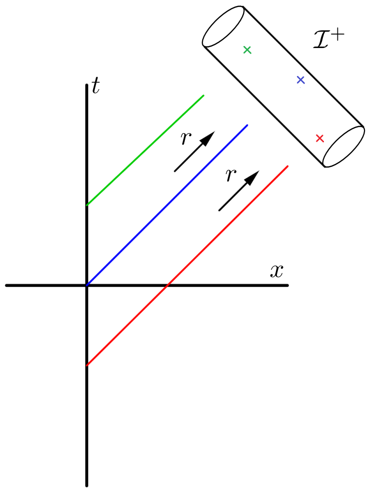

Figura 2.2: Diagrama de Minkowski 1+1 dimensional. En diferentes colores se muestras tres rayos de luz emitidos en distintos tiempos. En el límite $r \to \infty$, a u constante llegamos a la variedad denominada como $\mathcal{I}^+$. En esta representamos las direcciones angular como un círculo, de forma que cada rayo de luz llega a un punto determinado.

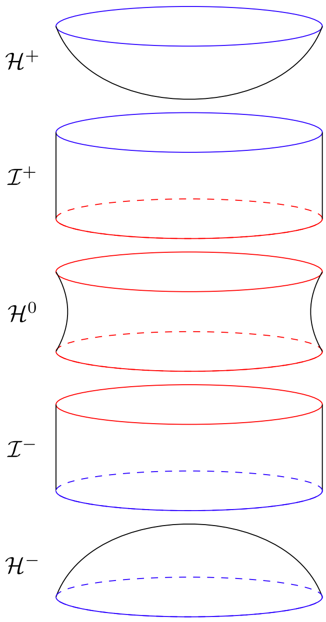

Figura 2.3: Representación esquemática de las diferentes regiones asintóticas. En azul, los bordes entre el infinito temporal futuro/pasado y el infinito nulo futuro/pasado. En rojo, los bordes entre el infinito espacial y ambos infinitos nulos. En todas estas regiones se deben imponer condiciones de continuidad sobre la métrica, creando un diagrama “puzzle”.

troducidos históricamente, que describimos en las secciones siguientes.

Como comentario final, vale la pena mencionar dos clases de espacio-tiempo que **no** son asintóticamente planos: los espacios-tiempos de *de Sitter* y *anti-de Sitter*. La presencia de una constante cosmológica en estos casos modifica la estructura asintótica de las soluciones de las ecuaciones de Einstein, por lo que se requiere un tratamiento diferente de la estructura asintótica en cada caso [17].

Teniendo en cuenta que la constante cosmológica en nuestro universo es positiva, uno podría cuestionarse la relevancia de los espacios asintóticamente planos. En la práctica, esto sugiere que el infinito relevante para estos análisis no se encuentra a distancias arbitrariamente grandes, sino a escalas mucho mayores que las del sistema considerado (por ejemplo, los parámetros de impacto en un proceso de scattering), pero todavía menores que la escala cosmológica de expansión (por ejemplo $L \sim \Lambda^{-1/2}$ para el caso del espacio-tiempo de de Sitter).

2.2. Infinito nulo: Métrica de Bondi

De manera resumida, el formalismo asintótico consiste en elegir, en cada una de las regiones mencionadas anteriormente, un conjunto de condiciones sobre las coordenadas (condiciones de gauge), así como condiciones de decaimiento para la métrica y los campos de materia. Luego se estudian las soluciones de las ecuaciones de Einstein compatibles con estas condiciones, y finalmente se impone consistencia entre las soluciones en las fronteras entre las distintas regiones.

Bondi, van der Burg y Metzner consideraron espacios asintóticamente planos en el infinito nulo. Para ello introducen un sistema de coordenadas definido en términos de las coordenadas esféricas estándar,

$$u = t - r + \cdots\ ,\ \ r,\ \ x^A = (\theta, \phi) + \cdots\ , \tag{2.3}$$

donde los puntos suspensivos indican términos subdominantes en r, que son determinados por las siguientes condiciones de gauge sobre la métrica

$$g_{rr} = g_{rA} = 0, \qquad \det g_{AB} = r^4 \det q_{AB}, \tag{2.4}$$

donde q_{AB} es la métrica de la esfera unidad. Si bien estas condiciones pueden motivarse geométricamente, es importante destacar que estas son arbitrarias. Denominaremos a estas coordenadas como *coordenadas de Bondi*.

Un detalle que sí vale la pena remarcar es el siguiente: si invertimos una métrica genérica que satisfaga (2.4), es posible ver que $g^{uu} = 0$. Esto implica que el covector tangente a las curvas $u = cte$, l_μ es nulo.

$$l \cdot l = g^{\mu\nu}\nabla_\mu u \nabla_\nu u = g^{uu} = 0, \tag{2.5}$$

por lo que estas curvas son geodésicas nulas. En consecuencia, al representar la trayectoria de una partícula sin masa en coordenadas de Bondi, esta no presenta desviaciones logarítmicas.

Posteriormente impondremos que la métrica se aproxime asintóticamente a la métrica de Minkowski de la siguiente manera:

$$\begin{aligned} ds^2 = &\Big(-1 + O(1/r)\Big)du^2 - 2\Big(1 + O(1/r)\Big)dudr \\ &+ O(r^0)dudx^A + \Big(r^2 q_{AB} + O(r)\Big)dx^A dx^B, \end{aligned} \tag{2.6}$$

donde los términos explícitos corresponden a la métrica de Minkowski en coordenadas de Bondi.

Por simplicidad asumiremos que no hay campos de materia sin masa presentes [1]. En ese caso podemos estudiar las ecuaciones de Einstein en vacío $G_{\mu\nu} = 0$, en torno al infinito nulo en coordenadas de Bondi. Imponiendo ya las ecuaciones $G_{ru} = G_{rr} = G_{rA} = G_{AB} = 0$ obtenemos la métrica de Bondi

$$\begin{aligned} ds^2 = &\Big(-1 + \frac{2G\mathcal{M}}{r} + O(1/r^2)\Big)du^2 - 2\Big(1 + O(1/r^2)\Big)dudr \\ &+ \Big(\nabla^B C_{AB} + \frac{2G}{3r}\mathcal{N}_A + O(1/r^2)\Big)dudx^A \\ &+ \Big(r^2 q_{AB} + rC_{AB} + O(r^0)\Big)dx^A dx^B, \end{aligned} \tag{2.7}$$

donde ahora ∇ es la derivada covariante compatible con la métrica q_{AB}. Notamos que esta depende de tres funciones de (u, x^A), a las cuales se les

[1]Esta suposición no se considera en los artículos de la tesis.

da el nombre de *aspecto de masa de Bondi* $\mathcal{M}$, *aspecto de momento angular de Bondi* $\mathcal{N}_A$, y *cizalla de Bondi* C_{AB}. Conceptualmente, estas cantidades corresponden a la densidad angular de energía y a la densidad angular de momento angular del espacio-tiempo para un valor dado de u, así como a la cizalla de una congruencia de geodésicas nulas. La condición de gauge sobre el determinante impone que esta última cantidad tenga traza cero.

$$q^{AB}C_{AB} =: C^A_A = 0. \tag{2.8}$$

Recordando que un tensor simétrico de traza cero en la esfera tiene dos componentes independientes, así como la interpretación geométrica de la cizalla (deformación que preserva el área), identificamos estas dos componentes con las dos polarizaciones de las ondas gravitacionales [1].

Estas tres cantidades no son independientes, sino que están relacionadas por las ecuaciones de Einstein que aún no impusimos, $G_{uu} = G_{uA} = 0$, que toman la forma

$$\partial_u \mathcal{M} = \frac{1}{4G}\nabla^A\nabla^B\partial_u C_{AB} - \frac{1}{8G}\partial_u C_{AB}\partial_u C^{AB}, \tag{2.9}$$

$$\partial_u \mathcal{N}_A = -\frac{2}{3}\nabla_A \mathcal{M} + \nabla^3 C_{AB}, \tag{2.10}$$

donde, esquemáticamente, $\nabla^3 C_{AB}$ corresponde a un operador diferencial con hasta tres derivadas en la esfera actuando sobre C_{AB}; su forma explícita no será necesaria.

Dado que ambas ecuaciones involucran derivadas primeras en u su solución está determinada salvo por el valor de los campos en cierto valor de $u = u_0$, es decir, $\mathcal{M}(u_0, x^A)$ y $\mathcal{N}_{\mathcal{A}}(u_0, x^A)$, así como C_{AB} para todo valor de u.

Ejemplos sencillos que toman la forma (2.7) son la solución de Schwarzschild, donde $\mathcal{M}$ es la masa del objeto esférico y $\mathcal{N}_A = C_{AB} = 0$, o la solución de Kerr, donde ahora $\mathcal{N}_A$ está relacionado con el momento angular del objeto. Sin embargo, puede surgir la duda de si estas soluciones pueden englobar situaciones de scattering.

A posteriori fue observado que esto es posible, pero es necesario una modificación sencilla, al menos al orden considerado; la incorporación de un término

[1] Estrictamente, es su derivada respecto al tiempo retardado la que implica la presencia de ondas gravitacionales, puesto que una cizalla constante puede interpretarse como un difeomorfismo trivial respecto al vacío.

logarítmico en r en una de las componentes de la métrica [36, 19] ,

$$g_{uA} \to g_{uA} + \frac{\log r}{r}\mathcal{N}_A^{(\log r)}. \tag{2.11}$$

Las ecuaciones de Einstein imponen que este término es independiente de u, por lo que es necesaria una nueva constante para determinar unívocamente una solución a las ecuaciones de movimiento.

Como se mencionó anteriormente, el formalismo asintótico requiere "pegar" las soluciones en las distintas regiones. Para ello es necesario especificar el comportamiento en los límites $u\pm\infty$ del campo C_{AB}. Este, mediante las ecuaciones (2.9) y (2.10), determina el comportamiento de los otros campos. Motivados por la expansión (1.6), y teniendo en cuenta las siguientes propiedades de las transformadas de Fourier

$$\lim_{u\to\infty}\int \omega^{-1}e^{i\omega u}d\omega \sim \theta(u), \quad \lim_{u\to\infty}\int \log\omega e^{i\omega u}d\omega \sim \frac{1}{u}, \tag{2.12}$$

consideraremos cizallas de la forma

$$\lim_{u\to\pm\infty} C_{AB}(u,x^A) = C_{AB}^{(0)}(x^A) + \frac{1}{u}C_{AB}^{(1)}(x^A) + \cdots . \tag{2.13}$$

Esto implica que

$$\lim_{u\to\pm\infty}\mathcal{N}_A = \log|u|\mathcal{N}_A^{(\log u)}(x^A) + \cdots . \tag{2.14}$$

Será la presencia de ambos términos logarítmicos (2.11), (2.14) la que dará cuenta de la divergencia del momento angular mencionada.

Por último, es posible realizar el mismo análisis en el infinito nulo pasado, salvo que ahora el papel de la coordenada u lo desempeña el tiempo avanzado $v = t + r$. Una sutileza que conviene mencionar es la siguiente: en procesos astrofísicos suele asumirse que no hay radiación gravitacional incidente. Esto se puede implementar en el análisis asintótico considerando simplemente cizallas constantes en el pasado

$$\partial_v C_{AB}^{(-)}(v,x^A) = 0, \tag{2.15}$$

donde $(-)$ hace referencia a que esta es la cantidad análoga a C_{AB} en el pasado.

2.3. Infinito espacial y temporal: Métrica de Beig-Schmidt

Inspirados en trabajos previos de Ashtekar y Hansen [6, 5], Beig y Schmidt consideran espacios-tiempos asintóticamente planos en el infinito espacial [8]. Para ello eligen coordenadas de la forma

$$\rho = \sqrt{r^2 - t^2} + \cdots, \quad x^a = (\tau, \theta, \phi), \tag{2.16}$$

donde x^a es una coordenada temporal, y dos coordenadas espaciales que folian el hiperboloide lorentziano unidad [1]. A su vez, impusieron las siguientes condiciones de gauge

$$g_{\rho\rho} = \left(1 + \frac{\sigma(x^a)}{\rho}\right)^2, \qquad g_{\rho a} = 0. \tag{2.17}$$

Posteriormente, asumieron un desarrollo en potencias de ρ para las componentes restantes

$$g_{ab} = \rho^2 h_{ab} + \rho h^{(1)}_{ab} + \rho^0 h^{(0)}_{ab} + \cdots, \tag{2.18}$$

donde el primer término corresponde a la métrica (fija) del hiperboloide de radio ρ, y los términos siguientes corresponden a desviaciones respecto a este espacio. Con esta hipótesis, mostraron que las ecuaciones de Einstein en vacío se convierten en una serie de ecuaciones en derivadas parciales hiperbólicas (ecuaciones de onda) para los campos σ y $h^{(n)}_{ab}$. La primera de ellas es

$$(D^2 + 3)\sigma = 0, \tag{2.19}$$

donde $D^2 = h^{ab} D_a D_b$ es el laplaciano covariante compatible con h_{ab}. A este punto la ecuación es lineal, lo cual no se cumple para todos los órdenes en ρ. Sin embargo, las ecuaciones se pueden resolver recursivamente.

Estas ecuaciones requieren datos iniciales de la forma $\{\sigma(\tau_0), h^{(n)}_{ab}(\tau_0)\}$ para determinar unívocamente una solución. Desde el punto de vista del formalismo asintótico, el rol de $\mathcal{H}^0$ es tomar una métrica de Bondi en $\mathcal{I}^-$ y conectarla con otra en $\mathcal{I}^+$ (o viceversa) mediante la evolución determinada por las ecuaciones de onda.

[1] En la sección 2.1 se denotó como $\tilde{\rho}$ y $\tilde{\tau}$ a estas coordenadas, para diferenciarlas de las utilizadas en $\mathcal{H}^+$. Omitiremos por ahora los tildes.

Al igual que con la métrica de Bondi, fue notado recientemente que la métrica de Beig-Schmidt puede generalizarse para describir procesos de scattering mediante la incorporación de términos logarítmicos en la coordenada radial [16]. Al orden que trabajará en los artículos de la tesis solo es necesario el primero de ellos que llamaremos i_{ab}. La presencia de estos términos no altera la estructura de las ecuaciones de onda, pero sí se agregan ecuaciones para los nuevos términos. La métrica de Beig-Schmidt generalizada toma la forma

$$ds^2 = \left(1 + \frac{\sigma}{\rho}\right)^2 d\rho^2 + \left(\rho^2 h_{ab} + \rho h^{(1)}_{ab} + \log \rho\, i_{ab} + \cdots\right) dx^a dx^b. \tag{2.20}$$

Compère, Gralla y Wei [18] propusieron reutilizar estas ideas en el estudio del infinito temporal, intercambiando el rol de las coordenadas ρ y τ. Las condiciones de gauge propuestas son análogas

$$g_{\tau\tau} = -\left(1 + \frac{\sigma}{\tau}\right)^2, \;\; g_{\tau a} = 0, \tag{2.21}$$

así como el ansatz

$$g_{ab} = \tau^2 h_{ab} + \tau h^{(1)}_{ab} + \log \tau\, i_{ab} + \cdots \tag{2.22}$$

con la diferencia de que ahora la métrica hiperbólica h_{ab} es espacial. Esta modificación implica que las ecuaciones obtenidas a partir de Einstein ya no son hiperbólicas, sino elípticas (ecuaciones de tipo Poisson). A su vez, en $\mathcal{H}^\pm$ consideraremos la presencia de materia masiva, lo que implica la necesidad de considerar términos de fuente en las ecuaciones. La primera de ellas es

$$(D^2 - 3)\sigma = 4\pi G \rho_m, \tag{2.23}$$

donde, de manera genérica, ρ_m corresponde a la densidad de energía de la materia. Notando la similitud con la ecuación de Poisson estándar, identificamos σ con una generalización del potencial newtoniano, en el infinito temporal.

Conceptualmente, una partícula en el infinito temporal se encuentra prácticamente libre, pero siente un potencial σ generado por el resto de las partículas. En consecuencia, su trayectoria se obtiene como una corrección a la trayectoria libre mediante la ecuación de geodésicas. Estas correcciones modifican el tensor de estrés-energía, dando lugar a nuevos términos de fuente para los campos subdominantes. En particular, la corrección logarítmica a las trayectorias mencionada anteriormente da lugar a términos de fuente para i_{ab}, por lo que este campo jugará un rol importante en el análisis asintótico del teorema soft logarítmico.

2.4. Gauge armónico: Análisis perturbativo

El análisis realizado mediante el formalismo asintótico será contrastado con el análisis perturbativo. En este último se estudian métricas de la forma

$$g_{\mu\nu} = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu} + G^2 h^{(2)}_{\mu\nu} + \cdots , \tag{2.24}$$

donde se asume que $G^n |h^{(n)}_{\mu\nu}| \ll 1$. Es decir, se consideran perturbaciones pequeñas respecto a la métrica de Minkowski. Las ecuaciones de Einstein pueden entonces expandirse en potencias de G, lo que permite resolverlas orden a orden en dicha expansión. Usualmente, se utiliza la condición de gauge armónico

$$\partial^\mu h^{(n)}_{\mu\nu} = \partial_\nu h^{(n)}, \tag{2.25}$$

donde $h = \eta^{\mu\nu} h_{\mu\nu}$. En este contexto resulta conveniente introducir la *perturbación de traza revertida*

$$e^{(n)}_{\mu\nu} := \frac{1}{2} h^{(n)}_{\mu\nu} - \frac{1}{4} h^{(n)} \eta_{\mu\nu}, \tag{2.26}$$

de modo que la ecuación de primer orden es la ecuación de ondas usual

$$\partial^2 e^{(1)}_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{2.27}$$

En la práctica, el análisis perturbativo permite calcular observables como una expansión en potencias de G, siempre que el campo gravitatorio sea débil. Una ventaja de este enfoque es que, al definirse un único sistema de coordenadas en todo el espacio-tiempo, basta con resolver un único conjunto de ecuaciones.

Por otro lado, el formalismo asintótico permite calcular observables mediante una expansión en el tiempo (o en la distancia), sin asumir que la gravedad sea débil. En particular no es necesario expandir en la constante de acoplamiento, por lo que los resultados obtenidos son “exactos”, en ese sentido. Sin embargo, este enfoque utiliza distintos sistemas de coordenadas y requiere trabajar con varios conjuntos de ecuaciones en diferentes regiones.

En los regímenes en que ambas aproximaciones son válidas, los dos métodos deben producir resultados equivalentes, siempre que se comparen cantidades invariantes de gauge. No obstante, y como se verá en el artículo presentado en la sección 3, los diferentes enfoques iluminan diferentes aspectos del cálculo, que a su vez tienen consecuencias en la comprensión de los resultados.

Capítulo 3

Invarianza bajo traslaciones logarítmicas

En este trabajo se da respuesta a una de las interrogantes que surgen en relación con el teorema soft logarítmico. En [33] los autores notan que la expresión (1.14), tras una serie de cancelaciones no triviales, puede reescribirse únicamente en términos de los momentos de los objetos entrantes (en su caso, todos masivos) y de los objetos masivos salientes. En otras palabras, la contribución proveniente de la radiación saliente se cancela completamente. Es natural preguntarse si existe algún principio físico detrás de esta cancelación.

Para explicar este resultado se realiza la siguiente observación. En el formalismo asintótico, si bien se imponen ciertas condiciones sobre las coordenadas, estas no quedan completamente determinadas, ya que existen transformaciones residuales que preservan dichas condiciones. Entre estas transformaciones se encuentran las llamadas *traslaciones logarítmicas*, que esquemáticamente actúan de la forma

$$X^\mu \to X^\mu + L^\mu \log R, \tag{3.1}$$

donde L^μ es un vector constante y R puede ser la coordenada temporal en el caso de $\mathcal{H}^\pm$, nula en el caso de $\mathcal{I}^\pm$, o radial en el caso de $\mathcal{H}^0$. Si consideramos trayectorias de la forma

$$X_i^\mu(s_i) = V_i^\mu s_i + c_i^\mu \log s_i + \cdots, \tag{3.2}$$

entonces las traslaciones logarítmicas actúan de la forma $c_i^\mu \to c_i^\mu + L^\mu$. Por otra parte, el análisis de las ecuaciones muestra que el vector c_i^μ es el mismo para todas las partículas sin masa. Esto permite fijarlo igual a cero mediante el uso de una traslación logarítmica. Como consecuencia, el momento angular divergente $c^{[\mu}V^{\nu]}$, y por lo tanto el teorema soft logarítmico, no reciben contribuciones de las partículas salientes sin masa.

Por otro lado, también se presenta una extensión de la fórmula del teorema soft que incluye la contribución de la radiación incidente. Si bien este caso no es particularmente relevante desde el punto de vista astrofísico, sí lo es desde el punto de vista teórico, ya que permite tratar en pie de igualdad a partículas masivas y no masivas.

Finalmente, asumiendo una expansión del campo de radiación en potencias de la frecuencia y logaritmos de la misma, se impone la misma condición de invariancia a todos los órdenes. Esto permite encontrar una relación de recurrencia consistente para algunos de estos términos, denominados logaritmos dominantes, que son de la forma

$$\omega^{n-1} \log \omega^n, \tag{3.3}$$

y se ha conjeturado que también presentan propiedades universales [24, 4]. Notamos que el primero de ellos ($n = 0$) es el término de Weinberg 1.7, donde la invarianza se satisface trivialmente, y el segundo ($n = 1$) corresponde al término logarítmico analizado en el artículo, cuya invariancia es demostrada explícitamente.

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Log translation invariance of log soft gravitational radiation

Gianni Boschetti[a] **and Miguel Campiglia**[b]

[a]*Instituto de Física, Facultad de Ingeniería, Universidad de la República, Julio Herrera y Reissig 565, Montevideo, Uruguay*
[b]*Instituto de Física, Facultad de Ciencias, Universidad de la República, Igúa 4225, Montevideo, Uruguay*

E-mail: gboschetti@fing.edu.uy, miguel.campiglia@fcien.edu.uy

Abstract: The gravitational radiation emitted during a classical scattering process is known to exhibit two universal logarithmic terms in its soft frequency expansion. We show that these terms can be written in a way that makes the action of *logarithmic translations* manifest. Invariance under log translations naturally explains a puzzling cancellation in the contribution from outgoing massless particles and leads to a recurrence relation for expected higher-order universal log soft terms.

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Contents

1 Introduction

In a series of remarkable papers [1–6] Laddha, Saha, Sahoo, and Sen identified universal logarithmic terms in the low-frequency expansion of gravitational radiation, along with the associated early/late-time tails; see [7] for a review. The universal nature of the first log term can be understood in terms of superrotations [8–10], within the broader setting of an "infrared triangle" that connects asymptotic symmetries with soft theorems and memory effects [11].

In this paper, we revisit these *classical soft theorems* from a different perspective. Our motivation lies in the seemingly "miraculous" cancellation of the outgoing massless contributions to the log soft factors [3, 6]. This cancellation has so far been observed in the first two known universal log terms but is believed to extend to conjectured higher-order terms [5–7, 12]. Here we provide a simple explanation of this cancellation by making explicit a gauge redundancy in the asymptotic description of the gravitational field known as logarithmic translations [13–15].[1] In a nutshell, the contribution from outgoing massless particles can be removed altogether by a log translation. But since the soft factors are invariant under such transformations, this contribution must vanish to begin with.

[1]These are distinct from log supertranslations, recently proposed as part of the symmetry algebra of asymptotically flat gravity [16] (see also [17, 18]). Although we do not make use of this extension, our perspective is consistent with [16] in treating log translations as pure gauge.

Logarithmic translations were first introduced in the context of spatial infinity, but they can also arise at timelike and even null infinity, as we discuss in the next section. Our analysis draws from Ashtekar's geometric treatment of spatial infinity [19–21], exported to the case of timelike infinity, following the spirit of the "asymptotic framework" of Compère, Gralla, and Wei [22] (see also [23–28]). The ideas presented here set the stage for a fully asymptotic derivation of the log soft theorems, an approach we develop in [29].

The organization of the paper is as follows. In section 2 we review the concept of log translations and introduce the notion of log deviation vector, which captures the leading deviation of asymptotic geodesics from straight lines. In section 3 we present the description of timelike infinity, and discuss how log translations arise in that context. We further argue that there is a single "global" log translation group that simultaneously acts on the asymptotic future and past. Given these preliminaries, in section 4 we show how the two known log soft theorems can be expressed in terms of the log deviation vector in a log translation invariant way. In section 5 we explore the consequences of log translation invariance on the putative higher order log soft theorems. We conclude in section 6 with a brief discussion of our results and future perspectives. The paper is complemented with two appendices containing supporting details for the main discussion.

2 Log translations and log deviation vector

Any notion of asymptotic flatness requires that the spacetime metric becomes flat as some distance parameter R goes to infinity. While the precise meaning of R varies with the setting — in particular, with the signature of the direction along which infinity is approached — the leading deviation from the flat metric typically scales as $1/R$,

$$g_{\mu\nu} \overset{R\to\infty}{=} \eta_{\mu\nu} + O(1/R), \tag{2.1}$$

where $g_{\mu\nu}$ is the spacetime metric in appropriate asymptotic Cartesian coordinates, and $\eta_{\mu\nu}$ the Minkowski metric. Depending on the gauge condition being used, there may exist a residual gauge freedom affecting the $1/R$ term in (2.1) due to *logarithmic translations* [13, 14],

$$\xi^\mu_L \overset{R\to\infty}{=} \log R\, L^\mu + O(R^0), \tag{2.2}$$

where L^μ are constant vectors. A well-studied instance where this occurs is in Beig-Schmidt gauge at spatial [15] and timelike [22, 27] infinity. By contrast, radiative gauges at null infinity — such as Bondi [30, 31] or Newman-Unti [32]—do not allow for log translations. A second relevant example where log translations are frozen is harmonic gauge (see appendix A.2).

We will consider the standard situation in which the limit in (2.1) is approached along asymptotic geodesics,

$$X^\mu(s) \overset{s\to\infty}{=} sV^\mu + \log s\, c^\mu + \cdots\,, \tag{2.3}$$

where V^μ is the geodesic asymptotic direction and s an affine parameter that, to leading order, is proportional to R. The logarithmic term in (2.3) arises from the $1/R$ piece in the metric (2.1), which leads to non-trivial $O(1/R^2)$ Christoffel symbols [3].[2] Indeed, using the geodesic equation and (2.3) one can show that

$$c^\mu \equiv \lim_{R\to\infty} \Gamma^\mu_{\nu\rho} X^\nu X^\rho. \tag{2.4}$$

Under logarithmic translations (2.2), this *log deviation vector* transforms as

$$\delta_L c^\mu = -L^\mu, \tag{2.5}$$

as can be inferred directly from (2.3), or through a short calculation using the transformation properties of the Christoffel symbols.[3]

In general, the log deviation vector obtained in the limit (2.4) depends on the asymptotic direction V^μ. A notable exception occurs for null rays, where c^μ turns out to be independent of the null direction [3], see section 3.2.[4] In this case, it can be gauged away by a logarithmic translation — precisely what is achieved in radiative coordinates. These considerations will play a key role in elucidating structural properties of the log soft theorem.

3 Timelike infinity and log frames

In order to apply the ideas of the previous section to a scattering setting, we consider the case in which infinity is approached along timelike directions.[5] In this limit, the leading structure of the gravitational field is captured by a scalar function on the unit hyperboloid that plays the role of a gravitational potential. This potential encodes the same information as the log deviation vector but is simpler to determine from Einstein equations. We present two complementary methods for obtaining it: an "asymptotic approach", formulated independently of any specific gauge choice (aside from assuming the form (2.1)), and a "bulk approach" that we implement in harmonic gauge. The asymptotic approach naturally exposes the freedom under log translations, allowing one to select any desired log frame, such as those associated with radiative or harmonic coordinates.

At this stage, we will essentially be recovering the leading term of the well-known coordinate transformation between harmonic and radiative gauges [33, 34]. The novelty in our treatment arises when considering both the incoming and outgoing versions of these transformations, ultimately allowing us to identify *global* log translations that simultaneously act at future and past infinities. This identification is a prerequisite for defining the action of log translations on soft factors, discussed in section 4.

[2] In the case of asymptotic null directions, we further require that $V^\mu V^\nu \partial_u g_{\mu\nu} = O(1/R^2)$, where u is the associated asymptotic retarded (or advanced) time. This condition is satisfied in solutions of Einstein equations in harmonic coordinates of the type discussed in [3] and reviewed in section 3.3.

[3] We follow conventions such that the action of a vector field ξ^μ on the coordinates and the metric is given by $\delta_\xi X^\mu = -\xi^\mu$ and $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$, respectively. The change of the $O(1/R)$ metric perturbation under log translations contains a piece from the $O(R^0)$ vector field in (2.2), whose precise form depends on the gauge-fixing condition (see appendix A.2). This dependence, however, drops out in the combination of Christoffel symbols given in (2.4).

[4] A similar contrast between null and spatial directions lies at the heart of the analysis in [19].

[5] By an appropriate limiting procedure, this will also allow us to describe asymptotic null geodesics, and hence massless particles.

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3.1 Spacetime metric at timelike infinity

For simplicity, we set the discussion in the context of future timelike infinity, but entirely analogous considerations apply to past time infinity. A comparison between the future and past equations is presented in subsection 3.4.

The role of the parameter R in (2.1) is now played by the asymptotic proper time τ, while the asymptotic direction V^μ is a unit timelike vector,

$$X^\mu \overset{\tau\to\infty}{=} \tau V^\mu + \cdots, \quad V^\mu V_\mu = -1, \tag{3.1}$$

where the dots indicate subleading terms and the norm condition is taken with respect to the Minkowski metric. The unit timelike hyperboloid is parametrized by coordinates x^a (see eq. (3.10) below for an explicit expression)

$$V^\mu = V^\mu(x), \tag{3.2}$$

and the associated hyperboloid metric is

$$h_{ab} = \partial_a V^\mu \partial_b V_\mu. \tag{3.3}$$

We denote by $\gamma_{\mu\nu}(x)$ the tensor capturing the leading deviation from the flat metric in the large time limit,

$$g_{\mu\nu}(\tau, x) \overset{\tau\to\infty}{=} \eta_{\mu\nu} + \frac{1}{\tau}\gamma_{\mu\nu}(x) + \cdots \tag{3.4}$$

As discussed in appendix A.2, this tensor contains both "pure gauge" and "non-gauge" pieces. To disentangle them, it is convenient to decompose it in terms of 3d hyperboloid fields

$$\gamma_{\mu\nu} \leftrightarrow \begin{cases} \sigma \\ \gamma_a \\ \gamma_{ab} \end{cases}, \tag{3.5}$$

where[6]

$$\sigma = -\tfrac{1}{2} V^\mu V^\nu \gamma_{\mu\nu}, \quad \gamma_a = \partial_a V^\mu V^\nu \gamma_{\mu\nu}, \quad \gamma_{ab} = \partial_a V^\mu \partial_b V^\nu \gamma_{\mu\nu}. \tag{3.6}$$

The gauge invariant information contained in these fields can be obtained by evaluating the asymptotic electric and magnetic part of the Weyl curvature [19]. Under typical asymptotic flatness conditions the latter vanishes, while the former is fully determined by σ. All remaining components in (3.5) are therefore either pure gauge or not independent.[7]

Summarizing, the non-trivial part of the metric perturbation at order $1/\tau$ is the scalar σ, which we refer to as the gravitational potential. Our interest in this quantity is that it determines the log deviation vector according to,

$$c^\mu = D^a V^\mu \partial_a \sigma - V^\mu \sigma, \tag{3.7}$$

as can be checked by direct computation from the definition (2.4) for the case $R = \tau$.

[6] In the notation of appendix A.1, $\sigma = -\frac{1}{2}\gamma_\parallel$.

[7] Modulo a supertranslation Goldstone field [35], which plays no role for our analysis. See appendix A.2 for a more detailed discussion of the material presented in this subsection.

Conversely, the potential can be recovered from the deviation vector via

$$\sigma = c^\mu V_\mu. \tag{3.8}$$

Thus, σ and c^μ encode the same information. Depending on the context, one or the other may be more convenient to work with. Like the deviation vector, σ is not fully gauge invariant due to logarithmic translations, which act as

$$\delta_L \sigma = -L^\mu V_\mu. \tag{3.9}$$

3.2 Null limit

It will be of interest to consider the limit in which the timelike direction V^μ becomes null. To be explicit, let us consider coordinates $x^a = (\rho, x^A)$ in (3.2) defined by

$$V^\mu = (\sqrt{\rho^2+1}, \rho\,\hat{n}), \quad V^\mu V_\mu = -1, \tag{3.10}$$

where ρ is a radial coordinate and x^A are coordinates on the unit sphere associated with the unit 3-vector $\hat{n}$. In this parametrization, the hyperboloid line element takes the form

$$h_{ab}dx^a dx^b = \frac{d\rho^2}{\rho^2+1} + \rho^2 d\Omega^2 \tag{3.11}$$

where $d\Omega^2$ is the unit sphere metric.

The null limit of V^μ is achieved by taking $\rho \to \infty$ in (3.10). To leading order one gets

$$V^\mu \overset{\rho\to\infty}{=} \rho\, n^\mu + O(1/\rho) \tag{3.12}$$

where

$$n^\mu = (1, \hat{n}) \tag{3.13}$$

is the null direction associated to the unit 3-vector $\hat{n}$. The divergent ρ factor multiplying the null direction (3.12) indicates the need to simultaneously rescale the affine parameter to properly describe the corresponding null trajectory. Alternatively, when looking at a particle's momentum, (3.12) should be multiplied by a $m \to 0$ mass such that the energy $E = m\rho$ is kept constant.

From a 3d hyperboloid perspective, the limit $\rho \to \infty$ brings us to its asymptotic boundary. As we shall review in the next section, Einstein equations imply that, in this limit, σ approaches [22]

$$\sigma \overset{\rho\to\infty}{=} \rho\, n_\mu c^\mu_{\mathcal{I}} + \cdots, \tag{3.14}$$

where $c^\mu_{\mathcal{I}}$ is a constant (i.e. independent of $\hat{n}$) vector. It then follows from (3.7) that this vector represents the large ρ limit of c^μ,

$$c^\mu_{\mathcal{I}} \equiv \lim_{\rho\to\infty} c^\mu. \tag{3.15}$$

In fact, this vector coincides with the log deviation vector of null trajectories. This is because the logarithmic coefficient in the asymptotic geodesics (2.3) is unaffected by rescalings of the affine parameter. The independence of $c^\mu_{\mathcal{I}}$ from $\hat{n}$ confirms the result quoted at the end of section 2, namely that for asymptotically null trajectories, the log deviation vector is independent of the null direction.

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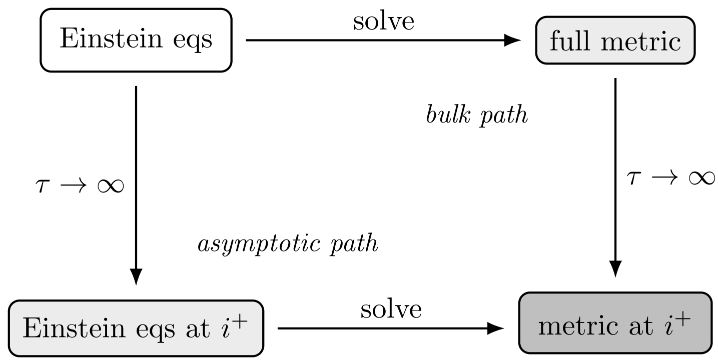

Figure 1. Two ways to find the asymptotic metric at timelike infinity i^+.

3.3 Solving Einstein equations

Our presentation so far has been mostly kinematical. We now discuss how Einstein equations determine the gravitational potential (and more generally, the spacetime metric) at timelike infinity. As illustrated in Figure 1, there are two paths one may follow, depending on whether the large time limit is taken before or after solving Einstein equations.

Below, we discuss these two approaches, focusing on the gravitational potential. As we will see, the two methods generally yield potentials that differ by a log translation. The asymptotic approach will be presented in a way that is agnostic to gauge-fixing conditions. For the bulk approach, however, a gauge-fixing condition is necessary. We will consider the case of harmonic gauge, as used in standard perturbative treatments. This will allow us to characterize the log translation gauge-fixing condition implicit in the analysis of [2, 3], and eventually to relax it in section 4.

Asymptotic approach

To leading order in the $\tau \to \infty$ limit, Einstein equations, applied to (3.4) leads to (see e.g. [10, 22, 29])

$$(D^2 - 3)\sigma = 4\pi G \rho_{\text{massive}}, \tag{3.16}$$

where D^2 is the hyperboloid Laplacian and

$$\rho_{\text{massive}}(x) = \sum_{\substack{i \\ m_i \neq 0}} m_i \delta(x, x_i) \tag{3.17}$$

is the energy density of massive particles at future timelike infinity. From the asymptotic perspective, σ is to be determined by solving (3.16). Let $\mathcal{G}(x,x')$ be a Green's function for this equation, i.e.

$$(D^2 - 3)\mathcal{G}(x,x') = \delta(x,x'). \tag{3.18}$$

If we demand that $\mathcal{G}(x,x')$ depends only on the geodesic distance $d(x,x')$ between x and x', one arrives at a one-parameter family of solutions (see eq. (C.7) of [10])

$$\mathcal{G}_\lambda(x,x') = -\frac{1}{4\pi}\left(\frac{2\chi^2 - 1}{\sqrt{\chi^2 - 1}} + \lambda\chi\right) \tag{3.19}$$

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where

$$\chi = -V(x) \cdot V(x') = \cosh d(x, x') \tag{3.20}$$

and λ is an undetermined constant. For a given choice of this constant, the resulting potential is then

$$\sigma^{(\lambda)}(x) = -G \sum_{\substack{i \\ m_i \neq 0}} m_i \left(\frac{2\chi_i^2 - 1}{\sqrt{\chi_i^2 - 1}} + \lambda \chi_i \right) \tag{3.21}$$

$$= -G \sum_{\substack{i \\ m_i \neq 0}} \frac{2(V \cdot p_i)^2 - m_i^2}{\sqrt{(V \cdot p_i)^2 - m_i^2}} + G\lambda V \cdot P_{\text{massive}} \tag{3.22}$$

$$\tag{3.23}$$

where

$$V^\mu \equiv V^\mu(x), \quad V_i^\mu \equiv V^\mu(x_i), \quad p_i^\mu \equiv m_i V_i^\mu, \tag{3.24}$$

and

$$P^\mu_{\text{massive}} = \sum_{\substack{i \\ m_i \neq 0}} p_i^\mu \tag{3.25}$$

is the total momentum of the massive particles. Comparing with (3.9), we see that different choices of λ are related to each other by log translations that are proportional to P^μ_{massive}.

A natural way to eliminate the log translation freedom is to require that σ vanishes asymptotically [22]

$$\lim_{\rho \to \infty} \sigma = 0 \iff c^\mu_{\mathcal{I}} = 0 \quad \text{(radiative log frame)}, \tag{3.26}$$

which amounts to choosing $\lambda = -2$ in (3.21). This condition singles out a specific log translation frame that we refer to as radiative frame.[8]

"Bulk" approach in harmonic gauge.

In harmonic gauge, the spacetime metric is written as

$$g^{\text{harm}}_{\mu\nu} = \eta_{\mu\nu}(1 - e) + 2e_{\mu\nu}, \tag{3.27}$$

where $\eta_{\mu\nu}$ is a reference Minkowski metric, with associated Cartesian coordinates X^μ. $e_{\mu\nu}$ is the so-called trace-reversed metric perturbation and $e = \eta^{\mu\nu} e_{\mu\nu}$ its trace. The gauge condition is

$$\partial_\nu e^{\mu\nu} = 0, \tag{3.28}$$

and allows Einstein equations to be written as

$$\Box e_{\mu\nu} = -8\pi G \mathcal{T}_{\mu\nu}, \quad \mathcal{T}_{\mu\nu} = T_{\mu\nu} + T^{\text{grav}}_{\mu\nu}, \tag{3.29}$$

[8] As discussed in subsection 3.5 there are in fact two distinct radiative frames, depending on whether condition (3.26) is imposed at the future or at the past.

where $\Box \equiv \partial^\mu \partial_\mu$ is the d'Alembertian w.r.t. the flat metric, $T_{\mu\nu}$ is the usual matter stress-energy tensor and $T^{\text{grav}}_{\mu\nu}$ is a "gravitational stress-energy tensor" that is constructed out of $e_{\mu\nu}$ and its derivatives. We emphasize that (3.29) is simply a rewriting of the full non-linear Einstein equations [7].

The formal solution to (3.29) with retarded boundary conditions is

$$e_{\mu\nu}(X) = 4G \int d^4X'\theta(X^0 - X'^0)\delta((X - X')^2)\mathcal{T}_{\mu\nu}(X'), \tag{3.30}$$

and can be evaluated perturbatively by regarding $e_{\mu\nu}$ as a series expansion in G. However, for the purposes of obtaining the leading late-time component

$$e_{\mu\nu} \overset{\tau\to\infty}{=} \frac{1}{\tau}\overset{1}{e}_{\mu\nu}(x) + \cdots\,, \tag{3.31}$$

it suffices to evaluate (3.30) at the zeroth perturbative order, for which the only contribution to $\mathcal{T}_{\mu\nu}$ is due to the freely propagating particles, see e.g. [42]. Setting

$$X'^\mu = \tau' V^\mu(x') + \cdots\,, \qquad X^\mu = \tau V^\mu(x) + \cdots\,, \qquad \tau \to \infty \tag{3.32}$$

in (3.30) leads to (see appendix A of [36] for an analogous computation)

$$\overset{1}{e}_{\mu\nu}(x) = 2G \int \frac{d^3x'}{\sqrt{\chi^2 - 1}} V_\mu(x')V_\nu(x')\,\rho(x') \tag{3.33}$$

where

$$\rho(x') = \sum_i m_i \delta(x', x_i) \tag{3.34}$$

is the energy density at timelike infinity *including* massless contributions, as we discuss below.

We now extract from (3.33) the gravitational potential σ. The $O(1/\tau)$ metric perturbation (3.4) is given by $\gamma_{\mu\nu} = 2\overset{1}{e}_{\mu\nu} - \overset{1}{e}\eta_{\mu\nu}$, and from the definition of σ (3.6) one gets

$$\sigma^{\text{harm}} = -\overset{1}{e}_{\mu\nu} V^\mu V^\nu - \frac{1}{2}\overset{1}{e} \tag{3.35}$$

$$= -G \int d^3x' \frac{2\chi^2 - 1}{\sqrt{\chi^2 - 1}}\,\rho(x'). \tag{3.36}$$

Upon using (3.34), this yields a potential that looks like $\sigma^{(\lambda=0)}$ in (3.21), but where the sum includes both massive and massless particles. To account for the latter, we express the potential in terms of the particle momenta, as in (3.22), leading to,

$$\sigma^{\text{harm}} = -G \sum_i \frac{2(V\cdot p_i)^2 - m_i^2}{\sqrt{(V\cdot p_i)^2 - m_i^2}} = \sigma^{(\lambda=0)} + 2GV \cdot P_{\text{massless}}, \tag{3.37}$$

where the two terms in the last equality correspond to the contributions from massive and massless particles to the sum, and where we used that $|V \cdot p_i| = -V \cdot p_i$.

On the other hand, according to the discussion from the previous section, the potential in the radiative log frame (3.26) can be written as

$$\sigma^{\text{rad}} = \sigma^{(\lambda=-2)} = \sigma^{(\lambda=0)} - 2GV \cdot P_{\text{massive}}. \tag{3.38}$$

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Solving for $\sigma^{(\lambda=0)}$ in (3.38) and substituting in (3.37) leads to

$$\sigma^{\text{harm}} = \sigma^{\text{rad}} + 2GV \cdot P_{\text{total}}, \tag{3.39}$$

where $P^\mu_{\text{total}} = P^\mu_{\text{massive}} + P^\mu_{\text{massless}}$.

Thus, the harmonic and radiative frame potentials are related by a log translation that is proportional to the total spacetime momentum. In particular, the null limit of the log deviation vector is now given by

$$c^\mu_{\mathcal{I}} = 2GP^\mu_{\text{total}} \quad \text{(harmonic log frame)}. \tag{3.40}$$

This reproduces, in our conventions, eq. (3.29) of [3]. The expression is also consistent with the well-known asymptotic diffeomorphism that interpolates between harmonic and radiative coordinates, see e.g. eq. (95) of [34]. We refer to (3.40) as the harmonic log frame.[9]

3.4 Future vs. past

The discussion of the previous subsections has a direct analogue at past timelike infinity. Below we summarize the equations and conventions that we shall use to describe both infinities. When needed, we will distinguish future and past quantities by a $+/-$ label.

- Asymptotic geodesics:

$$X^\mu(s) \overset{s\to\pm\infty}{=} sV^\mu_\pm + \log|s| c^\mu_\pm + \cdots, \quad V^\mu_\pm \text{ future-pointing}. \tag{3.41}$$

- Spacetime hyperbolic coordinates: $(\tau > 0, x^a)$ and $(\tau < 0, x^a)$ with

$$X^\mu \overset{\tau\to\pm\infty}{=} \tau V^\mu(x) + \cdots, \quad V^\mu V_\mu = -1, \quad V^\mu \text{ future-pointing}. \tag{3.42}$$

- Gravitational potential and log deviation vector:

$$\sigma_\pm = -\tfrac{1}{2} \lim_{\tau\to\pm\infty} V^\mu V^\nu |\tau| (g_{\mu\nu} - \eta_{\mu\nu}) \tag{3.43}$$

$$c^\mu_\pm = \pm(D^a V^\mu \partial_a \sigma_\pm - V^\mu \sigma_\pm). \tag{3.44}$$

$$\sigma_\pm = \pm c^\mu_\pm V_\mu. \tag{3.45}$$

$$c^\mu_{\mathcal{I}^\pm} = \lim_{\rho\to\infty} c^\mu_\pm \tag{3.46}$$

- Energy density and leading Einstein equation:

$$\rho_\pm = \lim_{\tau\to\pm\infty} |\tau^3| V^\mu V^\nu T_{\mu\nu} \tag{3.47}$$

$$(D^2 - 3)\sigma_\pm = 4\pi G \rho_\pm \tag{3.48}$$

[9] We are here presenting the harmonic log frame as a condition on the log deviation vector of outgoing null rays. One can alternatively write it as a condition on incoming null rays, as reviewed in the next subsection. There is a non-trivial relationship between the future and past null deviation vectors which we discuss in subsection 3.5.

- Gravitational potential in terms of outgoing/incoming momenta:

  We follow standard amplitude conventions in which the particles' momenta at past timelike infinity are taken with an overall minus sign (so that they point to the past):

$$p_i^\mu = \pm m_i V_{\pm i}^\mu. \tag{3.49}$$

  With these conventions, the gravitational potential at future/past timelike infinity in the various log frames discussed earlier is

$$\mathring{\sigma}_\pm = -G \sum_{\substack{i\in\pm \\ m_i\neq 0}} \frac{2(V\cdot p_i)^2 - m_i^2}{\sqrt{(V\cdot p_i)^2 - m_i^2}} \tag{3.50}$$

$$\sigma_\pm^{\text{rad}\pm} = \mathring{\sigma}_\pm \mp 2GV\cdot P_{\text{massive}\pm} \tag{3.51}$$

$$\sigma_\pm^{\text{harm}} = \sigma_\pm^{\text{rad}} \pm 2GV\cdot P_{\text{total}\pm} \tag{3.52}$$

  where $\mathring{\sigma}_\pm \equiv \sigma^{(\lambda=0)}$, $i\in\pm$ indicates outgoing/incoming particles and "rad±" refers to the radiative log frame at future/past infinity. $P^\mu_{\text{total}\pm} = P^\mu_{\text{massive}\pm} + P^\mu_{\text{massless}\pm}$ is the total outgoing/incoming momentum, with the conventions of (3.49) so that momentum conservation reads

$$P^\mu_{\text{total}+} + P^\mu_{\text{total}-} = 0. \tag{3.53}$$

  Since $P^\mu_{\text{total}+}$ is just the total momentum of the system, we shall occasionally omit the plus subscript and write it as

$$P^\mu_{\text{total}} \equiv P^\mu_{\text{total}+}. \tag{3.54}$$

- The log deviation vector can be obtained by applying (3.44) to the corresponding gravitational potential. In particular, the deviation vector associated to the $\lambda = 0$ potential (3.50) is

$$\mathring{c}^\mu_\pm = \mp G \sum_{\substack{i\in\pm \\ m_i\neq 0}} \frac{(2(V\cdot p_i)^3 - 3m_i^2 V\cdot p_i)p_i^\mu - m_i^4 V^\mu}{((V\cdot p_i)^2 - m_i^2)^{3/2}}. \tag{3.55}$$

  The deviation vector associated to the other potentials can be obtained from (3.55) by appropriate log translations. In the harmonic log frame the expression has the form (3.55) but with the sum running over all (massive and massless) outgoing/incoming particles. In particular, one finds[10]

$$c^\mu_{\mathcal{I}\pm} = 2GP^\mu_{\text{total}\pm} \quad \text{(harmonic log frame)}. \tag{3.56}$$

[10] The sign difference in (3.55) disappears in (3.56) because $|V\cdot p_i| = \mp V\cdot p_i$ for outgoing/incoming p_i, according to the conventions (3.41) and (3.49).

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3.5 Global log translations

A priori, there are two independent log translations at the future/past

$$\xi^{\mu}_{L^{\pm}} \overset{\tau\to\pm\infty}{=} \log|\tau| L^{\mu}_{\pm} + O(\tau^0), \tag{3.57}$$

which act according to

$$\delta_{L_{\pm}} c^{\mu}_{\pm} = -L^{\mu}_{\pm}, \quad \delta_{L_{\pm}} \sigma_{\pm} = \mp L^{\mu}_{\pm} V_{\mu}. \tag{3.58}$$

However, as in the case of ordinary translations (or more generally, BMS supertranslations [48]), these two transformations should not be regarded as independent, but instead satisfy

$$L^{\mu}_{+} = L^{\mu}_{-} \equiv L^{\mu}. \tag{3.59}$$

This condition arises naturally when working in harmonic coordinates, where a single patch covers all spacetime (at least outside the scattering bodies). Alternatively, as discussed in [29], it follows from the matching properties of the gravitational field across timelike, null and spatial infinity.

The identification (3.59) leads to a single log translation group acting simultaneously on both future and past log deviation vectors,

$$\delta_L c^{\mu}_{\pm} = -L^{\mu}. \tag{3.60}$$

This allows for a global notion of log frame, i.e., one that applies to all infinities. Among the possible choices, three global log frames are of special relevance: the future and past radiative frames, and the harmonic frame.

The first two are characterized by the vanishing of the corresponding log deviation vector of null geodesics,

$$c^{\mu}_{\mathcal{I}^{\pm}} = 0 \quad (\text{rad}^{\pm} \text{ log frame}). \tag{3.61}$$

The harmonic log frame corresponds to condition (3.56) on either outgoing or incoming null rays. In view of (3.59), however, the two conditions cannot be regarded as independent. Taking the sum and difference of (3.56) and using momentum conservation one finds,

$$c^{\mu}_{\mathcal{I}^{+}} + c^{\mu}_{\mathcal{I}^{-}} = 0 \quad (\text{harmonic log frame}) \tag{3.62}$$

$$c^{\mu}_{\mathcal{I}^{+}} - c^{\mu}_{\mathcal{I}^{-}} = 4GP^{\mu}_{\text{total}}. \tag{3.63}$$

Only the first combination is not invariant under (3.60), and thus serves to fix global log translations. By contrast, eq. (3.63) is an identity that holds in any global log frame, even though it was obtained from a harmonic gauge computation.[11]

Comment. It is interesting to note that there is a spatial infinity version of these equations, in which $c^{\mu}_{\mathcal{I}^{\pm}}$ is obtained from the null limit of asymptotic spacelike geodesics. In that context, (3.62) coincides with the log translation fixing proposed in [14], while (3.63) reproduces a long-known "discontinuity" of the gravitational field at spatial infinity [19, 22, 37]. See [29] for further details.

[11] In particular, we learn that in the future radiative frame $c^{\mu}_{\mathcal{I}^{-}} = -4GP^{\mu}_{\text{total}}$, while in the past radiative frame $c^{\mu}_{\mathcal{I}^{+}} = 4GP^{\mu}_{\text{total}}$.

4 Soft theorems and their log translation invariance

The classical soft theorems found in [1–5] constrain the soft limit of the gravitational waveform through certain universal terms. Specifically, the Fourier transform of the metric perturbation at null infinity satisfies

$$\tilde{h}_{\mu\nu}(r,\omega,\hat{n}) \overset{\omega\to 0}{=} \frac{1}{r}\left(\omega^{-1}h^{(0)}_{\mu\nu}(\hat{n}) + \log\omega\, h^{(\log)}_{\mu\nu}(\hat{n}) + \omega\log^2\omega\, h^{(\log^2)}_{\mu\nu}(\hat{n}) + \cdots\right) \tag{4.1}$$

where we only kept the $O(1/r)$ radiative field, ω is the frequency and $\hat{n}$ the direction on the celestial sphere from which the waveform is observed. The leading term[12]

$$h^{(0)}_{\mu\nu}(\hat{n}) = -4Gi\sum_i \frac{p^i_\mu p^i_\nu}{p_i\cdot n}, \tag{4.2}$$

is described by Weinberg's soft theorem [38] and captures what is known as the gravitational memory effect [39]. The results from [1–5] concern the subleading logarithmic terms. In this section we focus on the first two of what are believed to be infinitely many universal "leading logs" $\omega^{k-1}\log^k\omega$ with $k = 1, 2, \ldots$ [4–7, 12].[13]

The proofs of these soft theorems are based on rather involved perturbative computations. There is, however, a shortcut that reproduces these results more directly [1, 3]. Although it does not amount to a full proof, it provides insight into the origin of the various contributions. More importantly for our purposes, it reveals where the harmonic log frame is implicitly assumed and thus how it can be relaxed.

Following section 5 of [4], the starting point for the shortcut derivation is the tree-level (or $d > 4$) $O(G)$ soft expansion [42–45]

$$\tilde{h}^{\text{tree}}_{\mu\nu}(r,\omega,\hat{n}) \overset{\omega\to 0}{=} \frac{1}{r}\left(\omega^{-1}h^{(0)}_{\mu\nu}(\hat{n}) + h^{(1)}_{\mu\nu}(\hat{n}) + \omega\, h^{(2)}_{\mu\nu}(\hat{n}) + \cdots\right) \tag{4.3}$$

where[14]

$$h^{(1)}_{\mu\nu} = 4G\sum_i \frac{p^i_{(\mu}J^i_{\nu)\rho}n^\rho}{p_i\cdot n}, \quad h^{(2)}_{\mu\nu} = 2iG\sum_i \frac{J^i_{\mu\rho}J^i_{\nu\sigma}n^\rho n^\sigma}{p_i\cdot n}, \tag{4.4}$$

with $J^i_{\mu\nu}$ the particle angular momenta.

The expansion (4.3) ignores the effect of the logarithmic deviation in the asymptotic trajectories. This can be incorporated by taking into account: 1) The logarithmic deviation of the soft null rays and 2) The logarithmic deviation of the hard particles. The first effect, referred to as gravitational drag, is incorporated via the phase factor [1, 3]

$$\tilde{h}^{\text{tree}}_{\mu\nu} \to e^{-i\omega\log\omega\, n\cdot c_{\mathcal{I}^+}}\,\tilde{h}^{\text{tree}}_{\mu\nu}, \tag{4.5}$$

where, in harmonic gauge, $c^\mu_{\mathcal{I}^+} = 2GP^\mu_{\text{total}}$, see eq. (3.40).

[12] Here and in what follows the index i runs over all incoming and outgoing particles, with sign conventions as in (3.49), and n^μ is the future-pointing null direction associated to $\hat{n}$ (3.13). The sums include both massive and massless "particles", the latter capturing the contribution from radiation (gravitational or other) [7].

[13] See [40, 41] for results on non-leading soft logs.

[14] There could be non-universal terms at order $O(\omega)$ [44] which however play no role for the log soft theorems.

The second effect is due to the divergent nature of the particle angular momenta. For asymptotic trajectories of the form (3.41) one has

$$J^i_{\mu\nu} \overset{s\to\pm\infty}{=} \log|s| J^{\mathrm{div}\, i}_{\mu\nu} + \cdots , \tag{4.6}$$

where

$$J^{\mathrm{div}\, i}_{\mu\nu} := c^i_\mu p^i_\nu - c^i_\nu p^i_\mu . \tag{4.7}$$

The corresponding contribution to the log soft theorem is then obtained by substituting (4.6) in (4.3) and making the identification [1]

$$\log|s| \sim \log(\omega^{-1}). \tag{4.8}$$

Including both effects, one arrives at [4]

$$\tilde{h}_{\mu\nu} = \frac{1}{r} e^{-i\omega \log\omega\, n\cdot c_{\mathcal{I}^+}} \left[\omega^{-1} h^{(0)}_{\mu\nu} - \log\omega\, h^{(1)\mathrm{div}}_{\mu\nu} + \omega \log^2\omega\, h^{(2)\mathrm{div}}_{\mu\nu} + \cdots\right], \tag{4.9}$$

where the "div" label indicates that $J^i_{\mu\nu}$ is replaced by $J^{\mathrm{div}\, i}_{\mu\nu}$ in the expressions (4.4).

Finally, by expanding the exponential in (4.9), one recovers (4.1) with

$$h^{(\log)}_{\mu\nu} = -in\cdot c_{\mathcal{I}^+}\, h^{(0)}_{\mu\nu} - h^{(1)\mathrm{div}}_{\mu\nu}, \tag{4.10}$$

$$h^{(\log^2)}_{\mu\nu} = -\tfrac{1}{2}(n\cdot c_{\mathcal{I}^+})^2\, h^{(0)}_{\mu\nu} + in\cdot c_{\mathcal{I}^+} h^{(1)\mathrm{div}}_{\mu\nu} + h^{(2)\mathrm{div}}_{\mu\nu} \tag{4.11}$$

Remarkably, these expressions correctly reproduce the rigorously derived log soft factors [4, 5]. We recall that, in both the heuristic and rigorous derivations, the deviation vectors are written in the harmonic log frame. We now show that the above expressions are in fact valid in any log frame.

Under a log translation, both the "soft" null-ray deviation vector in (4.5) and the "hard" particles' deviation vector in (4.7) transform according to (3.60)

$$\delta_L c^\mu_{\mathcal{I}^+} = \delta_L c^\mu_i = -L^\mu . \tag{4.12}$$

Applying this to the factors $h^{(1)\mathrm{div}}_{\mu\nu}$ and $h^{(2)\mathrm{div}}_{\mu\nu}$ one finds

$$\delta_L h^{(1)\mathrm{div}}_{\mu\nu} = iL\cdot n\, h^{(0)}_{\mu\nu}, \quad \delta_L h^{(2)\mathrm{div}}_{\mu\nu} = iL\cdot n\, h^{(1)\mathrm{div}}_{\mu\nu} \tag{4.13}$$

where in the first relation we used momentum conservation $\sum_i p^i_\mu = 0$ and in the second one $\sum_i J^{\mathrm{div}\, i}_{\mu\nu} = 0$.[15] Using (4.12) and (4.13) one can verify the invariance of the log soft factors

$$\delta_L h^{(\log)}_{\mu\nu} = 0, \quad \delta_L h^{(\log^2)}_{\mu\nu} = 0. \tag{4.14}$$

The result just proven can be used to explain the cancellations among outgoing massless particles observed in [3, 6]. In the harmonic gauge setting used in these references, the dependence on outgoing massless particles appears through: 1) their total momentum in the drag term (4.5) and 2) their individual momenta in the factors $h^{(k)\mathrm{div}}_{\mu\nu}$. The cancellation of these two contributions can be traced back to the fact that $c^\mu_i \equiv c^\mu_{\mathcal{I}^+}$ for outgoing massless particles.

The independence from outgoing massless particles is however manifest if one writes the log soft factors in the future radiative log frame, where $c^\mu_{\mathcal{I}^+} = 0$. In that case, each contribution vanishes separately.

[15] In fact, the total incoming and outgoing divergent angular momentum vanishes separately.

Comments:

- As a consistency check and illustration, in appendix B we evaluate the first log term (4.10) both in harmonic and radiative log frames, recovering the expression as presented in [3, 7].

- In time domain, the soft expansion translates into early and late time tails in the gravitational waveform that can be extracted by giving the soft frequency a small positive/negative imaginary part [7]. Without such prescription, the expansion (4.1) only captures *differences* between these early/late time contributions, see appendix B of [12] for further details. The soft terms considered here and in [29] are associated to these differences. In particular, the log translation invariance does not apply to individual early/late time components.

- Invariance under log translations leads to non-trivial identities only when the soft factors are expressed in terms of the deviation vectors. Once the deviation vectors are written in terms of particle momenta (in any log frame), the invariance is trivially satisfied, since particle momenta are unaffected by log translations.

- The validity of (4.9) for the second log relies on the fact [4, 5] that the subleading deviation in the asymptotic trajectory (2.3) is of order $\ln s/s$, and therefore does not contribute to $\omega \log^2 \omega$ in the replacement $s \to \omega^{-1}$. This fall-off can be shown by expanding (3.30) to subleading order, which yields an $O(\ln \tau/\tau^2)$ term.

5 On higher order soft theorems

Following the success of the heuristic derivation of the first two log soft theorems, the general structure of higher-order leading log terms was described in [5, 12]. In this section we revisit these considerations from the perspective of the preceding discussion.

The starting point is again the $O(G)$ waveform. One can argue on general grounds that it admits a power expansion in the frequency that extends (5.1) to all orders

$$\tilde{h}^{\text{tree}}_{\mu\nu}(r,\omega,\hat{n}) = \frac{1}{r\omega}\sum_{k=0}^{\infty} \omega^k h^{(k)}_{\mu\nu}(\hat{n}). \tag{5.1}$$

In [46, 47] it was shown that the $k \geq 2$ coefficients take the form[16]

$$h^{(k)}_{\mu\nu}(\hat{n}) = -4Gi\frac{(-i)^k}{k!}\sum_i \frac{J^i_{\mu\rho}J^i_{\nu\sigma}n^\rho n^\sigma}{p_i\cdot n}(b_i\cdot n)^{k-2} + \tilde{r}^{(k)}_{\mu\nu}, \quad k\geq 2, \tag{5.2}$$

for $O(G^0)$ asymptotic trajectories $X^\mu_i(s) = \pm s p^\mu_i/m_i + b_i + \cdots$. These are referred to as partial soft theorems, since there is an undetermined piece $\tilde{r}^{(k)}_{\mu\nu}$. The "universal" term in (5.2) is distinguished from this remainder through its dependence on n^μ, the latter being of the form $\tilde{r}^{(k)}_{\mu\nu} = \tilde{r}^{(k)}_{\mu\nu\,\alpha_1\ldots\alpha_{k-1}} n^{\alpha_1}\ldots n^{\alpha_{k-1}}$. The remainders are generically non-trivial, except for the $k = 2$ case which vanishes according to (4.4).

[16] The analysis of [46, 47] is in the context of momentum-space scattering amplitudes. We adapt their expressions by making the replacement $\partial/\partial p^i_\mu \to -ib^\mu_i$.

Doing the replacement $b_i^\mu \to -\ln\omega c_i^\mu$, one arrives at the expansion [5, 12]

$$\tilde{h}_{\mu\nu}(r,\omega,\hat{n}) \approx -\frac{4Gi}{r\omega} e^{-i\omega\log\omega\, n\cdot c_{\mathcal{I}^+}} \sum_{k=0} \frac{1}{k!}(-i\omega\log\omega)^k\, a_{\mu\nu}^{(k)}, \tag{5.3}$$

where the wiggly equal sign means that we are ignoring terms of the form $\omega^k\log^n\omega$ with $n<k$ and where we adopt the normalization of [12] for the coefficients.[17] These are given by

$$a_{\mu\nu}^{(0)} = \sum_i \frac{p_\mu^i p_\nu^i}{p_i\cdot n}, \quad a_{\mu\nu}^{(1)} = \sum_i \frac{p^i_{(\mu} J^{\mathrm{div}\, i}_{\nu)\rho} n^\rho}{p_i\cdot n}, \tag{5.4}$$

$$a_{\mu\nu}^{(k)} = (-1)^k \sum_i \frac{J^{\mathrm{div}\, i}_{\mu\rho} J^{\mathrm{div}\, i}_{\nu\sigma} n^\rho n^\sigma}{p_i\cdot n} (c_i\cdot n)^{k-2} + r_{\mu\nu}^{(k)}, \quad k\geq 2, \tag{5.5}$$

with $r_{\mu\nu}^{(k)} = r^{(k)}_{\mu\nu\,\alpha_1\ldots\alpha_{k-1}} n^{\alpha_1}\ldots n^{\alpha_{k-1}}$ undetermined for $k\geq 3$.

In [12], a proposal is given for the remainders whose general applicability is yet to be established [7]. Unfortunately, the proposal of [12] does not appear to admit a rewriting in terms of deviation vectors and hence we cannot establish its consistency with log translation invariance.

We can, however, reverse the logic of the previous section and ask: if the remainders admit arbitrary log-frame expressions, what restrictions are imposed by log translation invariance? To answer this question, we first note that, by applying an infinitesimal log translation to (5.3) and requiring it to vanish leads to

$$\delta_L a_{\mu\nu}^{(k)} = k\,(n\cdot L)\, a_{\mu\nu}^{(k-1)}, \tag{5.6}$$

generalizing the relations (4.13) to arbitrary order. Using the general form (5.5) in (5.6) leads to,

$$\begin{aligned} \delta_L r_{\mu\nu}^{(k)} &= k\,(n\cdot L)\, r_{\mu\nu}^{(k-1)} \\ &\quad - (-1)^k \sum_i \Big(L_{[\mu} c^i_{\rho]} p^i_{[\nu} c^i_{\sigma]} + (\mu\leftrightarrow\nu)\Big)\, n^\rho n^\sigma (c_i\cdot n)^{k-3}, \quad k\geq 3, \end{aligned} \tag{5.7}$$

where $v_{[\mu}w_{\nu]} \equiv v_\mu w_\nu - v_\nu w_\mu$. We emphasize the non-trivial cancellation in the $1/(p_i\cdot n)$ dependence on both sides of (5.6), so that the recursion relation for the remainder is polynomial in n^μ.

We leave for future investigations the study of solutions of (5.7). The initial seed is given by $r_{\mu\nu}^{(2)} = 0$. Although not immediately obvious, the integrability of (5.7) (i.e. the condition that the r.h.s. of (5.7) is consistent with $[\delta_L, \delta_{L'}] = 0$) follows from that of (5.6), where it is trivially satisfied. Note also that (5.7) is consistent with ordinary gauge invariance, $r_{\mu\nu}^{(k)} n^\nu = 0$.

6 Discussion

In this paper, we uncovered the role of log translations in classical gravitational soft theorems. Unlike the by now standard link between asymptotic symmetries and soft theorems, log

[17] Except for the $k=0$ one, which here differs by an overall sign from [12]. To relate their expressions with ours, note that the log deviation vector of the i-th particle in harmonic gauge can be written as $c_i^\mu = -2G\sum_j \tau_{ij} p_j^\mu + (\cdots)p_i^\mu$ where the sum is over either incoming or outgoing particles and τ_{ij} is the "relative logarithmic drift" as defined in [12].

translations are pure gauge and lead to trivial conservation laws. However, by writing the soft factor in terms of log translation-dependent quantities one can extract non-trivial information on the form of these coefficients. In particular, this explains why the log soft factors cannot depend on outgoing massless particles.

We explicitly checked the log translation invariance of the first two known log soft coefficients. Although there exist specific proposals for higher order terms, these do not admit any obvious rewriting in terms of log translation-dependent quantities and hence we could not verify their consistency with log translations. Nevertheless, if general log frame expressions exist for such terms, they would obey a recurrence relation that constrains their structure.

From a broader perspective, our results represent a new example, among the many that followed [48], of the rich interplay between perturbative results and the geometric description of the gravitational field at infinity. In an upcoming paper [29] we deepen this interplay by providing a proof of the first log soft theorem purely from an asymptotic perspective.

There are many avenues for future research. It would be important to understand how the considerations presented here connect to the realization of soft theorems as Ward identities. These are typically discussed in a context where log translations are fixed (e.g. by working in radiative or harmonic coordinates). A first step in this direction would be to achieve a description of superrotations (and, more generally, asymptotic higher-spin symmetries [49–54]) that is valid in arbitrary log frames. Such a framework should be useful for an eventual extension of [10] to the higher-order log soft theorems.

Our focus was on the classical theory, but much of the motivation comes from quantum gravity. It would be interesting to explore the consequences of log translation invariance for the quantum soft theorems [3, 55–57], and, more ambitiously, for asymptotically flat holography [58–65].

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A More on timelike infinity

In this appendix, we expand the discussion of timelike infinity given in the body of the paper. We start by introducing notation and summarizing useful identities. We then present a systematic description of the first order gravitational field at timelike infinity, and specialize it to the harmonic gauge case. In this setting, we give a general argument showing that log translations are frozen, and recover the BMS diffeomorphisms at timelike infinity as described in [26]. Finally, we show that the metric (3.30) has a vanishing asymptotic magnetic Weyl tensor, consistent with standard assumptions in the asymptotic literature.

A.1 Hyperboloid decomposition of tensors

Given a Minkowski 4-vector F^μ, we write its decomposition with respect to the hyperbolic splitting as

$$F^\mu = fV^\mu + f^a\partial_a V^\mu \tag{A.1}$$

where

$$f = -V_\mu F^\mu, \quad f^a = D^a V_\mu F^\mu. \tag{A.2}$$

We note the identities:

$$F_\mu G^\mu = -fg + f^a g_a, \tag{A.3}$$
$$D^a V_\mu D_a V_\nu = \eta_{\mu\nu} + V_\mu V_\nu. \tag{A.4}$$

More generally, one can decompose any Minkowski tensor into longitudinal and transverse components by contracting its Lorentz indices with $\eta_{\mu\nu} = -V_\mu V_\nu + D^a V_\mu D_a V_\nu$. For instance, a symmetric tensor $T_{\mu\nu}$ can be written as

$$T_{\mu\nu} = t_\| V_\mu V_\nu - 2t_a D^a V_{(\mu} V_{\nu)} + t_{ab} D^a V_\mu D^b V_\nu \tag{A.5}$$

with

$$\begin{aligned} t_\| &= V^\mu V^\nu T_{\mu\nu}, \\ t_a &= D_a V^\mu V^\nu T_{\mu\nu}, \\ t_{ab} &= D_a V^\mu D_b V^\nu T_{\mu\nu}. \end{aligned} \tag{A.6}$$

The subscript in the 3-scalar is to distinguish it from the trace of the 3-tensor,

$$t_\perp := h^{ab} t_{ab} \tag{A.7}$$

so that

$$T \equiv \eta^{\mu\nu} T_{\mu\nu} = -t_\| + t_\perp. \tag{A.8}$$

Useful identities

$$D_a D_b V^\mu = h_{ab} V^\mu, \tag{A.9}$$

$$[D_a, D_b] f_c = f_a h_{bc} - f_b h_{ac}. \tag{A.10}$$

A.2 Asymptotic diffeos and asymptotic Weyl tensor

In this section we present several details of the discussion given in section 3.1. Our starting point is the metric (3.4)

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{\tau}\gamma_{\mu\nu} + \cdots \, . \tag{A.11}$$

The asymptotic vector fields preserving this form are given by

$$\xi^\mu = \tau R^{\mu\nu} V_\nu + \log\tau L^\mu + F^\mu + \cdots , \tag{A.12}$$

where $R^{\mu\nu}$ is a constant antisymmetric matrix representing Lorentz transformations and L^μ a constant vector representing log translations. F^μ, on the other hand, depends on the hyperboloid point. Within it, one finds regular translations, supertranslations, and pure gauge transformations, as we discuss below. From now on we set $R^{\mu\nu} = 0$; its only effect at this order being a Lorentz rotation on $\gamma_{\mu\nu}$.

To evaluate the action of the remaining components in (A.12), we write

$$\mathcal{L}_\xi g_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} + \cdots \tag{A.13}$$

where the dots indicate subleading terms that do not affect $\gamma_{\mu\nu}$. The Cartesian derivatives in terms of hyperbolic coordinates are given by

$$\partial_\mu = -V_\mu \partial_\tau + \tau^{-1} D^a V_\mu D_a, \tag{A.14}$$

from which one finds

$$\delta_\xi \gamma_{\mu\nu} = -2V_{(\mu}L_{\nu)} + 2D_a V_{(\mu} D^a F_{\nu)}. \tag{A.15}$$

We now consider the hyperboloid components of $\gamma_{\mu\nu}$, defined in (3.6). From (A.15) and the identities presented in section A.1 their transformation rules under (A.12) (with $R^{\mu\nu} = 0$) are found to be

$$\delta_\xi \sigma = l \tag{A.16}$$

$$\delta_\xi \gamma_a = l_a - f_a - \partial_a f \tag{A.17}$$

$$\delta_\xi \gamma_{ab} = 2D_{(a} f_{b)} + 2f h_{ab}, \tag{A.18}$$

with f and f_a given by (A.2) and similarly for l and l_a. Since the latter are constructed out of a *constant* vector, they satisfy the additional properties

$$l_a = -\partial_a l, \quad D_a D_b l - l h_{ab} = 0. \tag{A.19}$$

As mentioned in section 3.1, the gauge invariant content of the gravitational field at this order is captured by the leading electric and magnetic components of the Weyl tensor [19, 66]

$$\mathcal{E}_{ab} = D_{\langle a} D_{b\rangle} \sigma \tag{A.20}$$

$$\mathcal{B}_{ab} = \frac{1}{2} \epsilon_{amn} D^m k^n{}_b \tag{A.21}$$

where the angle brackets denote symmetrization and extraction of the trace and

$$k_{ab} = \gamma_{ab} + 2D_{(a}\gamma_{b)} + 2\sigma h_{ab}. \tag{A.22}$$

From (A.16) and (A.19) one can verify that $\delta_\xi \mathcal{E}_{ab} = 0$. On the other hand, it is easy to see that

$$\delta_\xi k_{ab} = -2(D_a D_b f - h_{ab} f), \tag{A.23}$$

which in turns implies $\delta_\xi \mathcal{B}_{ab} = 0$.

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The leading $\tau\tau$ component of Einstein equations constraint σ to satisfy (3.16), while the τa and ab components restrict, respectively, the divergence and laplacian of k_{ab} by *homogenous* equations, since the corresponding components of the matter stress tensor vanish at this order, see [22, 29]. For spacetimes of physical interest a stronger condition holds on k_{ab}, namely the vanishing of the magnetic Weyl curvature (A.21)

$$\mathcal{B}_{ab} = 0, \tag{A.24}$$

which in turn implies [19, 22]

$$k_{ab} = -2(D_a D_b \Phi - h_{ab}\Phi), \tag{A.25}$$

where Φ is an unconstrained scalar field on the hyperboloid. We will later verify that the asymptotic scattering metric (3.33) indeed satisfies (A.25).

From (A.23) one finds that Φ transforms as

$$\delta_\xi \Phi = f, \tag{A.26}$$

which is why it is identified as a Goldstone mode for supertranslations [35].

Gauge vs. non-gauge diffeos

Let us summarize our discussion so far. For spacetimes satisfying (A.24), the initial ten components in $\gamma_{\mu\nu}$ are reduced to five:[18]

$$\mathcal{B}_{ab} = 0 \implies \gamma_{\mu\nu} \leftrightarrow \begin{cases} \sigma \\ \gamma_a \\ \Phi \end{cases} . \tag{A.27}$$

Out of these, we need to isolate "gauge" vs "non-gauge" components, which requires us to take into account asymptotic diffeomorphisms (other than asymptotic Lorentz rotations and translation, which do not induce shifts in $\gamma_{\mu\nu}$). These are parametrized by

$$\xi^\mu \leftrightarrow \begin{cases} l \\ f_a \\ f \end{cases} , \tag{A.28}$$

with l satisfying (A.19). We are now faced with the subtle issue of disentangling pure gauge diffeos vs. asymptotic symmetries. The standard way to discern between the two is by studying the corresponding Hamiltonians [67]. We will not go into this type of analysis here, but instead summarize the different prescriptions discussed in the literature.

In all the literature we are aware of, the diffeos f_a are treated as pure gauge. For instance, in the geometric description of Ashtekar and Hansen [19] they do not feature at all, while in the Beig-Schmidt treatment [15], they are fixed by the condition

$$\gamma_a = 0 \implies f_a = l_a - \partial_a f \quad \text{Beig-Schmidt gauge.} \tag{A.29}$$

Likewise, the harmonic gauge condition fixes them, see (A.36) below.

[18] $\mathcal{B}_{ab}$ is symmetric and traceless and so it has five independent components.

The treatment of [19] does not either feature log translations and only deals with the diffeos spanned by f (referred to as “Spi supertranslations” in the spatial infinity context). In [19] these are frozen by imposing the vanishing of k_{ab},

$$k_{ab} = 0 \implies D_a D_b f - h_{ab} f = 0 \tag{A.30}$$
$$\implies f = -V_\mu T^\mu, \ \text{ with } \partial_a T^\mu = 0. \tag{A.31}$$

The “residual” f-diffeomorphisms left out by this condition are just spacetime translations, parametrized by constant vectors T^μ.

However, as argued in [26] (see also [22, 35, 66, 68]) in order to realize BMS supertranslations at timelike infinity, one needs to impose a condition weaker than (A.30). In its more general form, as given in [66], this condition fixes the trace of k_{ab} to a given function $\bar{k}$ on the hyperboloid

$$k^a_a = \bar{k} \implies (D^2 - 3)f = 0. \tag{A.32}$$

The family of functions f satisfying (A.32) can then be shown to be in one-to-one correspondence with BMS supertranslations [26].

From the Beig-Schmidt perspective, (A.32) appears as an additional condition on top of (A.29). Interestingly, as reviewed below, in harmonic gauge this condition appears (almost) automatically.

Harmonic gauge residual diffeos

Using (A.14), the harmonic gauge condition (3.28) on (3.31) reads

$$V_\nu \overset{1}{e}{}^{\mu\nu} + D^a V_\nu D_a \overset{1}{e}{}^{\mu\nu} = 0, \tag{A.33}$$

or, in terms of the hyperbolic components defined in section A.1:

$$D^b \overset{1}{e}_{ab} - 3\overset{1}{e}_a = 0, \tag{A.34}$$
$$-2\overset{1}{e}_{\parallel} - \overset{1}{e}_{\perp} + D^b \overset{1}{e}_b = 0. \tag{A.35}$$

Applying a general asymptotic diffeo on these equations leads to

$$g_a := (D^2 + 1) f_a + 2\partial_a f + 2\partial_a l = 0, \tag{A.36}$$
$$g := -2D^b f_b - D^2 f - 3f + 2l = 0. \tag{A.37}$$

These are the asymptotic harmonic gauge fixing equations. The first one is the analogue of eq. (A.29), as it determines f_a in terms of f and l (although this time through a differential equation). The second one constrains f and l. To find how, consider first the divergence of the first equation

$$D^a g_a = D^2 D^a f_a - D^a f_a + 2D^2 f + 6l. \tag{A.38}$$

Solving (A.37) for $D^b f_b$ and substituting in (A.38) one finds

$$D^a g_a|_{g=0} = -\frac{1}{2}\left(D^2 + 1\right)\left((D^2 - 3)f - 4l\right) = 0, \tag{A.39}$$

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where we used $D^2 l = 3l$ to factor out the operator (D^2+1). To study (A.39), let us rewrite it as a system of two equations for three unknowns

$$\left(D^2+1\right)\psi = 0 \tag{A.40}$$

$$(D^2-3)f = 4l - \psi. \tag{A.41}$$

By general arguments of Laplace-type equations in hyperbolic space (see e.g. [59]) one knows that solutions to (A.40) behave as $\psi = O(\ln\rho/\rho)$ for large ρ (see also appendix B.1 of [69] for explicit integral solutions). On the other hand, eq. (A.41) admits a formal integral solution for f in terms of the Green's function $\mathcal{G}$ (3.19), with $4l-\psi$ playing the role of source. However, since $l = O(\rho)$, $\mathcal{G} = O(1/\rho^3)$ and the volumen element scales as $\sqrt{h} = O(\rho)$, the integral is logarithmically divergent.[19] We thus conclude that l must vanish.

The resulting system of equations still allows for a non-trivial ψ. We do not know whether this possibility is allowed by the full original set of equations, (A.36) and (A.37). The $\psi = 0$ case leads to the vector fields considered in [26], which requires the asymptotic vanishing of their divergence.[20] In our notation this condition reads $0 = \delta_\xi\gamma = 2D^b f_b + 6f + 2l$, which, in conjunction to (A.37) leads to $(D^2-3)f = 4l = 0$.

Vanishing of the asymptotic magnetic Weyl curvature

We now show that the asymptotic metric (3.33) leads to a tensor k_{ab} of the form (A.25), which in turn implies $\mathcal{B}_{ab} = 0$. We start by rewriting (A.22) in terms of the hyperbolic components of $\overset{1}{e}_{\mu\nu}$,

$$k_{ab} = 2\overset{1}{e}_{ab} + 4D_{(a}\overset{1}{e}_{b)} - 2\overset{1}{e}_{\perp}h_{ab}. \tag{A.42}$$

Using (3.33) and treating separately the trace-free and trace part of (A.42) leads to

$$k_{\langle ab\rangle} = 4G\int d^3x' \frac{(\chi^2-3)\partial_{\langle a}\chi\partial_{b\rangle}\chi}{(\chi^2-1)^{3/2}}\rho(x'), \tag{A.43}$$

$$k = 16G\int d^3x' \frac{\chi^2}{\sqrt{\chi^2-1}}\rho(x'). \tag{A.44}$$

From the identities given in appendix A.1, one can show that $D_{\langle a}D_{b\rangle}f(\chi) = f''(\chi)\partial_{\langle a}\chi\partial_{b\rangle}\chi$ for any function $f(\chi)$. This allow us to pull out the derivatives in (A.43) and write

$$k_{\langle ab\rangle} = -2D_{\langle a}D_{b\rangle}\Phi \tag{A.45}$$

with

$$\Phi := -2G\int d^3x'\left(\sqrt{\chi^2-1} + \chi\tanh^{-1}\left(\tfrac{\chi}{\sqrt{\chi^2-1}}\right)\right)\rho(x'). \tag{A.46}$$

Finally, using that $D^2 f(\chi) = f''(\chi)(\chi^2-1) + 3f'(\chi)\chi$ one can show that

$$k = -2(D^2-3)\Phi. \tag{A.47}$$

Ecs. (A.45) and (A.47) imply (A.25), with Φ given by (A.46).

[19]The coefficient of this divergence is proportional to $\int d^2\hat{n}(L\cdot n)/(V\cdot n)^3 \sim L\cdot V$.

[20]We are excluding superrotations in the comparison with [26]. The analysis of [26] did not consider eq. (A.36), but the present discussion shows that this equation is consistent with the requirement $\delta_\xi\gamma = 0$.

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B Log soft factor in terms of particle momenta

In this appendix we make contact with the type of expressions presented in [3, 6, 7]. Let us start by unpacking (4.10). The explicit expression of $h^{(1)\mathrm{div}}_{\mu\nu}$ in terms of the deviation vector is

$$h^{(1)\mathrm{div}}_{\mu\nu} = 4G \sum_i \left(\frac{p^i_\mu p^i_\nu}{p_i \cdot n} c_i \cdot n - p^i_{(\mu} c^i_{\nu)} \right). \tag{B.1}$$

Combining this with the first term in (4.10) leads to

$$h^{(\log)}_{\mu\nu} = 4G \sum_i \frac{p^i_\mu p^i_\nu}{p_i \cdot n} (c_i - c_{\mathcal{I}^+}) \cdot n - 4G \sum_i p^i_{(\mu} c^i_{\nu)}. \tag{B.2}$$

Notice that in this form, the invariance under log translations is manifest: the first sum is invariant term by term, while the second sum is invariant thanks to momentum conservation. It is also clear that there is no individual contribution from outgoing massless particles in the first sum since $c^\mu_i = c^\mu_{\mathcal{I}^+}$ for them.

For simplicity, we now restrict attention to the case where there is incoming radiation. The log soft factor then takes the form

$$h^{(\log)}_{\mu\nu} = 4G \sum_{\substack{i \\ m_i \neq 0}} \frac{p^i_\mu p^i_\nu}{p_i \cdot n} (c_i - c_{\mathcal{I}^+}) \cdot n - 4G \sum_i p^i_{(\mu} c^i_{\nu)} \quad \text{(no incoming radiation)}. \tag{B.3}$$

In harmonic gauge, the deviation vectors in (B.3) are

$$\begin{aligned} c^\mu_{\mathcal{I}^+} &= 2G P^\mu_{\text{total}+} \\ c^\mu_\pm &= \mathring{c}^\mu_\pm \pm 2G P^\mu_{\text{massless}\pm} \end{aligned} \qquad \text{(harmonic log frame)} \tag{B.4}$$

where $\mathring{c}^\mu_\pm$ is given in eq. (3.55) and $P^\mu_{\text{massless}-} = 0$ in the current case of no incoming radiation. Substituting (B.4) in (B.3), and splitting the second sum in (B.3) into massive and massless contributions, one gets

$$h^{(\log)}_{\mu\nu} = 4G \sum_{\substack{i \\ m_i \neq 0}} \left(\frac{p^i_\mu p^i_\nu}{p_i \cdot n} \mathring{c}_i \cdot n - p^i_{(\mu} \mathring{c}^i_{\nu)} \right) \tag{B.5}$$

$$+ 8G^2 n \cdot P_{\text{massive}+} \sum_{\substack{i \in \text{out} \\ m_i \neq 0}} \frac{p^i_\mu p^i_\nu}{p_i \cdot n} + 8G^2 n \cdot P_{\text{massive}-} \sum_{\substack{i \in \text{in} \\ m_i \neq 0}} \frac{p^i_\mu p^i_\nu}{p_i \cdot n}$$

$$+ 8G^2 P^{\text{massive}+}_\mu P^{\text{massive}+}_\nu - 8G^2 P^{\text{massive}-}_\mu P^{\text{massive}-}_\nu. \tag{B.6}$$

The first two lines can be written in terms of double sums over massive particles, leading to an expression that has the same form as the full log soft factor, except that massless contributions are excluded. The last line then compensates for this exclusion. Eq. (B.5) corresponds to the Sahoo-Sen "rewritten" version of the log soft factor [6, 7].

It is also interesting to see how (B.5) is recovered by considering (B.2) in the future radiative frame. There, $c_{\mathcal{I}^+} = 0$ and (B.2) simplifies to

$$h^{(\log)}_{\mu\nu} = 4G \sum_{\substack{i \\ m_i \neq 0}} \left(\frac{p^i_\mu p^i_\nu}{p_i \cdot n} c_i \cdot n - p^i_{(\mu} c^i_{\nu)} \right) \quad \text{(rad}^+ \text{ log frame, no incoming radiation).} \tag{B.7}$$

The deviation vectors are now given by

$$c^\mu_\pm = \mathring{c}^\mu_\pm \mp 2GP^\mu_{\text{massive}\pm} \quad \text{(rad}^+ \text{ log frame).} \tag{B.8}$$

Substituting (B.8) in (B.7) one can check that the expression coincides with (B.5).

Data Availability Statement. This article has no associated data or the data will not be deposited.

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Capítulo 4

Prueba del teorema soft logarítmico

En este trabajo se presenta una prueba del teorema soft logarítmico (1.16) utilizando el formalismo asintótico descrito anteriormente. Para ello, dicho teorema se reinterpreta en términos de las variables de Bondi, en particular mediante la componente proporcional a $1/u$ de la cizalla C_{AB}. Posteriormente se resuelven las ecuaciones en todas las regiones asintóticas del espacio-tiempo. En particular, esto requiere resolver ecuaciones en derivadas parciales elípticas e hiperbólicas cuya solución general no se conocía previamente. Mediante condiciones de matching (o "pegado") se logra relacionar el campo C_{AB} con la materia presente en el scattering.

Finalmente, el trabajo responde a otra cuestión relacionada con el teorema soft logarítmico. Las expresiones obtenidas previamente para este teorema no respetan la invarianza bajo inversión temporal $t \to -t$. Esto resulta llamativo, ya que la teoría subyacente —la relatividad general— sí posee dicha simetría. La explicación radica en que son las condiciones iniciales, en particular la hipótesis de ausencia de radiación incidente, las que rompen esta invarianza. En el Apéndice A se muestra explícitamente que, al incluir la contribución de la radiación incidente, se recupera la invariancia bajo inversiones temporales.

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An asymptotic proof of the classical log soft graviton theorem

Gianni Boschetti[a] **Miguel Campiglia**[b]

[a] *Instituto de Física, Facultad de Ingeniería, Universidad de la República, Julio Herrera y Reissig 565, Montevideo, Uruguay*

[b] *Instituto de Física, Facultad de Ciencias, Universidad de la República , Igua 4225, Montevideo, Uruguay*

E-mail: gboschetti@fing.edu.uy, miguel.campiglia@fcien.edu.uy

Abstract: We present a derivation of the classical log soft graviton theorem within the asymptotic framework of Compère, Gralla, and Wei. The proof relies solely on Einstein equations near timelike, spatial, and null infinity, together with matching properties across these regions. The approach is fully covariant under time reversal and incorporates contributions from incoming soft radiation. In the absence of incoming memory one recovers the standard log soft factor, which features an asymmetry between future and past hard components. From an asymptotic perspective, the origin of this asymmetry lies in a long-known discontinuity of the gravitational field at spatial infinity.

Contents

1 Introduction

In [1] Penrose introduced his celebrated conformal description of spacetime, partly motivated by "a longer term aim . . . for a covariant S-matrix theory incorporating gravitation". This marked the beginning of what may be called an *asymptotic* approach to gravitational scattering, which avoids relying on the Minkowski background built into perturbative treatments. Traditionally, the successes of this approach were mostly kinematical, e.g. the characterization of asymptotic states and charges [2–4]. Over the past decade it has become clear that the asymptotic perspective is specially suited for addressing dynamical questions within the soft sector of gravitational radiation [5]. In particular, a variety of *soft theorems* admit a natural formulation in terms of matching properties of the gravitational field across timelike, null and spatial infinity [6]. In this paper we deepen this viewpoint by presenting an asymptotic proof of the classical log soft graviton theorem [7–10].[1] We build on and further develop the framework put forward by Compère, Gralla and Wei (CGW) [18] where the three types of infinities are treated on equal footing (see Figure 1[2]). We refer to [19, 20] for earlier work in this direction and [21] for a very recent treatment addressing similar aspects to those discussed here.

Although our contribution is limited to the soft sector, we are motivated by the hope that advancing the CGW framework may serve the longer-term aim of a fully covariant theory of gravitational scattering. This perspective is in line with recent efforts towards boundary descriptions of the S-matrix [22–25] in the context of flat space holography (see [26–28] and references therein).

The plan of the paper is as follows. Section 2 provides some basic preliminaries that permeate our discussion, including notation and conventions. In section 3 we review the leading and log classical soft graviton theorems, extending the latter to the case where there is incoming soft gravitational radiation. Section 4 is a self-contained exposition of the main points of the CGW framework that we shall need for the asymptotic proof, along with several extensions, notably explicit Green's functions solutions to the asymptotic metric coefficients at timelike and spatial infinities. The results of this analysis are put together in section 5 to provide a proof of the log soft theorem. The paper is supplemented with six appendices containing technical material.

2 Preliminaries

2.1 Asymptotic geodesics

In asymptotic Cartesian coordinates, the general asymptotic form of geodesics that escape to infinity is

$$X^\mu(s) \overset{|s|\to\infty}{=} sV^\mu + \log|s|\, c^\mu(V) + O(s^0), \tag{2.1}$$

[1]See [11–17] for related discussions in the context of electromagnetism and massless scalars.

[2]The diagrams in Figure 1 only intend to capture the exterior boundaries of spacetime, i.e. those associated to geodesics that escape to infinity. There can also be internal boundaries, notably in presence of black holes. In the asymptotic picture, black holes (or any other compact object) are registered as point-like sources at $\mathcal{H}^\pm$ with multipolar moments appearing at subleading order in the large time expansion. We thank G. Compère for discussions on this point.

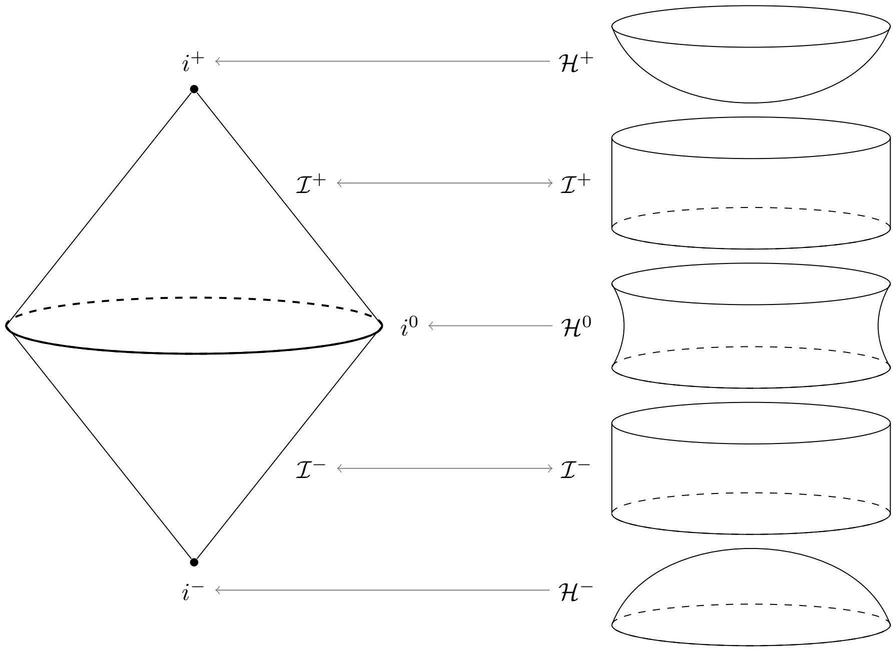

Figure 1. Left: Penrose diagram of an asymptotically flat spacetime. Right: A "democratic" representations of null, timelike and spatial infinities. In both cases the null infinities $\mathcal{I}^\pm$ are described by cylinders. The timelike $\mathcal{H}^\pm$ and spatial $\mathcal{H}^0$ infinities on the right diagram are hyperboloids parametrizing the directions of asymptotic geodesics at the points $i^\pm$ and i^0 on the left diagram.

where s is an affine parameter and V^μ the asymptotic direction. c^μ captures the leading deviation from a straight line, which we refer to as *log deviation vector*.[3] This vector is intimately tied with *logarithmic translations* [29, 30], diffeomorphisms that have the asymptotic form

$$\xi^\mu_L \overset{|s|\to\infty}{=} \log|s|\, L^\mu + \cdots\,, \tag{2.2}$$

and that act on c^μ according to

$$\delta_L c^\mu = -L^\mu. \tag{2.3}$$

The possible asymptotic directions V^μ fall into five categories: future/past null, future/past timelike and spacelike, corresponding to the five asymptotic boundaries pictured in the right panel of Figure 1.

For null rays, we parametrize the asymptotic directions by

$$n^\mu = (1, \hat{n}), \tag{2.4}$$

where $\hat{n}$ is a unit 3-vector. Eq. (2.1) then takes the form

$$X^\mu(s) \overset{s\to\pm\infty}{=} s\, n^\mu + \log|s|\, c^\mu_{\mathcal{I}^\pm} + O(s^0) \quad \text{(null geodesics)}, \tag{2.5}$$

[3] In terms of asymptotic Christoffel symbols, $c^\mu(V) = \lim_{|s|\to\infty} s^2 \Gamma^\mu_{\nu\rho}(sV)V^\nu V^\rho$. Our minimal assumption on asymptotic flatness is that this limit is well defined.

where incoming/outgoing null rays are distinguished by the sign of s. A key property of asymptotic null geodesics is that their log deviation vector $c^\mu_{\mathcal{I}^\pm}$ is independent of the asymptotic direction [8, 31]. This allows one to "gauge it away" by a log translation, as expected by general results on the existence of radiative coordinates [32]. In such (future/past) *radiative log frame*, the $O(s^0)$ piece of (2.5) can be used to distinguish different arrival/departure times. By an appropriate choice of $O(s^0)$ parametrization, Eq. (2.5) can be brought into the form

$$X^\mu(s) \overset{s\to\pm\infty}{=} s\, n^\mu + u\, t^\mu + \cdots \quad \text{(null geodesics in radiative log frame)}, \tag{2.6}$$

where $t^\mu = (1, \vec{0})$, u is a retarded/advanced time and the dots refer to terms that vanish in the large s limit. The "endpoints" of such null geodesics, parametrized by $(u, \hat{n})$, define the null infinities $\mathcal{I}^\pm$.

For non-null geodesics, we normalize the asymptotic velocity such that s becomes a proper time/distance. The log deviation vector now exhibits a non-trivial dependance on the asymptotic velocity and cannot be gauged away.[4] Eq. (2.1) then takes the form

$$X^\mu(s) \overset{s\to\pm\infty}{=} sV^\mu + \log|s| c^\mu_{\mathcal{H}^\pm}(V) + O(s^0), \quad V^\mu V_\mu = -1, \quad V^0 > 0, \tag{2.7}$$

$$X^\mu(s) \overset{s\to\infty}{=} sV^\mu + \log|s| c^\mu_{\mathcal{H}^0}(V) + O(s^0), \quad V^\mu V_\mu = 1, \tag{2.8}$$

for timelike/spacelike geodesics. Their corresponding "endpoints" are parametrized by the unit hyperboloids $\mathcal{H}^\pm$ and $\mathcal{H}^0$. Note that, as in the null case, our conventions are such that $V^0 > 0$ for both incoming and outgoing timelike geodesics.

In the following subsection we present the coordinates that we shall use on these "endpoints at infinity" spaces, see Table 1 for a summary. The extension of these coordinates into spacetime is discussed in section 4.

space	coordinates	infinity type
$\mathcal{I}^\pm$	(u, ϕ^A)	null
$\mathcal{H}^\pm$	$(\rho, \phi^A) = x^a$	timelike
$\mathcal{H}^0$	$(\tau, \phi^A) = x^a$	spatial

Table 1. Coordinates at the five infinities.

2.2 Coordinate conventions

All five infinities feature an angular direction. We denote by ϕ^A the corresponding 2d coordinates. These specify the unit 3-vector $\hat{n}$ and associated null direction n^μ (2.4). The null infinities are therefore parametrized by (u, ϕ^A) where u is a retarded/advanced time. For the most part the specific choice of ϕ^A will be irrelevant. However, some expressions

[4] This velocity-dependence is related to the non-smoothness of the Penrose metric at the points $i^0, i^\pm$ [19].

simplify considerably in holomorphic coordinates $(z,\bar z)$ where

$$\hat n = \frac{1}{1+|z|^2}\left(z+\bar z, -i(z-\bar z), 1-|z|^2\right). \tag{2.9}$$

For the unit timelike hyperboloids we define 3d coordinates $x^a = (\rho, \phi^A)$ by

$$V^\mu(x) = \left(\sqrt{\rho^2+1}, \rho\,\hat n(\phi)\right) \quad \iff \quad V^\mu V_\mu = -1, \quad V^0 > 0, \tag{2.10}$$

where $\rho \in [0,\infty)$ is a “boost radius”.

For the unit spatial hyperboloid we take $x^a = (\tau, \phi^A)$ with

$$V^\mu(x) = \left(\tau, \sqrt{\tau^2+1}\,\hat n(\phi)\right) \quad \iff \quad V^\mu V_\mu = 1, \tag{2.11}$$

where $\tau \in (-\infty,\infty)$ is a “boost time”.

A key element of our discussion will be to understand the behavior of fields on these spaces as one approaches their asymptotic boundaries. At timelike infinity this boundary is denoted as $\partial\mathcal{H}^\pm$ and reached when $\rho\to\infty$. In this limit the direction (2.10) asymptotes to,

$$V^\mu(\rho,\phi) \overset{\rho\to\infty}{=} \rho\, n^\mu(\phi) + O(1/\rho). \tag{2.12}$$

At spatial infinity there are two boundaries denoted by $\partial_\pm\mathcal{H}^0$, corresponding to $\tau \to \pm\infty$. In these limits, the direction (2.11) becomes

$$V^\mu(\tau,\phi) \overset{\tau\to+\infty}{=} \tau\, n^\mu(\phi) + O(1/\tau), \tag{2.13}$$

$$V^\mu(\tau,\phi) \overset{\tau\to-\infty}{=} \tau\, \mathcal{A}_* n^\mu(\phi) + O(1/\tau) \tag{2.14}$$

where $\mathcal{A}$ is the antipodal map on the the sphere and $\mathcal{A}_*$ its corresponding pullback,

$$\mathcal{A}_*\hat n(\phi) = \hat n(\mathcal{A}\phi) = -\hat n(\phi). \tag{2.15}$$

It is interesting to see how the null geodesics (2.5) are recovered from the timelike/spatial ones under the above limits. The divergent factors multiplying the null directions in (2.12), (2.13) and (2.14) indicate the need to simultaneously rescale the affine parameter s in (2.7), (2.8). Such rescaling, however, does not affect the corresponding log deviation vectors, from which one concludes that

$$\begin{aligned} \lim_{\rho\to\infty} c^\mu_{\mathcal{H}^\pm}(\rho,\phi) &= c^\mu_{\mathcal{I}^\pm}, \\ \lim_{\tau\to\pm\infty} c^\mu_{\mathcal{H}^0}(\tau,\phi) &= c^\mu_{\mathcal{I}^\pm}. \end{aligned} \tag{2.16}$$

These relations provide a first example of a matching property across infinities. In this case, the matching can be used to define a global notion of log translations that simultaneously act on all five infinities. In particular, it allows to discuss log translation invariance in a scattering setting, see [31] and subsection 3.

2.3 Log translation frames

In actual computations, log translations are usually gauge-fixed. A simple way to do so is by prescribing the value of *either* $c^\mu_{\mathcal{I}^+}$ or $c^\mu_{\mathcal{I}^-}$. There is an obstruction to assign independent values to these quantities, due to a well-known discontinuity at spatial infinity [18, 19, 33] that fixes their difference by

$$c^\mu_{\mathcal{I}^+} - c^\mu_{\mathcal{I}^-} = 4GP^\mu, \tag{2.17}$$

where P^μ is the total spacetime momentum, see [31] and section 4.2 for further details.

We shall deal with three fixings or *log translation frames* that are summarized in Table 2. The (future and past) radiative frames were already mentioned in Eq. (2.6). The harmonic frame is the one associated to harmonic coordinates and leads to a parity-even gravitational potential at spatial infinity [30].

Log frame	$c^\mu_{\mathcal{I}^+}$	$c^\mu_{\mathcal{I}^-}$
Future radiative	0	$-4GP^\mu$
Past radiative	$4GP^\mu$	0
Harmonic	$2GP^\mu$	$-2GP^\mu$

Table 2. Values of $c^\mu_{\mathcal{I}^\pm}$ on three different log translation frames.

Looking down either column in Table 2 we can identify the log translation (2.3) that interpolates between frames. Denoting by $c^{\text{rad}\pm}_\mu, c^{\text{harm}}_\mu$ the corresponding log deviation vectors, we have:

$$c^{\text{rad}+}_\mu - c^{\text{rad}-}_\mu = -4GP_\mu, \tag{2.18}$$
$$c^{\text{harm}}_\mu - c^{\text{rad}\pm}_\mu = \pm 2GP_\mu. \tag{2.19}$$

2.4 Asymptotic particles

Following standard practice [10], we treat as *particles* the elementary constituents in a scattering processes, even though they can refer to macroscopic bodies or radiation. This perspective is specially suited for the discussion of soft theorems, as they are insensitive to the internal structure of the scattering entities.

We use an index i to label all such "particles", including incoming, outgoing, massive and massless. The momentum of a particle with mass m_i and asymptotic trajectory (2.7) is defined as[5]

$$\begin{aligned} p^\mu_i &= m_i V^\mu_i \quad \text{if} \quad i \in \text{out}, \\ p^\mu_i &= -m_i V^\mu_i \quad \text{if} \quad i \in \text{in}, \end{aligned} \tag{2.20}$$

where, following amplitudes conventions, incoming momenta carry an overall negative sign. Massless particles can be described by sending $m_i \to 0$ and $\rho \to \infty$ with $E_i = m_i \rho$ held fixed. The total momentum is given by

$$P^\mu = \sum_{i \in \text{out}} p^\mu_i = -\sum_{i \in \text{in}} p^\mu_i. \tag{2.21}$$

[5] We will also use the notation "$i \in +$" and "$i \in -$" for outgoing/incoming particles respectively.

We denote by c_i the log deviation vector of the i-th particle, omitting the labels $\mathcal{H}^\pm$ or $\mathcal{I}^\pm$ used in Eqs. (2.5) and (2.7). A well known consequence of this vector is that it leads to a divergence in the particle's angular momentum [7],

$$J^i_{\mu\nu} \overset{s\to\pm\infty}{=} \log|s| J^{\mathrm{div}\,i}_{\mu\nu} + O(s^0), \tag{2.22}$$

where

$$J^{\mathrm{div}\,i}_{\mu\nu} = c^i_{[\mu}\, p^i_{\nu]} = c^i_\mu p^i_\nu - c^i_\nu p^i_\mu. \tag{2.23}$$

Important for our analysis is the fact that (2.23) is sensitive to log translations. We denote by $J^{\mathrm{rad}\pm\, i}_{\mu\nu}, J^{\mathrm{harm}\, i}_{\mu\nu}$ the value of this quantity in the corresponding log frame. According to Eqs. (2.18) and (2.19) these are related by

$$J^{\mathrm{rad}+\, i}_{\mu\nu} = J^{\mathrm{rad}-\, i}_{\mu\nu} - 4GP_{[\mu}\, p^i_{\nu]}, \tag{2.24}$$
$$J^{\mathrm{harm}\, i}_{\mu\nu} = J^{\mathrm{rad}\pm\, i}_{\mu\nu} \pm 2GP_{[\mu}\, p^i_{\nu]}. \tag{2.25}$$

3 Classical soft theorems

Gravitational radiation is encoded in the leading deviation from the flat metric along null rays. In asymptotic radiative coordinates of the type (2.6), the outgoing gravitational waveform is defined by

$$h^{\mathrm{out}}_{\mu\nu}(u,\hat{n}) := \lim_{r\to\infty} r(g_{\mu\nu} - \eta_{\mu\nu}), \tag{3.1}$$

where $r = s$ is the asymptotic radial distance. The classical soft theorems refer to universal components in the low frequency expansion of the Fourier transform of (3.1),

$$\tilde{h}^{\mathrm{out}}_{\mu\nu}(\omega,\hat{n}) \overset{\omega\to 0}{=} \omega^{-1}\tilde{h}^{(0)}_{\mu\nu}(\hat{n}) + \log\omega\, \tilde{h}^{(\log)}_{\mu\nu}(\hat{n}) + \cdots \tag{3.2}$$

The leading term is described by Weinberg's soft theorem [34]

$$\tilde{h}^{(0)}_{\mu\nu} = -4Gi\sum_i \frac{p^i_\mu p^i_\nu}{p_i\cdot n}, \tag{3.3}$$

and captures what is known as the gravitational memory effect [35, 36]. The logarithmic term in (3.2) can be written as a sum of two contributions [7, 8],

$$\tilde{h}^{(\log)}_{\mu\nu} = \tilde{h}^{(\mathrm{div})}_{\mu\nu} + \tilde{h}^{(\mathrm{drag})}_{\mu\nu}. \tag{3.4}$$

The first one originates from the divergence in the particle angular momenta (2.22) and reads[6]

$$\tilde{h}^{(\mathrm{div})}_{\mu\nu} = -4G\sum_i \frac{p^i_{(\mu} J^{\mathrm{div}\,i}_{\nu)\rho} n^\rho}{p_i\cdot n}. \tag{3.5}$$

[6]This is of the same form as the $O(G)$ subleading soft factor [37], except that it features the divergent angular momentum.

The second contribution arises from the propagation of the soft radiation on the curved background due to the "hard" particles and is given by [8]

$$\tilde{h}^{(\text{drag})}_{\mu\nu} = -i\, n \cdot c_{\mathcal{I}^+}\, \tilde{h}^{(0)}_{\mu\nu}. \tag{3.6}$$

The analysis of [7, 8] derive these results through perturbative calculations in harmonic gauge, where $J^{\text{div}\,i}_{\mu\nu}$ and $c^{\mu}_{\mathcal{I}^+}$ are evaluated in the harmonic log frame. It was however noted in [31] that the sum of (3.5) and (3.6) is invariant under log translations, and hence it holds in *any* log frame. In particular, in the future radiative frame where $J^{\text{div}}_{\mu\nu} = J^{\text{rad}+}_{\mu\nu}$ and $c^{\mu}_{\mathcal{I}^+} = 0$ one has

$$\tilde{h}^{(\log)}_{\mu\nu} = -4G \sum_i \frac{p^i_{(\mu} J^{\text{rad}+\,i}_{\nu)\rho} n^\rho}{p_i \cdot n}. \tag{3.7}$$

This rewriting makes explicit the independence of the log soft factor on outgoing massless particles [8, 39] since $J^{\text{rad}+}_{\mu\nu} = 0$ for them.

For the asymptotic proof of the log soft theorem we will use yet a different rewriting of $\tilde{h}^{(\log)}_{\mu\nu}$. In the asymptotic analysis of Einstein equations one works with future/past radiative frames for future/past timelike and null infinities. On the other hand, (3.5) and (3.7) express future and past contributions in the *same* log frame. Using (2.24) on the incoming terms in (3.7) allows to express the log soft factor as

$$\begin{aligned}\tilde{h}^{(\log)}_{\mu\nu} = -4G \sum_{\substack{i\in\text{out}\\ m_i\neq 0}} \frac{p^i_{(\mu} J^{\text{rad}+\,i}_{\nu)\rho} n^\rho}{p_i \cdot n} - 4G \sum_{\substack{i\in\text{in}\\ m_i\neq 0}} \frac{p^i_{(\mu} J^{\text{rad}-\,i}_{\nu)\rho} n^\rho}{p_i \cdot n} \\ - 16G^2 \left(P \cdot n \sum_{i\in\text{in}} \frac{p^i_\mu p^i_\nu}{p_i \cdot n} + P_\mu P_\nu \right),\end{aligned} \tag{3.8}$$

where the sums in the first line are restricted to massive particles since $J^{\text{rad}\pm\,i}_{\mu\nu} = 0$ for outgoing/incoming massless particles.

This is the form of the log soft theorem that naturally emerges in the asymptotic analysis (modulo incoming soft contributions discussed below). In that context, the terms in the first line of (3.8) arise from future/past contributions in the corresponding radiative frames, whereas the second line compensates for the mismatch (2.17) between future and past frames.

Incoming soft radiation

Incoming gravitational radiation is encoded in the past infinity version of (3.1), which we write as

$$h^{\text{in}}_{\mu\nu}(u, \hat{n}) := \lim_{r\to-\infty} r(g_{\mu\nu} - \eta_{\mu\nu}), \tag{3.9}$$

where $r = s$ is *minus* the radial distance in the asymptotic coordinates (2.6). With these conventions we have[7]

$$h^{\text{out}}_{\mu\nu}(u,\hat{n}) = h^{\text{in}}_{\mu\nu}(u,\hat{n}) \quad \text{for } O(G^0) \text{ (free-field) propagation.} \tag{3.10}$$

It is natural to assume, and we will do so, that the Fourier transform of (3.9) admits a soft expansion as (3.2)

$$\tilde{h}^{\text{in}}_{\mu\nu}(\omega,\hat{n}) \overset{\omega\to 0}{=} \omega^{-1}\tilde{h}^{(0)\text{in}}_{\mu\nu}(\hat{n}) + \log\omega\, \tilde{h}^{(\log)\text{in}}_{\mu\nu}(\hat{n}) + \cdots \tag{3.11}$$

However, in our previous discussion (and generally in the literature) incoming gravitational radiation, if present at all, is considered to be purely hard:

$$\text{Implicit assumption so far:} \quad \tilde{h}^{(0)\text{in}}_{\mu\nu} = \tilde{h}^{(\log)\text{in}}_{\mu\nu} = 0. \tag{3.12}$$

In presence of non-trivial incoming soft terms, the formulas (3.3) and (3.4) acquire additional contributions,

$$\tilde{h}^{(0)}_{\mu\nu} \to \tilde{h}^{(0)}_{\mu\nu} + \tilde{h}^{(0)\text{in}}_{\mu\nu}, \tag{3.13}$$
$$\tilde{h}^{(\log)}_{\mu\nu} \to \tilde{h}^{(\log)}_{\mu\nu} + \tilde{h}^{(\log)\text{in}}_{\mu\nu} - 4iG\, n\cdot P\, \tilde{h}^{(0)\text{in}}_{\mu\nu}. \tag{3.14}$$

From a perturbative perspective, the first corrections in (3.13) and (3.14) come from the free propagation of the incoming radiation (3.10). The second correction in (3.14) represents a drag term on the incoming soft radiation, and can be obtained by "completing" with $\tilde{h}^{(0)\text{in}}_{\mu\nu}$ the incoming leading soft factor in the second line of (3.8).[8] All the terms in (3.13) and (3.14) naturally arise in the asymptotic proofs of section 5.

A benefit of including incoming soft radiation is that it makes manifest the time-reversal symmetry of the soft theorems. Whereas this is straightforward for the leading soft theorem [40], it is less so for the logarithmic one, due to the second line in (3.8). However, upon including the corrections (3.14) *and* taking into account the leading soft theorem, one can establish the time-reversal covariance of the log soft theorem, see appendix A for details.

[7] Eq. (3.10) can be shown by a writing the free-field as a Fourier integral, see e.g. [40]. Notice there is no antipodal map on the celestial sphere since we use the same null vector (2.4) for both future and past null rays (2.6).

[8] A naive guess for this contribution is obtained by substituting (3.13) in (3.6). The result of this substitution, however, depends on the log frame in which is performed. In the harmonic log frame where $c^{\mu}_{\mathcal{I}^+} = 2GP^{\mu}$, this gives half of what is claimed in (3.14). The mismatch can be understood from the fact that (3.6) only accounts for outgoing drag, while incoming soft radiation also experiences incoming drag. The naive substitution yields the correct result if performed in the past radiative frame, for which there is no incoming drag and $c^{\mu}_{\mathcal{I}^+} = 4GP^{\mu}$.

4 Spacetime asymptotics

4.1 Timelike infinity

To describe events near future/past timelike infinity, we consider asymptotic coordinates (τ, x^a) obtained by setting $s = \tau$ and $V^\mu = V^\mu(x)$ in (2.7),

$$X^\mu(\tau, x) \overset{\tau\to\pm\infty}{=} \tau V^\mu(x) + \cdots , \quad V^\mu V_\mu = -1, \quad V^0 > 0, \tag{4.1}$$

with $V^\mu(x)$ given by Eq. (2.10). The dots in (4.1) depend on the choice of gauge. We adopt Beig-Schmidt (BS) conditions [18, 41],

$$g_{\tau a} = 0, \quad g_{\tau\tau} = -\left(1 + \frac{\sigma}{\tau}\right)^2, \tag{4.2}$$

as they allow to separate Einstein equations into a series of elliptic equations on $\mathcal{H}^\pm$ that can be solved recursively [18, 41].

The remaining metric components admit a large τ expansion [18]

$$g_{ab} = \tau^2 h_{ab} + \tau(k_{ab} - 2\sigma h_{ab}) + \log|\tau|\, i_{ab} + O(\tau^0), \tag{4.3}$$

where $h_{ab} = \partial_a V^\mu \partial_b V_\mu$ is the unit hyperboloid metric, used to raise and lower indices $a, b, \ldots$. In the coordinates $x^a = (\rho, \phi^A)$ (2.10) this metric reads

$$h_{ab}dx^a dx^b = \frac{d\rho^2}{\rho^2 + 1} + \rho^2 d\Omega^2, \tag{4.4}$$

with $d\Omega^2 = \partial_A \hat{n} \cdot \partial_B \hat{n}\, d\phi^A d\phi^B$ the unit sphere line element. The rest of the coefficients in (4.2) and (4.3) are fields on $\mathcal{H}^\pm$ that we describe below. To avoid clutter, we omit future/past labels on them unless required.[9]

The scalar σ in (4.2) plays the role of a potential for the asymptotic electric part of the Weyl tensor [19, 42]. It also captures the log deviation vector according to [31]

$$c^\mu = D^a V^\mu \partial_a \sigma - V^\mu \sigma. \tag{4.5}$$

The tensor k_{ab} is a potential for the magnetic part of the Weyl tensor [19, 42] which vanishes under standard asymptotic flatness conditions. This implies it takes a "pure gauge" form

$$k_{ab} = -2(D_a D_b - h_{ab})\Phi, \tag{4.6}$$

where Φ is a scalar that plays the role of a supertranslations Goldstone mode [43].

The fields of interest for our purposes will be σ and i_{ab}. Φ and k_{ab} play no role in the asymptotic derivation of soft theorems and thus will not be discussed any further.

[9] Our conventions are such that the asymptotic expansions at $\tau \to \pm\infty$ look identical. The distinction between future and past coefficients appear when expressing them in terms of asymptotic particles, see e.g. Eq. (4.8). We note that at $\mathcal{H}^-$ some coefficients carry opposite signs relative to the conventions of [18, 31].

Matter stress tensor and Einstein equations

As explained in section 2.4, the matter content at timelike infinity is described by massive particles. The resulting stress tensor admits the following large τ expansion (see appendix C)

$$\begin{aligned} T_{\tau\tau} &= \frac{1}{\tau^3}\rho - \frac{\log|\tau|}{\tau^4} D \cdot j + O(\tau^{-4}) \\ T_{\tau a} &= \frac{\log|\tau|}{\tau^3} j_a + O(\tau^{-3}) \\ T_{ab} &= O(\tau^{-3}\log^2|\tau|) \end{aligned} \tag{4.7}$$

where ρ is the (signed) energy density of massive particles at future/past timelike infinity,

$$\rho\big|_{\mathcal{H}^\pm}(x) = \pm \sum_{\substack{i\in\pm \\ m_i\neq 0}} m_i \delta(x, x_i) \tag{4.8}$$

and

$$j_a = \rho\, c_a, \quad c_a = D_a V^\mu c_\mu = \partial_a \sigma. \tag{4.9}$$

One can show that the expansion (4.7) is consistent with that of the Einstein tensor, and that Einstein equations lead to (see appendix C)

$$(D^2 - 3)\sigma = 4\pi G \rho, \tag{4.10}$$

and

$$\begin{aligned} h^{ab} i_{ab} &= 4\pi G D \cdot j \\ D^b i_{\langle ab\rangle} &= 8\pi G \left(\tfrac{1}{3} D_a D \cdot j - j_a\right) \\ (D^2 + 2) i_{\langle ab\rangle} &= 4\pi G \left(D_{\langle a} D_{b\rangle} D \cdot j - 4 D_{\langle a} j_{b\rangle}\right) \end{aligned} \tag{4.11}$$

where $i_{\langle ab\rangle}$ is the trace-free part of i_{ab}.

In appendix D.1 we construct the solutions to these elliptic equations by Green's functions methods. In the following we summarize the properties of such solutions that will be used in the asymptotic proof of the soft theorems. Before doing so, however, we need to make a few remarks on log translations.

Log translations

As originally discussed in [41] in the spatial infinity context, log translations are allowed by conditions (4.2). Their asymptotic form in BS coordinates is

$$\xi_L \overset{\tau\to\pm\infty}{=} l \ln|\tau| \partial_\tau + \cdots, \quad l := -L^\mu V_\mu, \tag{4.12}$$

where the dots are determined by requiring consistency with (4.2). By evaluating the Lie derivative on the asymptotic metric and stress tensor one finds[10]

$$\delta_L \sigma = l, \tag{4.13}$$
$$\delta_L i_{ab} = D^c(D_c l\,(D_a D_b - h_{ab})\sigma), \tag{4.14}$$
$$\delta_L j_a = D_a l\,\rho, \tag{4.15}$$
$$\delta_L h_{ab} = \delta_L k_{ab} = \delta_L \rho = 0. \tag{4.16}$$

Note these transformations are compatible with (2.3), (4.9) and (4.11). For the purposes of matching with null infinity, log translations will be fixed by imposing future/past radiative log frame conditions.

Asymptotic properties of σ

Let $\mathcal{G}(x, x')$ be the Green's function of the differential operator on the LHS of (4.10), so that

$$\sigma(x) = 4\pi G \int d^3x' \mathcal{G}(x, x')\rho(x'). \tag{4.17}$$

There are potential ambiguities in the definition of $\mathcal{G}$, due to homogeneous solutions to (4.10), which are eliminated by the radiative log frame condition. In terms of σ, this condition reads [18, 31]

$$\lim_{\rho\to\infty} \sigma = 0 \quad \text{(radiative log frame condition)}. \tag{4.18}$$

The resulting Green's function is reviewed in appendix D.1. For matching purposes, we will only need its asymptotic form at large ρ,

$$\mathcal{G}(x, x') \overset{\rho\to\infty}{=} \frac{1}{16\pi\rho^3}\frac{1}{(n\cdot V')^3} + \cdots . \tag{4.19}$$

Using (4.19) and (4.8) in (4.17) leads to

$$\sigma\big|_{\mathcal{H}^\pm}(\rho, \phi) \overset{\rho\to\infty}{=} \frac{\overset{0}{\sigma}\big|_{\partial\mathcal{H}^\pm}(\phi)}{\rho^3} + \cdots , \tag{4.20}$$

with

$$\overset{0}{\sigma}\big|_{\partial\mathcal{H}^\pm} = \frac{G}{4}\sum_{\substack{i\in\pm \\ m_i\neq 0}} \frac{m_i^4}{(p_i\cdot n)^3}, \tag{4.21}$$

where we expressed the result in terms of the asymptotic momenta (2.20). Note that the $\pm$ signs in (4.8) and (2.20) cancel each other so that Eq. (4.20) takes the same form at both infinities.

[10] Eq. (4.14) only holds when k_{ab} is "pure gauge" as in (4.6), otherwise there is an additional contribution proportional to the magnetic component of the asymptotic Weyl tensor. See Eq. (4.108) of [42] for the full expression in the spatial infinity case.

Asymptotic properties of i_{ab}

In appendix D.1 we show how the last two equations in (4.11) can be solved in terms of a Green's function $\mathcal{G}^{c'}_{ab}(x,x')$,

$$i_{\langle ab\rangle}(x) = 8\pi G \int d^3x' \mathcal{G}^{c'}_{ab}(x,x') j_{c'}(x'). \tag{4.22}$$

This time the log translation ambiguity does not manifest in the Green's function, but in the source term through Eq. (4.15). As before, we fix this freedom by requiring the radiative log frame condition (4.18).

For the matching with null infinity, all we shall need is the asymptotic form of the radial-sphere components of the Green's function, given by (see appendix D.1)

$$\mathcal{G}^{b'}_{\rho A}(x,x') \overset{\rho\to\infty}{=} \frac{3}{8\pi\rho^3} \frac{\partial_A n^\mu n^\nu D^{b'} V'_{[\mu} V'_{\nu]}}{(n\cdot V')^4} + \cdots \tag{4.23}$$

Using (4.23) and (4.9) leads to

$$i_{\rho A}\big|_{\mathcal{H}^\pm}(\rho,\phi) \overset{\rho\to\infty}{=} \frac{\overset{0}{i}_{\rho A}\big|_{\partial\mathcal{H}^\pm}(\phi)}{\rho^3} + \cdots, \tag{4.24}$$

with

$$\overset{0}{i}_{\rho A}\big|_{\partial\mathcal{H}^\pm} = 3G \sum_{\substack{i\in\pm \\ m_i\neq 0}} m_i^4 \frac{\partial_A n^\mu n^\nu c^i_{[\mu} p^i_{\nu]}}{(p_i\cdot n)^4}. \tag{4.25}$$

Comment

It is interesting to note the following parallel between the expressions for σ and i_{ab}: The asymptotic values of the Green's functions (4.19) and (4.23) can be interpreted as boundary-to-bulk Green's function for supertranslations and superrotations respectively [44]. At the same time, the asymptotic coefficients (4.21) and (4.25) can be interpreted as the particles' contribution to the mass and angular momentum aspects respectively (the latter for the case of an orbital angular momentum $J_{\mu\nu} = c_{[\mu} p_{\nu]}$, see e.g. [45]). This double interpretation underlies the symmetry realization of the leading and subleading soft graviton theorems in presence of massive particles [46, 47].

4.2 Spatial infinity

To describe events near spatial infinity, we consider asymptotic coordinates (ρ, x^a) such that

$$X^\mu(\rho,x) \overset{\rho\to\infty}{=} \rho V^\mu(x) + \cdots, \quad V^\mu V_\mu = 1, \tag{4.26}$$

where ρ is an asymptotic proper distance and $x^a = (\tau, \phi^A)$ parametrize unit spacelike directions according to (2.11). To extend these coordinates beyond leading order we again adopt Beig-Schmidt conditions [41],

$$g_{\rho a} = 0, \quad g_{\rho\rho} = \left(1 + \frac{\sigma}{\rho}\right)^2, \tag{4.27}$$

$$g_{ab} = \rho^2 h_{ab} + \rho(k_{ab} - 2\sigma h_{ab}) + \log\rho\, i_{ab} + O(\rho^0), \tag{4.28}$$

where the coefficients are now tensor fields on $\mathcal{H}^0$, with metric $h_{ab} = \partial_a V^\mu \partial_b V_\mu$ given by

$$h_{ab} dx^a dx^b = -\frac{d\tau^2}{1+\tau^2} + (1+\tau^2) d\Omega^2. \tag{4.29}$$

As before, σ and k_{ab} are potentials for the electric and magnetic parts of the asymptotic Weyl tensor, respectively. The former captures the log deviation vector according to

$$c^\mu = -D^a V^\mu \partial_a \sigma - V^\mu \sigma, \tag{4.30}$$

while the latter takes the form

$$k_{ab} = -2(D_a D_b + h_{ab})\Phi, \tag{4.31}$$

due to the vanishing of the asymptotic magnetic Weyl tensor.

We note that one can formally map the Beig-Schmidt expressions at timelike infinity to those at spatial infinity by doing the replacements [18][11]

$$\tau \to i\rho, \quad \sigma \to i\sigma, \quad h_{ab} \to -h_{ab}, \quad \Phi \to -i\Phi, \quad i_{ab} \to i_{ab}. \tag{4.32}$$

Einstein equations

Einstein equations for σ and i_{ab} can be obtained from those at timelike infinity by applying (4.32), and setting to zero the source terms. This leads to

$$(D^2 + 3)\sigma = 0, \tag{4.33}$$

and

$$h^{ab} i_{ab} = 0 \quad D^b i_{ab} = 0, \quad (D^2 - 2) i_{ab} = 0. \tag{4.34}$$

In appendix D.2 we construct the solutions to these hyperbolic equations in terms of initial (final) data at the asymptotic past (future) boundary of $\mathcal{H}^0$, assuming decaying boundary conditions as required for consistency with the Bondi expansion at null infinity [18]. After a few comments on log translations, we summarize below the key aspects of such solutions.

[11] Along with $\rho \to -i\sqrt{\tau^2+1}$, $V^\mu \to -iV^\mu$.

Log translations

At spatial infinity, log translations take the asymptotic form

$$\xi_L \overset{\rho\to\infty}{=} l \ln\rho\, \partial_\rho + \cdots, \quad l := L^\mu V_\mu, \tag{4.35}$$

and act on the metric components by [42]

$$\delta_L \sigma = l, \tag{4.36}$$
$$\delta_L i_{ab} = D^c\big(D_c l\, \mathcal{E}_{ab}\big), \tag{4.37}$$
$$\delta_L h_{ab} = \delta_L k_{ab} = 0, \tag{4.38}$$

where

$$\mathcal{E}_{ab} = (D_a D_b + h_{ab})\sigma \tag{4.39}$$

is the asymptotic electric Weyl curvature [19], itself invariant under log translations.

Of special importance for our analysis is the log translation that interpolates between future and past radiative frames (2.18). In this case $L^\mu = 4GP^\mu$ and relations (4.36), (4.37) become[12]

$$\sigma^{\text{rad}+} - \sigma^{\text{rad}-} = 4GP^\mu V_\mu, \tag{4.40}$$
$$i^{\text{rad}+}_{ab} - i^{\text{rad}-}_{ab} = 4GP^\mu D^c\big(D_c V_\mu\, \mathcal{E}_{ab}\big), \tag{4.41}$$

where rad+/rad− refers to the solution in the future/past log radiative frame.

Asymptotic properties of σ

As emphasized in [18] it is crucial to distinguish between solutions to (4.33) in different log frames. In particular, decaying conditions can only be imposed at either the future or past,

$$\sigma^{\text{rad}\pm}(\tau,\phi) \overset{\tau\to\pm\infty}{=} \frac{\overset{0}{\sigma}\big|_{\partial_\pm\mathcal{H}^0}(\phi)}{\tau^3} + \cdots, \tag{4.42}$$

where the coefficients $\overset{0}{\sigma}\big|_{\partial_\pm\mathcal{H}^0}$ provide final/initial data that parametrize solutions to (4.33) in future/past radiative log frame. For instance, in the future radiative frame (see appendix D.2 for details)

$$\sigma^{\text{rad}+}(x) = \frac{4}{\pi}\int d^2\phi'\, V(x)\cdot n(\phi')\theta\big(V(x)\cdot n(\phi')\big)\overset{0}{\sigma}\big|_{\partial_+\mathcal{H}^0}(\phi'), \tag{4.43}$$

[12] Although (4.36) and (4.37) refer to infinitesimal variations, they also hold for *finite* variations. For (4.36) this property is evident, while for (4.37) it follows because $\delta_L \mathcal{E}_{ab} = 0$.

where θ is the step function. Analogous expression holds for $\sigma^{\text{rad}-}$ in terms of $\overset{0}{\sigma}|_{\partial_-\mathcal{H}^0}$. The two are however not independent. Using the relation[13]

$$P^\mu = \frac{1}{\pi G}\int d^2\phi\, n^\mu(\phi)\overset{0}{\sigma}|_{\partial_+\mathcal{H}^0}(\phi), \tag{4.44}$$

one can verify that the decaying solution at $\tau \to -\infty$ can be obtained from (4.40) and satisfies

$$\overset{0}{\sigma}|_{\partial_-\mathcal{H}^0} = -\mathcal{A}_*\overset{0}{\sigma}|_{\partial_+\mathcal{H}^0}, \tag{4.45}$$

where $\mathcal{A}$ is the antipodal map (2.15). This well-known result [18, 51–53] underlies the matching of the Bondi mass aspect at spatial infinity [5, 54] that we shall encounter in Eq. (5.7).

Comment
Combining (4.30) and (4.43) one obtains an integral expression for the log deviation (in future radiative frame)

$$c_\mu^{\text{rad}+}(x) = -\frac{4}{\pi}\int d^2\phi'\, n_\mu(\phi')\,\theta\big(V(x)\cdot n(\phi')\big)\overset{0}{\sigma}|_{\partial_+\mathcal{H}^0}(\phi'). \tag{4.46}$$

Using (4.44) one can verify (4.46) satisfies (2.17). Similar considerations apply for the log deviation vector in other frames.

Asymptotic properties of i_{ab}

We now consider solutions to (4.34) in terms of initial/final data,

$$i_{\tau A}(\tau,\phi) \overset{\tau\to\pm\infty}{=} \frac{\overset{0}{i}_{\tau A}|_{\partial_\pm\mathcal{H}^0}(\phi)}{\tau^3} + \cdots . \tag{4.47}$$

Unlike the previous case, the Green's function for this problem is insensitive to log translation ambiguities. These instead manifest in the boundary values $\overset{0}{i}_{\tau A}|_{\partial_\pm\mathcal{H}^0}$, see below. In appendix D.2 we discuss the general solution of (4.34) under (4.47) and show it satisfies

$$\overset{0}{i}_{\tau A}|_{\partial_-\mathcal{H}^0} = -\mathcal{A}_*\overset{0}{i}_{\tau A}|_{\partial_+\mathcal{H}^0}. \tag{4.48}$$

For the asymptotic proof of the soft theorem, we will need to compare the future/past asymptotic values of the rad+/rad− log frames expressions for $i_{\tau A}$. To this end, consider the large τ expansion of (4.37), which we write as

$$\delta_L i_{\tau A}(\tau,\phi) \overset{\tau\to\pm\infty}{=} \frac{\delta_L\overset{0}{i}_{\tau A}|_{\partial_\pm\mathcal{H}^0}(\phi)}{\tau^3} + \cdots , \tag{4.49}$$

[13]This expression can be obtained by evaluating the general formula [19, 48, 49] $P^\mu = (1/8\pi G)\oint \mathcal{E}^{ab}\partial_b V^\mu dS_a$ on a $\tau\to\infty$ slice of $\mathcal{H}^0$, see also [33, 50].

with (see the end of appendix D.2)[14]

$$\begin{aligned}\delta_L \overset{0}{i}_{\tau A}\big|_{\partial_+\mathcal{H}^0} &= -4(n\cdot L\partial_A + 3\partial_A n\cdot L)\overset{0}{\sigma}\big|_{\partial_+\mathcal{H}^0},\\ \delta_L \overset{0}{i}_{\tau A}\big|_{\partial_-\mathcal{H}^0} &= -4(\mathcal{A}_* n\cdot L\partial_A + 3\partial_A \mathcal{A}_* n\cdot L)\overset{0}{\sigma}\big|_{\partial_-\mathcal{H}^0}.\end{aligned} \tag{4.50}$$

The large τ limit of Eq. (4.41) then implies

$$\overset{0}{i}{}^{\mathrm{rad}+}_{\tau A}\big|_{\partial_+\mathcal{H}^0} = -\mathcal{A}_* \overset{0}{i}{}^{\mathrm{rad}-}_{\tau A}\big|_{\partial_-\mathcal{H}^0} - 16G(n\cdot P\partial_A + 3\partial_A n\cdot P)\overset{0}{\sigma}\big|_{\partial_+\mathcal{H}^0}. \tag{4.51}$$

This "matching" property will allow us to relate the logarithmic angular momentum aspects at future and past infinity in Eq. (5.17). In contrast to Eq. (4.45) for σ, there is now a inhomogeneous term that captures a discontinuity across spatial infinity. In the analysis of section 5, this term will generate the "extra" contribution to the soft factor given in the second line of (3.8).

4.3 Null infinity

To describe events near future/past null infinity, we consider asymptotic coordinates (r, u, ϕ^A), obtained by setting $s = r$ in (2.6)

$$X^\mu(r,u,\phi) \overset{r\to\pm\infty}{=} r\, n^\mu(\phi) + u\, t^\mu + \cdots\,. \tag{4.52}$$

To fix the dots in (4.52) we adopt Bondi-Sachs gauge conditions [55, 56]

$$g_{rr} = g_{rA} = 0, \quad \det g_{AB} = r^4 \det q_{AB}, \tag{4.53}$$

where q_{AB} is the unit sphere metric, used to raised and lower $2d$ indices. Note that our definition of radial coordinate r is such that it is positive/negative for future/past coordinates; depending on the case, u represents either retarded or advanced time. These conventions will allow us to have identical expressions at future and past null infinities.

We assume Winicour's logarithmic asymptotic flatness conditions [57] for the large r expansion of the metric components, as required for generic scattering spacetimes [21, 58, 59]:[15]

$$\begin{aligned} g_{uu} &= -1 + \frac{2G\mathcal{M}}{r} + O(\ln r/r^2)\\ g_{ur} &= -1 + O(1/r^2)\\ g_{AB} &= r^2 q_{AB} + r\, C_{AB} + O(r^0)\\ g_{uA} &= \frac{1}{2}D^B C_{AB} + \log|r|\frac{2G}{3r}\overset{\log r}{\mathcal{N}}_A + \frac{2G}{3r}\tilde{\mathcal{N}}_A + O(\ln^2 r/r^2). \end{aligned} \tag{4.54}$$

The coefficients in this expansion are regarded as fields on $\mathcal{I}^\pm$, i.e. as functions of (u, ϕ^A), and have the following properties:

---

[14] We note that (4.50) is compatible with (4.48) thanks (4.45).

[15] The fall-offs of subleading terms are taken from [59] but their precise form is not important in our analysis.

- Bondi's shear C_{AB} is trace-free and unconstrained by Einstein equations. Its time derivative is denoted by

$$N_{AB} = \partial_u C_{AB}, \tag{4.55}$$

and referred to as the news tensor.

- The mass aspect $\mathcal{M}$ is constrained by Einstein equations to satisfy

$$\partial_u \mathcal{M} = \frac{1}{4G} D_A D_B N^{AB} - 4\pi \rho_{\text{massless}}, \tag{4.56}$$

where ρ_{massless} is the energy flux at future/past null infinity, including that due to gravitational radiation,

$$\rho_{\text{grav}} = \frac{1}{32\pi G} N_{AB} N^{AB}. \tag{4.57}$$

- The coefficient $\tilde{\mathcal{N}}_A$ contains in it the angular momentum aspect $\mathcal{N}_A$ (i.e. the angular momentum angular density at given u). There are various prescriptions in the literature for such quantity [62]. For our purposes we find it convenient to define it according to

$$\tilde{\mathcal{N}}_A = \mathcal{N}_A + u\partial_A \mathcal{M} - \frac{3}{32G}\partial_A C^2 + \frac{u}{4G} D^B D_{[B} D^C C_{A]C}, \tag{4.58}$$

where $C^2 \equiv C^{AB} C_{AB}$. This definition coincides with that of [18, 63], except for the last $O(u\, C/G)$ term. With this piece added, Einstein equations (written for simplicity in holomorphic coordinates (2.9)) imply

$$\partial_u \mathcal{N}_z = -\frac{1}{2G} u D_z^3 N^{zz} + \text{angular momentum flux}, \tag{4.59}$$

where the "angular momentum flux" can be due to massless matter or gravitational radiation, the latter being quadratic in the shear/news.[16]

- The coefficient $\overset{\log r}{\mathcal{N}_A}$ is an extra component allowed by Winicour that is excluded in the original work of Bondi and Sachs. It may be thought of as a "divergent in r" angular momentum aspect. Consistency with Einstein equations requires it is u-independent

$$\partial_u \overset{\log r}{\mathcal{N}_A} = 0. \tag{4.60}$$

This field comes along with a $O(r^0)$ trace-free component in g_{AB} [56, 57, 60] that will not be needed in the present work.

[16] Consisting in terms of the form $O(CN/G)$ and $O(uN^2/G)$. The angular momentum flux is needed for the asymptotic description of the tree-level $O(\omega^0)$ soft theorem, as it accounts for the massless "hard" contribution [64]. It is, however, irrelevant for the $O(\log \omega)$ soft theorem and this is why we do not display it explicitly; see Eq. (2.6) of [63] for its expression.

Large-u behavior

To fully characterize the spacetime metric at null infinity, we need to specify the large u behavior of the shear. Compatibility with the soft expansion (3.2) requires [38][17]

$$C_{AB}(u,\phi) \overset{u\to\pm\infty}{=} \overset{0}{C}_{AB}(\phi) + \frac{1}{u}\overset{1}{C}_{AB}(\phi) + \cdots , \tag{4.61}$$

with a "purely electric" leading term [5, 40]

$$D_{[A}D^C\overset{0}{C}_{B]C} = 0. \tag{4.62}$$

Eq. (4.61) implies the news tensor falls off as $1/u^2$ and

$$\partial_u\mathcal{M} = O(1/u^2), \quad \partial_u\mathcal{N}_z = O(1/u), \tag{4.63}$$

where the decaying rates in (4.63) are dictated by the linear-in-news terms in (4.56) and (4.59) respectively. Integrating (4.63) in u leads to

$$\mathcal{M}(u,\phi) \overset{u\to\pm\infty}{=} \overset{0}{\mathcal{M}}(\phi) + O(1/u) \tag{4.64}$$

$$\mathcal{N}_z(u,\phi) \overset{u\to\pm\infty}{=} \log|u|\,\overset{\log u}{\mathcal{N}_z}(\phi) + O(u^0), \tag{4.65}$$

where[18]

$$\overset{\log u}{\mathcal{N}_z} = \frac{1}{2G}D_z^3\overset{1}{C}{}^{zz}, \tag{4.66}$$

and where the difference of the $\overset{0}{\mathcal{M}}$ coefficients at $u\to\pm\infty$ is restricted by the total u-integral of (4.56).

Logarithmic angular momentum aspect

The Bondi metric (4.54) exhibits two distinct logarithmic divergences in the angular momentum aspect, captured by $\overset{\log r}{\mathcal{N}_A}$ and $\overset{\log u}{\mathcal{N}_A}$. For the discussion of the upcoming sections, it will be useful to combine these two terms in a single "log angular momentum aspect" defined by

$$\overset{\log}{\mathcal{N}_A} := \overset{\log r}{\mathcal{N}_A} + \overset{\log u}{\mathcal{N}_A}. \tag{4.67}$$

The proof of the soft theorem is actually insensitive to the value of $\overset{\log r}{\mathcal{N}_A}$, and one may therefore be tempted to take it to zero. However, consistency with the individual $u\to\pm\infty$ coefficients of $\overset{\log u}{\mathcal{N}_A}$ requires a non-trivial $\overset{\log r}{\mathcal{N}_A}$ [61].

We finally note that both logarithmic divergences vanish upon 2d smearing with Lorentz generators and hence they are consistent with a finite total angular momentum. For the $\log u$ term, this follows from (4.66) while for the $\log r$ term this is because it can be written as the divergence of a symmetric, traceless 2d tensor [57].

[17] We omit for now the labels that distinguish the coefficients at $u\to\pm\infty$.

[18] We note that, because of (4.62), Eq. (4.66) holds unchanged for the angular momentum aspect of [18].

4.4 Matching conditions

In the previous subsections we described the spacetime metric near each of the five asymptotic boundaries depicted in the right panel of Figure 1,

$$\mathcal{H}^-, \mathcal{I}^-, \mathcal{H}^0, \mathcal{I}^+, \mathcal{H}^+. \tag{4.68}$$

In order to complete the CGW framework, we need to describe the matching conditions at the adjacent boundaries of (4.68) (see also Figure 2)

$$\partial\mathcal{H}^- \sim \partial_-\mathcal{I}^-, \quad \partial_+\mathcal{I}^- \sim \partial_-\mathcal{H}^0, \quad \partial_+\mathcal{H}^0 \sim \partial_-\mathcal{I}^+, \quad \partial_+\mathcal{I}^+ \sim \partial\mathcal{H}^+. \tag{4.69}$$

Below we discuss the matching properties that are needed for the present work. A sketch of their proof is given in appendix E, and we refer the reader to [18] for further details.

Consistency between the Beig-Schmidt and Bondi expansions leads to the identification of the boundary values of the BS potential (Eqs. (4.20) and (4.42)) and Bondi mass aspect (4.64) according to

$$\overset{0}{\sigma}{}^{\mathrm{rad}\pm}|_{\partial\mathcal{H}^\pm} = -\frac{G}{4}\overset{0}{\mathcal{M}}|_{\partial_\pm\mathcal{I}^\pm}, \tag{4.70}$$

$$\overset{0}{\sigma}{}^{\mathrm{rad}+}|_{\partial_+\mathcal{H}^0} = \frac{G}{4}\overset{0}{\mathcal{M}}|_{\partial_-\mathcal{I}^+}, \tag{4.71}$$

$$\overset{0}{\sigma}{}^{\mathrm{rad}-}|_{\partial_-\mathcal{H}^0} = \frac{G}{4}\mathcal{A}_*\overset{0}{\mathcal{M}}|_{\partial_+\mathcal{I}^-}. \tag{4.72}$$

where we have now made explicit all labels.

Similarly, the leading component of the logarithmic BS metric component (4.24) and (4.47) can be identified with the logarithmic angular momentum aspect (4.67) according to

$$\overset{0}{i}{}^{\mathrm{rad}\pm}_{\rho A}|_{\partial\mathcal{H}^\pm} = -G\overset{\log}{\mathcal{N}}_A|_{\partial_\pm\mathcal{I}^\pm}, \tag{4.73}$$

$$\overset{0}{i}{}^{\mathrm{rad}+}_{\tau A}|_{\partial_+\mathcal{H}^0} = G\overset{\log}{\mathcal{N}}_A|_{\partial_-\mathcal{I}^+}, \tag{4.74}$$

$$\overset{0}{i}{}^{\mathrm{rad}-}_{\tau A}|_{\partial_-\mathcal{H}^0} = G\mathcal{A}_*\overset{\log}{\mathcal{N}}_A|_{\partial_+\mathcal{I}^-}. \tag{4.75}$$

5 Asymptotic proof of soft theorems

We finally show how the results from the previous section imply the leading and log soft theorems, reviewed in section 3. The general idea behind these *asymptotic proofs* is illustrated in Figure 2. The direction of the soft radiation is associated to a null ray that leaves its imprint at all five infinities. Using Einstein equations at each of these 3d hypersurfaces, together with matching conditions across their 2d boundaries, allows one to evaluate the soft component of the radiation in terms of energy fluxes at timelike and null infinities.

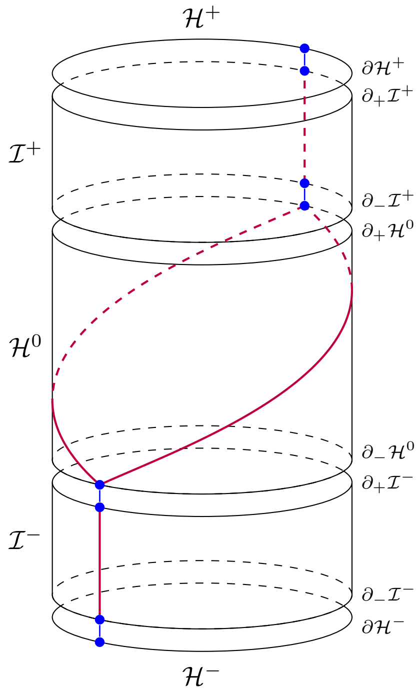

Figure 2. Simplified drawing of the five infinities and their boundaries, together with a null ray at infinity that connects asymptotic past and future null directions. The hyperbolic spaces $\mathcal{H}^\pm$ are depicted as disks, and the de Sitter space $\mathcal{H}^0$ as a cylinder. The blue dots represent points at each of the eight boundaries. There are four blue links indicating their matching across adjacent boundaries. The red vertical lines are null generators at $\mathcal{I}^\pm$. The curved red lines represent the evolution of null rays across $\mathcal{H}^0$.

For the leading soft theorem this type of analysis has already been discussed in [18, 40, 51–53]. The purpose of its exposition here is to present the overall rationale in the simplest setting, as well as to highlight the new features that appear in the log case.

The results we are about to discuss can additionally be interpreted as conservation laws for asymptotic charges [40, 47]. Here we will not rely on this interesting perspective but just limit ourselves to establish the connection between Einstein equations, matching conditions and soft theorems.

5.1 Soft factors in Bondi variables

The first step is to express the soft theorems in terms of the Bondi fields of section 4.3. The outgoing/incoming Cartesian waveform (3.1)/(3.9) is related to the future/past Bondi shear by

$$C_{AB}|_{\mathcal{I}^\pm} = \partial_A n^\mu \partial_B n^\nu h_{\mu\nu}|_{\mathcal{I}^\pm} \tag{5.1}$$

where $h_{\mu\nu}|_{\mathcal{I}^+} \equiv h^{\text{out}}_{\mu\nu}$ and $h_{\mu\nu}|_{\mathcal{I}^-} \equiv h^{\text{in}}_{\mu\nu}$. The soft-frequency coefficients in (3.2) and (3.11) can then be written in terms of differences of the large u coefficients [10] of the Bondi shear

(4.61),

$$\big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^\pm} = -i\partial_A n^\mu \partial_B n^\nu \tilde{h}^{(0)}_{\mu\nu}|_{\mathcal{I}^\pm}, \tag{5.2}$$

$$\big[\overset{1}{C}_{AB}\big]_{\mathcal{I}^\pm} = -\partial_A n^\mu \partial_B n^\nu \tilde{h}^{(\log)}_{\mu\nu}|_{\mathcal{I}^\pm}, \tag{5.3}$$

where

$$\big[\overset{n}{C}_{AB}\big]_{\mathcal{I}^\pm} := \overset{n}{C}_{AB}|_{\partial_+\mathcal{I}^\pm} - \overset{n}{C}_{AB}|_{\partial_-\mathcal{I}^\pm}, \quad n = 0, 1. \tag{5.4}$$

For the proof of the soft theorems, we will relate these coefficients to the mass and angular aspects respectively.

5.2 Leading soft theorem

In terms of the Bondi coefficients (5.4), the leading soft theorem (3.3) (including incoming memory according to Eq. (3.13)) takes the form

$$\big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^+} = -4G\partial_A n^\mu \partial_B n^\nu \sum_i \frac{p^i_\mu p^i_\nu}{p_i \cdot n} + \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^-}. \tag{5.5}$$

In order to relate this expression to the mass aspect, we take the double sphere divergence of (5.5). Using the identities reviewed in appendix F this leads to

$$D^A D^B \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^+} = -4G \sum_i \frac{m_i^4}{(p_i \cdot n)^3} + D^A D^B \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^-}, \tag{5.6}$$

where the sum includes both massive and massless particles, the latter understood by a $m_i \to 0$ limit (see below). So far, Eq. (5.6) is just a rewriting of the leading soft theorem.[19] We now show how this relation follows from the asymptotic Einstein equations and the matching conditions.

We start by noting that Eqs. (4.45), (4.71), (4.72) lead to Strominger's identification of the Bondi mass aspects at spatial infinity [5], which in our conventions reads

$$\overset{0}{\mathcal{M}}|_{\partial_-\mathcal{I}^+} = -\overset{0}{\mathcal{M}}|_{\partial_+\mathcal{I}^-}. \tag{5.7}$$

The next step is to express each side of (5.7) in terms of data at null and timelike infinity. To this end, consider the integrated version of (4.56)

$$\overset{0}{\mathcal{M}}|_{\partial_+\mathcal{I}^\pm} - \overset{0}{\mathcal{M}}|_{\partial_-\mathcal{I}^\pm} = \frac{1}{4G} D^A D^B \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^\pm} - 4\pi \int du \rho_{\text{massless}}\big|_{\mathcal{I}^\pm}. \tag{5.8}$$

The mass aspect at timelike infinity can be evaluated by combining Eqs. (4.70) and (4.21),

$$\overset{0}{\mathcal{M}}|_{\partial_\pm\mathcal{I}^\pm} = -\sum_{\substack{i\in\pm \\ m_i\neq 0}} \frac{m_i^4}{(p_i \cdot n)^3}. \tag{5.9}$$

[19] The original version (5.5) can be recovered from (5.6) and (4.62) [40].

Meanwhile, the massless radiated energy per solid angle can be written as [5, 35, 36],

$$\int du \rho_{\text{massless}}(u,\phi)\big|_{\mathcal{I}^\pm} = \pm \sum_{\substack{i\in\pm \\ m_i=0}} E_i \delta(\phi,\phi_i). \tag{5.10}$$

where (E_i, ϕ_i) parametrize the null momentum $p_i^\mu = E_i n^\mu(\phi_i)$.

Solving for $\overset{0}{\mathcal{M}}|_{\partial_\mp \mathcal{I}^\pm}$ in (5.8) and using Eqs. (5.9), (5.10) leads to

$$\begin{aligned}
\overset{0}{\mathcal{M}}|_{\partial_\mp \mathcal{I}^\pm} &= \mp \frac{1}{4G} D^A D^B \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^\pm} - \sum_{\substack{i\in\pm \\ m_i\neq 0}} \frac{m_i^4}{(p_i\cdot n)^3} + 4\pi \sum_{\substack{i\in\pm \\ m_i=0}} E_i \delta(\hat{n},\hat{p}_i) && (5.11)\\
&= \mp \frac{1}{4G} D^A D^B \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^\pm} - \sum_{i\in\pm} \frac{m_i^4}{(p_i\cdot n)^3}, && (5.12)
\end{aligned}$$

where in the second line we combined massive and massless contributions by means of the identity [46]

$$\lim_{m_i\to 0} \frac{m_i^4}{(p_i\cdot n(\phi))^3} = -4\pi E_i \delta(\phi,\phi_i). \tag{5.13}$$

Substituting (5.12) in (5.7) and solving for $D^A D^B \big[\overset{0}{C}_{AB}\big]_{\mathcal{I}^+}$ leads to (5.6).

5.3 Log soft theorem

As before, we start by expressing the log soft theorem, including incoming memory (3.14), in terms of the subleading Bondi coefficient (5.4),

$$\big[\overset{1}{C}_{zz}\big]_{\mathcal{I}^+} = -\partial_z n^\mu \partial_z n^\nu \tilde{h}^{(\log)}_{\mu\nu} + 4G\, n\cdot P \big[\overset{0}{C}_{zz}\big]_{\mathcal{I}^-} + \big[\overset{1}{C}_{zz}\big]_{\mathcal{I}^-}, \tag{5.14}$$

where $\tilde{h}^{(\log)}_{\mu\nu}$ stands for the expression (3.8) and we choose 2d holomorphic coordinates to simplify the analysis.

Let us further rewrite (5.14) in terms of differences of the logarithmic angular momentum aspect (4.67),

$$\big[\overset{\log}{\mathcal{N}}_z\big]_{\mathcal{I}^\pm} = \frac{1}{2G} D_z^3 \big[\overset{1}{C}{}^{zz}\big]_{\mathcal{I}^\pm}, \tag{5.15}$$

where we used Eq. (4.66) together with the fact that $\big[\overset{\log r}{\mathcal{N}}_z\big]_{\mathcal{I}^\pm} = 0$ due to its u-independence.

Using the identities presented in Appendix F, the third derivative in (5.15) can be evaluated on the various terms in (5.14) leading to

$$\left[\overset{\log}{\mathcal{N}}_z\right]_{\mathcal{I}^+} = -3 \sum_{\substack{i\in\text{out}\\ m_i\neq 0}} m_i^4 \frac{\partial_z n^\mu n^\nu J^{\text{rad}+i}_{\mu\nu}}{(p_i\cdot n)^4} - 3 \sum_{\substack{i\in\text{in}\\ m_i\neq 0}} m_i^4 \frac{\partial_z n^\mu n^\nu J^{\text{rad}-i}_{\mu\nu}}{(p_i\cdot n)^4}$$
$$+ (n\cdot P\partial_z + 3\partial_z n\cdot P)\left(4G\sum_{i\in\text{in}}\frac{m_i^4}{(p_i\cdot n)^3} - 2D_z^2\left[\overset{0}{C}{}^{zz}\right]_{\mathcal{I}^-}\right)$$
$$+ \left[\overset{\log}{\mathcal{N}}_z\right]_{\mathcal{I}^-}. \tag{5.16}$$

This expression provides a reformulation of the log soft theorem, analogous to (5.6) for the leading soft theorem.[20] Our goal now is to show that this identity follows from the matching properties of the asymptotic fields.

Let us start by considering the logarithmic angular momentum aspect at spatial infinity. Combining Eqs. (4.51), (4.71), (4.74), (4.75) and (5.7) leads to

$$\overset{\log}{\mathcal{N}}_z\big|_{\partial_-\mathcal{I}^+} = -\overset{\log}{\mathcal{N}}_z\big|_{\partial_+\mathcal{I}^-} + 4G(n\cdot P\partial_z + 3\partial_z n\cdot P)\overset{0}{\mathcal{M}}\big|_{\partial_+\mathcal{I}^-}. \tag{5.17}$$

This equation is the analogue of (5.7) for the Bondi mass aspect, except that now there is a discontinuity captured in the last term.

Next, we express (5.17) in terms of data at null and timelike infinity. The logarithmic angular momentum aspect at spatial infinity can be written as

$$\overset{\log}{\mathcal{N}}_z\big|_{\partial_\mp\mathcal{I}^\pm} = \mp\left[\overset{\log}{\mathcal{N}}_z\right]_{\mathcal{I}^\pm} + \overset{\log}{\mathcal{N}}_z\big|_{\partial_\pm\mathcal{I}^\pm} \tag{5.18}$$
$$= \mp\left[\overset{\log}{\mathcal{N}}_z\right]_{\mathcal{I}^\pm} - 3\sum_{\substack{i\in\pm\\ m_i\neq 0}} m_i^4 \frac{\partial_z n^\mu n^\nu J^{\text{rad}\pm i}_{\mu\nu}}{(p_i\cdot n)^4}, \tag{5.19}$$

where in the last equality we used Eqs. (4.25) and (4.73) with $J^{\text{rad}\pm i}_{\mu\nu} \equiv c^{\text{rad}\pm i}_{[\mu} p^i_{\nu]}$. The last term in (5.17) can be evaluated from (5.12),

$$\overset{0}{\mathcal{M}}\big|_{\partial_+\mathcal{I}^-} = -\sum_{i\in\text{in}}\frac{m_i^4}{(p_i\cdot n)^3} + \frac{1}{2G}D_z^2\left[\overset{0}{C}{}^{zz}\right]_{\mathcal{I}^-}. \tag{5.20}$$

where we used (4.62) to write $D^A D^B\left[\overset{0}{C}_{AB}\right]_{\mathcal{I}^-} = 2D_z^2\left[\overset{0}{C}{}^{zz}\right]_{\mathcal{I}^-}$. Substituting (5.19) and (5.20) in (5.17) and solving for $\left[\overset{\log}{\mathcal{N}}_z\right]_{\mathcal{I}^+}$ leads to (5.16).

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[20] One can recover (5.14) from (5.16) by inverting the D_z^3 differential operator [64, 65].

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A Time-reversal covariance of soft theorems

In a scattering context, time-reversal symmetry exchanges outgoing and incoming data through the map $t \mapsto -t$ between future and past asymptotic Cartesian times. Denoting this symmetry by $\mathcal{T}$ and with the conventions of Eqs. (3.1) and (3.9), one has[21]

$$\begin{aligned}
\mathcal{T} h^{\text{out}}_{ij}(u,\hat{n}) &= \quad -h^{\text{in}}_{ij}(-u,-\hat{n}) \\
\mathcal{T} h^{\text{out}}_{0i}(u,\hat{n}) &= \qquad h^{\text{in}}_{0i}(-u,-\hat{n}) \\
\mathcal{T} h^{\text{out}}_{00}(u,\hat{n}) &= \quad -h^{\text{in}}_{00}(-u,-\hat{n}),
\end{aligned} \tag{A.1}$$

with analogue expressions for $\mathcal{T} h^{\text{in}}_{\mu\nu}$.

Asymptotic momenta change by reversing their spatial components. Under the conventions of (2.20) this leads to

$$\mathcal{T} p^{\mu}_{\text{out}} = (-p^0_{\text{in}}, \vec{p}_{\text{in}}), \qquad \mathcal{T} p^{\mu}_{\text{in}} = (-p^0_{\text{out}}, \vec{p}_{\text{out}}). \tag{A.2}$$

For definitiveness we will discuss the purely spatial components of the soft theorems, but similar considerations go through for the remaining components. Eq. (A.1) implies the spatial components of the soft frequency coefficients transform as

$$\mathcal{T} \tilde{h}^{(0)\text{out}}_{ij}(\hat{n}) = \tilde{h}^{(0)\text{in}}_{ij}(-\hat{n}), \quad \mathcal{T} \tilde{h}^{(0)\text{in}}_{ij}(\hat{n}) = \tilde{h}^{(0)\text{out}}_{ij}(-\hat{n}). \tag{A.3}$$

and

$$\mathcal{T} \tilde{h}^{(\log)\text{out}}_{ij}(\hat{n}) = -\tilde{h}^{(\log)\text{in}}_{ij}(-\hat{n}), \quad \mathcal{T} \tilde{h}^{(\log)\text{in}}_{ij}(\hat{n}) = -\tilde{h}^{(\log)\text{out}}_{ij}(-\hat{n}). \tag{A.4}$$

As a warmup, consider first the leading soft theorem,

$$\frac{1}{4Gi} \tilde{h}^{(0)\text{out}}_{ij}(\hat{n}) + \sum_{\text{out}} \frac{p_i p_j}{p \cdot n} = \frac{1}{4Gi} \tilde{h}^{(0)\text{in}}_{ij}(\hat{n}) - \sum_{\text{in}} \frac{p_i p_j}{p \cdot n}, \tag{A.5}$$

where we separated incoming and outgoing terms and we left implicit the particle index to avoid confusion with the spatial indices. Using (A.2) and (A.3), the time-reversal of (A.5) is

$$\mathcal{T}(\text{A.5}): \quad \frac{1}{4Gi} \tilde{h}^{(0)\text{in}}_{ij}(-\hat{n}) - \sum_{\text{in}} \frac{p_i p_j}{p \cdot \bar{n}} = \frac{1}{4Gi} \tilde{h}^{(0)\text{out}}_{ij}(-\hat{n}) + \sum_{\text{out}} \frac{p_i p_j}{p \cdot \bar{n}} \tag{A.6}$$

where for convenience we defined

$$\bar{n}^{\mu} := (1, -\hat{n}). \tag{A.7}$$

Eq. (A.6) is just (A.5) evaluated at $-\hat{n}$.

[21] In this appendix, indices $i, j, \ldots$ are reserved to denote spatial Cartesian components.

Consider now the log soft theorem

$$
\begin{aligned}
&\frac{1}{4G}\tilde{h}^{(\log)\text{out}}_{ij}(\hat{n})+\sum_{\substack{\text{out}\\ m\neq 0}}\frac{p_{(i}J^{\text{rad}+}_{j)\rho}n^\rho}{p_i\cdot n}= \\
&\frac{1}{4G}\tilde{h}^{(\log)\text{in}}_{ij}(\hat{n})-\sum_{\substack{\text{in}\\ m\neq 0}}\frac{p_{(i}J^{\text{rad}-}_{j)\rho}n^\rho}{p\cdot n}+4G\Big(P\cdot n\Big(\frac{1}{4Gi}\tilde{h}^{(0)\text{in}}_{ij}(\hat{n})-\sum_{\text{in}}\frac{p_ip_j}{p\cdot n}\Big)-P_iP_j\Big),
\end{aligned}
\tag{A.8}
$$

where again, we have grouped incoming and outgoing terms and suppressed the particle index. Note that the "extra" term in the second line can alternatively be written in terms of outgoing data by using (A.5). This is the key observation that allows to establish the time-reversal covariance of (A.8).

To evaluate the time-reversal of (A.8) we still need to specify the transformation properties of the log deviation vector under $\mathcal{T}$. One can show (either from its expression in terms of momenta, or through its relation with the asymptotic metric at timelike infinity) that the future and past radiative frame log deviation vectors are mapped into each other according to

$$
\mathcal{T}c^\mu_{\text{rad}+,\text{out}}=\left(-c^0_{\text{rad}-,\text{in}},\vec{c}_{\text{rad}-,\text{in}}\right), \tag{A.9}
$$

with analogue expression for $\mathcal{T}c^\mu_{\text{rad}-,\text{in}}$. In particular, the angular momentum terms in (A.8) transform as

$$
\mathcal{T}J^{\text{rad}+}_{j\rho}n^\rho=-J^{\text{rad}-}_{j\rho}\bar{n}^\rho,\quad \mathcal{T}J^{\text{rad}-}_{j\rho}\bar{n}^\rho=-J^{\text{rad}+}_{j\rho}\bar{n}^\rho, \tag{A.10}
$$

where we have kept implicit the in/out labels. Using (A.2), (A.4), (A.10) and, importantly, (A.5), the time-reversal of (A.8) takes the form

$$
\begin{aligned}
&\mathcal{T}(\text{A.8}):\quad -\frac{1}{4G}\tilde{h}^{(\log)\text{in}}_{ij}(-\hat{n})+\sum_{\substack{\text{in}\\ m\neq 0}}\frac{p_{(i}J^{\text{rad}-}_{j)\rho}\bar{n}^\rho}{p_i\cdot\bar{n}}= \\
&-\frac{1}{4G}\tilde{h}^{(\log)\text{out}}_{ij}(-\hat{n})-\sum_{\substack{\text{out}\\ m\neq 0}}\frac{p_{(i}J^{\text{rad}+}_{j)\rho}\bar{n}^\rho}{p\cdot\bar{n}}+4G\Big(P\cdot\bar{n}\Big(\frac{1}{4Gi}\tilde{h}^{(0)\text{in}}_{ij}(-\hat{n})-\sum_{\text{in}}\frac{p_ip_j}{p\cdot\bar{n}}\Big)-P_iP_j\Big).
\end{aligned}
\tag{A.11}
$$

This relation is equivalent to (A.8) evaluated at $-\hat{n}$. We recall that the appearance of the antipodal map is due to our conventions in (3.9), see Footnote 7 for related comments.

B Hyperboloid geometry formulas

Timelike infinity

- Nontrivial Christoffel symbols of the metric (4.4) (without considering those associated to the 2-sphere):

$$\Gamma^{\rho}_{\rho\rho} = -\frac{\rho}{1+\rho^2}, \quad \Gamma^{\rho}_{AB} = -\rho(1+\rho^2)q_{AB}, \quad \Gamma^{A}_{B\rho} = \frac{1}{\rho}\delta^A_B \tag{B.1}$$

- Commutator of covariant derivatives:

$$[D_a, D_b]\omega_c = \omega_a h_{bc} - \omega_b h_{ac}. \tag{B.2}$$

- Contractions of derivatives of the unit vector (2.10)

$$D_a V^\mu D_b V_\mu = h_{ab}, \quad D^a V^\mu D_a V^\nu = \eta^{\mu\nu} + V^\mu V^\nu. \tag{B.3}$$

- Second derivative identities of the unit vector

$$(D_a D_b - h_{ab})V^\mu = 0 \implies (D^2 - 3)V^\mu = 0. \tag{B.4}$$

Spatial infinity

- Nontrivial Christoffel symbols of the metric (4.29) (without considering those associated to the 2-sphere):

$$\Gamma^{\tau}_{\tau\tau} = -\frac{\tau}{1+\tau^2}, \quad \Gamma^{\tau}_{AB} = \tau(1+\tau^2)q_{AB}, \quad \Gamma^{A}_{B\tau} = \frac{\tau}{1+\tau^2}\delta^A_B \tag{B.5}$$

- Commutator of covariant derivatives:

$$[D_a, D_b]\omega_c = -\omega_a h_{bc} + \omega_b h_{ac}. \tag{B.6}$$

- Contractions of derivatives of the unit vector (2.11)

$$D_a V^\mu D_b V_\mu = h_{ab}, \quad D^a V^\mu D_a V^\nu = \eta^{\mu\nu} - V^\mu V^\nu. \tag{B.7}$$

- Second derivative identities of the unit vector

$$(D_a D_b + h_{ab})V^\mu = 0 \implies (D^2 + 3)V^\mu = 0. \tag{B.8}$$

C Sourced Beig-Schmidt expansion at timelike infinity

In this appendix we extend the BS expansion at timelike infinity of [18], by explicitly including the matter stress tensor contribution. As in section 4.1, we treat simultaneously the $\tau \to \pm\infty$ cases, omitting $\pm$ labels.

Asymptotic trajectories

We start by describing the particles' asymptotic trajectories in BS coordinates, as they are the main input for the construction of the asymptotic stress tensor. In the notation of section 2, the particles' trajectories in asymptotic Cartesian coordinates take the form

$$X_i^\mu(s_i) \overset{s_i\to\pm\infty}{=} s_i V_i^\mu + \log|s_i| c_i^\mu + \cdots\,, \tag{C.1}$$

with s_i the proper time of the i-th particle. In BS coordinates (τ, x^a) the asymptotic geodesics take the form

$$\tau_i(s_i) = s_i + O(\log|s_i|), \quad x_i^a(s_i) = x_i^a + \frac{\log|s_i|}{s_i} c_i^a + \cdots\,, \tag{C.2}$$

where x_i^a (without the argument) is the hyperboloid coordinate associated to V_i^μ and

$$c_i^a = D^a V_\mu(x) c^\mu(x)|_{x=x_i}. \tag{C.3}$$

By studying the asymptotic geodesic equation [8, 31] one can recover $c_i^a = D^a\sigma(x)|_{x=x_i}$, in accordance with (4.5).

Stress tensor

The stress tensor due to outgoing/incoming massive particles is

$$T^{\mu\nu}(X) = \sum_{i\in\pm} m_i \int ds_i \frac{\delta^{(4)}(X - X_i(s_i))}{\sqrt{-g(X)}} \frac{dX_i^\mu}{ds_i}\frac{dX_i^\nu}{ds_i}, \tag{C.4}$$

where μ, ν stand for arbitrary spacetime coordinates, not necessarily Cartesian. Here and in the equations that follow we keep implicit the condition $m_i \neq 0$ in the sums.

We are interested in evaluating (C.4) in BS coordinates for large $|\tau|$. Substituting (C.2) in (C.4), the various terms can be expanded according to

$$\delta^{(4)}(X - X_i(s_i)) = \delta(\tau - \tau_i(s_i))\Big(\delta^{(3)}(x, x_i) - \frac{\log|s_i|}{s_i} c_i^a \partial_a \delta^{(3)}(x, x_i) + O(s_i^{-1})\Big), \tag{C.5}$$

$$\frac{1}{\sqrt{-g}} = |\tau|^{-3}\frac{1}{\sqrt{h}}\Big(1 + \frac{2\sigma}{\tau} + O(\tau^{-2}\log|\tau|)\Big), \tag{C.6}$$

$$\frac{dx_i^a(s_i)}{ds_i} = -c_i^a\frac{\log|s_i|}{s_i^2} + O(s_i^{-2}), \tag{C.7}$$

$$\frac{d\tau_i}{ds_i} = 1 + O(s_i^{-1}). \tag{C.8}$$

Performing the s_i integral leads to

$$T_{\tau\tau} = \frac{1}{\tau^3} T_{\tau\tau}^{(1)} + \frac{\log|\tau|}{\tau^4} T_{\tau\tau}^{(\log)} + O(\tau^{-4}), \tag{C.9}$$

$$T_{\tau a} = \frac{\log|\tau|}{\tau^3} T^{(\log)}_{\tau a} + O(\tau^{-3}), \tag{C.10}$$

$$T_{ab} = O(\tau^{-3}\log^2|\tau|), \tag{C.11}$$

with[22]

$$T^{(1)}_{\tau\tau} = \pm \sum_{i\in\pm} m_i \frac{\delta^{(3)}(x,x_i)}{\sqrt{h}} \equiv \rho(x) \tag{C.12}$$

$$T^{(\log)}_{\tau a} = \pm \sum_{i\in\pm} m_i \frac{\delta^{(3)}(x,x_i)}{\sqrt{h}} c_{ia} = \rho\, c_a(x) \equiv j_a(x) \tag{C.13}$$

$$T^{(\log)}_{\tau\tau} = \mp \sum_{i\in\pm} m_i D_a \Big(\frac{\delta^{(3)}(x,x_i)}{\sqrt{h}}\Big) c^a_i = -D_a\Big(\rho\, c^a(x)\Big) = -D\cdot j(x). \tag{C.14}$$

Einstein tensor

The Einstein tensor for the Beig-Schmidt metric (4.2), (4.3) has the asymptotic form

$$G_{\tau\tau} = \frac{1}{\tau^3} G^{(1)}_{\tau\tau} + \frac{\log|\tau|}{\tau^4} G^{(\log)}_{\tau\tau} + O(\tau^{-4}), \tag{C.15}$$

$$G_{\tau a} = \frac{1}{\tau^2} G^{(1)}_{\tau a} + \frac{\log|\tau|}{\tau^3} G^{(\log)}_{\tau a} + O(\tau^{-3}), \tag{C.16}$$

$$G_{ab} = \frac{1}{\tau} G^{(1)}_{ab} + \frac{\log|\tau|}{\tau^2} G^{(\log)}_{ab} + O(\tau^{-2}), \tag{C.17}$$

with

$$G^{(1)}_{\tau\tau} = 2(D^2-3)\sigma, \quad G^{(1)}_{\tau a} = -\frac{1}{2} D^b k_{ab}, \tag{C.18}$$

$$G^{(1)}_{ab} = -\frac{1}{2}(D^2+3)k_{ab} + D_{(a}D^c k_{b)c} - \frac{1}{2} D^c D^d k_{cd}\, h_{ab}, \tag{C.19}$$

$$G^{(\log)}_{\tau\tau} = -i - \frac{1}{2} D^2 i + \frac{1}{2} D^a D^b i_{ab}, \quad G^{(\log)}_{\tau a} = D_a i - D^b i_{ab}, \tag{C.20}$$

$$G^{(\log)}_{ab} = -i_{ab} - \frac{1}{2} D^2 i_{ab} - \frac{1}{2} D_a D_b i + D_{(a} D^c i_{b)c} + \frac{1}{2} D^2 i\, h_{ab} - \frac{1}{2} D^c D^d i_{cd}\, h_{ab}. \tag{C.21}$$

where

$$i = h^{ab} i_{ab}. \tag{C.22}$$

Einstein equations

Equating the asymptotic Einstein tensor to ($8\pi G$ times) the asymptotic stress tensor, one finds:

- Equations for k_{ab}:
$$G^{(1)}_{\tau a} = G^{(1)}_{ab} = 0. \tag{C.23}$$

- Equation for σ:
$$(D^2-3)\sigma = 4\pi G\rho. \tag{C.24}$$

[22] In the rest of the paper we keep implicit the factors of $\sqrt{h}$.

- Equations for i_{ab}:

$$i + \frac{1}{2}D^2 i - \frac{1}{2}D^a D^b i_{ab} = 8\pi G D \cdot j, \tag{C.25}$$

$$D_a i - D^b i_{ab} = 8\pi G j_a, \tag{C.26}$$

$$i_{ab} + \frac{1}{2}D^2 i_{ab} + \frac{1}{2}D_a D_b i - D_{(a}D^c i_{b)c} - \frac{1}{2}D^2 i\, h_{ab} + \frac{1}{2}D^c D^d i_{cd}\, h_{ab} = 0. \tag{C.27}$$

Equations (C.23) are automatically solved by the "pure gauge" form of k_{ab} given in Eq. (4.6) [18, 42]. The solution to (C.24) is well-known and will be reviewed in the next section. We now discuss how to simplify the system of equations (C.25), (C.26), (C.27) for i_{ab}.

Taking 1/2 of the divergence of (C.26) and subtracting the result to (C.25) leads to

$$i = 4\pi G D \cdot j. \tag{C.28}$$

Using (C.28), Eq. (C.26) becomes

$$D^b i_{ab} = 4\pi G\Big(D_a D \cdot j - 2 j_a\Big). \tag{C.29}$$

Substituting (C.28) and (C.29) in (C.27) leads to

$$(D^2 + 2) i_{ab} = 4\pi G\Big((D_a D_b + 2h_{ab}) D \cdot j - \frac{1}{2}D_{(a} j_{b)}\Big). \tag{C.30}$$

Finally, using (C.28) to extract the traces in (C.29) and (C.30) leads to (4.11). We note that when $j_a = 0$ the resulting set of equations for i_{ab} reduce to the ones presented in [18].

D Green's functions

In this appendix we construct the Green's functions that determine the required metric coefficients at timelike and spatial infinity.[23]

D.1 Timelike infinity

We start by reviewing the construction of the Green's function in (4.17) (see e.g. [47]), and then apply the same ideas for (4.22).

Green's function for σ

We want to find $\mathcal{G}(x, x')$ such that

$$(D^2 - 3)\mathcal{G}(x, x') = \delta(x, x'). \tag{D.1}$$

[23] See [66–68] for similar analysis in the context of electromagnetism and massless scalars; [21, 51, 69–71] for alternative treatments based on spherical harmonic decomposition; and [72–75] for related constructions in the context of flat-space holography.

Our basic assumption is that the Green's function is symmetric under exchange of x and x' so that[24]

$$\mathcal{G}(x,x') = g(\chi), \quad \chi := -V^\mu V'_\mu, \tag{D.2}$$

where V^μ and V'^μ are the unit vectors associated to the points x and x' according to Eq. (2.10). Using the identities given in appendix B one has

$$(D^2 - 3)g = (\chi^2 - 1)\ddot{g} + 3\chi\dot{g} - 3g, \tag{D.3}$$

where the dot in (D.3) denotes derivative with respect to χ. The vanishing of (D.3) for $\chi \neq 1$ (i.e. $x \neq x'$) leads to a second order differential equation, whose general solution is

$$g = A\left(\frac{2\chi^2 - 1}{\sqrt{\chi^2 - 1}} + B\chi\right), \tag{D.4}$$

with A, B integration constants. The constant A is fixed by requiring compatibility with the Dirac delta in (D.1) when $\chi \to 1$ [47] (see the last subsection of this appendix for similar considerations in the context of i_{ab}) from where one gets

$$A = -\frac{1}{4\pi}. \tag{D.5}$$

The value of B depends on the choice of log translation frame. In the radiative frame we are interested in, we have

$$g \overset{\chi\to\infty}{\to} 0 \implies B = -2. \tag{D.6}$$

The resulting Green's function is then

$$\mathcal{G}(x,x') = -\frac{1}{4\pi}\left(\frac{2\chi^2 - 1}{\sqrt{\chi^2 - 1}} - 2\chi\right). \tag{D.7}$$

We finally study the large ρ behaviour of (D.7). In this limit we have

$$\chi \overset{\rho\to\infty}{=} \rho\,\psi + O(1/\rho), \quad \psi := -n^\mu V'_\mu. \tag{D.8}$$

Substituting (D.8) in (D.7) we get

$$\mathcal{G}(x,x') \overset{\rho\to\infty}{=} -\frac{1}{16\pi\rho^3}\frac{1}{\psi^3} + \cdots, \tag{D.9}$$

which corresponds to Eq. (4.19).

[24] The geodesic distance between x and x' is given by $d(x,x') = \cosh^{-1}\chi$.

Green's function for $i_{\langle ab\rangle}$

We now turn to the problem given by the last two equations in (4.11),

$$\begin{aligned} D^b i_{\langle ab\rangle} &= 8\pi G\left(\tfrac{1}{3}D_a D\cdot j - j_a\right), \\ (D^2+2)i_{\langle ab\rangle} &= 4\pi G\left(D_{\langle a}D_{b\rangle}D\cdot j - 4D_{\langle a}j_{b\rangle}\right). \end{aligned} \tag{D.10}$$

Our starting point is to consider an ansatz of the form

$$i_{\langle ab\rangle} = D_{\langle a}i_{b\rangle}, \tag{D.11}$$

where i_a is a vector field on $\mathcal{H}^{\pm}$. It is a priori not obvious that the most general solution to (D.10) can be written in this form, but we shall later provide an argument as to why this must be the case. In order for (D.11) to satisfy (D.10), the vector field has to obey[25]

$$\begin{aligned} D^a i_a &= 4\pi G D\cdot j \\ (D^2-2)i_a &= 8\pi G\Big(\frac{1}{2}D_a D\cdot j - 2j_a\Big). \end{aligned} \tag{D.12}$$

We will first construct the Green's function for (D.12), and then use it to obtain the Green's function for the original problem (D.10).

Green's function for (D.12)

Let us write the solution to (D.12) as

$$i_a(x) = 8\pi G\int d^3x'\,\mathcal{G}_a^{b'}(x,x')j_{b'}(x'), \tag{D.13}$$

where $\mathcal{G}_a^{b'}(x,x')$ is defined by the conditions

$$\begin{aligned} D^a\mathcal{G}_a^{b'}(x,x') &= -\tfrac{1}{2}D^{b'}\delta(x,x'), \\ (D^2-2)\mathcal{G}_a^{b'}(x,x') &= -\big(\tfrac{1}{2}D_a D^{b'} + 2\delta_a^{b'}\big)\delta(x,x'). \end{aligned} \tag{D.14}$$

By Lorentz invariance, it should be possible to express $\mathcal{G}_a^{b'}$ in terms of $\chi \equiv -V\cdot V'$ and its derivatives. The most general form that is compatible with the tensorial structure is

$$\mathcal{G}_a^{b'}(x,x') = f(\chi)D_a\chi D^{b'}\chi + g(\chi)D_a D^{b'}\chi, \tag{D.15}$$

with f and g free functions. Using the identities given in appendix B one finds

$$D^a\mathcal{G}_a^{b'} = \Big[(\chi^2-1)\dot{f} + 4\chi f + \chi\dot{g} + 3g\Big]\,D^{b'}\chi \tag{D.16}$$

[25] As in the original set of equations (D.10), one can verify the compatibility of Eqs. (D.12) by comparing the Laplacian of the first one with the divergence of the second one.

$$(D^2-2)\mathcal{G}_a^{b'} = \left[(\chi^2-1)\ddot{f} + 7\chi\dot{f} + 2f + 2\dot{g}\right] D_a\chi D^{b'}\chi$$
$$+ \left[(\chi^2-1)\ddot{g} + 3\chi\dot{g} - g + 2\chi f\right] D_a D^{b'}\chi. \quad \text{(D.17)}$$

For $x \neq x'$ ($\chi \neq 1$) the coefficients in square brackets in (D.16) and (D.17) should vanish. These yields three self-consistent equations that can be used to determine f and g. The most general solution is

$$\begin{aligned} f(\chi) &= A\left(\frac{\chi\left(4\chi^4-10\chi^2+9\right)}{3\left(\chi^2-1\right)^{5/2}} + B\right) \\ g(\chi) &= -A\left(\frac{4\chi^4-6\chi^2+3}{3\left(\chi^2-1\right)^{3/2}} + B\chi\right), \end{aligned} \quad \text{(D.18)}$$

where A and B are integration constants. As in the case of σ, the constant A is fixed by ensuring the singularity of the Green's function at $\chi = 1$ is in accordance with the (derivatives of) Dirac deltas in (D.14). This leads to (see the end of this appendix for a proof)

$$A = \frac{3}{8\pi}. \quad \text{(D.19)}$$

The dependence on the constant B disappears upon taking the symmetrized derivative (D.11), and thus plays no role for the Green's function of i_{ab}.[26] From the point of view of i_a, however, it is natural to fix it by requiring decaying boundary condition. We will return to this point later on.

Green's function for (D.10)

Let us write the solution to (D.10) as

$$i_{\langle ab\rangle}(x) = 8\pi G \int d^3x' \mathcal{G}_{ab}^{c'}(x,x') j_{c'}(x'). \quad \text{(D.20)}$$

As before, we assume the Green's function is to be constructed out of χ and its derivatives. The most general expression compatible with the tensorial structure is

$$\mathcal{G}_{ab}^{c'}(x,x') = F(\chi) D_{\langle a}\chi D_{b\rangle}\chi D^{c'}\chi + G(\chi) D_{\langle a}\chi D_{b\rangle} D^{c'}\chi. \quad \text{(D.21)}$$

Let us now compare this expression with what one gets from (D.15) and (D.11):

$$D_{\langle a}\mathcal{G}_{b\rangle}^{c'}(x,x') = \dot{f}(\chi) D_{\langle a}\chi D_{b\rangle}\chi D^{c'}\chi + \left[f(\chi) + \dot{g}(\chi)\right] D_{\langle a}\chi D_{b\rangle} D^{c'}\chi. \quad \text{(D.22)}$$

We see that (D.21) can always be brought into the form (D.22), provided we do the

[26]The ambiguity associated to the constant B corresponds to the possibility of adding a term proportional to a global rotation $i_a \to i_a + L^{\mu\nu} V_{[\mu} D_a V_{\nu]}$ with constant $L^{\mu\nu}$. The situation is similar to the ambiguity of B in (D.4), although here we lack an interpretation in terms of residual diffeomorphisms.

identification,

$$\begin{aligned} F(\chi) &= \dot{f}(\chi), \\ G(\chi) &= f(\chi) + \dot{g}(\chi). \end{aligned} \tag{D.23}$$

So far we are not imposing any restrictions on F and G: Given any pair (F, G) we can always find (f, g) satisfying (D.23), so that (D.21) is given by the total derivative expression (D.22). The argument shows that the solution to (D.10) is indeed of the form (D.11). It then follows that the Green's function (D.20) is given by (D.22), with f and g given by (D.18), (D.19). As expected, the dependance on the constant B disappears in (D.23).

Asymptotic behavior

We finally study the $\rho \to \infty$ behavior of the Green's functions (D.13) and (D.20). We first focus on i_a.

For large enough ρ we can set to zero the source terms in (D.12). The resulting asymptotic equations admit only two possible asymptotic behaviors on i_A,

$$i_A \overset{\rho\to\infty}{=} O(\rho^{\pm 2}). \tag{D.24}$$

The plus case corresponds to homogenous solutions to (D.12) and represent superrotation vectors fields $i^a = h^{ab} i_b$ on $\mathcal{H}^{\pm}$, as defined in [46]. The minus case corresponds to solutions with source terms in the interior; these are the ones we are interested in. Let us explicitly see how they arise from the Green's function.

Consider first the large χ expansion of (D.18)

$$\begin{aligned} f(\chi) &\overset{\chi\to\infty}{=} A\left((4/3 + B) + \frac{1}{2\chi^4} + O(1/\chi^6)\right) \\ g(\chi) &\overset{\chi\to\infty}{=} -A\left((4/3 + B)\chi + \frac{1}{2\chi^3} + O(1/\chi^5)\right). \end{aligned} \tag{D.25}$$

Recalling that $O(\chi) = O(\rho)$ one can check that, indeed, these produces $O(\rho^{\pm 2})$ terms in i_A, consistent with (D.24). Furthermore, the $O(\rho^2)$ part can be eliminated by setting[27]

$$B = -4/3. \tag{D.26}$$

Using (D.25) and (D.26) in (D.15) along with (D.8)

$$\chi \overset{\rho\to\infty}{=} \rho\psi + O(1/\rho), \quad \psi \equiv -n \cdot V' \tag{D.27}$$

we get

$$\mathcal{G}_A^{b'}(x, x') \overset{\rho\to\infty}{=} -\frac{A}{2\rho^2\psi^4} \ell_A^{b'} + O(\rho^{-4}) \tag{D.28}$$

[27]This is analogous to condition (D.6), see footnote 26. We recall that $i_{\langle ab\rangle}$ is independent of B.

where

$$\ell_A^{b'} := -D_A\psi D^{b'}\psi + \psi D_A D^{b'}\psi \tag{D.29}$$
$$= \partial_A n^\mu n^\nu D^{b'} V'_{[\mu}V'_{\nu]}. \tag{D.30}$$

A short computation shows that, for a vector field with the above fall-offs one has[28]

$$i_A \overset{\rho\to\infty}{=} \frac{1}{\rho^2}\overset{0}{i}_A + \cdots \implies D_{\langle\rho}i_{A\rangle} \overset{\rho\to\infty}{=} -\frac{2}{\rho^3}\overset{0}{i}_A. \tag{D.31}$$

Applying (D.31) to (D.28) we obtain,

$$\mathcal{G}^{b'}_{\rho A}(x,x') \overset{\rho\to\infty}{=} \frac{A}{\rho^3\psi^4}\ell^{b'}_A + O(\rho^{-5}), \tag{D.32}$$

which corresponds to (4.23) upon using (D.19).

Eq. (D.19)

In this final subsection, we determine the constant A in (D.18). The idea is that in the limit where x is close to x', we can approximate (D.14) by their flat space counterparts, where they can easily be integrated. Let us fix

$$V'^\mu = (1,\vec{0}) \implies \chi = \sqrt{1+\rho^2}, \tag{D.33}$$

and define local Cartesian coordinates by,

$$\mathrm{x}^i := \rho\,\hat{n}^i, \tag{D.34}$$

in terms of which, the hyperboloid metric takes the form

$$h_{ij} = \delta_{ij} - \frac{1}{\rho^2+1}\mathrm{x}_i\mathrm{x}_j, \tag{D.35}$$

where $\mathrm{x}_i = \mathrm{x}^i$ and $\rho^2 = \mathrm{x}^i\mathrm{x}_i$.

We will focus on the second of Eqs. (D.14). In the $\rho\to 0$ limit and in the coordinates (D.34), the equation at $x'=0$ takes the form

$$\left(\partial^k\partial_k + O(\rho^0)\right)\mathcal{G}_i^{j'}(\mathrm{x},0) = -\left(\tfrac{1}{2}\partial_i\partial^{j'} + 2\delta_i^{j'}\right)\,\delta(\mathrm{x},\mathrm{x}')\big|_{\mathrm{x}'=0} \tag{D.36}$$

where $\partial^i = \partial_i = \partial/\partial\mathrm{x}^i$. We note that the $O(\rho^0)$ piece in (D.36) includes multiplicative terms as well as terms with one derivative, of the form $O(\rho)\partial_k$ since each derivative counts as $O(\rho^{-1})$.

Using the standard flat-space relation

$$\delta(\mathrm{x},0) = -\frac{1}{4\pi}\partial^k\partial_k\,\frac{1}{|\mathrm{x}-\mathrm{x}'|}\bigg|_{\mathrm{x}'=0}, \tag{D.37}$$

[28] The i_ρ component decays as $1/\rho^5$ and does not contribute to the leading term of $D_{\langle\rho}i_{A\rangle}$.

the solution to (D.36) is found to be

$$\mathcal{G}_i^{j'}(\mathrm{x},0) = \frac{1}{4\pi}\left(\tfrac{1}{2}\partial_i\partial^{j'} + 2\delta_i^{j'}\right)\frac{1}{|\mathrm{x}-\mathrm{x}'|}\bigg|_{\mathrm{x}'=0} + O(\rho^{-1}) \tag{D.38}$$

$$= \frac{1}{4\pi}\left(\frac{1}{2}\frac{\delta_i^{j'}}{\rho^3} - \frac{3}{2}\frac{\mathrm{x}_i\mathrm{x}^{j'}}{\rho^5}\right) + O(\rho^{-1}). \tag{D.39}$$

Let us now consider the $\rho \to 0$ expansion of (D.15). The derivatives of χ in the coordinates (D.34) at $x' = 0$ take the form[29]

$$\begin{aligned} &\partial_i'\chi = -\mathrm{x}_i, \quad \partial_i\partial_j'\chi = -\delta_{ij}, \\ &\partial_i\chi = \frac{\mathrm{x}_i}{\chi} = \mathrm{x}_i(1-\rho^2/2 + O(\rho^4)), \end{aligned} \tag{D.40}$$

while the short distance expansion of the functions (D.18) read

$$\begin{aligned} f(\chi) &\overset{\rho\to 0}{=} A\left(\frac{1}{\rho^5} - \frac{1}{6}\frac{1}{\rho^3} + \cdots\right), \\ g(\chi) &\overset{\rho\to 0}{=} -\frac{A}{3}\left(\frac{1}{\rho^3} + 2\frac{1}{\rho} + \cdots\right). \end{aligned} \tag{D.41}$$

Substituting (D.40), (D.41) in (D.15) yields

$$\mathcal{G}_i^{j'}(\mathrm{x},0) \overset{\rho\to 0}{=} A\left(-\frac{1}{\rho^5}\mathrm{x}_i\mathrm{x}^{j'} + \frac{1}{3}\frac{1}{\rho^3}\delta_i^{j'}\right) + O(\rho^{-1}). \tag{D.42}$$

By comparing with (D.39), we conclude that $A = 3/(8\pi)$.

D.2 Spatial infinity

In this subsection we construct Green's function for σ and i_{ab} that solve their respective equations in terms of asymptotic final or initial data.

Green's function for σ

Let us for concreteness focus on the future radiative frame solution. We want to find $\sigma^{\mathrm{rad}+}$ such that

$$(D^2+3)\sigma^{\mathrm{rad}+} = 0, \quad \sigma^{\mathrm{rad}+}(\tau,\phi) \overset{\tau\to+\infty}{=} \overset{0}{\sigma}{}^{+}(\phi)/\tau^3 + \cdots, \tag{D.43}$$

for a given final data $\overset{0}{\sigma}{}^{+}(\phi)$. This problem may be solved by first considering the finite-τ final value problem to the PDE (e.g. through a 3d "bulk-to-bulk" advanced Green's function) and then take $\tau \to \infty$.[30] It is however simpler to find directly the "boundary-to-bulk" Green's function

$$(D^2+3)G(x,\phi') = 0, \quad G(\tau,\phi,\phi') \overset{\tau\to+\infty}{=} \delta(\phi,\phi')/\tau^3 + \cdots, \tag{D.44}$$

[29] To evaluate the primed derivatives one needs the expression for χ away from $x' = 0$, $\chi = \sqrt{1+\rho^2}\sqrt{1+\rho'^2} - \mathrm{x}^k\mathrm{x}_k$.

[30] See [66] for an implementation of this approach in the context of electrodynamics.

in terms of which the solution to (D.43) reads

$$\sigma^{\text{rad}+}(x) = \int d^2\phi' G(x,\phi')\overset{0}{\sigma}{}^{+}(\phi'). \tag{D.45}$$

To solve for (D.44) we consider the ansatz

$$G(x,\phi') = g(\psi), \quad \psi := V^\mu(x) n_\mu(\phi'), \tag{D.46}$$

that ensures Lorentz covariance. From the identities reviewed in appendix B, g should satisfy

$$(D^2+3)g = -\psi^2\ddot{g} - 3\psi\dot{g} + 3g = 0, \tag{D.47}$$

where the dot denotes derivative with respect to ψ. The general *non-distributional* solution to this equation is $g = A\psi + B/\psi^3$, with A, B integration constants. The first term leads to a pure log translation while the second term leads to a Green's function with boundary value $1/\psi^3 \overset{\tau\to+\infty}{\sim} \tau\delta(\phi,\phi')$.[31] To find solutions with the desired decaying behavior one needs to consider distributional solutions. These can be obtained by doing the replacement $\psi \to \psi + i0$ in the regular solutions [77], leading to $g = C\psi\theta(\psi) + D\ddot{\delta}(\psi)$ where θ is the step function, $\ddot{\delta}$ the second derivative of the Dirac delta, and C, D new integration constants.[32] As we now discuss, the desired boundary condition is obtained by setting $D = 0$ and $C = 4/\pi$.

The asymptotic behavior of ψ at large positive τ is

$$\psi \overset{\tau\to+\infty}{=} \tau\, n(\phi)\cdot n(\phi') + \cdots\,. \tag{D.48}$$

One can show (see e.g. appendix E of [76]) that in this limit,

$$\psi\,\theta(\psi) \overset{\tau\to+\infty}{=} \frac{\pi}{4\tau^3}\delta(\phi,\phi') + \cdots\,, \tag{D.49}$$

and therefore

$$g(\psi) = \frac{4}{\pi}\psi\,\theta(\psi) \tag{D.50}$$

is the solution to (D.47) with the desired boundary value (D.44).

In order to establish the matching properties between future and past, let us now evaluate (D.50) at $\tau \to -\infty$. The asymptotic behavior of ψ for large negative τ is

$$\psi \overset{\tau\to-\infty}{=} \tau\, n(\mathcal{A}\phi)\cdot n(\phi') + \cdots \tag{D.51}$$

while the analogue of Eq. (D.49) in this limit is

$$\psi\,\theta(-\psi) \overset{\tau\to-\infty}{=} \frac{\pi}{4\tau^3}\delta(\mathcal{A}\phi,\phi') + \cdots \tag{D.52}$$

[31] Relevant for solving $\Phi(x)$ in (4.31) in terms of the supertranslation Goldstone $C(\hat{n})$ [43, 46, 51].

[32] Ec. (D.47) is a particular case of $(D^2+(n^2-1))g = 0, n\in\mathbb{N}^+$, with general solution $g = A\psi^{n-1} + B/\psi^{n+1} + C\psi^{n-1}\theta(\psi) + D\delta^{(n)}(\psi)$. It would be interesting to understand the relationship with the scalar harmonics studied in [69], which are defined through the same family of equations.

Writing $\theta(\psi) = 1 - \theta(-\psi)$ in (D.50) and using (D.52) we conclude that

$$g(\psi) \stackrel{\tau\to-\infty}{=} \frac{4\psi}{\pi} - \frac{1}{\tau^3}\delta(\mathcal{A}\phi, \phi') + \cdots . \tag{D.53}$$

In (D.53) we purposely did not expand the first term as in (D.51) since it is, by itself, a solution of (D.47). This means that the remaining pieces must add up to a solution of (D.47). We now show that such remaining terms are the potential in the past radiative log frame. To this end, let us smear (D.53) with $\overset{0}{\sigma}{}^{+}(\phi')$ as in (D.45). Using (4.44), the integrated version of (D.53) then reads

$$\sigma^{\text{rad}+}(x) \stackrel{\tau\to-\infty}{=} 4GP \cdot V(x) - \frac{1}{\tau^3}\overset{0}{\sigma}{}^{+}(\mathcal{A}\phi) + \cdots . \tag{D.54}$$

Comparing with (4.40) we see that indeed the second term in the RHS of (D.54), with dots included, is $\sigma^{\text{rad}-}(x)$, with boundary value at $\tau \to -\infty$ given by $\overset{0}{\sigma}{}^{-}(\phi) = -\overset{0}{\sigma}{}^{+}(\mathcal{A}\phi)$.

Green's function for i_{ab}

As in the timelike case, the solution to (4.34) can be written in terms of a "vector potential" i_a, such that

$$i_{ab} = D_{\langle a} i_{b\rangle}, \tag{D.55}$$

with i_a satisfying

$$D^a i_a = 0, \quad (D^2 + 2)i_a = 0. \tag{D.56}$$

For large τ, (D.56) implies $i_A = O(\tau^{\pm 2})$. We are interested in the decaying solutions (modulo a few growing solutions in the kernel of (D.55), see below). Consider for concreteness final value conditions

$$i_A(\tau, \phi) \stackrel{\tau\to\pm\infty}{=} \frac{\overset{0}{i}{}^{+}_{A}(\phi)}{\tau^2} + \cdots . \tag{D.57}$$

The solution to (D.56) under (D.57) can be written as

$$i_a(x) = \int d^2\phi' G_a^{B'}(x, \phi')\overset{0}{i}{}^{+}_{B'}(\phi'), \tag{D.58}$$

with $G_a^{B'}$ satisfying

$$D^a G_a^{B'} = 0, \tag{D.59}$$
$$(D^2 + 2)G_a^{B'} = 0, \tag{D.60}$$
$$G_A^{B'}(\tau, \phi, \phi') \stackrel{\tau\to\pm\infty}{=} \delta_A^{B'}\delta(\phi, \phi')/\tau^2 + \cdots . \tag{D.61}$$

As in (D.46), we consider an ansatz where the Green's function depends on ψ,

$$G_a^{B'}(x, \phi') = \ell_a^{B'} g(\psi) \tag{D.62}$$

with

$$\ell_a^{B'} := -D_a\psi D^{B'}\psi + \psi D_a D^{B'}\psi \tag{D.63}$$

capturing the tensorial structure.[33] For later reference, we note that the large τ limit of the sphere-sphere components of (D.63) is

$$\ell_A^{B'} \overset{\tau\to\pm\infty}{=} \tau^2\big(-(D_A n\cdot n')\,(n\cdot D^{B'}n') + n\cdot n' D_A n\cdot D^{B'}n'\big) + O(\tau^0). \tag{D.64}$$

where $n' \equiv n(\phi')$.

Using the identities of appendix B, one finds that the divergence-free condition (D.59) is automatically satisfied by (D.62), while (D.60) translates into

$$\psi^2\ddot{g} + 5\psi\dot{g} = 0. \tag{D.65}$$

The general non-distributional solution to (D.65) is $g = A + B/\psi^4$. The constant term represents a redundancy in the definition of i_a that goes away when evaluating i_{ab} in (D.55).[34] The second term can be shown to lead to a growing solution, $i_A = O(\tau^2)$, relevant for the boundary-to-bulk Green's function for superrotations. To get decaying boundary values we need to consider distributional solutions to (D.65), given by $g = C\theta(\psi)+D\delta^{(3)}(\psi)$. In both cases the large τ limit is proportional to (possibly derivatives of) $\delta(\phi,\phi')$, which implies the first term in (D.64) is subdominant with respect to the second one. Using this fact, along with (D.49) and $D_A n(\phi)\cdot D^{B'}n(\phi')\big|_{\phi'=\phi} = \delta_A^{B'}$ we find

$$\ell_A^{B'}\theta(\psi) \overset{\tau\to\pm\infty}{=} \frac{\pi}{4\tau^2}\delta_A^{B'}\delta(\phi,\phi') + \cdots\,. \tag{D.66}$$

We then conclude that the solution to (D.65) that is consistent with the boundary value (D.61) is given by

$$g(\psi) = \frac{4}{\pi}\theta(\psi). \tag{D.67}$$

Let us finally evaluate the $\tau\to-\infty$ limit of this Green's function. The analogue of (D.66) in this limit is

$$\ell_A^{B'}\theta(-\psi) \overset{\tau\to-\infty}{=} \frac{\pi}{4\tau^2}\mathcal{A}_*\delta_A^{B'}\delta(\mathcal{A}\phi,\phi') + \cdots\,, \tag{D.68}$$

where $\mathcal{A}_*\delta_A^{B'} = D_A n(\mathcal{A}\phi)\cdot D^{B'}n(\phi')\big|_{\phi'=\mathcal{A}\phi}$ implements the pull-back map on the index A.

The large negative τ limit of the sphere-sphere component of the Green's function can then be written as

$$G_A^{B'}(\tau,\phi,\phi') \overset{\tau\to-\infty}{=} \frac{4}{\pi}\ell_A^{B'} - \frac{1}{\tau^2}\mathcal{A}_*\delta_A^{B'}\delta(\mathcal{A}\phi,\phi') + \cdots\,, \tag{D.69}$$

where similarly to what we did in Eq. (D.53), we kept unexpanded the first term. Using

[33] One could consider the more general ansatz $f_1(\psi)D_a\psi D^{B'}\psi + f_2(\psi)D_a D^{B'}\psi$. The field equations however restrict these two terms to combine in the form (D.58). This is a spatial infinity counterpart to the timelike infinity expression (D.28).

[34] The same phenomenon occurs in the timelike infinity case, see footnote 26.

(D.69) in (D.58) we get

$$i_A(\tau,\phi) \overset{\tau\to-\infty}{=} R_A(x) - \frac{1}{\tau^2}\mathcal{A}_* \overset{0}{i}{}^{+}_{A}(\phi) + \cdots . \tag{D.70}$$

where R_A is the sphere component of the Killing vector field

$$R_a(x) = \frac{4}{\pi}\int d^2\phi' \ell_a^{B'} \overset{0}{i}{}^{+}_{B'}(\phi'). \tag{D.71}$$

This term vanishes when evaluating the symmetrized derivative (D.55), and hence we conclude that i_{ab} has decaying boundary conditions at both future and past infinities. Using (D.57) and (D.70) to evaluate $i_{\tau A} = D_{\langle\tau}i_{A\rangle}$ at these limits one finds[35]

$$i_{\tau A}(\tau,\phi) \overset{\tau\to\pm\infty}{=} \frac{\overset{0}{i}{}^{\pm}_{\tau A}(\phi)}{\tau^3} + \cdots \tag{D.72}$$

with

$$\overset{0}{i}{}^{+}_{\tau A} = -2\overset{0}{i}{}^{+}_{A}, \qquad \overset{0}{i}{}^{-}_{\tau A} = 2\mathcal{A}_*\overset{0}{i}{}^{+}_{A}. \tag{D.73}$$

Eq. (4.50)

It is simpler to work at the level of the vector i_a. Eq. (4.37) can be written as

$$\delta_L i_{ab} = D_{\langle a}\delta_L i_{b\rangle} \tag{D.74}$$

with

$$\delta_L i_a = -2lD_a\sigma + 3\sigma D_a l + D^c l D_c D_a \sigma. \tag{D.75}$$

Evaluating the large τ limit of (D.75) leads to

$$\delta_L i_A(\tau,\phi) \overset{\tau\to+\infty}{=} \frac{2}{\tau^2}(n(\phi)\cdot L\partial_A + 3\partial_A n(\phi)\cdot L)\overset{0}{\sigma}{}^{+}(\phi). \tag{D.76}$$

Using the relation $\overset{0}{i}{}^{+}_{\tau A} = -2\overset{0}{i}{}^{+}_{A}$ one recovers the $\tau\to+\infty$ identity in (4.50). The $\tau\to-\infty$ limit can be obtained from the previous one by doing the replacements $n^\mu \to \mathcal{A}_* n^\mu$ and $\overset{0}{\sigma}{}^{+} \to \overset{0}{\sigma}{}^{-}$.

E Matching infinities

In this appendix we review the asymptotic changes of coordinates [18] that interpolate between Beig–Schmidt and Bondi coordinates near each pair of adjacent spacetime boundaries. The analysis requires dealing with double limits: a first limit that takes us from the bulk spacetime into one of the $3d$ boundaries (4.68), and a second one that reaches the $2d$ boundaries (4.69). For instance, in Bondi coordinates we take

$$|r|\to\infty, \quad |u|\to\infty, \quad \text{with} \quad |r|/|u|\to\infty, \tag{E.1}$$

[35] Similar to the timelike case, the i_τ component decays as $1/\tau^5$ and does not contribute to $\overset{0}{i}_{\tau A}$.

where the last condition specifies the order of limits.

We will discuss separately the timelike and spatial infinity cases, with a focus on the pieces that are relevant for the derivation of soft theorems. In both cases, the starting point is the doubly-expanded Bondi metric, obtained by considering the large u expansion of (4.54),

$$ds^2_{\text{Bondi}} = (-2+\cdots)dudr + \Big(-1+\frac{2G\overset{0}{\mathcal{M}}}{r}+\cdots\Big)du^2 + \Big(r^2 q_{AB} + r\overset{0}{C}_{AB}+\cdots\Big)d\Omega^A d\Omega^B + \Big(D^B\overset{0}{C}_{AB} + \frac{4G}{3r}\big(\log|r|\overset{\log r}{\mathcal{N}_A} + \log|u|\overset{\log u}{\mathcal{N}_A}\big)+\cdots\Big)dud\Omega^A, \quad \text{(E.2)}$$

where the dots stand for terms that are subleading under (E.1). We denote by Ω^A the angular Bondi coordinates in order to distinguish them from the Beig-Schmidt angular coordinates ϕ^A.

Timelike to null

We start with the asymptotic change of coordinates between timelike and null infinities

$$(\tau,\rho,\phi^A)\to(r,u,\Omega^A), \tag{E.3}$$

within their overlapping region. On the future/past timelike infinity side, this region is characterized by

$$\tau\to\pm\infty, \quad \rho\to\infty, \quad \text{with} \quad \tau/\rho\to\pm\infty, \tag{E.4}$$

where the last condition specifies the required order of limits. Similarly, on the future/past null infinity side we consider

$$r\to\pm\infty, \quad u\to\pm\infty, \quad \text{with} \quad r/u\to\infty. \tag{E.5}$$

To leading order, the change of coordinates (E.3) can be identified by comparing the asymptotic Cartesian expressions (4.1) and (4.52). The large ρ expansion of the unit timelike vector (2.10) reads

$$V^\mu(\rho,\phi) \overset{\rho\to\infty}{=} \rho\, n^\mu(\phi) + \frac{1}{2\rho}t^\mu + O(1/\rho^3), \tag{E.6}$$

where $t^\mu=(1,\vec{0})$. Substituting (E.6) into (4.1) leads to

$$X^\mu = \tau\rho\, n^\mu + \frac{\tau}{2\rho}t^\mu+\cdots, \tag{E.7}$$

where the dots are terms that are subleading under (E.4). Comparing (E.7) with (4.52) we learn that

$$r=\tau\rho+\cdots, \quad u=\frac{\tau}{2\rho}+\cdots, \quad \Omega^A=\phi^A+\cdots. \tag{E.8}$$

To specify the subleading terms in (E.8) one needs to set up a generic double asymptotic expansion for the change of coordinates that maps the large-u Bondi metric (E.2) into the large-ρ BS metric in radiative log frame. From the analysis of section 4.1, one finds the latter is given by[36]

$$
\begin{aligned}
g_{\tau\rho} &= g_{\tau A} = 0 && \text{(E.9)}\\
g_{\tau\tau} &= -1 - \frac{2\overset{0}{\sigma}}{\tau\rho^3} + \cdots && \text{(E.10)}\\
g_{AB} &= \tau^2\rho^2 q_{AB} + \cdots && \text{(E.11)}\\
g_{\rho A} &= \tau O(\rho^0) + \frac{\ln|\tau|}{\rho^3}\overset{0}{i}_{\rho A} + \cdots && \text{(E.12)}\\
g_{\rho\rho} &= \tau^2 O(\rho^{-2}) + \tau O(\rho^{-3}) + \ln|\tau| O(\rho^{-2}) + \cdots . && \text{(E.13)}
\end{aligned}
$$

Consistency with (E.2) requires

$$
\begin{aligned}
r &= \tau\rho + \tau^0 O(\rho^{-1}) + \cdots \\
u &= \tau\rho(\sqrt{1+\rho^{-2}} - 1) + \tau^0 O(\rho^{-3}) + \cdots \\
\Omega^A &= \phi^A + \tau^{-1} O(\rho^{-3}) + \frac{\ln|\tau|}{\tau^2}\left(\frac{1}{\rho^4}\overset{\log}{\phi}{}^A + \cdots\right) + \cdots .
\end{aligned}
\tag{E.14}
$$

where we have only displayed the coefficients that are needed for our analysis, see [18] for a more complete treatment. The corresponding differentials are

$$
\begin{aligned}
dr &= d\tau\rho + \tau d\rho + \cdots \\
du &= \frac{d\tau}{2\rho} - \frac{\tau d\rho}{2\rho^2} + \cdots \\
d\Omega^A &= d\phi^A + \cdots + \ln|\tau|\overset{\log}{\phi}{}^A\Big(-2\frac{d\tau}{\tau^3\rho^4} - 4\frac{d\rho}{\tau^2\rho^5}\Big) + \cdots
\end{aligned}
\tag{E.15}
$$

Substituting in (E.2) and collecting coefficients of $d\tau^2$ we recover (E.28) with

$$
\overset{0}{\sigma} = -\frac{G}{4}\overset{0}{\mathcal{M}}, \tag{E.16}
$$

which corresponds to Eq. (4.70) after including the implicit labels.

Similarly, collecting $d\tau d\phi^A$ and $d\rho\, d\phi^A$ terms one finds

$$
\begin{aligned}
g_{\tau A} &= \cdots + \frac{\ln|\tau|}{\tau\rho^2}\left(\frac{G}{3}\left(\overset{\log r}{\mathcal{N}}_A + \overset{\log u}{\mathcal{N}}_A\right) - 2\overset{\log}{\phi}_A\right) + \cdots && \text{(E.17)}\\
g_{\rho A} &= \cdots + \frac{\ln|\tau|}{\rho^3}\left(-\frac{G}{3}\left(\overset{\log r}{\mathcal{N}}_A + \overset{\log u}{\mathcal{N}}_A\right) - 4\overset{\log}{\phi}_A\right) + \cdots , && \text{(E.18)}
\end{aligned}
$$

where, again, we only display terms that are needed for our analysis. The vanishing of (E.17) determines $\overset{\log}{\phi}_A$, which upon substitution in (E.18) leads to an expression consistent

[36]In addition to the fall-offs $\sigma = O(\rho^{-3})$ and $i_{\rho A} = O(\rho^{-3})$, we are using that $\Phi = O(\rho)$ and $i_{\rho\rho} = O(\rho^{-6})$.

with (E.30) with

$$\overset{0}{i}_{\rho A} = -G\left(\overset{\log r}{\mathcal{N}}_A + \overset{\log u}{\mathcal{N}}_A\right). \tag{E.19}$$

Recalling (4.67) and restoring labels, this leads to Eq. (4.73).

Spacelike to null

We now present the map between asymptotic coordinates at spatial and null infinities

$$(\rho, \tau, \phi^A) \to (r, u, \Omega^A), \tag{E.20}$$

within their common region

$$\rho \to \infty, \quad \tau \to \pm\infty, \quad \text{with} \quad \rho/\tau \to \pm\infty, \tag{E.21}$$

and

$$r \to \pm\infty, \quad u \to \mp\infty, \quad \text{with} \quad r/u \to -\infty. \tag{E.22}$$

As before, to leading order the change of coordinates can be determined by comparing the asymptotic Cartesian expressions (4.1) and (4.52). This leads to

$$r = \rho\tau + \cdots, \quad u = -\frac{\rho}{2\tau} + \cdots, \tag{E.23}$$

$$\Omega^A = \begin{cases} \phi^A + \cdots, & \text{for} \quad \tau \to +\infty \\ \mathcal{A}\phi^A + \cdots, & \text{for} \quad \tau \to -\infty \end{cases} \tag{E.24}$$

where $\mathcal{A}\phi^A$ denotes the antipodal point of ϕ^A. Condition (E.24) ensures the future and past null directions are consistent with (2.13) and (2.14) respectively. Including subleading terms, the asymptotic change of coordinates takes the form (we keep implicit for now the antipodal map (E.24) in the $\tau \to -\infty$ case)

$$\begin{aligned} r &= \rho\tau\sqrt{1+\tau^{-2}} + \rho^0 O(\tau^{-1}) + \cdots \\ u &= -\rho\tau(\sqrt{1+\tau^{-2}} - 1) + \rho^0 O(\tau^{-3}) + \cdots \\ \Omega^A &= \phi^A + \rho^{-1} O(\tau^{-3}) + \frac{\ln\rho}{\rho^2}\left(\frac{1}{\tau^4}\overset{\log}{\phi}{}^A + \cdots\right) + \cdots . \end{aligned} \tag{E.25}$$

with corresponding differentials given by

$$\begin{aligned} dr &= d\rho\,\tau + \rho\, d\tau + \cdots \\ du &= -\frac{d\rho}{2\tau} + \frac{\rho\, d\tau}{2\tau^2} + \cdots \\ d\Omega^A &= d\phi^A + \cdots + \ln\rho\,\overset{\log}{\phi}{}^A\big(-2\frac{d\rho}{\rho^3\tau^4} - 4\frac{d\tau}{\rho^2\tau^5}\big) + \cdots \end{aligned} \tag{E.26}$$

On the other hand, the BS metric coefficients (4.27), (4.28) in the limit (E.21) take the form

$$
\begin{aligned}
g_{\rho\tau} &= g_{\rho A} = 0 && \text{(E.27)}\\
g_{\rho\rho} &= 1 + \frac{2\overset{0}{\sigma}}{\rho\,\tau^3} + \cdots && \text{(E.28)}\\
g_{AB} &= \rho^2\tau^2 q_{AB} + \cdots && \text{(E.29)}\\
g_{\tau A} &= \rho\, O(\tau^0) + \frac{\ln\rho}{\tau^3}\,\overset{0}{i}_{\tau A} + \cdots && \text{(E.30)}\\
g_{\tau\tau} &= \rho^2 O(\tau^{-2}) + \rho\, O(\tau^{-3}) + \ln\rho\, O(\tau^{-2}) + \cdots\,. && \text{(E.31)}
\end{aligned}
$$

As before, one can recover this expansion by applying the change of coordinates (E.25) to the Bondi metric (E.2). We omit the steps as they are essentially the same as those presented in the timelike case. The resulting matching conditions are those presented in Eqs. (4.71), (4.72), (4.74), (4.75).

F Sphere derivatives of soft factors

In this section we discuss the identities used for the evaluation of Eqs. (5.6) and (5.16).

The identities are most easily established in a 2d frame in which the celestial metric is flat (see e.g. appendix A of [78]). This can be achieved by choosing the null vector n^μ to be

$$
n^\mu = \frac{1}{\sqrt{2}}\left(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2\right), \tag{F.1}
$$

where $(z,\bar{z})$ are stereographic coordinates on the celestial sphere.

We take

$$
k^\mu = \frac{1}{\sqrt{2}}\,(1,0,0,-1)\,, \tag{F.2}
$$

to be a complementary null direction such that $k\cdot n = -1$. Together with the polarization vectors $\partial_z n^\mu$ and $\partial_{\bar{z}} n^\mu$, these give a (complex) basis of four null vectors. The resolution of the identity in this basis can be written as

$$
\delta^\mu_\nu = \overset{+}{\delta}{}^\mu_\nu + \overset{-}{\delta}{}^\mu_\nu \tag{F.3}
$$

where

$$
\begin{aligned}
\overset{+}{\delta}{}^\mu_\nu &= \partial^z n^\mu \partial_z n_\nu - k^\mu n_\nu, && \text{(F.4)}\\
\overset{-}{\delta}{}^\mu_\nu &= \partial^{\bar{z}} n^\mu \partial_{\bar{z}} n_\nu - n^\mu k_\nu, && \text{(F.5)}
\end{aligned}
$$

are projections on the null planes $(k^\mu, \partial^z n^\mu)$ and $(n^\mu, \partial^{\bar{z}} n^\mu)$, and $\partial^z \equiv \partial_{\bar{z}}$.

All the identities we shall need are a consequence of the following "master" identity that may be checked by a short computation:

$$
\partial_z^2\left(\frac{\partial^z n^\mu \partial^z n^\nu}{p\cdot n}\right) = \frac{2\overset{+}{p}{}^\mu \overset{+}{p}{}^\nu}{(p\cdot n)^3}, \tag{F.6}
$$

where p^μ is a *massive* momentum and $\overset{\pm}{p}{}^\mu = \overset{\pm}{\delta}{}^\mu_\nu p^\nu$. For later use, we note that

$$p^\mu = \overset{+}{p}{}^\mu + \overset{-}{p}{}^\mu, \quad \overset{\pm}{p} \cdot \overset{\pm}{p} = 0, \quad \overset{+}{p} \cdot \overset{-}{p} = -m^2/2. \tag{F.7}$$

Consider first the contraction of (F.6) with $p_\mu p_\nu$. Using (F.7) one finds

$$\partial_z^2 \left(\frac{\partial^z n \cdot p \, \partial^z n \cdot p}{p \cdot n} \right) = \frac{m^4}{2} \frac{1}{(p \cdot n)^3}. \tag{F.8}$$

The massless limit of this relation has been discussed in [44, 46] and gives [40]

$$\partial_z^2 \left(\frac{\partial^z n \cdot p \, \partial^z n \cdot p}{p \cdot n} \right) \overset{m \to 0}{=} -2\pi E \delta(\hat{n}, \hat{p}), \tag{F.9}$$

where $\hat{p} = \vec{p}/E$, and E positive (negative) for outgoing (incoming) momentum.

For the log soft factor we have an extra derivative and an extra factor of the null vector n^ρ. Since $\partial_z^2 n^\rho = 0$ we have that

$$\partial_z^3 (n^\rho f) = (n^\rho \partial_z + 3\partial_z n^\rho)\, \partial_z^2 f \tag{F.10}$$

for any function f. In particular,

$$\begin{aligned} \partial_z^3 \left(n^\rho \frac{\partial^z n^\mu \partial^z n^\nu}{p \cdot n} \right) &= (n^\rho \partial_z + 3\partial_z n^\rho) \frac{2\overset{+}{p}{}^\mu \overset{+}{p}{}^\nu}{(p \cdot n)^3} & \text{(F.11)} \\ &= \frac{6\overset{+}{p}{}^\mu \overset{+}{p}{}^\nu \left(\partial_z n^\rho p \cdot n - n^\rho \, p \cdot \partial_z n\right)}{(p \cdot n)^4}, & \text{(F.12)} \end{aligned}$$

where in the first equality we used Eq. (F.6) and in the second equality we evaluated the derivative in (F.11).

To deal with the second line of (3.8), we contract (F.11) with $p_\mu p_\nu b_\rho$ with b_ρ a constant vector. This results in

$$\begin{aligned} \partial_z^3 \left(n \cdot b \frac{\partial^z n \cdot p \, \partial^z n \cdot p}{p \cdot n} \right) &= \frac{m^4}{2} \left(n \cdot b \partial_z + 3\partial_z n \cdot b\right) \frac{1}{(p \cdot n)^3} & \text{(F.13)} \\ &= \frac{3}{2} m^4 \frac{\partial_z n^\mu n^\nu b_{[\mu} p_{\nu]}}{(p \cdot n)^4}. & \text{(F.14)} \end{aligned}$$

This relation with $b_\mu = P_\mu$, explains the first term in the second line of (5.16) (the second term in that line follows simply from (F.10)).

To handle the "angular momentum" contribution in (3.8), we contract (F.11) with $p_\mu J_{\nu\rho}$ with

$$J_{\nu\rho} = b_\nu p_\rho - b_\rho p_\nu, \tag{F.15}$$

for a constant vector b_μ. It easy to see from (F.12) that only the second term in (F.15) contributes in the contraction, thus reducing the result to (minus) that in (F.14),

$$\partial_z^3\left(\frac{\partial^z n\cdot p\,\partial^z n^\nu n^\rho J_{\nu\rho}}{p\cdot n}\right)=-\frac{3}{2}m^4\frac{\partial_z n^\mu n^\nu b_{[\mu}p_{\nu]}}{(p\cdot n)^4}. \tag{F.16}$$

This relation, with $b_\mu = c_\mu^{\mathrm{rad}\pm}$, leads to the first two terms in (5.16).

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Capítulo 5

Conclusiones

La radiación gravitacional en el límite de baja frecuencia está gobernada por los denominados teoremas soft. En esta tesis se estudiaron diversos aspectos del teorema soft logarítmico en el marco de la teoría de la relatividad general.

Utilizando un análisis asintótico de las ecuaciones de Einstein acopladas a materia, fue posible dar una prueba de dicho teorema que no requiere un análisis perturbativo; es decir, no se asume en ningún momento que el campo gravitatorio sea débil.

A su vez, el hecho de que el cálculo utilice múltiples nociones geométricas otorga una perspectiva diferente, lo cual permitió resolver ciertas interrogantes que surgieron en análisis anteriores. En particular, se explica el hecho de que, tras ciertas cancelaciones no triviales, las expresiones finales no dependan de las partículas sin masa salientes. El formalismo asintótico reveló cierta ambigüedad geométrica (la invariancia bajo traslaciones logarítmicas) que, una vez impuesta, prueba la cancelación mencionada. Esto, a su vez, permitió conjeturar nuevas relaciones de consistencia para términos de mayor orden en la expansión en baja frecuencia, cuya expresión aún no ha sido determinada.

Por otra parte, fue posible extender la expresión del teorema soft logarítmico al caso en el que se considera radiación incidente en el proceso de scattering. Esto, a su vez, resolvió otra interrogante: el hecho de que las expresiones propuestas hasta el momento no respetaban la invariancia bajo inversiones temporales, ya que los estados inicial y final no estaban en pie de igualdad.

En cuanto a consideraciones formales, este trabajo proporciona evidencia de que las soluciones de la relatividad general en espacios-tiempo asintóticamente planos presentadas en el capítulo 2 efectivamente engloban problemas de scattering, lo cual ha sido motivo de debate en la literatura. También se muestra que el formalismo asintótico permite no solo estudiar aspectos formales de la teoría, como la definición de cargas conservadas, sino también funcionar como un método de cálculo de observables gravitatorios, que luego pueden contrastarse con resultados obtenidos mediante otras técnicas.

Interrogantes abiertas

La relación de recurrencia para los denominados logaritmos dominantes obtenida en [12] nos permite plantear diversas preguntas. A priori, no está claro si esta relación es suficiente para determinar completamente dichos términos, ni si admite una solución única. En [4], los autores proponen de manera heurística ciertas expresiones, por lo que un estudio más profundo de la relación obtenida podría verificar (o no) esta propuesta, y conducir a nuevos avances en el estudio de los teoremas soft logarítmicos de alto orden.

Si bien nuestra prueba del teorema soft se basa en condiciones de *matching* sobre el campo gravitatorio, existe una perspectiva complementaria [3, 14], en la cual se considera una "carga logarítmica" asociada al momento angular, cuya conservación es equivalente al teorema soft. Esta carga es divergente y por ende no está bien definida. Creemos que un análisis más profundo del espacio de fase podría proporcionar un mecanismo que permita definir estas cantidades correctamente. Esto es de particular relevancia para el estudio de los teoremas soft en el caso cuántico, donde las cargas corresponden a operadores que deben estar bien definidos, y ya es conocido que el término logarítmico recibe contribuciones adicionales.

Finalmente, creemos que es posible generalizar el formalismo asintótico a órdenes más altos. Esto permitiría determinar nuevos términos: por un lado, los logarítmicos dominantes que se conjetura están únicamente determinados por los momentos; y por otro, términos adicionales que reciben contribuciones dependientes de la estructura interna de las partículas involucradas [20].

Esto requiere estudiar objetos no puntuales en gravedad. Este aspecto resulta relevante, por ejemplo, en el caso de agujeros negros rotantes, cuya rotación induce deformaciones que usualmente se describen en términos de momentos multipolares [9].

Declaración de autoría

- Nombre del estudiante: Gianni Boschetti
- Posgrado: Maestría en Física
- Nombre de los tutores: Miguel Campiglia, Guzmán Hernández
- Título de la tesis: Radiación gravitacional clásica y teoremas soft logarítmicos
- Fecha: 19/03/2026

El autor y tutores declaran que se atribuyen la autoría exclusiva de las ideas, desarrollos, programas de computación y resultados de la presente tesis, salvo en los casos que se indique expresamente lo contrario. La presente Declaración de Autoría tiene valor de declaración jurada del estudiante y sus tutores de tesis.

La presente tesis está compuesta por la compilación de 1 artículo que ha sido publicado o aceptados para publicación y 1 artículo que ha sido remitido para publicación, todos ellos, en revistas arbitradas por pares, indexadas y reconocidas en el área específica de la tesis. El estudiante y sus tutores dejan constancia de que las contribuciones aquí declaradas no forman parte de otra tesis.

- Artículo: 1
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    - Autor de correspondencia: Gianni Boschetti
    - Co-autores: Miguel Campiglia
    - Correos electrónicos de los co-autores: campi@fisica.edu.uy
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▪ Artículo: 2

- Título: An asymptotic proof of the classical log soft graviton theorem
- Autor de correspondencia: Gianni Boschetti
- Co-autores: Miguel Campiglia
- Correos electrónicos de los co-autores: campi@fisica.edu.uy
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