Some Interactions Between Expansive Dynamics and Mathematical Logic

Author: Luis Alberto Rosa Ferrari

Advisors: Alfonso Artigue, Joan Bagaria

Examination Committee: José Rafael León, Octavio Malherbe, Matilde Martínez, Alexandre Miquel, Antonio Montalbán

Thesis Work
PhD in Mathematics, PEDECIBA
Facultad de Ciencias, Universidad de la República
December 23, 2025

Resumen

Esta tesis explora la propiedad de la *expansividad* en *sistemas dinámicos* desde tres perspectivas fundamentales: la cuantitativa, la conjuntista y la topológica.

En el contexto de los *espacios métricos compactos*, se establece un *criterio cuantitativo* que vincula la existencia de *pares doblemente asintóticos* con el decaimiento de las constantes de expansividad . Empleando *métricas hiperbólicas autosimilares*, se obtiene una caracterización precisa del decaimiento exponencial de dicha constante.

En una segunda línea, el estudio se extiende a *acciones de grupo* sobre *espacios ordinales.* A través de una formulación de *expansividad por cubrimientos* se caracteriza qué espacios ordinales admiten una *acción expansiva* (continua o CB–estable). Este resultado generaliza el teorema de Kato–Park a *ordinales no numerables* y permite establecer una cota inferior para la cardinalidad del grupo actuante.

Una de las contribuciones de esta tesis es relacionar la expansividad con la *Hipótesis Generalizada del Continuo (*GCH*).* A partir de esa conexión se define la *Expansive Generalized Continuum Hypothesis* EGCH(λ), demostrando que EGCH($\aleph_0$) es un *teorema de* ZFC.

Se aísla y formaliza el *núcleo combinatorio* que subyace a este vínculo. Además, se introduce la jerarquía DGCH$_n(\lambda)$ y se prueban sus propiedades de *consistencia relativa* con ZFC y de *monotonía.*

Por último, la investigación aborda la expansividad en *espacios métricos no compactos.* Se demuestra que, para espacios LCσ, la *independencia de la métrica* es equivalente a la *expansividad cocompacta* y a la existencia de una *extensión expansiva* a la *compactificación de Alexandroff*, concluyéndose con un análisis de su relación con la *compactificación no estándar.*

Palabras clave: expansividad; análisis no estándar; ordinales y espacios dispersos; acciones de grupo; GCH/EGCH/DGCH; compactificación de Alexandroff.

Abstract

This thesis explores the property of *expansivity* in *dynamical systems* from three fundamental perspectives: the quantitative, the set-theoretic, and the topological.

In the context of *compact metric spaces*, we establish a *quantitative criterion* that links the existence of *doubly asymptotic pairs* with the decay of the expansivity constants. Employing *self-similar hyperbolic metrics*, we obtain a precise characterization of the exponential decay of that constant.

In a second line of research, the study is extended to *group actions* on *ordinal spaces.* Through a formulation of *cover expansivity* we characterize which ordinal spaces admit an *expansive action* (continuous or CB–stable). This result generalizes the Kato–Park theorem to *uncountable ordinals* and allows one to establish a lower bound for the cardinality of the acting group.

One of the contributions of this thesis is to relate expansivity to the *Generalized Continuum Hypothesis (*GCH*).* From this connection we define the *Expansive Generalized Continuum Hypothesis* EGCH(λ), proving that EGCH($\aleph_0$) is a *theorem of* ZFC.

We isolate and formalize the *combinatorial core* underlying this connection. Moreover, we introduce the hierarchy DGCH$_n(\lambda)$ and prove its properties of *relative consistency* with ZFC and of *monotonicity.*

Finally, the research addresses expansivity in *non-compact metric spaces.* We show that, for LCσ spaces, *independence of the metric* is equivalent to *cocompact expansivity* and to the existence of an *expansive extension* to the *Alexandroff compactification*, concluding with an analysis of its relation to the *nonstandard compactification.*

Keywords: expansivity; nonstandard analysis; ordinals and scattered spaces; group actions; GCH/EGCH/DGCH; Alexandroff compactification.

Acknowledgements

I would like to begin by expressing my gratitude to my advisors, Alfonso Artigue and Joan Bagaria, for their invaluable support. Above all, I am grateful for the freedom they gave me to pursue unconventional paths—attempting to bring dynamical systems and logic into dialogue did not seem an easy undertaking, at least initially. Many thanks, Alfonso, and moltes gràcies, Joan, for your patience.

I also wish to thank Walter Ferrer, who has not only been a mathematician instrumental in shaping the modern mathematical community in Uruguay and the main promoter of the study of logic and the foundations of mathematics in the country, but who has also been a teacher to me since my undergraduate studies, encouraging me to value the importance of a broad mathematical education.

My sincere recognition also goes to Professor Jorge Lewowicz, who, in addition to being one of the founders of the dynamical systems community in Uruguay, was among the first to introduce the study of expansive dynamics in the country.

Mathematics is a collective endeavor, nourished and sustained by dialogue with others. I would like to acknowledge several individuals—though I fear I will forget many—who have contributed significantly at various stages of my academic development, some as former thesis advisors, others through valuable suggestions in more informal discussions. Among them, and again with the risk of omitting many: Jorge Groisman, Samuel Gomes Da Silva, Alexandre Miquel, José Rafael León, Matilde Martínez, and Mauricio Achigar.

I would also like to extend my sincere gratitude to the members of the examination committee: Alexandre Miquel, Antonio Montalban, José Rafael León, Matilde Martínez, and Octavio Malherbe. I am deeply grateful for their careful reading of this manuscript and for their insightful comments and useful recommendations.

Finally, I am grateful to the Department of Mathematics and Applications (DMA) at CURE, which provided me with excellent working conditions that allowed me to dedicate myself fully to this thesis, including the opportunity to spend a couple of months each year in Barcelona working with my advisor.

“The essence of mathematics lies precisely in its freedom.”

— Georg Cantor

Contents

Chapter 1

Introduction

Motivation and Context

The seminal works of Henri Poincaré laid the foundations for modern dynamical systems theory and chaos theory. In studying the stability of planetary systems, Poincaré observed that in a three-body system under gravitational attraction, small differences in initial conditions can lead to significant trajectory divergences.

Following Poincaré, the study of chaos largely paused until the advent of computing in the second half of the 20th century. Computational simulations then revealed its significance across fields such as meteorology, biology, and physics, with key concepts including the "Butterfly Effect" and "strange attractors" demonstrating unpredictable behaviors arising from initial sensitivities.

Discrete Dynamical Systems and Expansivity

An important class of dynamical systems is discrete systems, where time progresses in integer steps—suitable for modeling phenomena like population dynamics. In topological terms, a discrete dynamical system is a pair (X, f), where X is a metric space and $f\colon X \to X$ is a homeomorphism.

In 1950, Utz [36] introduced the concept of *unstable homeomorphisms*, later termed *expansive homeomorphisms*. An expansive dynamical system is a pair (X, f) with metric d, where there exists a constant $c > 0$ (the expansivity constant) such that for any distinct $x, y \in X$, there is an $n \in \mathbb{Z}$ with $d(f^n(x), f^n(y)) > c$. These systems exhibit strong sensitivity to initial conditions, as nearby points separate by more than c in the past or future.

Expansivity is a hallmark of hyperbolic sets, including Anosov diffeomorphisms, the non-wandering sets of Smale's Axiom A systems, and subshifts of finite type. Such systems typically feature doubly asymptotic points (see Definition 3.0.1), such as homoclinic points relative to periodic orbits. Expansive homeomorphisms on compact surfaces, conjugate to pseudo-Anosov maps, also possess these points. Moreover, all known expansive homeomorphisms on non-totally disconnected compact metric spaces exhibit doubly asymptotic points, whereas totally disconnected spaces may admit examples without them [2, 27], often conjugate to subshifts.

The existence of asymptotic pairs (doubly asymptotics points in the context of homeomorphisms) in expansive group actions has been actively studied recently, notably by Chung and Li [10], Meyerovitch [29] and S. Barbieri, F. García-Ramos, and H. Li [6].

One key contribution of this thesis is a necessary and sufficient condition for doubly asymptotic points in expansive homeomorphisms on compact metric spaces—framed in terms of expansivity constant decay—using nonstandard analysis. This framework is particularly well-suited for our purposes, as it allows us to translate asymptotic properties—such as the long-term behavior of trajectories—into precise statements about infinitesimal and infinite numbers, making the underlying geometry more tangible. This logical tool provides geometric intuition akin to that in Utz's theorem (Theorem 3.1.4), which guarantees asymptotic pairs in either the past or future, and proved essential for our results.

Intersections with Set Theory and the Generalized Continuum Hypothesis

Another principal interaction we develop between expansive dynamics and mathematical logic involves set theory, particularly the Generalized Continuum Hypothesis (GCH).

The Continuum Hypothesis (CH), the first of Hilbert's 23 problems, asserts that there is no cardinal strictly between $\aleph_0$ (the cardinality of the natural numbers) and $2^{\aleph_0}$ (the cardinality of its power set). Cantor's efforts to prove its truth failed. Gödel [19] showed that both CH and GCH ($\mathrm{GCH}(\lambda) := \neg\exists\kappa\ (\lambda < \kappa < 2^\lambda)$) are consistent with ZFC axioms, while Cohen [11] demonstrated the consistency of their negations. We establish an equivalence between GCH and the existence of certain expansive actions. This surprising connection arises from the strong cardinal constraints an action imposes on its space, a relationship that becomes particularly sharp for CB-stable actions. This is precisely the class of dynamics studied in the Kato and Park theorem (Theorem 2.2 in [26]), whose generalization was a key driver for our research. Ultimately, this link reveals an underlying combinatorial relation that inspires new questions in set theory.

Expansivity in Non-Compact Spaces

While expansivity has primarily been studied in compact spaces, its definition extends to general metric spaces. This thesis contributes results on expansive group actions in non-compact settings, particularly those independent of the chosen compatible metric.

Guiding questions

The following questions structure the research:

(Q1) Is there a quantitative criterion that characterizes when an expansive homeomorphism admits doubly asymptotic pairs? How is this reflected in the decay of $\gamma(f^n)$?

(Q2) Once we know that the Kato–Park Theorem is a dynamical theorem on countable ordinals, is it possible to generalize it to uncountable ordinals?

(Q3) What is the minimal cardinality (or structural size) a group must have in order to act expansively on a given compact ordinal (with the order topology)?

(Q4) What is the underlying combinatorial principle that links expansivity with the Generalized Continuum Hypothesis (GCH)? What hierarchies and generalizations arise when this principle is varied?

(Q5) In non-compact metric spaces, when is expansivity a topological property, i.e., independent of the particular metric chosen?

Main Contributions

- Chapter 3: **Expansive Dynamics and Nonstandard Analysis**

  The results presented in this chapter constitute the author's contributions to the paper [3], which has been accepted for publication.

  For an expansive homeomorphism f on a compact metric space and a *self-similar* hyperbolic metric d with factor $\lambda > 1$, the following theorem establishes a necessary and sufficient condition for the existence of doubly asymptotic points in terms of $\gamma(f^n)$, where $\gamma(f)$ is defined by

  $$\gamma(f) = \sup\{c \in \mathbb{R}^+ : \ c \text{ is an expansivity constant for } f\}.$$

  Theorem. *3.0.4 An expansive homeomorphism $f : X \to X$ admits doubly asymptotic points if and only if, for a self-similar metric d with expanding factor λ, there exist a standard real number $C > 0$ and an infinite natural number N such that $\gamma({}^*f^N) < \frac{C}{\lambda^{N/2}}$.*

  Moreover, we prove the following theorem

  Theorem. *3.0.5 Let $f : X \to X$ be an expansive homeomorphism. If X is countably infinite, then (X, f) has doubly asymptotic points.*

- Chapter 4 : **Expansive Actions and the Generalized Continuum Hypothesis.**

  The results presented in this chapter are essentially those contained in the paper [17], which was published by the author of this thesis.

  The motivation for extending to group actions comes from generalizing the following theorem:

Theorem (Kato and Park, 1999. Theorema 2.2 in [26])**.** *Let* (X, d) *be a countable compact metric space. It admits an expansive homeomorphism if and only if* $\deg(X)$ *is not an infinite limit ordinal, where* $\deg(X)$ *is the derived degree of* X*.*

From Mazurkiewicz and Sierpiński [30], Baker [5], and Semadeni [34] we know that a compact scattered first-countable topological space with characteristic (α, n) (see definition 2.2.21) is metrizable and homeomorphic to an ordinal of the form $\omega^\alpha n + 1$ with the order topology. Consequently, a countable compact metric space is homeomorphic to a successor ordinal equipped with the order topology. Therefore, the Kato and Park theorem is a theorem of dynamics on countable ordinals. For uncountable ordinals we obtain the following generalization.

Theorem. *4.2.2 Let* X *be a compact Hausdorff scattered space with characteristic* (α, n)*. Then* X *admits an expansive continuous* G*-action if and only if* α *is not an infinite limit ordinal or* $n \neq 1$*.*

- **EGCH and the DGCH hierarchy.**

Studying the cardinalities of continuous actions that can act on a compact ordinal, we arrive at the following results:

Theorem. *4.4.5 Let* X *be a compact ordinal space, and let* $\varphi : G \times X \to X$ *be an expansive continuous action. Then* $|G| \geq |X|$*.*

Theorem. *4.3.4 Let* λ *be an infinite cardinal. For every* κ *with* $\lambda \leq \kappa \leq 2^\lambda$*, there exist a compact Hausdorff space* X*, a group* G*, and an action* $\varphi : G \times X \to X$*,* CB-stable *, such that the action is expansive with* $|G| = \lambda$ *and* $|X| = \kappa$*.*

This yields an equivalence of GCH in terms of expansive actions:

$$\begin{aligned} \mathrm{GCH}(\lambda) \iff \neg\exists\, G \curvearrowright X,&\ \text{expansive CB-stable action,} \\ &|G| = \lambda < |X| \neq 2^\lambda, \quad X \text{ compact Hausdorff space.} \end{aligned}$$

As is customary in mathematics, once a concept has been reformulated in terms of a parameter that can vary—for example, that the action be CB-stable —we may ask what happens when we modify it. This motivates the following definition, which we call the Expansive Generalized Continuum Hypothesis (EGCH):

$$\begin{aligned} \mathrm{EGCH}(\lambda) := \neg\exists\, G \curvearrowright X,&\ \text{expansive continuous action,} \\ &|G| = \lambda < |X| \neq 2^\lambda, \quad X \text{ compact Hausdorff space.} \end{aligned}$$

In Theorem 4.4.8 we prove that

Theorem. EGCH($\aleph_0$) *is a theorem of ZFC.*

- Chapter 5 : **Combinatorial Expansiveness.**

  In this chapter we isolates and develops the combinatorial core underlying the relationship between expansivity and the Generalized Continuum Hypothesis (GCH) established in the previous chapter. The goal is to extract from that link the purely combinatorial ingredients and analyze them in a framework independent of dynamics, in order to open new questions about the structure of cardinals that violate GCH.

  To that end, we define, for all infinite cardinals λ and κ, and for a cardinal ρ with $0 < \rho \leq \aleph_0$, a type of combinatorial action denoted $\lambda \overset{\rho}{\curvearrowright} \kappa$. The central viewpoint will be to *vary the complexity of the graph that encodes* the relationship between expansivity and GCH. Indeed, we will see that GCH is related to a combinatorial problem on bipartite graphs, which have one or two orbits depending on the cardinality of the bipartition defining the graph. This provides a complexity parameter that organizes a hierarchy of statements that we call the Dual Generalized Continuum Hypothesis, defined for every infinite cardinal λ and every natural number n, with formula

  $$\mathrm{DGCH}_\rho(\lambda) := \neg\exists\kappa\big((\kappa \neq \lambda, 2^\lambda) \wedge (\lambda \overset{\rho}{\curvearrowright} \kappa)\big).$$

  When $n = 1$ we have

  $$\mathrm{GCH}(\lambda) \Leftrightarrow \mathrm{DGCH}_1(\lambda),$$

  and in general,

  $$\mathrm{GCH}(\lambda) \Rightarrow \mathrm{DGCH}_n(\lambda).$$

  In Theorem 5.2.16 we prove that

  $$\mathrm{DGCH}_2(\lambda) \Rightarrow \mathrm{DGCH}_\rho(\lambda).$$

  Under certain hypotheses, a monotonicity property holds; specifically, we will prove the following theorem.

  Theorem. *5.2.26 Let λ be an infinite cardinal such that* GCH(λ') *holds for all $\lambda' < \lambda$. Then, for all $i < j \leq \omega$,*

  $$\mathrm{DGCH}(\lambda)_i \Rightarrow \mathrm{DGCH}(\lambda)_j.$$

  Collecting the results proved in Theorems 5.2.13, 5.2.23, 5.2.24, and **??**, we obtain that for every infinite cardinal λ and every cardinal ρ with $2 \leq \rho \leq \aleph_0$, it holds that

  $$\neg\mathrm{DGCH}_\rho(\lambda) \quad \text{is consistent with ZFC.}$$

- Chapter 6 : **Metric-Independent Expansiveness.**

  The theory of expansive actions on compact metric spaces establishes that expansivity is a property intrinsic to the topology of the space, being independent of the chosen compatible metric. Motivated by this fact, in Chapter 6 we turn our attention to the behavior of expansive actions in a broader setting—namely (Definition 6.1.1), that of metric spaces which are not necessarily compact—investigating under what conditions expansivity remains independent of the metric.

  One of the contributions of this chapter is Theorem 6.1.12, which characterizes expansive actions on locally compact and σ-compact metric spaces (hereafter referred to as $LC\sigma$-spaces) in terms of a property we call *cocompactly expansive*. This property combines the cocompactness of the action with a separation condition involving the compact set that witnesses cocompactness. Furthermore, the theorem establishes the equivalence between this notion and the existence of an expansive extension to the one-point (Alexandroff) compactification.

  To examine more precisely the equivalences stated in Theorem 6.1.12, we will use the notion of *cover expansivity* (Definition 4.1.4), which was introduced in Chapter 4. This tool allows us to refine the analysis of the logical relationships involved in the theorem. Additionally, we will explore the behavior of *metrically independent dynamics* in specific contexts, such as ordinal and totally bounded spaces, and analyze the role played by the *completeness of the ambient space*.

  In the final section, we will once again use *nonstandard analysis* to present a generalization of Utz's theorem for *metrically independent expansive homeomorphisms* and to study the relationship between *nonstandard compactification* and the property of being *metrically independent* in the case of group actions.

Transversal Ideas and Tools

Two methodological ideas unify the different chapters:

Nonstandard analysis as an asymptotic microscope. Nonstandard analysis provides a language where asymptotic behaviors become ≪visible≫ through infinite and infinitesimal elements. This allows us to encode the decay of $\gamma(f^n)$ and translate the existence of asymptotic pairs into properties of infinitely close points.

Extraction of the combinatorial core. Cover expansivity can be reinterpreted as a problem about graphs and partitions. This reading reveals that expansivity and GCH share a combinatorial problem on bipartite graphs, whose complexity parametrizes a natural hierarchy of GCH-type principles.

Thesis Organization

Chapter 2 The preliminaries present a concise background in nonstandard analysis, general topology, and set theory to facilitate the reading of the thesis.

Chapter 3: Expansive Dynamics and Nonstandard Analysis. We develop the nonstandard framework to prove the quantitative criterion that connects doubly asymptotic pairs with the decay of $\gamma(f^n)$.

Chapter 4: Expansive Actions and the Generalized Continuum Hypothesis. We extend expansivity to group actions, characterize the scattered spaces that admit them, and discover the connection with GCH.

Chapter 5: Combinatorial Expansivity. We isolate the combinatorial core, introduce the $\mathrm{DGCH}_n(\lambda)$ hierarchy, and prove consistency and monotonicity results.

Chapter 6: Metric-Independent Expansivity. We characterize when expansivity is independent of the metric in non-compact spaces and its relationship with the Alexandroff compactification.

Chapter 7: The final chapter of the thesis contains the concluding remarks and a summary of the open problems that remain unresolved.

Chapter 2

Preliminaries

To facilitate the reading of the thesis, we present a brief introduction to the mathematical tools that will be used. The reader may consult the cited references for further details.

2.1 Nonstandard Analysis

2.1.1 Superstructures, Transfer Principle, and Compactification

Superstructure and language

Definition 2.1.1 (Superstructure)**.** Let X be a base set. Define

$$V_0(X) = X, \qquad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)), \qquad V(X) = \bigcup_{n\in\mathbb{N}} V_n(X).$$

We call $V(X)$ the *superstructure over* X. An element of $V_n(X) \setminus V_{n-1}(X)$ has *rank* n (with the convention $V_{-1}(X) = \varnothing$).

Definition 2.1.2 (Language L_X)**.** The language L_X contains the relational symbols $\in$ and $=$, countably many variables, and a constant symbol for each $a \in V(X)$. We use bounded quantifiers $(\forall x \in t)\,\varphi$ and $(\exists x \in t)\,\varphi$ as abbreviations. Terms and formulas are defined by the standard inductive clauses. In our usage, it should be clear from context when we are dealing with a formula and when we refer to the *concept* of a formula. For a rigorous definition of these notions in nonstandard analysis, see Definition 2.2.1 in [28].

Motivation. We work inside $V(X)$ because it uniformly hosts the mathematical objects in this manuscript (sets, functions, relations, metrics, topological spaces, etc.). The nonstandard passage will be a map $^*\colon V(X) \to V(Y)$ that preserves first-order structure and allows us to *reason with infinite/infinitesimal quantities* without losing truths from $V(X)$.

The $*$-transform, extensions, and Transfer

Definition 2.1.3 ($*$-transform of formulas)**.** Given a formula φ in L_X, its transform $^*\varphi$ is obtained by replacing each standard constant $a \in V(X)$ with the symbol $^*a$; connectives, quantifiers, and variables are left unchanged.

Definition 2.1.4 (Elementary $*$-extension)**.** A map $^* : V(X) \to V(Y)$ is a $*$-*extension* if it preserves rank and membership: for all $x, y \in V(X)$,

$$x \in y \iff {}^*x \in {}^*y.$$

We say that * is *elementary* if, for every formula $\varphi(x_1, \ldots, x_n)$ of L_X and all $a_1, \ldots, a_n \in V(X)$, one has

$$V(X) \models \varphi(a_1, \ldots, a_n) \iff V(Y) \models {}^*\varphi({}^*a_1, \ldots, {}^*a_n).$$

Remark 2.1.5 (Trivial extension vs. nontrivial enlargement)**.** The identity $a \mapsto a$ is a *trivial* $*$-extension with $^*X = X$; then $^*\varphi = \varphi$, no infinite or infinitesimal elements appear, and the equivalence above adds no practical power. From now on we **fix** a *nontrivial* extension * with $^*\mathbb{N} \setminus \mathbb{N} \neq \varnothing$ and such that $^*\mathbb{R}$ contains positive infinitesimals. A standard way to construct a nontrivial *nonstandard extension* is via the *ultraproduct method*; see, for instance, [28, Chapter 2].

Theorem 2.1.6 (Existence of a Nonstandard Extension)**.** *There exists a $*$-extension $^*\colon V(X) \to V(Y)$ that is both* elementary *and* nontrivial*.*

Proof. A nontrivial extension—the one assumed throughout this thesis—is obtained via the ultraproduct construction. See Theorem 2.4.5 in [28]. □

Corollary 2.1.7 (Transfer Principle)**.** *As a consequence of the previous theorem, for every formula $\varphi(x_1, \ldots, x_n)$ of L_X and all $a_1, \ldots, a_n \in V(X)$, the following equivalence—known as the* Transfer Principle*—holds:*

$$V(X) \models \varphi(a_1, \ldots, a_n) \iff V(Y) \models {}^*\varphi({}^*a_1, \ldots, {}^*a_n).$$

Definition 2.1.8 (Standard and internal)**.** We will call *standard sets* those of the form $^*A$ with $A \in V(X)$. A set E in the nonstandard universe is *internal* if $E \in {}^*B$ for some $B \in V(X)$.

Remark 2.1.9. For every $A \in V(X)$, the set $^*A$ is internal, since $^*A \in {}^*\{A\}$.

Definition 2.1.10 (Enlargement and countable saturation)**.** We assume that the nonstandard extension * is an *enlargement*: whenever $\{A_i\}_{i \in I} \subset V(X)$ is indexed by a standard set I and has the finite intersection property (FIP), one has

$$\bigcap_{i\in I} {}^*A_i \neq \varnothing.$$

We also assume *countable saturation*: every countable family of internal sets with the FIP has nonempty intersection.

Remark 2.1.11. Both hypotheses above are ensured by the usual ultrapower construction (e.g., over a nonprincipal ultrafilter on $\mathbb{N}$); see Theorem 11.10.1 in [20].

For completeness, we record the following general notion of saturation (parametrized by an infinite cardinal).

Definition 2.1.12 (κ-saturation)**.** Let κ be an infinite cardinal. A nonstandard universe ${}^*V(X)$ is said to be *κ-saturated* if for every family of *internal sets* $\{A_i\}_{i\in I}$ with $|I| < \kappa$ that has the *finite intersection property* (i.e., every finite subfamily has nonempty intersection), the total intersection is nonempty:

$$\Big(\forall\, J \subseteq I \text{ finite}, \bigcap_{j\in J} A_j \neq \varnothing\Big) \implies \bigcap_{i\in I} A_i \neq \varnothing.$$

Remark 2.1.13 (Scope)**.** Definition 2.1.12 is included only for completeness. In what follows we rely exclusively on $\aleph_1$-*saturation* (countable saturation), introduced earlier in Definition 2.1.10. Stronger degrees of saturation can be convenient, e.g., for nonstandard compactifications of spaces with very large topologies, but they will not be needed here.

Remark 2.1.14 (Working convention)**.** We identify X with a subset of ${}^*X$ via $x \mapsto {}^*x$ and, by abuse of notation, write x instead of ${}^*x$ for standard constants. We set $\mathbb{Z}_\infty = {}^*\mathbb{Z} \setminus \mathbb{Z}$.

Nonstandard extension of metric spaces

Basic definitions. Let (X, d) be a standard metric space. Its *nonstandard extension* is $({}^*X, {}^*d)$, where ${}^*X$ and ${}^*d$ are the images under the fixed extension *.

The nonstandard extension of the distance is a function

$${}^*d : {}^*X \times {}^*X \longrightarrow {}^*\mathbb{R}_{\geq 0}, \qquad (x, y) \longmapsto {}^*d(x, y),$$

where ${}^*\mathbb{R}_{\geq 0} = \{r \in {}^*\mathbb{R} : r \geq 0\}$. In particular, if $x, y \in X$ are standard, then ${}^*d(x, y) = d(x, y)$.

Proposition 2.1.15 (Metric properties by Transfer)**.** *For all $x, y, z \in {}^*X$ the following hold:*

*1) (*Nonnegativity*)* ${}^*d(x, y) \geq 0$.

2) (Identity of indiscernibles) ${}^*d(x,y) = 0 \iff x = y$.

3) (Symmetry) ${}^*d(x,y) = {}^*d(y,x)$.

4) (Triangle inequality) ${}^*d(x,z) \le {}^*d(x,y) + {}^*d(y,z)$.

Proof. Each property is a first-order sentence of the language L_X valid for (X,d). By the Transfer Principle (Theorem 2.1.7), the same sentences hold for $({}^*X, {}^*d)$. □

Define *infinitesimal proximity* on ${}^*X$ by

$$x \sim y \quad \iff \quad {}^*d(x,y) \text{ is infinitesimal.}$$

For $x \in X$ (standard), the *monad* of x is $\mu(x) = \{y \in {}^*X : \; y \sim x\}$. The set of *near-standard* points is

$$\mathrm{ns}(X) = \bigcup_{x\in X} \mu(x) \;\subseteq\; {}^*X,$$

and the set of points at *finite distance* is

$$\mathrm{fin}({}^*X) = \{y \in {}^*X : \; {}^*d(y,p) \text{ is finite for some (equivalently, for every) } p \in X\}.$$

One has $\mathrm{ns}(X) \subseteq \mathrm{fin}({}^*X)$.

Proposition 2.1.16 (Internal balls and monads)**.** *For standard $\varepsilon > 0$ and $x \in X$, the ball ${}^*B(x,\varepsilon) = \{y \in {}^*X : \; {}^*d(x,y) < \varepsilon\}$ is an* internal *subset of ${}^*X$ and satisfies*

$$\bigcap_{\varepsilon>0} {}^*B(x,\varepsilon) = \mu(x).$$

Standard part on $\mathrm{ns}(X)$. If $y \in \mathrm{ns}(X)$, there is a unique $x \in X$ with $y \sim x$; define the *standard part*

$$\mathrm{st} : \mathrm{ns}(X) \longrightarrow X, \qquad \mathrm{st}(y) = x \;\text{ such that }\; y \sim x.$$

For $y, z \in \mathrm{ns}(X)$,

$$d(\mathrm{st}(y), \mathrm{st}(z)) = \mathrm{st}\left({}^*d(y,z)\right).$$

Theorem 2.1.17 (Robinson's compactness criterion)**.** *Let (X,d) be a metric space. Then X is compact if and only if for every $y \in {}^*X$ there exists $x \in X$ such that $x \sim y$.*

Proof. See [28], p. 86. □

Proposition 2.1.18. *Let (X,d) be a metric space and $f : X \to X$ a map. If $x_0 \in X$, then f is continuous at x_0 if and only if for every $x \in {}^*X$ with $x \sim x_0$ one has ${}^*f(x) \sim f(x_0)$.*

Proof. See Theorem 1.9.2 in [28]. □

Nonstandard compactification

Definition 2.1.19 (Nonstandard compactification)**.** Let (X, τ) be a topological space and ${}^*X$ its nonstandard extension constructed over a saturated universe.

The *S-topology* on ${}^*X$ is generated by the base of opens $\{\, {}^*O : O \in \tau \,\}$. A set $U \subseteq {}^*X$ is open in this topology if for each $x' \in U$ there exists $O \in \tau$ such that $x' \in {}^*O \subseteq U$.

Define an equivalence relation $x' \sim_S y'$ on ${}^*X$ by

$$x' \sim_S y' \iff \forall\, O \in \tau, \quad \big(x' \in {}^*O \iff y' \in {}^*O\big).$$

The *nonstandard compactification* of X, denoted SX, is the quotient ${}^*X/\sim_S$ endowed with the quotient topology induced by the canonical projection $\pi : {}^*X \to SX$.

Remark 2.1.20. The nonstandard compactification SX is related to the Stone–Čech compactification βX. If X is **Tychonoff** (completely regular and Hausdorff), then $SX \cong \beta X$. This result is contained in [32, Proposition 2.1]; for an explicit statement see Proposition 4.4 in [33].

Proposition 2.1.21 (Fundamental properties)**.** *Let SX be the nonstandard compactification of X. Then:*

1) *The canonical map $i_X : X \to SX$, $x \mapsto [x]$, is a topological embedding with dense image.*

2) *SX is compact.*

3) *SX is Hausdorff if and only if X is Hausdorff.*

Proof. (1) **Dense embedding:** The map i_X is an embedding because the preimage of a basic open $\pi({}^*O)$ is exactly O. To see that the image is dense, let $[x'] \in SX$ and U be a neighborhood of it. By definition, there exists $O \subseteq X$ open with $[x'] \in \pi({}^*O) \subseteq U$. Since $x' \in {}^*O$, by Transfer the set O is nonempty. Taking $p \in O$, we have $i_X(p) = [p] \in \pi({}^*O)$, so the image of X meets U.

(2) **Compactness:** Compactness of SX follows from the fact that ${}^*X$ endowed with the S-topology is compact. This property (sometimes called *S-compactness*) is proved using saturation. As the canonical projection $\pi : {}^*X \to SX$ is continuous and surjective, and $({}^*X, S\text{-topology})$ is compact, its image SX is compact as well. For details, see, e.g., [28, Theorem 4.1.13].

(3) **Hausdorff:** If SX is Hausdorff, then X (viewed as a subspace) is Hausdorff. Conversely, if X is Hausdorff and $[x'] \neq [y']$ in SX, by definition of the equivalence there exists an open O_x with $x' \in {}^*O_x$ but $y' \notin {}^*O_x$ (or vice versa). Using standard separation in X, one can find disjoint opens

$O_1, O_2 \subseteq X$ such that $x' \in {}^*O_1$ and $y' \in {}^*O_2$. Then $\pi({}^*O_1)$ and $\pi({}^*O_2)$ are disjoint neighborhoods of $[x']$ and $[y']$ in SX. □

2.2 Set Theory and General Topology

In this thesis we will work within the framework of the classical set theory ZFC.

2.2.1 Ordinals and Cardinals

Ordinals

Definition 2.2.1. A set α is an *ordinal* if it is transitive (that is, $x \in y \in \alpha \Rightarrow x \in \alpha$) and well-ordered by $\in$. We define $0 = \varnothing$ and the successor $\alpha + 1 = \alpha \cup \{\alpha\}$. We say that α is a *limit* if it is not a successor. Equivalently, α is a limit if and only if

$$\alpha = \bigcup \alpha = \bigcup_{\beta<\alpha} \beta,$$

that is, α is the union of all its predecessors.

Definition 2.2.2 (Order topology on an ordinal)**.** Let α be an ordinal with its natural order $<$ (defined by $\beta < \gamma \iff \beta \in \gamma$). The *order topology* on α is generated by the base

$$\mathcal{B} = \{\, (\beta, \gamma) : \beta < \gamma < \alpha \,\} \cup \{\, [0, \gamma) : 0 < \gamma \leq \alpha \,\} \cup \{\, (\beta, \alpha) : \beta < \alpha \,\},$$

where

$$(\beta, \gamma) = \{x \in \alpha : \ \beta < x < \gamma\}, \quad [0, \gamma) = \{x \in \alpha : \ x < \gamma\}, \quad (\beta, \alpha) = \{x \in \alpha : \ \beta < x\}.$$

Remark 2.2.3. It is easy to verify that an ordinal α, with the order topology, is compact if and only if α is the zero ordinal or a successor ordinal. This is because compactness of an ordinal is equivalent to the existence of a maximum element.

Theorem 2.2.4 (Cantor normal form)**.** *Every ordinal $\alpha > 0$ can be written uniquely as*

$$\alpha \;=\; \omega^{\beta_1} \cdot n_1 \;+\; \omega^{\beta_2} \cdot n_2 \;+\; \cdots \;+\; \omega^{\beta_k} \cdot n_k,$$

where $k \in \mathbb{N}$, $\beta_1 > \beta_2 > \cdots > \beta_k$ are ordinals, and $n_i \in \mathbb{N} \setminus \{0\}$.

Proof. See Theorem 2.26 [23]. □

Cardinals and cardinal arithmetic

Definition 2.2.5 (Cardinal)**.** A *cardinal* is an initial ordinal, that is, an ordinal κ that is not equipotent to any smaller ordinal: $\forall \alpha < \kappa$, $|\alpha| \neq |\kappa|$. We identify the *cardinality* of a set X with the unique cardinal $|X|$ equipotent to X.

Definition 2.2.6 (Cardinal sum, product, and exponentiation)**.** Let A, B be sets such that $|A| = \kappa$ and $|B| = \lambda$.

- *Sum:* $\kappa + \lambda := |A \sqcup B|$.
- *Product:* $\kappa \cdot \lambda := |A \times B|$.
- *Exponentiation:* $\kappa^\lambda := |A^B|$, where $A^B := \{f : B \to A\}$ is the set of all functions from B to A.

These definitions do not depend on the choice of A and B.

Proposition 2.2.7 (Basic cardinal arithmetic)**.** *Let κ, λ, μ be infinite cardinals. The following properties hold.*

1) Commutativity and associativity

$$\kappa + \lambda = \lambda + \kappa, \quad (\kappa + \lambda) + \mu = \kappa + (\lambda + \mu),$$

$$\kappa \cdot \lambda = \lambda \cdot \kappa, \quad (\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu).$$

2) Distributivity

$$\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu.$$

3) Monotonicity *If $\kappa \leq \kappa'$ and $\lambda \leq \lambda'$, then*

$$\kappa + \lambda \leq \kappa' + \lambda', \qquad \kappa \cdot \lambda \leq \kappa' \cdot \lambda', \qquad \kappa^\lambda \leq (\kappa')^{\lambda'}.$$

Proposition 2.2.8 (Arithmetic of infinite cardinals)**.** *If $\kappa, \lambda > 0$ are cardinals and at least one is infinite, then*

$$\kappa + \lambda \; = \; \kappa \cdot \lambda \; = \; \max\{\kappa, \lambda\}.$$

In particular, if $0 < \kappa \leq \lambda$ and λ is infinite, then

$$\kappa + \lambda = \lambda, \qquad \kappa \cdot \lambda = \lambda.$$

For the proofs of the preceding propositions, the reader is referred to page 29 of [23].

Definition 2.2.9 (Cofinality)**.** The *cofinality* of a limit ordinal α, denoted $\mathrm{cf}(\alpha)$, is the least ordinal β such that there exists a cofinal function $f : \beta \to \alpha$. For a cardinal κ, $\mathrm{cf}(\kappa)$ is the cofinality of κ viewed as an ordinal.

Definition 2.2.10 (Regular and singular cardinals)**.** An infinite cardinal κ is *regular* if $\mathrm{cf}(\kappa) = \kappa$. It is *singular* if $\mathrm{cf}(\kappa) < \kappa$.

Definition 2.2.11 (Successor cardinal)**.** For a cardinal κ, the *successor cardinal* κ^+ is the least cardinal strictly greater than κ.

Remark 2.2.12. Every successor cardinal is regular. In particular, $\aleph_{\alpha+1}$ is regular for every ordinal α. The cardinal $\aleph_\omega = \sup\{\aleph_n : n < \omega\}$ is singular with $\mathrm{cf}(\aleph_\omega) = \omega$.

Definition 2.2.13 (The map $\alpha \mapsto \aleph_\alpha$)**.** We define by transfinite recursion:

$$\aleph_0 = \omega, \qquad \aleph_{\alpha+1} = (\aleph_\alpha)^+, \qquad \aleph_\delta = \sup_{\beta<\delta} \aleph_\beta \ \text{ if } \delta \text{ is a limit ordinal.}$$

(Generalized) Continuum Hypothesis

Definition 2.2.14 (CH and GCH)**.** The *Continuum Hypothesis* (CH) asserts $2^{\aleph_0} = \aleph_1$.

The *Generalized Continuum Hypothesis* (GCH) asserts that for every infinite cardinal κ,

$$2^\kappa = \kappa^+.$$

2.2.2 Forcing

The consistency results in this work are obtained by forcing. Specifically, we use Easton's theorem to build models of ZFC with a prescribed behavior of the power-set function on regular cardinals.

Easton's theorem

Theorem 2.2.15 (Easton, 1970)**.** *Assume the GCH holds.Let F be a function defined on the regular cardinals such that:*

(E1) If $\kappa \le \lambda$ are regular, then $F(\kappa) \le F(\lambda)$.

(E2) For every regular κ, $\mathrm{cf}(F(\kappa)) > \kappa$.

Then there is a forcing extension that preserves cardinals and cofinalities, in which $2^\kappa = F(\kappa)$ for every regular cardinal κ.

Proof. See Easton [23, Theorem 15.18] and Cummings [12, Lemmas 11.1–11.4, pp. 811–813]. □

Remark 2.2.16 (Standing assumption for the remainder)**.** Throughout this chapter we do *not* assume GCH below a given λ. When needed, we force only *below* λ to make 2^λ as large as required, preserving all cardinals and cofinalities; this suffices for the constructions used later (including the "limits of limits" patterns inside $(\lambda, 2^\lambda)$). For regular λ we use Easton iterations *above* λ as in Subsec. 2.2.2; for singular λ we follow a lightweight route (Remark 2.2.17). If one instead insists on assuming GCH below λ, more sophisticated techniques may be required; exploring that direction lies beyond the scope of this work and is left for future research.

Application to $\mathrm{DGCH}_\rho(\lambda)$

To obtain $\neg\mathrm{DGCH}_\rho(\lambda)$ with λ regular, we write $\lambda = \aleph_\alpha$ and work in two scenarios, depending on the combinatorial requirements of Chapter 5:

(A) Minimal version (sufficient for $\neg\mathrm{DGCH}_2(\lambda)$). We take, for regular $\aleph_\alpha$,

$$F(\aleph_\alpha) = \aleph_{\alpha+\omega+1}.$$

The Easton support iteration (Cummings [12, §11, pp. 811–813]) over of a model of GCH produces:

- An intermediate cardinal
$$\kappa = \aleph_{\alpha+\omega}$$
with $\mathrm{cf}(\kappa) = \omega$ and $\aleph_\alpha < \kappa < 2^{\aleph_\alpha}$.
- A sequence of cardinals
$$\{\aleph_{\alpha+n} : 1 \leq n < \omega\},$$
useful for the basic constructions in Chapter 5.

(B) Version with a cofinal sequence indexed by λ. When the proofs require a cofinal chain of cardinals above a regular $\aleph_\alpha$ and below $2^{\aleph_\alpha}$ *of length* λ, we set $F(\aleph_\alpha) = \aleph_{\alpha+\omega^\omega\cdot\lambda+1}$ and after Easton Forcing over a model of GCH,

$$2^{\aleph_\alpha} = \aleph_{\alpha+\omega^\omega\cdot\lambda+1}, \qquad \kappa := \aleph_{\alpha+\omega^\omega\cdot\lambda}.$$

Then there is a strictly increasing sequence $\{k_i\}_{i<\lambda}$ with

$$\lambda < k_i < \kappa, \quad \sup_{i<\lambda} k_i = \kappa,$$

and for each $i < \aleph_\alpha$ an ω-cofinal sequence of *limit cardinals* $\{k_{ij}\}_{j<\omega}$ above $\aleph_\alpha$ and below k_i with $\sup_{j<\omega} k_{ij} = k_i$. See Lemma 5.2.21 for an explicit realization of these choices under Easton.

Remark 2.2.17 (Lightweight route for singular λ)**.** Let $\lambda = \aleph_\alpha$ be singular with $\mu = \mathrm{cf}(\lambda) = \mathrm{cf}(\alpha)$. Assuming the GCH holds at λ, we can add $\aleph_{\alpha+\Theta+1}$ Cohen subsets of ω, where $\Theta = \mu^\omega.\lambda$, so that, in the forcing extension:

- $2^{\aleph_\alpha} = \aleph_{\alpha+\Theta+1}$ for some fixed limit ordinal Θ (e.g. $\Theta = \mu^\omega \cdot \lambda$);
- with $\kappa := \aleph_{\alpha+\Theta}$ we have $\lambda < \kappa < 2^\lambda$ and $\mathrm{cf}(\kappa) = \omega \leq \lambda$;
- there exists a strictly increasing sequence $\{k_i\}_{i<\lambda} \subset (\lambda, \kappa)$ with $\sup_{i<\lambda} k_i = \kappa$, and, for each i, a sequence $\{k_{ij}\}_{j<\omega} \subset (\lambda, k_i)$ of limit cardinals with $\mathrm{cf}(k_{ij}) = \omega$ and $\sup_{j<\omega} k_{ij} = k_i$.

For concreteness, when we work in the minimal version (A) we use as intermediates

$$\kappa = \aleph_{\alpha+\omega}, \qquad \kappa_n = \aleph_{\alpha+n}\ (1 \leq n < \omega),$$

whereas when a cofinal chain indexed by λ is required (version (B)), we use $\kappa = \aleph_{\alpha+\omega^\omega\cdot\lambda}$ and a family $\{k_i\}_{i<\lambda}$ as above.

2.2.3 Metric Spaces and General Topology

Definition 2.2.18. A topological space X is said to be σ-compact if there exists a sequence of compact subsets $K_1, K_2, \ldots$ such that

$$X = \bigcup_{n=1}^{\infty} K_n.$$

Definition 2.2.19. A topological space X is *locally compact* if, for every $x \in X$, there exists an open neighborhood U of x such that its closure $\overline{U}$ is compact.

Definition 2.2.20. Let X be a topological space. For every ordinal α we define $X^{(\alpha)}$, the *Cantor–Bendixson* derivative of X of order α, by transfinite induction.
$X^{(0)} = X$.
If $\alpha = \beta + 1$, $X^{(\beta+1)} = (X^{(\beta)})'$, the set of accumulation points of $X^{(\beta)}$.
If α is a limit ordinal, $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

Definition 2.2.21. If there exists an ordinal α such that $X^{(\alpha)} = \{x_1, \ldots, x_n\}$, we say that (α, n) is the *characteristic* of X, and α is the *derived degree* of X ($\deg(X)$).

Definition 2.2.22. Let X be a topological space. We say that it is scattered if every nonempty subspace has an isolated point.

Remark 2.2.23. Every ordinal space is scattered because, for every nonempty subspace, its minimal element is an isolated point.

Theorem 2.2.24 (Semadeni [34])**.** *Every compact scattered first-countable topological space with characteristic (α, n) is metrizable and homeomorphic to an ordinal of the form $\omega^\alpha n + 1$ with the order topology.*

Proposition 2.2.25. *Let (X, d) be a metric space and let $h : Y \to X$ be a bijection. Then (Y, d_h) is a metric space, where $d_h : Y \times Y \to \mathbb{R}$ is defined by $d_h(y, y') := d(h(y), h(y'))$.*

Proof. This is a straightforward verification. □

Proposition 2.2.26. *Let (X, d) be a metric space and let $h : Y \to X$ be a bijection such that h^{-1} is continuous. Then $d_h : Y \times Y \to \mathbb{R}$ is a metric compatible with the topology of Y.*

Proof. Since

$$B_{d_h}(y, r) = h^{-1}(B_d(h(y), r)),$$

the balls in the metric d_h are open sets in Y. Let U be an open set in Y, and let $y \in U$. By continuity of h^{-1}, there exists $\epsilon > 0$ such that $h^{-1}(B_d(h(y), \epsilon)) \subset U$. Since

$$h^{-1}(B_d(h(y), \epsilon)) = B_{d_h}(y, \epsilon),$$

it follows that d_h induces the topology of Y. □

Proposition 2.2.27 (*Conjugation of an action*)**.** *Let $\varphi : G \times X \to X$ be an action and let $h : Y \to X$ be a bijection. The map h induces an action $\varphi_h : G \times Y \to Y$ defined by*

$$\varphi_h(g, y) := h^{-1}(\varphi(g, h(y))),$$

which we call the conjugation of φ by h.

Proof. We verify the action axioms. Let $e \in G$ be the identity of G. Then

$$\varphi_h(e, y) = h^{-1}(\varphi(e, h(y))) = h^{-1}(h(y)) = y.$$

For $g, g' \in G$ we have

$$\varphi_h(gg', y) = h^{-1}(\varphi(gg', h(y))) = h^{-1}(\varphi(g, \varphi(g', h(y)))).$$

Since

$$\varphi(g, \varphi(g', h(y))) = \varphi\big(g, h(h^{-1}(\varphi(g', h(y))))\big),$$

it follows that

$$h^{-1}\big(\varphi(g, h(\varphi_h(g', y)))\big) = \varphi_h(g, \varphi_h(g', y)).$$

Thus, φ_h satisfies the second axiom. □

Remark 2.2.28. If X and Y are topological spaces and $h : Y \to X$ is a homeomorphism, then φ is continuous if and only if φ_h is continuous.

2.3 Graph Theory

2.3.1 Graphs, Orbits, and Direct Limits

In this section we fix notation and elementary definitions from graph theory and actions by automorphisms that are used in Chapter 5.

Simple graphs and degrees

A *graph* Y is a pair $(V(Y), E(Y))$ where $V(Y)$ is the vertex set and $E(Y) \subseteq [V(Y)]^2$ is the set of edges. We work with simple, undirected, loopless graphs. For $x \in V(Y)$ we define the neighborhood

$$N_Y(x) := \{y \in V(Y) : \{x, y\} \in E(Y)\},$$

and the degree $\deg_Y(x) := |N_Y(x)|$. We say that Y is d-regular if $\deg_Y(x) = d$ for all $x \in V(Y)$. We write $V_d(Y) := \{x \in V(Y) : \deg_Y(x) = d\}$.

Remark 2.3.1. Every automorphism of Y preserves degrees; in particular, each orbit (see §2.3.1) is contained in some class $V_d(Y)$.

Bipartite product and the operator $D(\cdot)$

Let Y be a graph and $A \subseteq V(Y)$. Set $A^c := V(Y) \setminus A$. The bipartite product is expressed via the macro defined in the preamble:

$$A \circledast B \ := \ \big\{\{x, y\} \in [A \cup B]^2 : x \in A,\ y \in B\big\}.$$

Define

$$D(A) \ := \ A \circledast A^c.$$

Automorphisms and orbits

Let $\mathrm{Aut}(Y)$ denote the automorphism group of Y. The natural action

$$\mathrm{Aut}(Y) \curvearrowright V(Y), \qquad g \cdot x := g(x),$$

generates, for each $x \in V(Y)$, the orbit

$$O_Y(x) := \{g \cdot x : g \in \mathrm{Aut}(Y)\}.$$

The set of orbits is written

$$O(\mathrm{Aut}(Y) \curvearrowright Y) \ := \ \{O_Y(x) : x \in V(Y)\}.$$

When indices are needed, we write $\{O_j(Y)\}_{j<J}$ with $V(Y) = \bigsqcup_{j<J} O_j(Y)$. We define the orbit complexity by $\rho(Y) := |O(\mathrm{Aut}(Y) \curvearrowright Y)|$.

Remark 2.3.2. If $\mathrm{Aut}(Y)$ acts transitively on $V(Y)$, then $\rho(Y) = 1$.

Direct systems and direct limit of graphs

Let $(I, \leq)$ be a directed set. A direct system of graphs is a family

$$\{(Y_n,\ f_{n,m} : Y_n \to Y_m)\}_{n \leq m \in I},$$

where $f_{n,n} = \mathrm{id}$ and $f_{n,k} = f_{m,k} \circ f_{n,m}$ for all $n \leq m \leq k$. The direct limit is defined by

$$\varinjlim_{n \in I} Y_n \ := \ \bigsqcup_{n \in I} V(Y_n) \ / \sim,$$

with $x_n \sim x_m$ if there exists $k \in I$ with $f_{n,k}(x_n) = f_{m,k}(x_m)$. Adjacency in the colimit is induced as follows: $[x_n]$ and $[y_m]$ are joined by an edge if there exist representatives and $k \in I$ such that $\{f_{n,k}(x_n), f_{m,k}(y_m)\} \in E(Y_k)$.

Chapter 3

Expansive Dynamics and Nonstandard Analysis

In this chapter we apply techniques from nonstandard analysis to study expansive dynamical systems. the principal results of this chapter are a necessary and sufficient condition for an expansive homeomorphism on a compact metric space to admit doubly asymptotic points, expressed via the decay of expansivity constants of the iterates.

Definition 3.0.1 (Asymptotic and doubly asymptotic points)**.** Let (X, d) be a metric space and let $f\colon X \to X$ be a homeomorphism.

- An ordered pair $(x, y) \in X \times X$ with $x \neq y$ is *forward asymptotic* (asymptotic in the future) if
$$\lim_{n\to+\infty} d\big(f^n(x), f^n(y)\big) = 0.$$
- An ordered pair $(x, y) \in X \times X$ with $x \neq y$ is *backward asymptotic* (asymptotic in the past) if
$$\lim_{n\to+\infty} d\big(f^{-n}(x), f^{-n}(y)\big) = 0.$$
- An ordered pair $(x, y) \in X \times X$ with $x \neq y$ is *doubly asymptotic* if it is both forward and backward asymptotic, that is,
$$\lim_{n\to+\infty} d\big(f^n(x), f^n(y)\big) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d\big(f^{-n}(x), f^{-n}(y)\big) = 0.$$

Remark 3.0.2. In this text we will often refer to such ordered pairs simply as *doubly asymptotic points.*

Definition 3.0.3. Let (X, d) be a metric space and let $f\colon X \to X$ be a homeomorphism. We say that f is *expansive* if there exists a constant $c > 0$, called an *expansivity constant*, such that for every pair of distinct points $x, y \in X$ there exists an integer $n \in \mathbb{Z}$ with

$$d\big(f^n(x),\, f^n(y)\big) > c.$$

We prove that every expansive homeomorphism on a countably infinite compact metric space admits doubly asymptotic points.

Theorem 3.0.4. *An expansive homeomorphism $f\colon X \to X$ admits doubly-asymptotic points if and only if for a self-similar metric d with expanding factor λ, there exists a standard real number $C > 0$ and an infinite natural number N such that $\gamma({}^*f^N) < \frac{C}{\lambda^{N/2}}$.*

and we will provide a necessary and sufficient condition for an expansive homeomorphism on a compact metric space to admit doubly-asymptotic points.

Theorem 3.0.5. *Let $f\colon X \to X$ be an expansive homeomorphism. If X is countable infinite then (X, f) has doubly-asymptotic points.*

3.1 Applications of Nonstandard Analysis

In this section, we begin applying nonstandard analysis to the study of expansivity. The content is essentially the same as that of Chapter 4 of our Master's thesis [16], and it is included here to facilitate the reader's understanding of the new results that will be introduced in the following sections. What is novel in this chapter is an alternative proof of a classical result by Utz concerning the existence of asymptotic pairs for an expansive homeomorphism on a compact metric space. The graphical intuition provided by the nonstandard analysis framework plays a crucial role in the proof we present of Theorem 3.3.7.

First we show a nonstandard characterization of asymptotic points for a continuous map $f\colon X \to X$. Its proof is similar to [24, Theorem 3.1].

Proposition 3.1.1. *Two points $x, y \in X$ are asymptotic if and only if for every infinitely positive integer $m \in {}^*\mathbb{Z}$ we have ${}^*f^m(x) \sim {}^*f^m(y)$.*

Proof. To prove the direct part suppose that $\lim\limits_{n\to+\infty} d(f^n(x), f^n(y)) = 0$. Then, for any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{Z}^+$ such that the following formula holds:

$$(\forall m \in \mathbb{Z}^+)(m \geq n_\varepsilon \to d(f^m(x), f^m(y)) < \varepsilon).$$

By the Transfer Principle we have that the following formula is also true:

$$(\forall m \in {}^*\mathbb{Z}^+)(m \geq n_\varepsilon \to {}^*d({}^*f^m(x), {}^*f^m(y)) < \varepsilon).$$

If m is an infinitely positive integer, then $m > n_\varepsilon$ for any ε, thus ${}^*d({}^*f^m(x), {}^*f^m(y)) < \varepsilon$ holds for any ε, implying ${}^*f^m(x) \sim {}^*f^m(y)$.

To prove the converse suppose $\lim_{n\to+\infty} d(f^n(x), f^n(y)) \neq 0$. Then, there exists $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, the following holds:

$$(\exists m \in \mathbb{Z}^+)((m \geq n) \wedge d(f^m(x), f^m(y)) > \varepsilon).$$

Then, taking a function $\psi : \mathbb{Z}^+ \to \mathbb{Z}^+$ we have

$$(\forall n \in \mathbb{Z}^+)(\psi(n) > n) \wedge (d(f^{\psi(n)}(x), f^{\psi(n)}(y)) > \varepsilon).$$

By the Transfer Principle we obtain:

$$(\forall n \in {}^*\mathbb{Z}^+)({}^*\psi(n) > n) \wedge ({}^*d({}^*f^{{}^*\psi(n)}(x), {}^*f^{{}^*\psi(n)}(y)) > \varepsilon).$$

If m is an infinitely positive integer then ${}^*\psi(m)$ is also an infinitely positive integer. Since $d(f^{\psi(n)}(x), f^{\psi(n)}(y)) > \varepsilon$ it follows that $f^m(x) \not\sim f^m(y)$. □

The next result characterizes asymptoticity for expansive homeomorphisms and is well known. We give a nonstandard proof.

Lemma 3.1.2. *Suppose that f is an expansive homeomorphism. If $x, y \in X$ and there exists an expansivity constant δ such that for every $n \in \mathbb{Z}^+$, $d(f^n(x), f^n(y)) \leq \delta$, then x and y are asymptotic.*

Proof. Applying the Transfer Principle to the formula $(\forall n \in \mathbb{Z}^+)(d(f^n(x), f^n(y)) \leq \delta)$ we have

$$(\forall n \in {}^*\mathbb{Z}^+)({}^*d({}^*f^n(x), {}^*f^n(y)) \leq \delta).$$

Arguing by contradiction and applying Proposition 3.1.1 we can take an infinitely positive integer N such that ${}^*f^N(x) \not\sim {}^*f^N(y)$. Therefore, there exists a positive standard real number r such that $r < {}^*d({}^*f^N(x), {}^*f^N(y))$. By the compactness criterion of Robinson, there exist $x', y' \in X$ such that ${}^*f^N(x) \sim x'$ and ${}^*f^N(y) \sim y'$. By continuity, for every $n \in \mathbb{Z}$, we have ${}^*f^{N+n}(x) \sim f^n(x')$ and ${}^*f^{N+n}(y) \sim f^n(y')$. Therefore, $r < d(f^n(x), f^n(y)) \leq \delta$, which contradicts the expansivity hypothesis. □

Suppose that $x, y \in {}^*X$, $x \neq y$ and $x \sim y$. By continuity, for any standard integer $n \in \mathbb{Z}$ we have ${}^*f^n(x) \sim {}^*f^n(y)$. That is, ${}^*d({}^*f^n(x), {}^*f^n(y))$ is infinitesimal for all $n \in \mathbb{Z}$, even if f is expansive. The following result shows that the expansiveness of the dynamics (X, f) essentially extends to the dynamics $({}^*X, {}^*f)$.

Remark 3.1.3. Assuming expansiveness, for any two distinct points $x, y \in X$, there exists a time, minimal in absolute value, at which they are separated by a distance greater than the expansiveness constant. Consider the formula

$$\varphi(x,y) := \big(x \neq y \to \big(\exists n_{\min} \in \mathbb{Z}\ \big(d(f^{n_{\min}}(x), f^{n_{\min}}(y)) > c\big) \\ \wedge\ (\forall n \in \mathbb{Z},\ d(f^n(x), f^n(y)) > c \to |n| \geq |n_{\min}|)\big)\big).$$

By the Transfer Principle, the formula

$$(\forall x, y \in {}^*X)\ {}^*\varphi(x,y)$$

also holds true. Therefore, the same result applies to any two points in ${}^*X$ with respect to the dynamics $({}^*X, {}^*f)$, using the same expansiveness constant as the dynamics (X, f). In particular, if (X, f) is an expansive dynamical system with expansiveness constant $c > 0$, then $({}^*X, {}^*f)$ is also "expansive" in the sense that, for any distinct points $x, y \in {}^*X$, there exists $n \in {}^*\mathbb{Z}$ such that ${}^*d({}^*f^n(x), {}^*f^n(y)) > c$. Note that this implies that if $x \sim y$, then there exists an infinite $n \in {}^*\mathbb{Z}$ such that ${}^*d({}^*f^n(x), {}^*f^n(y)) > c$.

We present below a nonstandard proof of a result due to Utz. The graphical intuition shown in Figure 6.1 is the key.

Theorem 3.1.4 (Theorem 2.1 in [36])**.** *If f is expansive and X has infinitely many points then there exist different asymptotic points $x, y \in X$ for f or f^{-1}.*

Proof. Since X is infinite and compact there exists an accumulation point, which implies the existence of $x \in X$ and $y \in {}^*X$ such that $x \sim y$. From Remark 3.1.3 there exists $m \in {}^*\mathbb{Z}$, the minimum in absolute value, such that ${}^*d({}^*f^m(x), {}^*f^m(y)) > c$. Assuming that m is positive we will show that there are asymptotic points for f^{-1} (for m negative the same argument gives asymptotic points for f). By continuity, for every $n \in \mathbb{Z}$ we have ${}^*f^n(x) \sim {}^*f^n(y)$, thus m is infinite. Since X is compact, by Theorem 2.1.17 there exist $x', y' \in X$ such that $x' \sim {}^*f^m(x)$ and $y' \sim {}^*f^m(y)$. See Figure 6.1.

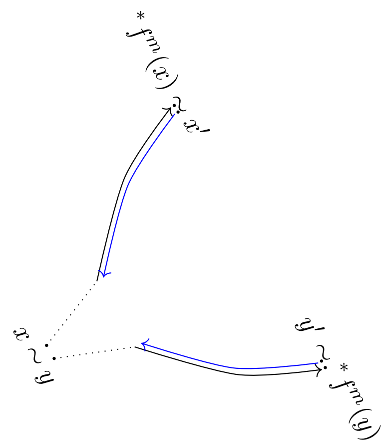


Figure 3.1: Construction of the asymptotic points x' and y'.

By continuity, for every $n \in \mathbb{Z}^+$, we have $^*f^{m-n}(x) \sim f^{-n}(x')$ and $^*f^{m-n}(y) \sim f^{-n}(y')$, but

$$^*d(^*f^{m-n}(x), ^*f^{m-n}(y)) \leq c$$

for every $n \in \mathbb{Z}^+$, therefore $d(f^{-n}(x'), f^{-n}(y')) \leq c$ for every $n \in \mathbb{Z}^+$. Hence, by Lemma 3.1.2, we can conclude that x' and y' are asymptotic for f^{-1}. $\square$

3.2 Expansivity on Countable Metric Spaces

In this section we consider expansivity on a compact and countable (infinite) metric space X. For this kind of space we have some particular tools to use. For every ordinal α, we define $X^{(\alpha)}$ as the *Cantor–Bendixson derivative* of X by transfinite induction:

- $X^{(0)} = X$,
- if $\alpha = \beta + 1$, then $X^{(\beta+1)} = (X^{(\beta)})'$, the subset of accumulation points of $X^{(\beta)}$,
- if α is an infinite limit ordinal, then $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

If there exists an ordinal α such that $X^{(\alpha)}$ is finite, we define the *derived degree* of X as the minimal such ordinal, denoted $\deg(X) = \alpha$. In [26] it is shown that a countable and compact metric space admits an expansive homeomorphism if and only if $\deg(X)$ is not an infinite limit ordinal.

Proof of Theorem 3.0.5. By [26, Theorem 2.2] we know that $\deg(X) = \beta + 1$, so $X^{(\beta+1)} = \{x_1, \ldots, x_n\}$. By transfinite induction, it is easy to prove that for every ordinal α, $f(X^{(\alpha)}) = X^{(\alpha)}$.

This implies that the points $\{x_1, \ldots, x_n\}$ are periodic. Notice that each of these points is fixed by $f^{n!}$. Therefore, we can restrict f to $X^{(\beta)}$, and $f|_{X^{(\beta)}} \colon X^{(\beta)} \to X^{(\beta)}$ is an expansive homeomorphism. As $X^{(\beta)}$ has infinitely many points and is compact, we know from Theorem 3.1.4 that there are asymptotic points in $X^{(\beta)}$. Thus, not all points in $X^{(\beta)}$ are periodic. Therefore, there exists $x \in X^{(\beta)} \setminus X^{(\beta+1)}$ such that $\alpha(x), \omega(x) \subset X^{(\beta+1)}$. Then, for every positive infinite integer N and for every negative infinite integer M, there exist x_i and x_j in $X^{(\beta+1)}$ such that ${}^*f^N(x) \sim x_i$ and ${}^*f^M(x) \sim x_j$.

Let $y = f^{n!}(x)$. We will prove that x and y are asymptotic pairs. Due to the continuity of f, if ${}^*f^N(x) \sim x_i$, then $f^{n!}({}^*f^N(x)) \sim f^{n!}(x_i)$. Therefore, ${}^*f^N(f^{n!}(x)) \sim x_i$, which implies ${}^*f^N(y) \sim x_i$. Hence, ${}^*f^N(x) \sim {}^*f^N(y)$. Thus, by Proposition 3.1.1, x and y are asymptotic. Similarly, ${}^*f^M(x) \sim {}^*f^M(y)$, and x, y are asymptotic for f^{-1}. □

3.3 Doubly Asymptotic Points and the Decay of Expansivity Constants

In this section we will prove Theorem 3.0.4 . he proof is divided into two theorems. Suppose that f is expansive, and recall

$$\gamma(f) = \sup\{c \in \mathbb{R}^+ : c \text{ is an expansivity constant of } f\}.$$

Remark 3.3.1. If f is an expansive homeomorphism with expansivity constant c, we know that $({}^*X, {}^*f)$ is an expansive dynamical system with the same expansivity constant c in the sense explained in Remark 3.1.3. For more generality, we can regard ${}^*f$ as an expansive dynamical system as in the following definition.

Definition 3.3.2. Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism. We say that ${}^*f : {}^*X \to {}^*X$ is *expansive* if there exists $c \in {}^*\mathbb{R}^+$ such that for all distinct $x, y \in {}^*X$, there exists $n \in {}^*\mathbb{Z}$ such that

$${}^*d(f^n(x), f^n(y)) > c.$$

In the following remark we explain the meaning of $\gamma({}^*f)$ and $\gamma({}^*f^N)$ for all $N \in {}^*\mathbb{Z}$.

Remark 3.3.3. It follows directly from the Transfer Principle that if f is expansive, then ${}^*f$ is also expansive. Thus we can define

$$\mathrm{SupExp}(f,s) := \left(\underbrace{(\forall c \in \mathbb{R}^+)(\mathrm{Exp}(f,c) \to c \leq s)}_{\text{upper bound}}\right)$$
$$\wedge \left(\underbrace{(\forall s' \in \mathbb{R})((\forall c \in \mathbb{R}^+)(\mathrm{Exp}(f,c) \to c \leq s') \to s \leq s')}_{\text{least upper bound}}\right).$$

Therefore, $\gamma(f) = s$ if and only if $\mathrm{SupExp}(f,s)$ holds. By applying the Transfer Principle, we obtain

$$\mathrm{SupExp}(^*f,s) := \left(\underbrace{(\forall c \in {}^*\mathbb{R}^+)(\mathrm{Exp}(^*f,c) \to c \leq s)}_{\text{upper bound}}\right)$$
$$\wedge \left(\underbrace{(\forall s' \in {}^*\mathbb{R})((\forall c \in {}^*\mathbb{R}^+)(\mathrm{Exp}(^*f,c) \to c \leq s') \to s \leq s')}_{\text{least upper bound}}\right).$$

Hence, $\mathrm{SupExp}(^*f,s)$ holds for $\gamma(^*f) = s$. In particular we see that $\gamma(^*f) = \gamma(f)$.

If f is expansive then f^n is also expansive for all $n \in \mathbb{Z}$, $n \neq 0$. Then, arguing as before, $\gamma(^*f^N)$ exists for all $N \in {}^*\mathbb{Z}$.

Lemma 3.3.4. *Let (X,d) be a compact metric space where d is a hyperbolic metric with expanding factor λ, and $f\colon X \to X$ an expansive homeomorphism with expansivity constant c. Assume that for some $x, y \in X$, $d(f^n(x), f^n(y)) \leq c$ for all $n \geq 0$ (or for all $n \leq 0$). Then*

$$d(f^n(x), f^n(y)) \leq \frac{d(x,y)}{\lambda^{|n|}}.$$

for all $n \geq 0$ (or for all $n \leq 0$).

Proof. We will prove the case $n \geq 0$; for $n \leq 0$ it is analogous. We see that for all $n \geq 0$,

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \frac{d(f^n(x), f^n(y))}{\lambda}.$$

Suppose not, that is, there exists n such that $\lambda d(f^{n+1}(x), f^{n+1}(y)) > d(f^n(x), f^n(y))$. Since

$$\max\{d(f^n(x), f^n(y)), d(f^{n+2}(x), f^{n+2}(y))\} \geq \lambda d(f^{n+1}(x), f^{n+1}(y)),$$

then $d(f^{n+2}(x), f^{n+2}(y)) \geq \lambda d(f^{n+1}(x), f^{n+1}(y))$. By induction, we deduce that for all $j \geq 0$,

$$d(f^{n+j}(x), f^{n+j}(y)) \geq \lambda^j d(f^n(x), f^n(y)).$$

This contradicts the fact that $d(f^n(x), f^n(y)) \leq c$ for all $n \geq 0$. □

Theorem 3.3.5. *Let $f\colon X \to X$ be an expansive homeomorphism with $x \neq y$ being doubly asymptotic. If d is a hyperbolic metric with an expansion factor λ, then there exists $C > 0$ (a standard real) such that for every infinite natural number N it holds that*

$$\gamma({}^*f^N) < \frac{C}{\lambda^{N/2}}.$$

Proof. As x and y are doubly asymptotic, there exists a standard natural number $l > 0$ such that for all $k \in \mathbb{Z}$, with $|k| \geq l$, the following holds:

$$d(f^k(x), f^k(y)) \leq c,$$

where c is the expansivity constant. Therefore, for $k > l$

$$d(f^k(x), f^k(y)) = d(f^{k-l}(f^l(x)), f^{k-l}(f^l(y))) \leq c.$$

By Lemma 3.3.4, we have:

$$d(f^{k-l}(f^l(x)), f^{k-l}(f^l(y))) \leq \frac{d(f^l(x), f^l(y))}{\lambda^{k-l}} \leq \frac{c}{\lambda^{k-l}}.$$

Similarly, for $k < -l$, we get:

$$d(f^k(x), f^k(y)) = d(f^{k+l}(f^{-l}(x)), f^{k+l}(f^{-l}(y))) \leq \frac{c}{\lambda^{-k-l}}.$$

Thus, for $|k| > l$, we obtain:

$$d(f^k(x), f^k(y)) \leq \frac{c\lambda^l}{\lambda^{|k|}}.$$

Then, for every infinite integer k, the following holds:

$${}^*d({}^*f^k(x), {}^*f^k(y)) \leq \frac{c\lambda^l}{\lambda^{|k|}}.$$

Let N be an infinite natural number. We have $|Ni + (N-r)/2| \geq (N-r)/2$ for all $i \in {}^*\mathbb{Z}$ and for $r = 0$ and $r = 1$. If N is even, we set $r = 0$, and if N is odd, we set $r = 1$. In either case, we have:

$${}^*d({}^*f^{Ni}(f^{(N-r)/2}(x)), {}^*f^{Ni}(f^{(N-r)/2}(y))) \leq \frac{c\lambda^l}{\lambda^{|Ni+(N-r)/2|}} \leq \frac{c\lambda^{l+r/2}}{\lambda^{N/2}}$$

for all $i \in {}^*\mathbb{Z}$. Thus, $\gamma({}^*f^N) < \frac{C}{\lambda^{N/2}}$, where $C = c\lambda^l$ or $C = c\lambda^{l+1/2}$. □

In standard terms this result has the following meaning.

Corollary 3.3.6. *Let $f\colon X \to X$ be an expansive homeomorphism with $x \neq y$ being doubly asymptotic. If d is a hyperbolic metric with an expansion factor λ, then there exists $C > 0$ such that*

$$\gamma(f^n) < \frac{C}{\lambda^{n/2}} \tag{3.1}$$

for all $n \geq 1$.

Proof. From Theorem 3.3.5 we have that there is $k \geq 1$ such that (3.1) is true for all $n \geq k$. Thus, changing the constant C if needed, we have that (3.1) is true for all $n \geq 1$. □

Theorem 3.3.7. *Let $f\colon X \to X$ be an expansive homeomorphism and d a bi-Lipschitz metric for f with Lipschitz constant λ. If there exist a standard real $C > 0$ and an infinite natural number N such that*

$$\gamma({}^*f^N) < \frac{C}{\lambda^{N/2}},$$

then f has doubly-asymptotic points.

Proof. We choose $x \neq y \in {}^*X$ such that

$${}^*d({}^*f^{Nk}(x), {}^*f^{Nk}(y)) < \frac{C}{\lambda^{N/2}} \tag{3.2}$$

for all $k \in {}^*\mathbb{Z}$.

We define

$$m = \min\big\{\, |n| \in {}^*\mathbb{Z}^+ : {}^*d({}^*f^n(x), {}^*f^n(y)) > c/2 \big\},$$

which is the first instant (in absolute value) at which the pair separates by more than $c/2$.

We know that the following formula holds:

$$(\forall m \in {}^*\mathbb{Z})(\forall N \in {}^*\mathbb{Z}^+)(\exists k \in {}^*\mathbb{Z}^+)(N(k-1) < m \leq Nk).$$

Possibly replacing x and y with ${}^*f^{N(k-1)}(x)$ and ${}^*f^{N(k-1)}(y)$ we can assume that $k = 1$ for both m and N, with $0 < m < N$ (notice that m cannot be 0 nor N because (3.2) for $k = 0$ and $k = 1$). Then, we have the configuration given by Figure 3.2.

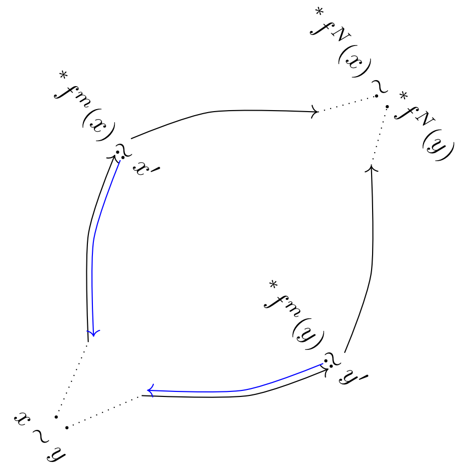


Figure 3.2: Asymptotic points x', y'.

Let us now take

$$m' = \min\left\{n \in {}^*\mathbb{Z}^+ : 0 < n \leq N - m,\ {}^*d\big({}^*f^{\,N-n}(x), {}^*f^{\,N-n}(y)\big) > c/2\right\}.$$

Notice that $N - m' - m \geq 0$. Now let x' and y' in X be such that ${}^*f^m(x) \sim x'$ and ${}^*f^m(y) \sim y'$. Arguing as in the proof of Theorem 3.1.4, we see that x' and y' are asymptotic for f. By the same argument, if x'' and y'' in X satisfy ${}^*f^{N-m'}(x) \sim x''$ and ${}^*f^{N-m'}(y) \sim y''$, they will also be asymptotic. If we can prove that the iterates between ${}^*f^m(x)$ and ${}^*f^{N-m'}(x)$ are finite, i.e., there exists a finite h such that ${}^*f^{m+h}(x) = {}^*f^{N-m'}(x)$, then by continuity we will have ${}^*f^{m+h}(x') \sim {}^*f^{N-m'}(x)$ and ${}^*f^{m+h}(y') \sim {}^*f^{N-m'}(y)$. Therefore, x' and y' will be doubly-asymptotic.

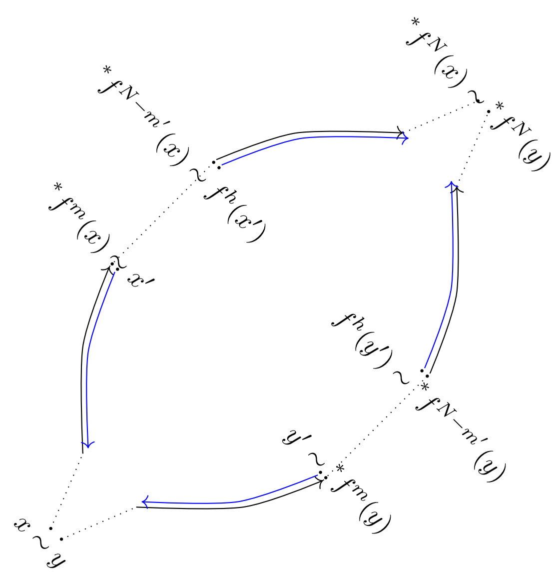


Figure 3.3: Proving doubly-asymptoticity.

Let us show that such finite h exists. Indeed, since f is bi-Lipschitz, the following formula holds

$$(\forall x, y \in X)(\forall n \in \mathbb{Z})(d(f^n(x), f^n(y)) \leq \lambda^n d(x, y))$$

and by the Transfer Principle we conclude

$$(\forall x, y \in {}^*X)(\forall n \in {}^*\mathbb{Z})({}^*d({}^*f^n(x), {}^*f^n(y)) \leq \lambda^n \, {}^*d(x, y)).$$

From the inequality $c/2 < {}^*d({}^*f^m(x), {}^*f^m(y))$ and the bi-Lipschitz character of the metric, it follows that:

$$\frac{c}{2} < {}^*d({}^*f^m(x), {}^*f^m(y)) \leq \lambda^m \, {}^*d(x, y),$$

but, by (3.2) for $k = 0$ we have

$${}^*d(x, y) \leq C\lambda^{-N/2}.$$

Taking logarithms, we obtain

$$\log\left(\frac{c}{2}\right) < \left(m - \tfrac{N}{2}\right) \log \lambda + \log C.$$

Solving for N, we get $N < 2m + 2\dfrac{\log(2C/c)}{\log \lambda}$, where $2\dfrac{\log(2C/c)}{\log \lambda}$ is a *standard* quantity. Arguing as before, for $a = f^N(x)$ and $b = f^N(y)$, in place of x and y, we obtain

$$\begin{cases} \frac{c}{2} < {}^*d({}^*f^{-m'}(a), {}^*f^{-m'}(b)) \leq \lambda^{m'} \, {}^*d(a, b) \text{ and} \\ {}^*d(a, b) \leq C\lambda^{-N/2}. \end{cases}$$

Therefore $N < 2m' + 2\dfrac{\log(2C/c)}{\log \lambda}$ and both inequalities imply $N < m + m' + 2\dfrac{\log(2C/c)}{\log \lambda}$. Now, since $h = N - m' - m$, it follows that $h \leq 2\dfrac{\log(2C/c)}{\log \lambda}$, and therefore h is finite. This completes the proof. □

In standard terms we obtain the following consequence.

Corollary 3.3.8. *Let $f\colon X \to X$ be an expansive homeomorphism and d a bi-Lipschitz metric for f, with Lipschitz constant λ. If there exist $C > 0$ such that*

$$\gamma(f^n) < \frac{C}{\lambda^{n/2}},$$

for infinitely many natural numbers $n \geq 1$ then f has doubly-asymptotic points.

We can combine Theorems 3.3.5 and 3.3.7 to prove the Theorem 3.0.4 stated in §**??**.

Proof of Theorem 3.0.4. If there are doubly-asymptotic points we can take a self-similar metric d, which in particular is hyperbolic with expanding factor λ, and by Theorem 3.3.5, we have

$$\gamma({}^*f^N) < \frac{C}{\lambda^{N/2}}.$$

Conversely, if we have the above inequality for a self-similar metric, which is in particular bi-Lipschitz, then by Theorem 3.3.7 we ensure the existence of doubly-asymptotic points. $\square$

Chapter 4

Expansive Actions and the Generalized Continuum Hypothesis

In Chapter 3, we studied expansive dynamics using tools from nonstandard analysis. In particular, we showed that every countable compact space admits doubly asymptotic points. To this end, we relied on a result from [26]. In this chapter, we show that by interpreting this result as a theorem of topological dynamics on ordinal spaces, we not only encounter new questions, but also uncover a connection between expansivity and the Generalized Continuum Hypothesis.

Definition 4.0.1. We say that $f : X \to X$ is **Cantor-Bendixson stable** (CB-stable) if $f(X^{(\alpha)}) \subset X^{(\alpha)}$ for all ordinals α.

Proposition 4.0.2. *Every injective and continuous function $f : X \to X$ is* CB-stable

Proof. By transfinite induction: If $\alpha = 0$, the result is trivial. For the case where $\alpha = \beta + 1$, for any $x \in X^{(\alpha)} = X^{(\beta+1)} = \left(X^{(\beta)}\right)'$ and any neighborhood V of $f(x)$, we have $f(y) \in V$ for some $y \in X^{(\beta)} \backslash \{x\}$. By the induction hypothesis and the injectivity of f, we obtain $f(y) \in X^{(\beta)} \backslash \{f(x)\}$, thus $f(x) \in \left(X^{(\beta)}\right)' = X^{(\beta+1)} = X^{(\alpha)}$, which implies $f(X^{(\alpha)}) \subset X^{(\alpha)}$.
Let $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$, then $f(X^{(\alpha)}) = f(\bigcap_{\beta<\alpha} X^{(\beta)}) \subset \bigcap_{\beta<\alpha} f(X^{(\beta)})$. But by induction hypothesis $f(X^{(\beta)}) \subset X^{(\beta)}$ for all $\beta < \alpha$. Therefore $f(X^{(\alpha)}) \subset X^{(\alpha)}$. □

4.1 Expansive Actions

Definition 4.1.1. Let X be a set and G a group. We say that $\varphi : G \times X \to X$ is an action if it satisfies the following axioms:

1) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

2) $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ for all $x \in X$, and for all $g, h \in G$.

We use the shorthand notation $g.x$ for $\varphi(g, x)$ and $G \curvearrowright X$ for an action φ.

Remark 4.1.2. An action $\varphi : G \times X \to X$ can be viewed as a group homomorphism $\varphi : G \to \mathrm{Bij}(X)$, where $\varphi(g)(x) = g.x$, and $\mathrm{Bij}(X)$ is the group of bijective functions on X with composition as the group operation.
If X is a topological space, we say that the action is continuous when $\varphi(g)$ is continuous for all $g \in G$.

Definition 4.1.3. Let $\mathcal{U}$ be a cover of X. We say that $A \subset X$ *refines* $\mathcal{U}$ (denoted $A \prec \mathcal{U}$) if there exists $U \in \mathcal{U}$ such that $A \subset U$.
A cover $\mathcal{V}$ is a refinement of $\mathcal{U}$ (denoted $\mathcal{V} \prec \mathcal{U}$) if for all $A \in \mathcal{V}$, $A \prec \mathcal{U}$.

Definition 4.1.4. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be an open cover of X, and $\varphi : G \times X \to X$ an action. We say that $\mathcal{U}$ is a *cover of expansivity* for φ, if for all $x, y \in X$

$$\{g.x, g.y\} \prec \mathcal{U} \text{ for all } g \in G \text{ implies } x = y.$$

In this case, we say that φ is an expansive action, and that G acts expansively on X.

This is a straightforward generalization of orbital expansiveness for homeomorphisms (see Definition 2.1 in [1]). It is not difficult to demonstrate that these concepts align with the usual ones when X is a compact metric space with a metric d, and $T_g : X \to X$, defined by $T_g(x) := g.x$, is continuous for all $g \in G$. Specifically, there exists a constant $c > 0$ such that, for any pair of distinct points $x, y \in X$, there exists $g \in G$ for which $d(g.x, g.y) > c$.

Definition 4.1.5. Let $\varphi : G \times X \to X$ be an action. We say that it is a continuous expansive action if it is expansive and $T_g(x) := g.x$ is continuous for all $g \in G$.

4.2 Expansive Actions on Ordinals

It is known from the work of Kato and Park [26] that a countable compact metric space (a countable compactum) admits an expansive homeomorphism if and only if the degree of the space, $\deg(X)$, is not an infinite limit ordinal. Semadeni ([34]) (see Corollary 2 in [5] for a clearer formulation) has shown that a compact scattered first-countable topological space with characteristic (α, n) is metrizable and homeomorphic to an ordinal of the form $\omega^\alpha n + 1$ with the order topology. A well-known consequence of this is that a countable compactum is homeomorphic to an ordinal. Bryant's paper [8] demonstrates that expansiveness is preserved under homeomorphism, that is, if X admits an expansive homeomorphism f and $g : X \to Y$ is a homeomorphism, then $g \circ f \circ g^{-1} : Y \to Y$ is an expansive homeomorphism. Hence, we can restate Kato and Park's theorem as follows:

Theorem. *Let X be a countable compact ordinal space. X admits an expansive homeomorphism if and only if $deg(X)$ is not an infinite limit ordinal.*

A natural question arises: Is it possible to remove the countability hypothesis?
In principle, the following theorem answers negatively.

Theorem 4.2.1. *(Theorem 2.7 [1]) If a compact Hausdorff topological space admits an orbit expansive homeomorphism, then it is metrizable.*

But if we rewrite Kato and Park's theorem in the following way we can generalize the result.

Theorem. *Let X be a countable compact ordinal space. X admits a continuous $\mathbb{Z}$-action if and only if $deg(X)$ is not an infinite limit ordinal.*

Now, we can generalize this theorem to actions over a general group. We will prove the following theorem.

Theorem 4.2.2. *Let X be a compact Hausdorff scattered space with characteristic (α, n). X admits an expansive continuous G-action if and only if α is not an infinite limit ordinal or $n \neq 1$.*

To prove this theorem, we will establish the following lemma.

Lemma 4.2.3. *Let X be a compact topological space with $X^{(\alpha)} = \{x_1, \ldots, x_n\}$. If α is an infinite limit ordinal, then for any neighborhoods $V_1, \ldots, V_n$ of $x_1, \ldots, x_n$ respectively, there exists an ordinal $\beta_0 < \alpha$ such that for all $\beta \geq \beta_0$, $X^{(\beta)} \subset \bigcup_{i=1}^{n} V_i$.*

Proof. The proof is a direct consequence of the equation $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ and the compactness of $X^{(\beta)}$ for all $\beta < \alpha$. □

In fact, we will prove something more general. For this, we introduce the following definition.

Definition 4.2.4. Let X be a topological space, G a group, and $\varphi : G \times X \to X$ an action. We say that φ is CB-stable if for every $g \in G$, the function $T_g : X \to X$, defined by $T_g(x) := g.x$, is CB-stable .

Remark 4.2.5. It is a direct consequence of Proposition 4.0.2 that a continuous action is CB-stable
.

Proof of Theorem 4.2.2. Assume $X^{(\alpha)} = \{x_1\}$ and α is an infinite limit ordinal. Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be a cover of X by open sets. If V is a neighborhood of x such that $V \subset U_j$ for some j, then by Lemma 4.2.3, there exists $\beta_0 < \alpha$ such that for all $\beta \geq \beta_0$, $X^{(\beta)} \subset V$. Let $y \in X^{(\beta)}$ with $y \neq x$. If φ is a CB-stable action, then $\{g.x, g.y\} \subset V$ for all g, implying that $\{g.x, g.y\} \prec \mathcal{U}$ for all g. Therefore, φ is not expansive.

The proof of the converse: By hypothesis, the characteristic of X is either of the form $(\alpha+1, n)$, with α an arbitrary ordinal and n a natural number, or of the form (α, n), with $n > 1$ and α an arbitrary ordinal. Hence, by Semenedi's theorem, X is homeomorphic to either $\omega^{\alpha+1}n + 1$ or $\omega^{\alpha}n + 1$ with the order topology. Let us consider the case where X is homeomorphic to $\omega^{\alpha+1}n + 1$. Of course, the non-trivial case is when $n > 0$. Assume now that $\omega^{\alpha+1}n + 1$, but $\omega^{\alpha+1}n + 1 = (\omega^{\alpha+1} + 1) + \ldots + (\omega^{\alpha+1} + 1)$, repeated n times. Then, if each of these summands admits an expansive action, so does X, and therefore, without loss of generality, we may assume that X is homeomorphic to $(0, \omega^{\alpha+1}]$ with the order topology. For each $i \geq 0$, let us define $X_i = (\omega^{\alpha} i, \omega^{\alpha}(i+1)]$. Note that X_i is isomorphic as an ordered set and, in particular, homeomorphic to X_0 for all $i \geq 0$. Indeed, we define $copy_i : X_i \to X_0$ as follows:

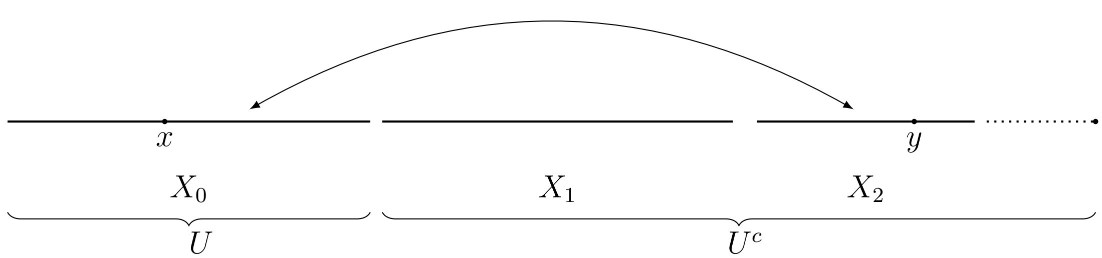
$$copy_i(x) = \begin{cases} \beta & \text{if } x = \omega^{\alpha} i + \beta, \\ \omega^{\alpha} & \text{if } x = \omega^{\alpha}(i+1). \end{cases}$$

It is clear that this is an order isomorphism.

Now take the set $\mathcal{U} = \{U, U^c\}$, where $U = X_0$. We will construct a group of homeomorphisms G and an action φ such that $\mathcal{U}$ is an expansive covering for φ.

First case: For each $i > 0$, a natural number, we define the following function:

$$F_i(z) = \begin{cases} copy_i(z) & \text{if } z \in X_i, \\ copy_i^{-1}(z) & \text{if } z \in X_0, \\ z & \text{if } z \notin X_i \cup X_0. \end{cases}$$

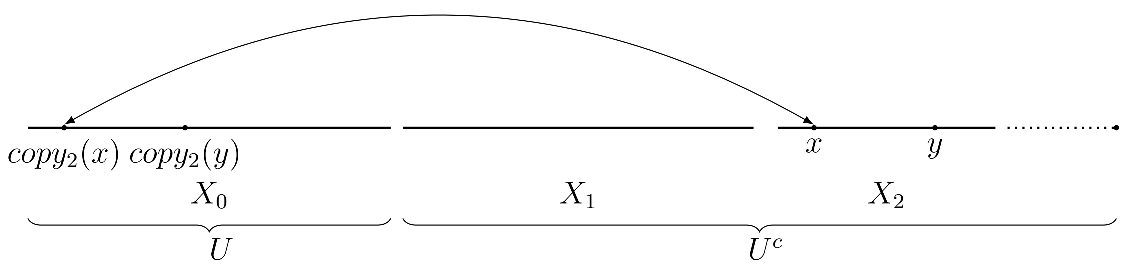


It is clear that F_i is a homeomorphism of X. Furthermore, for all $x, y \in X$ such that $x \in X_i$

and $y \in X_j$ with $i < j$, it is easy to verify that $\{F_j(x), F_j(y)\} \nsubseteq U$ and $\{F_j(x), F_j(y)\} \nsubseteq U^c$.

Second case: Let $x < y$, $x, y \in X_i$, and suppose that $i > 0$. We define the function $T_{x,y} : X \to X$ as follows.

$$T_{x,y}(z) = \begin{cases} copy_i(z) & \text{if } z \leq x \text{ and } z \in X_i, \\ copy_i^{-1}(z) & \text{if } z \leq copy_i(x) \text{ and } x \in X_0, \\ z & \text{otherwise.} \end{cases}$$



It is clear that $T_{x,y}$ is a homeomorphism. For all $i > 0$ such that x and y are distinct elements of X_i, we have:

$$\{T_{x,y}(x), T_{x,y}(y)\} \nsubseteq U \quad \text{and} \quad \{T_{x,y}(x), T_{x,y}(y)\} \nsubseteq U^c.$$

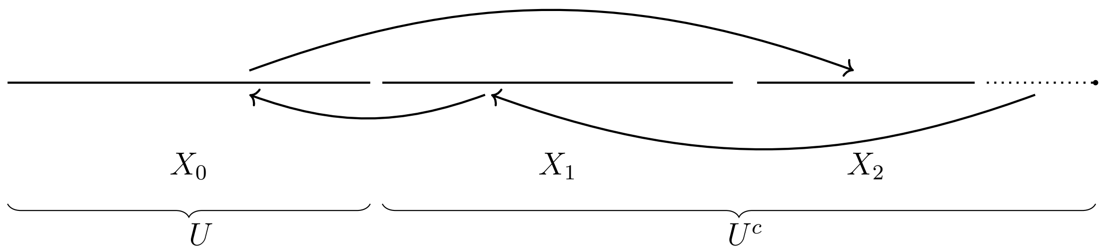
In the case where x and y are distinct elements of X_0, observe that:

$$\{T^{-1}_{copy_1^{-1}(x),copy_1^{-1}(y)}(x), T^{-1}_{copy_1^{-1}(x),copy_1^{-1}(y)}(y)\} \nsubseteq U$$
$$\text{and}$$
$$\{T^{-1}_{copy_1^{-1}(x),copy_1^{-1}(y)}(x), T^{-1}_{copy_1^{-1}(x),copy_1^{-1}(y)}(y)\} \nsubseteq U^c.$$

In fact, any $copy_i^{-1}$ could have been used instead of $copy_1^{-1}$.

Third case: Let $x = \omega^{\alpha+1}$ and $y \neq x$. We define the function $H : X \to X$ as follows.

$$H(z) = \begin{cases} copy_{i+2}(z) & \text{if } z \in X_i \text{ and } i \text{ is even,} \\ copy_{i-2}(z) & \text{if } z \in X_i \text{ and } i \text{ is odd and } i \geq 3, \\ copy_0(z) & \text{if } z \in X_1, \\ z & \text{if } z = \omega^{\alpha+1}. \end{cases}$$



It is clear that H is a homeomorphism. For all $y \neq \omega^{\alpha+1}$, we have

$$\{H(x), H(y)\} \nsubseteq U \text{ and } \{H(x), H(y)\} \nsubseteq U^c.$$

Now, if we take the group G generated by the functions F_i, $T_{x,y}$, and H with composition as the operation, defining for all $g \in G$, $g.x := g(x)$, the result follows.

It is clear that H is a homeomorphism, and for all $y \neq \omega^{\alpha+1}$, we have $\{H(x), H(y)\} \nsubseteq U$ and $\{H(x), H(y)\} \nsubseteq U^c$. Now, if we take the group G generated by the functions F_i, $T_{x,y}$, and H with composition as the operation, defining for all $g \in G$, $g.x := g(x)$, the result follows.
For the case where X is homeomorphic to $\omega^\alpha n + 1$, with $n > 1$, the process is very similar to what we did previously. We may assume that X is homeomorphic to $(0, \omega^\alpha n]$, and by defining for each $i \in \{0, \ldots n-1\}$ the sets $X_i = (\omega^\alpha i, \omega^\alpha (i+1)]$, we observe that for every i, X_i is isomorphic to X_1, and we define $copy_i : X_i \to X_1$ as before:

$$copy_i(x) = \begin{cases} \beta & \text{if } x = \omega^\alpha i + \beta, \\ \omega^\alpha & \text{if } x = \omega^\alpha (i+1). \end{cases}$$

Since we have a finite number of X_i, we only need to construct homeomorphisms similar to those in the first and second types of the previous case.

First case: For each $i \in \{1, \ldots n\}$, we define the following function:

$$F_i(z) = \begin{cases} copy_i(z) & \text{if } z \in X_i, \\ copy_i^{-1}(z) & \text{if } z \in X_0, \\ z & \text{if } z \notin X_i \cup X_0. \end{cases}$$

As in the previous case, for all $x, y \in X$ such that $x \in X_i$ and $y \in X_j$ with $i < j$, it is easy to verify that

$$\{F_j(x), F_j(y)\} \nsubseteq U \text{ and } \{F_j(x), F_j(y)\} \nsubseteq U^c.$$

Second case: If $x < y$, $x, y \in X_i$, and suppose that $i \neq 1$, we define $T_{x,y} : X \to X$ as follows:

$$T_{x,y}(z) = \begin{cases} copy_i(z) & \text{if } z \leq x \text{ and } z \in X_i, \\ copy_i^{-1}(z) & \text{if } z \leq copy_i(x) \text{ and } x \in X_1, \\ z & \text{otherwise.} \end{cases}$$

Similarly to the previous case, we take the group G generated by the functions F_i and H with composition as the operation, and by defining for all $g \in G$, $x \in X$, $g.x := g(x)$, the theorem is proven. □

Observe that in the proof of the forward implication of Theorem 4.2.2 the only property used is that the action is CB-stable ; hence we obtain the following theorem.

Theorem 4.2.6. *Let X be a compact Hausdorff scattered space with characteristic (α, n). X admits an expansive* CB-stable *action if and only if α is not an infinite limit ordinal or $n \neq 1$.*

Remark 4.2.7. Note that $|G| = |X|$.

4.3 $\mathcal{A}_{CB}(X)$ and the GCH

In the previous section, we demonstrated that for every compact ordinal space X, there exists a group G that acts expansively in a continuous and particularly CB-stable manner. In order to characterize the groups acting on X, a natural question arises about the cardinality they can have. We observed that the group constructed in the demonstration has the same cardinality as X. Of course, the group can be trivially extended to increase its cardinality, making the interesting question whether there exists a G acting expansively CB-stable on X with $|G| < |X|$. Let's define the following set:

$$\mathcal{A}_{CB}(X) = \{\lambda < |X| : \text{there exists } G \curvearrowright X \text{ CB-stable expansive action with } |G| = \lambda\}.$$

We will prove that: $\text{GCH}(\lambda) \leftrightarrow \neg\exists G \curvearrowright X$, CB-stable expansive action, with $|G| = \lambda < |X| \neq 2^\lambda$, with X compact Hausdorff space, and that

$$\mathcal{A}_{CB}(X) = \{\lambda : \lambda < |X| \leq 2^\lambda\}.$$

Proposition 4.3.1. *If $\varphi : G \times X \to X$ is an expansive action, then $|X| \leq 2^{|G|}$.*

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a of expansivity for φ. By taking intersections and complements, we can find a partition $\mathcal{V} = \{X_1, \ldots, X_m\}$ of X, such that $\mathcal{V} \prec \mathcal{U}$. Define $F : X \to \{1, \ldots, m\}^G$ as $F(x)(g) = i$ if $g.x \in X_i$.

F is injective: Let $x \neq y$, the expansivity of φ implies that there exists $g \in G$ such that $\{g.x, g.y\} \not\prec \mathcal{U}$, thus there exists $j \in \{1, \ldots, m\}$, such that $g.x \in X_j$ and $g.y \notin X_j$, then $F(x)(g) \neq F(y)(g)$. $\square$

Corollary 4.3.2. *If there exists a $G \curvearrowright X$ expansive and* CB-stable *on X, with $|G| = \lambda < |X| \neq 2^\lambda$, where X is a compact Hausdorff space, then* $\neg\mathrm{GCH}(\lambda)$.

Before proving the next theorem, we will prove the following lemma.

Lemma 4.3.3. *Let λ be an infinite cardinal and X a set such that $\lambda \leq |X| \leq 2^\lambda$. Then there exists a set $\mathcal{B}$ of cardinality λ consisting of bipartitions of X; that is, $P \in \mathcal{B}$ if $P = \{A, A^c\}$, where A and A^c are non-empty and $X = A \cup A^c$. Moreover, for any distinct $x, y \in X$, there exists $\{A, A^c\} \in \mathcal{B}$ such that $\{x, y\} \nsubseteq A$ and $\{x, y\} \nsubseteq A^c$.*

Proof. Without loss of generality, we can assume that X is a subset of ${}^\lambda 2 = \{f : \lambda \to \{0, 1\}\}$. For each $i \in \lambda$, we define the set $\chi_i = \{f \in {}^\lambda 2 : f(i) = 1\}$, and the functions $e_i \in {}^\lambda 2$ as follows:

$$e_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For each $i < \lambda$, we take a set $K_i \subset \chi_i$, and a set $K_i' \subset \chi_i^c$, such that $e_i \in K_i$ and $|K_i| = |K_i'| = |X|$. We consider the set $K := \bigcup_{i<\lambda}(K_i \cup K_i')$. It is clear that $|K| = |X|$, and we identify A with K.

For each $i < \lambda$, we define $A_i := K \cap \chi_i$, and then $A_i^c := K \cap \chi_i^c$ (where $A_i^c = K \setminus A_i$). It is clear that for all $i < \lambda$, $\{A_i, A_i^c\}$ is a bipartition of A. Now, let $f, g \in A$ be distinct, then there exists some $i_0 < \lambda$ such that $f(i_0) \neq g(i_0)$. Hence, either $f \in \chi_{i_0}$ and $g \in \chi_{i_0}^c$, or $f \in \chi_{i_0}^c$ and $g \in \chi_{i_0}$. Therefore, $\{f, g\} \nsubseteq A_{i_0}$ and $\{f, g\} \nsubseteq A_{i_0}^c$. Since $e_i \in A_i$ and $e_i \notin A_j$ for $i \neq j$, it follows that $A_i \neq A_j$ when $i \neq j$. Therefore, the set $\mathcal{B} := \{\{A_i, A_i^c\} : i < \lambda\}$ has cardinality λ. $\square$

Theorem 4.3.4. *Let λ be an infinite cardinal. For every κ with $\lambda \leq \kappa \leq 2^\lambda$, there exists a compact Hausdorff space X, a group G, and an action $\varphi : G \times X \to X$,* CB-stable *, such that the action is expansive with $|G| = \lambda$ and $|X| = \kappa$.*

Proof. Let κ be such that $\lambda \leq \kappa \leq 2^\lambda$, and let us take an ordinal α such that $|\alpha| = \kappa$. Consider $X = (0, \omega^{\alpha+1}]$, which with the order topology is a compact Hausdorff space, and $|X| = \kappa$. As in Theorem 4.2.2, for each $i \geq 0$, we define the set $X_i = (\omega^\alpha i, \omega^\alpha(i + 1)]$ and the isomorphisms $copy_i : X_i \to X_0$. We take the set $\mathcal{U} = \{U, U^c\}$, where $U = X_0$. We will construct a group G and a φ-action, which is CB-stable , such that $\mathcal{U}$ is an expansive covering for φ.
For each natural $i \geq 0$, by Lemma 4.3.3, we know that there exists a sequence of sets $\{X_{i,\beta}, X_{i,\beta}^c\}$ which bipartitions X_i. For all distinct $x, y \in X_i$, there exists β_0 such that $\{x, y\} \nsubseteq X_{i,\beta}$ and

$\{x, y\} \nsubseteq X_{i,\beta}^c$. Similarly to the first case in the proof of Theorem 4.2.2, we need to construct functions that map distinct elements x and y, where $x \in X_i$ and $y \in X_j$ with $i < j$, to different open sets in the covering $\mathcal{U}$. In this case, let us define the same functions as in the proof of Theorem 4.2.2 for each natural number $i > 0$:

$$F_i(z) = \begin{cases} copy_j(z) & \text{if } z \in X_i, \\ copy_i^{-1}(z) & \text{if } z \in X_0, \\ z & \text{if } z \notin X_i \cup X_0. \end{cases}$$

By the same reasoning, F_i are homeomorphisms and, in particular, CB-stable . Additionally, we have $\{F_j(x), F_j(y)\} \nsubseteq U$ and $\{F_j(x), F_j(y)\} \nsubseteq U^c$.

Second case: Let $x < y$, $x, y \in X_i$. Given a natural i, by Lemma 4.3.3, we know that for every $\beta < \lambda$, there exists a bipartition $\{X_{i,\beta}, X_{i,\beta}^c\}$ of X_i. For all distinct $x, y \in X_i$, there exists $\beta_0 < \lambda$ such that $\{x, y\} \nsubseteq X_{i,\beta_0}$ and $\{x, y\} \nsubseteq X_{i,\beta_0}^c$. Then for every natural $i > 0$ and $\beta < \lambda$, we define the following functions:

$$T_{i,\beta}(z) = \begin{cases} copy_i(z) & \text{if } z \in X_{i,\beta}, \\ copy_i^{-1}(z) & \text{if } z \in copy_i(X_{i,\beta}), \\ z & \text{otherwise.} \end{cases}$$

Note that in general, $T_{i,\beta}$ is not continuous but is CB-stable . Furthermore, for every natural $i > 0$, if $x, y \in X_i$ are distinct, there exists $\beta_0 < \lambda$ such that

$$\{x, y\} \nsubseteq X_{i,\beta_0} \text{ and } \{x, y\} \nsubseteq X_{i,\beta_0}^c,$$

and thus

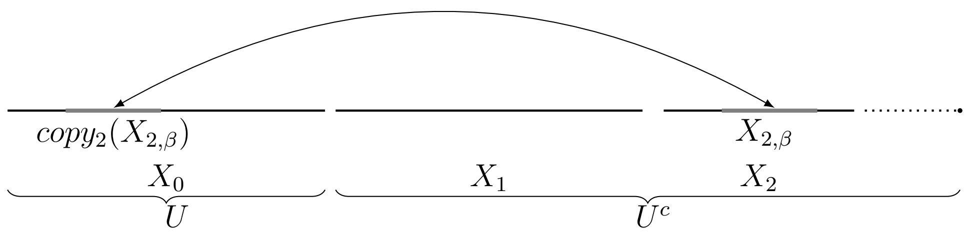
$$\{T_{i,\beta_0}(x), T_{i,\beta_0}(y)\} \nsubseteq U \text{ and } \{T_{i,\beta_0}(x), T_{i,\beta_0}(y)\} \nsubseteq U^c.$$

For the case where $x, y \in X_0$, we know that there exists $\beta_0 < \lambda$ such that

$$\{copy_1^{-1}(x), copy_1^{-1}(y)\} \nsubseteq X_{i,\beta_0} \text{ and } \{copy_1^{-1}(x), copy_1^{-1}(y)\} \nsubseteq X_{i,\beta_0}^c,$$

and thus

$$\{T_{1,\beta_0}^{-1}(x), T_{1,\beta_0}^{-1}(y)\} \nsubseteq U \text{ and } \{T_{1,\beta_0}^{-1}(x), T_{1,\beta_0}^{-1}(y)\} \nsubseteq U^c.$$



Note that for every natural i and distinct $\beta, \beta' < \lambda$, there exists $z \in X$ such that $T_{i,\beta}(z) = z$ and $T_{i,\beta'}(z) \neq z$. Therefore, the set of functions $T_{i,\beta}$ has cardinality λ.

Third case: Let $x = \omega^{\alpha+1}$ and $y \neq x$. The function H is exactly the same as in Theorem 4.2.2:

$$H(z) = \begin{cases} copy_{i+2}(z) & \text{if } z \in X_i \text{ and } i \text{ is even,} \\ copy_{i-2}(z) & \text{if } z \in X_i \text{ and } i \text{ is odd and } i \geq 3, \\ copy_0(z) & \text{if } z \in X_1, \\ z & \text{if } z = y. \end{cases}$$

If we take the group G generated by the functions $T_{i,\beta}$, F_i, and H with composition as the operation, we have that $|G| = \lambda$. By defining for every $g \in G$ and $x \in X$, $g.x := g(x)$, the theorem is proven. □

Corollary 4.3.5. $\mathrm{GCH}(\lambda) \leftrightarrow \neg\exists G \curvearrowright X$, CB-stable *expansive action, with* $|G| = \lambda < |X| \neq 2^\lambda$, *with* X *compact Hausdorff space.*

Corollary 4.3.6. $\mathcal{A}_{CB}(X) = \{\lambda : \lambda < |X| \leq 2^\lambda\}$

4.4 Expansive Generalized Continuum Hypothesis

A natural generalization of the formula in the corollary 4.3.5 is to replace CB-stable with continuous.

Definition 4.4.1. (Expansive Generalized Continuum Hypothesis) Let λ be an infinite cardinal.

$$\mathrm{EGCH}(\lambda) := \neg\exists G \curvearrowright X, \text{ continuous expansive action, with } |G| = \lambda < |X| \neq 2^\lambda,$$
with X compact Hausdorff space.

Remark 4.4.2. For every infinite cardinal λ,

$$\mathrm{GCH}(\lambda) \to \mathrm{EGCH}(\lambda)$$

Lemma 4.4.3. *Let $\varphi : G \times X \to X$ be a continuous expansive action, $\mathcal{U} = \{U_1, \dots, U_n\}$ an expansivity cover for φ, and $\mathcal{V} = \{V_1, \dots, V_m\}$ a set such that $\mathcal{V} \prec \mathcal{U}$. Then, for all $h \in \{1, \dots, m\}^G$, we have $|\bigcap_{g\in G} g \cdot V_{h(g)}| \leq 1$.*

Proof. Suppose there exists an $h \in \{1, \dots, m\}^G$ such that $|\bigcap_{g\in G} g \cdot V_{h(g)}| > 1$. Then, we can find distinct $x, y \in \bigcap_{g\in G} g \cdot V_{h(g)}$ with $\{g^{-1} \cdot x, g^{-1} \cdot y\} \subset V_{h(g)}$ for every $g \in G$. Given that $\mathcal{V} \prec \mathcal{U}$, for each $V_{h(g)}$, there exists a $U_{h'(g)} \in \mathcal{U}$ such that $V_{h(g)} \subset U_{h'(g)}$ for all $g \in G$. Consequently, $\{g^{-1}.x, g^{-1}.y\} \prec \mathcal{U}$ for every $g \in G$. This implies φ is not expansive. □

Theorem 4.4.4. *Let X be an infinite compact Hausdorff space, and let $\varphi : G \times X \to X$ be an expansive continuous action. Then X has a topology basis of cardinality at most $|G|$.*

Proof. Let $\mathcal{U}$ be a covering of expansivity for φ. For every $x \in X$, we select an open neighborhood V_x of x such that the closure of V_x ($\overline{V}_x$) is contained within $\mathcal{U}$. Due to compactness, we can choose a finite subcovering $\mathcal{V}$. Since $\mathcal{V} \prec \mathcal{U}$, according to lemma 4.4.3, it is established that $\left|\bigcap_{g\in G} g \cdot \overline{V}_{h(g)}\right| \leq 1$ for all $h \in \{1, \dots, m\}^G$. Let $x \in X$, and consider $g \cdot \mathcal{V}$ as an open cover for every $g \in G$. There exists some $h \in \{1, \dots, m\}^G$ such that x is contained in $\bigcap_{g\in G} g \cdot V_{h(g)}$, which is a subset of $\bigcap_{g\in G} g \cdot \overline{V}_{h(g)}$. Consequently, $\bigcap_{g\in G} g \cdot \overline{V}_{h(g)} = \{x\}$. Let W be a neighborhood of x, and let $A_{h(g)} = (g \cdot \overline{V}_{h(g)})^c$. Thus, $\{A_{h(g)}\}_{g\in G} \cup \{W\}$ forms an open cover of X, allowing for the selection of $g_1, \dots, g_l$ such that $\{A_{h(g_i)}\}_{i=1}^l \cup \{W\}$ covers X. Therefore, $\bigcap_{i=1}^l g_i \cdot \overline{V}_{h(g_i)} \subseteq W$, implying that $\bigcap_{i=1}^l g_i \cdot V_{h(g_i)} \subseteq W$. □

Theorem 4.4.5. *Let X be a compact ordinal space, and $\varphi : G \times X \to X$ be an expansive continuous action. Then $|G| \geq |X|$.*

Proof. Suppose that $\lambda = |G| < |X|$. By the above theorem, every $x \in X$ has a base of neighborhoods with cardinality less than or equal to λ. However, in $[0, x_0]$, where

$$x_0 = \min\{x \in X : \lambda < |[0, x]|\},$$

x_0 does not have a base of neighborhoods with cardinality $\leq \lambda$. □

Corollary 4.4.6. *Let X be an infinite compact ordinal space. Then $\mathcal{A}_c(X) = \emptyset$.*

A consequence of a classical result of Čech-Pospišil [9] is the following theorem.

Theorem 4.4.7. *(Theorem 7.20 [21]) Every compact, first-countable space is countable or has cardinality $2^{\aleph_0}$.*

As an immediate consequence of Theorems 4.4.4 and 4.4.7, we obtain the following theorem.

Theorem 4.4.8. EGCH($\aleph_0$)

Chapter 5

Combinatorial Expansiveness.

Let's recall the definition of expansiveness by coverings that we saw in the previous chapter.

Let $\mathcal{U}$ be a cover of X. We say that $A \subset X$ *refines* $\mathcal{U}$ (denoted $A \prec \mathcal{U}$) if there exists $U \in \mathcal{U}$ such that $A \subset U$.

A cover $\mathcal{V}$ is a refinement of $\mathcal{U}$ (denoted $\mathcal{V} \prec \mathcal{U}$) if for all $A \in \mathcal{V}$, $A \prec \mathcal{U}$.

Let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open cover of X, and $\varphi : G \times X \to X$ an action. We say that $\mathcal{U}$ is a *cover of expansivity* for φ, if for all $x, y \in X$:

$$\{g.x, g.y\} \prec \mathcal{U} \text{ for all } g \in G \text{ implies } x = y.$$

To an expansive action, we can associate a graph coloring problem. Specifically, for each $g \in G$, we define:

$$X_g := \big\{\{x, y\} \in [X]^2 : \{g.x, g.y\} \not\prec \mathcal{U}\big\}$$

We can interpret X_g as a graph whose vertex set $V(X_g)$ is X, and its edge set $E(X_g)$ is X_g. Then, we have that $\varphi : G \times X \to X$ is an expansive action if, for all distinct $x, y \in X$, there exists $g \in G$ such that $\{x, y\} \in E(X_g)$.

Furthermore, given an infinite cardinal λ, every cardinal κ such that $\lambda \leq \kappa \leq 2^\lambda$ is associated with the following combinatorial problem: for each $i \in \lambda$, there exists a bipartition $\{A_i, A_i^c\}$ of κ such that for all distinct $x, y \in \kappa$, there exists $i_0 \in \lambda$ such that either $x \in A_i$ and $y \in A_i^c$, or $x \in A_i^c$ and $y \in A_i$.

Indeed, for each $i \in \lambda$, we define the set:

$$X_i = \left\{h \in {}^{\lambda}2 : h(i) = 1\right\}$$

and the functions $e_i \in {}^{\lambda}2$ as follows:

$$e_i(j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

For each $i < \lambda$, we take a set $K_i \in X_i$, and a set $K_i' \in X_i^c$, such that $|K_i| = |K_i'| = \kappa$. We consider the set:

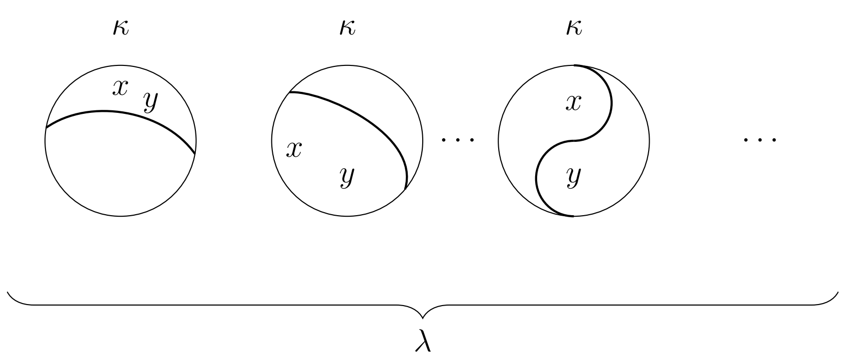
$$K := \left(\bigcup_{i<\lambda} K_i \cup K_i' \cup \{e_i\}\right) \cup \{\texttt{nul}\},$$

where $\texttt{nul} \in {}^{\lambda}2$ is the null function. It is clear that $|K| = \kappa$, and we identify κ with K.

For every $i < \lambda$, we define:

$$A_i := K \cap X_i, \quad A_i^c := K \cap X_i^c$$

Let f and g in K be distinct functions, then there exists $i_0 < \lambda$ such that $f(i_0) \neq g(i_0)$. Either $f \in X_{i_0}$ and $g \in X_{i_0}^c$, or $f \in X_{i_0}^c$ and $g \in X_{i_0}$, implying that either $f \in A_{i_0}$ and $g \in A_{i_0}^c$, or $f \in A_{i_0}^c$ and $g \in A_{i_0}$. Moreover, there are no redundant bipartitions; that is, given $j < \lambda$, for all $i \neq j$, we have $\{e_j, \texttt{nul}\} \subset A_i^c$.



In Proposition 5.2.8, we will see that if κ satisfies this combinatorial problem, then $\lambda \leq \kappa \leq 2^{\lambda}$.

Given a bipartition P of X, it is possible to generate a dual bipartition $(D(P))$ in $[X]^2$ as follows: if $P = \{A, A^c\}$, we define $D(P) = \{D(A), D(A)^c\}$, where $D(A) = \{\{x, y\} \in [X]^2 : x \in A, y \in A^c\}$. Therefore, we can define the dual graph $(D(P))$ such that $V(D(P)) = X$ and $E(D(P)) = \{\{x, y\} : x \in P \text{ and } y \in P^c\}$.

Thus, the cardinals κ that satisfy $\lambda \leq \kappa \leq 2^\lambda$ are characterized by the same combinatorial problem associated with expansivity in the case of graphs derived from bipartitions. Schematically:

$$\text{Expansivity} \Leftrightarrow \text{ A graph problem}$$

$$\lambda \leq \kappa \leq 2^\lambda \Leftrightarrow \text{ A bipartite graph problem}$$

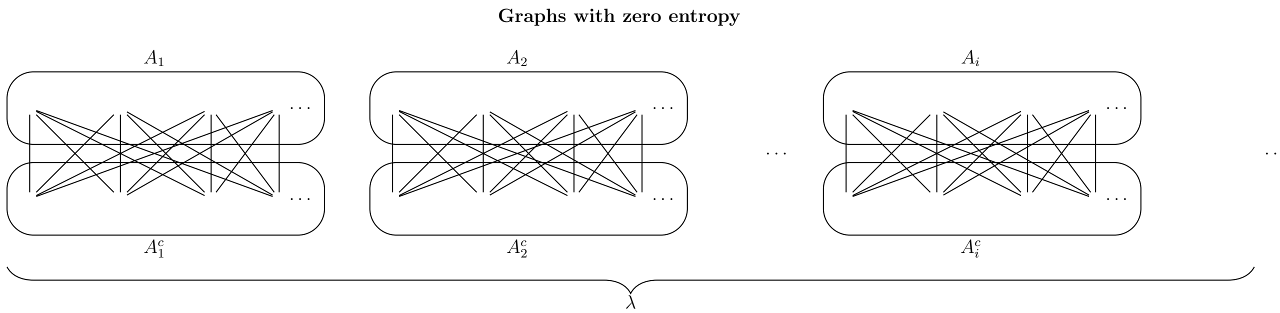
This relationship forms the combinatorial core for the equivalence between CB-stable expansive actions and the generalized continuum hypothesis found in [17] and in Chapter 4.

Once we have characterized the cardinals κ that satisfy $\lambda \leq \kappa \leq 2^\lambda$ in terms of a combinatorial bipartition problem, and in particular those cardinals κ that violate the generalized continuum hypothesis for λ, the question arises as to what is the natural generalization of this combinatorial problem. Consequently, we wonder whether new questions arise about the generalized continuum hypothesis. This work aims to explore this idea. We will see that there are several ways to generalize the bipartite graph problem and define some problems related to the GCH.

The most natural generalization of the partition problem is to remove the restriction that they are bipartitions. It is easy to see that if we consider partitions of any cardinality, the combinatorial problem is equivalent to that of bipartitions. In the next section, we will present a way to extend the combinatorial problem in a non-trivial manner.

5.1 Graph Complexity

Observe that the bipartite graphs found have the particular property that all their points are interchangeable by a graph automorphism; that is, the cardinality of the set of graph orbits is 1. If the graph were finite and we considered the entropy due to Trucco [39], the graph would have zero entropy.



The idea of this section is to generalize the combinatorial problem in terms of orbits under automorphisms.

5.1.1 Symmetry Groups in Graphs

Definition 5.1.1. Let X be a set, and G a group, an *action* of G on X is a function $\varphi : G \times X \to X$, which satisfies:

1) For all $x \in X$, $\varphi(e, x) = x$, where e is the identity element of G.

2) For all $g, h \in G$, and for all $x \in X$, $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$.

We denote $\varphi(g, x)$ as $g.x$.
If there exists an action of G on X, we say that G acts on X, when there is no risk of ambiguity, we denote the action φ as $G \curvearrowright X$.

Definition 5.1.2. Let $G \curvearrowright X$ be an action. We define the *orbit* of x as the set

$$\mathcal{O}(x) := \{g.x : g \in G\}$$

We define the set of orbits of the action as $\mathcal{O}(G \curvearrowright X) := \{\mathcal{O}(x) : x \in X\}$.
We define $Fix(G \curvearrowright X) := \{x \in X : \forall g \in G, g.x = x\}$.

Definition 5.1.3. A graph Y is a pair of sets $(V(Y), E(Y))$, where $V(Y)$ is a set that we will call the *vertices* of Y and $E(Y) \subset [V(Y)]^2$ a set that we will call the *edges* of Y. For us, graphs will be an irreflexive relation.

Definition 5.1.4. Let Y be a graph, and $x, y \in V(Y)$, a *path* between x and y is a sequence $\{x_0, x_1\}, \ldots, \{x_{n-1}, x_n\}$ of edges, with $x_0 = x$ and $x_n = y$.

Definition 5.1.5. A graph Y is *connected* if for every $x, y \in V(Y)$, distinct, there exists a path between x and y.

Definition 5.1.6. Let Y and Y' be graphs. A graph isomorphism (denoted as $f : Y \to Y'$) is a bijective function $f : V(Y) \to V(Y')$, such that for all $y_1, y_2 \in V(Y)$, $\{y_1, y_2\} \in E(Y)$ if and only if $\{f(y_1), f(y_2)\} \in E(Y')$.

Remark 5.1.7. Let $\mathrm{AUT}(Y) := \{f : V(Y) \to V(Y), \quad f \text{ is a graph isomorphism}\}$.
$\mathrm{AUT}(Y)$ is a group under composition and acts on $V(Y)$ with the action $g.x := g(x)$, where $x \in V(Y)$ and $g \in \mathrm{AUT}(Y)$.
When we consider $G \subseteq \mathrm{AUT}(Y)$ acting on $V(Y)$, we denote it as $G \curvearrowright V(Y)$ or as $G \curvearrowright Y$.
Unless stated otherwise, all actions we consider will be of this type.

Remark 5.1.8. Every $g \in \mathrm{AUT}(Y)$ naturally induces a bijection $g : E(Y) \to E(Y)$, defined as $g(\{x, y\}) := \{g(x), g(y)\}$, if $\{x, y\} \in E(Y)$.

Remark 5.1.9. Let X be a set. Given a bipartition P of X, it is possible to generate a dual bipartition $(D(P))$ in $[X]^2$ as follows: If $P = \{A, A^c\}$, we define $D(P) = \{D(A), D(A)^c\}$, with $D(A) = \{\{x, y\} \in [X]^2 : x \in A, y \in A^c\}$.

Definition 5.1.10. Given a set $A \subsetneq X$, non-empty. We define the dual graph associated to the set A, as the $D(A) \subset [X]^2$, defined by the remark 5.1.9

Remark 5.1.11. If X is a set, $\emptyset \subsetneq A \subsetneq X$, and $Y = D(A)$. Then $V(Y) = X$.

5.2 Dual Generalization Continuum Hypothesis

5.2.1 Expansive Actions Among Cardinals

Proposition 5.2.1. *Let Y be a graph such that $|E(Y)| > 1$. The following statements are equivalent.*

1) $Y = D(A)$ for some $A \subset V(Y)$.

2) Y satisfies the following properties:

(a) Y is connected.

(b) Y does not contain triangles (closed paths of three edges).

(c) For every $x \in V(Y)$, for every $\{y, y'\} \in E(Y)$, there exists $g \in AUT(Y)$ such that

$$\begin{cases} g.y = x \\ g.y' = y' \end{cases}$$

or

$$\begin{cases} g.y = y \\ g.y' = x \end{cases}$$

3) *There exists $G \subset AUT(Y)$, a subgroup, such that for every $x, y \in V(Y)$, $\{x, y\} \in E(Y)$ if and only if*

$$|\{x, y\} \cap Fix(G \curvearrowright Y)| = 1$$

.

4) *Y satisfies the following properties.*

   (a) *Y is connected.*

   (b) *$Y \neq [V(Y)]^2$.*

   (c) *For every $\{x, y\}, \{x', y'\}$, if $(x \neq y'), (x' \neq y), (\mathcal{O}(x) = \mathcal{O}(x'))$ and $(\{x, y\}, \{x', y'\} \in E(Y))$, then it is verified that there exists $g \in AUT(Y)$ such that*

$$(g^2 = Id) \wedge (g.x = x') \wedge (g.y = y')$$

.

5) *Y satisfies the following properties.*

   (a) *Y is connected.*

   (b) *$Y \neq [V(Y)]^2$.*

   (c) *For every $A \in \mathcal{O}(AUT(Y) \curvearrowright Y)$, for every $x, y \in A$, if $\{x, y\} \in E(Y)$ then $A = V(Y)$.*

   (d) *For every $\{x, y\}, \{x', y'\} \in E(Y)$, if $(x \neq y') \wedge (x' \neq y)$ and $(|\mathcal{O}(x)| = |\mathcal{O}(x')| \leq |\mathcal{O}(y)| \leq |\mathcal{O}(y')|)$, then there exists $g \in AUT(Y)$ such that*

$$(g^2 = Id) \wedge (g.x = x') \wedge (g.y = y')$$

.

   (e) *$|\mathcal{O}(AUT(Y) \curvearrowright Y)| \leq 2$.*

Proof. $1 \Rightarrow 2$: Let $Y = D(A)$, $x \in V(Y)$, $\{y, y'\} \in E(Y)$. Then either $y \in A$ and $y' \in A^c$, or $y \in A^c$ and $y' \in A$. Suppose that $x \in A$, if $x \in A^c$ the reasoning is the same. We define the function $g : V(Y) \to V(Y)$, as $g(x) = y$, $g(y) = x$, and all the remaining points are fixed by g. It is clear that g is a graph isomorphism. If $y \in A^c$ we define g similarly, $g(x) = y'$, $g(y') = x$ and g

fixes the remaining points. Additionally, it is clear that Y is connected and does not have triangles.

$2 \Rightarrow 1$: Let $\{y, y'\} \in E(Y)$, if $E(Y) = \{y, y'\}$ there is nothing to prove. Otherwise, there exists $x_0 \neq y, y'$, and due to connectivity, we can assume that $\{x_0, y\} \in E(Y)$. Since Y does not have triangles, we have that $\{x_0, y'\} \notin E(Y)$. Therefore, $A := \{x \in V(Y) : x \neq x_0, \{x_0, x\} \notin E(Y)\}$ verifies that A and its complement are non-empty. We will prove that $Y = D(A)$. Let $x \in A$, and $x' \in A^c$, since $\{x', x_0\} \in E(Y)$ by 2) we know that either there exists $g \in \mathrm{AUT}(Y)$ such that

$$\begin{cases} g.x' = x \\ g.x_0 = x_0 \end{cases}$$

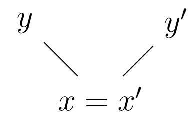
or

$$\begin{cases} g.x_0 = x \\ g.x' = x' \end{cases}$$

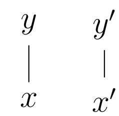
By the definition of the set A, we have that $x' \in A^c$ implies that $\{x', x_0\} \in E(Y)$, therefore if the first option occurred we would have that $\{x, x_0\} = \{g(x'), g(x_0)\} \in E(Y)$, and then $\{x, x_0\} \in E(Y)$, absurd. Thus, we have that $\{x, x'\} = \{g(x_0), g(x')\} \in E(Y)$. Therefore, $\{x, x'\} \in E(Y)$.
Let $y, z \in A^c$, distinct, we know that $\{y, x_0\} \in E(Y)$ and $\{z, x_0\} \in E(Y)$. But since there are no triangles in Y we can conclude that $\{y, z\} \notin E(Y)$. Similarly, if $y, z \in A$, are distinct, then $\{x_0, y\}$, $\{x_0, z\} \notin E(Y)$. We have to see $\{y, z\} \in E(Y)$. If $\{y, z\} \in E(Y)$ there exists g such that $g(y) = x_0$ and $g(z) = z$ or well $g(y) = y$ and $g(z) = x_0$. Therefore, either $\{x_0, y\} \in E(Y)$ or $\{x_0, z\} \in E(Y)$. Absurd.
$3 \Rightarrow 1$: If for every $x, y \in V(Y)$, $\{x, y\} \in E(Y)$ if and only if $|\{x, y\} \cap Fix(G \curvearrowright Y)| = 1$, then for every $x, y \in V(Y)$, $\{x, y\} \in E(Y)$ if and only if $x \in Fix(G \curvearrowright Y)$, and $y \in (Fix(G \curvearrowright Y))^c$, or vice versa. Therefore, $Y = D(Fix(G \curvearrowright Y))$.

$1 \Rightarrow 3$: Assume $Y = D(A)$, suppose $|A^c| > 1$, define $G := \{g \in \mathrm{AUT}(Y) : \forall x \in A, g.x = x\}$, clearly, G is a subgroup of $\mathrm{AUT}(Y)$. Let $\{x, y\} \in E(Y)$, with $x \in A$, and $y \in A^c$. Take a $y' \in A^c$, different from y. Define the function $g : V(Y) \to V(Y)$, as $g(y) = y'$, $g(y') = y$, and all other points fixed by g, then $g \in G$, and we can conclude that $y \notin Fix(G \curvearrowright Y)$, but $x \in Fix(G \curvearrowright Y)$. Let $x, y \in V(Y)$ such that $|\{x, y\} \cap Fix(G \curvearrowright Y)| = 1$, meaning that $x \in A$ and $y \in A^c$, or vice versa.

$1 \Rightarrow 4$: As $|E(Y)| > 1$ and $Y = D(A)$ for some $A \subset V(Y)$, then Y is connected and $Y \neq [V(Y)]^2$. It remains to prove 4c). Condition c) tells us is that if we have a graph of the following type:



then there exists a $g \in \mathrm{AUT}(Y)$ such that $g(x) = x$, $g(y) = y'$, $g(y') = y$. And if we have a graph of the type:



with x and x' in the same orbit, then there exists $g \in \mathrm{AUT}(Y)$ such that $g(y) = g(y')$, $g(y') = g(y)$, $g(x) = g(x')$ and $g(x') = g(x)$.
But if $Y = D(A)$, either $|A| = |A^c|$, or $|A| \neq |A^c|$. In the first case, all vertices are interchangeable, that is, for every $x, x' \in V(Y)$, a function g that swaps x and x' and fixes the remaining points is a $g \in \mathrm{AUT}(Y)$ that verifies what we need. In the second case, we have two orbits, A and A^c, indeed, if $x, x' \in A$, a function g that swaps x and x' and fixes the remaining points is a $g \in \mathrm{AUT}(Y)$ that verifies what we need. If $x, x' \in A^c$, reasoning as before, we find that x, x' are in the same orbit. If $x \in A$ and $x' \in A^c$, there is no $g \in \mathrm{AUT}(Y)$ that maps x to x', since such an automorphism must preserve incidences, and at x incidences are $|A|$ edges and at x' incidences are $|A^c|$ edges, with $|A| \neq |A^c|$.

$4 \Rightarrow 1$: First, let's see that $|\mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)| \leq 2$. If there is more than one orbit due to connectivity, there must be at least two distinct orbits A and B such that there exist $a \in A$ and $b \in B$ with $\{a, b\} \in E(Y)$. Let C be an orbit; due to connectivity, there must exist a $c \in C$ such that either $\{a, c\} \in E(Y)$ or $\{b, c\} \in E(Y)$. If $\{a, c\} \in E(Y)$, then by c), there exists $g \in \mathrm{AUT}(Y)$ such that $g(a) = a$, $g(c) = b$, and $g(b) = c$, but then $c \in B$ and therefore $B = C$. Similarly, if $\{b, c\} \in E(Y)$, then by c), there exists $g \in \mathrm{AUT}(Y)$ such that $g(b) = b$, $g(c) = a$, and $g(a) = c$, but then $c \in A$ and therefore $A = C$.

Suppose $|\mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)| = 1$. Since $Y \neq [V(Y)]^2$, there must exist $x_0, z \in V(Y)$, distinct, such that $\{z, x_0\} \notin E(Y)$. Let's take such an x_0 and define the set $A := \{y \in V(Y) : \{x_0, y\} \in E(Y)\}$. Since $Y \neq \emptyset$ and is connected, we know that $A \neq \emptyset$, and A^c contains an element different from x_0. Therefore, $V(Y) = A \cup A^c$. We will prove that $Y = D(A)$.

Let $x \in A^c$ and $y \in A$. Since Y is connected, we know that there exists a path from x to y, i.e., there exist $x_1, \ldots, x_n \in X$ such that $x_1 = x$, $x_n = y$, with $\{x_i, x_{i+1}\} \in E(Y)$ for every $i = 1, \ldots, n-1$. Let's take the path of minimum length. If the length of the path is greater than two, then we have $\{x, x_2\}$, $\{x_2, x_3\}$, $\{x_3, x_4\} \in E(Y)$. Let's visualize the graph as follows:

$$\begin{array}{ccc} x_2 & & x_4 \\ | & \diagdown & | \\ x & & x_3 \end{array}$$

Since there is only one orbit, in particular, x and x_3 are in the same orbit, so by 4) c), we know that the parallel edges in the drawing are interchangeable. In particular, there exists $g \in \mathrm{AUT}(Y)$ such that $g(x_2) = x_4$, $g(x_3) = x$, but $\{x_2, x_3\} \in E(Y)$, so $\{x_4, x\} \in E(Y)$.

$$\begin{array}{ccc} x_2 & & x_4 \\ | & \times & | \\ x & & x_3 \end{array}$$

But this contradicts that the taken path is of minimum length, so the path of minimum length must be of length less than or equal to two. If it is of length 2, we have the following configuration:

$$\begin{array}{ccc} x_2 & & x_0 \\ | & \diagdown & | \\ x & & y \end{array}$$

Reasoning as we did before, we deduce that $\{x, x_0\} \in E(Y)$, which is absurd since $x \in A^c$. From this, we deduce that the minimum length path is one and therefore $\{x, y\} \in E(Y)$. We still need to prove that those are all the edges. Suppose there exist $y, y' \in A$ with $\{y, y'\} \in E(Y)$, consider $z \in A^c$ and the following configuration:

$$\begin{array}{ccc} y' & & x_0 \\ | & \diagdown & | \\ z & & y \end{array}$$

By a similar reasoning to the previous ones, we deduce that $\{z, x_0\} \in E(Y)$, which is absurd. Now let's suppose that there exist $x, x' \in A^c$ with $\{x, x'\} \in E(Y)$. Let's take a $y \in A$ and consider the following configuration:

$$\begin{array}{ccc} x' & & x_0 \\ | & \diagdown & | \\ x & & y \end{array}$$

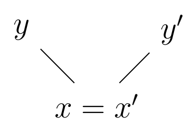
And reasoning as before, we deduce that $\{x, x_0\} \in E(Y)$, which is absurd. Therefore, $Y = D(A)$.

Suppose $|\mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)| = 2$.

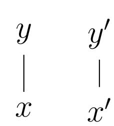
Let A be the set such that $\{A, A^c\} = \mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)$. Since Y is connected, there exist $x_0 \in A$

and $y_0 \in A^c$ such that $\{x_0, y_0\} \in E(Y)$. For every $x \in A$, it holds that $\{x, x_0\} \notin E(Y)$; otherwise, by c), there would exist $g \in \mathrm{AUT}(Y)$ such that $g.x_0 = x_0$, and $g.x = y_0$, implying that x and y_0 belong to the same orbit, which is absurd. Similarly, for every $y' \in A^c$, it is verified that $\{y, y'\} \notin E(Y)$; otherwise, by b), there would exist $g \in \mathrm{AUT}(Y)$ such that $g.y = y$, and $g.y' = x_0$, implying that x_0 and y' belong to the same orbit, which is absurd. Thus, by connectivity, every element z in A^c must be connected to x_0 by a path. However, by the previously proven fact, this path cannot have any vertex in A other than x_0, and it cannot have any vertex in A^c other than z. Therefore, $A^c = \{z \in X : \{z, x_0\} \in E(Y)\}$. By a similar argument, it is verified that for every y and y' in A^c, $\{y, y'\} \notin E(Y)$. Indeed, if $\{y, y'\} \in E(Y)$, by b), there would exist $g \in \mathrm{AUT}(Y)$ such that $g.y = y$, and $g.x_0 = y'$, implying that x_0 and y' are in the same orbit, which is absurd. Let $x \in A$ and $y \in A^c$. By connectivity, there exists a path connecting them. However, by the same argument used multiple times before, this path cannot have length greater than one; otherwise, we would have an element in an orbit with two incident edges coming from different orbits, which would lead to an absurdity by b). Therefore, we can conclude that $\{x, y\} \in E(Y)$, and by the same argument, we can affirm that for every $x, x' \in A$, $\{x, x'\} \notin E(Y)$. Hence, $Y = D(A)$.

1) $\Rightarrow$ 5): Let's observe that what e) tells us is that if we have a graph of the following type:



with $|\mathcal{O}(x)| \leq |\mathcal{O}(y)| \leq |\mathcal{O}(y')|$ then there exists a $g \in \mathrm{AUT}(Y)$ such that $g(x) = x$, $g(y) = y'$, $g(y') = y$. And if we have a graph of the type:



with $|\mathcal{O}(x)| = |\mathcal{O}(x')| \leq |\mathcal{O}(y)| \leq |\mathcal{O}(y')|$, then there exists $g \in \mathrm{AUT}(Y)$ such that $g(y) = g(y')$, $g(y') = g(y)$, $g(x) = g(x')$ and $g(x') = g(x)$, but this is fulfilled in the case where $Y = D(A)$, since if it has a unique orbit, it is trivial, and if it has two, these are A and A^c, so either $x, x' \in A$ and $y, y' \in A^c$, or $x, x' \in A^c$ and $y, y' \in A$. It is straightforward to observe that the remaining conditions are satisfied.

5) $\Rightarrow$ 1): If $|\mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)| = 1$ then the hypotheses of 5) imply 4) and therefore we have that $Y = D(A)$. Let's see the case where $|\mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)| = 2$. Let A be the set such that $\{A, A^c\} = \mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)$, since Y is connected there are $x_0 \in A$ and $y_0 \in A^c$ such that

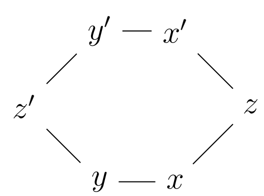
$\{x_0, y_0\} \in E(Y)$. By c) it is fulfilled that for every $x, x' \in A$, $\{x, x'\} \notin A$, and that for every $y, y' \in A$, $\{y, y'\} \notin A^c$. Let $x \in A$ and $y \in A^c$, by connectivity there exists a path that connects them, but by c) it cannot have length greater than one, then $\{x, y\} \in E(Y)$, and therefore $Y = D(A)$. □

Proposition 5.2.2. *The hypotheses of 5) in the previous proposition are independent.*

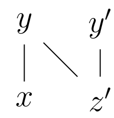
Proof. If we take a non-empty set X and consider the graph whose vertex set is X and its edges as $[X]^2$, it is clear that it does not satisfy b) but does satisfy the rest. Consider the following graph:

Clearly it is not connected and $Y \neq V(Y)$, therefore it does not satisfy a) but it does satisfy b). Its orbits are $\{x, z, x', z'\}$ and $\{y, y'\}$ and there is no edge whose vertices belong to the same orbit, so c) , d) and e) are satisfied. Consider the following graph:

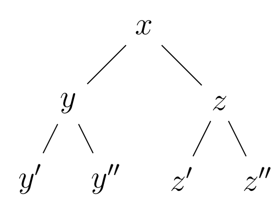
It is easy to see that it satisfies all the conditions of 5) except c).
Consider the following graph:



As it is a closed path, there is a unique orbit, therefore it satisfies e), it is clear that it is connected, that it is not the complete graph and since it has a unique orbit it also satisfies b). The condition c) is also clearly satisfied. Let's see that it does not satisfy d): Let's observe the following arrangement of edges:



Since there is a unique orbit, it is particularly true that x and y are in the same orbit and that y' and z' are in the same orbit, furthermore $y \neq z'$ and $x \neq y'$, so if d) were fulfilled we should have an isomorphism of graphs g such that $g.y = y'$, $g.x = z'$ and with $g^2 = Id$, but $\{y, z'\}$ is an edge, therefore $\{x, y'\}$ should also be an edge, but this is not the case. So d) is not satisfied.
Consider the following graph:



There are three orbits which are $\{x\}$, $\{y, z\}$, and $\{y', y'', z', z''\}$, therefore e) is not satisfied, but it is easy to verify that the rest are satisfied. □

The proposition 5.2.1 leads us to suspect that the natural generalization of the bipartition problem comes from the side of group actions. I believe the advantage of this approach is that it allows defining the problem in terms of more "canonical" mathematical concepts. Condition 5) of the proposition 5.2.1 has the advantage that each of its hypotheses is independent and introduces a parameter that is easy to modify, which is the number of orbits. Therefore, let's see what happens if we opt for this path.

Definition 5.2.3. Let λ and κ be infinite cardinals, and ρ a cardinal, $0 < \rho \leq \kappa$. We say that λ *acts expansively* on κ through ρ orbits. We denote this as $\lambda \overset{\rho}{\curvearrowright} \kappa$, if there exists a sequence of connected graphs $\{Y_i\}_{i<\lambda}$ with $V(Y_i) = \kappa$ for every $i < \lambda$ that verify the following properties:

1) For all $i < \lambda$, it holds that:

   (a) $|\mathcal{O}(\mathrm{AUT}(Y_i) \curvearrowright Y_i)| = \rho$.

   (b) For every $A \in \mathcal{O}(\mathrm{AUT}(Y_i) \curvearrowright Y_i)$, for every $x, y \in A$, if $\{x, y\} \in E(Y_i)$ then $A = V(Y_i)$.

   (c) For every $\{x, y\}, \{x', y'\} \in E(Y)$, if $(x \neq y') \wedge (x' \neq y)$ and $(|\mathcal{O}(x)| = |\mathcal{O}(x')| \leq |\mathcal{O}(y)| \leq |\mathcal{O}(y')|)$, then there exists $g \in \mathrm{AUT}(Y)$ such that

$$(g^2 = Id) \wedge (g.x = x') \wedge (g.y = y')$$

.

2) $[\kappa]^2 = \bigcup_{i<\lambda} Y_i$, and for every $j < \lambda$ it is true that $[\kappa]^2 \neq \bigcup_{i<\lambda/\{j\}} Y_i$.

Remark 5.2.4. If $\lambda \overset{\rho}{\curvearrowright} \kappa$ and $\{Y_i\}_{i<\lambda}$ is the sequence of connected graphs, then for every $i < \lambda$, Y_i satisfies the conditions of 5) from the proposition 5.2.1. Indeed, by 2) of Definition 5.2.3 we have that $\emptyset \neq Y_i \neq [V(Y_i)]^2 = [\kappa]^2$.

Remark 5.2.5. Given a graph Y, unless stated otherwise, when we talk about orbits we will refer to $\mathcal{O}(\mathrm{AUT}(Y) \curvearrowright Y)$ and we will simply denote it as $\mathcal{O}(Y)$.

Proposition 5.2.6. *Let $\lambda \overset{\rho}{\curvearrowright} \kappa$, and let $\{Y_i\}_{i<\lambda}$ be the sequence of graphs defined by the expansiveness condition. There exists a subset $J \subseteq \omega$ such that for every $i < \lambda$, there exists a strictly increasing sequence $\{|\mathcal{O}_j(Y_i)|\}_{j<J}$ satisfying $\mathcal{O}(Y_i) = \{\mathcal{O}_j(Y_i) : j < J\}$. In particular, $1 \leq \rho \leq \omega$.*

Proof. Let $i < \lambda$, and consider $\mathcal{O}_0(Y_i)$ as an orbit of minimal cardinality. If there exist two orbits, A and B, connected to $\mathcal{O}_0(Y_i)$, then by (1), (c), we have $A = B$. We denote this unique orbit as $\mathcal{O}_1(Y_i)$.

Now, let A be an orbit connected to $\mathcal{O}_1(Y_i)$ that is different from $\mathcal{O}_0(Y_i)$. If $|\mathcal{O}_0(Y_i)| = |\mathcal{O}_1(Y_i)|$, then by (1), (c), we would have $A = \mathcal{O}_0(Y_i)$, which is a contradiction. Therefore, it follows that

$$|\mathcal{O}_0(Y_i)| < |\mathcal{O}_1(Y_i)| \, .$$

Proceeding by induction, suppose that we have constructed a sequence

$$\{|\mathcal{O}_j(Y_i)|\}_{j\in\{0,\ldots,n\}}$$

such that

$$|\mathcal{O}_j(Y_i)| < |\mathcal{O}_{j+1}(Y_i)| \, ,$$

where $\mathcal{O}_j(Y_i)$ is connected to $\mathcal{O}_{j+1}(Y_i)$ for every $j \in \{0, \ldots, n\}$.

If there exists an orbit A connected to $\mathcal{O}_n(Y_i)$, then by (1), (c), we must have $|A| > |\mathcal{O}_n(Y_i)|$. We then define $\mathcal{O}_{n+1}(Y_i) := A$. In this way, we construct the desired sequence.

Indeed, for any orbit B, there exists some $j_0 \in J$ such that B is connected to $\mathcal{O}_{j_0}(Y_i)$. If J is finite and B is neither the first nor the last element, then by (1), (c), it follows that $B = \mathcal{O}_{j_0+1}(Y_i)$. The same reasoning applies if $J = \omega$ and B is not the first element.

□

Proposition 5.2.7. *Let λ, κ be infinite cardinals, and ρ a cardinal, with $2 \leq \rho \leq \kappa$. Then,*

$$\lambda \overset{\rho}{\curvearrowright} \kappa \Rightarrow \lambda \leq \kappa \leq \rho^\lambda$$

.

Proof. Let's see that $\lambda \leq \kappa$: By 2) of the definition 5.2.3 we know that for every $j < \lambda$ the set $M_j = \{v \in [\kappa]^2 : v \in Y_j \setminus \bigcup_{i<\lambda\setminus\{j\}} Y_i\}$ is non-empty. Endowing $[\kappa]^2$ with a good order, for example the lexicographic, we can define the function $f : \lambda \to [\kappa]^2$ as $f(j) := \min M_j$. f is injective, indeed, let $j, j' < \lambda$, if $j \neq j'$ then $f(j) \in Y_j \setminus Y_{j'}$ and $f(j') \in Y_{j'} \setminus Y_j$, therefore $f(j) \neq f(j')$. Then $\lambda \leq |[\kappa]^2| = \kappa$.
Let's see that $\kappa \leq \rho^\lambda$: We define the function $f : \kappa \to {}^\lambda\rho$ as follows: $f(x)(i) = j$, where j verifies that x belongs to the j-th orbit of Y_i. Let's see that f is injective, indeed, given $x \neq x'$, $\{x, x'\} \in [\kappa]^2 = \bigcup_{i<\lambda} Y_i$, then there exists i_0 such that $\{x, x'\} \in E(Y_{i_0})$, but by 1), b) of the definition 5.2.3 x and x' have to be in different orbits, therefore $f(x)(i_0) \neq f(x')(i_0)$. Then $\kappa \leq \left|{}^\lambda\rho\right| = \rho^\lambda$. □

Proposition 5.2.8. *Let λ and κ be infinite cardinals.*

$$\lambda \overset{1}{\curvearrowright} \kappa \Leftrightarrow \lambda \leq \kappa \leq 2^\lambda$$

Proof. $\lambda \leq \kappa$: By 2) of the definition 5.2.3 we know that for every $j < \lambda$ the set $M_j = \{v \in [\kappa]^2 : v \in Y_j \setminus \bigcup_{i<\lambda\setminus\{j\}} Y_i\}$ is non-empty. Endowing $[\kappa]^2$ with a good order, for example the lexicographic, we can define the function $f : \lambda \to [\kappa]^2$ as $f(j) := \min M_j$. f is injective, indeed, let $j, j' < \lambda$, if $j \neq j'$ then $f(j) \in Y_j \setminus Y_{j'}$ and $f(j') \in Y_{j'} \setminus Y_j$, therefore $f(j) \neq f(j')$.
$\kappa \leq 2^\lambda$: We know that $Y_i = D(A_i)$ (Proposition 5.2.1, for every $i < \lambda$. We define a function $f : \kappa \to {}^\lambda 2$ as:

$$f(x)(i) = \begin{cases} 1 & \text{if } x \in A_i \\ 0 & \text{if } x \notin A_i \end{cases}$$

f is injective, indeed, given $x \neq x'$, $\{x, x'\} \in [\kappa]^2 = \bigcup_{i<\lambda} Y_i$, then there exists i_0 such that $\{x, x'\} \in E(Y_{i_0})$, but since $Y_{i_0} = D(A_{i_0})$, we have that either $x \in A_{i_0}$ and $x' \in A^c_{i_0}$, or $x \in A^c_{i_0}$ and $x' \in A_{i_0}$. In any case, we have that $f(x)(i_0) \neq f(x')(i_0)$. Then $|\kappa| \leq \left|{}^\lambda 2\right| = 2^\lambda$.
Reciprocal: Let κ be such that $\lambda \leq \kappa \leq 2^\lambda$. For each $i \in \lambda$, we define the set $X_i = \{h \in {}^\lambda 2 : h(i) = 1\}$, and the functions $e_i \in {}^\lambda 2$ as follows:

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

For each $i < \lambda$, we take a set $K_i \in X_i$, and a set $K'_i \in X^c_i$, such that $|K_i| = |K'_i| = \kappa$. We consider the set $K := (\bigcup_{i<\lambda} K_i \cup K'_i \cup \{e_i\}) \cup \{nul\}$, where $nul \in {}^\lambda 2$ is the null function. It is clear that $|K| = \kappa$, we identify κ with K.
For every $i < \lambda$, we define $A_i := K \cap X_i$, $A^c_i := K \cap X^c_i$, and $Y_i := D(A_i)$. By construction, we have that $|A_i| = |A^c_i| = \kappa$, therefore $|\mathcal{O}(\mathrm{AUT}(Y_i) \curvearrowright Y_i)| = 1$ for every $i < \lambda$. Let $\mathcal{B} = \{Y_i\}_{i<\lambda}$. Let's see

that this sequence satisfies the definition of expansiveness: By the proposition 5.2.1 the only thing we need to prove is that it satisfies property 2) of 5.2.3, that is, $[K]^2 = \bigcup_{i<\lambda} Y_i$, and that for every $j < \lambda$, $[K]^2 \neq \bigcup_{i<\lambda\setminus\{j\}} Y_i$. Let f and g in K be different functions, then there exists $i_0 < \lambda$ such that $f(i_0) \neq g(i_0)$ either $f \in X_{i_0}$ and $g \in X_{i_0}^c$, or $f \in X_{i_0}^c$ and $g \in X_{i_0}$, then either $f \in A_{i_0}$ and $g \in A_{i_0}^c$ or $f \in A_{i_0}^c$ and $g \in A_{i_0}$. Therefore $\{f, g\} \in E(Y_{i_0})$, then $[K]^2 = \bigcup_{i<\lambda} Y_i$. For every $j < \lambda$, $\{e_j, nul\} \in [K]^2$, but $nul \in A_i^c$ and $e_j \notin A_i$ for every $i \neq j$, therefore $\{e_j, nul\} \notin \bigcup_{i<\lambda\setminus\{j\}} Y_i$. Then for every $j < \lambda$, $[K]^2 \neq \bigcup_{i<\lambda\setminus\{j\}} Y_i$. □

Remark 5.2.9. From the previous proposition, we trivially have the following equivalence:

$$\neg\mathrm{GCH}(\lambda) \Leftrightarrow \exists\kappa((\kappa \neq \lambda, 2^\lambda) \wedge (\lambda \overset{1}{\curvearrowright} \kappa))$$

.

The previous observation shows us that the negation of the Generalized Continuum Hypothesis, and therefore the Generalized Continuum Hypothesis itself, can be defined in terms of expansive actions by 1 orbit. However, from the previous propositions, it is deduced that if λ and κ are infinite cardinals, and ρ is a cardinal such that $0 < \rho \leq \omega$, then:

$$\lambda \overset{\rho}{\curvearrowright} \kappa \Rightarrow \lambda \leq \kappa \leq 2^\lambda$$

. This suggests the following question: What happens if we consider expansive actions by more orbits?

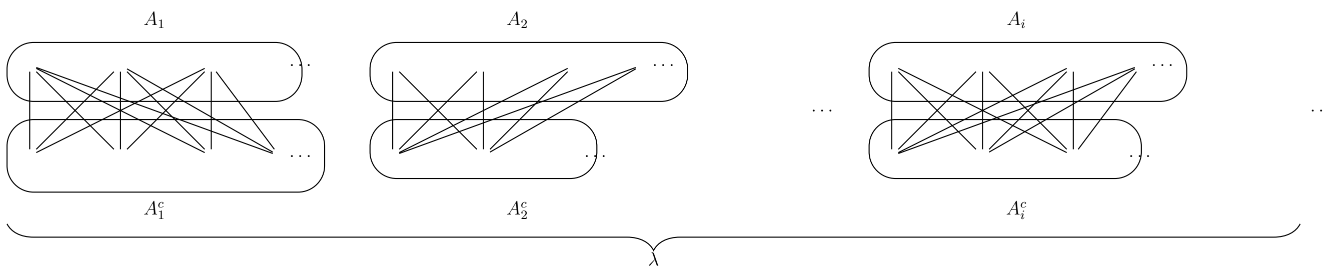
Definition 5.2.10. Let λ be an infinite cardinal and ρ a cardinal such that $0 < \rho \leq \omega$. We define the dual Generalized Continuum Hypothesis as

$$\mathrm{DGCH}_\rho(\lambda) := \neg\exists\kappa((\kappa \neq \lambda, 2^\lambda) \wedge (\lambda \overset{\rho}{\curvearrowright} \kappa))$$

Remark 5.2.11. For every infinite cardinal λ and for every cardinal ρ with $0 < \rho \leq \omega$, it is verified that $\mathrm{GCH}(\lambda) \Rightarrow \mathrm{DGCH}_\rho(\lambda)$.

A natural question is whether the converse of the previous observation holds. The following proposition shows that, in the case $\rho = 2$, the answer is negative. For this case, we have a set of graphs with "entropy" greater than zero.

Graphs with entropy greater than zero



To do this, let's first prove a lemma.

Lemma 5.2.12. *Let κ and λ be infinite cardinals, if $cof(\kappa) \leq \lambda < \kappa$, then there exists a sequence of pairwise disjoint sets $\{A_i\}_{i<\lambda}$ such that $\kappa = \bigcup_{i<\lambda} A_i$ and for every $j < \lambda$, $\left|\bigcup_{i<j} A_i\right| < \kappa$.*

Proof. We know there exists a sequence $\{B_i\}_{i<cof(\kappa)}$ of pairwise disjoint sets such that $\kappa = \bigcup_{i<cof(\kappa)} B_i$, and for every $j < cof(\kappa)$, $\left|\bigcup_{i<j} B_i\right| < \kappa$. Let δ be the ordinal defined by $\delta = \lambda + cof(\kappa)$. Let $\{x_i\}_{i<\lambda}$ be a sequence such that $x_i \notin \kappa$, for every $i < \lambda$. We define $\{A_i\}_{i<\delta}$ as follows:

$$A_i := \begin{cases} \{x_i\} \ if \ i < \lambda \\ B_j \ if \ i = \lambda + j \end{cases}$$

Then $K = \bigcup_{i<\delta} A_i$, verifies that $\{A_i\}_{i<\delta}$ is a sequence of pairwise disjoint sets, $|K| = \kappa$, $|\delta| = \lambda$, and for every $j < \lambda$, $\left|\bigcup_{i<j} A_i\right| < \kappa$. □

Theorem 5.2.13. *Let λ be an infinite cardinal. Then,*

$$\neg \mathrm{DGCH}_2(\lambda) \Leftrightarrow \exists \kappa (\lambda < \kappa < 2^\lambda) \wedge (cof(\kappa) \leq \lambda)$$

Proof. Direct: Suppose there exists κ with $\lambda < \kappa < 2^\lambda$ and $\lambda \overset{2}{\curvearrowright} \kappa$. Then (Proposition 5.2.1 4)) there exists $\{Y_i\}_{i<\lambda}$ a sequence of connected graphs with $V(Y_i) = \kappa$, $Y_i = D(A_i)$. This implies (Proposition 5.2.1) that $|A_i| \neq |A_i^c|$. Then $|A_i| < \kappa$ for every $i < \lambda$. Moreover, $[\kappa]^2 = \bigcup_{i<\lambda} Y_i$. Suppose there exists an $x_0 \in \kappa$ with $x_0 \notin \bigcup_{i<\lambda} A_i$. For every $y \neq x_0$ it is the case that $\{x_0, y\} \in [\kappa]^2 = \bigcup_{i<\lambda} Y_i$, then there exists an $i_0 < \lambda$ such that $\{x_0, y\} \in Y_{i_0}$, but $x_0 \in A_{i_0}^c$, then $y \in A_{i_0}$. Then either $\kappa \setminus \{x_0\} = \bigcup_{i<\lambda} A_i$ or $\kappa = \bigcup_{i<\lambda} A_i$. In either case, we can conclude that $cof(\kappa) \leq \lambda$.

Reciprocal: Let κ, with $cof(\kappa) \leq \lambda < \kappa < 2^\lambda$. For each $i < \lambda$ consider the set $B_i := \{x \in {}^\lambda 2 : \forall j < i, x(j) = 0, x(i) = 1\}$, observe that $|B_i| = 2^\lambda$. By Lemma 5.2.12 we can take a sequence $\{A_i\}_{i<\lambda}$ such that $\left|\bigcup_{i<\lambda} A_i\right| = \kappa$ and for every $j < \lambda$, $\left|\bigcup_{i<j} A_i\right| < \kappa$. Without loss of generality, we can assume that for every $i < \lambda$, $A_i \subset B_i$ and $e_i \in A_i$ where

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for every $i < \lambda$.
To $K = \bigcup_{i<\lambda} A_i$ we can write it as $K = (K \cap X_i) \cup (K \cap X_i^c)$, where $X_i := \{h \in {}^{\lambda} : h(i) = 1\}$ for every $i < \lambda$. But $A_i \subset K \cap X_i \subset \bigcup_{j \leq i} A_j$, moreover, by Lemma 5.2.12 we know that $\left|\bigcup_{j\leq i} A_j\right| < \kappa = |K|$. Consider $Y_i := D(K \cap X_i)$, we already saw that for this case it is verified that $|\mathcal{O}(\mathrm{AUT}(Y_i) \curvearrowright Y_i)| = 2$, for every $i < \lambda$. And we have that $[K]^2 = \bigcup_{i<\lambda} Y_i$, moreover, we can assume without loss of generality that the null function belongs to K, with which we have that $\{e_j, null\} \in Y_j \setminus \bigcup_{i\neq j} Y_i$. Then $\lambda \overset{2}{\curvearrowright} \kappa$. □

Remark 5.2.14. Observe that in the previous proof, if κ is an infinite cardinal such that $\lambda < \kappa < 2^\lambda$, with $\mathrm{cof}(\kappa) \leq \lambda$, it is possible to take the sequence $\{Y_i\}_{i<\lambda}$, with $Y_i = D(A_i)$ for all $i < \lambda$ defined by the action in such a way that $\bigcup_{i<\lambda} A_i^c = \kappa$.

Proposition 5.2.15. *Let λ and κ be infinite cardinals such that $\lambda \overset{\rho}{\curvearrowright} \kappa$. Then $cof(\kappa) \leq \lambda$.*

Proof. Let $\{Y_i\}_{i<\lambda}$ be the sequence of graphs defined by the action. Suppose there exists $j_0 < \rho$ such that for some $i < \lambda$, $\kappa = |\mathcal{O}_{j_0}(Y_i)|$. Then for every $i < \lambda$ there exists $i_{\max} < \rho$ such that $k = |\mathcal{O}_{i_{\max}}(Y_i)|$. If there exist $x, y \in \kappa$, distinct, such that for every $i < \lambda$, $x, y \in \mathcal{O}_{i_{\max}}(Y_i)$, then $\{x, y\} \notin E(Y_i)$ for every $i < \lambda$. Therefore, $[\kappa]^2 \neq \bigcup_{i<\lambda} E(Y_i)$, absurd. Thus, there can be at most one point from κ that does not belong to $\bigcup_{i<\lambda} \bigcup_{j\neq i_{\max}} \mathcal{O}_j(Y_i)$. Therefore, $\kappa = |\bigcup_{i<\lambda} \bigcup_{j\neq i_{\max}} \mathcal{O}_j(Y_i)|$. In which case $\mathrm{cof}(\kappa) \leq \lambda$. Conversely, if there is no $j_0 < \rho$ such that for some $i < \lambda$, there exists an $i_{\max} < \rho$ such that $k = |\mathcal{O}_{i_{\max}}(Y_i)|$. Then for every $i < \lambda$, in particular for $i = 1$, $\kappa = \bigcup_{j<\rho} \mathcal{O}_j(Y_1)$, with $|\mathcal{O}_j(Y_1)| < \kappa$ for every $j < \rho$. Hence, we can assert that in any case $\mathrm{cof}(\kappa) \leq \lambda$. □

Proposition 5.2.16. *For every infinite cardinal λ and for every $\rho \geq 2$, it holds that*

$$\mathrm{DGCH}_2(\lambda) \Rightarrow \mathrm{DGCH}_\rho(\lambda)$$

Proof. If $\neg\mathrm{DGCH}_\rho(\lambda)$ holds, then there exists a cardinal κ such that $\lambda < \kappa < 2^\lambda$, with $\lambda \overset{\rho}{\curvearrowright} \kappa$. Then by Proposition 5.2.15 we have that $\mathrm{cof}(\kappa) \leq \lambda$, and by Proposition 5.2.13 we can conclude that $\neg\mathrm{DGCH}_2(\lambda)$ holds. □

Remark 5.2.17 (Consistency of $\neg\mathrm{DGCH}_2(\lambda)$)**.** Let λ be an infinite cardinal.

- **Regular $\lambda = \aleph_\alpha$.** In the extension given by Theorem 2.2.15 (see Subsec. 2.2.2) with

$$2^{\aleph_\alpha} = \aleph_{\alpha+\omega+1},$$

setting

$$\kappa := \aleph_{\alpha+\omega}$$

yields $\lambda < \kappa < 2^\lambda$ and $\mathrm{cf}(\kappa) = \omega \leq \lambda$. By Theorem 5.2.13 (the characterization for $\rho = 2$), this witnesses $\neg\mathrm{DGCH}_2(\lambda)$. (For the implementation of the iteration, see [12].)

- **Singular $\lambda = \aleph_\alpha$.** By the lightweight route (Remark 2.2.17), force only below λ so that,

$$2^{\aleph_\alpha} = \aleph_{\alpha+\Theta+1},$$

where $\Theta = \mathrm{cf}(\alpha)^\omega.\lambda$. Choose β with $\alpha < \beta < \alpha + \Theta$ and $\mathrm{cf}(\beta) = \omega$, and set $\kappa := \aleph_\beta$. Then $\lambda < \kappa < 2^\lambda$ and $\mathrm{cf}(\kappa) = \omega \leq \lambda$, so by Theorem 5.2.13 we conclude $\neg\mathrm{DGCH}_2(\lambda)$.

5.2.2 Consistency of $\neg\mathrm{DGCH}_\rho$ for $\rho > 3$

Definition 5.2.18. Let A and B be sets, we define $A \circledast B := \{\{x, y\} \in [A \cup B]^2 : (x, y) \in A \times B\}$.

Remark 5.2.19. $Y = D(A)$ if only if $Y = A \circledast A^c$.

Definition 5.2.20. Let Y be a graph such that $|\mathcal{O}(Y)| = \rho$, and let A be a set. With $\{\mathcal{O}_j(Y)\}_{j<\rho}$ strictly increasing. For each $j < \rho$, we define $Y \circledast^j A := Y \cup (\mathcal{O}_j(Y) \circledast A)$.

Lemma 5.2.21 (Chain of "limits of limits" under Easton)**.** *Assume the GCH holds. Let $\lambda = \aleph_\alpha$ be a regular cardinal. In the model of Subsec. 2.2.2 with*

$$2^{\aleph_\alpha} = \aleph_{\alpha+\omega^\omega.\lambda+1},$$

define

$$\kappa := \aleph_{\alpha+\omega^\omega.\lambda}.$$

Then there exists a strictly increasing sequence $\{k_i\}_{i<\lambda}$ of cardinals with

$$\lambda < k_0 < k_1 < \cdots < \sup_{i<\lambda} k_i = \kappa < 2^\lambda,$$

such that for each $i < \lambda$:

1) *k_i is a* limit *cardinal with $\mathrm{cf}(k_i) = \omega$;*

2) *there exists a strictly increasing sequence $\{k_{ij}\}_{j<\omega}$ of* limit cardinals *with $\lambda < k_{ij} < k_i$ and $\sup_{j<\omega} k_{ij} = k_i$.*

In particular, κ witnesses $\neg\mathrm{DGCH}_2(\lambda)$, and the family $\{k_i, k_{ij}\}$ allows one to keep unchanged the proof steps that require "limits of limits".

Proof. We work with the function $\beta \mapsto \aleph_{\alpha+\beta}$. For $i < \lambda$ set

$$k_i := \aleph_{\alpha+\ \omega^\omega\cdot(i+1)}.$$

Since $\omega^\omega\cdot(i+1) = \omega^\omega\cdot i + \omega^\omega$ is a *limit* ordinal with $\mathrm{cf}(\,\omega^\omega\cdot(i+1)\,) = \omega$, each k_i is a *limit* cardinal with $\mathrm{cf}(k_i) = \omega$. Also $k_i < \aleph_{\alpha+\omega^\omega\cdot\lambda} = \kappa$ and

$$\sup_{i<\lambda} k_i = \aleph_{\alpha+\ \sup_{i<\lambda}\omega^\omega\cdot(i+1)} = \aleph_{\alpha+\ \omega^\omega\cdot\lambda} = \kappa,$$

since $\sup_{i<\lambda}\big(\omega^\omega\cdot(i+1)\big) = \omega^\omega\cdot\lambda$.

For each i define, for $j < \omega$,

$$k_{ij} := \aleph_{\alpha+\ \omega^\omega\cdot i+\ \omega^{j+1}}.$$

Because ω^{j+1} is limit with cofinality ω, each k_{ij} is a *limit cardinal*; and

$$\sup_{j<\omega} k_{ij} = \aleph_{\alpha+\ \omega^\omega\cdot i+\ \sup_{j<\omega}\omega^{j+1}} = \aleph_{\alpha+\ \omega^\omega\cdot i+\ \omega^\omega} = \aleph_{\alpha+\ \omega^\omega\cdot(i+1)} = k_i.$$

Preservation of cardinals and cofinalities follows from Easton support iteration (Subsec. 2.2.2). □

Lemma 5.2.22 (Chain of "limits of limits" for singular λ)**.** *Assume the GCH holds. Let $\lambda = \aleph_\alpha$ be a* singular *cardinal and $\mu = \mathrm{cf}(\lambda)$. Let $\Theta = \mu^\omega.\lambda$. Then by adding $\aleph_{\alpha+\Theta+1}$ many Cohen subset of ω, we get a model where:*

1) $2^{\aleph_\alpha} = \aleph_{\alpha+\Theta+1}$ *for some limit ordinal Θ (e.g. $\Theta = \mu^\omega\cdot\lambda$);*

2) *if we define $\kappa := \aleph_{\alpha+\Theta}$, there exists a strictly increasing sequence $\{k_i\}_{i<\lambda}$ of cardinals with*

$$\lambda < k_0 < k_1 < \cdots < \sup_{i<\lambda} k_i = \kappa < 2^\lambda;$$

3) *for each $i < \lambda$, k_i is* limit *with $\mathrm{cf}(k_i) = \omega$, and there is also a strictly increasing*

$$\{k_{ij}\}_{j<\omega} \subset (\lambda, k_i)$$

sequence of limit cardinals *with $\mathrm{cf}(k_{ij}) = \omega$ such that $\sup_{j<\omega} k_{ij} = k_i$.*

In particular, $\lambda < \kappa < 2^\lambda$ and $\mathrm{cf}(\kappa) = \omega \le \lambda$, so (by Theorem 5.2.13) we obtain $\neg\mathrm{DGCH}_2(\lambda)$, and the family $\{k_i, k_{ij}\}$ allows one to reuse unchanged the steps that require "limits of limits".

Proof. Choose β with $\alpha < \beta < \alpha + \Theta$ and $\mathrm{cf}(\beta) = \omega$, and set $\kappa := \aleph_\beta$. Fix strictly increasing sequences $\{\beta_i\}_{i<\lambda} \uparrow \beta$ and, for each i, $\{\beta_{i,j}\}_{j<\omega} \uparrow \beta_i$, all of cofinality ω, and define $k_i := \aleph_{\beta_i}$ and

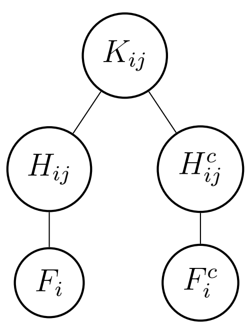
$k_{i,j} := \aleph_{\beta_{i,j}}$. Then $\lambda < k_{i,j} < k_i < \kappa < 2^\lambda$, each k_i and $k_{i,j}$ is limit of cofinality ω, $\sup_{j<\omega} k_{i,j} = k_i$, and $\sup_{i<\lambda} k_i = \kappa$. □

Theorem 5.2.23. *For every infinite cardinal λ it is consistent that $\neg\mathrm{DGCH}_3(\lambda)$ holds.*

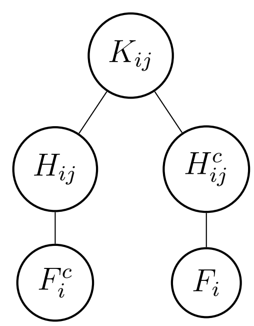
Proof. Fix an infinite cardinal λ. By Lemma 5.2.21 (regular λ) and Lemma 5.2.22 (singular λ), it is consistent that there exist a cardinal κ and families $\{k_i\}_{i<\lambda}$, $\{K_i\}_{i<\lambda}$, and, for each $i < \lambda$, $\{k_{ij}\}_{j<\omega}$ such that:

- $\lambda < \kappa < 2^\lambda$,
- $\{k_i\}_{i<\lambda}$ is strictly increasing with $\lambda < k_i < 2^\lambda$ for all $i < \lambda$, and $\kappa = \bigcup_{i<\lambda} k_i$,
- $\{K_i\}_{i<\lambda}$ are pairwise disjoint with $\kappa = \bigcup_{i<\lambda} K_i$ and $|K_i| = k_i$ for all $i < \lambda$,
- for each $i < \lambda$, $k_i = \bigcup_{j<\omega} k_{ij}$ where $\{k_{ij}\}_{j<\omega}$ is strictly increasing (of limit cardinals) and $\sup_{j<\omega} k_{ij} = k_i$.

As $\mathrm{cof}(\kappa_i) \leq \lambda$ and $\lambda < \kappa_i < 2^\lambda$, by the proposition and the remark, we know there exists a sequence of partitions $\{K_{ij}, K_{ij}^c\}_{j<\lambda}$ of K_i, such that $|K_{ij}| < |K_{ij}^c| = \kappa_i$, and $\kappa_i = \bigcup_{j<\lambda} K_{ij}^c$. Such that the sequence of graphs $\{D(K_{ij})\}_{j<\lambda}$ is the sequence in $\lambda \overset{2}{\curvearrowright} \kappa_i$. Let us define the following sets: for all $i < \omega$, and $j < \lambda$, H_{ij} such that $\{H_{ij}, H_{ij}^c\}$ is a partition of K_{ij}^c, with $|H_{ij}| = |H_{ij}^c| = |K_{ij}^c$. And for all $j < \lambda$. For every $i < \omega$, we take $\{F_i, F_i^c\}$ a partition of $\bigcup_{l\neq i} \kappa_l$, such that $|F_i| = |F_i^c| = \bigcup_{l\neq i} \kappa_l$. Consider the sequence of graphs $Y_{ij} := (K_{ij} \circledast K_{ij}^c) \cup (H_{ij} \circledast F_i) \cup (H_{ij}^c \circledast F_i^c)$. $Y'_{ij} := (K_{ij} \circledast K_{ij}^c) \cup (H_{ij} \circledast F_i^c) \cup (H_{ij}^c \circledast F_i)$.



Y_{ij}



Y'_{ij}

Observe that the orbits of Y_{ij} and Y'_{ij} are K_{ij}, K_{ij}^c, and $\bigcup_{l\neq i} K_l$. Similarly, it is easy to see that $[\kappa]^2 = (\bigcup_{(i,j)\in\omega\times\lambda} Y_{ij}) \cup (\bigcup_{(i,j)\in\omega\times\lambda} Y'_{ij})$. If we reindex the previous sequences, we have that $[\kappa]^2 = \bigcup_{i<\lambda} Y''_i$. It is easy to verify that all properties of the definition of expansiveness are met, except, a priori, the one that says for every $j \neq i$, $[\kappa]^2 \neq \bigcup_{i\neq j} Y''_i$. This is because there may exist many superfluous graphs. However, if we remove the graphs that are not necessary, we can be sure that the resulting set always has cardinality λ. Indeed, for every $i < \lambda$, particularly, for example for $i = 1$, Y_{1j} contains as a subgraph $K_{1j} \circledast K_{1j}^c$, therefore it is not possible to remain with a subset in $\{Y''_i : i < \lambda\}$ of cardinality less than the set $\{K_{1j} \circledast K_{1j}^c : j < \lambda\}$. □

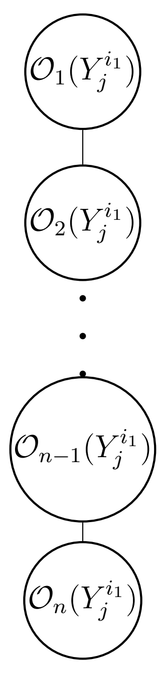
Theorem 5.2.24. *For every infinite cardinal λ, it is consistent that $\neg\mathrm{DGCH}_n(\lambda)$ holds for every n, $3 < n < \omega$.*

Proof. Fox λ. Let $\varphi(n)$ be the property stating that there exists a sequence $\{\kappa_{i_1,\ldots,i_{n-1}} : i_1, \ldots, i_n \leq \omega\}$ that verifies the following properties:

1) For every $h = 2, \ldots, n-1$, it holds that $\kappa_{i_1,\ldots,i_{h-1},\omega,\ldots,\omega} = \bigcup_{i_h<\omega} \kappa_{i_1,\ldots,i_{h-1},i_h,\ldots,\omega}$, and the sequence $\{\kappa_{i_1,\ldots,i_{h-1},i_h,\ldots,\omega}\}_{i_h<\omega}$ is strictly increasing.

2) $\lambda < \kappa_{i_1,\ldots,i_{n-2},\omega} < 2^\lambda$.

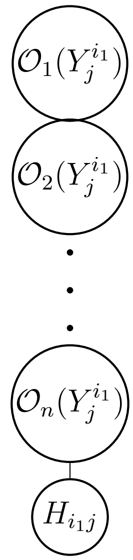
By Lemma 5.2.21 (regular λ case) and Remark 2.2.17 (singular λ case), together with the Easton preparation in Subsec. 2.2.2, we can fix in a forcing extension a *family* $\{k_{i_1,\ldots,i_{n-1}}\}$ indexed exactly as in (1)–(2), where each $k_{i_1,\ldots,i_h}$ is a *limit cardinal of cofinality* ω and the unions in (1) hold. We choose representatives as *disjoint* sets for these cardinals so that the unions in (1) are disjoint unions. Then, for $\kappa_{\omega,\ldots,\omega} = \bigcup_{i_1<\omega} \kappa_{i_1,\omega,\ldots,\omega}$, it holds that $\lambda \overset{n}{\curvearrowright} \kappa_{\omega,\ldots,\omega}$. And if $\{Y_i\}_{i<\lambda}$ is the sequence of graphs from the definition of expansiveness, it is verified that $\kappa = \bigcup_{i<\lambda} \mathcal{O}_n(Y_i)$.
We will prove by induction that $\varphi(n)$ holds for all $n \geq 3$. In Theorem 5.2.23, we proved that $\varphi(3)$ holds. Suppose it holds for n and let's see that it holds for $n+1$. Take a sequence $\{k_{i_1,\ldots,i_n} : i_1, \ldots, i_n \leq \omega\}$ that verifies the previous properties. By induction hypothesis, we have that $\lambda \overset{n}{\curvearrowright} \kappa_{i_1,\omega,\ldots,\omega}$ for every $i_1 < \omega$. And if for $i_1 < \omega$, $\{Y_j^{i_1}\}_{j<\lambda}$ are the graphs that define the action, we have that $\kappa_{i_1,\omega,\ldots,\omega} = \bigcup_{j<\lambda} \mathcal{O}_n(Y_j^{i_1})$ for every $i_1 < \omega$.



Graph $Y_j^{i_1}$

For every $i_1 < \omega$, we define $H_{i_1} := \bigcup_{l \neq i_1} \kappa_{l,\omega,\ldots,\omega}$.

For every $j < \lambda$ and for every $i_1 < \omega$, we define $Z_{i_1 j} := Y_j^{i_1} \circledast^n H_{i_1}$.



Graph $Z_{i_1 j}$

Consider the sequence $\{Z_{i_1 j} : i_1 < \omega, j < \lambda\}$. It is easy to see that we can take a subset that satisfies the definition of expansiveness such that $\lambda \overset{n+1}{\curvearrowright} \kappa_{\omega,\omega,\ldots,\omega}$. $\square$

Theorem 5.2.25. *Let λ be an infinite cardinal. Then,*

$$\neg DGCH_\omega(\lambda) \text{ is consistent.}$$

Proof. Consider a model of ZFC such that for every $n \geq 3$ there exist cardinals κ_n of the type constructed in Lemma 5.2.21 and Lemma 5.2.22, with $\lambda < \kappa_n < 2^\lambda$ for all $n \geq 3$. For instance, consider a model where, if $\lambda = \aleph_\alpha$, then $2^\lambda = \aleph_{\alpha+\omega^\omega+1}$.

For each $n \geq 3$, let $\{Y_i^n\}_{i<\lambda}$ be the sequence defined by expansiveness such that $\lambda \overset{n}{\curvearrowright} \kappa_n$. By the construction given in Theorem 5.2.24, we can define, for every $i < \lambda$, an embedding (injective graph homomorphism) $g^i_{n,n+1} : Y_i^n \to Y_i^{n+1}$ for all $n \geq 3$. This embedding $g^i_{n,n+1}$ maps an indispensable element of Y_i^n to an indispensable element of Y_i^{n+1}.

Now, for each $i < \lambda$, we define graph homomorphisms $f^i_{n,m} : Y_i^n \to Y_i^m$, for all $3 \leq n \leq m$, as follows:

$$f^i_{n,m} = \begin{cases} g_{m-1,m} \circ g_{m-2,m-1} \circ \ldots \circ g_{n,n+1}, & \text{if } n \neq m, \\ \text{the identity}, & \text{if } n = m. \end{cases}$$

Thus, it is clear that for every $i < \lambda$, the collection $\{(Y_i^n, f^i_{n,m}) : n, m \in I\}$, where $I = \{n \in \mathbb{N} : n \geq 3\}$, forms a direct system.

Define:

$$Y_i = \varinjlim_{n \in I} Y_i^n.$$

Finally, let $\kappa := \sup\{\kappa_n : n \geq 3\}$, and consider the collection $\{Y_i\}_{i<\lambda}$. It is not difficult to show that $\lambda \overset{\omega}{\curvearrowright} \kappa$. □

Proposition 5.2.26. *Let λ be an infinite cardinal such that* GCH(λ') *holds for all $\lambda' < \lambda$. Then, for all $i < j \leq \omega$,*

$$\mathrm{DGCH}(\lambda)_i \Rightarrow \mathrm{DGCH}(\lambda)_j.$$

Proof. Let us assume that $\neg\mathrm{DGCH}_{n+1}(\lambda)$, with $n \geq 2$. Then, there exists κ such that $\lambda < \kappa < 2^\lambda$, where $\lambda \overset{n+1}{\curvearrowright} \kappa$. In Proposition 5.2.6, we observed that if $\lambda \overset{\rho}{\curvearrowright} \kappa$, then there exists a subset $J \subseteq \omega$ such that for every $i < \lambda$, there exists a strictly increasing sequence $\{|\mathcal{O}_j(Y_i)|\}_{j<J}$ satisfying

$$\bigcup_{j<J} \mathcal{O}_j(Y_i) = V(Y_i).$$

Moreover, the set $\{\mathcal{O}_j(Y_i) : j < J\}$ has order type ω, with $\mathcal{O}_j(Y_i) < \mathcal{O}_{j'}(Y_i)$ if $|\mathcal{O}_j(Y_i)| < |\mathcal{O}_{j'}(Y_i)|$.

For every $i < \lambda$ and $j \in \{1, \ldots, n\}$, we define the graph $Y_{i,j}$ as follows:

$$V(Y_{i,j}) := V(Y_i),$$

and

$$\begin{aligned} E(Y_{i,j}) := &\Big(E(Y_i) \setminus \big\{\{x,y\} : x \in \mathcal{O}_j(Y_i),\ y \in \mathcal{O}_{j+1}(Y_i)\big\}\Big) \\ &\cup\ \Big(\big(\mathcal{O}_{j-1}(Y_i) \circledast \mathcal{O}_j(Y_i)\big) \cup \mathcal{O}_j(Y_i)\Big) \\ &\cup\ \big(\mathcal{O}_j(Y_i) \circledast \mathcal{O}_{j+2}(Y_i)\big) \end{aligned}$$

if $1 < j < n-1$,

$$E(Y_{i,1}) := \big(E(Y_i) \setminus \{\{x,y\} : x \in \mathcal{O}_1(Y_i), y \in \mathcal{O}_2(Y_i)\}\big) \cup \big((\mathcal{O}_1(Y_i) \cup \mathcal{O}_2(Y_i)) \circledast \mathcal{O}_3(Y_i)\big),$$

for $j = 1$, and

$$E(Y_{i,n-1}) := \big(E(Y_i) \setminus \{\{x,y\} : x \in \mathcal{O}_{n-1}(Y_i), y \in \mathcal{O}_n(Y_i)\}\big) \cup \big(\mathcal{O}_{n-2}(Y_i) \circledast (\mathcal{O}_{n-1}(Y_i) \cup \mathcal{O}_n(Y_i))\big),$$

for $j = n-1$.

It is straightforward to verify that the set $\{Y_{i,j} : i < \lambda, j < J\}$ satisfies the properties of the definition of expansiveness. Moreover, $|\mathcal{O}(AUT(Y_{i,j}) \curvearrowright Y_{i,j})| = n$. However, during the process, we may have introduced many superfluous graphs, meaning property 2) of Definition 5.2.3 might not hold. To address this, we define the following recursive function.

Let $f : \lambda \to V$ such that:

$$f(0) := Y',$$

where $Y' = \{Y'_i\}_{1 \le i < \lambda}$ is an ordering of the set $\{Y_{i,j} : i < \lambda, j < J\}$.

If $\alpha = \beta + 1$, we define

$$f(\beta+1) = \begin{cases} f(\beta) \setminus \{Y_{\beta+1}\} & \text{if } E(Y_{\beta+1}) \subset \bigcup\{E(Y) : Y \in f(\beta)\}, \\ f(\beta) & \text{otherwise.} \end{cases}$$

If α is a limit ordinal, then

$$f(\alpha) = \bigcup_{\beta<\alpha} f(\beta).$$

Finally, $|f(\lambda)| \overset{n}{\curvearrowright} \kappa$, and it follows that $|f(\lambda)| \le \lambda < \kappa \le 2^{f(\lambda)} < 2^\lambda$. Hence, if $|f(\lambda)| < \lambda$, we would have $\neg\mathrm{GCH}(|f(\lambda)|)$, which contradicts $|f(\lambda)| < \lambda$. Therefore, we conclude that $|f(\lambda)| = \lambda$, and hence $\lambda \overset{n}{\curvearrowright} \kappa$, implying $\neg\mathrm{DGCH}_n(\lambda)$.

For the case $\neg\mathrm{DGCH}_\omega(\lambda)$, assume that there exists κ such that $\lambda < \kappa < 2^\lambda$, with $\lambda \overset{\omega}{\curvearrowright} \kappa$. For every $i < \lambda$, consider $\{\mathcal{O}_j(Y_i) : j < \omega\}$. Let $n \in \omega$, and define the following graphs:

$$V(Y_{i,j}) = V(Y_i),$$

and

$$\begin{aligned} E(Y_{i,j}) := \big(E(Y_i) \setminus \big[\big(\bigcup_{l<j}\{\{x,y\} : x \in \mathcal{O}_l(Y_i), y \in \mathcal{O}_{l+1}(Y_i)\}\big) \\ \cup \big(\bigcup_{l \ge n-1} \{\{x,y\} : x \in \mathcal{O}_l(Y_i), y \in \mathcal{O}_{l+1}(Y_i)\}\big)\big]\big) \\ \cup \big(\bigcup_{l<j}\{\{x,y\} : x \in \mathcal{O}_l(Y_i), y \in \mathcal{O}_{l+1}(Y_i)\}\big) \circledast \mathcal{O}_j(Y_i) \\ \cup\, \mathcal{O}_{n-1}(Y_i) \circledast \big(\bigcup_{l<j}\{\{x,y\} : x \in \mathcal{O}_l(Y_i), y \in \mathcal{O}_{l+1}(Y_i)\}\big). \end{aligned}$$

Using the same reasoning as before, we conclude that $\lambda \overset{n}{\curvearrowright} \kappa$, and hence $\neg\mathrm{DGCH}_n(\lambda)$. □

Chapter 6

Metric-Independent Expansiveness

The theory of expansive actions on compact metric spaces establishes that expansivity is a property intrinsic to the topology of the space, being independent of the chosen compatible metric. Motivated by this fact, in the present chapter we turn our attention to the behavior of expansive actions in a broader setting—namely, that of metric spaces which are not necessarily compact—investigating under what conditions expansivity remains independent of the metric.

One of the contributions of this chapter is encapsulated in Theorem 6.1.12, which characterizes expansive actions on locally compact and σ-compact metric spaces (hereafter referred to as LCσ-spaces) in terms of a property we call *cocompactly expansive*. This property combines the cocompactness of the action with a separation condition involving the compact set that witnesses cocompactness. Furthermore, the theorem establishes the equivalence between this notion and the existence of an expansive extension to the one-point (Alexandroff) compactification.

To examine with greater precision the equivalences asserted in the main theorem, we employ the notion of cover expansivity introduced in the previous chapter; this framework sharpens our analysis of the logical dependencies involved. We also investigate metrically independent dynamics in specific settings—namely, ordinal spaces and totally bounded spaces—and assess the role played by completeness of the ambient space. In addition, we show that if an expansive action extends to the nonstandard compactification, then it is metric-independent (MIE). The converse, however, fails.

6.1 Metric-Independent Expansive Actions

In this section, we define *metric-independent expansivity* (MIE), present examples and basic properties, and give a characterization of MIE in terms of the *Alexandroff compactification* and a property we call *cocompactly expansive*, for locally compact and σ-compact spaces (LCσ spaces).

Definition 6.1.1. Let X be a metrizable topological space, and $\varphi : G \times X \to X$ be an action. We say that the action is *expansive independent of the metric* (MIE) if for every metric d compatible with the topology of X, there exists $c > 0$ such that for all $x \neq y$ in X, there exists $g \in G$ such that

$$d(g \cdot x, g \cdot y) > c.$$

Example 6.1.2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(x) = x + 1$. Endowing $\mathbb{Z}$ with the discrete topology, f becomes an expansive homeomorphism (a $\mathbb{Z}$-action) that is independent of the metric.
Indeed, given any metric d compatible with the topology, since $\{0\}$ is open, there exists $\epsilon > 0$ such that $B_d(0, \epsilon) \subset \{0\}$. Consequently, ϵ serves as an expansivity constant for φ with respect to the metric d.

Example 6.1.3. Let $f : \mathbb{R} \to \mathbb{R}$ be the homothety with ratio $k > 1$, that is, $f(x) = kx$. It is clear that f is an expansive homeomorphism with respect to the Euclidean metric. We define the metric

$$d'(x, y) = \frac{2\,|x - y|}{\sqrt{(1 + x^2)(1 + y^2)}}$$

given by the stereographic projection of S^1 onto $\mathbb{R}$. It is clear that f is not an expansive homeomorphism with this metric.

Proposition 6.1.4. *Let X be a metric space. MIE is preserved under conjugacy and restriction to a closed invariant subset $Y \subset X$.*

Proof. Conjugation: It is straightforward to verify that $\varphi_h : G \times Y \to Y$ is also an action MIE, where $h : Y \to X$ is a homeomorphism.

Restriction to closed invariant subsets: Let $G \curvearrowright X$ be an MIE action, and let $Y \subset X$ be a closed subset invariant under the action. Denote $d_Y = d|_{Y \times Y}$. Then the restricted action

$$G \curvearrowright Y$$

is MIE.

By Hausdorff's metric extension theorem [37] (see [38] for a short proof), there exists a (complete) metric D on X that coincides with d_Y on Y. Since φ is MIE, there exists $\varepsilon > 0$ such that for every distinct pair $u, v \in X$, there exists $g \in G$ such that

$$D(g \cdot u,\, g \cdot v) \;\geq\; \varepsilon.$$

If $y \neq y' \in Y$, then $g \cdot y,\, g \cdot y' \in Y$ and

$$d_Y\big(g \cdot y,\, g \cdot y'\big) = D\big(g \cdot y,\, g \cdot y'\big) \;\geq\; \varepsilon.$$

Since d_Y is the restriction of an arbitrary compatible metric on Y, the restricted action is MIE.

□

The following example shows that metric-independent expansiveness is not preserved under restriction to subgroups.

Example 6.1.5. Let $G = \mathbb{Z}^2 \curvearrowright X = \{0,1\}^{\mathbb{Z}^2}$ be the shift action with the metric

$$d(x,y) = \sum_{(i,j) \in \mathbb{Z}^2} 2^{-(|i|+|j|)} \big|x(i,j) - y(i,j)\big|.$$

Although the action of G is metric-independent expansive (MIE), its restriction to the subgroup

$$H = \mathbb{Z} \times \{0\} \subset G$$

is not.

Let x_0 be the zero configuration. For each $n \geq 1$, define

$$y_n(i,j) = \begin{cases} 1, & \text{if } (i,j) = (0,n), \\ 0, & \text{otherwise.} \end{cases}$$

For $h = (h_1, 0) \in H$, we have

$$h \cdot x_0 = x_0, \quad h \cdot y_n(i,j) = y_n(i - h_1, j),$$

and therefore,

$$d(h \cdot x_0, h \cdot y_n) = 2^{-(|h_1|+n)} \leq 2^{-n}.$$

Given any $\varepsilon > 0$, choose N such that $2^{-N} < \varepsilon$. Then for this pair,

$$d(h \cdot x_0, h \cdot y_N) \leq 2^{-N} < \varepsilon \quad \forall\, h \in H,$$

which shows that there is no positive expansive constant for H. Therefore, the action $H \curvearrowright X$ is not MIE.

Remark 6.1.6. In Section 6.1.2, we will see that if the space is a LCσ-space (see Definition 2.2.18) and the subgroup is syndetic, MIE is preserved under passage to subgroups.

Note that if a metric space is σ-compact but not locally compact, then its one-point compactification fails to be metrizable. Nevertheless, by employing the notion of cover expansivity introduced in the previous chapter, the following definition becomes entirely well-defined.

Definition 6.1.7. Let X be a metrizable topological space, and let $\varphi : G \times X \to X$ be an action. We say that φ is *expansively extendible at a point* if there exists an expansive action, in the sense of cover expansivity, $\varphi' : G \times X' \to X'$, where $X' = X \cup \{p\}$ is the one-point compactification of X, such that $\varphi'|_{G\times X} = \varphi$ and $g \cdot p = p$ for all $g \in G$.

Definition 6.1.8. Let X be a topological space, and let $\varphi : G \times X \to X$ be an action. We say that φ is *cocompact* if there exists a compact subset K such that $G.K = X$.

Remark 6.1.9. Observe that in Example 6.1.3, the homeomorphism is not independent of the metric, although it is cocompact. To characterize those actions on LCσ-spaces,that are independent of the metric, cocompactness alone is not sufficient: an additional condition must be imposed. This is precisely the purpose of the first definition in the following section.

6.1.1 Metric-Independent Expansiveness and LCσ-spaces

Definition 6.1.10. Let X be a topological space, $\varphi : G \times X \to X$ an action, and let $\mathcal{U} = \{U_1, \dots, U_n\}$ be an open cover of X. We say that φ is *cocompactly expansive*, with $\mathcal{U}$ as an expansivity cover, if there exists a compact set $K \subset X$ such that:

1) $G \cdot K = X$; that is, φ is cocompact with respect to K.

2) $\{g \cdot x, g \cdot y\} \prec \mathcal{U} \cup (X \setminus K)$ for all $g \in G$ implies $x = y$.

Note that the property of a metric space being an LCσ-space is the minimal hypothesis required for its Alexandroff compactification to be metrizable.

Remark 6.1.11. For homeomorphisms, the implication (1) $\Rightarrow$ (2) in the following theorem is already implicit in Bryant's work [8].

Theorem 6.1.12. *Let X be a LCσ metric space, and let $\varphi : G \times X \to X$ be an action. The following statements are equivalent:*

1) φ is MIE.

2) φ is expansively extendible at a point.

3) φ is cocompactly expansive.

Proof. (1) ⇒ (2)**:** Let $\varphi' : G \times X' \to X'$ be the extension of φ to the Alexandroff compactification X' of X. Suppose that there exists a metric $\delta : X' \times X' \to \mathbb{R}^+$ compatible with the topology of X' such that φ' is not expansive. Then, for every $n \in \mathbb{N}$, there exist distinct points $x_n, y_n \in X'$ such that for all $g \in G$,

$$\delta(g \cdot x_n, g \cdot y_n) \leq 2^{-n}.$$

Clearly, if this inequality holds for some subsequence $(x_{n_k}, y_{n_k}) \subset X$, it contradicts the expansiveness of φ with respect to the metric $\delta|_{X\times X}$, and thus φ would not be MIE. Therefore, the remaining case to consider is when $y_n = p$ for all $n \in \mathbb{N}$. But then

$$\delta|_{X\times X}(g \cdot x_n, g \cdot x_m) \leq \delta(g \cdot x_n, p) + \delta(p, g \cdot x_m) \leq 2^{-n} + 2^{-m},$$

which also contradicts the expansiveness of φ with respect to the metric $\delta|_{X\times X}$.

As mentioned in Chapter 4, metric expansiveness is equivalent to expansiveness by coverings in the case where the space is metrizable. Therefore, the implication is proved.

(2) ⇒ (1)**:** Let $\varphi' : G \times X' \to X'$ be the one-point extension of φ that is expansive by coverings. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be an expansivity covering, where $U_n = X' \setminus K$ with $K \subset X$ compact, and set $\mathcal{V} = \{U_1, \ldots, U_{n-1}\}$, a covering of K.

Let d be a metric compatible with the topology of X, and choose a compact $K' \subset X$ with $K \subset K'$ such that

$$\text{dist}(K, X \setminus K') = \epsilon > 0.$$

Define $\mathcal{U}' = \{U_1, \ldots, U_{n-1}, U_n'\}$, where $U_n' := U_n \setminus \{p\}$. Let δ be the Lebesgue number of $\mathcal{V}$ and δ' that of $\mathcal{U}'$ over K'.

Let $x, y \in X$ with $x \neq y$. Then:

- If $x, y \in K$, there exists $g \in G$ such that $\{g \cdot x, g \cdot y\} \not\prec \mathcal{U}$, so $d(g \cdot x, g \cdot y) > \delta$. - If $x \in K$ and $y \in U_n'$, then $d(x, y) > \delta'$. - If $y \notin K'$, then $y \in U_n$, and there exists $g \in G$ such that $\{g \cdot y, g \cdot p\} \not\prec \mathcal{U}$, so $g \cdot y \in K$. If $g \cdot x \in K'$, we fall into a previous case; otherwise $d(g \cdot x, g \cdot y) > \epsilon$.

Taking $c = \min\{\epsilon, \delta, \delta'\}$, we conclude that c is an expansivity constant for φ.

(2) ⇒ (3)**:** If φ is expansively extendible at a point, then by definition it is expansive. To satisfy the definition of cocompact expansiveness, it suffices to take as K the complement of the open set in the expansivity covering of φ that contains p.

(3) ⇒ (2)**:** Let $\{U_1, \ldots, U_n\}$ and K be the expansivity cover and compact set given by the

definition of cocompactly expansive action. Then

$$\{U_1, \dots, U_n, X' \setminus K\}$$

is an expansivity cover for $\varphi' : G \times X' \to X'$. □

If the property of cocompact expansivity is expressed purely in metric terms, the following equivalence follows immediately.

Remark 6.1.13. Let X be a LCσ metric space, and let $\varphi : G \times X \to X$ be an action. The following statements are equivalent:

1) φ is cocompactly expansive.

2) There exists a metric d compatible with the topology, a compact set $K \subset X$ such that $G \cdot K = X$, and a constant $c > 0$ such that for all distinct $x, y \in X$, there exists $g \in G$ satisfying $d(g \cdot x, g \cdot y) > c$ and $\{g \cdot x, g \cdot y\} \cap K \neq \emptyset$.

Remark 6.1.14. Observe that in Example 6.1.3, the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1, which is known not to admit expansive homeomorphisms. Therefore, by Theorem 6.1.12, we can conclude that the action is not expansive independently of the metric, without the need to explicitly construct such a metric.

Remark 6.1.15. Observe that in the proof of Theorem 6.1.12, we did not use the continuity of the action.

Remark 6.1.16. Let X be a LCσ metrizable space, and let $\varphi : G \times X \to X$ be an action. The following statements are equivalent:

1) φ is continuous and expansive independently of the metric.

2) φ is continuous and expansively extendible at a point.

3) φ is continuous and cocompactly expansive.

Remark 6.1.17. In $(2 \Rightarrow 1)$, and $(2 \Leftrightarrow 3)$ we used expansivity by coverings, which is equivalent to metric expansivity in the case where X' is metrizable. Therefore, we obtain the following result.

Theorem 6.1.18. *Let X be a metrizable topological space, and let $\varphi : G \times X \to X$ be an action. Then φ is expansively extendible at a point if and only if φ is cocompactly expansive. Moreover, if either of these equivalent conditions holds, then φ is expansive independently of the metric.*

6.1.2 Preservation of Cocompactly Expansivity under Subgroups, Products, and Coproducts

Definition 6.1.19 (Syndetic subgroup)**.** A subgroup $H \le G$ is called *syndetic* if there exists a finite set $F \subset G$ such that $G = FH$.

Proposition 6.1.20 (Inheritance to syndetic subgroups)**.** *Let $\varphi : G \times X \to X$ be a continuous and cocompactly expansive action, and let H be a syndetic subgroup of G. Then the restricted action $\varphi|_{H\times X} : H \times X \to X$ is also continuous and cocompactly expansive.*

Proof. Let K be the compact set from the cocompactness of φ, $\mathcal{U} = \{U_1, \ldots, U_N\}$ be the expansivity covering, and $F = \{f_1, \ldots, f_m\} \subset G$ be the finite set such that $G = FH$.

We define

$$\mathcal{W} := \left\{ \bigcap_{f\in F_0} f^{-1}.U_{i(f)} \;\middle|\; \varnothing \neq F_0 \subseteq F,\ i(f) \in \{1, \ldots, N\} \right\}.$$

Each $f_j^{-1}.K$ is compact, so $K' := \bigcup_{j=1}^{m} f_j^{-1}.K$ is compact as well. For any $x \in X$, choose $g \in G$, $k \in K$ such that $x = g.k$ (since $G \cdot K = X$). Writing $g = f_j h$ with $f_j \in F$, $h \in H$, we obtain

$$x = f_j.(h.k) \in H \cdot K'.$$

Hence, $H \cdot K' = X$.

Since $\mathcal{U}$ and F are finite, the family $\mathcal{W}$ is also finite. Each element of $\mathcal{W}$ is open (being a finite intersection of open sets) and contains all the sets $f^{-1}.U_i$, so it covers X. Adding $X \setminus K'$, we obtain a finite open cover:

$$\mathcal{W} \cup \{X \setminus K'\}.$$

Let $x \neq y \in X$. Since φ is cocompactly expansive, there exists $g \in G$ such that:

$$\{g.x, g.y\} \cap K \neq \varnothing, \quad \text{and} \quad \forall U \in \mathcal{U},\ \{g.x, g.y\} \not\subset U.$$

We write $g = f_j h$ with $f_j \in F$, $h \in H$, and define $x' := h.x$, $y' := h.y$.

Then $\{g.x, g.y\} = f_j.\{x', y'\}$, and one of these points lies in K, so its preimage lies in $f_j^{-1}.K \subset K'$. Thus,

$$\{x', y'\} \cap K' \neq \varnothing.$$

If $\{x', y'\} \subset X \setminus K'$, then applying f_j gives $\{g.x, g.y\} \subset X \setminus K$, contradicting the assumption.

Now take $W = \bigcap_{f\in F_0} f^{-1}.U_{i(f)} \in \mathcal{W}$ and fix $f_k \in F_0$. Then $W \subset f_k^{-1}.U_{i(f_k)}$. If $\{x', y'\} \subset W$, then applying f_k yields:

$$\{f_k.x', f_k.y'\} \subset U_{i(f_k)} \subset \mathcal{U},$$

which contradicts the expansivity assumption. Therefore, $\{x', y'\} \not\subset W$, and $\varphi|_{H\times X}$ is cocompactly expansive. □

Corollary 6.1.21. *The subgroup $n\mathbb{Z}$ is syndetic in $\mathbb{Z}$. Therefore, if $f : X \to X$ is a cocompactly expansive homeomorphism, then f^n is also cocompactly expansive for every $n \in \mathbb{Z}$. Moreover, if the space X is σ-compact, then by Theorem 6.1.12, if f is MIE, so is f^n for all $n \in \mathbb{Z}$, with $n > 0$.*

Proposition 6.1.22 (Stability under coproducts)**.** *Let G be a group acting cocompactly expansively on two topological spaces X and Y. Then the coproduct (disjoint union) of the actions is also cocompactly expansive. That is, consider the disjoint union $X \sqcup Y$ with the componentwise action*

$$g \cdot z := \begin{cases} g.x & \text{if } z = x \in X, \\ g.y & \text{if } z = y \in Y. \end{cases}$$

Then the induced action on $X \sqcup Y$ is cocompactly expansive.

Proof. Assume the data $(\mathcal{U}_X, K_X)$ and $(\mathcal{U}_Y, K_Y)$ satisfy the definition of cocompact expansiveness for X and Y, respectively.

1. Cocompact set. Let

$$K := K_X \sqcup K_Y \subset X \sqcup Y.$$

Since $G \cdot K_X = X$ and $G \cdot K_Y = Y$, it follows that $G \cdot K = X \sqcup Y$.

2. Finite covering. Let

$$\mathcal{U} := \mathcal{U}_X \cup \mathcal{U}_Y.$$

This is a finite open covering of $X \sqcup Y$, since $\mathcal{U}_X$ covers X and $\mathcal{U}_Y$ covers Y.

3. Universal expansivity condition. Let $z \neq w \in X \sqcup Y$. We consider two cases:

Case A: $z, w \in X$ or $z, w \in Y$. By cocompact expansiveness in X, there exists $g \in G$ such that

$$\{g.z, g.w\} \cap K_X \neq \varnothing, \quad \forall U \in \mathcal{U}_X,\ \{g.z, g.w\} \not\subset U.$$

Since $\{g.z, g.w\} \subset X$, this pair is not contained in any $U \in \mathcal{U}_Y$, and we also have $\{g.z, g.w\} \cap K = \{g.z, g.w\} \cap K_X \neq \varnothing$. So the pair satisfies the condition with respect to $(\mathcal{U}, K)$ in $X \sqcup Y$.

Case B: $z \in X$, $w \in Y$ or (vice versa). Let $g \in G$ be such that $g.z \in K_X$. Then $\{g.z, g.w\} \cap K = \{g.z\} \neq \varnothing$. Since $g.z \in X$ and $g.w \in Y$, the pair cannot be contained in any $U \in \mathcal{U}_X$ or

$U \in \mathcal{U}_Y$, because all such sets are contained either in X or in Y, respectively. Hence, the universal expansivity condition is satisfied.

In all cases, there exists $g \in G$ such that the pair (z, w) satisfies the expansivity condition with respect to the data $(\mathcal{U}, K)$. Therefore, the action on $X \sqcup Y$ is cocompactly expansive. □

Corollary 6.1.23. *If X and Y are LCσ metric spaces, and if there exist actions $\varphi : G \times X \to X$ and $\phi : G \times Y \to Y$ that are metric-independent expansive (MIE), then the direct sum $\varphi + \phi$ defines an MIE action on the coproduct $X \sqcup Y$.*

Proof. This follows directly from applying Proposition 6.1.22 and Theorem 6.1.12. □

The following example shows that the property of being cocompactly expansive is not preserved under products.

Example 6.1.24. Let $G = \mathbb{Z}$, and consider the shift action

$$n \cdot x = x + n \qquad \text{on} \qquad X := Y := (\mathbb{Z}, \tau_{\text{dis}}),$$

with the discrete topology.

1) Each action $G \curvearrowright X$ and $G \curvearrowright Y$ is cocompactly expansive.

2) The diagonal action on $X \times Y$,

$$n \cdot (x, y) := (x + n,\ y + n),$$

is not cocompact, and therefore not cocompactly expansive.

Proof. We already saw that each individual action is metric-independent expansive. Since the space is LCσ-space, Theorem 6.1.12 implies that the actions are cocompactly expansive. However, the product action is not cocompact, and thus cannot be cocompactly expansive.

Indeed, the orbit of a point (x, y) under the diagonal action is

$$\mathcal{O}(x, y) = \{(x + n,\ y + n) : n \in \mathbb{Z}\},$$

which is an infinite straight line in $\mathbb{Z}^2$.

Let $K \subset \mathbb{Z}^2$ be compact (i.e., finite). The saturated set

$$G \cdot K = \bigcup_{(x,y) \in K} \mathcal{O}(x, y)$$

is a finite union of such lines.

However, covering $\mathbb{Z}^2$ requires *infinitely many* distinct lines—one for each value of $y - x$. Therefore, for any compact K, we have $G \cdot K \neq X \times Y$, and the diagonal action is not cocompact. □

6.2 Metric-Independent Expansivity in Ordinal Spaces

Proposition 6.2.1. *Let X be a LCσ metric space, and let $\varphi : G \times X \to X$ be a continuous expansive action that is independent of the metric. If $|G| \leq \aleph_0$, then the Alexandroff compactification $X' = X \cup \{p\}$ satisfies the second countability axiom.*

Proof. By Remark 6.1.16, φ can be extended to an expansive and continuous action on the one-point compactification X'. Moreover, by Theorem 4.4.4 (Theorem 6.4 in [17]), if $|G| \leq \aleph_0$, then X' satisfies the second countability axiom. □

A consequence of the theorem by Kato and Park is the following theorem.

Theorem 6.2.2. *Let X be a countable scattered metric space (see Definition 2.2.22). Then X admits a homeomorphism $f\colon X \to X$ that is MIE if and only if the derived degree of X is not an infinite limit ordinal.*

Proof. If X is a countable scattered metric space, then its one-point compactification is also a countable scattered metric space. By Baker, X' is homeomorphic to an ordinal of the form $\omega^\alpha + 1$ with the order topology. Since X admits an expansive homeomorphism independent of the metric by Remark 6.1.16, X' also admits an expansive homeomorphism. Then, by Theorem 2.2 in [26], $\deg(X') = \alpha$ is not an infinite limit ordinal. However, X is homeomorphic to $\omega^\alpha n + 1$, and thus $\deg(X') = \alpha$ or $\alpha + 1$. In any case, $\deg(X)$ is not an infinite limit ordinal.

Conversely, since X' is homeomorphic to $\omega^\alpha n + 1$, then X is homeomorphic to $\omega^\alpha n$. If α is an infinite limit ordinal, then $\deg(X) = \alpha$; otherwise, $\deg(X) = \alpha + 1$. Thus, α is not an infinite limit ordinal. Since $\deg(X') = \alpha$, by Theorem 2.2 in [26], X' admits an expansive homeomorphism f. However, in the construction of the expansive homeomorphism, x_∞ is the point of the one-point compactification and is fixed by the homeomorphism. Therefore, by Remark 6.1.16, the restriction $f \restriction X : X \to X$ is an expansive homeomorphism independent of the metric. □

Theorem 6.2.3. *A countable scattered metric space X admits a metric-independent* CB-stable *action if and only if the characteristic (α, n) of its one-point compactification satisfies either α is not an infinite limit ordinal or $n > 1$.*

Proof. By Theorem 4.2.6 (Theorem 4.2 in [17]), we know that X' admits an expansive CB-stable action if and only if the characteristic (α, n) satisfies either α is not an infinite limit ordinal or $n > 1$. However, the action can always be constructed with the point p of the compactification as a fixed point under the action. □

6.3 Metric-Independent Expansivity in Totally Bounded Spaces

In this section we investigate MIE in totally bounded spaces, focusing in particular on those that arise as open subsets of a compact space.

Definition 6.3.1 (Totally Bounded Space)**.** A metric space X is said to be *totally bounded* if, for every $\varepsilon > 0$, there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$X \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon),$$

where $B(x_i, \varepsilon) = \{y \in X : d(x_i, y) < \varepsilon\}$ is the open ball of radius ε centered at x_i.

Definition 6.3.2. Let X be a topological space and $\varphi : G \times X \to X$ an action. We say that $K \subset X$ is *dynamically isolated* if there exists an open set U with $K \subset U$ such that

$$\bigcap_{g \in G} g.U = K.$$

Before proving the following theorem, we recall some fundamental facts about equivalence relations and prove a few lemmas.

Definition 6.3.3. Let X be a set and $\sim$ an equivalence relation on X. A subset $A \subseteq X$ is said to be *saturated* (with respect to $\sim$) if for every $x \in A$, the equivalence class $[x] \subseteq A$; that is, A contains entirely each equivalence class it intersects. Equivalently, A is saturated if there exists a subset $B \subseteq X/{\sim}$ such that $A = \pi^{-1}(B)$, where $\pi : X \to X/{\sim}$ is the canonical projection sending each point $x \in X$ to its equivalence class $[x] \in X/{\sim}$.

Lemma 6.3.4. *Let (X, d) be a compact metric space, $\varphi \colon G \times X \to X$ a continuous action, and $K \subset X$ a closed φ-invariant subset. Define the equivalence relation*

$$x \sim y \quad \Longleftrightarrow \quad (\, x = y \,) \text{ or } (\, x, y \in K \,),$$

and let $\pi \colon X \to Y := X/{\sim}$ be the canonical projection, with $p = [K]$ denoting the equivalence class of all points in K. Then:

1) Y is compact.

2) The induced action

$$\varphi' : G \times Y \longrightarrow Y, \qquad \varphi'(g,[x]) := [\,\varphi(g,x)\,]$$

is well defined and continuous.

3) The restriction

$$\pi\big|_{X\setminus K} : X \setminus K \xrightarrow{\cong} Y \setminus \{p\}$$

is a homeomorphism. In particular, $X \setminus K$ and $Y \setminus \{p\}$ are homeomorphic, and π maps K to the point p.

Proof. 1) Since $Y = \pi(K)$ is the image of the compact set K under the continuous map π, the quotient $Y = X/K$ is compact.

2) The induced action φ' is well defined because if $x \in K$, then $\varphi(g,x) \in K$ for all g, and thus $[\varphi(g,x)] = p$. The continuity of φ' follows from the fact that π is open and saturated.

3) The restriction $\pi|_{X\setminus K}$ is continuous, bijective, and open onto $Y \setminus \{p\}$. Its inverse maps $[x] \mapsto x$ for $x \notin K$. This implies that it is a homeomorphism.

□

Lemma 6.3.5. *Under the notation of Lemma 6.3.4, the subset $K \subset X$ is dynamically isolated in X* if and only if *$\{p\} \subset Y$ is dynamically isolated for the action φ'. That is,*

$$\begin{aligned} &\exists\, U \subset X \text{ open with } K \subset U \text{ and } \\ &\quad \bigcap_{g\in G} \varphi(g,U) = K \\ &\iff \quad \exists\, V \subset Y \text{ open with } p \in V \text{ and } \\ &\quad \bigcap_{g\in G} \varphi'(g,V) = \{p\}. \end{aligned}$$

Proof. $\Rightarrow$ Suppose that K is dynamically isolated in X. Then there exists an open set $U \subset X$ such that $K \subset U$ and $\bigcap_g \varphi(g,U) = K$. Define $V := \pi(U) \subset Y$. Since π is continuous and saturated, V is open and contains p. For each $y \neq p$ in Y, choose $x \notin K$ with $\pi(x) = y$. Then $x \notin \bigcap_g \varphi(g,U)$, so there exists g such that $\varphi(g,x) \notin U$. Hence, $\varphi'(g,y) = [\varphi(g,x)] \notin V$. Consequently, $\bigcap_g \varphi'(g,V) = \{p\}$.

$\Leftarrow$ Suppose that $\{p\}$ is dynamically isolated in Y. Then there exists an open set $V \subset Y$ with $p \in V$ and $\bigcap_g \varphi'(g,V) = \{p\}$. Let $U := \pi^{-1}(V) \subset X$. Then U is open and contains

$K = \pi^{-1}(\{p\})$. For any $x \notin K$, there exists g such that $\varphi'(g, [x]) \notin V$, that is, $[\varphi(g, x)] \notin V$, which implies $\varphi(g, x) \notin U$. Thus, we conclude that $\bigcap_g \varphi(g, U) = K$.

□

Theorem 6.3.6. *Let (X, d) be a compact metric space, and let $\varphi : G \times X \to X$ be an expansive action. Let $K \subset X$ be a closed, φ-invariant subset. Then:*

$$\varphi\big|_{X\setminus K} \text{ is MIE on } X \setminus K \iff K \text{ is dynamically isolated.}$$

Proof. By Proposition 6.1.4, we know that

$$\varphi\big|_{X\setminus K} \text{ is MIE on } X \setminus K \iff \varphi'\big|_{Y\setminus\{p\}} \text{ is MIE on } Y \setminus \{p\}.$$

But $X \setminus K$ is σ-compact, and therefore so is $Y \setminus \{p\}$. Then, by Theorem 6.1.12,

$$\varphi'\big|_{Y\setminus\{p\}} \text{ is MIE on } Y \setminus \{p\} \iff \varphi' \text{ is MIE on } Y.$$

Moreover, since Y is compact, this is equivalent to φ' being expansive on Y, that is,

$$\varphi' \text{ is expansive on } Y \iff \{p\} \text{ is dynamically isolated in } Y.$$

Finally, by Lemma 6.3.5,

$$\{p\} \text{ is dynamically isolated in } Y \iff K \text{ is dynamically isolated in } X.$$

This concludes the proof. □

Remark 6.3.7. The previous theorem provides a method for constructing MIE actions as well as actions that are not MIE. It suffices to consider an expansive action on a compact space and a closed invariant subset K. If K is dynamically isolated, then the restriction of the action to its complement is MIE. If K is not dynamically isolated, then the restricted action is not MIE. The following example illustrates this mechanism.

Example 6.3.8. One of the classical examples of an expansive homeomorphism (in fact, a diffeomorphism) is the Anosov automorphism on T^2.

We denote

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2$$

the quotient space of $\mathbb{R}^2$ by the lattice $\mathbb{Z}^2$. Each point in T^2 is denoted $[x]$, with $x \in \mathbb{R}^2$, and $[x]$ its equivalence class modulo $\mathbb{Z}^2$. We equip T^2 with the distance

$$d([x], [y]) = \inf_{m\in\mathbb{Z}^2} \|x - y + m\|,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$.

Consider the hyperbolic matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

We define the linear diffeomorphism

$$f : T^2 \to T^2, \qquad f([x]) = [Ax].$$

Let K be the orbit of a periodic point. It is easy to verify that K is dynamically isolated. Then, by Theorem 6.1.12, the Anosov map restricted to $X \setminus K$ is MIE.

We can construct another elementary example by defining a dynamical system on $[0, \omega + 1]$.

Example 6.3.9. Let $X = [0, \omega]$, let $x_\infty := \omega$, and define $f : X \to X$ as follows:

$$f(x) = \begin{cases} x + 2 & \text{if } x \text{ is even,} \\ x - 2 & \text{if } x \text{ is odd and greater than 1,} \\ 0 & \text{if } x = 1, \\ x_\infty & \text{if } x = x_\infty. \end{cases}$$

It is easy to verify that f is an expansive homeomorphism and that $K = \{x_\infty\}$ is a closed dynamically isolated set. Then, by the previous theorem, $f \restriction X \setminus K$ is metric-independent. We observe that $f \restriction X \setminus K$ is conjugate to the homeomorphism introduced in Example 6.1.2. This yields an alternative proof that the latter is an MIE homeomorphism.

Proposition 6.3.10. *Let X be a compact scattered topological space, and let $\varphi : G \times X \to X$ be a* CB-stable *action. Then there exists a closed, invariant, and proper dynamically isolated set $K \subset X$.*

Proof. We have already seen that X is homeomorphic to $\omega^\alpha k+1$ with the order topology. Therefore, we may assume, up to conjugation, that $X = \omega^\alpha k + 1$.

Let $Y = \{\omega^\alpha i + 1 : i = 1, \dots, k\}$. Since φ is CB-stable , it follows that X is invariant under φ.

If for every family of open-and-closed neighborhoods U_i of $\omega^\alpha i + 1$, there exists $g \in G$ such that

$$g\left(\bigcup_{i=1}^{k} U_i\right) \neq \bigcup_{i=1}^{k} U_i,$$

then $K = Y$ is the closed dynamically isolated set. Otherwise,

$$K = \left(\bigcup_{i=1}^{k} U_i \right)^c$$

is the closed dynamically isolated set. □

Using a similar reasoning as before, we obtain the following proposition. Note that in [3] we showed that in every countable compact space there exist doubly asymptotic points. The following result shows that in certain cases we can guarantee the existence of homoclinic points.

Proposition 6.3.11. *Let X be a countable compact metric space with characteristic $(\alpha + 1, 1)$, and let $f : X \to X$ be an expansive homeomorphism. Then f has nontrivial homoclinic points.*

Proof. As we have already seen, we may assume that $X = \omega^{\alpha+1} + 1$. Let

$$Y = \{\omega^\alpha i + 1 : i \in \mathbb{N}\}.$$

Since f is continuous, it is CB-stable ; therefore, Y is invariant under f. Observe that, due to the expansiveness of f, not all points can be fixed. Hence, there exists

$$x = \omega^\alpha j + 1, \quad \text{with } j \in \mathbb{N},$$

such that

$$\lim_{n \to +\infty} f^n(x) = \lim_{n \to -\infty} f^n(x) = \omega + 1.$$

□

6.4 Metric-Independent Expansivity and Completion of Metric Spaces

Definition 6.4.1. Let X be a metric space and $\varphi : G \times X \to X$ an action. We say that it is uniformly continuous if the map $T_g : X \to X$, defined by $T_g(x) := g.x$, is uniformly continuous for all $g \in G$.

Definition 6.4.2. Let (X, d) be a metric space and $\varphi : G \times X \to X$ a continuous action. We say that φ is *Cauchy expansive* if there exists a constant $c > 0$ such that for every sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset X \times X$ satisfying:

1) the sequence $(x_n, y_n)_{n\in\mathbb{N}} \subset X \times X$ is Cauchy in $X \times X$,

2) $\inf_{n\in\mathbb{N}} d(x_n, y_n) > 0$,

there exist $n_0 \in \mathbb{N}$ and $g \in G$ such that for all $n \geq n_0$, one has

$$d(\varphi(g, x_n), \varphi(g, y_n)) > c.$$

Remark 6.4.3. It is clear that Cauchy expansiveness implies expansiveness, simply by taking, for any $x, y \in X$, the constant sequences $x_n = x$ and $y_n = y$.

Theorem 6.4.4. *Let (X, d) be a metric space and $\varphi : G \times X \longrightarrow X$ a Cauchy expansive action and uniformly continuous. Let $\hat{X}$ denote the completion of X. Then there exists a unique continuous and expansive action*

$$\hat{\varphi} : G \times \hat{X} \longrightarrow \hat{X},$$

such that $\hat{\varphi}\restriction_X = \varphi$; moreover, the expansivity constant can be taken equal to that of φ.

Proof. For each $g \in G$, the map $\varphi_g : X \to X$ is uniformly continuous, and X is dense in $\hat{X}$; the uniform extension theorem provides a *unique* continuous function $\hat{\varphi}_g : \hat{X} \to \hat{X}$ with $\hat{\varphi}_g\restriction_X = \varphi_g$.

For the identity element e, we obtain $\hat{\varphi}_e = \mathrm{id}_{\hat{X}}$ by uniqueness. If $g, h \in G$, since $\varphi_{gh} = \varphi_g \circ \varphi_h$ on X, the uniqueness of the extensions gives $\hat{\varphi}_{gh} = \hat{\varphi}_g \circ \hat{\varphi}_h$. Defining $\hat{\varphi}(g, \hat{x}) := \hat{\varphi}_g(\hat{x})$, we obtain a continuous action of G on $\hat{X}$.

Let $\varepsilon > 0$ be the expansivity constant of φ. Take $\hat{x} \neq \hat{y}$ in $\hat{X}$. There exist sequences $x_n, y_n \in X$ with $x_n \to \hat{x}$ and $y_n \to \hat{y}$. Since $\inf_n d(x_n, y_n) \geq \frac{1}{2} d(\hat{x}, \hat{y}) > 0$, the pair (x_n, y_n) satisfies the conditions to apply the hypothesis of Cauchy expansiveness, and thus there exist $g \in G$ and $n_0 \in \mathbb{N}$ such that $d(\varphi_g(x_n), \varphi_g(y_n)) > \varepsilon$ for all $n \geq n_0$. Passing to the limit and using the continuity of $\hat{\varphi}_g$, we obtain:

$$d\big(\hat{\varphi}_g(\hat{x}), \hat{\varphi}_g(\hat{y})\big) = \lim_{n\to\infty} d\big(\varphi_g(x_n), \varphi_g(y_n)\big) \;\geq\; \varepsilon.$$

Therefore, $\hat{\varphi}$ is expansive. □

Corollary 6.4.5. *Let X be a totally bounded metric space, $\varphi : G \times X \to X$ a uniformly continuous Cauchy expansive action, and suppose that $\hat{X} \setminus X$ is closed.*
If $\hat{X} \setminus X$ is dynamically isolated, then φ is metric-independent.

Proof. Let $\hat{\varphi} : G \times \hat{X} \to \hat{X}$ be the expansive extension of φ. Then $\hat{\varphi} \restriction (\hat{X} \setminus (\hat{X} \setminus X)) \equiv \varphi$. By the previous theorem, φ is metric-independent if $\hat{X} \setminus X$ is dynamically isolated. □

6.5 Metric-Independent Expansivity and Nonstandard Analysis

Recall that in Chapter 3 we established that nonstandard analysis is a valuable framework for addressing dynamical problems and for developing alternative proofs that provide new insights. For instance, it allows us to develop visual intuitions, as in the proof of Theorem 3.1.4, which was essential for establishing Theorem 3.0.4, the main result of that chapter. We shall now employ this framework to generalize Utz's result for MIE homeomorphisms.

Theorem 6.5.1. *Let X be a metric* LCσ *space such that there exists $x_0 \in X$ which is an accumulation point, and let $f : X \to X$ be a metric-independent expansive (MIE) homeomorphism. Then there exist distinct points $x, y \in X$ such that, for every metric compatible with the topology of X, the points x and y are asymptotic either in the past or in the future.*

Proof. Let $x_0 \in X$ be an accumulation point and $X' = X \cup \{p\}$ the one-point compactification of X. Since X is an LCσ-space, we know that X' is metrizable. Fix a metric d_0 compatible with the topology of X'.

By Theorem 6.1.12, the map f extends to an expansive homeomorphism $f' : X' \to X'$; thus, there exists an expansivity constant $c > 0$. Consider the nonstandard extension

$$^*f' : {}^*X' \to {}^*X'.$$

Since x_0 is an accumulation point, there exists $y \in {}^*X \subset {}^*X'$ such that $x_0 \sim y$.

Reasoning as in Theorem 3.1.4, we know that there exists a minimal $m \in {}^*\mathbb{N}$, which we may assume is positive, such that

$$^*d_0\big({}^*f'^m(x_0), {}^*f'^m(y)\big) > c.$$

Because X' is compact, by Robinson's compactness criterion there exist $x', y' \in X'$ such that $x' \sim {}^*f'^m(x_0)$ and $y' \sim {}^*f'^m(y)$. Since $^*d_0\big({}^*f'^m(x_0), {}^*f'^m(y)\big) > c$, at least one of the points is different from p. Without loss of generality, assume $y' \neq p$. Then $y' \in X$ and $^*d_0(x_0, y') = r - 1$ for some standard real number $r > 0$.

By the same argument as in Theorem 3.1.4, we deduce that x' and y' are asymptotic under f'^{-1}, which coincides with f on X. Therefore, there exists $n \in \mathbb{N}$ such that

$$\{f^{-n}(x'), f^{-n}(y')\} \subset B_{d_0|_{X\times X}}(x_0, r) \subset \overline{B_{d_0|_{X\times X}}(x_0, r)}.$$

Because X is an LCσ-space, the set $\overline{B_{d_0|_{X\times X}}(x_0, r)}$ is compact in X. On compact sets, all compatible metrics are uniformly equivalent, and therefore $f^{-n}(x')$ and $f^{-n}(y')$ are asymptotic with respect to any metric d compatible with the topology of X. It follows that x' and y' are also asymptotic.

The accompanying figure illustrates the geometric intuition behind this proof.

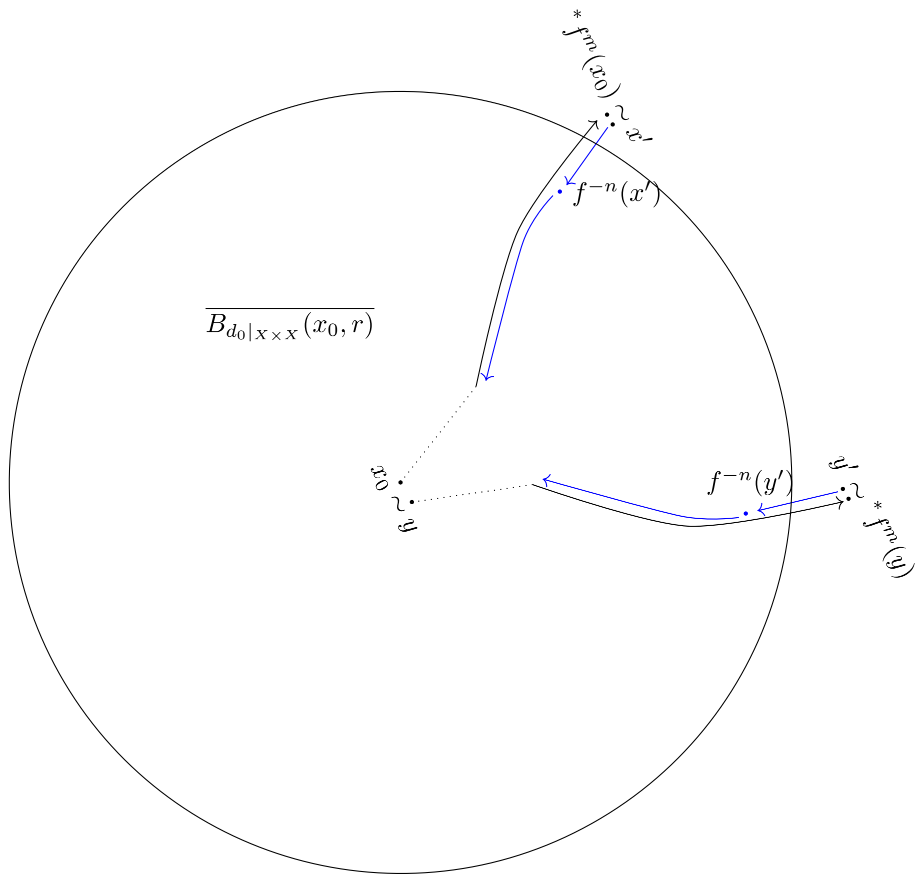


Figure 6.1: Construction of the asymptotic points x' and y'.

□

Definition 6.5.2 (S-expansivity)**.** Let X be a topological space and let $\varphi : G \times X \to X$ be an action of a group G. Consider the natural extension

$$S\varphi : G \times SX \longrightarrow SX, \qquad S\varphi(g, [x']) := [^*\varphi(g, x')],$$

where SX denotes the nonstandard compactification of X.

We say that the action φ is S-*expansive* if $S\varphi$ is expansive by coverings (see Definition 4.1.4).

Remark 6.5.3. According to Corollary 3.6.7 in Ryszard Engelking's *General Topology* [14], if X is compact and $K \subseteq X$ is such that $X \setminus K$ is dense, then $\beta X \cong X$. Therefore, the homeomorphism in Example 6.3.8 is S-expansive.

Remark 6.5.4. Suppose that the action $\varphi : G \times X \to X$ is such that for each $g \in G$, the map

$$x \mapsto \varphi(g, x)$$

is continuous on X. Then the extension

$$S\varphi : G \times SX \longrightarrow SX, \qquad S\varphi(g, [x']) = [{}^*\varphi(g, x')]$$

is continuous for each $g \in G$.

Proof. Fix $g \in G$ and consider the map $T_g : X \to X$, $T_g(x) = \varphi(g, x)$. By hypothesis, T_g is continuous.

Let $O \subseteq X$ be open. By the definition of the S-topology on ${}^*X$,

$${}^*T_g^{-1}({}^*O) = {}^*(T_g^{-1}(O)).$$

If $x' \sim y'$, then for every open set $O \subseteq X$:

$$x' \in {}^*O \iff y' \in {}^*O.$$

Applying the previous equality we obtain:

$${}^*T_g(x') \in {}^*O \iff x' \in {}^*(T_g^{-1}(O)) \iff y' \in {}^*(T_g^{-1}(O)) \iff {}^*T_g(y') \in {}^*O.$$

Therefore,

$${}^*T_g(x') \sim {}^*T_g(y').$$

This shows that $S\varphi(g, [x']) = [{}^*T_g(x')]$ is well-defined and that

$$S\varphi(g, \cdot) : SX \to SX$$

is continuous (being the quotient of a continuous map on ${}^*X$ with respect to the S-topology). □

Theorem 6.5.5. *Let X be a metric space and $\varphi : G \times X \to X$ an action of a group G. If φ is S-expansive, then φ is MIE.*

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a finite open cover of SX that makes $S\varphi$ expansive by coverings; that is:

for every distinct $x, y \in SX$, there exists $g \in G$ such that

$$\{\ S\varphi(g, x), S\varphi(g, y)\ \} \not\subseteq U, \quad \text{for every } U \in \mathcal{U}.$$

Every open subset of SX is a union of sets $\pi({}^*O)$, with $O \subseteq X$ open (where $\pi : {}^*X \twoheadrightarrow SX$ is the projection).

For each i, define $V_i := U_i \cap X$. If $x \in V_i \subseteq U_i$, there exists an open set $O \subseteq X$ such that $x \in \pi({}^*O) \subseteq U_i$; but $\pi({}^*O) \cap X = O$, so $x \in O \subseteq V_i$. Therefore V_i is open in X, and $\mathcal{V} = \{V_1, \dots, V_n\}$ covers X.

Now let $x \neq y \in X$. By the expansivity of $S\varphi$, we know that there exists $g \in G$ such that

$$\{\ \varphi(g,x), \varphi(g,y)\ \} \nsubseteq V, \quad \text{for every } V \in \mathcal{V}.$$

Hence $\mathcal{V}$ is an expansivity covering for φ.

Moreover, since the space is LCσ, its one-point compactification is metrizable. This implies that φ is expansively extendable to the one-point compactification. By Theorem 6.1.12, it then follows that φ is MIE. □

Theorem 6.5.6. *Metric-independent expansivity does not imply S-expansivity.*

Proof. Recall that the homeomorphism $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$ is MIE. Consider $Sf : S\mathbb{Z} \to S\mathbb{Z}$, the extension of f to its nonstandard compactification $S\mathbb{Z}$. Since the equivalence classes are $[n] = \{n\}$ for every $n \in \mathbb{Z}$, we have $Sf = {}^*f$. We now show that ${}^*f$ is not expansive by coverings.

Let ${}^*\mathcal{A} = \{{}^*A_1, \dots, {}^*A_k\}$ be a covering of $\mathbb{Z}$, with $\mathcal{A} = \{A_1, \dots, A_k\}$ a covering of $\mathbb{Z}$. We shall prove that there exist distinct $x, y \in {}^*\mathbb{Z}$ such that, for every $n \in \mathbb{Z}$ and every $i \in \{1, \dots, k\}$,

$$\chi_{{}^*A_i}(x+n) = \chi_{{}^*A_i}(y+n),$$

where χ_{A_i} denotes the characteristic function of A_i.

Consider the finite alphabet

$$\Sigma \subseteq \{0,1\}^k, \qquad \Sigma := \big\{(\chi_{A_1}(n), \dots, \chi_{A_k}(n)) : n \in \mathbb{Z}\big\}.$$

Define $s : \mathbb{Z} \to \Sigma$ by $s(n) := \big(\chi_{A_1}(n), \dots, \chi_{A_k}(n)\big)$.

For each $m \in \mathbb{N}$, consider the set of pairs that coincide on the *centered window* of radius m:

$$T_m = \{(i,j) \in \mathbb{Z}^2 :\ i \neq j \ \text{ and } \ s(i+t) = s(j+t) \text{ for all } |t| \leq m\}.$$

Claim 1: $T_m \neq \varnothing$ for every $m \in \mathbb{N}$. Indeed, the number of possible words of length $2m+1$ over Σ is $|\Sigma|^{2m+1} < \infty$, while there are infinitely many positions $i \in \mathbb{Z}$. By the pigeonhole principle, there exist $i \neq j$ with the same centered block, hence $(i,j) \in T_m$.

Claim 2: The family is decreasing: $T_{m+1} \subseteq T_m$ for every m (clear by definition).

By transfer, ${}^*T_m \neq \varnothing$ for every m, and the family $\left({}^*T_m\right)_{m\in\mathbb{N}}$ is decreasing, with

$$ {}^*T_m = \{(i,j) \in {}^*\mathbb{Z}^2 : \ i \neq j \ \text{ and } \ {}^*s(i+t) = {}^*s(j+t) \text{ for all } |t| \leq m\}. $$

By countable saturation (recall that we work under the standing assumption that the nonstandard extension is countably saturated; see Remark 2.1.11), there exists $(x,y) \in \bigcap_{m\in\mathbb{N}} {}^*T_m$, with $x \neq y$. This means that for every $m \in \mathbb{N}$ and every t with $|t| \leq m$, one has $s(x+t) = s(y+t)$.

Let $n \in \mathbb{Z}$ (standard); taking $m > |n|$ and $t = n$ we obtain $s(x+n) = s(y+n)$. Therefore $\chi_{{}^*A_i}(x+n) = \chi_{{}^*A_i}(y+n)$ for every $i = 1,\ldots,k$ and every standard $n \in \mathbb{Z}$. It follows that ${}^*\mathcal{A} = \{{}^*A_1,\ldots,{}^*A_k\}$ is not an expansivity covering for ${}^*f$ (Sf).

Since ${}^*\mathcal{A}$ is an arbitrary covering, we conclude that Sf is not S-expansive. □

Chapter 7

Conclusions and Open Problems

Main Findings

The aim of this thesis has been to show that a **dialogue** between **dynamical systems** and **logic** (in particular, nonstandard analysis and set theory) is not only possible but also highly fruitful. Although these areas have historically interacted less than their potential would suggest, this work is grounded in a conviction—also shared by Hilbert—regarding the **unity of mathematics**: the boundaries between disciplines are permeable, and the conceptual tools of one branch can illuminate central problems in another.

From the dynamical side, **expansiveness** provides an ideal laboratory for measuring how small differences amplify over time. The **nonstandard translation** of this phenomenon makes it possible to study the behavior of orbits at infinite scales and then return with quantitative conclusions to the standard world. In this back-and-forth, it becomes natural to relate qualitative properties, such as the existence of doubly asymptotic pairs, to the **decay rates** of the expansivity constants of the iterates of the system. In this way, the nonstandard language furnishes a common grammar for formulating and resolving questions that are simultaneously geometric and arithmetic.

On the other hand, when studying expansive actions on ordinal spaces, a landscape emerges in which topological dynamics intertwines with **cardinal arithmetic**. In this context, principles of set theory such as the **Generalized Continuum Hypothesis (GCH)** and its "expansive" version (EGCH) are not mere external curiosities, but frameworks that constrain which **sizes** of groups can act expansively on certain compacta. This bridge between cardinality and expansiveness suggests a general message: global dynamical invariants are also modulated by the choices of set theory. The discovery that expansiveness and GCH **share** a combinatorial problem has been one of the most pleasant surprises of this thesis, a point we believe deserves further exploration and which motivated Chapter 5.

The notion of **metric independence of expansiveness** is another avenue for future interactions between logic and dynamical systems. As seen in Chapter 3, nonstandard analysis has proved to be a useful tool, and the **nonstandard compactification** provides a powerful alternative to that of Stone–Čech. The interaction between the dynamics of a space and its various compactifications may also be a promising route for linking cardinality and dynamics. In fact, in Chapter 4 we show that if the space is σ-compact and locally compact, an action is MIE if and only if it extends to the one-point compactification as an expansive action, whereas this fails for the Stone–Čech compactification, the largest one. It seems natural and interesting to ask about possible relationships between the dynamics of a space and the "size" of a compactification that will inherit dynamical properties such as expansiveness.

The exploration of these connections opens the door to a series of unresolved problems that merit study. Below are some of the most relevant questions that have arisen from this research.

Open Problems

Question 7.0.1. For $\lambda > \aleph_0$, what can be said about EGCH(λ)? Our conjecture is that it is independent of the axioms of ZFC. At the time of completing this thesis we are working on the design of a forcing to show that $\neg$EGCH(λ) is consistent with ZFC. We believe that at least for regular λ this is within reach.

Question 7.0.2. In Proposition 5.2.26, is it possible to drop the hypothesis GCH?

Question 7.0.3. In Proposition 5.2.26, does $\mathrm{DGCH}(\lambda)_i \Leftrightarrow \mathrm{DGCH}(\lambda)_j$ hold?

Question 7.0.4. Is metric independence preserved under finite coproducts?

Question 7.0.5. What happens to Theorem 6.1.12 when the space is not LCσ?

Question 7.0.6. Does expansiveness imply Cauchy–expansiveness?

Bibliography

[1] M. Achigar, A. Artigue, I. Monteverde, *Expansive homeomorphisms on non-Hausdorff spaces*, Topology and its Applications 207 (2016), 109–122.

[2] E. Akin, A. Artigue, L. Ferrari, *Asymptotic Pairs for Interval Exchange Transformations*, preprint, arXiv:2009.02592, 2020.

[3] A. Artigue, L. Ferrari, J. Groisman, *Nonstandard analysis of asymptotic points and exponential decay of expansivity constants*, Fundamenta Mathematicae, to appear.

[4] A. Artigue, *Self-similar hyperbolicity*, Ergod. Th. & Dynam. Sys., 38(7), 2422–2446, 2017.

[5] J. Baker, *Compact spaces homeomorphic to a ray of ordinals*, Fundamenta Mathematicae 76 (1972), 19–27.

[6] S. Barbieri, F. García-Ramos, H. Li, *Markovian properties of continuous group actions: algebraic actions, entropy and the homoclinic group*, Advances in Mathematics, (397), 108–196, 2022.

[7] M. Brin, G. Stuck, *Introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 2003.

[8] B. F. Bryant, *Expansive Self-Homeomorphisms of a Compact Metric Space*, The American Mathematical Monthly 69 (1962), 386–391.

[9] E. Čech, B. Pospišil, *Sur les espaces compacts*, Publ. Fac. Sci. Univ. Masaryk 258 (1938), 1–14.

[10] N. Chung, H. Li, *Homoclinic groups, IE groups, and expansive algebraic actions*, Invent. Math., 199(3):805–858, 2015.

[11] P. Cohen, *The Independence of the Continuum Hypothesis*, Proceedings of the National Academy of Sciences of the United States of America 50 (1963), 1143–1148.

[12] J. Cummings, *Iterated forcing and elementary embeddings*, in *Handbook of Set Theory*, eds. M. Foreman and A. Kanamori, Springer, Dordrecht, 2010, pp. 775–883.

[13] T. Eisworth, *Successors of singular cardinals*, in *Handbook of Set Theory*, eds. M. Foreman and A. Kanamori, Springer, Dordrecht, 2010, pp. 1229–1350.

[14] R. Engelking, *General Topology*, revised and completed edition, Heldermann Verlag, Berlin, 1989.

[15] A. Fathi, *Expansiveness, Hyperbolicity and Hausdorff Dimension*, Commun. Math. Phys., 126, 249–262, 1989.

[16] L. Ferrari, *Expansividad Non-Standard*, Tesis de Maestría, Universidad de la República, Facultad de Ciencias, 2019. Disponible en línea: https://www.cmat.edu.uy/biblioteca/monografias-y-tesis/tesis-de-maestria/tesismaestria.pdf/view.

[17] L. Ferrari, *Expansive actions and the GCH*, Topology and its Applications, 361, 109190, 2025.

[18] M. Gitik, *Prikry-type forcings*, in *Handbook of Set Theory*, eds. M. Foreman and A. Kanamori, Springer, Dordrecht, 2010, pp. 1351–1447.

[19] K. Gödel, *Consistency-proof for the generalized continuum-hypothesis*, Proceedings of the U.S. National Academy of Sciences 25 (1938).

[20] R. Goldblatt, *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*, Graduate Texts in Mathematics, vol. 188, Springer-Verlag, New York, 1998.

[21] K. Kunen, J. Vaughan, *Handbook of set theoretic topology*, North-Holland, Amsterdam, 1984.

[22] K. Hiraide, *Expansive homeomorphisms of compact surfaces are pseudo-Anosov*, Osaka J. Math., 27, 117–162, 1990.

[23] T. Jech, *Set Theory: The Third Millennium Edition, Revised and Expanded*, Springer-Verlag, Berlin, 2003.

[24] A. E. Hurd, *Nonstandard analysis of dynamical systems. I: limit motions, stability*, Transactions of the American Mathematical Society, vol. 160, 1971.

[25] A. E. Hurd, P. A. Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, New York, 1985.

[26] H. Kato, J.-J. Park, *Expansive homeomorphisms of countable compacta*, Topol. Appl., 95, 207–216, 1999.

[27] J. L. King, *A map with topological minimal self-joinings in the sense of del Junco*, Ergod. Th. and Dynam. Sys., vol. 10, pp. 745–761, 1990.

[28] P. A. Loeb, M. P. H. Wolff, *Nonstandard analysis for the working mathematician*, Springer, 2000.

[29] T. Meyerovitch, *Pseudo-orbit tracing and algebraic actions of countable amenable groups*, Ergodic Theory and Dynamical Systems, 39(9):2570–2591, (2019).

[30] S. Mazurkiewicz, W. Sierpiński, *Contribution à la topologie des ensembles dénombrables*, Fundamenta Mathematicae 1 (1920), 17–27.

[31] A. Robinson, *Non-Standard Analysis*, North-Holland Publishing Company, Amsterdam, 1966.

[32] A. Robinson, *Compactification of Groups and Rings and Nonstandard Analysis*, The Journal of Symbolic Logic, 34(4):576–588, 1969.

[33] S. Salbany, T. Todorov, *Nonstandard Analysis in Topology: Nonstandard and Standard Compactifications*, The Journal of Symbolic Logic, 65(4):1836–1840, 2000.

[34] Z. Semadeni, *Sur les ensembles Clairsemés*, Rozpr. Matem. 19 (1959), 1–39.

[35] P. Sun, *Exponential decay of expansive constants*, Sci. China Math., 56, 2063–2067, 2013.

[36] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc., 1(6), 769–774, 1950.

[37] F. Hausdorff, *Erweiterung einer Homöomorphie*, Fund. Math. **16** (1930), pp. 353-360.

[38] H. Toruńczyk, *A simple proof of Hausdorff's theorem on extending metrics*, Fundamenta Mathematicae **77** (1972), no. 2, 191–193.

[39] E. Trucco, *A note on the information content of graphs*, Bull. Math. Biophys. **18** (1956), no. 2, 129–135.