Monotonicity formulas via parabolic-to-elliptic transformations: applications to the Ricci flow and fractional heat operators

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Abstract

This thesis contributes to the development of a unified theory of parabolic-to-elliptic transformations, which interprets parabolic partial differential equations as high-dimensional limits of their elliptic counterparts. Our work advances this framework through two principal contributions:

First, we extend this connection to fractional operators, enabling new derivations of monotonicity formulae for fractional parabolic equations from known elliptic results. As a central result, we establish the first monotonicity formula for the semilinear fractional parabolic equation

$$(\partial_t - \Delta)^s u = |u|^{p-1}u,$$

yielding a fractional analogue of the Giga-Kohn monotonicity formula and thereby extending these techniques beyond their original local setting.

Second, we deepen the geometric understanding of the relationship between Colding's monotonic volume and Perelman's entropy functional for the Ricci flow. While Perelman's reduced volume was previously known to emerge as a high-dimensional limit of the Bishop-Gromov relative volume, the geometric origins of the entropy functional $\mathcal{W}$ had remained elusive. We demonstrate that both functionals naturally arise from a unified high-dimensional framework via Perelman's N-space, providing a complete elliptic foundation for these fundamental parabolic quantities.

Resumen

Esta tesis contribuye al desarrollo de una teoría unificada de transformaciones parabólico-elípticas, que interpreta ecuaciones diferenciales parciales parabólicas como límites en alta dimensión de sus contrapartes elípticas. Nuestro trabajo avanza este marco mediante dos contribuciones principales:

En primer lugar, extendemos esta conexión a operadores fraccionarios, permitiendo nuevas derivaciones de fórmulas de monotonía para ecuaciones parabólicas fraccionarias a partir de resultados elípticos conocidos. Como resultado central, establecemos la primera fórmula de monotonía para la ecuación parabólica fraccionaria semilineal

$$(\partial_t - \Delta)^s u = |u|^{p-1} u,$$

obteniendo un análogo fraccionario de la célebre fórmula de monotonía de Giga-Kohn y extendiendo así estas técnicas más allá de su contexto original local.

En segundo lugar, profundizamos la comprensión geométrica de la relación entre el volumen monotónico introducido por Colding y el funcional de entropía de Perelman en el flujo de Ricci. Si bien es sabido que el volumen reducido de Perelman emerge como un límite en dimensión alta del volumen relativo de Bishop-Gromov, los orígenes geométricos de su funcional de entropía $\mathcal{W}$ no poseían semejante explicación. Aquí, demostramos que ambos funcionales surgen naturalmente de un marco unificado en dimensión alta a través del N-espacio de Perelman, proporcionando así una base elíptica completa para estas cantidades parabólicas fundamentales.

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Chapter 1

Introduction

Many central problems in geometric analysis and partial differential equations (PDEs) rely on understanding how quantities evolve under scaling, deformation, or time-dependent processes. A powerful technique for such problems, employed, for example, in Almgren's study of harmonic functions [3], Alt-Caffarelli-Friedman's analysis of free boundaries [4], and Huisken's work on mean curvature flow [39], involves constructing integral functionals that exhibit monotonic behavior along natural parameter families. Over the past decades, this approach has become increasingly prominent, with monotonicity formulas yielding diverse applications including compactness, regularity, and rigidity theorems in elliptic and parabolic PDEs, as well as in geometric flows [19].

An intriguing pattern emerges when comparing parabolic and elliptic settings: many parabolic monotonic quantities appear to have elliptic counterparts. For instance, Huisken's monotonicity for the mean curvature flow parallels Allard's monotonicity for minimal surfaces [2], while Struwe's monotonicity for the harmonic map heat flow [61] corresponds to Schoen-Uhlenbeck's monotonicity for harmonic maps [55]. Further examples include Hamilton's monotonicity for the Yang-Mills heat flow [37], which relates to Price's formula for the elliptic Yang-Mills equation [49], and Almgren's frequency for harmonic functions [3], contrasted with Poon's parabolic frequency for the heat equation [48] (later extended to manifolds by Colding-Minicozzi [22]). These parabolic monotonicity formulas are often more challenging to derive than their elliptic counterparts, as they typically rely on subtle applications of backward heat-type equations.

Given the prevalence of this parabolic-elliptic duality, it is natural to ask whether it reflects a general principle. While elliptic theory is often viewed as the stationary case of parabolic theory, recent work has explored the reverse perspective: expressing parabolic theory as a limiting case of elliptic theory when the spatial dimension tends to infinity. The many parallels between these theories suggest a connection beyond mere analogy, raising the possibility of systematically deriving parabolic monotonicity formulas from their elliptic counterparts. This novel idea was first employed by Perelman [47] in his completion of Hamilton's program for the Ricci flow on 3-manifolds [17, 38].

The present work aims to deepen the understanding of the connection between parabolic theory and high-dimensional elliptic theory. To this end, we first demonstrate that the same underlying principle extends to fractional operators, where fractional parabolic monotonicity emerges as a high-dimensional limit of fractional elliptic monotonicity. As an application, we derive a monotonicity formula for a semilinear fractional parabolic equation.

Furthermore, building on Perelman's work on the Ricci flow, we show that Perelman's entropy functional (commonly known as the $\mathcal{W}$-functional) arises as a high-dimensional limit of Colding's monotonic volume for level sets of appropriately scaled Green functions on non-parabolic manifolds with nonnegative Ricci curvature.

We begin by introducing the general framework for heat-type equations, followed by a review of Perelman's original derivation of his celebrated reduced volume. We then provide a detailed exposition of the results obtained in this work.

1.0.1 Parabolic theory as a limit of elliptic theory

To formalize the parabolic-to-elliptic perspective, consider a smooth, ancient solution $u : \mathbb{R}^d \times (-\infty, 0] \to \mathbb{R}$ of the heat equation on Euclidean space,

$$\partial_t u - \Delta_x u = 0.$$

Let $y \in \mathbb{R}^N$, and express it in polar coordinates as $y = (r, \theta)$, where $r > 0$ and θ is a coordinate on the $(N-1)$-dimensional sphere of radius 1, $\mathbb{S}_1^{N-1} \subset \mathbb{R}^N$. In these coordinates, the Laplacian becomes,

$$\Delta_y f = \partial_r^2 f + \frac{N-1}{r}\partial_r f + \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}} f,$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace-Beltrami operator on the $(N-1)$-dimensional sphere. For spherically symmetric functions $f = f(r)$, this simplifies to

$$\Delta_y f = \partial_r^2 f + \frac{N-1}{r}\partial_r f.$$

As $N \to \infty$, the first-order term dominates. To use this asymptotic behavior, we introduce

$$\tau = \frac{r^2}{2N} = \frac{y_1^2 + \cdots + y_N^2}{2N},$$

which represents the average squared norm of y, and set $t = -\tau$. A direct application of the chain rule yields,

$$\Delta_y f = -\frac{2t}{N}\partial_t^2 f - \partial_t f.$$

In regions where $r^2 = O(N)$ and thus $\tau = O(1)$, we have

$$\Delta_y f \to -\partial_t f \quad \text{as} \quad N \to \infty.$$

This convergence suggests the following construction. Given an ancient solution u for the heat equation, we consider the lift of u which we denote by u_N, defined as,

$$u_N(x,y) := u(x,t) = u\left(x, -\frac{y_1^2 + \cdots + y_N^2}{2N}\right).$$

A calculation shows that u_N satisfies,

$$\Delta_{x,y} u_N = \frac{r^2}{N^2}\partial_t^2 u - \partial_t u + \Delta_x u = \frac{r^2}{N^2}\partial_t^2 u = O(1/N),$$

thus becoming approximately harmonic as N grows large.

The geometric interpretation of this transformation becomes clearer when we express the Euclidean metric g_E on $\mathbb{R}^{N+1} \times \mathbb{R}^d$ in polar coordinates with respect to y,

$$g_E = dr^2 + r^2 g_{\mathbb{S}^N} + g_{\mathbb{R}^d} = \frac{N}{2\tau} d\tau^2 + \tau g_{\tilde{\mathbb{S}}^N} + g_{\mathbb{R}^d}, \tag{1.0.1}$$

where $g_{\tilde{\mathbb{S}}^N}$ is the standard metric on $\mathbb{S}^N$ with constant curvature $1/2N$. This metric decomposition mirrors Perelman's construction for the Ricci flow.

The elliptic approximation relates parabolic theory to elliptic theory as $N \to \infty$. To illustrate its utility, we apply this transformation to the mean value property for harmonic functions as follows. Since u_N is almost-harmonic we expect the mean value formula to hold for u_N, and therefore we have

$$u_N(0,0) \approx \frac{1}{|B^{N+d}(0,r_0)|} \int_{|y|^2+|x|^2 \le r_0^2} u_N(x,y)\, dy\, dx,$$

where $|B^{N+d}(0,r_0)| = C_{N,d} r_0^{N+d}$ is the volume of the ball in $\mathbb{R}^{N+d}$. Rewriting in polar coordinates gives

$$u_N(0,0) \approx c_{N,d} r_0^{-(N+d)} \int_{\mathbb{R}^d} \int_{0 \le r \le \sqrt{r_0^2 - |x|^2}} u_N(x, -r^2/2N) r^{N-1}\, dr\, dx,$$

for some constant $c_{N,d} > 0$.

The key step involves setting $r_0^2 = 2N\tau$ and analyzing the regime where $x = O(1)$ and $\tau = O(1)$. The term r^{N-1} localizes the integral near its endpoint, leading to the approximation

$$\int_{\mathbb{R}^d}\int_{0\le r\le\sqrt{r_0^2-|x|^2}} u_N(x,-r^2/2N)r^{N-1}\,dr\,dx \approx \frac{1}{N}\left(\sqrt{r_0^2-|x|^2}\right)^N u(x,-(r_0^2-|x|^2)/2N).$$

Since $r_0^2 = O(N)$ and $x = O(1)$, we can further approximate $u(x,-(r_0^2-|x|^2)/2N) \approx u(x,-\tau)$. Moreover, the term $\left(\sqrt{r_0^2-|x|^2}\right)^N$ behaves asymptotically as

$$\left(\sqrt{r_0^2-|x|^2}\right)^N \approx r_0^N \exp\left(-\frac{N|x|^2}{2r_0^2}\right).$$

Substituting $r_0^2 = 2N\tau$ and combining these approximations yields the familiar heat kernel representation,

$$u(0,0) \approx \lim_{\tau\to 0^+}\frac{\tilde{c}_{N,d}}{\tau^{d/2}}\int_{\mathbb{R}^d} e^{-|x|^2/4\tau}u(x,-\tau)\,dx.$$

This connection has been further developed by several authors: the exposition here follows Tao's lectures on the Ricci flow [62], while Svérak [68] offers a complementary probabilistic perspective by recalling several ideas dating back to Weiner. Next, we discuss this probabilistic formalism.

1.0.2 A probabilistic approach

Following Svérak [61] (see also [23]), we model a random walk in $\mathbb{R}$, in which a particle begins at $(x,t) = (0,0)$ and takes N steps $y_1,\dots,y_N$. Instead of fixing step sizes, we enforce a global constraint: for the particle located at $(x,t) = (0,0)$, we assume that up to time t the random steps, $y_1,\dots,y_N$ are subject to the constraint

$$y_1^2 + \cdots + y_N^2 = t.$$

After completing all steps, the particle's final position is

$$x = y_2 + \cdots + y_N.$$

We now define the probability law governing $(y_1,\dots,y_N)$. The natural assumption is that the vectors $(y_1,\dots,y_N)$ are uniformly distributed over the $(N-1)$ dimensional sphere of radius $\sqrt{t}$ with respect to the canonical surface measure on the sphere. Let μ_N^t denote the normalized measure, with unit mass. We may write

$$\mu_t^N = \frac{1}{|\mathbb{S}^{N-1}|t^{(N-1)/2}}\sigma_{N-1}^t = \frac{\Gamma(N/2)}{2\pi^{N/2}t^{(N-1)/2}}\sigma_{N-1}^t,$$

where σ_{N-1}^t is the canonical surface measure. Define

$$f_N(y) = y_1 + \cdots + y_N,$$

as the projection of the high-dimensional space onto $\mathbb{R}$ by the rule previously described. The pushforward measure $f_{N\#}(\mu_N^t)$ is explicitly computed as

$$f_{N\#}(\mu_N^t) = \frac{1}{\sqrt{2\pi t}}\frac{\Gamma(N/2)}{\Gamma((N-1)/2)\sqrt{N/2}}\left(1 - \frac{x^2}{\frac{N}{2}2t}\right)^{N/2-3/2} dx.$$

Stirling's formula gives,

$$\lim_{N\to\infty}\frac{\Gamma(N/2)}{\Gamma((N-1)/2)\sqrt{N/2}} = 1,$$

and, combined with the identity

$$\lim_{N\to\infty}\left(1 + \frac{a}{N}\right)^N = e^a,$$

we recover the limiting measure,

$$f_{N\#}(\mu_N^t) \to \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx$$

as $N \to \infty$, which is the heat kernel for

$$\partial_t u - \frac{1}{2}\partial_x^2 u = 0.$$

The same procedure can be carried out for $\mathbb{R}^d$, allowing us to relate integral quantities in the high-dimensional and original spaces via the pushforward, with the heat kernel appearing naturally as the pushforward of our probabilistic law. Davey [23] (see also Davey-Smit [25]) recently used this interpretation to systematically derive parabolic monotonicity formulae from elliptic counterparts, recovering classical results by applying elliptic monotonicity to high-dimensional equations and passing to the limit as $N \to \infty$. These high-dimensional equations are equivalent to the parabolic case (holding if and only if the parabolic equation does) and converge formally to the elliptic equation as $N \to \infty$.

1.0.3 Perelman's reduced volume

Using similar ideas, we now outline Perelman's original derivation of his celebrated reduced volume for the Ricci flow [47]. Let M^n be a closed n-dimensional manifold, and let $g(\tau)$ be a backward solution to the Ricci flow

$$\partial_\tau g = 2\,\mathrm{Ric}, \tag{1.0.2}$$

on M defined on the interval $[0,T]$. Let $r^2 = 2N\tau$, and define Perelman's N-space,

$$\hat{M}^m := (0, \sqrt{2NT})_r \times \mathbb{S}^N_\theta \times M^n_x \subset \mathbb{R}^{N+1} \times M^n, \quad \text{where} \quad m = N+n+1,$$

endowed with the metric

$$\hat{g} := r^2 g_{\mathbb{S}^N} + \left(1 + \frac{Rr^2}{N^2}\right) dr^2 + g. \tag{1.0.3}$$

where R is the scalar curvature of g, and at a point $(r,\theta,x) \in \hat{M}$, $g_{\mathbb{S}^N}$ is evaluated at θ, and R and g are evaluated at $(\tau = r^2/2N, x) \in (0,T) \times M$. Note that (1.0.3) reduces to (1.0.1) when $M = \mathbb{R}^n$ endowed with the Euclidean metric. One can show that $\hat{M}$ becomes asymptotically flat as $N \to \infty$, with its Ricci tensor satisfying $|\hat{\mathrm{Ric}}| = O(1/N)$.

Stationary solutions to (1.0.2) satisfy the Ricci-flat equation,

$$\mathrm{Ric} = 0, \tag{1.0.4}$$

for which the Bishop-Gromov relative volume serves as a fundamental monotonic quantity, and states the following.

Let M be a complete n-dimensional Riemannian manifold whose Ricci curvature satisfies the lower bound $\mathrm{Ric} \geq (n-1)K$ for some constant $K \in \mathbb{R}$. Let M^n_K denote the complete, simply connected n-dimensional space form of constant sectional curvature K, that is, the n-sphere of radius $1/\sqrt{K}$ when $K > 0$, Euclidean space for $K = 0$, or a rescaled hyperbolic space when $K < 0$. Then, for any $p \in M$ and $p_K \in M^n_K$, the ratio

$$\phi(r) = \frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}\, B(p_k,r)},$$

is a non-increasing function of r. This result has found broad applications throughout differential geometry.

The near Ricci-flatness of $(\hat{M}, \hat{g})$ enables a heuristic derivation of monotonic quantities for the Ricci flow via the Bishop-Gromov inequality, which we apply assuming $K = 0$. We examine metric balls centered at $(p, s, 0) \in \hat{M}$, where the $\mathbb{S}^N$ fiber degenerates to a point at $\tau = 0$. Length-minimizing geodesics $\gamma(\tau)$ connecting $(p, s, 0)$ to $(q, \bar{s}, \bar{\tau}) \in \hat{M}$ must be orthogonal to the spherical fibers, since the angular variable only influences the term $r^2 g_{\mathbb{S}^N}$ of the metric. The length of such a geodesic must then be given by

$$\ell(\gamma) = \int_0^{\bar{\tau}} \sqrt{\left(\frac{N}{2\tau} + R\right) + |\dot{\gamma}(\tau)|^2_{g_{ij}(\tau)}}\, d\tau.$$

A Taylor expansion yields the asymptotic behavior

$$\ell(\gamma) = \sqrt{2N\bar{\tau}} + \frac{1}{\sqrt{2N}} \int_0^{\bar{\tau}} \sqrt{\tau} \left(R + |\dot{\gamma}(\tau)|^2_{g_{ij}}\right) d\tau + O\left(N^{-\frac{3}{2}}\right),$$

and therefore, a shortest geodesic should minimize the N-independent functional

$$\mathcal{L}(\gamma) = \int_0^{\bar{\tau}} \sqrt{\tau} \left(R + |\dot{\gamma}(\tau)|^2_{g_{ij}}\right) d\tau.$$

Let $L(q, \bar{\tau})$ denote the infimum of this quantity over all paths joining both endpoints.

A metric sphere $\mathbb{S}_{\hat{M}}(\sqrt{2N\bar{\tau}})$ in $\hat{M}$ of radius $\sqrt{2N\bar{\tau}}$ centered at $(p, s, 0) \in M \times \mathbb{S}^N \times \mathbb{R}_+$ is $O\left(N^{-1}\right)$ close to the hypersurface $\{\tau = \bar{\tau}\}$. Indeed, for $(x, s', \tau(x)) \in \mathbb{S}_{\hat{M}}(\sqrt{2N\bar{\tau}})$, the distance between $(x, s', \tau(x))$ and $(p, s, 0)$ is

$$\sqrt{2N\bar{\tau}} = \sqrt{2N\tau(x)} + \frac{1}{\sqrt{2N}} L(x, \tau(x)) + O\left(N^{-\frac{3}{2}}\right).$$

Rearranging gives,

$$\sqrt{\tau(x)} - \sqrt{\bar{\tau}} = -\frac{1}{2N} L(x, \tau(x)) + O\left(N^{-2}\right) = O\left(N^{-1}\right),$$

Since the metric $2N g_{\alpha\beta}$ on $\mathbb{S}^N$ has constant sectional curvature $1/2N$,

$$\begin{aligned}
\mathrm{Vol}\left(\mathbb{S}_{\tilde{M}}(\sqrt{2N\bar{\tau}})\right) &\approx \int_M \left(\int_{\mathbb{S}^N} dV_{\tau(x) g_{\alpha\beta}}\right) dV_{g_{ij}}(x) \\
&= \int_M (\tau(x))^{\frac{N}{2}} \,\mathrm{Vol}\left(\mathbb{S}^N\right) dV_M \\
&\approx (2N)^{\frac{N}{2}} \omega_N \int_M \left(\sqrt{\bar{\tau}} - \frac{1}{2N} L(x, \tau(x)) + O\left(N^{-2}\right)\right)^N dV_M \\
&\approx (2N)^{\frac{N}{2}} \omega_N \int_M \left(\sqrt{\bar{\tau}} - \frac{1}{2N} L(x, \bar{\tau}) + O\left(N^{-1}\right)\right)^N dV_M,
\end{aligned}$$

where ω_N is the volume of the standard N-dimensional sphere. Comparing with Euclidean sphere volumes,

$$\operatorname{Vol}\left(\mathbb{S}_{\mathbb{R}^{n+N+1}}(\sqrt{2N\bar{\tau}})\right) = (2N\bar{\tau})^{\frac{N+n}{2}}\omega_{n+N},$$

yields the ratio

$$\frac{\operatorname{Vol}\left(\mathbb{S}_{\tilde{M}}(\sqrt{2N\bar{\tau}})\right)}{\operatorname{Vol}\left(\mathbb{S}_{\mathbb{R}^{n+N+1}}(\sqrt{2N\bar{\tau}})\right)} \approx \text{const} \cdot N^{-\frac{n}{2}} \cdot \int_M (\bar{\tau})^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sqrt{\bar{\tau}}}L(x,\bar{\tau})\right\} dV_M.$$

Consequently, the asymptotic Ricci-flatness of $\tilde{M}$ and Bishop-Gromov theorem suggest the monotonicity of

$$\bar{V}(\bar{\tau}) := \int_M (4\pi\bar{\tau})^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sqrt{\bar{\tau}}}L(x,\bar{\tau})\right\} dV_M,$$

known as Perelman's reduced volume. This fact was rigorously proved in [47].

1.1 Overview and results

Having established the general framework, we now present the specific problems addressed in this thesis.

1.1.1 Fractional operators

Fractional operators provide the mathematical foundation for modeling anomalous diffusion (that is, stochastic processes deviating from Brownian motion), a field of growing research interest. These processes are characterized by nonlocal integro-differential operators, chief among them the fractional Laplacian introduced by Riesz [52], defined for $v : \mathbb{R}^d \to \mathbb{R}$ as

$$(-\Delta)^s v(x) := \frac{4^s\Gamma(d/2+s)}{\pi^{d/2}|\Gamma(-s)|} \lim_{r\to 0^+} \int_{\mathbb{R}^d \setminus B_r(x)} \frac{v(x)-v(z)}{|x-z|^{d+2s}}\, dz, \quad \text{where } 0 < s < 1. \tag{1.1.1}$$

This operator of order $2s$ continuously interpolates between the identity ($s \to 0^+$) and the standard Laplacian ($s \to 1^-$), making it particularly versatile for capturing a wide range of diffusive behaviors. Its nonlocal nature allows it to model systems with long-range interactions and memory effects that elude classical diffusion operators, while maintaining many useful analytic properties of elliptic operators. The combination of such features explains its growing importance across physics, biology, and engineering applications where complex, multiscale transport phenomena occur.

Notable applications include modified gravity theories, where, for example, Benetti et al. [7] investigate whether known galactic dark matter observations could arise from fractional gravity effects. In this context, while the standard law of inertia remains valid, the gravitational potential is determined by a modified Poisson equation incorporating fractional derivatives that capture nonlocal interactions. Subsequent work [6] has extended this approach to galaxy clusters, and in [35,65,66] a fractional-dimension gravity model that successfully reproduces flat rotation curves through modifications to the Newtonian potential was also explored.

Additional applications include the study of anomalous diffusion and transport processes. In porous media, fractional diffusion equations describe contaminant transport exhibiting heavy-tailed particle jumps [8], while in neuroscience, fractional cable equations provide more accurate modeling of electrodiffusion of ions in nerve cells [41]. Turbulent systems and plasma physics similarly benefit from fractional dynamics in describing anomalous particle motion [44]. Beyond the natural sciences, the fractional Laplacian finds applications in financial mathematics, where it models extreme market movements through Lévy processes [54] and in image processing, where fractional order filters have been studied for enhanced edge detection and noise reduction [34]. In social sciences, it has also been employed to construct superdiffusive models explaining criminal activity patterns through long-range jump processes [56]. Several applications are also available in geometric analysis and probability theory [32].

The natural parabolic counterpart to the fractional Laplacian, the fractional heat operator, is defined as

$$(\partial_t - \Delta)^s u(x,t) := \int_{-\infty}^{t} \int_{\mathbb{R}^d} (u(x,t) - u(z,\tau)) K_s(x-z, t-\tau)\, dz\, d\tau, \tag{1.1.2}$$

where

$$K_s(z,\tau) = \frac{1}{(4\pi)^{d/2}|\Gamma(-s)|} \frac{e^{-|z|^2/4\tau}}{\tau^{d/2+1+s}}, \tag{1.1.3}$$

and is non-local in both space and time. This operator has also found several interesting applications, since it models systems exhibiting anomalous diffusion and memory effects. For example, in statistical physics, it arises in the study of continuous-time random walks with Lévy flights or subdiffusive trapping events, providing a model for particle motion that deviates from classical Brownian behavior due to heavy-tailed jump distributions or waiting times [44]. In ecology, it models population dynamics in heterogeneous environments, where species propagation can be accelerated by fast-diffusion channels such as river networks or transportation corridors [9]. In finance, it has been used to model scenarios where the waiting time between transactions is correlated with ensuing price jumps [51]. The flat parabolic Signorini problem has also been shown to be equivalent to the obstacle problem for $(\partial_t - \Delta)^{1/2}$. Additional applications in physics include the modeling of viscoelastic materials and non-Newtonian fluids, where it describes memory-dependent stress-strain relationships [28], and chaotic Hamiltonian systems, where it characterizes anomalous transport phenomena arising from fractal phase-space structures [67].

Given the significant similarities shared by both operators with their local counterparts, a natural question is whether parabolic monotonicity can also be recovered from elliptic monotonicity in fractional contexts. This work establishes that the parabolic-to-elliptic framework extends to both fractional operators, allowing to recover fractional parabolic monotonicity formulae from elliptic ones. This is illustrated by studying solutions of the semilinear fractional equation,

$$(\partial_t - \Delta)^s u = |u|^{p-1}u. \tag{1.1.4}$$

The local analogue of this equation,

$$\partial_t u - \Delta u = |u|^{p-1}u \tag{1.1.5}$$

is well-understood, with established results for its well-posedness, regularity theory, and blow-up profiles (see [50] and references therein). For the local case, Giga and Kohn [33] derived a fundamental monotonicity formula: if u is a solution of (1.1.5), applying a time reversal $t \mapsto -t =: \tau$ to u, the function

$$\mathcal{D}(\tau) := \int_{\mathbb{R}^d} \left(\frac{|\nabla u|^2}{2} - \frac{1}{p+1}|u|^{p+1} \right) \Phi dx + \frac{1}{p-1} \int_{\mathbb{R}^d} \frac{u^2}{2\tau} \Phi \, dx, \tag{1.1.6}$$

is non-decreasing for the time-reversed variable τ, and its derivative is explicitly given by,

$$\frac{d}{d\tau}\mathcal{D}(\tau) = \int_{\mathbb{R}^d} \left(\partial_\tau u + \frac{x}{2\tau} \cdot \nabla u + \frac{2}{p-1} \frac{u}{2\tau} \right)^2 \Phi \, dx. \tag{1.1.7}$$

Here,

$$\Phi(x,\tau) = (4\pi\tau)^{\frac{p+1}{p-1}} \frac{1}{(4\pi\tau)^{\frac{d}{2}}} e^{-|x|^2/4\tau}$$

is an appropriate rescaling of the backward heat kernel. This formula plays a crucial role in characterizing the blow-up profiles of solutions [33].

Notably, nonlocal monotonicity formulas are derived via their corresponding extension problems for both the fractional Laplacian and the fractional heat operator. These results, first established for the fractional Laplacian in Cafarelli and Silvestre's seminal work [14], interprets the fractional Laplacian as a Dirichlet-to-Neumann opreator for a degenerate but local PDE on the half space $\mathbb{R}^{d+1}_+$. Similarly, these techniques were adapted to the fractional heat operator by Stinga and Torrea [60] and independently, by Nyström and Sande [45], allowing us to reinterpret the fractional heat operator as a local but degenerate parabolic problem on the half space $\mathbb{R}^{d+1}_+$.

Using the extension problem for the fractional Laplacian, several examples of monotonicity formulas were found for fractional elliptic problems. Some of those are an Almgren frequency-type parabolic monotonicity formula due to Caffarelli and Silvestre [14] and an Alt-Caffarelli-Friedman type monotonicity formula proved by Terracini, Verzini and Zilio [63]. For the fractional heat operator, an Almgren frequency-type parabolic monotonicity formula was found by Stinga and Torrea [60], by adapting the techniques discussed in [14].

Going back to equation (1.1.4), we observe that its stationary solutions correspond to solutions of the fractional Lane-Emden equation,

$$(-\Delta)^s u = |u|^{p-1}u, \tag{1.1.8}$$

since, as shown in [60], although the fractional heat operator is nonlocal in both space and time, it reduces to the fractional Laplacian $(-\Delta)^s$ when applied to a function that solely depends on x.

The fractional Lane-Emden equation (1.1.8) has been extensively studied, with many classical results extended to the nonlocal setting (see [16, 29, 42] and references therein). The Dávila-Dupaigne-Wei monotonicity formula [29] for this equation, which was used to classify solutions of finite Morse index, proves particularly relevant in our setting. This quantity, which we will discuss in Section 2.1, can be thought of as the fractional analogue to the local monotonicity formula for the equation $-\Delta u = |u|^{p-1}u$, as discussed by Fazly and Shahgholian [30] (see also the article by Pacard [46] for a similar monotonicity formula in the case $-\Delta u = u^p$).

By using the monotonic quantity for the fractional Lane-Emden equation , we develop a new monotonicity formula for solutions of (1.1.4). Specifically, let $u = u(x,t)$ be a solution to (1.1.4) on a time interval $(-T_I, T_F)$, where $T_I, T_F > 0$. Due to the nonlocal nature of (1.1.2), u needs to be defined in $(-\infty, T_F)$, so we may either prescribe $u(.,t) = f(.,t)$ for $t \leq -T_I$, or consider ancient solutions instead. Let $U(x_0, x, t)$ be its parabolic Caffarelli-Silvestre extension, also defined in $(-T_I, T_F)$. Then, under appropriate growth and regularity assumptions, by applying a time-reversal $t \mapsto -t$ to both u and U, the function

$$\mathcal{J}(t) := \int_{\mathbb{R}^{d+1}_+} x_0^{1-2s} \frac{|\nabla U|^2}{2} \mathcal{G}_s \, dX - \frac{\eta_s}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \tilde{\mathcal{G}}_s \, dx + \frac{s}{p-1} \int_{\mathbb{R}^{d+1}_+} x_0^{1-2s} \frac{U^2}{2t} \mathcal{G}_s \, dX, \tag{1.1.9}$$

where $X = (x_0, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, is non-decreasing for the time-reversed variable t, and its derivative is explicitly given by,

$$\frac{d}{dt}\mathcal{I}(t) = \sqrt{2t}\int_{\mathbb{R}^{d+1}_+} x_0^{1-2s}\left(\partial_t U + \frac{X}{2t}\cdot\nabla U + \frac{2s}{p-1}\frac{U}{2t}\right)^2 \mathcal{G}_s\, dX. \tag{1.1.10}$$

Here, the functions

$$\mathcal{G}_s(X,t) := t^{\frac{2s}{p-1}+1}\mathcal{G}(X,t) \quad \text{and} \quad \tilde{\mathcal{G}}_s(x,t) := t^{\frac{2s}{p-1}+1}\mathcal{G}((0,x),t),$$

are appropriate rescalings of the fundamental solution $\mathcal{G}$ for the extension problem of the equation $(\partial_t - \Delta)^s u = 0$,

$$\mathcal{G}(X,t) = \frac{1}{(4\pi)^{d/2}\Gamma(s)}\frac{e^{-|X|^2/4t}}{t^{d/2+1-s}}, \tag{1.1.11}$$

where $X \in \mathbb{R}^{d+1}_+$, $t > 0$, and η_s is a constant given by,

$$\eta_s := \frac{2s|\Gamma(-s)|}{4^s\Gamma(s)}. \tag{1.1.12}$$

Our result establishes the fractional analogue to the Giga-Kohn monotonicity formula, and is, to the best of our knowledge, the first such formula for semilinear fractional parabolic equations. In order to achieve this, we develop and apply a similar parabolic-to-elliptic procedure to the one we previously described: we use the elliptic monotonicity for the extension problem of the nonlocal equation (1.1.8) to construct our parabolic monotonicity, but extra difficulties arise when treating the boundary in the extension problem.

We remark that, during the final preparation stages of this work, a new article by Davey and Smit was made public on ArXiv also discussing the extension of Perelman's ideas to the fractional framework [24]. Though related to [24], our methodology primarily follows the methods proposed by Perelman [47] and later explored in [23,25]. Through careful variable changes and a redefinition of the dimension for the high-dimensional limit procedure, our high-dimensional space can be reinterpreted within the framework discussed in these earlier works. Another key distinction is that [24] focuses on the equation $(\partial_t - \Delta)^s u = 0$, whereas our nonlinear case presents additional challenges, particularly when addressing the boundary of the space in which the extension problem is defined. By studying the geometry of the high-dimensional space and precisely controlling volume elements on integration domains, we also obtain explicit formulas for the derivative of our monotonic quantity. The existence of multiple viable approaches to this high-dimensional transformation underscores the method's versatility.

The techniques discussed here naturally extend to derive the classical Giga-Kohn monotonicity formula for $\partial_t u - \Delta u = |u|^{p-1}u$ from the monotonicity formula for $-\Delta u = |u|^{p-1}u$ introduced by Fazly and Shahgholian [30]. Since the computations are carried out in a similar manner, we omit them in the interest of brevity. The methods we present hold potential for application in other nonlinear settings, as well as for systems of equations.

1.1.2 A unified framework for Perelman's Ricci flow

The second part of this thesis revisits Perelman's framework to derive new monotonicity formulas for the Ricci flow. In his groundbreaking work [47], Perelman introduced two fundamental quantities: the reduced volume and the entropy. As established earlier, the reduced volume arises from a careful analysis of the Bishop-Gromov inequality in the high-dimensional limit of Perelman's N-space. Perelman's work, however, left the entropy's geometric origin unexplained. We therefore extend this approach to the entropy, revealing both quantities as manifestations of a unified high-dimensional limit in Perelman's N-space.

Perelman's entropy is defined as follows. For a backward solution to the Ricci flow (1.0.2), consider u a solution to,

$$\partial_\tau u = \Delta u - Ru, \tag{1.1.13}$$

positive at the initial time $\tau = 0$, and hence, also positive for all times by the maximum principle. Define f by $u = \tau^{-n/2}e^{-f}$ so that f satisfies,

$$\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}. \tag{1.1.14}$$

Then, the entropy $\mathcal{W}$ (for the function f) is given by,

$$\mathcal{W}(\tau) = \int_M \big(\tau(|\nabla f|^2 + R) + f - n\big)\,(4\pi\tau)^{-n/2}e^{-f}d\nu, \tag{1.1.15}$$

and its derivative takes the form,

$$\frac{d}{d\tau}\mathcal{W} = -\int_M 2\tau \left|\mathrm{Ric} + \nabla\nabla f - \frac{1}{2\tau}g\right|^2 (4\pi\tau)^{-n/2}e^{-f}d\nu, \tag{1.1.16}$$

from where it follows that it is monotonically decreasing in τ.

Deriving the entropy as a high-dimensional limit requires identifying suitable elliptic monotonicity, thus enabling the application of our earlier framework. The elliptic monotonicity we employ is Colding's monotonic volume, introduced in [18]. This quantity, defined via level sets of positive Green functions, was used to study asymptotic cones on Ricci flat non-parabolic manifolds [18, 20]. Generalizations of Colding's monotonic volume were later given by Colding and Minicozzi in [21] and applications to General Relativity were explored by Agostiniani, Mazzieri and Oronzio in [1].

The monotonic volume is defined as follows: on a manifold $(N, \bar{g})$ admitting a a positive and proper Green function G, define $b = G^{1/(2-m)}$. Then, we define 'area' A and the 'volume' V, on the level sets of b as,

$$A(s) = \frac{1}{s^{m-1}} \int_{b=s} (|\nabla b|^2 - 1)|\nabla b| \, dA, \tag{1.1.17}$$

and,

$$V(s) = \frac{1}{s^m} \int_{b \leq s} (|\nabla b|^2 - 1)|\nabla b|^2 \, dV. \tag{1.1.18}$$

Then, the monotonic volume is defined as

$$W(s) = 2(m-1)V(s) - A(s), \tag{1.1.19}$$

and the derivative of W is given by the expression (see Theorem 2.4 in [18]),

$$\frac{d}{ds} W(s) = -\frac{1}{2s^{m+1}} \int_{b \leq s} \left(\left| \nabla\nabla b^2 - \frac{\Delta b^2}{m} g \right|^2 + \mathrm{Ric}(\nabla b^2, \nabla b^2) \right) dV, \tag{1.1.20}$$

from which it follows that it is monotonically decreasing in s.

The flat Euclidean space $\mathbb{R}^m$, provides a computable example for these quantities, where the Green function at the origin is given by $G(x) = 1/|x|^{m-2}$. If we define $b = G^{1/(2-m)}$, then $b = |x|$, and therefore, $|\nabla b| = 1$. Consequently, in $\mathbb{R}^m$, the area, volume, and monotonic volume are all identically zero.

The ideas outlined in Section 1.0.1 can also be applied here. Following this procedure, we analyze the level sets $b = \text{const}$, and show that Perelman's entropy emerges a high dimensional limit of Colding's monotonic volume on Perelman's N-space. From (1.1.20), we recover the known expression for the derivative of the $\mathcal{W}$ functional, showing that Perelman's volume and entropy can be thought of as emerging from a single, unified high dimensional elliptic framework from its elliptic counterpart, the Ricci flat equation. As a byproduct of our proof, we show that the entropy also emerges as a high-dimensional limit of Colding's area. This behavior also reflects the boundary mass concentration phenomenon discussed in Section 1.0.1 in the high-dimensional limit of Perelman's N-space.

This approach could also lead to new, previously unknown monotonic quantities for the Ricci flow, derived from known elliptic quantities on manifolds with nonnegative Ricci curvature.

1.1.3 Contributions

Chapter 2 collects results from the work,

[12] I. Bustamante - *A monotonicity formula for a semilinear fractional parabolic equation*, ArXiv, (2025).

Chapter 3 is an expanded version of the results on the article,

[13] I. Bustamante, M. Reiris - *Deriving Perelman's entropy from Colding's monotonic volume,* J. Reine Angew. Math., (2025).

Chapter 2

Applications of parabolic-to-elliptic transformations to fractional operators

The parabolic-to-elliptic procedure outlined in Chapter 1, which connects parabolic evolution equations to high-dimensional elliptic problems, can be extended to nonlocal operators. In this chapter, we show that this is indeed the case, by applying a similar procedure to the fractional Laplacian and fractional heat operator instead of their classical counterparts. The extension problems associated with these operators play a central role here since they localize their behavior via degenerate elliptic and parabolic equations on a half-space respectively, allowing us to proceed as in Section 1.0.1.

We begin by introducing the fractional Laplacian and its parabolic counterpart, the fractional heat operator, and recalling their extension properties. We then derive the corresponding parabolic-to elliptic procedure, and illustrate how to use it by obtaining a monotonicity formula for solutions of the fractional semilinear parabolic equation (1.1.4), adapting the elliptic monotonicity of its stationary analogue, the fractional Lane-Emden equation (1.1.8).

2.1 Elliptic and parabolic fractional operators

Both the fractional Laplacian (1.1.1) and the fractional heat operator (1.1.2) have several interesting properties as well as multiple definitions. Here, we focus on the ones relevant to our approach. For convenience and to distinguish between the different setups, we will work on $\mathbb{R}^N$ for elliptic problems, and on $\mathbb{R}^d \times \mathbb{R}$ for parabolic problems, and we will later take $N = d + n$ as $n \to \infty$.

2.1.1 The fractional Laplacian

The fractional Laplacian admits multiple equivalent definitions; for example, [40] catalogs ten distinct characterizations and establishes their equivalence. In this work, we use the pointwise definition (1.1.1), which is well-defined provided that $v \in C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$ and v satisfies

$$\int_{\mathbb{R}^N} \frac{|v(z)|}{(1+|z|)^{N+2s}} dz < +\infty, \tag{2.1.1}$$

as shown in [27].

It is particularly useful to employ different definitions to showcase different properties of this operator. For this reason, and to motivate the pointwise definition we adopt, we start by showcasing the definition of the fractional Laplacian as a pseudo-differential operator acting on the Schwartz class $\mathcal{S}$

$$\mathcal{S} := \left\{ v \in C^\infty(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N} |x^\alpha \partial^\beta v(x)| < \infty \forall \alpha, \beta \in \mathbb{N}^N \right\},$$

via the Fourier transform.

We start by recalling the Fourier transform $\mathfrak{F}$, defined as

$$\mathfrak{F}(v)(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi.x} v(x)\, dx,$$

for any $v \in \mathcal{S}$. Similarly, the inverse Fourier transform of v is defined for $x \in \mathbb{R}^N$ as

$$\mathfrak{F}^{-1}(v)(x) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i\xi.x} v(x)\, dx. \tag{2.1.2}$$

Observe that $\mathfrak{F}(v)(\xi) = \mathfrak{F}^{-1}(v)(-\xi)$, so the properties of $\mathfrak{F}(v)$ also hold for $\mathfrak{F}^{-1}(v)$. A thorough discussion of the Fourier transform can be found in [59].

Proposition 2.1.1. *Let $v \in L^1(\mathbb{R}^N)$. For any $z \in \mathbb{R}^N$, define $T_z v(x) := v(x-z)$. Then,*

$$\mathfrak{F}(T_z v)(\xi) = e^{-i\xi.z} \mathfrak{F}(v)(\xi). \tag{2.1.3}$$

If $v \in \mathcal{S}$,

$$\mathfrak{F}(\partial^\alpha v)(\xi) = (i\xi)^\alpha \mathfrak{F}(v)(\xi), \tag{2.1.4}$$

for any multiindex $\alpha \in \mathbb{N}^k$.

If $v \in L^2(\mathbb{R}^N)$, then the inversion formula holds,

$$\mathfrak{F}^{-1}(\mathfrak{F}(v))(x) = v(x) \quad a.e.\ x \in \mathbb{R}^N. \tag{2.1.5}$$

In particular, if $\mathfrak{F}(v) \in L^1(\mathbb{R}^N)$*, then* v *is continuous and*

$$v(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \mathfrak{F}(v)(\xi) e^{ix.\xi} d\xi.$$

Moreover, the Foruier transform is an isomorphism of the Schwartz class $\mathcal{S}$.

From (2.1.4) we observe that

$$\mathfrak{F}(-\Delta v)(\xi) = |\xi|^2 \mathfrak{F}(v)(\xi),$$

and from (2.1.5),

$$-\Delta v(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i\xi.x} |\xi|^2 \mathfrak{F}(v)(\xi)\, d\xi \tag{2.1.6}$$

follows. Therefore, the Laplacian can be represented by an integral formula in the frequency space, where its symbol is $\sigma(\xi) = |\xi|^2$. In a similar manner, one can prove the following.

Proposition 2.1.2. *Let* $s \in (0,1)$*, and let* $(-\Delta)^s$ *denote the operator defined in* (1.1.1)*. If* $v \in \mathcal{S}$*, then*

$$(-\Delta)^s v = \mathfrak{F}^{-1}(|\xi|^{2s} \mathfrak{F}(v)).$$

In particular, the fractional Laplacian is an elliptic pseudo-differential operator of order $2s$*.*

Proof. See Proposition 3.3 in [27]. □

Proposition 2.1.2 shows that we can define the fractional Laplacian $(-\Delta)^s$ as the pseudo-differential operator with symbol $\sigma(\xi) = |\xi|^{2s}$. We remark that the constant

$$C_{N,s} := \frac{4^s \Gamma(N/2 + s)}{\pi^{N/2} |\Gamma(-s)|},$$

appearing on the definition (1.1.1) of the fractional Laplacian acting on functions $v : \mathbb{R}^N \to \mathbb{R}$, ensures consistency between the pointwise and the Fourier definition of this operator. Furthermore, it can also be shown that, when $s \to 0^+$ and $s \to 1^-$, we have

$$C_{N,s} \sim s(1-s) \text{ for } s \to \{0^+, 1^-\}.$$

Using this asymptotic behavior of $C_{N,s}$, we can also show the following.

Proposition 2.1.3. *For any* $u \in \mathcal{S}$*, the following statements hold:*

$$\lim_{s \to 0^+} (-\Delta)^s u = u,$$

and

$$\lim_{s\to 1^+}(-\Delta)^s u = -\Delta u.$$

Proof. The case for $s \to 1^+$ can be found in [11], and the case $s \to 0^-$ in [43]. For a direct computation, see Proposition 4.4 in [27]. □

We now discuss two other equivalent definitions that will become useful for our analysis. First, we recall the semigroup definition of the fractional Laplacian, since it will also be relevant when discussing fractional heat operators.

Recall that for any $0 < s < 1$, we have the formula,

$$\Gamma(-s) = \frac{\Gamma(1-s)}{-s} = \int_0^\infty (e^{-w}-1)\frac{dw}{w^{1+s}} < 0.$$

Let $\lambda > 0$, and define $w = r\lambda$. We then get,

$$\lambda^s = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{r\lambda}-1)\frac{dt}{t^{1+s}}, \tag{2.1.7}$$

which also holds when $\lambda = 0$. Now let $\lambda = |\xi|^2$, multiply by $\mathfrak{F}(v)$ and use the Fourier definition of the fractional Laplacian to obtain

$$\mathfrak{F}((-\Delta)^s v)(\xi) = |\xi|^{2s}\mathfrak{F}(v)(\xi) = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{-r|\xi|^2}\mathfrak{F}(v)(\xi) - \mathfrak{F}(v)(\xi))\frac{dr}{r^{1+s}}.$$

After applying the inverse Fourier transform, we have

$$(-\Delta)^s v(x) = \frac{1}{(2\pi)^{N/2}\Gamma(-s)}\int_{\mathbb{R}^N}\int_0^\infty (e^{-r|\xi|^2}\mathfrak{F}(v)(\xi)e^{ix.\xi} - \mathfrak{F}(v)(\xi)e^{ix.\xi})\frac{dr}{r^{1+s}}d\xi, \tag{2.1.8}$$

which is absolutely convergent since, by (2.1.7),

$$\int_{\mathbb{R}^N}\int_0^\infty (1-e^{-r|\xi|^2})|\mathfrak{F}(v)(\xi)|\frac{dr}{r^{1+s}}d\xi = |\Gamma(-s)|\int_{\mathbb{R}^N}|\xi|^{2s}|\mathfrak{F}(v)(\xi)|\,d\xi.$$

Applying Fubini's Theorem in (2.1.8) and recalling (2.1.2), we obtain the semigroup definition of the fractional Laplacian,

$$(-\Delta)^s v(x) = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{r\Delta}v(x) - v(x))\frac{dr}{r^{1+s}}, \tag{2.1.9}$$

for $v \in \mathcal{S}$.

The family of operators $\{e^{r\Delta}\}_{r\geq 0}$ is the heat difussion semigroup generated by Δ. If we consider the solution u of the heat equation on the whole space with initial temperature v,

$$\begin{cases} \partial_r u = \Delta u & \text{for } x \in \mathbb{R}^N, r > 0 \\ u(x,0) = v(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{2.1.10}$$

we may apply the Fourier transform in the x-variable for each fixed r, to obtain

$$\mathfrak{F}(u)(\xi, r) = e^{-r|\xi|^2}\mathfrak{F}(v)(\xi) = \mathfrak{F}(e^{r\Delta}v)(\xi).$$

Moreover, it is well-known that u may be written as a convolution with the Gauss-Weierstrass heat kernel,

$$W_r(x) := (4\pi r)^{-N/2} e^{-|x|^2/4r}, \tag{2.1.11}$$

that is,

$$u(x, r) = W_r * v(x).$$

Substituting this expression back onto (2.1.9) and using that

$$\int_{\mathbb{R}^N} W_r(x)\, dx = 1, \tag{2.1.12}$$

we recover expression (1.1.1), as shown in Theroem 12.1 of [59]. Moreover, the semigroup definition of the fractional Laplacian (2.1.9) holds for a more general class than the Schwartz class, since it can also be applied to compute the fractional Laplacian of any $v \in C^{2s+\varepsilon}(\mathbb{R}^N)$ obeying (2.1.1), as mentioned in Remark 2 of [58] (see also Chapter 12 in [59]).

The final definition of the fractional Laplacian we address in this work is the one given through harmonic extensions. This definition, which will serve as our main tool in later sections, characterizes the fractional Laplacian via a Dirichlet-to-Neumann operator using an extension problem in the half-space $\mathbb{R}^{N+1}_+$, as established by Caffarelli and Silvestre [14], and is given as follows.

Let $v : \mathbb{R}^N \to \mathbb{R}$ be such that $v \in C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$, and assume that v obeys (2.1.1). Set

$$a := 1 - 2s \in (-1, 1),$$

and let $V : \mathbb{R}^{N+1}_+ \to \mathbb{R}$ be defined as

$$V(z_0, z) := \int_{\mathbb{R}^N} v(z - y) P(z_0, y) dy,$$

where we denote $(z_0, z) \in \mathbb{R}^{N+1}_+$, $z_0 > 0$, and P is the Poisson kernel,

$$P(z_0, z) := C_{N,a} z_0^{1-a} |(z_0, z)|^{-(N+1-a)}.$$

The constant $C_{N,a} > 0$ is chosen so that $\int_{\mathbb{R}^N} P(z_0, z) dz = 1$. Then, $V \in C^2(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+})$, $z_0^a \partial_{z_0} V \in C(\overline{\mathbb{R}^{N+1}_+})$ and V is a solution of the extension problem,

$$\begin{cases} \nabla \cdot (z_0^a \nabla V) = 0 & \text{for } (z_0, z) \in \mathbb{R}^{N+1}_+, \\ \lim_{z_0 \to 0^+} V(z_0, z) = v(z) & \text{for } z \in \mathbb{R}^N. \end{cases} \tag{2.1.13}$$

Observe that the first equation of (2.1.13) is equivalent to,

$$\Delta_z V + \frac{a}{z_0} \partial_{z_0} V + \partial^2_{z_0} V = 0 \quad \text{for } (z_0, z) \in \mathbb{R}^{N+1}_+. \tag{2.1.14}$$

Then, the function V obeys,

$$-\lim_{z_0 \to 0^+} z_0^a \partial_{z_0} V(z_0, z) = \kappa_s (-\Delta)^s v,$$

with

$$\kappa_s := \frac{\Gamma(1-s)}{2^{2s-1} \Gamma(s)}.$$

In particular, this procedure allows us to recover $(-\Delta)^s v$ from the normal derivative of the extension V.

2.1.2 The monotonicity formula for the fractional Lane-Emden equation

In order to apply the previous dicussion to the Lane-Emden equation (1.1.8), we let $0 < s < 1$ and $v : \mathbb{R}^N \to \mathbb{R}$ such that $v \in C^{2s+\varepsilon}(\mathbb{R}^N)$ for some $\varepsilon > 0$, and assume v obeys (2.1.1) so that its fractional Laplacian, $(-\Delta)^s v$, is well defined. Then, its Caffarelli-Silvestre extension V obeys (2.1.13), and moreover, it satisfies

$$-\lim_{z_0 \to 0^+} z_0^a \partial_{z_0} V(z_0, z) = \kappa_s |v|^{p-1} v\,(z), \tag{2.1.15}$$

as discussed in [29]. For such V, the following monotonicity formula is known.

Theorem 2.1.4 (Theorem 1.4 in [29]). *Let $V(z_0, z) \in C^2(\mathbb{R}^{N+1}_+) \cap C(\overline{\mathbb{R}^{N+1}_+})$, such that V*

obeys (2.1.15) and (2.1.14), and suppose $z_0^a \partial_{z_0} V \in C(\overline{\mathbb{R}^{N+1}_+})$. *For* $R > 0$, *let*

$$\begin{aligned} E(R) :=& R^{2s\frac{p+1}{p-1}-N} \left(\frac{1}{2} \int_{\mathbb{R}^{N+1}_+ \cap B_R^{N+1}} z_0^a |\nabla V|^2 dz dz_0 - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}^{N+1}_+ \cap B_R^{N+1}} |V|^{p+1} dz \right) \\ &+ R^{2s\frac{p+1}{p-1}-N-1} \frac{s}{p-1} \int_{\partial B_R^{N+1} \cap \mathbb{R}^{N+1}_+} z_0^a V^2 d\sigma. \end{aligned} \tag{2.1.16}$$

Then, E *is a non-decreasing function of* R. *Moreover,*

$$\frac{dE}{dR} = R^{2s\frac{p+1}{p-1}-N+1} \int_{\partial B_R^{N+1} \cap \mathbb{R}^{N+1}_+} z_0^a \left(\frac{\partial V}{\partial r} + \frac{2s}{p-1} \frac{V}{r} \right)^2 d\sigma. \tag{2.1.17}$$

Here, B_R^{N+1} denotes the Euclidean ball in $\mathbb{R}^{N+1}$ centered at the origin of radius R, σ is the N-dimensional Hausdorff measure restricted to the hypersurface ∂B_R^{N+1}, $r = |(z_0, z)|$ the Euclidean norm of a point $(z_0, z) \in \mathbb{R}^{N+1}_+$, and $\partial_r = \frac{(z_0,z)}{r} \cdot \nabla$ is the radial derivative.

2.1.3 The fractional heat operator

We now turn to the fractional heat operator (1.1.2). Analogously to the fractional Laplacian, this operator admits a definition via the Fourier transform, given by

$$\mathfrak{F}((\partial_t - \Delta)^s u)(\xi, \rho) := (i\rho + |\xi|^2)^s \mathfrak{F}(u)(\xi, \rho),$$

for a given function $u = u(x,t) : \mathbb{R}^d \times \mathbb{R}$. However, a limitation of this definition is that it only applies to functions defined on all of space-time. Therefore, if we want to apply the operator to functions defined up to a finite time T_F, we need a pointwise formula. For this reason, we define the fractional heat operator via the pointwise formula (1.1.2), where, following [60], we additionally require that $u : \mathbb{R}^d \times (-\infty, T_F) \to \mathbb{R}$ is parabolic Hölder continuous of order $2s + \varepsilon$ for some $\varepsilon > 0$. The space of parabolic Hölder continuous functions is defined as follows.

Definition 2.1.5. *We say* $u : \Omega \to \mathbb{R}$ *is parabolic Hölder continuous of order* $0 < \gamma \leq 1$ *if* $u \in L^\infty(\Omega)$, *and there exists* $\gamma > 0$ *such that*

$$|u(x,t) - u(z,s)| \leq C(|x-z|^2 + |t-s|)^{\gamma/2},$$

for every $(x,t), (z,s) \in \Omega$. *In this case, we write* $u \in C^\gamma_{t,x}(\Omega)$.

For $1 < \gamma \leq 2$, *we say that* $u \in C^\gamma_{t,x}(\Omega)$ *if* $u \in L^\infty(\Omega)$, u *is* $\gamma/2$*-Hölder continuous in* t *uniformly in* x *and its gradient* $\nabla_x u$ *is* $(\gamma - 1)-$*Hölder continuous in* x *uniformly in* t.

We notice that, although our operator cannot be the fractional heat operator in the

Fourier sense, their pointwise formulas coincide whenever the Fourier definition holds [60]. As in the case of the fractional Laplacian, it can also be shown that our operator is the fractional heat operator in the sense of semigroups [31]. To see this, let $\mathcal{L} = \partial_t - \Delta$ be the classical heat operator. The formula (2.1.7) used for the semigroup definition of the fractional Laplacian holds for general operators, as shown in [10, 57]. Specifically, given an operator L, the semigroup formula for L^s reads,

$$L^s v = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-rL} v - v) \frac{dr}{r^{1+s}}, \quad 0 < s < 1. \tag{2.1.18}$$

Then, $\mathcal{L}^s$ may be written as,

$$\mathcal{L}^s = \frac{1}{|\Gamma(-s)|} \int_0^\infty \left(I - e^{-r\mathcal{L}} \right) \frac{dr}{r^{1+s}}, \tag{2.1.19}$$

where, as before, $v := e^{-r\mathcal{L}} u$ is the solution of

$$\begin{cases} \partial_r v = -\mathcal{L} v, & r > 0 \\ v|_{r=0} = u. \end{cases}$$

Now, v obeys

$$\begin{cases} (\partial_r + \partial_t) v = \Delta v, & x \in \mathbb{R}^N, t, r > 0 \\ v(x, t, 0) = u(x, t), & x \in \mathbb{R}^N, t > 0 \end{cases}$$

and therefore we may write,

$$v(x, t, s) = \int_{\mathbb{R}^d} W_r(z) u(x - z, t - r)\, dz,$$

where W_r is the d-dimensional Gauss-Weierstrass kernel (2.1.11). We may now use this expression together with (2.1.19) and (2.1.12) to obtain

$$\begin{aligned} \mathcal{L}^s u(x, t) &= \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty (u(x, t) - v(x, t, r)) \frac{ds}{s^{1+\sigma}} \\ &= \frac{1}{|\Gamma(-\sigma)|} \int_0^\infty \int_{\mathbb{R}^N} (u(x, t) - u(x - z, t - r)) W_r(z) dz \frac{dr}{r^{1+s}}, \end{aligned}$$

which coincides with our pointwise expression (1.1.2) for the fractional heat operator. Since this calculation remains valid regardless of whether u is defined for all positive times, and since the integral is well-defined for any $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-\infty, T_F))$, we conclude that our operator coincides with $(\partial_t - \Delta)^s$ in the semigroup sense. Henceforth, we will make no distinction between these two operators.

General properties of the fractional heat operator $(\partial_t - \Delta)^s$ can be found in [60] (see also [45]). Some relevant properties are its invariance under translations in space and time and its homogeneity of order $2s$ under the scaling $x \to \lambda x, t \to \lambda^2 t$. We also remark that a strong maximum principle holds for this operator [60].

When applied to functions depending only on t or x, space or time, the fractional heat operator simplifies to the Marchaud derivative or the fractional Laplacian respectively. In fact we have, for $u = u(t)$,

$$\begin{aligned}(\partial_t - \Delta)^s u(t) &= \frac{1}{|\Gamma(-s)|} \int_{-\infty}^{t} \frac{u(t) - u(\tau)}{(t-\tau)^{1+s}} d\tau, \\ &= (\partial_t)^s u(t),\end{aligned} \tag{2.1.20}$$

which is the Marchaud derivative of order s [53].

On the other hand, if $u(x,t) = u(x)$, we define $r = t - \tau$ and observe that

$$\begin{aligned}(\partial_t - \Delta)^s u(x) &= \int_0^\infty \int_{\mathbb{R}^d} \big(u(x) - u(z)\big) K_s(x-z,r)\, dz\, dr, \\ &= \int_{\mathbb{R}^d} \big(u(x) - u(z)\big) \left(\int_0^\infty K_s(x-z,r)\, dr\right) dz.\end{aligned}$$

Since

$$\int_0^\infty K_s(x-z,r)\, dr = \frac{4^s \Gamma\left(\frac{d}{2}+s\right)}{\pi^{d/2}|\Gamma(-s)|} \cdot \frac{1}{|x-z|^{d+2s}},$$

we find,

$$\begin{aligned}(\partial_t - \Delta)^s u(x) &= \int_{\mathbb{R}^d} \big(u(x) - u(z)\big) \left(\int_0^\infty K_s(x-z,r)\, dr\right) dz \\ &= C_{d,s} \int_{\mathbb{R}^d} \frac{u(x)-u(z)}{|x-z|^{d+2s}} dz \\ &= (-\Delta)^s u(x).\end{aligned} \tag{2.1.21}$$

As in the case of the fractional Laplacian, we can also define an extension problem the fractional heat operator. Given u a solution to (2.1.26), define the parabolic extension U of u as

$$U(x_0,x,t) := \int_0^\infty \int_{\mathbb{R}^d} P^s_{x_0}(z,\tau) u(x-z,t-\tau) dz d\tau, \tag{2.1.22}$$

where $X = (x_0, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $P^s_{x_0}(z,\tau)$ is the fractional Poisson kernel,

$$P^s_{x_0}(z,\tau) = \frac{1}{4^{d/2+s}\pi^{d/2}\Gamma(s)} \frac{x_0^{2s}}{\tau^{d/2+1+s}} e^{-(x_0^2+|z|^2)/4\tau},$$

and the constant is chosen so that

$$\int_0^\infty \int_{\mathbb{R}^d} P^s_{x_0}(z,\tau)dzd\tau = 1. \tag{2.1.23}$$

As shown in [60], U is well defined whenever $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-\infty, T_F))$ is parabolic Hölder continuous and, moreover, it satisfies two key properties: first, it solves the extension problem,

$$\begin{cases} \partial_t U = \Delta_x U + \frac{a}{x_0}\partial_{x_0} U + \partial^2_{x_0} U, & \text{for } (X,t) \in \mathbb{R}^{d+1}_+ \times (-T_I, T_F), \\ \lim\limits_{x_0 \to 0^+} U(x_0,x,t) = u(x,t), & \text{for } (x,t) \in \mathbb{R}^d \times (-T_I, T_F), \end{cases} \tag{2.1.24}$$

and second, the fractional heat operator can be recovered using the normal derivative at the boundary $\partial \mathbb{R}^{d+1}_+$,

$$\eta_s |u|^{p-1}u = \eta_s(\partial_t - \Delta)^s u = -\lim_{x_0 \to 0^+} x_0^a \partial_{x_0} U(x_0,x,t),$$

where η_s is the constant defined in (1.1.12). The proof of this fact presented in [60] relies on the Fourier transform. Nevertheless, as discussed in Section 2 of [31], U obeys these properties whenever the integrals involved are well defined.

The extension (2.1.22) can alternatively be expressed in terms of the fundamental solution $\mathcal{G}$. If u is a solution to the master equation

$$(\partial_t - \Delta)^s u = h,$$

for some h regular enough, the solution of the parabolic extension problem (2.1.24) for u can be written as,

$$U(x_0,x,t) := \int_0^\infty \int_{\mathbb{R}^d} \mathcal{G}(x_0,z,\tau) h(x-z,t-\tau) dzd\tau, \tag{2.1.25}$$

where $\mathcal{G}$ is defined in (1.1.11), see [60]. As before, U obeys

$$\eta_s h(x,t) = -\lim_{x_0 \to 0^+} x_0^{1-2s} \partial_{x_0} U(x_0,x,t).$$

Moreover, it can be checked that the function $\mathcal{G}$ obeys

$$\lim_{x_0 \to 0^+} x_0^a \partial_{x_0} \mathcal{G}(x_0,x,t) = 0,$$

for any $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, and

$$\partial_t \mathcal{G} = \Delta_x \mathcal{G} + \frac{a}{x_0}\partial_{x_0}\mathcal{G} + \partial_{x_0}^2 \mathcal{G},$$

for any positive time t.

2.1.4 A fractional semilinear parabolic equation

Let $0 < s < 1$, and assume that $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-\infty, T_F))$ solves

$$(\partial_t - \Delta)^s u = |u|^{p-1}u \quad \text{for } (x,t) \in \mathbb{R}^d \times (-T_I, T_F), \tag{2.1.26}$$

where $T_I, T_F > 0$. The data $u|_{(-\infty,-T_I]}$ may be prescribed, thereby considering the problem,

$$\begin{cases} (\partial_t - \Delta)^s u = |u|^{p-1}u & \text{for } (x,t) \in \mathbb{R}^d \times (-T_I, T_F) \\ u(x,t) = f(x,t) & \text{for } (x,t) \in \mathbb{R}^d \times (-\infty, -T_I]. \end{cases} \tag{2.1.27}$$

In [31], a similar problem has been studied, considering instead $(\partial_t - \Delta)^s u = u^p$ with nonnegative memory data f. For this problem, they show the following: if the memory data f is nonnegative, f is C^1 in time and both f and $\partial_t f$ decay as $|t|^{-\sigma}$ for some $\sigma > s$, then the problem is well-posed. We note that this is a special case of our equation since, by the maximum principle, for such nonnegative memory data f, we have $u \geq 0$ and therefore our problem reduces to the case discussed there.

Another case of interest is ancient solutions to the equation (1.1.4), that is, functions $u : \mathbb{R}^d \times (-\infty, T_F) \to \mathbb{R}$ such that u solves $(\partial_t - \Delta)^s u = |u|^{p-1}u$ for $(x,t) \in \mathbb{R}^d \times (-\infty, T_F)$. For any of the two problems the following discussion holds, where, in the case of ancient solutions, we set $T_I = \infty$.

We consider backward solutions of equation (1.1.4), which are defined as follows: for any function $g : \mathbb{R}^d \times \mathbb{R}$, we let $\bar{g}(x,t) := g(x,-t)$ denote its time reversal. Then, given a function $u : (-\infty, T_F) \to \mathbb{R}$ which solves (2.1.26), we have that $\bar{u} : \mathbb{R}^d \times (-T_F, +\infty)$ solves

$$(-\partial_t - \Delta)^s u = |u|^{p-1}u \ \text{ for } (x,t) \in \mathbb{R}^d \times (-T_F, T_I), \tag{2.1.28}$$

where

$$\begin{aligned} (-\partial_t - \Delta)^s u(x,t) &= (\partial_t - \Delta)^s \bar{u}(x,-t) \\ &= \int_t^\infty \int_{\mathbb{R}^d} (u(x,t) - u(z,\tau)) \overline{K_s}(x-z, t-\tau)\, dz d\tau, \end{aligned}$$

see [24, 60]. To show this, notice that if u solves (2.1.26), then

$$(-\partial_t - \Delta)^s \bar{u}(x,t) = (\partial_t - \Delta)^s u(x,-t) = |u|^{p-1}u(x,-t) = |\bar{u}|^{p-1}\bar{u}(x,t). \tag{2.1.29}$$

Similarly, starting from a function $u : \mathbb{R}^d \times (-T_F, +\infty) \to \mathbb{R}$ which solves the backward fractional heat equation (2.1.28), its time reversal $\bar{u}$ is a solution of the forward fractional heat equation (2.1.26), and from (2.1.22) it follows that the extension associated to u is

$$U(x_0,x,t) = \bar{U}(x_0,x,-t) = \int_{-\infty}^{0}\int_{\mathbb{R}^d} \overline{P^s_{x_0}}(z,\tau)u(x-z,t-\tau)dzd\tau, \tag{2.1.30}$$

where $\bar{U}$ is the extension (2.1.22) associated to the (forward) solution $\bar{u}$. By (2.1.24), it follows that U is a solution to the backward extension problem,

$$\begin{cases} \partial_t U + \Delta_x U + \frac{a}{x_0}\partial_{x_0}U + \partial^2_{x_0}U = 0, & \text{for } (X,t) \in \mathbb{R}^{d+1}_+ \times (-T_F, T_I), \\ \lim\limits_{x_0\to 0^+} U(x_0,x,t) = u(x,t), & \text{for } (x,t) \in \mathbb{R}^d \times (-T_F, T_I). \end{cases} \tag{2.1.31}$$

As before, we find that

$$-\lim_{x_0\to 0^+} x_0^a \partial_{x_0} U(x_0,x,t) = \eta_s(-\partial_t - \Delta)^s u(x,t) = \eta_s |u|^{p-1}u(x,t), \tag{2.1.32}$$

for every $(x,t) \in \mathbb{R}^d \times (-T_F, T_I)$. We remark that the solution U also admits a heat kernel representation similar to (2.1.25) for the memory problem, see [31].

From this point forward, we will exclusively consider backward solutions, defined in $\mathbb{R}^d \times (-T_F, +\infty)$ for some $T_F > 0$, and obey equation (2.1.29) in $(-T_F, T_I)$. We will restrict our attention to compact time intervals which, by a time translation, we may assume to be $[0,T]$ for some $T \in (0, T_I)$. Backward solutions will be denoted by u, and their extensions will be denoted by U. We will work within the class of functions we now define.

Definition 2.1.6. *We say that $U : \mathbb{R}^{d+1}_+ \times [0,T] \to \mathbb{R}$ belongs to the function class $\mathcal{U}([0,T])$ if $U \in C^2(\mathbb{R}^{d+1}_+ \times [0,T]) \cap C(\overline{\mathbb{R}^{d+1}_+} \times [0,T])$ and the following hold:*

(a) $\lim\limits_{x_0\to 0^+} x_0^{1+a}\partial_t U(x_0,x,t) = 0$ *for* $(x,t) \in \mathbb{R}^d \times [0,T]$ *and* $x_0^a \partial_{x_0} U \in C(\overline{\mathbb{R}^{d+1}_+} \times [0,T])$.

(b) The functions f_U belong to $C((0,T); L^2(\mathbb{R}^{d+1}_+, x_0^a dX))$, and

$$\sup_{t\in(0,T)} ||f_U(X,t)||^2_{L^2(\mathbb{R}^{d+1}_+, x_0^a dX)} < \infty,$$

for any of the following f_U*:*

$$e^{-|X|^2/8t}\nabla U, \quad e^{-|X|^2/8t}U, \quad e^{-|X|^2/8t}X.\nabla U, \quad e^{-|X|^2/8t}\partial_t U.$$

(*c*) *We have*

$$\sup_{t\in(0,T)} \left|\left|g_U(X,t)e^{-|X|^2/4t}\right|\right|_{L^1(\mathbb{R}^{d+1}_+,\, x_0^a dX)} < +\infty,$$

for any of the following g_U*:*

$$(X\cdot\nabla U)\partial_t U, \quad UH, \quad \partial_t UH, \quad H(X\cdot\nabla U),$$

where

$$H := 2(X,t)\cdot\nabla_{(X,t)}\partial_t U + (d+1+a)\partial_t U. \tag{2.1.33}$$

Here ∇ *denotes the gradient with respect to the* X *variables, and* $\nabla_{(X,t)}$ *denotes the gradient with respect to the* (X,t) *variables.*

Items (*b*) and (*c*) of Definition 2.1.6 impose moderate growth controls over U and its derivatives. While these conditions suffice for our proof, the results may remain valid under less restrictive assumptions. We also note that related classes of functions (adapted to the method of proof employed and the hypotheses needed for each problem) have been considered when deriving monotonicity formulae for variable coefficient parabolic operators in [25] and, recently, for solutions of the extension problem of the fractional parabolic equation $(-\partial_t - \Delta)^s u = 0$ in [24]. The following proposition presents a class of functions for which its backward extensions obey Definition 2.1.6.

Proposition 2.1.7. *Assume* $u : \mathbb{R}^d \times [0,+\infty) \to \mathbb{R}$ *is a parabolic Hölder continuous function of order* $2s+\varepsilon$ *such that its time reversal* $\bar{u} : \mathbb{R}^d \times (-\infty, 0] \to \mathbb{R}$ *is an ancient solution of* (1.1.4)*. Assume also that* $u \in C^2(\mathbb{R}^d \times [0,+\infty))$ *and that its first and second derivatives are bounded. Then,* $U \in \mathfrak{U}([0,T])$ *for any* $T > 0$*.*

Similarly, let $u : \mathbb{R}^d \times (-\infty, T_F) \to \mathbb{R}$ *be a solution of the problem with memory* (2.1.27)*, such that the memory data is twice differentiable and satisfies* $|(\partial_t - \Delta)^s f(x,t)| \le C$*. Then, its backward extension obeys* $U \in \mathfrak{U}([0,T])$ *for any* $0 < T < T_I$*.*

Proof. Since $|u| < C$, using the bounds on the derivatives together with expressions (2.1.30) and (2.1.23), we can show that $|U| < C$, $|\partial_{x_i} U| < C$ for $i \in \{1,\dots,d\}$, and $|\partial_t U| < C$. The fact that $|\partial_{x_0} U| < C x_0^{-a}$ follows by using the last line of the representation formula (1.5) in [60].

For the problem with memory, we may use the representation formula (2.23) in [31] to prove the last bound instead. The other bounds for U and its derivatives are also valid in this case, and they can be obtained using expression (2.1.30) directly.

From the previous estimates, we see that for any fixed $t > 0$, $f_U(\cdot, t) \in L^2(\mathbb{R}^{d+1}_+, x_0^a dX)$, the supremum of their squared norms is finite and the bounds are explicitly computable. Continuity of the functions $f_U(\cdot, t)$ to $L^2(\mathbb{R}^{d+1}_+,\, x_0^a dX)$ follows from the Dominated Convergence Theorem, by employing the Gaussian decay. Condition (c) follows in a similar manner. The regularity properties of U discussed in [60] ensure the ones in our definition hold. $\square$

2.2 Parabolic-to-elliptic transformations for fractional operators

Next, we adapt the parabolic-to-elliptic transformations discussed in the Section 1.0.1 to the fractional setting as follows. First, we note that the extension problems arising for fractional operators are local (yet degenerate) problems. Then, we work with the extension problems, and interpret the variable x_0 as another variable that needs to be lifted, and perform the lift. Observe that, if t is the backward time (as is the case when considering backward solutions), Perelman's original variables require $t = r^2/2N$, where the variable x remains unchanged. Here we perform a slight modification of the variables for convenience, by absorbing the N-dependence to the high dimensional variables instead of t.

Let $(z_0, z, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^n$. Then, set

$$\begin{cases} x_0 &= \sqrt{n} z_0, \\ x &= \sqrt{n} z, \\ 2t &= R^2 = z_0^2 + |z|^2 + |y|^2. \end{cases} \tag{2.2.1}$$

We now write

$$\mathcal{F}_n(z_0, z, y) := (\sqrt{n} z_0, \sqrt{n} z, R^2/2).$$

Definition 2.2.1. *Let $U : \mathbb{R}^{d+1}_+ \times \mathbb{R} \to \mathbb{R}$. Then, its n-dimensional lift $V_n : \mathbb{R}^{d+n+1}_+ \to \mathbb{R}$ of U is,*

$$V_n(z_0, z, y) := U \circ \mathcal{F}_n(z_0, z, y) = U(x_0, x, t). \tag{2.2.2}$$

Observe that, if U is defined on a region $\mathbb{R}^{d+1}_+ \times [0, T)$, then V_n is defined on the region $\mathbb{R}^{d+n+1}_+ \cap B^{d+n+1}_{\sqrt{2T}}$. A direct application of the chain rule yields the following.

Lemma 2.2.2. *Let $V_n : \mathbb{R}^{n+d+1}_+ \to \mathbb{R}$ as in (2.2.2). Then V_n satisfies:*

$$\begin{aligned}
\partial_{z_i} V_n &= \sqrt{n}\partial_{x_i} U + z_i \partial_t U, \\
\partial_{y_j} V_n &= y_j \partial_t U, \\
\partial^2_{z_i} V_n &= n\partial^2_{x_i} U + 2x_i \partial^2_{x_i t} U + z_i^2 \partial_t^2 U + \partial_t U, \\
\partial^2_{y_j} V_n &= \partial_t U + y_j^2 \partial_t^2 U, \\
|\nabla V_n|^2 &= n|\nabla U|^2 + 2(X \cdot \nabla U)\partial_t U + 2t(\partial_t U)^2,
\end{aligned}$$

for any $i = 0, 1, \ldots, d$, and $j = 1, \ldots, n$. In particular,

$$\frac{a}{z_0}\partial_{z_0} V_n = \frac{a}{z_0}\left(\sqrt{n}\partial_{x_0} U + z_0 \partial_t U\right) = \frac{na}{x_0}\partial_{x_0} U + a\partial_t U, \tag{2.2.3}$$

and

$$\Delta_{(z,y)} V_n + \frac{a}{z_0}\partial_{z_0} V_n + \partial^2_{z_0} V_n = n\left(\partial_t U + \Delta_x U + \frac{a}{x_0}\partial_{x_0} U + \partial^2_{x_0} U\right) + H, \tag{2.2.4}$$

where H is defined in (2.1.33).

Proof. We start with the first derivatives. For $i = 0, 1, \ldots, d$, we have

$$\begin{aligned}
\partial_{z_i} V_n &= \partial_{x_i} U \cdot \partial_{z_i} x_i + \partial_t U \cdot \partial_{z_i} t \\
&= \sqrt{n}\partial_{x_i} U + z_i \partial_t U.
\end{aligned}$$

For $j = 1, \ldots, n$,

$$\partial_{y_j} V_n = \partial_t U \cdot \partial_{y_j} t = y_j \partial_t U.$$

Now, we compute $|\nabla V_n|^2$ using the previously computed first derivatives.

$$\begin{aligned}
|\nabla V_n|^2 &= (\partial_{z_0} V_n)^2 + \sum_{i=1}^{d} (\partial_{z_i} V_n)^2 + \sum_{j=1}^{n} (\partial_{y_j} V_n)^2 \\
&= \left(\sqrt{n}\partial_{x_0} U + z_0 \partial_t U\right)^2 + \sum_{i=1}^{d}\left(\sqrt{n}\partial_{x_i} U + z_i \partial_t U\right)^2 + \sum_{j=1}^{n} (y_j \partial_t U)^2 \\
&= n(\partial_{x_0} U)^2 + 2\sqrt{n} z_0 \partial_{x_0} U \partial_t U + z_0^2 (\partial_t U)^2 \\
&\quad + \sum_{i=1}^{d}\left[n(\partial_{x_i} U)^2 + 2\sqrt{n} z_i \partial_{x_i} U \partial_t U + z_i^2 (\partial_t U)^2\right] + \sum_{j=1}^{n} y_j^2 (\partial_t U)^2.
\end{aligned}$$

Rearranging the previous terms, we find,

$$\begin{aligned}|\nabla V_n|^2 = n\left[(\partial_{x_0}U)^2 + \sum_{i=1}^{d}(\partial_{x_i}U)^2\right] \quad &+ 2\sqrt{n}\partial_t U\left(z_0\partial_{x_0}U + \sum_{i=1}^{d} z_i\partial_{x_i}U\right)\\ &+ (\partial_t U)^2\left(z_0^2 + \sum_{i=1}^{d} z_i^2 + \sum_{j=1}^{n} y_j^2\right)\\ &= n|\nabla U|^2 + 2\sqrt{n}\partial_t U(z\cdot\nabla_x U + z_0\partial_{x_0}U) + R^2(\partial_t U)^2\\ &= n|\nabla U|^2 + 2(X\cdot\nabla U)\partial_t U + 2t(\partial_t U)^2,\end{aligned}$$

where in the last step we used (2.2.1).

Now, we compute the second derivatives. For $i = 0, 1, \ldots, d$,

$$\begin{aligned}\partial_{z_i}^2 V_n &= \partial_{z_i}(\sqrt{n}\partial_{x_i}U + z_i\partial_t U)\\ &= \sqrt{n}(\sqrt{n}\partial_{x_i}^2 U + z_i\partial_{x_i t}U) + \partial_t U + z_i(\sqrt{n}\partial_{t x_i}U + z_i\partial_t^2 U)\\ &= n\partial_{x_i}^2 U + 2x_i\partial_{x_i t}U + \partial_t U + z_i^2\partial_t^2 U.\end{aligned}$$

In particular,

$$\partial_{z_0}^2 V_n = n\partial_{x_0}^2 U + 2x_0\partial_{x_0 t}U + \partial_t U + z_0^2\partial_t^2 U. \tag{2.2.5}$$

For $j = 1, \ldots, n$,

$$\partial_{y_j}^2 V_n = \partial_{y_j}(y_j\partial_t U) = \partial_t U + y_j^2\partial_t^2 U.$$

We now compute the term $\frac{a}{z_0}\partial_{z_0}V_n$.

$$\begin{aligned}\frac{a}{z_0}\partial_{z_0}V_n &= \frac{a}{z_0}(\sqrt{n}\partial_{x_0}U + z_0\partial_t U)\\ &= \frac{na}{x_0}\partial_{x_0}U + a\partial_t U.\end{aligned} \tag{2.2.6}$$

For the Laplacian, we first compute $\Delta_{(z,y)}V_n$,

$$\begin{aligned}\Delta_{(z,y)}V_n &= \sum_{i=1}^{d}\partial_{z_i}^2 V_n + \sum_{j=1}^{n}\partial_{y_j}^2 V_n\\ &= n\Delta_x U + 2\sum_{i=1}^{d} x_i\partial_{x_i t}U + d\partial_t U + \sum_{i=1}^{d} z_i^2\partial_t^2 U + n\partial_t U + \sum_{j=1}^{n} y_j^2\partial_t^2 U.\end{aligned}$$

Simplifying,

$$\Delta_{(z,y)} V_n = n\Delta_x U + 2\sum_{i=1}^{d} x_i \partial_{x_i t} U + (d+n)\partial_t U + (2t - z_0^2)\partial_t^2 U. \tag{2.2.7}$$

We can then then combine, (2.2.5), (2.2.6) and (2.2.7) to obtain,

$$\begin{aligned}
\Delta_{(z,y)} V_n + \frac{a}{z_0}\partial_{z_0} V_n + \partial_{z_0}^2 V_n &= \big[n\Delta_x U + 2\sum_{i=1}^{d} x_i \partial_{x_i t} U + (d+n)\partial_t U + (2t - z_0^2)\partial_t^2 U\big] \\
&\quad + \big[\frac{na}{x_0}\partial_{x_0} U + a\partial_t U\big] + \big[n\partial_{x_0}^2 U + 2x_0\partial_{x_0 t} U + \partial_t U + z_0^2 \partial_t^2 U\big] \\
&= n(\Delta_x U + \partial_{x_0}^2 U + \frac{a}{x_0}\partial_{x_0} U) \\
&\quad + 2(x_0 \partial_{x_0 t} U + \sum_{i=1}^{d} x_i \partial_{x_i t} U + \partial_t U) + (d+a+1)\partial_t U + 2t\partial_t^2 U \\
&= n(\partial_t U + \Delta_x U + \frac{a}{x_0}\partial_{x_0} U + \partial_{x_0}^2 U) + H,
\end{aligned}$$

where

$$H = 2(X,t)\cdot\nabla_{(X,t)}\partial_t U + (d+1+a)\partial_t U,$$

and $X = (x_0, x) \in \mathbb{R}_+^{d+1}$. □

Let $u \in C_{t,x}^{2s+\varepsilon}(\mathbb{R}^d \times (-T_F, +\infty))$ such that u solves (2.1.28), and let $[0,T] \subset (-T_F, T_I)$. Let $U : \mathbb{R}_+^{d+1} \times [0,T] \to \mathbb{R}$ be its associated extension, and assume that $U \in \mathfrak{U}([0,T])$. We now establish the extension problems satisfied by the lifts V_n of U. First, notice that since U is a solution of (2.1.31), by (2.2.4) we have,

$$\Delta_{(z,y)} V_n + \frac{a}{z_0}\partial_{z_0} V_n + \partial_{z_0}^2 V_n = H, \tag{2.2.8}$$

and H does not depend on n. Then, notice that

$$\lim_{z_0\to 0^+} V_n(z_0, z, y) = \lim_{z_0\to 0^+} U(\sqrt{n}z_0, \sqrt{n}z, R^2/2) = \lim_{z_0\to 0^+} U(\sqrt{n}z_0, \sqrt{n}z, (z_0^2 + |z|^2 + |y|^2)/2).$$

Since $U \in \mathfrak{U}([0,T])$, U is continuous in $\overline{\mathbb{R}_+^{d+1}} \times [0,T]$ and therefore,

$$\lim_{z_0\to 0^+} V_n(z_0, z, y) = U(0, \sqrt{n}z, (|z|^2 + |y|^2)/2).$$

We define t_1 as

$$t_1 := \frac{|z|^2 + |y|^2}{2}, \tag{2.2.9}$$

and since $R^2 = 2t$, we have $R^2 = 2t_1$ for any $(0, z, y) \in \partial \mathbb{R}^{d+1}_+$. Then, using (2.1.31),

$$\lim_{z_0 \to 0^+} V_n(z_0, z, y) = U(0, x, t_1) = u(x, t_1), \tag{2.2.10}$$

and, by Lemma 2.2.2,

$$\begin{aligned} z_0^a \partial_{z_0} V_n(z_0, z, y) &= z_0^a \left(\sqrt{n}\partial_{x_0} U + z_0 \partial_t U\right) \\ &= \left(\frac{x_0}{\sqrt{n}}\right)^a \left(\sqrt{n}\partial_{x_0} U + \frac{x_0}{\sqrt{n}}\partial_t U\right) \\ &= n^{\frac{1-a}{2}} x_0^a \left(\partial_{x_0} U + \frac{x_0}{n}\partial_t U\right). \end{aligned}$$

Using (a) of Definition 2.1.6 and (2.1.32),

$$\begin{aligned} \lim_{z_0 \to 0^+} z_0^a \partial_{z_0} V_n(z_0, z, y) &= \lim_{x_0 \to 0^+} n^{\frac{1-a}{2}} x_0^a \left(\partial_{x_0} U + \frac{x_0}{n}\partial_t U\right)(x_0, x, t_1 + |x_0|^2/2n) \\ &= \lim_{x_0 \to 0^+} n^{\frac{1-a}{2}} x_0^a \partial_{x_0} U\,(x_0, x, t_1) \\ &= -\eta_s n^{\frac{1-a}{2}} |u|^{p-1} u(x, t_1). \end{aligned}$$

Finally, observe that

$$-\eta_s n^{\frac{1-a}{2}} |u|^{p-1} u(x, t_1) = -\eta_s n^{\frac{1-a}{2}} |V_n|^{p-1} V_n(0, z, y),$$

since $2t_1 = |z|^2 + |y|^2$. Combining the previous observations, we have the following.

Proposition 2.2.3. *Let $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-T_F, +\infty))$ such that u solves (2.1.28), and let $[0, T] \subset (-T_F, T_I)$. Let U be its associated extension. If $U \in \mathfrak{U}([0, T])$, then V_n obeys*

$$\begin{cases} \nabla \cdot (z_0^a \nabla V_n) = z_0^a H \circ \mathfrak{F}_n = n^{-a/2} x_0^a H & \text{in } \mathbb{R}^{d+n+1}_+ \cap B^{d+n+1}_{\sqrt{2T}} \\ \lim_{z_0 \to 0^+} V_n(z_0, z, y) = u(x, t_1) & \text{for } (z, y) \in B^{d+n}_{\sqrt{2T}}, \end{cases} \tag{2.2.11}$$

and

$$-\lim_{z_0 \to 0^+} z_0^a \partial_{z_0} V_n = \eta_s n^{\frac{1-a}{2}} |V_n|^{p-1} V_n(0, z, y) \quad \text{for } (z, y) \in \partial \mathbb{R}^{d+n+1}_+ \cap B^{d+n+1}_{\sqrt{2T}}. \tag{2.2.12}$$

Moreover, since $V_n(z_0, z, y) = U(x_0, x, t)$, we have

$$V_n \in C^2(\mathbb{R}^{d+n+1}_+ \cap B^{d+n+1}_{\sqrt{2T}}) \cap C(\overline{\mathbb{R}^{d+n+1}_+} \cap B^{d+n+1}_{\sqrt{2T}}),$$

and

$$z_0^a \partial_{z_0} V_n \in C(\overline{\mathbb{R}^{d+n+1}_+} \cap B^{d+n+1}_{\sqrt{2T}}).$$

2.3 An almost monotonicity formula for V_n.

Proposition 2.2.3 enables us to derive an "almost monotonicity formula" for the lifts V_n, by leveraging the similarities between (2.2.11) and (2.1.13). The term "almost monotonic" reflects the fact that the formula becomes strictly monotonic in the limit $n \to \infty$. This will become clear later in our analysis, where we demonstrate that the contribution of the source terms vanishes asymptotically. We will relate the integral quantities defined for the high-dimensional lift to those associated with U in $\mathbb{R}_+^{d+1} \times [0, T]$. To prove our formula, we adapt the arguments used in the proof of Theorem 1.4 in [29].

Theorem 2.3.1. *Let $u \in C_{t,x}^{2s+\varepsilon}(\mathbb{R}^d \times (-T_F, +\infty))$ such that u solves (2.1.28), and let $[0,T] \subset (-T_F, T_I)$. Let $U \in \mathcal{U}([0,T])$ be its associated extension, and V_n its n-dimensional lift. Then the function $\mathcal{E}_n : (0, \sqrt{2T}) \to \mathbb{R}$ defined as,*

$$\begin{aligned} \mathcal{E}_n(R) :=& R^{2s\frac{p+1}{p-1} - N} \left(\frac{1}{2} \int_{\mathbb{R}_+^{N+1} \cap B_R^{N+1}} z_0^a |\nabla V_n|^2 \, dz_0 dz dy - n^{\frac{1-a}{2}} \frac{\eta_s}{p+1} \int_{\partial \mathbb{R}_+^{N+1} \cap B_R^{N+1}} |V_n|^{p+1} \, dz dy \right) \\ &+ R^{2s\frac{p+1}{p-1} - N - 1} \frac{s}{p-1} \int_{\partial B_R^{N+1} \cap \mathbb{R}_+^{N+1}} z_0^a V_n^2 \, d\sigma, \end{aligned} \tag{2.3.1}$$

obeys,

$$\begin{aligned} \frac{d\mathcal{E}_n}{dR} =& R^{2s\frac{p+1}{p-1} - N + 1} \int_{\partial B_R^{N+1} \cap \mathbb{R}_+^{N+1}} z_0^a \left(\frac{\partial V_n}{\partial r} + \frac{2s}{p-1} \frac{V_n}{r} \right)^2 d\sigma \\ &- R^{2s\frac{p+1}{p-1} - N - 1} \int_{\mathbb{R}_+^{N+1} \cap B_R^{N+1}} \left(\frac{2s}{p-1} V_n + r \frac{\partial V_n}{\partial r} \right) (z_0^a H \circ \mathcal{F}_n) \, dz_0 dz dy, \end{aligned} \tag{2.3.2}$$

where $N = n + d$, $r = |(z_0, z, y)|$, and $\partial_r = \frac{(z_0,z,y)}{r} \cdot \nabla$ is the radial derivative.

Proof. Since $U \in \mathcal{U}((0,T])$, Proposition 2.2.3 holds. Now, For $(z_0, z, y) \in \mathbb{R}_+^{N+1}$, let

$$W(z_0, z, y; R) := R^{\frac{2s}{p-1}} V_n(Rz_0, Rz, Ry).$$

Then,

$$\begin{aligned} \nabla \cdot (z_0^a \nabla W)(z_0, z, y; R) &= R^{2s/(p-1)+2-a} \left(\nabla \cdot (z_0^a \nabla V_n) \right) (Rz_0, Rz, Ry) \\ &= R^{2s/(p-1)+2-a} (z_0^a H \circ \mathcal{F}_n)(Rz_0, Rz, Ry), \end{aligned} \tag{2.3.3}$$

and,

$$
\begin{aligned}
-\lim_{z_0\to 0^+} z_0^a \partial_{z_0} W(z_0,z,y;R) &= -R^{2s/(p-1)+1} \lim_{z_0\to 0^+} z_0^a \partial_{z_0} V_n(Rz_0,Rz,Ry) \\
&= -R^{2s/(p-1)+1-a} \lim_{z_0\to 0^+} (Rz_0)^a \partial_{z_0} V_n(Rz_0,Rz,Ry) \\
&= -R^{2s/(p-1)+1-a} \lim_{z_0\to 0^+} (z_0^a \partial_{z_0} V_n)(Rz_0,Rz,Ry) \\
&= R^{2sp/(p-1)} \eta_s n^{\frac{1-a}{2}} (|V_n|^{p-1} V_n)(0,Rz,Ry) \\
&= n^{\frac{1-a}{2}} \eta_s (|W|^{p-1} W)(0,z,y;R).
\end{aligned}
$$

Therefore, W obeys,

$$
\begin{cases}
\nabla\cdot(z_0^a \nabla W)((z_0,z,y);R) = R^{2s/(p-1)+2-a}(z_0^a H\circ \mathcal{F}_n)(Rz_0,Rz,Ry), \\
-\lim_{z_0\to 0^+}(z_0^a\partial_{z_0} W)(z_0,z,y;R) = n^{\frac{1-a}{2}}\eta_s |W|^{p-1} W(0,z,y;R).
\end{cases}
\tag{2.3.4}
$$

Define,

$$
\tilde{\mathcal{E}}_n(V_n;R) := R^{2s\frac{p+1}{p-1}-N}\left(\int_{\mathbb{R}_+^{N+1}\cap B_R^{N+1}} z_0^a \frac{|\nabla V_n|^2}{2}\,dydzdz_0 - \frac{n^{\frac{1-a}{2}}\eta_s}{p+1}\int_{\partial\mathbb{R}_+^{N+1}\cap B_R^{N+1}} |V_n|^{p+1}\,dydz\right).
\tag{2.3.5}
$$

It is straightforward to show that

$$
\tilde{\mathcal{E}}_n(V_n;R) = \tilde{\mathcal{E}}_n(W;1).
$$

Now,

$$
\tilde{\mathcal{E}}_n(W;1) = \int_{\mathbb{R}_+^{N+1}\cap B_1^{N+1}} z_0^a \frac{|\nabla W|^2}{2}\,dydzdz_0 - \frac{n^{\frac{1-a}{2}}\eta_s}{p+1}\int_{\partial\mathbb{R}_+^{N+1}\cap B_1^{N+1}} |W|^{p+1}\,dydz. \tag{2.3.6}
$$

Since,

$$
RW_R = \frac{2s}{p-1}W + rW_r,
$$

differentiating $\tilde{\mathcal{E}}_n(W;1)$, we find

$$
\frac{d\tilde{\mathcal{E}}_n}{dR}(V_n;R) = \int_{\mathbb{R}_+^{N+1}\cap B_1^{N+1}} z_0^a \nabla W\cdot\nabla W_R\,dydzdz_0 - n^{\frac{1-a}{2}}\eta_s\int_{\partial\mathbb{R}_+^{N+1}\cap B_1^{N+1}} |W|^{p-1}WW_R\,dydz.
$$

Now, observe that,

$$\begin{aligned} z_0^a \nabla W \cdot \nabla W_R &= \nabla \cdot (z_0^a W_R \nabla W) - W_R \nabla \cdot (z_0^a \nabla W) \\ &= \nabla \cdot (z_0^a W_R \nabla W)(z_0, z, y; R) - R^{2s/(p-1)+2-a} (z_0^a H \circ \mathfrak{F}_n)(Rz_0, Rz, Ry), \end{aligned} \tag{2.3.7}$$

and, to simplify notation, define

$$A(z_0, z, y; R) := R^{2s/(p-1)+2-a} (z_0^a H \circ \mathfrak{F}_n)(Rz_0, Rz, Ry).$$

We integrate by parts, and use the boundary condition to write,

$$\begin{aligned} \frac{d\tilde{\mathcal{E}}_n}{dR}(V_n; R) &= \int_{\partial B_1^{N+1} \cap \mathbb{R}_+^{N+1}} z_0^a W_r W_R \, d\sigma - \int_{\mathbb{R}_+^{N+1} \cap B_1^{N+1}} W_R A \, dz_0 dy dz \\ &= R \int_{\partial B_1^{N+1} \cap \mathbb{R}_+^{N+1}} z_0^a W_R^2 \, d\sigma - \frac{2s}{p-1} \int_{\partial B_1^{N+1} \cap \mathbb{R}_+^{N+1}} z_0^a W W_R \, d\sigma - \int_{\mathbb{R}_+^{N+1} \cap B_1^{N+1}} W_R A \\ &= R \int_{\partial B_1^{N+1} \cap \mathbb{R}_+^{N+1}} z_0^a W_R^2 \, d\sigma - \frac{s}{p-1} \partial_R \left(\int_{\partial B_1^{N+1} \cap \mathbb{R}_+^{n+1}} z_0^a W^2 \, d\sigma \right) - \int_{\mathbb{R}_+^{N+1} \cap B_1^{N+1}} W_R A. \end{aligned} \tag{2.3.8}$$

Now, since

$$\begin{aligned} W_R(z_0, z, y; R) &= R^{2s/(p-1)} \left(\frac{2s}{p-1} \frac{V_n(Rz_0, Rz, Ry)}{R} + (z_0, z, y) \cdot \nabla V_n(Rz_0, Rz, Ry) \right) \\ &= R^{2s/(p-1)-1} \left(\frac{2s}{p-1} V_n + (z_0, z, y) \cdot \nabla V_n \right) (Rz_0, Rz, Ry), \end{aligned} \tag{2.3.9}$$

we have,

$$\begin{aligned} &\int_{\mathbb{R}_+^{N+1} \cap B_1^{N+1}} W_R A \, dz_0 dz dy \\ &\quad = R^{4s/(p-1)+1-a} \int_{\mathbb{R}_+^{N+1} \cap B_1^{N+1}} \left(\frac{2s}{p-1} V_n + (z_0, z, y) \cdot \nabla V_n \right) (z_0^a H \circ \mathfrak{F}_n)(Rz_0, Rz, Ry) \\ &\quad = R^{2s\frac{p+1}{p-1} - N - 1} \int_{\mathbb{R}_+^{N+1} \cap B_R^{N+1}} \left(\frac{2s}{p-1} V_n + (z_0, z, y) \cdot \nabla V_n \right) (z_0^a H \circ \mathfrak{F}_n) \, dz_0 dz dy. \end{aligned} \tag{2.3.10}$$

The result follows after similarly scaling back the first two terms on the last line of equation (2.3.8). □

2.4 Volume elements and convergence lemmas

Lemma 2.2.2 shows that all the quantities appearing in Theorem 2.3.1 can be represented in terms of the functions U and u. To effectively express integral quantities in terms of the radial variable and $X = (x_0, x) \in \mathbb{R}^{d+1}_+$, we first examine the volume form on the subsets where integration takes place.

2.4.1 Induced volume elements

We now let $(z_0, z, y) \in \mathbb{R}^{d+1} \times \mathbb{R}^n$. Notice that here $z_0 \in \mathbb{R}$. We will restrict these variables to different subsets of interest later. Now let $(l, \phi) \in \mathbb{R}_+ \times \mathbb{S}^d_1$ denote the polar coordinates in the $\mathbb{R}^{d+1}$ factor, i.e., $(z_0, z) = (l, \phi)$. Similarly, for the $\mathbb{R}^n$ factor, we denote the polar coordinates $\mathbb{R}_+ \times \mathbb{S}^{n-1}_1$ by $y = (s, \theta)$.

The Euclidean metric in $\mathbb{R}^{n+d+1}$, $g_E = dz_0^2 + dz^2 + dy^2$, can be written in these coordinates as,

$$g_E = dl^2 + l^2 d\Omega_d^2 + ds^2 + s^2 d\Omega_{n-1}^2, \tag{2.4.1}$$

where $d\Omega_m^2$ denotes the standard metric in $\mathbb{S}^m_1$. Notice that if we define $r = |(z_0, z, y)|$, then $s = (r^2 - l^2)^{1/2}$. We introduce coordinates (r, X, θ) in $\mathbb{R}^{n+d+1}$ via the map

$$F(r, X, \theta) = \left(\frac{X}{\sqrt{n}}, \sqrt{\left(r^2 - \frac{|X|^2}{n} \right)}, \theta \right), \tag{2.4.2}$$

where $r \in \mathbb{R}_+$, $X \in \mathbb{R}^{d+1}$, $\theta \in \mathbb{S}^{n-1}_1$. Here, $(z_0, z) = X/\sqrt{n}$, and $l^2 = |X|^2/\sqrt{n}$ ensuring $r^2 \geq l^2$ and thus $s \geq 0$. Then, the following relations hold:

$$\begin{cases} (l, \phi) & = (z_0, z) = X/\sqrt{n}, \\ (s, \theta) & = y = (\sqrt{(r^2 - |X|^2/n)}, \theta). \end{cases} \tag{2.4.3}$$

Since $r^2 = s^2 + l^2$, on the $(n+d)$-dimensional spheres $\{r = R = \text{const}\} \subset \mathbb{R}^{n+d+1}$, we have $l^2 \leq R^2$ and

$$ds\big|_{T\{r=R\}} = -\frac{l}{s} dl\big|_{T\{r=R\}},$$

where $T\{r = R\}$ denotes the tangent space to the $(n+d)$-dimensional sphere $\{r = R\}$. The metric induced by (2.4.1) is,

$$g_E|_{\{r=R\}} = (1 + \frac{l^2}{R^2 - l^2}) dl^2 + l^2 d\Omega_d^2 + (R^2 - l^2) d\Omega_{n-1}^2. \tag{2.4.4}$$

Therefore,

$$\begin{aligned}\sqrt{g_E}|_{\{r=R\}} &= \sqrt{(1+\frac{l^2}{r^2-l^2})l^{2d}(r^2-l^2)^{n-1}}\\ &= rl^d(r^2-l^2)^{(n-2)/2}\\ &= r^{n-1}l^d\left(1-\frac{l^2}{r^2}\right)^{(n-2)/2},\end{aligned} \tag{2.4.5}$$

where $l^2 \leq r^2 = R^2$. Then, the Euclidean volume form is,

$$dV = r^{n-1}l^d\left(1-\frac{l^2}{r^2}\right)^{(n-2)/2} dl\wedge d\phi\wedge dr\wedge d\theta.$$

Since the Euclidean metric in $\mathbb{R}^{d+1}$ is given by $\tilde{g}_E = dz_0^2 + dz^2 = dl^2 + l^2 d\Omega_d^2$, we have,

$$dz_0\wedge dz_1\wedge\cdots\wedge dz_d = l^d dl\wedge d\phi.$$

Now, using that $X = (x_0, x) = \sqrt{n}(z_0, z)$, it is straightforward to show,

$$dz_0\wedge dz_1\wedge\cdots\wedge dz_d = n^{-\frac{d+1}{2}}dx_0\wedge dx,$$

where $dx = dx_1\wedge\cdots\wedge dx_d$. We can combine this with $l^2 = |X|/n \leq r^2$ to obtain,

$$dz_0\wedge dz\wedge dy = n^{-(d+1)/2}r^{n-1}\left(1-\frac{|X|^2}{nr^2}\right)^{(n-2)/2} dx_0\wedge dx\wedge dr\wedge d\theta. \tag{2.4.6}$$

Recall that $\partial\,\mathbb{R}^{n+d+1} = \{z_0 = 0\} = \{x_0 = 0\}$, and denote by B_R^m the ball of radius R in $\mathbb{R}^m$. We now write $\partial\,\mathbb{R}_+^{n+d+1}\cap B_R^{n+d+1}$ in the coordinates (r, X, θ) as,

$$\partial\,\mathbb{R}_+^{n+d+1}\cap B_R^{n+d+1} = \{(r,X,\theta) : 0\leq r\leq R,\ X=(x_0,x),\ x_0=0,\ x\in B^d_{\sqrt{nr^2}},\ \theta\in\mathbb{S}_1^{n-1}\}, \tag{2.4.7}$$

and since $\partial_{z_0} = \sqrt{n}\partial_{x_0}$, the induced volume form over this region is given by,

$$\begin{aligned}dz\wedge dy &= \iota_{\partial_{z_0}}dV|_{\{z_0=0\}\cap B_R^{n+d+1}}\\ &= n^{-d/2}r^{n-1}\left(1-\frac{|x|^2}{nr^2}\right)^{(n-2)/2} dx\wedge dr\wedge d\theta.\end{aligned}$$

Similarly, we obtain

$$\partial B_R^{n+d+1}\cap\mathbb{R}_+^{n+d+1} = \{(r,X,\theta) : r=R,\ X\in\mathbb{R}_+^{d+1}\cap B^{d+1}_{\sqrt{nR^2}},\ \theta\in\mathbb{S}_1^{n-1}\}. \tag{2.4.8}$$

The induced volume form on this region is,

$$\begin{aligned} d\sigma &= \iota_{\partial_r} dV|_{\{r=R\}} \\ &= n^{-(d+1)/2} R^{n-1} \left(1 - \frac{|X|^2}{nR^2}\right)^{(n-2)/2} dx_0 \wedge dx \wedge d\theta. \end{aligned}$$

Finally,

$$B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1} = \{(r, X, \theta) : 0 \le r \le R,\ X \in \mathbb{R}_+^{d+1} \cap B_{\sqrt{nr^2}}^{d+1},\ \theta \in \mathbb{S}_1^{n-1}\}, \tag{2.4.9}$$

and the volume form on this region is,

$$dz_0 \wedge dz \wedge dy = n^{-(d+1)/2} r^{n-1} \left(1 - \frac{|X|^2}{nr^2}\right)^{(n-2)/2} dx_0 \wedge dx \wedge dr \wedge d\theta.$$

Define $G_n : \mathbb{R}^{d+1} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$G_n(X,t) := \left(1 - \frac{|X|^2}{2nt}\right)^{(n-2)/2} \chi_{\mathbb{R}_+^{d+1} \cap B_{\sqrt{2nt}}^{d+1}}(X),$$

and $\tilde{G}_n : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\tilde{G}_n(x,t) := \left(1 - \frac{|x|^2}{2nt}\right)^{(n-2)/2} \chi_{B_{\sqrt{2nt}}^d}(x).$$

Using G_n and $\tilde{G}_n$, by (2.4.7), (2.4.8) and (2.4.9) we can write the resulting volume elements as,

$$dzdy|_{\partial \mathbb{R}_+^{n+d+1} \cap B_R} = n^{-d/2} r^{n-1} \tilde{G}_n(x, r^2/2)\, dxdrd\theta, \tag{2.4.10}$$

$$d\sigma|_{\partial B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1}} = n^{-(d+1)/2} R^{n-1} G_n(X, R^2/2)\, dXd\theta, \tag{2.4.11}$$

and

$$dz_0dzdy|_{B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1}} = n^{-(d+1)/2} r^{n-1} G_n(X, r^2/2)\, dXdrd\theta, \tag{2.4.12}$$

where $dX = dx_0 dx$.

2.4.2 Convergence lemmas

Now, we establish some convergence lemmas for the integral quantities defined from V_n. Our analysis begins with an observation about the limits of the functions G_n and $\tilde{G}_n$.

Define

$$G(X,t) := e^{-|X|^2/4t} \quad \text{and} \quad \tilde{G}(x,t) := e^{-|x|^2/4t}, \tag{2.4.13}$$

where G is a function on $\mathbb{R}^{d+1}_+ \times \mathbb{R}_+$,and $\tilde{G}$ is a function on $\mathbb{R}^d \times \mathbb{R}_+$. As shown in Proposition A.1.1 in the Appendix, we have

$$G_n(X,t) \to G(X,t) \quad \text{and} \quad \tilde{G}_n(x,t) \to \tilde{G}(x,t),$$

uniformly in $\mathbb{R}^{d+1}_+ \times \{t \geq t_0\}$ and $\mathbb{R}^d \times \{t \geq t_0\}$ for any $t_0 > 0$, respectively. Moreover, we can show

$$G_n(X,t) \leq eG(X,t),$$

for every $(X,t) \in \mathbb{R}^{d+1}_+ \times \mathbb{R}_+$, and, similarly we have,

$$\tilde{G}_n(x,t) \leq e\tilde{G}(x,t),$$

for every $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, see Lemma A.1.2 in the Appendix for a proof. The Dominated Convergence Theorem now ensures that, if $f(\cdot,t)$ is integrable with respect to $G(\cdot,t)\, x_0^a dX$, it is also integrable with respect to $G_n(\cdot,t)\, x_0^a dX$ for every $n \in \mathbb{N}, t > 0$, and similarly for the measures $\tilde{G}(\cdot,t)\, dx$ and $\tilde{G}_n(\cdot,t)\, dx$ respectively.

Lemma 2.4.1. *Let $0 < t_0 < t_1$. Suppose $f : \mathbb{R}^{d+1}_+ \times [t_0,t_1] \to \mathbb{R}$ is continuous and $fG \in C([t_0,t_1]; L^1(\mathbb{R}^{d+1}_+, d\mu))$, where $d\mu = x_0^a dX$. Define*

$$h_n(t) := \int_{\mathbb{R}^{d+1}_+} f(X,t) G_n(X,t)\, d\mu,$$

and

$$h(t) := \int_{\mathbb{R}^{d+1}_+} f(X,t) G(X,t)\, d\mu.$$

Then $h_n \to h$ uniformly in $[t_0,t_1]$.

The proof of Lemma 2.4.1 can be found in Section A.1.2 of the Appendix.

Remark 2.4.2. *By examining the proof of Lemma 2.4.1, we see that the result also holds if $\tilde{f} : \mathbb{R}^d \times [t_0,t_1] \to \mathbb{R}$ is continuous and $\tilde{f}\tilde{G} \in C([t_0,t_1]; L^1(\mathbb{R}^d, dx))$, where the functions h_n and h are replaced by $\tilde{h}_n(t) := \int_{\mathbb{R}^d} \tilde{f}(x,t)\tilde{G}_n(x,t)\, dx$ and $\tilde{h}(t) := \int_{\mathbb{R}^d} \tilde{f}(x,t)\tilde{G}(x,t)\, dx$ respectively.*

In particular, if $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-T_F, +\infty))$, this holds for $\tilde{f} = |u|^{p+1}$ on compact intervals of $(0,T)$. To show this, first observe that $|u|^{p+1}\tilde{G} < C\tilde{G}$ for some positive constant C. Continuity of $|u|^{p+1}\tilde{G}(\cdot,t)$ in $L^1(\mathbb{R}^d, dx)$ follows from the continuity of u and the Dominated Convergence Theorem applied to sequences $t_n \to t$. Moreover, $|u|^{p+1}\tilde{G} < C\tilde{G}$ also implies that

$$\sup_{t\in(0,T)} \int_{\mathbb{R}^d} |u|^{p+1}\tilde{G}(x,t)\, dx < \infty.$$

We will use both properties in the following section.

Remark 2.4.3. *Note that if $U \in \mathcal{U}([0,T])$, Lemma 2.4.1 applies to the functions $|\nabla U|^2$, U^2, $|X.\nabla U|^2$, and $(\partial_t U)^2$ on compact intervals of $(0,T)$, since by property (b) of Definition 2.1.6 and the Cauchy-Schwarz inequality, the functions $|\nabla U|^2 G$, $U^2 G$, $|X.\nabla U|^2 G$, and $(\partial_t U)^2 G$ belong to $C((0,T); L^1(\mathbb{R}^{d+1}_+, d\mu))$. In particular, it also applies to*

$$f(X,t) = \left(X \cdot \nabla U + 2t\partial_t U + \frac{2s}{p-1} U \right)^2, \tag{2.4.14}$$

since

$$g(X,t) = \left(X \cdot \nabla U + 2t\partial_t U + \frac{2s}{p-1} U \right) e^{-|X|^2/8t} \tag{2.4.15}$$

is a sum of continuous functions from $(0,T)$ to $L^2(\mathbb{R}^{d+1}_+, d\mu)$, and therefore it must also be continuous as a function from $(0,T)$ to $L^2(\mathbb{R}^{d+1}_+, d\mu)$.

Lemma 2.4.4. *Let $\varepsilon \in (0,1)$. Define $F_n : C([\varepsilon, 1]) \mapsto \mathbb{R}$ such that*

$$F_n(f) := n \int_\varepsilon^1 t^{n-1} f(t)\, dt,$$

and

$$F(f) := f(1).$$

Then $F_n(f) \to F(f)$.

Proof. Clearly F_n is linear for every $n \in \mathbb{N}$. To see that F_n is bounded, we compute

$$|F_n(f)| \leq n|f|_\infty \int_\varepsilon^1 t^{n-1}\, dt = n\left(\frac{1}{n} - \frac{\varepsilon^n}{n}\right)|f|_\infty \leq |f|_\infty,$$

and thus $|F_n| \leq 1$ for every $n \geq 1$. Since

$$F_n(t^k) = n\left(\frac{1}{n+k} - \frac{\varepsilon^{n+k}}{n+k}\right),$$

we see that

$$\lim_{n\to\infty} F_n(t^k) = 1 = t^k(1).$$

Since F_n is uniformly bounded and converges on the dense subset of polynomials, a standard approximation argument shows that the limit

$$\lim_{n\to\infty} F_n = F$$

exists, $|F| < +\infty$ and $F(f) = f(1)$ for every $f \in C([\varepsilon, 1])$. $\square$

Lemma 2.4.5. *Let $\varepsilon > 0$ and suppose $h_n : [\varepsilon, 1] \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly to $h : [\varepsilon, 1] \to \mathbb{R}$. Then $F_n(h_n) \to F(h)$.*

Proof.

$$\begin{aligned} |F(h) - F_n(h_n)| &\leq |F(h) - F_n(h)| + |F_n(h) - F_n(h_n)| \\ &\leq |F - F_n||h|_\infty + |F_n|_\infty |h - h_n|_\infty \to 0 \end{aligned} \tag{2.4.16}$$

as $n \to \infty$, since $|F_n|$ is uniformly bounded by 1 if $n \geq 1$. $\square$

2.5 A monotonicity formula for the fractional semilinear heat equation

Now, we derive the main result of this chapter. First, observe that if $(r, X, \theta) \in \mathbb{R}_+ \times \mathbb{R}^{d+1}_+ \times \mathbb{S}^{n-1}_1$,

$$V_n(z_0, z, y) = V_n \circ F(r, X, \theta) = U(X, r^2/2),$$

and the same applies for the quantities listed in Lemma 2.2.2. Therefore, whenever we are integrating with respect to the X and r variables, the functions will be evaluated in $(X, r^2/2)$ and, if $r = R$ is fixed, the quantities will be evaluated in $(X, R^2/2) = (X, t)$, by (2.2.1). Similarly, if we are integrating with respect to the x and r variables, observe that $V(0, z, y) = V \circ F(r, 0, x, \theta) = u(x, r^2/2) = u(x, t_1)$, where t_1 is defined in (2.2.9). Then, the functions are evaluated in $(x, r^2/2)$ and, for any fixed $r = R$ the quantities are evaluated in $(x, R^2/2) = (x, t)$, by (2.2.1). We follow this convention by default unless explicit evaluations are provided.

Let

$$C_n := \frac{n^{(d+1+a)/2}}{|\mathbb{S}^{n-1}_1|}. \tag{2.5.1}$$

We have the following proposition.

Proposition 2.5.1. *Let $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-T_F, +\infty))$ such that u solves (2.1.28) in $[0, T] \subset (-T_F, T_I)$ and suppose its associated extension satisfies $U \in \mathfrak{U}([0, T])$. Let V_n be its n-*

dimensional lift. Define,

$$\begin{aligned}\mathcal{E}(R) :=& R^{2s\frac{p+1}{p-1}-d}\frac{1}{2}\int_{\mathbb{R}^{d+1}_+} x_0^a|\nabla U|^2(X,R^2/2)G(X,R^2/2)\,dX \\ &- \frac{\eta_s}{p+1}R^{2s\frac{p+1}{p-1}-d}\int_{\mathbb{R}^d}|u|^{p+1}(x,R^2/2)\tilde{G}(x,R^2/2)\,dx \\ &+ R^{2s\frac{p+1}{p-1}-d-2}\frac{s}{p-1}\int_{\mathbb{R}^{d+1}} x_0^a U^2(X,R^2/2)G(X,R^2/2)\,dX.\end{aligned} \tag{2.5.2}$$

Then, for any $0<\varepsilon<T/2$,

$$C_n\mathcal{E}_n(R)\to\mathcal{E}(R),$$

for every $R\in[\sqrt{2\varepsilon},\sqrt{2(T-\varepsilon)}]$*, where* $\mathcal{E}_n$ *is defined in* (2.3.1)*.*

Proof. Let $0<\varepsilon<T/2$. We write

$$C_n\mathcal{E}_n(R) = \mathcal{E}_n^1(R)+\mathcal{E}_n^2(R)+\mathcal{E}_n^3(R), \tag{2.5.3}$$

where,

$$\mathcal{E}_n^1(R) = C_nR^{2s\frac{p+1}{p-1}-n-d}\frac{1}{2}\int_{\mathbb{R}^{n+d+1}_+\cap B_R^{n+d+1}} z_0^a|\nabla V_n|^2\,dydzdz_0, \tag{2.5.4}$$

$$\mathcal{E}_n^2(R) = -n^{\frac{1-a}{2}}C_n\frac{\eta_s}{p+1}R^{2s\frac{p+1}{p-1}-n-d}\int_{\partial\mathbb{R}^{n+d+1}_+\cap B_R^{n+d+1}}|V_n|^{p+1}\,dzdy, \tag{2.5.5}$$

and,

$$\mathcal{E}_n^3(R) = C_nR^{2s\frac{p+1}{p-1}-n-d-1}\frac{s}{p-1}\int_{\partial B_R^{n+d+1}\cap\mathbb{R}^{n+d+1}_+} z_0^aV_n^2\,d\sigma. \tag{2.5.6}$$

We start by computing the limit of $\mathcal{E}_n^1$. By (2.4.9) and (2.4.12) we have,

$$\begin{aligned}&C_nR^{-(n-1)}\frac{1}{2}\int_{\mathbb{R}^{N+1}_+\cap B_R^{N+1}} z_0^a|\nabla V_n|^2\,dydzdz_0 \\ &= C_n\frac{|\mathbb{S}_1^{n-1}|}{2n^{(d+1)/2}}\int_0^R\int_{\mathbb{R}^{d+1}_+}\left(\frac{x_0}{\sqrt{n}}\right)^a\left(n|\nabla U|^2+2(X\cdot\nabla U)\partial_tU+r^2(\partial_tU)^2\right)\left(\frac{r}{R}\right)^{n-1}G_n\,dXdr \\ &= \frac{1}{2}n\int_0^R\int_{\mathbb{R}^{d+1}_+}\left(\frac{r}{R}\right)^{n-1}x_0^a|\nabla U|^2G_n\,dXdr \\ &\quad+\frac{1}{2}\int_0^R\int_{\mathbb{R}^{d+1}_+}\left(\frac{r}{R}\right)^{n-1}x_0^a\left(2(X\cdot\nabla U)\partial_tU+r^2(\partial_tU)^2\right)G_n\,dXdr.\end{aligned} \tag{2.5.7}$$

In particular, we can write,

$$\mathcal{E}_n^1(R) = R^{2s\frac{p+1}{p-1}-d-1} n \int_0^R \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{I}}(r^2/2)\, dr + R^{2s\frac{p+1}{p-1}-d-1} \int_0^R \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{II}}(r^2/2)\, dr, \tag{2.5.8}$$

where

$$h_n^{\mathrm{I}}(t) := \frac{1}{2}\int_{\mathbb{R}_+^{d+1}} x_0^a |\nabla U|^2 G_n(X,t)\, dX,$$

and

$$h_n^{\mathrm{II}}(t) := \int_{\mathbb{R}_+^{d+1}} x_0^a \left(2(X\cdot\nabla U)\partial_t U + 2t(\partial_t U)^2\right) G_n(X,t)\, dX.$$

Using that $U \in \mathcal{U}([0,T])$ we can apply Lemma 2.4.1 to show that $h_n^{\mathrm{I}}(t)$ converges uniformly to

$$h^{\mathrm{I}}(t) := \frac{1}{2}\int_{\mathbb{R}_+^{d+1}} x_0^a |\nabla U|^2 G(X,t)\, dX,$$

for $t \in [\varepsilon/2, T-\varepsilon]$. Now, since $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$,

$$n\int_0^R \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{I}}(r^2/2)\, dr = n\int_{\sqrt{\varepsilon}}^R \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{I}}(r^2/2)\, dr + n\int_0^{\sqrt{\varepsilon}} \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{I}}(r^2/2)\, dr. \tag{2.5.9}$$

Given that the convergence $h_n^{\mathrm{I}}(t) \to h^{\mathrm{I}}(t)$ is uniform in $[\varepsilon/2, T-\varepsilon]$, the convergence $h_n^{\mathrm{I}}(R^2/2) \to h^{\mathrm{I}}(R^2/2)$ must be uniform for $R \in [\sqrt{\varepsilon}, \sqrt{2(T-\varepsilon)}]$. We perform the change of variables $\tilde{r} = r/R$, and apply Lemma 2.4.5 to show,

$$n\int_{\sqrt{\varepsilon}}^R \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{I}}(r^2/2)\, dr \to Rh^{\mathrm{I}}(R^2/2) = \frac{R}{2}\int_{\mathbb{R}_+^{d+1}} x_0^a |\nabla U|^2 G(X, R^2/2)\, dX, \tag{2.5.10}$$

for every $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Moreover, since $U \in \mathcal{U}([0,T])$, we have,

$$|h_n^{\mathrm{I}}(t)| \leq \frac{e}{2}\sup_{t\in(0,T)}\int_{\mathbb{R}_+^{d+1}} x_0^a |\nabla U|^2 G(X,t)\, dX < C, \tag{2.5.11}$$

for every $t \in (0,T)$ and some constant $C > 0$. Therefore,

$$n\left|\int_0^{\sqrt{\varepsilon}} \left(\frac{r}{R}\right)^{n-1} h_n^{\mathrm{I}}(r^2/2)\, dr\right| \leq CR^{1-n}(\sqrt{\varepsilon})^n \leq C\sqrt{2T}\left(\frac{1}{2}\right)^{n/2} \to 0, \tag{2.5.12}$$

for every $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$, where we used $r^2/2 < T$ for $r \in [0, \sqrt{\varepsilon}]$. Combining both computations we obtain,

$$n\int_0^R\left(\frac{r}{R}\right)^{n-1}h_n^{\mathrm{I}}(r^2/2)\,dr\to\frac{R}{2}\int_{\mathbb{R}^{d+1}_+}x_0^a|\nabla U|^2G(X,R^2/2)\,dX. \tag{2.5.13}$$

In order to control the term involving h_n^{II}, a similar argument to the one discussed in (2.5.11) shows

$$|h_n^{\mathrm{II}}(t)|<C, \tag{2.5.14}$$

for some constant $C>0$ and for every $t\in(0,T)$.

Therefore,

$$\begin{aligned}\left|\int_0^R\left(\frac{r}{R}\right)^{n-1}h_n^{\mathrm{II}}(r^2/2)\,dr\right|&\le C\int_0^R\left(\frac{r}{R}\right)^{n-1}dr\\&\le C\frac{R}{n}\\&\le C\frac{\sqrt{2T}}{n}\to 0,\end{aligned} \tag{2.5.15}$$

for every $R\in[\sqrt{2\varepsilon},\sqrt{2(T-\varepsilon)}]$. Combining (2.5.13) and (2.5.15), we find,

$$\mathcal{E}_n^1(R)\to R^{2s\frac{p+1}{p-1}-d}\frac{1}{2}\int_{\mathbb{R}^{d+1}_+}x_0^a|\nabla U|^2G(X,R^2/2)\,dX. \tag{2.5.16}$$

For the second term, $\mathcal{E}_n^2$, we use (2.4.7) and (2.4.10) to compute,

$$\begin{aligned}-n^{\frac{1-a}{2}}C_n\frac{\eta_s}{p+1}R^{-(n-1)}&\int_{\partial\mathbb{R}^{n+1}_+\cap B_R^{N+1}}|V_n|^{p+1}\,dzdy\\&=-n^{\frac{1-a-d}{2}}C_n\frac{\eta_s}{p+1}|\mathbb{S}^{n-1}|\int_0^R\int_{\partial\mathbb{R}^{d+1}_+}|u|^{p+1}\left(\frac{r}{R}\right)^{n-1}\tilde{G}_n\,dxdr\\&=-n\frac{\eta_s}{p+1}\int_0^R\int_{\mathbb{R}^d}|u|^{p+1}\left(\frac{r}{R}\right)^{n-1}\tilde{G}_n\,dxdr.\end{aligned} \tag{2.5.17}$$

Since u is parabolic Hölder continuous of order $2s+\varepsilon$, by Remark 2.4.2 we have that

$$\int_{\mathbb{R}^d}|u|^{p+1}\tilde{G}_n(x,t)\,dx\to\int_{\mathbb{R}^d}|u|^{p+1}\tilde{G}(x,t)\,dx$$

uniformly for $t\in[\varepsilon/2,T-\varepsilon]$ and,

$$\sup_{t\in(0,T)}\int_{\mathbb{R}^d}|u|^{p+1}\tilde{G}\,dx<+\infty.$$

Then, we proceed as for h_n^{I} in (2.5.9), (2.5.11) and (2.5.12) to obtain,

$$\mathcal{E}_n^2(R) \to -\frac{\eta_s}{p+1} R^{2s\frac{p+1}{p-1}-d} \int_{\mathbb{R}^d} |u|^{p+1} \tilde{G}\, dx, \tag{2.5.18}$$

for every R in $[\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$.

For the third term, $\mathcal{E}_n^3$, we use (2.4.8) and (2.4.11) to deduce,

$$\begin{aligned}
\mathcal{E}_n^3(R) &= C_n R^{2s\frac{p+1}{p-1}-n-d-1} \frac{s}{p-1} \int_{\partial B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1}} z_0^a V_n^2\, d\sigma, \\
&= R^{2s\frac{p+1}{p-1}-n-d-1} \frac{s}{p-1} n^{(d+a+1)/2} \int_{\mathbb{R}_+^{d+1}} \left(\frac{x_0}{\sqrt{n}}\right)^a U^2 n^{-(d+1)/2} R^{n-1} G_n\, dX \\
&= R^{2s\frac{p+1}{p-1}-d-2} \frac{s}{p-1} \int_{\mathbb{R}_+^{d+1}} x_0^a U^2 G_n\, dX.
\end{aligned} \tag{2.5.19}$$

Since $U \in \mathcal{U}([0,T])$, we have $R^{2s\frac{p+1}{p-1}-d-2} U^2 G(\cdot, R^2/2) \in C([\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]; L^1(R_+^{d+1}, x_0^a dX))$, and we can directly apply Lemma 2.4.1 to show,

$$\mathcal{E}_n^3(R) \to R^{2s\frac{p+1}{p-1}-d-2} \frac{s}{p-1} \int_{\mathbb{R}_+^{d+1}} x_0^a U^2 G\, dX, \tag{2.5.20}$$

for $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Combining (2.5.16), (2.5.18) and (2.5.20) the result follows. □

We now examine the convergence of the derivatives of $C_n \mathcal{E}_n$.

Proposition 2.5.2. *Let $U \in \mathcal{U}([0,T])$ and let $0 < \varepsilon < T/2$. Define,*

$$\mathcal{D}(R) := R^{2s\frac{p+1}{p-1}-d-2} \int_{\mathbb{R}_+^{d+1}} x_0^a \left(X \cdot \nabla U + R^2 \partial_t U + \frac{2s}{p-1} U \right)^2 G(X, R^2/2)\, dX. \tag{2.5.21}$$

Then

$$\frac{d}{dR} C_n \mathcal{E}_n(R) \to \mathcal{D}(R),$$

uniformly in $[\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$.

Proof. Fix $0 < \varepsilon < T/2$. Using (2.3.2), we write

$$\frac{d}{dR} C_n \mathcal{E}_n(R) = A_n(R) - B_n(R), \tag{2.5.22}$$

where

$$A_n(R) := C_n R^{2s\frac{p+1}{p-1}-n-d+1} \int_{\partial B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1}} z_0^a \left(\frac{\partial V_n}{\partial r} + \frac{2s}{p-1} \frac{V_n}{r} \right)^2 d\sigma, \tag{2.5.23}$$

and

$$B_n(R) := C_n R^{2s\frac{p+1}{p-1}-1-n-d} \int_{\mathbb{R}_+^{n+d+1} \cap B_R^{n+d+1}} \left(\frac{2s}{p-1} V_n + R(z_0, z, y) \cdot \nabla V_n \right) (z_0^a H \circ \mathcal{F}_n) \, dz_0 dz dy. \tag{2.5.24}$$

To examine the limit of A_n, first observe that for any $(z_0, z, y) \in \partial B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1}$, $|(z_0, z, y)| = R$, and therefore

$$\frac{\partial V_n}{\partial r} = \frac{1}{R}(z_0, z, y) \cdot \nabla V_n = \frac{1}{R}(X \cdot \nabla U + R^2 \partial_t U).$$

Then,

$$\begin{aligned} z_0^a \left(\frac{\partial V_n}{\partial r} + \frac{2s}{p-1} \frac{V_n}{r} \right)^2 &= \left(\frac{x_0}{\sqrt{n}} \right)^a \left(\frac{1}{R}(X \cdot \nabla U + R^2 \partial_t U) + \frac{2s}{p-1} \frac{U}{R} \right)^2 \\ &= n^{-a/2} \frac{x_0^a}{R^2} \left(X \cdot \nabla U + R^2 \partial_t U + \frac{2s}{p-1} U \right)^2. \end{aligned} \tag{2.5.25}$$

By (2.4.8) and (2.4.11),

$$\begin{aligned} A_n(R) &= C_n R^{2s\frac{p+1}{p-1}-n-d+1} \int_{\partial B_R^{n+d+1} \cap \mathbb{R}_+^{n+d+1}} z_0^a \left(\frac{\partial V_n}{\partial r} + \frac{2s}{p-1} \frac{V_n}{r} \right)^2 d\sigma, \\ &= R^{2s\frac{p+1}{p-1}-n-d+1} \int_{\mathbb{R}_+^{d+1}} \frac{x_0^a}{R^2} \left(X \cdot \nabla U + R^2 \partial_t U + \frac{2s}{p-1} U \right)^2 R^{n-1} G_n \, dX. \\ &= R^{2s\frac{p+1}{p-1}-d-2} \int_{\mathbb{R}_+^{d+1}} x_0^a \left(X \cdot \nabla U + R^2 \partial_t U + \frac{2s}{p-1} U \right)^2 G_n \, dX. \end{aligned} \tag{2.5.26}$$

Since $U \in \mathcal{U}([0,T])$, Remark 2.4.3 and Lemma 2.4.1 imply,

$$A_n \to \mathcal{D},$$

uniformly for $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Finally, we examine the non-homogeneous contribution, and show that it converges uniformly to zero. First notice that, by (2.2.1), we can rewrite

$$\frac{2s}{p-1} V_n + (z_0, z, y) \cdot \nabla V_n = \frac{2s}{p-1} U + (x_0, x) \cdot \nabla U + r^2 \partial_t U. \tag{2.5.27}$$

Then,

$$B_n(R) = R^{2s\frac{p+1}{p-1}-2-d} \int_0^R \int_{\mathbb{R}^{d+1}_+} x_0^a H \left(\frac{2s}{p-1} U + X \cdot \nabla U + r^2 \partial_t U \right) \left(\frac{r}{R} \right)^{n-1} G_n \, dX dr. \tag{2.5.28}$$

Let

$$s_n(t) := \int_{\mathbb{R}^{d+1}_+} x_0^a H \left(\frac{2s}{p-1} U + X \cdot \nabla U + 2t \partial_t U \right) G_n \, dX.$$

Using that $U \in \mathcal{U}([0,T])$,

$$|s_n(t)| \leq C,$$

for $t \in (0,T)$ and some constant $C > 0$. Now,

$$\begin{aligned} |B_n(R)| &\leq R^{2s\frac{p+1}{p-1}-2-d} \int_0^R \left(\frac{r}{R} \right)^{n-1} |s_n(r^2/2)| \, dr \\ &\leq R^{2s\frac{p+1}{p-1}-2-d} C \int_0^R \left(\frac{r}{R} \right)^{n-1} dr \\ &\leq \frac{C}{n} (\sqrt{2T})^{2s\frac{p+1}{p-1}-1-d}, \end{aligned} \tag{2.5.29}$$

for every $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Since the last bound is independent of R, it converges uniformly to zero in $[\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. □

Combining the previous results, we obtain the following.

Theorem 2.5.3. *Let $u \in C^{2s+\varepsilon}_{t,x}(\mathbb{R}^d \times (-T_F, +\infty))$ such that u solves (2.1.28), and let $[0,T] \subset (-T_F, T_I)$. Let $U \in \mathcal{U}([0,T])$ be its associated extension, and V_n be its n-dimensional lift. Then, the quantity (1.1.9) is non-decreasing in $(0,T)$. Furthermore, its derivative obeys (1.1.10).*

Proof. Let $0 < \varepsilon < T/2$, and let $t \in [\varepsilon, T-\varepsilon]$. Then, $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$ and, by Proposition 2.5.2 the convergence

$$\frac{d}{dR} C_n \mathcal{E}_n(R) \to \mathcal{D}(R)$$

is uniform in $[\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Furthermore, by Proposition 2.5.1 we have

$$\lim_{n\to\infty} C_n \mathcal{E}_n(R) = \mathcal{E}(R), \tag{2.5.30}$$

for every $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Then, a standard argument shows,

$$\frac{d}{dR}\mathcal{E}(R) = \lim_{n\to\infty} \frac{d}{dR} C_n \mathcal{E}_n(R) = \mathcal{D}(R), \tag{2.5.31}$$

for every $R \in [\sqrt{2\varepsilon}, \sqrt{2(T-\varepsilon)}]$. Since $\varepsilon > 0$ is arbitrary, (2.5.30) and (2.5.31) must hold for every $R \in (0, \sqrt{2T})$.

We may now define

$$\mathcal{J}(t) := \frac{1}{(4\pi)^{d/2}\Gamma(s)} \frac{1}{2^{s\frac{p+1}{p-1}-d/2}} \mathcal{E}(\sqrt{2t}),$$

and since $d/dt = (2t)^{-1/2} d/dR$,

$$\frac{d}{dt}\mathcal{J}(t) = \frac{1}{(4\pi)^{d/2}\Gamma(s)} \frac{1}{2^{s\frac{p+1}{p-1}-d/2}} \frac{1}{\sqrt{2t}} \frac{d\mathcal{E}}{dR}(\sqrt{2t}).$$

Using the expression (1.1.11) for the fundamental solution, and denoting

$$\tilde{\mathcal{G}}(x,t) = \mathcal{G}((0,x),t),$$

we find,

$$\begin{aligned} \mathcal{J}(t) =& t^{\frac{2s}{p-1}+1} \left(\frac{1}{2} \int_{\mathbb{R}^{d+1}_+} x_0^{1-2s} |\nabla U|^2 \mathcal{G}\, dX - \frac{\eta_s}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \tilde{\mathcal{G}}\, dx \right) \\ &+ t^{\frac{2s}{p-1}} \frac{s}{2(p-1)} \int_{\mathbb{R}^{d+1}_+} x_0^{1-2s} U^2 \mathcal{G}\, dX, \end{aligned} \tag{2.5.32}$$

and

$$\frac{d}{dt}\mathcal{J}(t) = \frac{1}{2\sqrt{2}} t^{\frac{2s}{p-1}-\frac{1}{2}} \int_{\mathbb{R}^{d+1}_+} x_0^{1-2s} \left(2t\partial_t U + X \cdot \nabla U + \frac{2s}{p-1} U \right)^2 \mathcal{G}\, dX. \tag{2.5.33}$$

The result readily follows. □

Chapter 3

A unified high-dimensional framework for the Ricci flow

The final part of this thesis focuses on the Ricci flow, particularly on how entropy emerges as a high-dimensional limit of Colding's monotonic volume. We begin by revisiting Perelman's original derivation of his entropy formula for the Ricci flow.

3.1 Perelman's modified Ricci flow

Since Hamilton introduced the Ricci flow [36], it was widely believed that it lacked a variational characterization as the gradient flow of any natural geometric functional. This view was further supported by the work of Bryant and Hamilton, who showed that, except in dimension two, the Ricci flow cannot be realized as a gradient flow on the space of smooth Riemannian metrics $\mathfrak{Met}$ under the standard L^2 metric. As discussed in Chapter 1, variational methods are fundamental in the study of heat-type equations, making it natural to expect that a geometric flow as natural as the Ricci flow should admit such a characterization. Remarkably, Perelman uncovered this structure by working on the extended space $\mathfrak{Met} \times C^\infty(M)$. Below, we outline this approach, which laid the foundation for the $\mathcal{W}$-functional (1.1.15).

The gradient flow associated to an energy functional is defined as follows: for a Hilbert space $\mathcal{H}$ and a smooth functional $E : \mathcal{H} \to \mathbb{R}$, the gradient vector field $\nabla E : \mathcal{H} \to \mathcal{H}$ is defined at $u \in \mathcal{H}$ as the unique vector $\nabla E(u) \in \mathcal{H}$ that obeys

$$\langle \nabla E(u), v \rangle = dE(u)(v),$$

for every $v \in \mathcal{H}$. Then, ∇E defines a gradient flow ϕ, given by

$$\begin{aligned} \frac{d}{dt}\phi_u(t) &= -\nabla E(\phi_u(t)), \\ \phi_u(0) &= u, \end{aligned}$$

for any given $u \in \mathcal{H}$. Observe that the flow lines ϕ_u are paths of steepest descent with respect to the graph $\Gamma(E) = \{(u, E(u)) \in \mathcal{H} \times \mathbb{R}\}$. To see this, notice that if $u = u(t) \in \mathcal{H}$,

$$\frac{d}{dt}E(u(t)) = dE(u)(\dot{u}) = \langle \nabla E(u), \dot{u} \rangle \geq -|\nabla E||\dot{u}|,$$

and the equality holds when $\dot{u} = -\lambda \nabla E(u)$ for $\lambda > 0$.

For the Ricci flow on a closed Riemannian manifold (M, g), the natural candidate for the functional E is the Einstein-Hilbert functional $E : \mathfrak{Met} \to \mathbb{R}$ given by,

$$E(g) = \int_M R\, d\nu,$$

where R is the scalar curvature of the metric g, and $d\nu$ is the measure induced by g. Nevertheless, the variation of E in the direction of h gives,

$$\delta_h E(g) = \int_M \left\langle \frac{R}{2} g - \mathrm{Ric}, h \right\rangle d\nu,$$

with the scalar curvature term coming from the variation of the volume element. From this, it follows that (twice) the metric obeys,

$$\frac{d}{dt} g_{ij} = 2(\nabla E)_{ij} = R g_{ij} - 2R_{ij},$$

which appears similar to the Ricci flow, but its symbol does not possess a definite real part and is therefore not parabolic, see Chapter 10 in [5]. In particular, short-time existence is not ensured.

To overcome this problem, Perelman considers an expanded space $\mathfrak{Met} \times C^\infty(M)$ and defines the functional,

$$\mathcal{F}(g, f) := \int_M (R + |\nabla f|^2) e^{-f}\, d\nu, \tag{3.1.1}$$

where f is known as the dilaton field in the String Theory literature (see for example Section 6 in [26]). The variation of $\mathcal{F}$ can be explicitly computed to yield the following.

Proposition 3.1.1. *On a closed manifold M, the variation of $\mathcal{F}$ is*

$$\begin{aligned}\delta_{(h,k)}\mathcal{F}(g,f) = &-\int_M \langle \operatorname{Ric} + \nabla\nabla f, h\rangle e^{-f}\, d\nu \\ &+ \int_M (\frac{1}{2}tr_g h - k)(2\Delta f - |\nabla f|^2 + R)e^{-f}\, d\nu.\end{aligned} \tag{3.1.2}$$

Proof. See Proposition 10.4 in [5]. □

As a corollary, it also follows,

Corollary 3.1.2. *For measure preserving variations, that is, $\delta_{(h,k)}e^{-f}d\nu(g,f) = 0$, the variation of the $\mathcal{F}$-functional is*

$$\delta_{(h,k)}\mathcal{F}(g,f) = -\int_M \langle \operatorname{Ric} + \nabla\nabla f, h\rangle e^{-f}\, d\nu. \tag{3.1.3}$$

Now, fix a smooth background measure $d\omega$ on M, and define $X : \mathfrak{Met} \to \mathfrak{Met} \times C^\infty(M)$ as,

$$X : g \mapsto \left(g, \log\frac{d\nu(g)}{d\omega}\right),$$

where $d\nu/d\omega$ is the Radon-Nykodym derivative. Then, the composition $\mathcal{F}^\omega = \mathcal{F}\circ X : \mathfrak{Met} \to \mathbb{R}$ is given by

$$\mathcal{F}^\omega(g) = \int_M \left(R + \left|\nabla \log\frac{d\nu}{d\omega}\right|^2\right) d\omega = \int_M (R + |\nabla f|^2)\, d\omega, \tag{3.1.4}$$

where we define $f = \log\frac{d\nu}{d\omega}$. From Corollary 3.1.2,it follows that

$$\delta_h\mathcal{F}^\omega(g) = -\int_M h^{ij}(R_{ij} + \nabla_i\nabla_j f)\, d\omega,$$

since $d\omega$ is fixed and $d\omega = e^{-f}d\nu$. Then, (twice) the gradient structure of $\mathcal{F}^\omega$ on the space $\mathfrak{Met}$ results,

$$\frac{d}{dt}g_{ij} = 2\nabla\mathcal{F}^\omega(g) = -2(R_{ij} + \nabla_i\nabla_j f),$$

and the evolution equation for f reads,

$$\frac{d}{dt}f = \frac{1}{2}g^{ij}\frac{d}{dt}g_{ij} = -\Delta f - R.$$

Therefore, for the coupled modified Ricci flow,

$$\begin{cases} \partial_t g = -2(\operatorname{Ric} + \nabla\nabla f) \\ \partial_t f = -\Delta f - R, \end{cases} \tag{3.1.5}$$

we have the following.

Corollary 3.1.3. *If* $(g(t), f(t))$ *is a solution to the coupled modified Ricci flow* (3.1.5)*, then*

$$\frac{d}{dt}\mathcal{F}^\omega(g) = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 \, d\omega. \tag{3.1.6}$$

The solution to the coupled modified Ricci flow can be related to the solution of the coupled Ricci flow,

$$\begin{cases} \partial_t g = -2\operatorname{Ric} \\ \partial_t f = -\Delta f + |\nabla f|^2 - R. \end{cases} \tag{3.1.7}$$

To see this, observe that given a solution $(\bar{g}, \bar{f})$ to the system (3.1.5), we can construct a solution (g, f) to the coupled system (3.1.7), by flowing along the gradient $\nabla_{\bar{g}(t)}\bar{f}(t)$. Now we show how to construct solutions to (3.1.5) from a solution of the Ricci flow.

Let $g(t)$ be a solution to the Ricci flow $\partial_t g = -2\operatorname{Ric}$ in $[0, T]$ forward in time. Next, we show that can use $g(t)$ to solve the backward equation

$$\partial_t f = -\Delta f + |\nabla f|^2 - R,$$

in $[0, T]$. In order to do this, we set

$$u := e^{-f}, \tag{3.1.8}$$

and note that, by reparametrising time by $\tau = T - t$, we have

$$\partial_\tau u = -\partial_t u = u\partial_t f = (\Delta f + |\nabla f|^2 - R)u = \Delta u - Ru,$$

and therefore u solves the forward, linear equation

$$\partial_\tau u = \Delta u - Ru,$$

for which there exists a unique solution u given initial data $u(0)$. Then we solve this equation for u, and we recover f from (3.1.8). Now that we have constructed a solution (g, f) to (3.1.7) from a solution to the Ricci flow, we can flow by the gradient $\nabla_{g(t)} f$ backwards to define a one-parameter family of diffeomorphisms Φ, and compute the pullbacks of $\Phi^* g(t)$ and

$f \circ \Phi(t)$ to obtain a solution to (3.1.5). Details of this construction can be found in Section 1.5 in [15], or Section 10.4.1 in [5]. From this, we obtain the following.

Proposition 3.1.4. *Given a solution $g(t)$ to the Ricci flow in $[0,T]$ on a closed manifold M, let $(g(t), f(t))$ be the unique solution to* (3.1.7). *Then,*

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |\operatorname{Ric} + \nabla\nabla f|^2 e^{-f}\, d\nu. \tag{3.1.9}$$

Thus, the Ricci flow admits a gradient flow structure on the expanded space $\mathfrak{Met} \times C^\infty(M)$.

3.1.1 The $\mathcal{W}$-functional

To analyze the flow near developing singularities, Perelman introduced the $\mathcal{W}$-functional, which can be thought of as a scale invariant version of the $\mathcal{F}$-functional. On a closed n-dimensional manifold M, define Perelman's $\mathcal{W}$-functional (or entropy functional) on the extended space $\mathcal{W} : \mathfrak{Met} \times C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathcal{W}(g, f, \tau) := \int_M \left(\tau\left(|\nabla f|^2 + R\right) + f - n\right) u\, d\nu, \tag{3.1.10}$$

where R is the scalar curvature of g, $u := (4\pi\tau)^{-n/2}e^{-f}$ and $\tau > 0$ is the scale parameter. Note that u differs from (3.1.8) by a factor of $(4\pi\tau)^{-n/2}$. The $\mathcal{W}$-functional is diffeomorphism-invariant: If $\Phi \in \operatorname{Diff}(M)$ then $\mathcal{W}(\Phi^* g, \Phi^* f, \tau) = \mathcal{W}(g, f, \tau)$, where $\Phi^* g$ is the pullback metric and $\Phi^* f = f \circ \Phi$. Moreover, under the scaling transformation $(g, f, \tau) \mapsto (cg, f, c\tau)$, we have $\mathcal{W}(cg, f, c\tau) = \mathcal{W}(g, f, \tau)$.

we can also relate $\mathcal{F}$ and $\mathcal{W}$ by the identity

$$\mathcal{W}(g, f, \tau) = \frac{1}{(4\pi\tau)^{n/2}}\left(\tau\mathcal{F}(g, f) + \int_M (f - n)e^{-f} d\nu\right). \tag{3.1.11}$$

We now show how to compute the τ-derivative of $\mathcal{W}$ in the classical way.

Proposition 3.1.5. *On a closed manifold M, the variation of $\mathcal{W}$ is*

$$\begin{aligned}\delta_{(h,k,\zeta)}\mathcal{W}(g, f, \tau) &= \int_M \left\langle \operatorname{Ric} + \nabla\nabla f - \frac{1}{2\tau}g, -\tau h + \zeta g\right\rangle u d\nu \\ &+ \int_M \tau\left(\frac{1}{2}\operatorname{tr}_g h - k - \frac{n}{2\tau}\zeta\right)\left(R + 2\Delta f - |\nabla f|^2 + \frac{f - n - 1}{\tau}\right) u d\nu\end{aligned} \tag{3.1.12}$$

Proof. We separate the variation $\delta_{(h,k,\zeta)}\mathcal{W}(g, f, \tau)$ of $\mathcal{W}$ at (g, f, τ) in the direction (h, k, ζ)

in two parts, since

$$\delta_{(h,k,\zeta)}\mathcal{W}(g,f,\tau)=\delta_{(h,k,0)}\mathcal{W}(g,f,\tau)+\delta_{(0,0,\zeta)}\mathcal{W}(g,f,\tau)$$

In order to compute $\delta_{(h,k,0)}\mathcal{W}(g,f,\tau)$ with τ fixed, use (3.1.11) together with Proposition 3.1.1 We obtain,

$$\begin{aligned}\delta_{(h,k,0)}\left(\frac{\tau}{(4\pi\tau)^{n/2}}\mathcal{F}(g,f)\right)(g,f,\tau)=&-\int_M \tau\langle \operatorname{Ric}+\nabla\nabla f,h\rangle u d\nu\\&+\int_M \tau\left(\frac{1}{2}\operatorname{tr}_g h-k\right)\left(2\Delta f-|\nabla f|^2+R\right)ud\nu,\end{aligned}$$

and by direct computation

$$\delta_{(h,k,0)}\left(\frac{1}{(4\pi\tau)^{n/2}}\int_M (f-n)e^{-f}d\nu\right)(g,f,\tau)=\int_M\left(k+\left(\frac{1}{2}\operatorname{tr}_g h-k\right)(f-n)\right)ud\nu.$$

To compute $\delta_{(0,0,\zeta)}\mathcal{W}(g,f,\tau)$ with g and f fixed, we see,

$$\delta_{(0,0,\zeta)}\mathcal{W}(g,f,\tau)=\int_M\left(\zeta\left(1-\frac{n}{2}\right)\left(R+|\nabla f|^2\right)-\frac{n\zeta}{2\tau}(f-n)\right)ud\nu.$$

Combining the two variations, we find

$$\begin{aligned}\delta_{(h,k,\zeta)}\mathcal{W}(g,f,\tau)=\int_M\Bigg[&\langle\operatorname{Ric}+\nabla\nabla f,-\tau h+\zeta g\rangle\\&+\tau\left(\frac{1}{2}\operatorname{tr}_g h-k\right)\left(2\Delta f-|\nabla f|^2+R+\frac{f-n}{\tau}\right)\\&+k+\zeta\left(|\nabla f|^2-\Delta f\right)-\frac{n\zeta}{2\tau}(f-n)-\frac{n\zeta}{2}\left(R+|\nabla f|^2\right)\Bigg]ud\nu.\end{aligned}$$

Now reorder the terms by absorbing $-\frac{n}{2\tau}\zeta$ into the first bracket of the terms on the second line. We get,

$$\begin{aligned}\delta_{(h,k,\zeta)}\mathcal{W}(g,f,\tau)=\int_M[&\langle\operatorname{Ric}+\nabla\nabla f,-\tau h+\zeta g\rangle\\&+\tau\left(\frac{1}{2}\operatorname{tr}_g h-k-\frac{n}{2\tau}\zeta\right)\left(R+2\Delta f-|\nabla f|^2+\frac{f-n}{\tau}\right)\\&+k+(n-1)\zeta\left(\Delta f-|\nabla f|^2\right)]\,ud\nu.\end{aligned}$$

Similarly, reorder to absorb $-\frac{1}{2\tau}g$ into the angled bracket terms, and combine this with

$$\left\langle -\frac{1}{2\tau}g, -\tau h + \zeta g \right\rangle = \frac{1}{2}\operatorname{tr}_g h - \frac{n}{2\tau}\zeta$$

to obtain

$$\begin{aligned}\delta_{(h,k,\zeta)}\mathcal{W}(g,f,\tau) = \int_M \Bigg[&\left\langle \operatorname{Ric} + \nabla\nabla f - \frac{1}{2\tau}g, -\tau h + \zeta g \right\rangle \\ &+ \tau\left(\frac{1}{2}\operatorname{tr}_g h - k - \frac{n}{2\tau}\zeta\right)\left(R + 2\Delta f - |\nabla f|^2 + \frac{f-n-1}{\tau}\right) \\ &+ (n-1)\zeta\left(\Delta f - |\nabla f|^2\right)\Bigg] u d\nu.\end{aligned}$$

The result now follows after observing that

$$\int_M \left(\Delta f - |\nabla f|^2\right) e^{-f} d\nu = -\int_M \Delta e^{-f} d\nu = 0.$$

□

Next, observe that defining

$$dm := u d\nu,$$

we may fix dm so that the variation $\delta_{(h,k,\zeta)} dm(g,f,\tau)$ vanishes, that is,

$$\frac{1}{2}\operatorname{tr}_g h - k - \frac{n}{2\tau}\zeta = 0.$$

We solve for f to find,

$$f = \log\frac{d\nu}{dm} - \frac{n}{2}\log(4\pi\tau). \tag{3.1.13}$$

As a corollary, we obtain

Corollary 3.1.6 (Measure-preserving variations of the entropy)**.** *For any variation* (h,k,ζ) *satisfying* $\delta_{(h,k,\zeta)} u d\nu(g,f,\tau) = 0$*, the variation of* $\mathcal{W}$ *obeys*

$$\delta_{(h,k,\zeta)}\mathcal{W}(g,f,\tau) = \int_M \left\langle \operatorname{Ric} + \nabla\nabla f - \frac{1}{2\tau}g, -\tau h + \zeta g \right\rangle u d\nu.$$

We now heuristically formulate an appropriate gradient flow for $\mathcal{W}$ so that $\mathcal{W}$ is also monotonic. Using the function f defined in (3.1.13) and considering the gradient flow for

g_{ij} previously used in (3.1.5), we obtain the following coupled gradient flow for (g, f, τ):

$$\frac{\partial}{\partial t} g = -2(\operatorname{Ric} + \nabla\nabla f) \tag{3.1.14}$$

$$\frac{\partial f}{\partial t} = -\Delta f - R + \frac{n}{2\tau} \tag{3.1.15}$$

$$\frac{d\tau}{dt} = -1, \tag{3.1.16}$$

where the last condition is imposed in order to ensure monotonicity, since, by Corollary 3.1.6,

$$\begin{aligned} \frac{d}{dt}\mathcal{W} &= \int_M \left(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right) \left(-\tau \dot{g}_{ij} + \dot{\tau} g_{ij} \right) dm \\ &= 2\tau \int_M \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} \right|^2 dm, \end{aligned}$$

whenever $dm = u d\mu$ is fixed, $\dot{g}_{ij} = -2\left(R_{ij} + \nabla_i \nabla_j f\right)$ and $\dot{\tau} = -1$. A similar diffeomorphism change to the one already discussed for (3.1.5) allows us to rewrite the coupled system of equations as,

$$\begin{aligned} \frac{\partial}{\partial t} g &= -2 \operatorname{Ric} \\ \frac{\partial f}{\partial t} &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \\ \frac{d\tau}{dt} &= -1. \end{aligned} \tag{3.1.17}$$

To simplify notation, we introduce the following.

Definition 3.1.7. *Given a solution* $(g(t), f(t), \tau(t))$ *to the coupled system* (3.1.17) *on a closed manifold* M*, we define the entropy* $\mathcal{W}(\tau)$ *as*

$$\mathcal{W}(\tau) := \mathcal{W}(g(t), f(t), \tau(t)), \tag{3.1.18}$$

as introduced in (1.1.15).

Putting all together, we obtain the following formula for the derivative of $\mathcal{W}(\tau)$.

Proposition 3.1.8. *The derivative of the entropy obeys* (1.1.16)*, that is,*

$$\frac{d}{dt}\mathcal{W}(\tau) = \int_M 2\tau \left| \operatorname{Ric} + \nabla\nabla - \frac{1}{2\tau} g \right|^2 u \, d\nu.$$

Interestingly, the monotonicity of Perelman's $\mathcal{W}$-functional can also be derived from a pointwise estimate using the conjugate heat operator. To see this, consider a Ricci flow defined for $t \in [0,T]$. By defining the heat operator,

$$\Box := \frac{\partial}{\partial t} - \Delta,$$

acting on $C^\infty(M \times [0,T])$ we see, by evaluating

$$\frac{d}{dt}\int_M vw\,d\nu,$$

that the conjugate heat operator

$$\Box^* := -\frac{\partial}{\partial t} - \Delta + R,$$

is conjugate to $\Box$ in the following sense.

Proposition 3.1.9. *If $g(t), t \in [0,T]$ is a solution to the Ricci flow and $v, w \in C^\infty(M \times [0,T])$, then*

$$\int_0^T \left(\int_M (\Box v) w d\mu\right) dt = \left[\int_M vwd\mu\right]_0^T + \int_0^T \left(\int_M v\,(\Box^* w)\,d\mu\right) dt.$$

Proof. See Lemma 11.5 in [5]. □

Now, let

$$w := \left(\tau\left(R + 2\Delta f - |\nabla f|^2\right) + f - n\right) u.$$

Then,

$$\mathcal{W}(\tau) = \int_M wd\nu, \tag{3.1.19}$$

since $\int_M \left(\Delta f - |\nabla f|^2\right) ud\nu = -\int_M \Delta ud\nu = 0$. The function w obeys the following.

Proposition 3.1.10 ([47], Proposition 9.1, [64], p. 77)**.** *Let (f, g, τ) be a solution of* (3.1.17)*. Then w satisfies*

$$\Box^* w = -2\tau \left|\mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{2\tau} g\right|^2 u. \tag{3.1.20}$$

The monotonicity of $\mathcal{W}$ now follows immediately since, setting $v = 1$ in Proposition 3.1.9, by (3.1.19) and (3.1.20) we have,

$$\frac{d}{dt}\mathcal{W} = \frac{d}{dt}\int_M wd\nu = -\int_M \Box^* wd\nu.$$

3.2 Colding's Monotonic Volume

Similarly to Perelman's derivation of his reduced volume from the Bishop-Gromov inequality discussed in Section 1.0.1, we want to derive the $\mathcal{W}$-functional as a high dimensional limit of an elliptic monotonic quantity. The appropriate quantity for this process is Colding's monotonic volume, which we now discuss.

Let $(N, \bar{g})$ be an m-dimensional manifold admitting a a positive and proper Green function G, and set

$$b := G^{1/(2-m)}.$$

Definition 3.2.1. *We define the area A on the level sets of b as,*

$$A(s) := \frac{1}{s^{m-1}} \int_{b=s} (|\nabla b|^2 - 1)|\nabla b| \, dA, \tag{3.2.1}$$

and the 'volume' V as,

$$V(s) := \frac{1}{s^m} \int_{b \leq s} (|\nabla b|^2 - 1)|\nabla b|^2 \, dV, \tag{3.2.2}$$

and the 'monotonic volume' as,

$$W(s) := 2(m-1)V(s) - A(s). \tag{3.2.3}$$

The main purpose of this section is to provide an explicit computation of the derivative of W, and show that W is non-decreasing and obeys (1.1.20). We begin with a few routine computations.

Lemma 3.2.2. *We have,*

$$\nabla b = \frac{1}{2-m} b^{m-1} \nabla G, \tag{3.2.4}$$

$$\Delta b = (m-1) b^{-1} |\nabla b|^2, \tag{3.2.5}$$

$$\Delta b^2 = 2m |\nabla b|^2, \tag{3.2.6}$$

Proof. Equation (3.2.4) is immediate. To see (3.2.5), use (3.2.4) and note that $\Delta G = 0$. For (3.2.6), use that

$$\Delta b^2 = 2|\nabla b|^2 + 2b\Delta b, \tag{3.2.7}$$

and combine it with (3.2.5). □

In order to compute the derivative of W, we begin by computing the derivative of two auxilliary functions, I and J.

Proposition 3.2.3. *The function*

$$I(s) := s^{1-m} \int_{b=s} |\nabla b| dA \tag{3.2.8}$$

is constant.

Proof. First, observe that

$$I' = \frac{1-m}{s} I(s) + s^{1-m} \partial_s \left(\int_{b=s} |\nabla b| dA \right). \tag{3.2.9}$$

Since

$$|\nabla b| = g\left(\nabla b, \frac{\nabla b}{|\nabla b|} \right),$$

by the coarea formula we have,

$$\begin{aligned} \partial_s \left(\int_{b=s} |\nabla b| dA \right) &= \partial_s \left(\int_{b \leq s} \Delta b dV \right) \\ &= \partial_s \left(\int_0^s \int_{b=r} \frac{\Delta b}{|\nabla b|} dA \, dr \right) \\ &= \int_{b=s} \frac{\Delta b}{|\nabla b|} dA. \end{aligned} \tag{3.2.10}$$

Using (3.2.5), we find,

$$I'(s) = 0.$$

□

Moreover, it can also be shown that the constant is

$$I(s) = I(1) = \text{ Vol}(\partial B_1(0)),$$

where $B_1(0) \subset \mathbb{R}^m$ is the Euclidean unit ball, see [18].

Proposition 3.2.4. *Let*

$$J(s) := s^{-m} \int_{b \leq s} |\nabla b|^2 dV.$$

Then, J is constant and moreover,

$$J(s) = Vol(B_1(0)).$$

Proof. To see this, notice that by the coarea formula,

$$\begin{aligned} J(s) &= s^{-m} \int_0^s \int_{b=r} |\nabla b| \, dA \, dr \\ &= s^{-m} \int_0^s r^{m-1} I(r) dr \\ &= \frac{I(1)}{m}. \end{aligned} \tag{3.2.11}$$

Now, since

$$\mathrm{Vol}(B_1(0)) = \frac{1}{m} \mathrm{Vol}(\partial B_1(0)),$$

the result follows. □

The previous computations show that, if we consider

$$\bar{A}(s) := \frac{1}{s^{m-1}} \int_{b=s} |\nabla b|^3 \, dA, \tag{3.2.12}$$

we have

$$A = \bar{A} - \mathrm{Vol}(\partial B_1(0))$$

and, similarly, if we define

$$\bar{V}(s) := \frac{1}{s^m} \int_{b \le s} |\nabla b|^4 \, dV, \tag{3.2.13}$$

then,

$$V = \bar{V} - \mathrm{Vol}(B_1(0)).$$

Therefore, the derivative of the function

$$\bar{W} := 2(m-1)\bar{V} - \bar{A}$$

coincides with the derivative of W. We now proceed to obtain an explicit formula for the derivative of this quantity. We will need the following lemma.

Lemma 3.2.5. $\bar{V}$ *obeys,*

$$\bar{V}'(s) = \frac{1}{s}(\bar{A}(s) - m\bar{V}(s)).$$

Proof. By the coarea formula, we can rewrite $\bar{V}(s)$ as

$$\bar{V}(s) = s^{-m} \int_{-\infty}^s \int_{b=r} |\nabla b|^3 dA \, dr.$$

Deriving, the lemma follows. □

We now compute the derivative of $\bar{W}$.

Theorem 3.2.6 (Theorem 2.4 in [18]). *$\bar{W}$ obeys,*

$$\bar{W}'(s) = \frac{s^{-1-m}}{2} \int_{b \leq s} \left(\left| \nabla\nabla b^2 - \frac{\Delta b^2}{n} g \right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right) \right) dV.$$

Proof. First, observe that

$$|\nabla b|^2 = |2b\nabla b|^2 = 4b^2|\nabla b|^2,$$

and therefore we can rewrite $A(s)$ as

$$\bar{A}(s) = s^{1-m} \int_{b=s} |\nabla b|^3 d\bar{A} = \frac{s^{-1-m}}{4} \int_{b=s} \left|\nabla b^2\right|^2 |\nabla b| dA.$$

We now compute,

$$\begin{aligned} s^{-2}\left(s^2 \bar{A}\right)'(s) =& \frac{s^{-1-m}}{4} \int_{b=s} \frac{d}{dn} \left|\nabla b^2\right|^2 dA \\ =& \frac{s^{-1-m}}{4} \int_{b \leq s} \Delta \left|\nabla b^2\right|^2 dV. \end{aligned}$$

Using Bochner's formula, we obtain

$$\begin{aligned} \frac{s^{-1-m}}{4} \int_{b \leq s} \Delta \left|\nabla b^2\right|^2 dV =& \frac{s^{-1-m}}{2} \int_{b \leq s} \left(\left|\nabla\nabla b^2\right|^2 + \left\langle \nabla \Delta b^2, \nabla b^2 \right\rangle + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right) \right) dV \\ =& \frac{s^{-1-m}}{2} \int_{b \leq s} \left(\left|\nabla\nabla b^2\right|^2 - \left|\Delta b^2\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right) \right) dV \\ &+ \frac{s^{-1-m}}{2} \int_{b=s} \left(\Delta b^2\right) \frac{d}{dn} b^2 dA \\ =& \frac{s^{-1-m}}{2} \int_{b \leq s} \left(\left|\nabla\nabla b^2\right|^2 - \left|\Delta b^2\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right) \right) dV \\ &+ 2ms^{-m} \int_{b=s} |\nabla b|^3 dA \\ =& \frac{s^{-1-m}}{2} \int_{b \leq s} \left(\left|\nabla\nabla b^2\right|^2 - \left|\Delta b^2\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right) \right) dV + \frac{2m}{s} A(s). \end{aligned}$$

Now, since

$$\left| \nabla\nabla b^2 - \frac{\Delta b^2}{m} g \right|^2 = \left|\nabla\nabla b^2\right|^2 + \frac{\left|\Delta b^2\right|^2}{m} - \frac{2\left|\Delta b^2\right|^2}{m} = \left|\nabla\nabla b^2\right|^2 - \frac{\left|\Delta b^2\right|^2}{m},$$

we have,

$$\left|\nabla\nabla b^2\right|^2 - \left|\Delta b^2\right|^2 = \left|\nabla\nabla b^2 - \frac{\Delta b^2}{n} g\right|^2 - \left(1 - \frac{1}{m}\right)\left|\Delta b^2\right|^2$$
$$= \left|\nabla\nabla b^2 - \frac{\Delta b^2}{m} g\right|^2 - 4m^2\left(1 - \frac{1}{m}\right)|\nabla b|^4.$$

Therefore,

$$s^{-2}\left(s^2\bar{A}\right)'(s) = \frac{s^{-1-m}}{2}\int_{b\le s}\left(\left|\nabla\nabla b^2 - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right)\right) dV$$
$$-2\left(1 - \frac{1}{m}\right) m^2 s^{-1-m}\int_{b\le s}|\nabla b|^4 d\,\mathrm{Vol} + \frac{2m}{s}\bar{A}(s).$$

We can now rewrite the expression above as

$$s^{-2}\left(s^2\bar{A}\right)'(s)$$
$$= \frac{s^{-1-m}}{2}\int_{b\le s}\left(\left|\nabla\nabla b^2 - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right)\right) dV - \frac{2(1-1/m)m^2}{s}\bar{V}(s) + \frac{2m}{s}\bar{A}(s)$$
$$= \frac{s^{-1-m}}{2}\int_{b\le r}\left(\left|\nabla\nabla b^2 - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right)\right) dV + \frac{2m}{s}(\bar{A}(s) - m\bar{V}(s)) + \frac{2m}{s}\bar{V}(s).$$

Now, since $s^{-2}\left(s^2\bar{A}\right)' = \bar{A}' + 2\bar{A}/s$,

$$\bar{A}' = \frac{s^{-1-m}}{2}\int_{b\le s}\left(\left|\nabla\nabla b^2 - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right)\right) dV + \frac{2(m-1)}{s}(\bar{A} - m\bar{V}),$$

and therefore, by Lemma 3.2.5,

$$(\bar{A} - 2(m-1)\bar{V})' = \frac{s^{-1-m}}{2}\int_{b\le s}\left(\left|\nabla\nabla b^2 - \frac{\Delta b^2}{n} g\right|^2 + \mathrm{Ric}\left(\nabla b^2, \nabla b^2\right)\right) dV.$$

□

Now, since $\bar{W}$ and W only differ by a constant, this also shows that W obeys (1.1.20).

3.3 High dimensional limits on Perelman's N-space: Preliminaries

The rest of the chapter is devoted to showing that the $\mathcal{W}$-functional arises as a limit of Colding's monotonic volume W on Perelman's N-space, and that we can effectively recover the derivative of $\mathcal{W}$ from the formula (1.1.20) by explicitly using the derivation of Theorem 3.2.6. In this section, we recall the setup as well as Perelman's N-space, and then proceed to outline some necessary results regarding orders of convergence as the dimension approaches infinity.

3.3.1 Perelman's N-space and the rescaled area, volume and monotonic volume

On a closed manifold M^n of dimension n, let $g(\tau)$ be a solution to the backward Ricci flow equation,

$$\partial_\tau g = 2\,\mathrm{Ric},$$

where $\tau \in [0, T]$. Similarly to the approach discussed for the coupled system of equations (3.1.17), we let u be a solution to,

$$\Box^* u = 0, \tag{3.3.1}$$

positive at time $\tau = 0$. We can trivially rewrite this as

$$\partial_\tau u = \Delta u - Ru,$$

and hence, u is also positive for all times by the maximum principle, see Section A.2.1 in the Appendix. Next, define f by the relation

$$u := \tau^{-n/2} e^{-f}, \tag{3.3.2}$$

in a similar manner to that of Section 3.1.1, but here we drop the constant $(4\pi)^{-n/2}$ in the definition of u for convenience. Then, f satisfies,

$$\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}, \tag{3.3.3}$$

and in particular, (g, f, τ) obeys (3.1.17).

Next, we recall Perelman's N-space.

Definition 3.3.1. *Let $g(\tau)$ be a solution to the backward Ricci flow equation on $[0, T]$. Denote by $\mathbb{S}^N$ the N-dimensional unit-sphere of $\mathbb{R}^{N+1}$. Let r be the distance to the origin in $\mathbb{R}^{N+1}$ and let θ denote points in $\mathbb{S}^N$. Then, the Perelman N-space $(\hat{M}, \hat{g})$ is the manifold,*

$$\hat{M}^m := (0, \sqrt{2NT})_r \times \mathbb{S}^N_\theta \times M^n_x \subset \mathbb{R}^{N+1} \times M^n, \quad \text{where} \quad m = N + n + 1,$$

endowed with the metric,

$$\hat{g} := r^2 g_{\mathbb{S}^N} + (1 + \frac{Rr^2}{N^2})dr^2 + g, \tag{3.3.4}$$

where R is the scalar curvature of g and where at a point $(r, \theta, x) \in \hat{M}$, $g_{\mathbb{S}^N}$ is evaluated at θ and R and g at $(\tau = r^2/2N, x) \in (0, T) \times M$.

Define $h : \hat{M} \to \mathbb{R}$ by,

$$h = r^{-(m-2)} e^{-f(\tau,x)}, \tag{3.3.5}$$

where f satisfies (3.3.3) and $\tau = r^2/2N$. The function h will serve as the analog to the Green function when applying Colding's monotonic formula to Perelman's N-space. The fact that h is almost harmonic, which justifies the replacement of the Green function G in Colding's argument for the function h considered here, is essentially due to Perelman: In Section 6.1 of [47], it is observed that

$$\tilde{u}^* := \tau^{-\frac{N-1}{2}} u = (2N)^{\frac{m-2}{2}} h,$$

is harmonic modulo N^{-1} if and only if f satisfies (3.3.3) (the precise statement is $\hat{\Delta}\tilde{u}^* = \tau^{(2-m)/2} O(1/N)$, see Proposition 3.5.2). From this definition, and following the arguments by Colding discussed in Section 3.2, we define $b : \hat{M} \to \mathbb{R}$ as,

$$b = h^{1/(2-m)} = re^{f/(m-2)}. \tag{3.3.6}$$

We will need to adapt the area, volume and monotonic volume from Definition 3.2.1 in order to take high dimensional limits. We define the following rescaled quantities.

Definition 3.3.2. *Let*

$$c_N := \frac{(4\pi)^{-n/2}(2N)^{n/2+1}}{4|\mathbb{S}^N|}.$$

We define the area $\mathcal{A}_N : (0, \sqrt{2NT}) \to \mathbb{R}$ (resp. the raw area $\overline{\mathcal{A}}_N : (0, \sqrt{2NT}) \to \mathbb{R}$) as,

$$\mathcal{A}_N(s) = \frac{c_N}{s^{m-1}} \int_{b=s} (|\hat{\nabla} b|^2 - 1)|\hat{\nabla} b| \, d\hat{A}, \ \left(\text{resp. } \overline{\mathcal{A}}_N(s) = \frac{c_N}{s^{m-1}} \int_{b=s} |\hat{\nabla} b|^3 \, d\hat{A} \right), \tag{3.3.7}$$

the volume $\mathcal{V}_N : (0, \sqrt{2NT}) \to \mathbb{R}$ as,

$$\mathcal{V}_N(s) = \frac{c_N}{s^m} \int_{b \le s} (|\hat{\nabla} b|^2 - 1)|\hat{\nabla} b|^2 \, d\hat{V}, \tag{3.3.8}$$

and the monotonic volume $\mathcal{W}_N : (0, \sqrt{2NT}) \to \mathbb{R}$ as,

$$\mathcal{W}_N(s) = (2(m-1)\mathcal{V}_N - \mathcal{A}_N)(s). \tag{3.3.9}$$

Notice that these quantities only differ from the ones discussed in Definition 3.2.1 by a constant c_N depending only on the dimension N.

Remark 3.3.3. *We associate $\lambda \in (0,T)$ with $s = \sqrt{2N\lambda} \in (0, \sqrt{2NT})$. Quantities that are functions of s, for instance $\mathcal{A}_N(s)$, will be considered sometimes as functions of $\lambda \in (0,T)$, in which case we will write, for example, $\mathcal{A}_N(\lambda)$ instead of $\mathcal{A}_N(s = \sqrt{2N\lambda})$. Convergence of $\mathcal{A}_N$ or $\mathcal{V}_N$ to the entropy $\mathcal{W}$ occurs when considering them as functions of $\lambda \in (0,T)$.*

3.3.2 Orders of convergence

Most of the quantities employed depend on the dimension of the N-space. For example, the metric $\hat{g}$ depends on N, but we will omit the subscript N to simplify notation. In a similar manner, we will omit the subscript N in most parts of the upcoming sections. For instance, the Ricci curvature of $\hat{g}$ will be denoted by $\widehat{\text{Ric}}$, the covariant derivative will be $\hat{\nabla}$, and so on.

The metric coefficient $(1 + Rr^2/N^2) = (1 + 2\tau R/N)$ will appear often, so to simplify notation we define,

$$v := 1 + \frac{Rr^2}{N^2}. \tag{3.3.10}$$

Note that $\hat{g}$ is invariant under rotations in $\mathbb{R}^{N+1}$ and therefore, scalar quantities like the scalar curvature $\hat{R}$, or the norm of the Ricci tensor $|\widehat{\text{Ric}}|$, are also invariant under rotations and thus θ-independent. For this reason, they will often be considered as functions on $(0,T) \times M$. For instance, v can be viewed as a function on $\hat{M}$ or $(0,T) \times M$, depending on the context, and similarly for any other function depending only on $x \in M$ and $\tau \in [0,T]$.

In [47], Perelman showed that $|\widehat{\text{Ric}}| = O(1/N)$, (see Section A.2.3 in the Appendix for details). This means that the sequence of functions $N|\widehat{\text{Ric}}|$, as functions on $(0,T) \times M$, are uniformly bounded on compact sets.

Regarding the definition of order, we will need to adapt it in the following way.

Definition 3.3.4. *Let $i \geq 0$ be an integer. A sequence of real-valued functions $F_N(\tau, \theta, x)$ is an $O_0(1/N^i)$ if for every $0 < \tau_1 < T$ there exists $K > 0$ such that,*

$$N^i |F_N(\tau, \theta, x)| \leq K,$$

for all $N > 0$ and for all (τ, θ, x) such that $\tau_1 \leq \tau \leq T$.

We say that $F_N(\tau, \theta, x)$ is an $O_k(1/N^i)$ if $\partial^\alpha F_N$ is an $O_0(1/N^i)$, for any multi-index $|\alpha| \leq k$, where the derivatives are taken in the τ or x^i variables.

Definition 3.3.5. *We say that a sequence of real-valued functions $F_N(\lambda)$, defined in $(0,T)$, is an $O_0(e^{-cN})$ if for each $\lambda_0 \in (0,T)$, given $\lambda_1 > \lambda_0$, there exists $K > 0$ and $c > 0$ such* *that,*

$$e^{cN}|F_N(\lambda)| \leq K, \tag{3.3.11}$$

for all $N > 0$ and for all $\lambda_1 \leq \lambda \leq T$. We say that $F_N(\lambda)$ is an $O_k(e^{-cN})$ if $d^jF_N/d\lambda^j$ is an $O_0(e^{-cN})$ for every $j \leq k$.

A direct application of the Taylor expansion with the remainder in integral form allows us to derive the following lemma, which we will frequently reference.

Lemma 3.3.6. *Let $F : (a, b) \to \mathbb{R}$ be a smooth, real-valued function. Let $w(\tau, x, N)$ be a smooth real-valued function with range in (a, b), such that,*

$$w(\tau, x, N) = w_0(\tau, x) + \delta(\tau, x, N),$$

where,

$$\delta(\tau, x, N) = O_k(\frac{1}{N^j}),$$

for some integers $k \geq 0$ and $j \geq 0$. Then, for any $l \geq 1$, $F(w(\tau, x, N))$ has the following decomposition (where we omit the τ, x and N dependence for notational convenience),

$$F(w) = F(w_0) + F'(w_0)\delta + F''(w_0)\frac{\delta^2}{2} + \ldots + F^{(l-1)}(w_0)\frac{\delta^{l-1}}{(l-1)!} + R_l,$$

and,

$$R_l(\tau, x, N) = O_k(\frac{1}{N^{jl}}).$$

Proof. See Section A.2.2 in the Appendix. □

Note that the usual rules of orders hold: the sum

$$O_k(1/N^i) + O_l(1/N^j),$$

is an $O_{\min(k,l)}(1/N^{\min(i,j)})$ and the product

$$O_k(1/N^i)O_l(1/N^j),$$

is an $O_{\min(k,l)}(1/N^{i+j})$. For instance, in Lemma 3.3.6, if $\delta = O_k(1/N^j)$, then $\delta^i = O_k(1/N^{ji})$.

As an example, of Lemma 3.3.6, we observe that

$$v = 1 + 2\tau R/N = 1 + \delta, \quad \text{where } \delta = 2\tau R/N = O_k(1/N),$$

and therefore,

$$1/\sqrt{v} = 1 - \tau R/N + O_2(\frac{1}{N^2}).$$

We also note that, while as in this and other cases, $\delta = O_k(1/N)$ for any $k \geq 0$, for our purposes it will be enough to use $k \leq 2$. Similarly,

$$e^{f/(m-2)} = 1 + \frac{f}{N} + O_2(\frac{1}{N^2}) = 1 + O_2(\frac{1}{N}).$$

We will not delve into detail for most order computations, as they primarily involve combinations of compositions and products of the previous examples, along with straightforward applications of Lemma 3.3.6.

3.4 Derivation of Perelman's entropy from Colding's monotonic volume

We start our derivation of the entropy by making some key observations about the level sets of b. Since $b/\sqrt{2N} = \sqrt{\tau}e^{f/(m-2)}$, for every $k \geq 0$ we have,

$$\frac{b}{\sqrt{2N}} = \sqrt{\tau} + O_k(\frac{1}{N}). \tag{3.4.1}$$

Therefore, for every small $\delta > 0$ and $k \geq 0$, the sequence of functions $b^2/2N : [\delta, T-\delta]_\tau \times M_x \to \mathbb{R}$ converge in C^k to the function (coordinate) $\tau : [\delta, T-\delta] \times M \to \mathbb{R}$. By standard calculus, it follows that there exists $N_0 > 0$ such that for every $N > N_0$ and $\lambda \in [2\delta, T-2\delta]$, the level set $b/\sqrt{2N} = \sqrt{\lambda}$ is given by an immersion

$$x \in M \mapsto (\phi_{N,\lambda}(x), x) \in (0,T) \times M, \tag{3.4.2}$$

for some smooth function $\phi_{N,\lambda} : M \to \mathbb{R}$. Furthermore, from (3.4.1) we get,

$$\lambda = \frac{b^2}{2N}(\phi_{N,\lambda}(x), x) = \tau(\phi_{N,\lambda}(x), x) + O_k(\frac{1}{N}) = \phi_{N,\lambda}(x) + O_k(\frac{1}{N}), \tag{3.4.3}$$

that is, the immersion approaches the level set $\tau = \lambda$. Since we will be interested in taking limits, we will always assume that N is sufficiently big such that (3.4.2) holds inside a region $\lambda \in [\delta, T-\delta]$, where δ is sufficiently small.

We will make repeated use of the following results.

Lemma 3.4.1. *Let $f = f(\tau, x)$ be any smooth real-valued function, that we consider as a function on $\hat{M}$. Then,*

$$|\hat{\nabla} b|^2 = 1 + \frac{1}{N}\left(2f - 2\tau R + 2\tau|\nabla f|^2 + 4\tau\partial_\tau f\right) + O_2(\frac{1}{N^2}). \tag{3.4.4}$$

In particular, if f satisfies (3.3.3), then,

$$\frac{N}{2}(|\hat{\nabla} b|^2 - 1) = \tau(2\Delta f - |\nabla f|^2 + R) + f - n + O_2(\frac{1}{N}), \tag{3.4.5}$$

and therefore,

$$|\hat{\nabla} b| = 1 + O_2(\frac{1}{N}). \tag{3.4.6}$$

Proof. We compute,

$$\hat{\nabla} b = \frac{e^{f/(m-2)}}{\sqrt{v}} \left(1 + \frac{r\partial_r f}{m-2}\right) \frac{\partial_r}{\sqrt{v}} + \frac{re^{f/(m-2)}}{m-2} \nabla f. \tag{3.4.7}$$

Thus,

$$|\hat{\nabla} b|^2 = \frac{e^{2f/(m-2)}}{v} \left(1 + \frac{r\partial_r f}{m-2}\right)^2 + \frac{r^2 e^{2f/(m-2)}}{(m-2)^2} |\nabla f|^2. \tag{3.4.8}$$

Now, $r\partial_r f = 2\tau\partial_\tau f$. Therefore,

$$\left(1 + \frac{r\partial_r f}{m-2}\right)^2 = 1 + \frac{4\tau\partial_\tau f}{N} + O_2(\frac{1}{N^2}). \tag{3.4.9}$$

Also, by virtue of Lemma 3.3.6 (see also the discussion below the Lemma),

$$\frac{e^{2f/(m-2)}}{v} = \left(1 + \frac{2f}{m-2} + O_2(\frac{1}{N^2})\right)\left(1 - \frac{2\tau R}{N} + O_2(\frac{1}{N^2})\right) = 1 + \frac{2f}{N} - \frac{2\tau R}{N} + O_2(\frac{1}{N^2}),$$

and similarly,

$$\frac{r^2 e^{2f/(m-2)}}{(m-2)^2} |\nabla f|^2 = \frac{2N\tau}{(m-2)^2} e^{2f/(m-2)} |\nabla f|^2 = \frac{2\tau|\nabla f|^2}{N} + O_2(\frac{1}{N^2}).$$

Combining these expressions with (3.4.8), we obtain (3.4.4). Finally, we can replace $\partial_\tau f$ by (3.3.3) to get (3.4.5). □

Proposition 3.4.2. *For any $\lambda \in (0, T)$, the volume element for the level set $b = s = \sqrt{2N\lambda}$ can be expressed as,*

$$d\hat{A} = s^N e^{-f(\lambda,x)}(1 + O_2(\frac{1}{N}))d\nu d\nu_{\mathbb{S}^N}, \tag{3.4.10}$$

where $d\nu_{\mathbb{S}^N}$ is the standard volume element in $\mathbb{S}^N$, and $d\nu$ is the volume element in $(M, g(\lambda))$. In particular, we have

$$d\hat{A} = (2N\lambda)^{N/2} e^{-f(\lambda,x)}(1 + O_2(\frac{1}{N}))d\nu d\nu_{\mathbb{S}^N}. \tag{3.4.11}$$

Proof. Since $\lambda \in (0, T)$, there exists $\delta > 0$ such that $\lambda \in [\delta, T - \delta]$. Then, using (3.4.2), for every $N > N_0$ we define a map $\psi_{N,\lambda} : \mathbb{S}^N \times M \to \hat{M}$ as

$$\psi_{N,\lambda}(\theta, x) = (\sqrt{2N\phi_{N,\lambda}}(x), \theta, x). \tag{3.4.12}$$

Let θ^α be coordinates in $\mathbb{S}^N$ and x^i coordinates on M. Since $r = \sqrt{2N\phi_{N,\lambda}}(x)$ and by (3.4.3) we have $\phi_{N,\lambda} = \lambda + O_k(N^{-1})$, we compute,

$$\begin{aligned}
\psi^*_{N,\lambda}\hat{g}(\partial_i, \partial_j) &= \hat{g}(d\psi_{N,\lambda}(\partial_i), d\psi_{N,\lambda}(\partial_j)) \\
&= \hat{g}\left(\frac{\sqrt{N}}{\sqrt{2\phi_{N,\lambda}}}\partial_i\phi_{N,\lambda}\partial_r + \partial_i, \frac{\sqrt{N}}{\sqrt{2\phi_{N,\lambda}}}\partial_j\phi_{N,\lambda}\partial_r + \partial_j\right) \\
&= g_{ij} + O_2(\frac{1}{N}), \\
\psi^*_{N,\lambda}\hat{g}(\partial_\alpha, \partial_\beta) &= \hat{g}(\partial_\alpha, \partial_\beta) = (2N\phi_{N,\lambda})g_{\mathbb{S}^N\alpha\beta}, \\
\psi^*_{N,\lambda}\hat{g}(\partial_\alpha, \partial_i) &= 0,
\end{aligned}$$

for every $N > N_0$. Therefore, we can express the volume element $d\hat{A}$ as,

$$d\hat{A} = (2N\phi_{N,\lambda})^{N/2}(1 + O_2(\frac{1}{N}))d\nu d\nu_{\mathbb{S}^N},$$

where we are using Lemma 3.3.6 when computing $\sqrt{\psi^*_{N,s}\hat{g}}$. Since $r^2/2N = \tau = \phi_{N,\lambda}(x)$ for any $(r, \theta, x) \in b = s$, we have,

$$r = \sqrt{2N\phi_{N,\lambda}}(x) = se^{f(\phi_{N,\lambda}(x),x)/(2-m)}, \tag{3.4.13}$$

and by (3.4.3) and Lemma 3.3.6 we can write,

$$\begin{aligned}
(2N\phi_{N,\lambda})^{N/2} &= s^N e^{-f(\lambda+O_k(1/N),x)+O_2(1/N)} \\
&= s^N e^{-f(\lambda,x)}(1 + O_2(\frac{1}{N})),
\end{aligned}$$

which completes the proof. □

We now demonstrate that Perelman's $\mathcal{W}$-functional and its derivative emerge as the limits of the area $\mathcal{A}_N$ and its derivative, respectively, when considered as functions of λ (see Remark 3.3.3).

Theorem 3.4.3. *The following equality holds:*

$$\mathcal{A}_N(\lambda) = \mathcal{W}(\lambda) + O_2(\frac{1}{N}), \tag{3.4.14}$$

where $\mathcal{W}$ *is Perelman's entropy functional* (*1.1.15*)*. In particular,*

$$\mathcal{A}_N \to \mathcal{W}, \tag{3.4.15}$$

and

$$\frac{d\mathcal{A}_N}{d\lambda} \to \frac{d\mathcal{W}}{d\lambda}, \tag{3.4.16}$$

uniformly on compact sets in $(0, T)$.

Proof. Let $s = \sqrt{2N\lambda}$. We compute,

$$\begin{aligned}
\mathcal{A}_N &= \frac{c_N}{s^{m-1}} \int_{b=s} (|\hat{\nabla} b|^2 - 1)|\hat{\nabla} b| d\hat{A} \\
&= (4\pi\lambda)^{-n/2} \int_M \big(\lambda(2\Delta f - |\nabla f|^2 + R) + f - n\big)\, e^{-f(\lambda,x)}(1 + O_2(\frac{1}{N}))d\nu \\
&= (4\pi\lambda)^{-n/2} \int_M \big(\lambda(2\Delta f - |\nabla f|^2 + R) + f - n\big)\, e^{-f(\lambda,x)} d\nu + O_2(\frac{1}{N}),
\end{aligned}$$

where we used (3.4.5) and (3.4.11) together with the fact that the integrand does not depend on $\theta \in \mathbb{S}^N$. Since M is a closed manifold,

$$\int_M (\Delta f - |\nabla f|^2) e^{-f} d\nu = - \int_M \Delta e^{-f} d\nu = 0,$$

and we use it to rewrite,

$$\mathcal{A}_N = (4\pi\lambda)^{-n/2} \int_M \big(\lambda(|\nabla f|^2 + R) + f - n\big)\, e^{-f} d\nu + O_2(\frac{1}{N}),$$

from which (3.4.14) follows. The previous expression also shows that $\mathcal{A}_N \to \mathcal{W}$ uniformly on compact sets in $(0, T)$, and differentiating with respect to λ on both sides of (3.4.14),

$$\frac{d\mathcal{A}_N}{d\lambda} = \frac{d\mathcal{W}}{d\lambda} + O_1(\frac{1}{N}), \tag{3.4.17}$$

which also shows that the convergence of the derivatives is uniform on compact sets in $(0, T)$. □

Now, we proceed to study the convergence of the volume functional (3.3.8). To begin, we introduce the following lemma.

Lemma 3.4.4. *Let* $\lambda_0 < \lambda \in (0, T)$*. Define* $\bar{s} = \sqrt{2N\lambda_0}$ *and* $s = \sqrt{2N\lambda}$*. Then,*

$$\mathcal{V}_N(s) = \frac{c_N}{s^m} \int_{\bar{s} \le b \le s} (|\hat{\nabla} b|^2 - 1)|\hat{\nabla} b|^2\, d\hat{V} + O_1(e^{-cN}). \tag{3.4.18}$$

Proof. We write,

$$\mathcal{V}_N(s) = \frac{c_N}{s^m}\int_{\bar{s}\le b\le s}(|\hat{\nabla} b|^2-1)|\hat{\nabla} b|^2\,d\hat{V} + \frac{c_N}{s^m}\int_{0\le b\le \bar{s}}(|\hat{\nabla} b|^2-1)|\hat{\nabla} b|^2\,d\hat{V}.$$

Since $f = -\ln u - (n/2)\ln\tau$, a straightforward computation using (3.4.7) shows that we can write $|\hat{\nabla} b|^2$ in terms of u as,

$$\begin{aligned}|\hat{\nabla} b|^2 &= (2N)^{n/(m-2)}(ur^n)^{2/(2-m)}\left[\left(1-\frac{2\tau\partial_\tau u - nu}{u(m-2)}\right)^2 + \frac{r^2}{(m-2)^2}\frac{|\nabla u|^2}{u^2}\right]\\ &= (2N)^{n/(m-2)}(ur^n)^{2/(2-m)}H_N(r^2/2N, x),\end{aligned}$$

and, since u is everywhere positive in $[0,\lambda_0]\times M$ and its first derivatives are bounded, it follows that $|H_N|\le C$ and $0<c_1<u<c_2$ in $[0,\lambda_0]\times M$, for some constants $C, c_1, c_2>0$. In particular, the same bounds hold for the region $0\le b\le \bar{s}$, and since $d\hat{V} = r^N dr d\nu d\nu_{\mathbb{S}^N}$, we use the previous expression for $|\hat{\nabla} b|^2$ to find,

$$\left|\frac{c_N}{s^m}\int_{0\le b\le\bar{s}}|\hat{\nabla} b|^4 d\hat{V}\right| \le K_1\left(\frac{\bar{s}}{s}\right)^{N+1}, \tag{3.4.19}$$

for some constant $K_1>0$. Since $\bar{s}/s = \lambda_0/\lambda < 1$, it decays exponentially fast. Similarly,

$$\left|\frac{c_N}{s^m}\int_{0\le b\le\bar{s}}|\hat{\nabla} b|^2 d\hat{V}\right| \le K_2\left(\frac{\bar{s}}{s}\right)^{N+1}, \tag{3.4.20}$$

for some constant $K_2>0$, which shows

$$\mathcal{V}_N = \frac{c_N}{s^m}\int_{\bar{s}\le b\le s}(|\hat{\nabla} b|^2-1)|\hat{\nabla} b|^2\,d\hat{V} + O_0(e^{-cN}).$$

Finally, in order to see that the term $O_0(e^{-cN})$ is in fact $O_1(e^{-cN})$, we note that

$$\begin{aligned}\frac{d}{d\lambda}\left(\frac{c_N}{s^m}\int_{0\le b\le\bar{s}}(|\hat{\nabla} b|^2-1)|\hat{\nabla} b|^2\,d\hat{V}\right) &= \frac{ds}{d\lambda}\frac{d}{ds}\left(\frac{c_N}{s^m}\int_{\bar{0}\le b\le\bar{s}}(|\hat{\nabla} b|^2-1)|\hat{\nabla} b|^2\,d\hat{V}\right)\\ &= -\left(\frac{N}{2\lambda}\right)^{1/2}\frac{m}{s}\frac{c_N}{s^m}\int_{\bar{0}\le b\le\bar{s}}(|\hat{\nabla} b|^2-1)|\hat{\nabla} b|^2 d\hat{V},\end{aligned}$$

and apply (3.4.19) and (3.4.20) to see that its derivative is also an $O_0(e^{-cN})$. □

Proposition 3.4.5. *The following equality holds:*

$$2(m-1)\mathcal{V}_N - \mathcal{A}_N = \mathcal{A}_N + O_1(\frac{1}{N}). \tag{3.4.21}$$

In particular,

$$\frac{d}{ds}(2(m-1)\mathcal{V}_N - \mathcal{A}_N) = \frac{d}{ds}\mathcal{A}_N + O_0(\frac{1}{N^{3/2}}). \tag{3.4.22}$$

Proof. Let $\lambda \in (0,T)$, and define $s = \sqrt{2N\lambda}$. Next, we choose $\lambda_0 < \lambda$ and define $\bar{s} = \sqrt{2N\lambda_0}$. By the coarea formula and equation (3.4.18) we obtain,

$$\mathcal{V}_N = \frac{c_N}{s^m}\int_{\bar{s}\le b\le s}(|\hat{\nabla}b|^2-1)|\hat{\nabla}b|^2 d\hat{V} + O_1(e^{-cN}) = \frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\mathcal{A}_N(w)dw + O_1(e^{-cN}).$$

Using (3.4.17), we deduce,

$$\begin{aligned}\frac{d\mathcal{A}_N}{ds} &= \frac{d\lambda}{ds}\frac{d\mathcal{A}_N}{d\lambda}\\ &= (4\pi)^{n/2}\frac{d\lambda}{ds}\left(\frac{d\mathcal{W}}{d\lambda} + \frac{d}{d\lambda}\left(O_2(\frac{1}{N})\right)\right)\\ &= (4\pi)^{n/2}\left(\frac{2\lambda}{N}\right)^{1/2}\left(\frac{d\mathcal{W}}{d\lambda} + O_1(\frac{1}{N})\right)\\ &= O_1(\frac{1}{N^{1/2}}),\end{aligned} \tag{3.4.23}$$

and,

$$\frac{d^2\mathcal{A}_N}{ds^2} = \frac{d\lambda}{ds}\frac{d}{d\lambda}\left(O_1(\frac{1}{N^{1/2}})\right) = O_0(\frac{1}{N}), \tag{3.4.24}$$

since $d\mathcal{W}/d\lambda = O_2(1)$. A Taylor expansion around s shows,

$$\mathcal{A}_N(w) = \mathcal{A}_N(s) + (w-s)\frac{d\mathcal{A}_N}{ds}(\xi_{w,s}),$$

for some $\sqrt{2N\lambda_0} \le \xi_{w,s} \le \sqrt{2N\lambda}$. The bound on $\xi_{w,s}$ implies the respective bound for its associated $\tau-$coordinate between λ_0 and λ, and therefore by (3.4.23),

$$\mathcal{A}_N(w) = \mathcal{A}_N(s) + (w-s)O_0(\frac{1}{N^{1/2}}),$$

for any $w \in [\bar{s}, s]$. Now, we notice that, since $|O_0(N^{-1/2})| \le CN^{-1/2}$,

$$\begin{aligned}-\frac{s^{m+1}}{s^m(m+1)m}\frac{C}{N^{1/2}} + O_0(e^{-cN}) &\le \frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\left((w-s)O_0(\frac{1}{N^{1/2}})\right)dw\\ &\le \frac{s^{m+1}}{s^m(m+1)m}\frac{C}{N^{1/2}} + O_0(e^{-cN}),\end{aligned} \tag{3.4.25}$$

which shows,

$$\frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\left((w-s)O_0(\frac{1}{N^{1/2}})\right)dw = O_0(\frac{1}{N^2}). \tag{3.4.26}$$

Then,

$$\begin{aligned}\mathcal{V}_N &= \frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\left(\mathcal{A}_N(s) + (w-s)O_0(\frac{1}{N^{1/2}})\right)dw + O_1(e^{-cN}) \\ &= \frac{1}{m}\mathcal{A}_N + O_0(\frac{1}{N^2}).\end{aligned} \tag{3.4.27}$$

In order to see that the term $O_0(N^{-2})$ is actually an $O_1(N^{-2})$, we use a Taylor expansion of order 2 to compute,

$$\begin{aligned}\frac{d\mathcal{V}_N}{ds} &= \frac{d}{ds}\left(\frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\mathcal{A}_N(w)dw\right) + O_0(e^{-cN}) \\ &= -\frac{m}{s^{m+1}}\int_{\bar{s}}^{s} w^{m-1}\mathcal{A}_N(w)dw + \frac{1}{s}\mathcal{A}_N(s) + O_0(e^{-cN}) \\ &= -\frac{m}{s^{m+1}}\int_{\bar{s}}^{s} w^{m-1}\left(\mathcal{A}_N(s) + (w-s)\frac{d}{ds}\mathcal{A}_N(s) + \frac{(w-s)^2}{2}\frac{d^2\mathcal{A}_N}{ds^2}(\xi_{w,s})\right)dw \\ &\quad + \frac{1}{s}\mathcal{A}_N(s) + O_0(e^{-cN}),\end{aligned} \tag{3.4.28}$$

where $\xi_{w,s} \in [w,s]$. Now, since $d^2\mathcal{A}_N/ds^2 = O_0(1/N)$, we proceed as in (3.4.25) and show,

$$\begin{aligned}&\int_{\bar{s}}^{s} w^{m-1}\left(\mathcal{A}_N(s) + (w-s)\frac{d\mathcal{A}_N}{ds}(s) + \frac{(w-s)^2}{2}\frac{d^2\mathcal{A}_N}{ds^2}(\xi_{w,s})\right)dw \\ &\qquad = \frac{s^m}{m}\mathcal{A}_N(s) - \frac{s^{m+1}}{m(m+1)}\frac{d\mathcal{A}_N}{ds}(s) + \frac{s^{m+2}}{m(m+1)(m+2)}O_0(\frac{1}{N}) + O_0(e^{-cN}).\end{aligned} \tag{3.4.29}$$

Use this in (3.4.28) and the fact that $d\mathcal{A}_N/ds = O_1(N^{-1/2})$ to write,

$$\frac{d\mathcal{V}_N}{ds} = \frac{1}{m+1}\frac{d\mathcal{A}_N}{ds}(s) + O_0(\frac{1}{N^{5/2}}) = \frac{1}{m}\frac{d\mathcal{A}_N}{ds}(s) + O_0(\frac{1}{N^{5/2}}). \tag{3.4.30}$$

This implies,

$$\frac{d\mathcal{V}_N}{d\lambda} = \frac{ds}{d\lambda}\frac{d\mathcal{V}_N}{ds} = \frac{1}{m}\frac{d\mathcal{A}_N}{d\lambda} + \frac{ds}{d\lambda}O_0(\frac{1}{N^{5/2}}) = \frac{1}{m}\frac{d\mathcal{A}_N}{d\lambda} + O_0(\frac{1}{N^2}),$$

which we combine with (3.4.27) to obtain,

$$\mathcal{V}_N = \frac{1}{m}\mathcal{A}_N + O_1(\frac{1}{N^2}). \tag{3.4.31}$$

From here, a straightforward computation shows (3.4.21). We finish the proof by differentiating (3.4.21) with respect to s and noting that

$$\frac{d}{ds}\left(O_1(\frac{1}{N})\right) = \frac{d\lambda}{ds}\frac{d}{d\lambda}\left(O_1(\frac{1}{N})\right) = O_0(\frac{1}{N^{3/2}}).$$

□

As a consequence of Theorem 3.4.3 and Proposition 3.4.5, and recalling the definition of the monotonic volume (3.3.9) we immediately derive the following result.

Corollary 3.4.6. *The monotonic volume obeys,*

$$\mathcal{W}_N(\lambda) = \mathcal{W}(\lambda) + O_1(\frac{1}{N}), \tag{3.4.32}$$

and,

$$\frac{d}{d\lambda}\mathcal{W}_N(\lambda) = \frac{d}{d\lambda}\mathcal{W}(\lambda) + O_0(\frac{1}{N}). \tag{3.4.33}$$

In particular, $\mathcal{W}_N \to \mathcal{W}$ and $d\mathcal{W}_N/d\lambda \to d\mathcal{W}/d\lambda$ uniformly on compact sets in $(0, T)$.

3.5 The derivative of the entropy

We now recover Perelman's formula for the derivative of $\mathcal{W}$ as a limit of derivatives of $\mathcal{W}_N$. To achieve this, several auxiliary results are in order.

Proposition 3.5.1. *Let $h = h(\tau, x)$ be any smooth, real-valued function, that we consider as a function on $\hat{M}$. Then,*

$$\hat{\Delta} h = \frac{1}{v}\left(\left(1 + \frac{1 + 2\tau R}{N} - \frac{2(\tau R + \tau^2\partial_\tau R)}{N^2 v}\right)\partial_\tau h + \frac{2\tau}{N}\partial_\tau^2 h\right) \tag{3.5.1}$$

$$+ \Delta h + \frac{1}{Nv}\langle \nabla R, \nabla h\rangle. \tag{3.5.2}$$

Proof. Let θ^α be coordinates in $\mathbb{S}^N$ and x^i coordinates on M. Then, the inverse metric components of $\hat{g}$ are,

$$\hat{g}^{ri} = \hat{g}^{r\alpha} = \hat{g}^{\alpha i} = 0, \quad \hat{g} = \frac{1}{v}, \quad \hat{g}^{ij} = g^{ij}, \quad \hat{g}^{\alpha\beta} = \frac{1}{r^2}g^{\alpha\beta}_{\mathbb{S}^N}, \tag{3.5.3}$$

and the Laplacian therefore is,

$$\hat{\Delta} h = \frac{1}{\sqrt{\hat{g}}}\partial_r(\sqrt{\hat{g}}\hat{g}^{rr}\partial_r h) + \frac{1}{\sqrt{\hat{g}}}\partial_i(\sqrt{\hat{g}}\hat{g}^{ij}\partial_j h) + \frac{1}{\sqrt{\hat{g}}}\partial_\alpha(\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\partial_\beta h). \tag{3.5.4}$$

A straightforward computation shows that the first term of the r.h.s of (3.5.4) is equal to the r.h.s of (3.5.1), the second term is equal to the two summands in (3.5.2) and the last term is equal to zero. □

The following assertion is drawn from Perelman (as mentioned in p.13 of [47]), where we needed to include factor of r^{2-m}.

Proposition 3.5.2 (Perelman, [47])**.** *Let $f = f(\tau, x)$ be any smooth real valued function that we consider as a function on $\hat{M}$, and let $h = r^{2-m}e^{-f}$. Then,*

$$\hat{\Delta} h = r^{2-m}\left(\partial_\tau f - \Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}\right)e^{-f} + r^{2-m}O_2(\frac{1}{N}). \tag{3.5.5}$$

Therefore, if f satisfies, (3.3.3), then

$$\hat{\Delta} h = r^{2-m}O_2(\frac{1}{N}). \tag{3.5.6}$$

Proof. Let $u = \tau^{-(n-1)/2}e^{-f}$ so that $h = (2N)^{(2-m)/2}\tau^{-N/2}u$. To simplify notation, we will we will disregard the multiplicative factor $(2N)^{(2-m)/2}$ in h during the computations, and add it back in afterwards. We compute,

$$\partial_\tau h = -\frac{N}{2}\tau^{-N/2-1}u + \tau^{-N/2}\partial_\tau u, \tag{3.5.7}$$

$$\partial_\tau^2 h = \frac{N}{2}\left(\frac{N}{2}+1\right)\tau^{-N/2-2}u - N\tau^{-N/2-1}\partial_\tau u + \tau^{-N/2}\partial_\tau^2 u. \tag{3.5.8}$$

We can write $\partial_\tau^2 u = \tau^{-(n+1)/2}O_2(1)$. Therefore,

$$\frac{2\tau}{N}\partial_\tau^2 h = \tau^{-N/2}\left(\frac{N}{2\tau}u + \frac{1}{\tau}u - 2\partial_\tau u\right) + \tau^{-m/2}O_2(\frac{1}{N}). \tag{3.5.9}$$

Similarly, writing (3.5.7) as,

$$\partial_\tau h = \tau^{-N/2}\left(-\frac{N}{2\tau}u + \partial_\tau u\right), \tag{3.5.10}$$

we obtain,

$$\left(1 + \frac{1+2\tau R}{N} - \frac{2(\tau R + \tau^2\partial_\tau R)}{N^2 v}\right)\partial_\tau h = \tag{3.5.11}$$

$$-\tau^{-N/2}\left(\frac{N}{2\tau}u + (\frac{1}{2\tau} + R)u - \partial_\tau u\right) + \tau^{-m/2}O_2(\frac{1}{N}). \tag{3.5.12}$$

Summing (3.5.9) and (3.5.11), and after the crucial mutual cancellation of the terms $\tau^{-N/2-1}Nu/2$, we deduce,

$$\frac{2\tau}{N}\partial_\tau^2 h + \left(1 + \frac{1+2\tau R}{N} - \frac{2(\tau R + \tau^2\partial_\tau R)}{N^2 v}\right)\partial_\tau h = \tag{3.5.13}$$

$$\tau^{-N/2}(-\frac{1}{2\tau}u - Ru - \partial_\tau u) + \tau^{-m/2}O_2(\frac{1}{N}). \tag{3.5.14}$$

Going back to the expression (3.5.1) for $\hat{\Delta}h$, and taking into account that $1/v = 1+O_2(1/N)$ and $\langle\nabla R, \nabla h\rangle/Nv = O_2(1/N)$, we arrive at,

$$\hat{\Delta}h = \tau^{-N/2}(-\partial_\tau u + \Delta u - Ru + \frac{1}{2\tau}u) + \tau^{-m/2}O_2(\frac{1}{N}). \tag{3.5.15}$$

Finally, recalling that $u = \tau^{-(n-1)/2}e^{-f}$ and multiplying by the factor $(2N)^{(2-m)/2}$, we deduce (3.5.5). $\square$

From the previous proposition we obtain the following.

Proposition 3.5.3. *Let h be defined as in (3.3.5) and b as in (3.3.6). Then,*

$$\hat{\Delta}b^2 = 2m|\hat{\nabla}b|^2 + O_2(\frac{1}{N}). \tag{3.5.16}$$

Proof. Direct computation shows,

$$\hat{\Delta}b^2 = 2m|\hat{\nabla}b|^2 + \frac{2}{2-m}h^{m/(2-m)}\hat{\Delta}h.$$

Using that $\hat{\Delta}h = r^{2-m}O_2(1/N)$ and that $b = h^{1/(2-m)}$, we get,

$$\frac{2}{2-m}h^{m/(2-m)}\hat{\Delta}h = \frac{2}{2-m}b^m r^{2-m}O_2(\frac{1}{N}) = \frac{2}{2-m}r^2 e^{mf/(m-2)}O_2(\frac{1}{N}) =$$
$$= \frac{4N}{2-m}\tau e^{mf/(m-2)}O_2(\frac{1}{N}) = O_2(\frac{1}{N}).$$

$\square$

The following formula for the derivative of the raw area $\overline{\mathcal{A}}_N$ follows essentially from Theorem 3.2.6, but with two key differences: (i) additional terms appear due to the fact that $\hat{\Delta}h \neq 0$, and (ii) when integrating applying Gauss's theorem, the integration is performed over the region on $\bar{s} \leq b \leq s$, for some $0 < \bar{s} < s$, rather than on $b \leq s$.

Proposition 3.5.4.

$$\begin{aligned}\frac{d}{ds}\overline{\mathcal{A}}_N =&\frac{c_N}{2s^{m+1}}\int_{\bar{s}\le b\le s}\left(\left|\hat{\nabla}\hat{\nabla}b^2-\frac{\hat{\Delta}b^2}{m}\hat{g}\right|^2+\hat{\mathrm{Ric}}(\hat{\nabla}b^2,\hat{\nabla}b^2)\right)d\hat{V}\\ &+\frac{2(m-1)}{s}(\overline{\mathcal{A}}_N-m\overline{\mathcal{V}}_N)+3[A_N]+[B_N]+[C_N]+[D_N],\end{aligned} \tag{3.5.17}$$

where,

$$\overline{\mathcal{V}}_N=\frac{c_N}{s^m}\int_{\bar{s}\le b\le s}|\hat{\nabla}b|^4\,d\hat{V}, \tag{3.5.18}$$

$$[A_N]=\frac{c_N}{2-m}\int_{b=s}|\hat{\nabla}b|\hat{\Delta}h\,d\hat{A}, \tag{3.5.19}$$

$$[B_N]=\frac{c_N}{4s^{m+1}}\int_{b=\bar{s}}\hat{\nabla}_{\hat{\mathrm{n}}}(|\hat{\nabla}b^2|^2)\,d\hat{A}, \tag{3.5.20}$$

$$[C_N]=-\frac{c_N}{2s^{m+1}}\int_{b=\bar{s}}(\hat{\Delta}b^2)\hat{\nabla}_{\hat{\mathrm{n}}}b^2\,d\hat{A}, \tag{3.5.21}$$

$$[D_N]=-(1-\frac{1}{m})\frac{c_N}{2s^{m+1}}\int_{\bar{s}\le b\le s}\left(\frac{8m}{2-m}|\hat{\nabla}b|^2b^m\hat{\Delta}h+\frac{4b^{2m}}{(2-m)^2}(\hat{\Delta}h)^2\right)d\hat{V}, \tag{3.5.22}$$

and where $\hat{\mathrm{n}}=\hat{\nabla}b/|\hat{\nabla}b|$ *is the unit-normal to* $b=\bar{s}$.

Proof. The proof mirrors that of Theorem 3.2.6, but, when computing $(s^2\overline{\mathcal{A}}_N)'/s^2$, use that $\hat{\Delta}b^2=2m|\hat{\nabla}b|^2+2b^m(\hat{\Delta}h)/(2-m)$ instead of $\hat{\Delta}b^2=2m|\hat{\nabla}b|^2$. Furthermore, when using Gauss's theorem, integrate on $\bar{s}\le b\le s$ rather than on $b\le s$. □

Proposition 3.5.5. *We have,*

$$\frac{d}{ds}\left(\frac{c_N}{s^{m-1}}\int_{b=s}|\hat{\nabla}b|d\hat{A}\right)=[A_N]+O_2(\frac{1}{N^{3/2}}). \tag{3.5.23}$$

Proof. Since $\hat{\nabla}_{\hat{n}}b=|\hat{\nabla}b|$, we compute,

$$\frac{d}{ds}\left(\frac{1}{s^{m-1}}\int_{b=s}|\hat{\nabla}b|\,d\hat{A}\right)=\frac{d}{ds}\left(\int_{b=s}\frac{\hat{\nabla}_{\hat{\mathrm{n}}}b}{b^{m-1}}\,d\hat{A}\right)=\frac{1}{2-m}\frac{d}{ds}\left(\int_{b=s}\hat{\nabla}_{\hat{\mathrm{n}}}h\,d\hat{A}\right) \tag{3.5.24}$$

$$=\frac{1}{2-m}\int_{b=s}\frac{\hat{\Delta}h}{|\hat{\nabla}b|}\,d\hat{A}. \tag{3.5.25}$$

Appeal now to (3.4.6) to obtain,

$$\frac{1}{|\hat{\nabla}b|}=|\hat{\nabla}b|\frac{1}{|\hat{\nabla}b|^2}=|\hat{\nabla}b|\frac{1}{1+O_2(\frac{1}{N})}=|\hat{\nabla}b|(1+O_2(\frac{1}{N}))=|\hat{\nabla}b|+O_2(\frac{1}{N}). \tag{3.5.26}$$

Then, using (3.4.13) in (3.5.6), we get $\hat{\Delta} h = s^{2-m} O_2(1/N)$ on the level set $b = s$. Now, combining this, (3.5.26) and (3.4.10) all in (3.5.25), we deduce,

$$\frac{d}{ds}\left(\frac{c_N}{s^{m-1}}\int_{b=s} |\hat{\nabla} b| d\hat{A}\right) = \\ = [A_N] + \frac{c_N}{2-m}\int_M s^{2-m} O_2(\frac{1}{N}) O_2(\frac{1}{N}) s^N (1 + O_2(\frac{1}{N})) e^{-f}\, d\nu d\nu_{\mathbb{S}^N}.$$

But, as $c_N = (4\pi)^{-n/2}(2N)^{n/2+1}/(4|\mathbb{S}^N|)$ and $s = (2N\lambda)^{1/2}$, we obtain,

$$\frac{c_N}{2-m}\int_M s^{2-m} O_2(\frac{1}{N}) O_2(\frac{1}{N}) s^N (1 + O_2(\frac{1}{N})) e^{-f}\, d\nu d\nu_{\mathbb{S}^N} = \\ = \frac{(2N)^{n/2+1}}{2-m}\frac{1}{N^2}(2N\lambda)^{1/2-n/2} O_2(1) = O_2(\frac{1}{N^{3/2}}),$$

as wished. □

We now proceed to simplify the expression (3.5.17). From this point onward, we set $\bar{s} = \sqrt{2N\lambda_0}$, where $\lambda_0 < \lambda$.

Proposition 3.5.6. *The following equality holds,*

$$3[A_N] + [B_N] + [C_N] + [D_N] = -[A_N] + O_0(\frac{1}{N^{3/2}}). \tag{3.5.27}$$

Proof. We first show that $[B_N]$ and $[C_N]$ decay exponentially fast. In order to control $[B_N]$, observe that $\hat{\nabla}_{\hat{\mathrm{n}}}(|\hat{\nabla} b^2|^2) = \hat{g}(\hat{\nabla}|\hat{\nabla} b^2|^2, \hat{n})$. Now, we write $|\hat{\nabla} b^2|^2 = 4b^2|\hat{\nabla} b|^2$ and use (3.4.4) to obtain,

$$\hat{\nabla}|\hat{\nabla} b^2|^2 = 8b|\hat{\nabla} b|^2\hat{\nabla} b + 4b^2\hat{\nabla}\left(O_2(\frac{1}{N})\right).$$

Using that $\partial_r = r/N\partial_\tau$ and $b = O_2(N^{1/2})$, we get,

$$\hat{\nabla}|\hat{\nabla} b^2|^2 = O_1(N^{1/2})\partial_r + \sum_{i=1}^n O_1(N^{1/2})\partial_{x^i}.$$

Since by (3.4.6) and (3.4.7) we have $\hat{n} = O_2(1)\partial_r + O_2(N^{-1})\nabla f$, we see,

$$|\hat{\nabla}_{\hat{\mathrm{n}}}(|\hat{\nabla} b^2|^2)| = O_1(N^{1/2}).$$

We integrate using (3.4.10) to find,

$$[B_N] = \frac{c_N}{4s^{m+1}}\int_{b=\bar{s}} O_1(N^{1/2}) d\hat{A} = O_1(N)\left(\frac{\bar{s}}{s}\right)^N = O_1(N)\left(\frac{\lambda_0}{\lambda}\right)^{N/2},$$

and since $\lambda_0 < \lambda$, it decays exponentially fast.

In order to control $[C_N]$, we observe that since $(\hat{\Delta} b^2)\hat{\nabla}_{\hat{n}} b^2 = (\hat{\Delta} b^2) 2b|\hat{\nabla} b|$, we can combine this with (3.5.16) and (3.4.6) to show that,

$$(\hat{\Delta} b^2)\hat{\nabla}_{\hat{n}} b^2 = O_2(N^{3/2}),$$

and now a similar computation as the one peformed for $[B_N]$ shows that $[C_N]$ also decays exponentially fast.

For $[D_N]$ we proceed as follows. By the coarea formula,

$$\begin{aligned}\frac{c_N}{2s^{m+1}} &\int_{\bar{s}\le b\le s} \frac{8m}{2-m} |\hat{\nabla} b|^2 b^m \hat{\Delta} h \, d\hat{V} \\ &= \frac{1}{s^{m+1}} \frac{4m}{2-m} \int_{\bar{s}}^{s} w^m \left(c_N \int_{b=w} |\hat{\nabla} b| \hat{\Delta} h \, d\hat{A} \right) dw.\end{aligned} \tag{3.5.28}$$

Then, perform a Taylor expansion on the term multiplying w^m around s to obtain,

$$\begin{aligned} c_N &\int_{b=w} |\hat{\nabla} b| \hat{\Delta} h \, d\hat{A} \\ &= c_N \int_{b=s} |\hat{\nabla} b| \hat{\Delta} h \, d\hat{A} + (2-m)(w-s) \frac{d}{ds} \left(\frac{c_N}{(2-m)} \int_{b=\xi_{w,N}} |\hat{\nabla} b| \hat{\Delta} h \, d\hat{A} \right), \end{aligned} \tag{3.5.29}$$

for some $\xi_{w,N} \in [w,s]$ (in particular, $\xi_{w,N} \in [\bar{s},s]$), and observe that the term inside parentheses is $[A_N]$ evaluated at $s = \xi_{w,N}$. Using (3.4.6) and (3.5.6), we see that $|\hat{\nabla} b|\hat{\Delta} h = r^{2-m} O_2(1/N)$. Then, by (3.4.13) and (3.4.10),

$$\begin{aligned} [A_N] = \frac{c_N}{2-m} \int_{b=s} |\hat{\nabla} b| \hat{\Delta} h d\hat{A} &= \frac{(2N)^{n/2}}{2(4\pi)^{n/2}} \int_M s^{2-m+N} e^{-f} O_2(\frac{1}{N})(1 + O_2(\frac{1}{N})) d\nu \\ &= O_2(\frac{1}{N^{1/2}}). \end{aligned} \tag{3.5.30}$$

We use this to compute,

$$\frac{d}{ds}[A_N] = \frac{d}{ds}\left(\frac{c_N}{2-m} \int_{b=s} |\hat{\nabla} b| \hat{\Delta} h d\hat{A} \right) = \frac{d\lambda}{ds} \frac{d}{d\lambda} \left(O_2(\frac{1}{N^{1/2}}) \right) = O_1(\frac{1}{N}), \tag{3.5.31}$$

and combine it with (3.5.29) to find,

$$c_N \int_{b=w} |\hat{\nabla} b| \hat{\Delta} h \, d\hat{A} = c_N \int_{b=s} |\hat{\nabla} b| \hat{\Delta} h \, d\hat{A} + (w-s) O_1(1).$$

Using this expression together with (3.5.28), we can proceed as in (3.4.25) and (3.4.27) to show,

$$\frac{c_N}{2s^{m+1}}\int_{\bar{s}\le b\le s}\frac{8m}{2-m}|\hat{\nabla} b|^2 b^m \hat{\Delta} h\, d\hat{V} = 4[A_N] + O_0(\frac{1}{N^{3/2}}). \tag{3.5.32}$$

In order to control the remaining term, we write

$$\begin{aligned}\frac{c_N}{2s^{m+1}}&\int_{\bar{s}\le b\le s}\frac{4b^{2m}}{(2-m)^2}(\hat{\Delta} h)^2\, d\hat{V}\\ &= \frac{2}{s^{m+1}}\int_{\bar{s}}^{s} w^m\left(\frac{c_N}{(2-m)^2}\int_{b=w} w^m\frac{(\hat{\Delta} h)^2}{|\hat{\nabla} b|}\, d\hat{A}\right) dw,\end{aligned}$$

and using (3.5.6), (3.4.6), (3.4.13) and (3.4.10), we notice that the term in parentheses is of order $O_2(N^{-3/2})$. Therefore, we proceed as in (3.4.25) and (3.4.27) to show,

$$\begin{aligned}\frac{c_N}{2s^{m+1}}\int_{\bar{s}\le b\le s}\frac{4b^{2m}}{(2-m)^2}(\hat{\Delta} h)^2\, d\hat{V} &= \frac{1}{m+1}O_0(\frac{1}{N^{3/2}})\\ &= O_0(\frac{1}{N^{5/2}}).\end{aligned} \tag{3.5.33}$$

Combining (3.5.32) and (3.5.33), it follows that

$$[D_N] = -4[A_N] + O_0(\frac{1}{N^{3/2}}). \tag{3.5.34}$$

Given that $[B_N]$ and $[C_N]$ decay exponentially fast, substituting (3.5.34) into the l.h.s. of (3.5.27) completes the proof. □

Proposition 3.5.7. *We have,*

$$\frac{2(m-1)}{s}(\overline{\mathcal{A}}_N - m\overline{\mathcal{V}}_N) = 2\frac{d}{ds}\overline{\mathcal{A}}_N(s) + O_0(\frac{1}{N^{3/2}}). \tag{3.5.35}$$

Proof. We use the coarea formula to write,

$$\overline{\mathcal{V}}_N = \frac{c_N}{s^m}\int_{\bar{s}\le b\le s}|\hat{\nabla} b|^4\, d\hat{V} = \frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\overline{\mathcal{A}}_N(w)dw,$$

and perform a Taylor expansion of $\overline{\mathcal{A}}_N$ around s, to obtain,

$$\overline{\mathcal{A}}_N(w) = \overline{\mathcal{A}}_N(s) + (w-s)\frac{d}{ds}\overline{\mathcal{A}}_N(s) + \frac{(w-s)^2}{2}\frac{d^2\overline{\mathcal{A}}_N}{ds^2}(\xi_{w,N}), \tag{3.5.36}$$

for some $\xi_{w,N}\in[w,s]$. In order to control the second derivative, we notice that

$$\overline{\mathcal{A}}_N = \mathcal{A}_N + \frac{c_N}{s^{m-1}}\int_{b=s}|\hat{\nabla} b|d\hat{A}. \tag{3.5.37}$$

By (3.4.23) and (3.4.24) respectively, we have $d\mathcal{A}_N/ds = O_1(N^{-1/2})$ and $d^2\mathcal{A}_N/ds^2 = O_0(N^{-1})$. Using Proposition 3.5.5, we compute the second derivative of the second term on the r.h.s of (3.5.37) as,

$$\begin{aligned}\frac{d^2}{ds^2}\left(\frac{c_N}{s^{m-1}}\int_{b=s}|\hat{\nabla}b|d\hat{A}\right) &= \frac{d}{ds}[A_N] + \frac{d\lambda}{ds}\frac{d}{d\lambda}\left(O_2(\frac{1}{N^{3/2}})\right)\\ &= O_1(\frac{1}{N}),\end{aligned} \tag{3.5.38}$$

since $d[A_N]/ds$ was already computed in (3.5.31). This shows,

$$\frac{d^2}{ds^2}\overline{\mathcal{A}}_N = O_0(\frac{1}{N}),$$

and therefore,

$$\overline{\mathcal{A}}_N(w) = \overline{\mathcal{A}}_N(s) + (w-s)\frac{d}{ds}\overline{\mathcal{A}}_N(s) + \frac{(w-s)^2}{2}O_0(\frac{1}{N}).$$

Now integrate using the bound $|O_0(1/N)| \leq C/N$ for some $C > 0$ and proceed as in (3.4.25) and (3.4.27) to show,

$$\frac{1}{s^m}\int_{\bar{s}}^{s} w^{m-1}\overline{\mathcal{A}}_N(w)dw = \frac{1}{m}\overline{\mathcal{A}}_N(s) - \frac{s}{m(m+1)}\frac{d}{ds}\overline{\mathcal{A}}_N(s) + O_0(\frac{1}{N^3}),$$

where we have absorbed the term decaying exponentially fast into $O_0(1/N^3)$. We finish the proof by expanding the l.h.s. of (3.5.35) to show,

$$\begin{aligned}\frac{2(m-1)}{s}(\overline{\mathcal{A}}_N - m\overline{\mathcal{V}}_N) &= \frac{2(m-1)}{s}\left(\frac{s}{(m+1)}\frac{d}{ds}\overline{\mathcal{A}}_N(s) + O_0(\frac{1}{N^2})\right)\\ &= 2\frac{d}{ds}\overline{\mathcal{A}}_N(s) + O_0(\frac{1}{N^{3/2}}).\end{aligned}$$

□

Combining the previous results, we obtain the following theorem.

Theorem 3.5.8. *We have,*

$$\frac{d}{ds}\mathcal{A}_N = -\frac{c_N}{2s^{m+1}}\int_{\bar{s}\leq b\leq s}\left(\left|\hat{\nabla}\hat{\nabla}b^2 - \frac{\hat{\Delta}b^2}{m}\hat{g}\right|^2 + \hat{\mathrm{Ric}}(\hat{\nabla}b^2, \hat{\nabla}b^2)\right)d\hat{V} + O_0(\frac{1}{N^{3/2}}). \tag{3.5.39}$$

In particular,

$$\frac{d}{ds}\big(2(m-1)\mathcal{V}_N - \mathcal{A}_N\big)(s)$$
$$= -\frac{c_N}{2s^{m+1}}\int_{\bar{s}\le b\le s}\left(\left|\hat{\nabla}\hat{\nabla}b^2 - \frac{\hat{\Delta}b^2}{m}\hat{g}\right|^2 + \hat{\mathrm{Ric}}(\hat{\nabla}b^2,\hat{\nabla}b^2)\right)d\hat{V} + O_0(\frac{1}{N^{3/2}}). \tag{3.5.40}$$

Proof. To prove (3.5.39), subtract (3.5.23) from (3.5.17), and simplify the expression using (3.5.27) and (3.5.35). To prove (3.5.40), apply (3.4.22). □

The integrand in (3.5.39) and (3.5.40) can be written as follows.

Proposition 3.5.9. *The following equality holds,*

$$\left|\hat{\nabla}\hat{\nabla}b^2 - \frac{\hat{\Delta}b^2}{m}\hat{g}\right|^2 + \hat{\mathrm{Ric}}(\hat{\nabla}b^2,\hat{\nabla}b^2) = \frac{4b^4}{(2-m)^2}\left|\nabla\nabla f + \mathrm{Ric} - \frac{1}{2\tau}g\right|^2 + O_1(\frac{1}{N}). \tag{3.5.41}$$

Proof. Computing the partial derivatives of b^2, we see

$$\partial_i b^2 = -\frac{2b^2}{2-m}\partial_i f, \quad \partial_r b^2 = \frac{2b^2}{r} + O_2(\frac{1}{N^{1/2}}), \quad \partial_i\partial_j b^2 = -\frac{2b^2}{2-m}\partial_i\partial_j f + O_2(\frac{1}{N}).$$

Using (3.5.16) and the fact that $b^2/(2-m) = -2\tau + O_2(1/N)$ we write,

$$\frac{\hat{\Delta}b^2}{m}\hat{g} = -\frac{2b^2}{2-m}\left(\frac{1}{2\tau} + O_2(\frac{1}{N})\right)\hat{g}.$$

To calculate the components of $\hat{\nabla}\hat{\nabla}b^2$, we can compute the Christoffel symbols of the metric $\hat{g}$. Using that

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}),$$

where we are using abstract index notation, we find that the Christoffel symbols are,

$$\hat{\Gamma}^r_{ij} = -\frac{r}{N}\hat{g}^{rr}R_{ij}, \quad \hat{\Gamma}^k_{ij} = \Gamma^k_{ik}, \quad \hat{\Gamma}^\alpha_{ij} = 0 \tag{3.5.42}$$

$$\hat{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}, \quad \hat{\Gamma}^r_{\alpha\beta} = -r\hat{g}^{rr}g_{\alpha\beta}, \quad \hat{\Gamma}^i_{\alpha\beta} = 0 \tag{3.5.43}$$

$$\hat{\Gamma}^r_{rr} = \frac{1}{2}\hat{g}^{rr}\left(\frac{r^3R_\tau}{N^3} + \frac{2Rr}{N^2}\right), \quad \hat{\Gamma}^i_{rr} = -\frac{r^2}{2N^2}g^{ij}\partial_i R, \quad \hat{\Gamma}^\alpha_{rr} = 0 \tag{3.5.44}$$

$$\hat{\Gamma}^\gamma_{\alpha r} = 2r\delta^\alpha_\gamma, \quad \hat{\Gamma}^k_{ir} = g^{kj}R_{ij}, \quad \hat{\Gamma}^r_{ir} = \frac{r^2}{2N^2}\hat{g}^{rr}\partial_i R, \tag{3.5.45}$$

$$\hat{\Gamma}^j_{i\alpha} = \hat{\Gamma}^\gamma_{i\beta} = \hat{\Gamma}^r_{i\beta} = \hat{\Gamma}^i_{r\beta} = \hat{\Gamma}^\alpha_{rj} = \hat{\Gamma}^0_{r\alpha} = 0, \tag{3.5.46}$$

where $g_{\alpha\beta}$ is the standard metric on $\mathbb{S}^N$, $\Gamma^\gamma_{\alpha\beta}$ are its Christoffel symbols, and Γ^k_{ij} are the

Christoffel symbols of the metric g on M. A lengthy yet straightforward inspection shows that the only contributions of order $O(1)$ to the squared norm of

$$S := \hat{\nabla}\hat{\nabla}b^2 - \frac{\hat{\Delta}b^2}{m}\hat{g},$$

which, on its expanded form reads,

$$\begin{aligned}\hat{g}^{ac}\hat{g}^{bd}S_{ab}S_{cd} = \hat{g}^{ik}\hat{g}^{jl}S_{ij}S_{kl} + \hat{g}^{ik}\hat{g}^{00}S_{i0}S_{k0} + \hat{g}^{ik}\hat{g}^{\alpha\beta}S_{i\alpha}S_{k\beta} \\ + (\hat{g}^{00}S_{00})^2 + \hat{g}^{00}\hat{g}^{jl}S_{0j}S_{0l} + \hat{g}^{00}\hat{g}^{\alpha\beta}S_{0\alpha}S_{0\beta} \\ + \hat{g}^{\alpha\gamma}\hat{g}^{\beta\delta}S_{\alpha\beta}S_{\gamma\delta} + \hat{g}^{\alpha\gamma}\hat{g}^{ij}S_{\alpha i}S_{\gamma j} + \hat{g}^{\alpha\gamma}\hat{g}^{00}S_{\alpha 0}S_{\gamma 0}.\end{aligned} \tag{3.5.47}$$

arise from the $\{ij\}$ indices during the norm computation. Using the previously computed Christoffel symbols, we show,

$$-\hat{\Gamma}^r_{ij}\partial_r b^2 = \frac{2b^2}{N}R_{ij} + O_2(\frac{1}{N}) = -\frac{2b^2}{2-m}R_{ij} + O_2(\frac{1}{N}),$$

and therefore,

$$\begin{aligned}S_{ij} &= \partial_i\partial_j b^2 - \hat{\Gamma}^k_{ij}\partial_k b^2 - \hat{\Gamma}^r_{ij}\partial_r b^2 - \frac{\hat{\Delta}b^2}{m}\hat{g}_{ij} \\ &= -\frac{2b^2}{2-m}\left(\nabla_i\nabla_j f + R_{ij} - \frac{1}{2\tau}g_{ij}\right) + O_2(\frac{1}{N}).\end{aligned}$$

Given that $v = 1 - 2\tau R/N + O_2(1/N^2)$, it is straightforward to verify that, after crucial cancellations in $S_{\alpha\beta}$ and S_{rr} between $\hat{\nabla}\hat{\nabla}b^2$ and $\hat{\Delta}b^2\hat{g}/m$, the the remaining components of S are,

$$S_{\alpha\beta} = O_2(1), \quad S_{rr} = O_2(\frac{1}{N}), \quad S_{ir} = O_2(\frac{1}{N^{1/2}}), \quad S_{i\alpha} = S_{r\alpha} = 0,$$

from where we compute,

$$\left|\hat{\nabla}\hat{\nabla}b^2 - \frac{\hat{\Delta}b^2}{m}\hat{g}\right|^2 = \hat{g}^{ik}\hat{g}^{jl}S_{ij}S_{kl} + O_2(\frac{1}{N}) \tag{3.5.48}$$

$$= \frac{4b^4}{(2-m)^2}\left|\nabla\nabla f + \mathrm{Ric} - \frac{1}{2\tau}g\right|^2 + O_2(\frac{1}{N}). \tag{3.5.49}$$

Now, using the coordinate expression for $\hat{\mathrm{Ric}}$ in terms of the Christoffel symbols, a straightforward computation (see Section A.2.3 in the Appendix) reveals that its components with respect to the (τ, θ, x)-coordinates are of order $O_1(1/N)$. Using (3.4.7) and $\partial_r = (r/N)\partial_\tau$, it follows that

$$\hat{\mathrm{Ric}}(\hat{\nabla} b^2, \hat{\nabla} b^2) = O_1(\frac{1}{N}). \tag{3.5.50}$$

□

Finally, we reproduce Perelman's formula for the derivative of the $\mathcal{W}$-functional as the limit of the derivatives of $\mathcal{W}_N$.

Theorem 3.5.10. *The derivative of Perelman's* $\mathcal{W}-$*functional is,*

$$\frac{d\mathcal{W}}{d\lambda}(\lambda) = -\int_M 2\lambda \left(\left| \nabla\nabla f + \mathrm{Ric} - \frac{1}{2\tau} g \right|^2 \right) (4\pi\lambda)^{-\frac{n}{2}} e^{-f} d\nu. \tag{3.5.51}$$

Proof. Define

$$F_N := \frac{c_N}{s^{N+4}} \int_{b=s} \left(\left| \hat{\nabla}\hat{\nabla} b^2 - \frac{\hat{\Delta} b^2}{m} \hat{g} \right|^2 + \hat{\mathrm{Ric}}(\hat{\nabla} b^2, \hat{\nabla} b^2) \right) \frac{d\hat{A}}{|\hat{\nabla} b|}. \tag{3.5.52}$$

Then, using (3.4.10) and (3.5.41), it is straightforward to show that

$$F_N = 4(4\pi)^{-n/2}(2N)^{n/2-1} \left(\int_M \left| \nabla\nabla f + \mathrm{Ric} - \frac{1}{2\tau} g \right|^2 e^{-f} d\nu + O_1(\frac{1}{N}) \right). \tag{3.5.53}$$

It follows that $F_N = (2N)^{n/2-1} O_1(1)$ and therefore,

$$\frac{d}{ds} F_N = \frac{d\lambda}{ds}\frac{d}{d\lambda} \left((2N)^{n/2-1} O_1(1) \right) = (2N)^{n/2-3/2} O_0(1). \tag{3.5.54}$$

Then, a Taylor expansion of order one centered at $s = \sqrt{2N\lambda}$ shows,

$$\begin{aligned}
&\frac{d}{ds}\big(2(m-1)\mathcal{V}_N - \mathcal{A}_N\big) \\
&= -\frac{c_N}{2s^{m+1}} \int_{\bar{s} \le b \le s} \left(\left| \hat{\nabla}\hat{\nabla} b^2 - \frac{\hat{\Delta} b^2}{m} \hat{g} \right|^2 + \hat{\mathrm{Ric}}(\hat{\nabla} b^2, \hat{\nabla} b^2) \right) d\hat{V} + O_0(\frac{1}{N^{3/2}}) \\
&= -\frac{1}{2s^{m+1}} \int_{\bar{s}}^{s} w^{N+4} F_N(w) dw + O_0(\frac{1}{N^{3/2}}) \\
&= -\frac{1}{2s^{m+1}} \int_{\bar{s}}^{s} w^{N+4} \left(F_N(s) + (w-s)\frac{d}{ds} F_N(\xi_{w,s}) \right) dw + O_0(\frac{1}{N^{3/2}}) \\
&= -\frac{1}{2s^{m+1}} \int_{\bar{s}}^{s} w^{N+4} \left(F_N(s) + (w-s)(2N)^{n/2-3/2} O_0(1) \right) dw + O_0(\frac{1}{N^{3/2}}).
\end{aligned}$$

Integrating, bounding the integral of the $O_0(1)$ term by above and below as in (3.4.25), and absorbing the terms that decay exponentially fast into $O_0(N^{-3/2})$, we find,

$$\begin{aligned}
&\frac{d}{ds}\big(2(m-1)\mathcal{V}_N - \mathcal{A}_N\big) \\
&\qquad = -\frac{s^{3-n}}{2(N+5)}F_N + \frac{s^{4-n}}{(N+6)(N+5)}(2N)^{n/2-3/2}O_0(1) + O_0(\frac{1}{N^{3/2}}) \\
&\qquad = -\frac{s^{3-n}}{2(N+5)}F_N + O_0(\frac{1}{N^{3/2}}).
\end{aligned}$$

Finally, since

$$\frac{d\mathcal{W}_N}{d\lambda} = \frac{d}{d\lambda}\big(2(m-1)\mathcal{V}_N - \mathcal{A}_N\big) = \frac{ds}{d\lambda}\frac{d}{ds}\big(2(m-1)\mathcal{V}_N - \mathcal{A}_N\big),$$

we use the expression (3.5.53) and $s = \sqrt{2N\lambda}$ to obtain,

$$\begin{aligned}
\frac{d\mathcal{W}_N}{d\lambda} &= \left(\frac{N}{2\lambda}\right)^{1/2}\left(-\frac{(\sqrt{2N\lambda})^{3-n}}{2(N+5)}F_N + O_0(\frac{1}{N^{3/2}})\right) + O_0(\frac{1}{N^{3/2}}) \\
&= -2(4\pi)^{-\frac{n}{2}}(\sqrt{2N\lambda})^{2-n}(2N)^{n/2-1}\left(\int_M \left|\nabla\nabla f + \mathrm{Ric} - \frac{1}{2\tau}g\right|^2 e^{-f}d\nu + O_1(\frac{1}{N})\right) \\
&\quad + O_0(\frac{1}{N}) \\
&= -2(4\pi)^{-\frac{n}{2}}\lambda^{1-n/2}\left(\int_M \left|\nabla\nabla f + \mathrm{Ric} - \frac{1}{2\tau}g\right|^2 e^{-f}d\nu + O_1(\frac{1}{N})\right) + O_0(\frac{1}{N}) \\
&= -\int_M 2\lambda\left(\left|\nabla\nabla f + \mathrm{Ric} - \frac{1}{2\lambda}g\right|^2\right)(4\pi\lambda)^{-\frac{n}{2}}e^{-f}d\nu + O_0(\frac{1}{N}).
\end{aligned}$$

By Corollary 3.4.6,

$$\frac{d\mathcal{W}}{d\lambda}(\lambda) = \lim_{N\to\infty}\frac{d\mathcal{W}_N}{d\lambda}(\lambda),$$

which finishes the proof. □

Appendix A

Auxiliary results

Here we prove some technical and computational results cited in the main text. We divide it in two sections: the first one is devoted to results used in Chapter 2, and the second one for those needed in Chapter 3.

A.1 Auxiliary results for Chapter 2

A.1.1 Uniform convergence and bounds for the functions $\tilde{G}_n$

Recall that

$$\tilde{G}_n(x,t) := \left(1 - \frac{|x|^2}{2nt}\right)^{(n-2)/2} \chi_{B^d_{\sqrt{2nt}}}(x),$$

$$G_n(X,t) := \left(1 - \frac{|X|^2}{2nt}\right)^{(n-2)/2} \chi_{\mathbb{R}^{d+1}_+ \cap B^{d+1}_{\sqrt{2nt}}}(X),$$

and

$$G(X,t) = e^{-|X|^2/4t}, \quad \tilde{G}(x,t) = e^{-|x|^2/4t}.$$

We will prove the following results for $\tilde{G}_n$ and $\tilde{G}$. Identical arguments hold for G_n and G respectively, replacing $\mathbb{R}^d$ with $\mathbb{R}^{d+1}_+$ where appropriate.

Proposition A.1.1 (Lemma 3.1 of [25]). *$\tilde{G}_n \to \tilde{G}$ uniformly in $\mathbb{R}^d \times \{t \geq t_0\}$, for any $t_0 > 0$.*

Proof. Since $\tilde{G}_n$ is radial for every n, we may change variables, and consider $w = |x|^2/4t$, $m = n/2$. With this, we rewrite $\tilde{G}_n$ as,

$$\tilde{G}_n(x,t) = f_m(w) := \left(1 - \frac{w}{m}\right)^{m-1/2} \chi_{\{0 \leq w \leq m\}}(w).$$

We will show

$$f_m(w) \to f(w) := e^{-w}\chi_{\mathbb{R}_+}(w),$$

from where the result will follow.

We first observe that if $w > m$, then $|f(w) - f_m(w)| = e^{-w} \leq e^{-m} \to 0$ as $m \to \infty$. Now, if $0 \leq w \leq m$, we estimate

$$|f(w) - f_m(w)| \leq \left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right| + \left| \left(1 - \frac{w}{m}\right)^{m-\frac{1}{2}} - \left(1 - \frac{w}{m}\right)^m \right|$$

In order to work with

$$\left(1 - \frac{w}{m}\right)^{m-\frac{1}{2}} - \left(1 - \frac{w}{m}\right)^m = \left(1 - \frac{w}{m}\right)^m \left(\left(1 - \frac{w}{m}\right)^{-\frac{1}{2}} - 1\right),$$

a simple computation with the derivative shows that $w = 1 - \left(1 - \frac{1}{2m}\right)^2$ is a global maximum, and the maximum is bounded by

$$\max_{w \in \mathbb{R}_+} \left(1 - \frac{w}{m}\right)^m \left(\left(1 - \frac{w}{m}\right)^{-\frac{1}{2}} - 1\right) \leq \frac{c}{2m}.$$

We now estimate

$$\left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right|.$$

Since $\log u < u - 1$ for every $u \in (0, 1)$, we see that

$$\log\left(1 - \frac{w}{m}\right) < -\frac{w}{m},$$

and therefore,

$$\log\left(1 - \frac{w}{m}\right)^m < -w$$

for every $w \in (0, m)$. On the other hand, a standard Taylor expansion shows that

$$\log(1 - u) = -u - u^2/2m + O(u^3).$$

Setting $u = -w/m$ we find that

$$m \log\left(1 - \frac{w}{m}\right) = -w - w^2/2m + O(w^3/m^2).$$

Exponentiating from both sides, we arrive to the expression,

$$\left(1 - \frac{w}{m}\right)^m = e^{-w - w^2/2m + O(w^3/m^2)}.$$

In order to find the appropriate bound for

$$\max_{w\in[0,m]} \left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right|,$$

we observe that previous step implies

$$\left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right| = e^{-w}\left(1 - e^{-w^2/2m + O(w^3/m^2)}\right).$$

Using the Taylor expansion for the exponential at 0,

$$\left(1 - e^{-w^2/2m + O(w^3/m^2)}\right) = O\left(\frac{w^2}{2m}\right) \le C\frac{w^2}{2m}.$$

Now, by Taylor approximation of e^{-w} again, we also find

$$e^{-w}\left(1 - e^{-w^2/2m + O(w^3/m^2)}\right) \le C\frac{w^2}{2m} = C\frac{w^2}{m}, \tag{A.1.1}$$

where we absorbed the 2 in the constant C. Now observe that if $a > 0$,

$$\max_{w\in[a,m]} \left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right| \le \max_{w\in[a,m]} e^{-w} \le e^{-a}.$$

Finally, taking $a = \log m$ and using (A.1.1), we see that

$$\max_{w\in[0,a]} \left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right| \le e^{-a} \le C\frac{(\log m)^2}{2m} = C\frac{(\log m)^2}{m}.$$

Therefore

$$\left| e^{-w} - \left(1 - \frac{w}{m}\right)^m \right|_\infty \le e^{-m} + \frac{1}{m} + C\frac{(\log m)^2}{m}.$$

Combining the previous bounds, we find

$$|f_m - f|_\infty \le e^{-m} + \frac{1}{m} + C\frac{(\log m)^2}{m},$$

which concludes the proof. □

Lemma A.1.2. *We have,*

$$\tilde{G}_n(x,t) \le e\tilde{G}(x,t),$$

for every $x \in \mathbb{R}^d$, $t > 0$ *and* $n \ge 1$.

Proof. Fix $n > 2$. First, observe that $\tilde{G}_n(x,t)$ is supported in the ball $|x| \le \sqrt{2nt}$, so the result trivially holds outside $|x| \le \sqrt{2nt}$. Within this region, we can express $\tilde{G}_n$ as

$$\tilde{G}_n(x,t) = \left(1 - \frac{|x|^2}{2nt}\right)^{\frac{n-2}{2}}.$$

Let $y = \frac{|x|^2}{4t}$, noting that $y \le \frac{n}{2}$ since $|x| \le \sqrt{2nt}$. The function $\tilde{G}_n$ can then be rewritten in terms of y as,

$$\tilde{G}_n(x,t) = \left(1 - \frac{2y}{n}\right)^{\frac{n-2}{2}}.$$

Taking logarithms from both sides,

$$\log \tilde{G}_n(X,t) = \frac{n-2}{2} \log\left(1 - \frac{2y}{n}\right).$$

Using the inequality $\ln(1-z) \le -z$ for $z \in [0,1)$, which follows from a Taylor expansion we obtain,

$$\log\left(1 - \frac{2y}{n}\right) \le -\frac{2y}{n}.$$

Therefore,

$$\log \tilde{G}_n(X,t) \le \frac{n-2}{2}\left(-\frac{2y}{n}\right) = -y + \frac{2y}{n}.$$

Exponentiating both sides yields

$$\tilde{G}_n(x,t) \le e^{-y+\frac{2y}{n}} = e^{-y} e^{\frac{2y}{n}}.$$

Since $y \le \frac{n}{2}$, the term $\frac{2y}{n}$ is bounded above by 1, and thus $e^{\frac{2y}{n}} \le e$. Consequently,

$$\tilde{G}_n(x,t) \le e e^{-y} = e\tilde{G}(x,t).$$

Since this bound is independent of n, the lemma follows. □

A.1.2 Proof of Lemma 2.4.1

Here we prove Lemma 2.4.1. We will use the following proposition.

Proposition A.1.3. *Let $g \in C([t_0,t_1]; L^1(\mathbb{R}^{d+1}_+, d\mu))$. Then, for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^{d+1}_+$ such that*

$$\int_{\mathbb{R}^{d+1}_+ \setminus K} |g(x,t)|\, d\mu < \varepsilon,$$

for every $t \in [t_0, t_1]$.

Proof. Fix $\epsilon > 0$. Since g is continuous, $g([t_0, t_1]) \subset L^1(\mathbb{R}_+^{d+1}, d\mu)$ is compact. In particular it must be totally bounded and we can find $g_1, \ldots, g_n \in L^1(\mathbb{R}_+^{d+1}, d\mu)$ such that for any $t \in [t_0, t_1]$, there exists $i \in \{1, \ldots, N\}$ with

$$||g(\cdot, t) - g_i||_{L^1(\mathbb{R}_+^{d+1}, d\mu)} < \varepsilon/2.$$

Now, since g_i is integrable for every i, for each g_i we may find a compact set K_i such that

$$\int_{\mathbb{R}_+^{d+1} \setminus K_i} |g_i(X)| \, d\mu < \varepsilon/2.$$

Then, let

$$K := \bigcup_{i=1}^{N} K_i.$$

We have,

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1} \setminus K} |g(x,t)| \, d\mu &\leq \int_{\mathbb{R}_+^{d+1} \setminus K} |g(x,t) - g_i(x)| \, d\mu + \int_{\mathbb{R}_+^{d+1} \setminus K} |g_i(x)| \, d\mu \\ &< ||g(\cdot, t) - g_i||_{L^1(\mathbb{R}_+^{d+1}, d\mu)} + \varepsilon/2,\end{aligned}$$

for every $i \in \{1, \ldots, N\}$. Choosing i such that $||g(\cdot, t) - g_i||_{L^1(\mathbb{R}_+^{d+1}, d\mu)} < \varepsilon/2$ the result follows. □

Proof of Lemma 2.4.1. Let $\varepsilon > 0$. We first observe that since $G_n \leq eG$,

$$h_n(t) \leq \int_{\mathbb{R}_+^{d+1}} |f| G_n \, d\mu \leq e \int_{\mathbb{R}_+^{d+1}} |f| G \, d\mu < \infty.$$

Now,

$$|h_n(t) - h(t)| \leq \int_K |f||G_n - G| \, d\mu + \int_{\mathbb{R}_+^{d+1} \setminus K} |f||G_n - G| \, d\mu, \tag{A.1.2}$$

for every compact set K. Since $fG(\cdot, t) \in C([t_0, t_1]; L^1(\mathbb{R}_+^{d+1}, d\mu))$, by Proposition A.1.3 there exists a compact set K such that

$$\int_{\mathbb{R}_+^{d+1} \setminus K} |fG| \, d\mu < \frac{\varepsilon}{2(1+e)},$$

for every $t \in [t_0, t_1]$. Then,

$$\int_{\mathbb{R}_+^{d+1} \setminus K} |f||\tilde{G}_n - G| \, d\mu < (1+e) \int_{\mathbb{R}_+^{d+1} \setminus K} |fG| \, d\mu < \frac{\varepsilon}{2}.$$

Now,

$$\int_K |f||\tilde{G}_n - G| \, d\mu < |K| \max_{(x,t) \in K \times [t_0, t_1]} |f| \max_{(x,t) \in K \times [t_0, t_1]} |\tilde{G}_n - G|.$$

Since $\tilde{G}_n \to G$ uniformly, the lemma follows. □

A.2 Auxiliary results for Chapter 3

A.2.1 The Scalar Maximum Principle

Here we prove the Scalar Maximum Principle. Generalizations to vector bundles can be found, for example, in [5].

Theorem A.2.1 (Scalar Maximum Principle). *Let M be a closed Riemannian manifold and $g(t)$ a family of metrics on M. Suppose that $u : M \times [0, T] \to \mathbb{R}$ satisfies the differential inequality*

$$\frac{d}{dt} u \geq \Delta_{g(t)} u + g(X(t), \nabla u) + F(u), \tag{A.2.1}$$

where $X(t)$ is a time-dependent vector field and F is a locally Lipschitz function. Let $h(t)$ be a solution of the associated ODE $\frac{d}{dt} h = F(h)$ with $u(\cdot, 0) \geq h(0)$. Then $u \geq h$ for all $x \in M$ and $t \in [0, T]$.

Proof. This theorem follows from the fact that at a local minimum, the Laplacian is non-negative and the gradient vanishes. Consider a function $u_\varepsilon := u + \varepsilon(\delta + t)$. Note that $M \times [0, T]$ is compact, so we can choose a uniform Lipschitz constant K for F. We select a small δ that depends only on K such that $u_\varepsilon - h > 0$ for $t \in [0, \delta]$; and we can let $\varepsilon \to 0$ to prove the result on $[0, \delta]$ and then repeat the argument with the same δ to cover the interval $[0, T]$.

Note that $u_\varepsilon > h$ at $t = 0$. Suppose there exists a first time t_0 such that $u_\varepsilon = h$ at a point x_0. Since for all times $t < t_0$ we have $(u_\varepsilon - h)(x_0, t_0) > 0$, the time derivative is non-positive and we are at a spatial minimum. Then at (x_0, t_0) we have,

$$\begin{aligned} 0 &\geq \frac{\partial}{\partial t}(u_\varepsilon - h) \\ &\geq \varepsilon + \Delta_{g(t)}(u_\varepsilon - h) + g(X(t), \nabla(u_\varepsilon - h)) + F(u_\varepsilon - \varepsilon(\delta + t)) - F(h) \\ &\geq \varepsilon - K|u_\varepsilon - h - \varepsilon(\delta + t)| \\ &= \varepsilon(1 - K|\delta + t|). \end{aligned} \tag{A.2.2}$$

Taking $\delta < \frac{1}{2K}$, this expression is strictly positive on $[0, \delta]$, which is a contradiction. □

It is important to note that these results also hold for the minimum; that is, if we consider the differential inequality

$$\frac{d}{dt}u \leq \Delta_{g(t)}u + g(X(t), \nabla u) + F(u), \tag{A.2.3}$$

where $X(t)$ is a time-dependent vector field and F is a locally Lipschitz function. Let $h(t)$ be a solution of the associated ODE $\frac{d}{dt}h = F(h)$ with $u(\cdot, 0) \leq h(0)$. Then $u \leq h$ for every $x \in M$ and $t \in [0, T]$.

A.2.2 Proof of Lemma 3.3.6

Proof of Lemma 3.3.6. We prove the lemma by analyzing the Taylor expansion of $F(w)$ around w_0, and estimating the remainder term R_l.

For a smooth function F, the Taylor expansion of $F(w)$ around w_0 up to order $l-1$ with remainder is given by,

$$F(w) = \sum_{m=0}^{l-1} \frac{F^{(m)}(w_0)}{m!}(w - w_0)^m + R_l,$$

where the remainder term R_l in integral form is,

$$R_l = \frac{1}{(l-1)!} \int_{w_0}^{w} (w - t)^{l-1} F^{(l)}(t)\, dt.$$

Substituting $w = w_0 + \delta$, we obtain,

$$R_l = \frac{1}{(l-1)!} \int_{w_0}^{w_0+\delta} (w_0 + \delta - t)^{l-1} F^{(l)}(t)\, dt.$$

And writing $u = t - w_0$, we find

$$R_l = \frac{1}{(l-1)!} \int_{0}^{\delta} (\delta - u)^{l-1} F^{(l)}(w_0 + u)\, du.$$

We now bound the remainder. Since F is smooth, $F^{(l)}$ is continuous and hence bounded on compact subsets of (a,b). Let M be a bound for $|F^{(l)}(w_0+u)|$ where $u \in [0,\delta]$. Then,

$$|R_l| \leq \frac{M}{(l-1)!}\left|\int_0^\delta (\delta-u)^{l-1}\,du\right| = \frac{M}{l!}|\delta|^l.$$

We now estimate the derivatives of the remainder. By assumption, $\delta = O_k(1/N^j)$, meaning that for any multi-index $|\alpha| \leq k$,

$$N^j|\partial^\alpha \delta| \leq K,$$

for some constant $K > 0$ and for all $N > 0$, $\tau_1 \leq \tau \leq T$, and $x \in M$.

To show that $R_l = O_k(1/N^{jl})$, we need to bound the derivatives $\partial^\alpha R_l$ for $|\alpha| \leq k$. Using the Leibniz rule, we see that the derivative of the integral expression for R_l involves terms of the form,

$$\partial^\alpha R_l = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\frac{1}{(l-1)!}\int_0^\delta \partial^{\alpha-\beta}\left[(\delta-u)^{l-1}\right]\partial^\beta F^{(l)}(w_0+u)\,du.$$

Each term in the sum can be bounded by

$$\left|\partial^{\alpha-\beta}\left[(\delta-u)^{l-1}\right]\right| \leq C|\delta|^{l-1-|\alpha-\beta|},$$

where C depends on l and the multi-index α. Since $F^{(l)}$ is smooth, its derivatives are bounded, and we obtain,

$$|\partial^\alpha R_l| \leq C'|\delta|^{l-|\alpha|} \leq C'\left(\frac{K}{N^j}\right)^l = \frac{C'K^l}{N^{jl}},$$

for some constant $C' > 0$. This shows that

$$R_l = O_k\left(\frac{1}{N^{jl}}\right),$$

completing the proof. □

A.2.3 The Riemann tensor of Perelman's N-space

Here we compute the Riemann curvature tensor of the manifold $\hat{M} = M \times \mathbb{S}^N \times \mathbb{R}_+$. For convenience, we doit on the (x,θ,τ) coordinates. For these coordinates, the metric takes the form,

$$\hat{g}_{ij} = g_{ij},$$
$$\hat{g}_{\alpha\beta} = \tau g_{\alpha\beta},$$
$$\hat{g}_{oo} = \frac{N}{2\tau} + R,$$
$$\hat{g}_{i\alpha} = \hat{g}_{io} = \hat{g}_{\alpha o}, = 0$$

where i, j are coordinate indices on M, α, β are coordinate indices on $\mathbb{S}^N$, and the coordinate τ on R_+ has index o. Since g_{ij} evolves by the backward Ricci flow

$$\frac{\partial}{\partial \tau} g_{ij} = 2R_{ij},$$

and the metric $g_{\alpha\beta}$ on $\mathbb{S}^N$ is a metric with constant sectional curvature $\frac{1}{2N}$, we can cdirectly ompute the Christoffel symbols of the metric $\hat{g}$, which are given by the following list:

$$\begin{aligned}
&\hat{\Gamma}^k_{ij} = \Gamma^k_{ij}\\
&\hat{\Gamma}^k_{i\beta} = 0 \quad \text{and} \quad \hat{\Gamma}^\gamma_{ij} = 0\\
&\hat{\Gamma}^k_{\alpha\beta} = 0 \quad \text{and} \quad \hat{\Gamma}^\gamma_{i\beta} = 0\\
&\hat{\Gamma}^k_{io} = g^{kl} R_{li} \quad \text{and} \quad \hat{\Gamma}^o_{ij} = -\hat{g}^{oo} R_{ij}\\
&\hat{\Gamma}^k_{oo} = -\frac{1}{2} g^{kl} \frac{\partial}{\partial x^l} R \quad \text{and} \quad \hat{\Gamma}^o_{io} = \frac{1}{2} \hat{g}^{oo} \frac{\partial}{\partial x^i} R\\
&\hat{\Gamma}^o_{i\beta} = 0, \hat{\Gamma}^k_{o\beta} = 0 \quad \text{and} \quad \hat{\Gamma}^\gamma_{oj} = 0\\
&\hat{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}\\
&\hat{\Gamma}^\gamma_{\alpha o} = \frac{1}{2\tau} \delta^\gamma_\alpha \quad \text{and} \quad \hat{\Gamma}^o_{\alpha\beta} = -\frac{1}{2} \hat{g}^{oo} g_{\alpha\beta}\\
&\hat{\Gamma}^\gamma_{oo} = 0 \quad \text{and} \quad \hat{\Gamma}^o_{o\beta} = 0\\
&\hat{\Gamma}^o_{oo} = \frac{1}{2} \hat{g}^{oo} \left(-\frac{N}{2\tau^2} + \frac{\partial}{\partial \tau} R \right)
\end{aligned}$$

Fix a point $(p, s, \tau) \in M \times \mathbb{S}^N \times \mathbb{R}^+$and choose normal coordinates around $p \in M$ and normal coordinates around $s \in \mathbb{S}^N$ such that $\Gamma^k_{ij}(p) = 0$ and $\Gamma^\gamma_{\alpha\beta}(s) = 0$ for all i, j, k and

α, β, γ. We compute the curvature tensor $\hat{R}m$ of the metric $\hat{g}$ at the point as follows:

$$\hat{R}_{ijkl} = R_{ijkl} + \hat{\Gamma}^k_{io}\hat{\Gamma}^o_{jl} - \hat{\Gamma}^k_{jo}\hat{\Gamma}^o_{il} = R_{ijkl} + O\left(\frac{1}{N}\right)$$
$$\hat{R}_{ijk\delta} = 0$$
$$\hat{R}_{ij\gamma\delta} = 0 \text{ and } \hat{R}_{i\beta k\delta} = \hat{\Gamma}^k_{io}\hat{\Gamma}^o_{\beta\delta} - \hat{\Gamma}^k_{\beta o}\hat{\Gamma}^o_{i\delta} = -\frac{1}{2}\hat{g}^{oo}g_{\beta\delta}g^{kl}R_{li} = O\left(\frac{1}{N}\right)$$
$$\hat{R}_{i\beta\gamma\delta} = 0$$
$$\hat{R}_{ijko} = \frac{\partial}{\partial x^i}R_{jk} - \frac{\partial}{\partial x^j}R_{ik} + \hat{\Gamma}^k_{io}\hat{\Gamma}^o_{jo} - \hat{\Gamma}^k_{jo}\hat{\Gamma}^o_{io} = P_{ijk} + O\left(\frac{1}{N}\right)$$

$$\begin{aligned}\hat{R}_{ioko} &= -\frac{1}{2}\frac{\partial^2}{\partial x^i \partial x^k}R - \frac{\partial}{\partial \tau}\left(R_{il}g^{lk}\right) + \hat{\Gamma}^k_{io}\hat{\Gamma}^o_{oo} - \hat{\Gamma}^k_{oj}\hat{\Gamma}^j_{io} - \hat{\Gamma}^k_{oo}\hat{\Gamma}^o_{io} \\ &= -\frac{1}{2}\nabla_i\nabla_k R - \frac{\partial}{\partial \tau}R_{ik} + 2R_{ik}R_{lk} - \frac{1}{2\tau}R_{ik} - R_{ij}R_{jk} + O\left(\frac{1}{N}\right) \\ &=: M_{ik} + O\left(\frac{1}{N}\right),\end{aligned}$$

and,

$$\hat{R}_{ij\gamma o} = 0 \text{ and } \hat{R}_{i\gamma jo} = 0$$
$$\hat{R}_{i\beta\gamma o} = -\tau\hat{\Gamma}^\gamma_{\beta o}\hat{\Gamma}^o_{io} = O\left(\frac{1}{N}\right) \quad \text{and} \quad \hat{R}_{io\gamma\delta} = 0$$
$$\hat{R}_{io\gamma o} = 0$$
$$\hat{R}_{\alpha\beta\gamma o} = 0$$
$$\hat{R}_{\alpha o\gamma o} = \left(\frac{1}{2\tau^2}\delta^\gamma_\alpha + \hat{\Gamma}^\gamma_{\alpha o}\hat{\Gamma}^o_{oo} - \hat{\Gamma}^\gamma_{o\beta}\hat{\Gamma}^\beta_{\alpha o}\right)\tau = O\left(\frac{1}{N}\right)$$
$$\hat{R}_{\alpha\beta\gamma\delta} = O\left(\frac{1}{N}\right)$$

Taking traces, we arrive at the following.

Corollary A.2.2 (Corollary 3.1.2 of [15])**.** *The components of the Ricci tensor of $\hat{g}$ are $O(N^{-1})$.*

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