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Superfluous edges and exponential expansions of De Bruijn and Kautz graphs

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Abstract

A new way to expand De Bruijn and Kautz graphs is presented. It consists of deleting *superfluous sets of edges* (i.e., those whose removal does not increase the diameter) and adding new vertices and new edges preserving the maximum degree and the diameter. The number of vertices added to the Kautz graph, for a fixed maximum degree greater than four, is exponential on the diameter. Tables with lower bounds for the order of superfluous sets of edges and the number of vertices that can be added, are presented.

Keywords: Large graphs; (Δ, D)-problem; De Bruijn graphs; Kautz graphs; Superfluous edges; Edge-critical graphs

1. Introduction

De Bruijn and Kautz digraphs have attracted the attention of researchers for a long time. Their properties have been intensively studied and many families of graphs have been constructed inspired on them.

One outstanding property is that the De Bruijn and Kautz digraphs have large orders for a given maximum out-degree and diameter. The optimization problem that consists of finding large graphs [digraphs] with fixed maximum [out-]degree Δ and diameter D

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is known as the (Δ, D)-problem. In this context, Kautz digraphs are the largest known digraphs (except for $(\Delta, D) = (4, D)$ with $D \geqslant 4$). Their underlying graphs are also large as graph, and they were the largest known for some time. Besides, both families have easy routing algorithms and low vulnerability, a fact that motivates their study as models for the design of interconnecting networks.

In spite of their high orders, it is possible to add new vertices to the original graphs preserving their diameter and maximum degree. For instance, Bond [4] was able to add $\Delta+1$ vertices to Kautz graphs preserving their maximum degree Δ and diameter (when the latter is greater than 2). The possibility of adding new nodes to a graph is known as expansiveness or scalability and has also been considered by many authors, since it is a desirable property for the modularity of computer networks. Some instances of scalability of the De Bruijn and Kautz graphs have been studied in [1,10,19], but in these cases the authors allow the graphs to change their maximum degree or diameter.

Our main results are the following. We proved the existence in these graphs of large sets of edges whose removal does not increase the diameter, sets we call *superfluous sets of edges*. We shown that it is possible to add a large number of vertices to the graph obtained by the deletion of these superfluous sets of edges, maintaining the maximum degree and the diameter. For instance, in the case of Kautz graphs, we added $2(\Delta+1)$ vertices (Proposition 13) or even an exponentially growing amount (Theorem 26) when the diameter goes to infinity.

The study of graphs without superfluous edges and the possibility of expanding them were introduced by Gliviak [11] and Gliviak et al. [13]. However, they allow the maximum degree to grow, so their results fall outside the frame of the (Δ, D)-problem.

2. Definitions

Let $G=(V,E)$ $[G=(V,A)]$ denotes a [di]graph with vertex set $V=V(G)$ and edge set $E=E(G)$ [arc set $A=A(G)$]. If $uv=vu=\{u,v\}\in E$ $[uv=(u,v)\in A]$ we write $u\overset{G}{\sim}v$ or simply $u\sim v[u\overset{G}{\rightsquigarrow}v$ or simply $u\rightsquigarrow v]$.

The *union* of two vertex disjoint graphs G and G' is denoted by $G\cup G'$ and has as vertex and edge sets the union of the corresponding vertex and edge sets; i.e.,

$$G\cup G'=(V(G)\cup V(G'),E(G)\cup E(G')).$$

If $G=(V,Æ)$ is a [di]graph and F is a subset of $V^{(2)}=\{U\subset V(G)\colon |U|=2\}$ [of $V^2=V\times V$], then $G+F$ and $G-F$ denote the [di]graphs

$$G+F=(V, Æ\cup F),$$

$$G-F=(V, Æ\setminus F).$$

The *underlying graph* of a digraph $G=(V,A)$ is the graph $UG=(V,E)$ obtained from G by forgetting orientations and eliminating loops and multiple edges; that is, $E=E(UG)=UA(G)=\{\{u,v\}\mid u\neq v,\ (u,v)\in A\}$. For example, the underlying graph of a *tournament* over V (i.e., a digraph $T=(V,A)$ with no loops such that uv is in A if and only if vu is not in A), is the complete graph over V.

A sequence of vertices $u_0, u_1, \ldots, u_{n-1}, u_n$ of a [di]graph G, such that $u_{i-1}u_i$ is an edge [arc] of G, is called a $u_0 - u_n$ *walk* or a [*directed*] *walk from* u_0 *to* u_n of *length* n. The walk is *closed* if $u_0 = u_n$. Directed closed walks of length one are called *loops* and those of length two *digons* (whenever $u_0 \neq u_1$). We will write $DN(G)$ for the set of digons of G; i.e.,

$$DN(G) = \{uv \in A(G) \mid vu \in A(G)\}.$$

Notice that, with this definition, any directed walk (without loops) of a digraph G is a walk of its underling graph UG, as well. We will work with [*strongly*] *connected* [di]*graphs*; that is, those having some [directed] walk between any pair of different vertices. In such graphs, it is possible to define a *distance* between two vertices, as the length of a shortest walk joining them. Analogously, the *distance* in a digraph *from* a vertex *to* another is the length of a shortest directed walk from the first to the second (despite its name, this function is not a metric because it is not symmetric). The *diameter* $D(G)$ of a [di]graph G is the maximum of its distance function. If we write $d_G(u,v)$ or simply $d(u,v)$ for the distance between [from] u and [to] v, then

$$D(G) = \max_{u,v \in G} d(u,v).$$

The *degree* $\deg_G(u)$ [*in* and *out-degrees*, $\deg_G^-(u)$ and $\deg_G^+(u)$ resp.] of a vertex u of G is the cardinality of the set $\Gamma(u)$ [sets $\Gamma^-(u)$ and $\Gamma^+(u)$ resp.] of the vertices adjacent with u [to and from u resp.]. The maximum [in and out] degree of a [di]graph G is denoted by $\Delta(G)$. In Section 4.4, we will use the following particular extension of Γ^+ and Γ^-: given a subset F of arcs of a digraph G, we define $\Gamma^+(F)$ and $\Gamma^-(F)$ to be the sets of vertices incident from and to an arc of F, respectively; i.e.,

$$\Gamma^+(F) = \{v : \exists u \in V(G) \mid uv \in F\},$$

$$\Gamma^-(F) = \{u : \exists v \in V(G) \mid uv \in F\}.$$

From a set-theoretical point of view, these functions are the projections to the second and first coordinate of the elements of F, respectively.

Following Bollobás [3], $\mathscr{H}_D(n,\Delta)$ stands for the set of graphs G of *order* $n = |V(G)|$ with maximum degree Δ and diameter D. Also in [3], the minimum number of edges of a graph in $\mathscr{H}_D(n,\Delta)$ is denoted by $e_D(n,\Delta)$; that is,

$$e_D(n,\Delta) = \min\{|E(G)|: G \in \mathscr{H}_D(n,\Delta)\}.$$

The exact value of $e_D(n,\Delta)$ is not known, except for some special cases. A graph with maximum degree Δ and diameter D is usually called a (Δ,D)-graph. We will denote by $\mathscr{H}_{\Delta,D}$ the class of all (δ,d)-graphs with $\delta \leqslant \Delta$ and $d \leqslant D$; that is,

$$\mathscr{H}_{\Delta,D} = \bigcup_{\substack{d \leqslant D \\ n \geqslant 1 \\ \delta \leqslant \Delta}} \mathscr{H}_d(n,\delta).$$

Our purpose in working with graphs in $\mathscr{H}_{\Delta,D}$, and not simply with (Δ,D)-graphs, is to simplify the formulation of the statements or proofs since the deletion of the superfluous edges of a graph might decrease its maximum degree, and the addition of vertices might decrease its diameter.

A graph G is a *subgraph* of G' and written $G \subset G'$, if $V(G) \subset V(G')$ and $E(G) \subset E(G')$. This relationship is an order on the class of all graphs and its restriction to the family $\mathscr{H}_{\Delta,D}$ gives rise to a new order that we will denote by $\sqsubset_{\Delta,D}$ or simply $\sqsubset$; more precisely,

$$G \sqsubset G' \text{ in } \mathscr{H}_{\Delta,D} \quad \text{if } G \subset G' \text{ and } G, G' \in \mathscr{H}_{\Delta,D}.$$

We will also work with the restriction $\lhd$ of the order $\sqsubset$ to the set of graphs with the same vertex set; that is

$$G \lhd G' \text{ in } \mathscr{H}_{\Delta,D} \quad \text{if } G \sqsubset_{\Delta,D} G' \text{ and } V(G) = V(G').$$

If $D(G') = D$, the edges of G' not in G are called *superfluous edges* of G'. Moreover, we say that a nonempty set of edges F of G' is a *superfluous set of edges* if both G' and the graph $G' - F$ have the same diameter; i.e., $D(G' - F) = D(G')$. The minimal elements of the order $\lhd$ are known as *diameter-minimal* or *edge-critical* graphs (see [7,12]) and are those having no superfluous edges.

We say that a sequence $G_0, \ldots, G_{2n}$ of graphs is a (Δ, D)-*admissible expansion of* G_0 *in* $k = |G_{2n}| - |G_0|$ *vertices*, if:

$$G_0 \rhd G_1 \sqsubset G_2 \rhd \cdots \rhd G_{2n-1} \sqsubset G_{2n} \quad \text{in } \mathscr{H}_{\Delta(G_0),D(G_0)}.$$

When two or more consecutive graphs of the sequence are equal, we will just write the first one. We will make an abuse of notation saying that G_{2n} is a (Δ, D)-*admissible expansion of* G_0.

In [1], the authors use the name *extension* when the increments in the orders are "atomic", that is, by at most one unit. Following them, we will say that the previous considered (Δ, D)-admissible expansion is a (Δ, D)-*admissible extension* if $|G_{2i}| - |G_{2i-1}| = 1$ for $i = 1, \ldots, n$.

When a graph is not regular, it is possible, in many cases, to add new vertices without increasing its maximum degree and diameter, taking into account the existence of vertices with "free valences". In order to compute the total amount of these "valences" on a given graph G, let us call the *total deficiency* of G respect to $k \geqslant \Delta(G)$, the integer $\mathrm{td}_k(G)$, defined by

$$\mathrm{td}_k(G) = \left\lfloor \tfrac{1}{2} k|G| \right\rfloor - |E(G)|.$$

Graphs with high total deficiency have been used by some authors [2,4,14] in order to obtain large graphs. In this context, one can wonder if it is actually possible to increase the total deficiency of a graph. In order to measure this potential increment, we call *deficiency reserve* of a graph G the maximum cardinality $\mathrm{der}(G)$ of a superfluous edge set; that is,

$$\mathrm{der}(G) = \max\{|F|: F \text{ superfluous edge set of } G\}.$$

Notice that if we order the superfluous edge sets of G by inclusion, the maximal elements are those sets F that make $G - F$ an edge-critical graph. Besides, if F has maximum order, i.e., if $|F| = \mathrm{der}(G)$, then

$$\mathrm{td}_{\Delta(G)}(G - F) = \mathrm{td}_{\Delta(G)}(G) + \mathrm{der}(G).$$

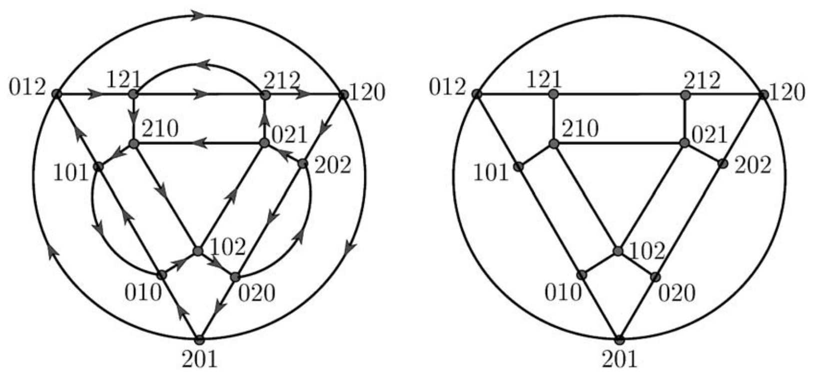

Fig. 1. $K(2,3)$ and $UK(2,3)$.

The concept of (Δ, D)-admissible expansion is related to the concept of routing. A *routing* ρ in $G=(V,E)$ is a function $\rho: V \times V \rightarrow V^*$ where V^* is the set of all walks of G. The routing ρ is *bounded* by k if the lengths of its walks is not greater than k and if $k=D(G)$, ρ is said to be *compatible* with G. A routing *avoids* F if its paths contain no edge of F. It is possible to redefine the previous concepts in terms of routing:

- A set of edges F of G is superfluous if and only if there exists a compatible routing in G avoiding F.
- $G \sqsubset G'$ in $\mathscr{H}_{\Delta,D}$ iff there exists routing ρ' in G' bounded by D, which restricted to $V(G) \times V(G)$ is also a routing $\rho = \rho'|_{V(G)\times V(G)}$ in G.

The former formulation tells us that, when a computer network is extended to a new one which is a (Δ, D)-admissible expansion of it, the traffic between nodes needs not to be stopped. In fact, in some steps of the expansion (those of the form $G_{2i-1} \sqsubset G_{2i}$), the routing algorithms need not be changed, but extended to the new nodes.

The basic graphs we will work with are the *De Bruijn* and *Kautz* graphs $UB(d,D)$ and $UK(d,D)$, respectively, which are the underlying graphs of the De Bruijn and Kautz digraph $B(d,D)$ and $K(d,D)$ of maximal in and out-degree d and diameter D. One way to define $B(d,D)$ $[K(d,D)]$ is as a digraph whose vertices are the words $x_0 \dots x_{D-1}$ of length D over an alphabet $\Sigma(B(d,D)) = \mathbb{Z}_d$ [alphabet $\Sigma(K(d,D)) = \mathbb{Z}_{d+1}$ with the restriction that $x_i \neq x_{i+1}$] and whose arcs are the pairs $(x_0 \dots x_{D-1}, x_1 \dots x_D)$. Both graphs, $UB(d,D)$ and $UK(d,D)$, have diameter D and maximum degree $\Delta = 2d$ if $D \geqslant 3$ and $2d-1$ if $D=2$. Fig. 1 shows Kautz digraph $K(2,3)$ and its underlying graph $UK(2,3)$.

In order to simplify the notation for edges [arcs] and [directed] walks of our basic graphs, instead of writing edges [arcs] as sets $\{x_0w, wx_D\}$ [pairs (x_0w, wx_D)] of words we will write them as single words x_0wx_D of length $D+1$. Similarly, if a [directed] walk $w_0, w_1, \dots, w_l$ is such that $w_i = x_ix_{i+1}\dots x_{i+D-1}$, then we will write it as a single word $x_0 \dots x_{l+D-1}$ of length $D+l$.

Since we will work frequently with words that have periodic strings, e.g., 010101 or 3201010, we will write "$\overline{01}$" for the former and "$32\overline{01}$" for the latter. We make the following three remarks about this notation:

- There is an ambiguity about the number of repetition, that will depend on the context. For example, if $\overline{01}$ is a vertex of $UK(2,3)$, then it represents the word 010, but if it is a vertex of $UK(2,4)$ or an edge of $UK(2,3)$ it represents the word 0101.
- If a directed walk $W = x_0 \ldots x_{l+D-1}$ is closed, the sequence $x_0 \ldots x_{l+D-1}$ will be periodic of period a divisor of l, thus we can represent it by the word $\overline{x_0 \ldots x_{l-1}}$ of length l.
- We include the empty string as a periodic word of any period, e.g., $34\overline{01}$ could represent the word 34.

For any other terminology or notation not defined above see [8].

3. Superfluous sets of underlying graphs of digraphs

As remarked in [6], the diameter of the underlying graph of a digraph does not grow by the deletion of any edge that is not the underlying edge of a digon; i.e., given a digraph G, then

$$D(UG - uv) \leqslant D(G)$$

for any edge $uv \in E(UG)$ such that $uv \notin DN(G)$. This remark almost characterizes the superfluous edges of the underlying graph of a digraph:

Lemma 1. *Let $G=(V,A)$ be a digraph with underlying graph UG, both with diameter D; i.e., $D(G) = D(UG) = D$. Then the underlying edge of any arc which does not belong to a digon is superfluous; that is*

$$uv \notin DN(G) \Rightarrow D(UG - uv) = D.$$

In what follows, we will call "*ex-digons*" the underlying edges of digons.

The next result will be extremely useful in order to find superfluous sets of edges of underlying graphs of digraphs.

Lemma 2. *Let $G = (V,A)$ be a digraph such that $D(G) = D(UG) = D$. If $F \subset A$ is a subset of arcs of G such that:*

(1) *no arc of F belongs to a digon of G (i.e., $F \cap DN(G) = \emptyset$),*
(2) *for any directed closed walk W of G of length bounded by $2D$, the number of times that W goes through F is, at most, one; i.e., if*

$$W = v_0, v_1, \ldots, v_{l-1}, v_l$$

with $v_l = v_0$ and $l \leqslant 2D$, then

$$|\{i\colon\ v_i v_{i+1} \in F,\ i = 0, \ldots, l-1\}| \leqslant 1.$$

Under these conditions, the set UF of underlying edges of F is a superfluous set of edges of UG.

Proof. Since $D(UG)=D$, given two different vertices u and v of UG, there exist two directed walks W_u^v and W_v^u of G, from u to v and from v to u, respectively, both of length at most D. Thus, the walk W constructed by concatenating W_u^v with W_v^u, is a directed closed walk of G of length at most $2D$. Applying hypothesis (2) to W, we infer that at most one of the first two walks contains an arc of F, then the other one, say W_u^v contains no arc of F. Therefore, W_u^v is a walk of UG joining u and v in at most D steps and, by hypothesis (1), avoiding UF. □

3.1. The superfluous edges of the De Bruijn and Kautz graphs

In the next results, we study the behavior of the diameter of De Bruijn and Kautz graphs under the deletion of its ex-digons. In particular, the next lemma give us some details in order to construct further expansions.

Lemma 3. *Let $u=x_1\dots x_D$ be a vertex of a De Bruijn or Kautz graph of diameter greater than 1 and 2, respectively, and let G be one of these graphs without its ex-digons. Then the distance in G between u and the vertex $\overline{01}$ is at most D. Furthermore, if $x_D=0$ or $x_{D-1}x_D=01$ this distance is less than D provided u is not $\overline{10}$. More precisely, if*

$$G_1=B(d,D),\quad D\geqslant 2$$

or

$$G_1=K(d,D),\quad D\geqslant 3$$

and

$$G=UG_1-U(DN(G_1))$$

then

$$d_G(u,\overline{01})\leqslant\begin{cases}D-1 & if\ u\neq\overline{10}\ and,\ either\ x_D=0\ or\ x_{D-1}x_D=01,\\ D & otherwise.\end{cases}$$

Proof. For $B(d,2)$, if $x\neq 1$ and $y\neq 0$ then the result follows from the fact that the sequences $x0,01$; 01; $yx,1y,01$; $0x,00,01$ and $y1,11,01$ are walks in G. Thus, in what follows we suppose that $D\geqslant 3$.

If $x_D\neq 0,1$, we distinguish two cases. First when $x_Dx_{D-1}\dots x_1=\overline{x_D\,0}$. In this case, if D is even [odd], the word $W=1\ \overline{0\,x_D}\ \overline{0\,1}$ [word $\overline{0\,1}\ \overline{0\,x_D}\ 1$] represents a walk in UG_1 of length $D-1$ joining vertex $\overline{0\,1}$ and vertex $v=1\,\overline{0\,x_D}$ [vertex $\overline{0\,x_D}\ 1$]. Besides, its edges are not ex-digons since they contain the triple x_D01 [triple $0x_D1$] which has three different symbols. Thus, W represents in fact a walk in G and $d_G(u,\overline{1\,0})\leqslant D$, since u and v are adjacent.

Second, when $x_D x_{D-1} \dots x_1 \neq \overline{x_D\, 0}$, the edge of UG_1 represented by $e = x_1 \dots x_D\, 0$ is not an ex-digon. Hence, the walk of length D in UG_1 represented by $w = x_1 \dots x_D\, \overline{01}$ has no ex-digons since, besides e, its other edges contain the triple $x_D 01$. Thus, w is also a walk in G and $d_G(u, \overline{01}) \leqslant D$.

If $x_D = 0, 1$, then we distinguish three cases. First, when there exists $i \in \{1, \dots, D-1\}$ such that $x_i \neq 1$ and $u = x_1 \dots x_i\, 0\, \overline{10}$, then by an argument similar to the previous one, $x_1 \dots x_i\, \overline{01}$ is a walk in G between u and $\overline{01}$ of length $i \leqslant D-1$. Notice that for De Bruijn graphs x_i could be 0, but this is not a obstacle, since no digon contains the pair 00.

Second, when $u = x_1 \dots x_i\, 1\, \overline{01}$ for some $x_i \neq 0$, then, as before, the distance in G between u and $x_i\, \overline{10}$ is at most $i-1$. Now, for Kautz graphs $x_i \neq 0, 1$ and we have that

$$x_i\, \overline{10} \overset{G}{\sim} \overline{10}\, x_i \overset{G}{\sim} \overline{01}.$$

For De Bruijn graphs, if s is the last symbol of vertex $x_i\, \overline{10}$, we have that

$$x_i\, \overline{10} \overset{G}{\sim} \overline{10}\, s \overset{G}{\sim} \overline{01}.$$

And then, for both cases,

$$d_G(u, \overline{01}) \leqslant (i-1) + 2 \leqslant (D-1-1) + 2 = D.$$

Furthermore, if $x_D = 0$ or $x_{D-1} x_D = 01$, then $i \leqslant D-2$ and $i \leqslant D-3$ respectively, and then $d_G(u, \overline{01}) \leqslant D-1$ in both cases.

Third and last, when $u = \overline{10}$, the sequence

$$\overline{10},\ x\, \overline{10},\ \overline{10}\, y,\ \overline{01}$$

is a walk in G of length $3 \leqslant D$ joining u and $\overline{01}$, for any $x, y \neq 0, 1$ in the case of Kautz graphs, and for $x = 1$ and $y = x_D$ in the case of De Bruijn graphs. □

Notice that if we consider the symmetric group over the alphabet of any Kautz or De Bruijn graph acting on the words that represent vertices, we obtain a subgroup (in fact, the whole group) of automorphisms of the graph. It can be verified that the same happened for the graph G of the previous lemma, so we have the following remark.

Remark 4. Under the hypothesis of Lemma 3, for any vertex $u = x_1 \dots x_D$ and any vertex $\overline{a\, b}$, the distance in G between u and $\overline{ab}$ verifies

$$d_G(u, \overline{ab}) \leqslant \begin{cases} D-1 & \text{if } u \neq \overline{b\, a} \text{ and either } x_D = a \text{ or } x_{D-1} x_D = ab, \\ D & \text{otherwise.} \end{cases}$$

In [6], Bond and Peyrat proved (Theorem 8) that any edge of De Bruijn and Kautz graphs of diameter at least three is superfluous. Another way to prove it is, by Lemma 1, to show that any ex-digons of these graphs is superfluous. The next proposition states that not only each isolated ex-digon is superfluous, but the entire set of ex-digons is superfluous, as well.

Proposition 5. *The set of digons of any De Bruijn or Kautz digraph gives rise to a superfluous edge set in the corresponding De Bruijn and Kautz graph if and only if the diameter is at least* 2 *or* 3, *resp.*; *furthermore if*

$$G_1 = B(d,D), \quad G_2 = K(d,D), \quad (d \geqslant 2)$$

then for $i = 1, 2$,

$$D \geqslant i+1 \Rightarrow D(U(G_i - DN(G_i))) = D$$

and

$$1 \leqslant D \leqslant i \Rightarrow D(UG_i - e) > D, \quad \forall e \in U(DN(G_i)).$$

Proof. First, we show that if $i = 1, 2$ and $1 \leqslant D \leqslant i$, then $D(UG_i - e) > D$ for any ex-digon e. Indeed, $UB(d,1)$ and $UK(d,1)$ are complete graphs and thus, have no superfluous edges. If $G = UK(d,2) - e$, with $e = aba = bab$, we have that $\Gamma(ab) = \{xa \mid x \neq a, b\} \cup \{bx \mid x \neq a, b\}$ and $\Gamma(ba) = \{xb \mid x \neq a, b\} \cup \{ax \mid x \neq a, b\}$, then $\Gamma(ab) \cap \Gamma(ba) = \emptyset$, and $d_G(ab, ba) > 2$.

Let $D \geqslant i+1$. By way of contradiction, let us suppose that $D(U(G_i - DN(G_i))) > D$, then there exist two vertices $u = x_1 \ldots x_D$ and $v = y_1 \ldots y_D$ such that any walk of length at most D joining them in $U(G_i)$, contains an ex-digon $e = \overline{ab}$. We claim that u or v (maybe both) must be incident with e, and then by Remark 4, it must be at distance at most D from any other vertex of the graph, contradicting our assumption. Indeed, the words $uv = x_1 \ldots x_D y_1 \ldots y_D$ and $vu = y_1 \ldots y_D x_1 \ldots x_D$ represent two walks of length D in $U(G_i)$ between u and v, then both must contain e as a substring. In other terms, there exist i and j such that $e = x_i \ldots x_D y_1 \ldots y_i = y_j \ldots y_D x_1 \ldots x_j$. But then our claim is true since if $j \leqslant i$, vertex v is a substring of $e = \overline{ab}$, and then, incident with it. Otherwise if $i < j$, it is u which is a substring of e, thus incident with it. □

From this result and Lemma 1 we can characterize the superfluous edges of De Bruijn and Kautz graphs.

Corollary 6. *An edge of a De Bruijn graph is superfluous if and only if the diameter of the graph is greater than* 1. □

Corollary 7. *An edge of a Kautz graph is superfluous if and only if the diameter of the graph is* 2 *and the edge is not the underlying edge of a digon or the diameter is greater than* 3. □

3.2. A large superfluous edge set for the De Bruijn and Kautz graphs

We now present the largest superfluous sets of edges for $UB(d,D)$ and $UK(d,D)$ we have been able to find (for $d \geqslant 3$).

Proposition 8. *Let G be $B(d,D)$ or $K(d,D)$, and let $s \in \Sigma(G)$ be an element of its alphabet ($\mathbb{Z}_d$ and $\mathbb{Z}_{d+1}$ resp.); then the set $UF_G(s)$ of underlying edges of*

$$F_G(s) = \{x_0 \dots x_{D-1}\, s \in A(G) \,|\, x_i \neq s \ \forall i\}$$

is a superfluous set of edges of UG for $D \geqslant 2$.

Proof. First of all notice that the arcs of $F_G(s)$ do not belong to any digon since $x_{D-2} \neq s$. Hence, by Lemma 2, it suffices to prove that any directed closed walk of length $l \leqslant 2D$ cannot cross $F_G(s)$ more than once. By way of contradiction, let us suppose that there exists such a directed closed walk,

$$W = \overline{x_0 \dots x_{l-1}}$$

of length $l \leqslant 2D$, and two arcs e and e' of $F_G(s)$, such that

$$e = x_{(i \bmod l)} \cdots x_{(i+D \bmod l)}, \tag{1}$$

$$e' = x_{(i' \bmod l)} \cdots x_{(i'+D \bmod l)} \tag{2}$$

for two different integers $i, i' \in \{0, \dots, l-1\}$. We claim that

$$i' \notin I = \{(i \bmod l), (i+1 \bmod l), \dots, (i+D \bmod l)\}.$$

Indeed, if $i' \equiv i + j \pmod l$ for some j with $1 \leqslant j \leqslant D$, then

$$x_{(i'-j+D \bmod l)} = x_{(i+D \bmod l)} = s,$$

where the last equality follows from (1) since $e \in F_G(s)$. Then e' has an s in its $(D-j)$th position, but as $e' \in F_G(s)$ it has an s only in its Dth position and $D > D-1 \geqslant D-j$. By symmetry, we infer that

$$i \notin I' = \{(i' \bmod l), (i'+1 \bmod l), \dots, (i'+D \bmod l)\}$$

thus I and I' are disjoint sets, both of cardinality $D+1$, but included in a set $\mathbb{Z}_l$ of cardinality $l \leqslant 2D$. □

From this proposition we can compute lower bounds for the deficiency reserve of the De Bruijn and Kautz graphs, and thus, upper bounds for $e_D(n,\Delta)$ when n is the order of these graphs.

Corollary 9. *If $D \geqslant 2$ and $d \geqslant 2$, then*

$$\mathrm{der}(UB(d,D)) \geqslant (d-1)^D,$$

$$\mathrm{der}(UK(d,D)) \geqslant d(d-1)^{D-1}.$$

Proof. Let G be $B(d,D)$ or $K(d,D)$ and G' be $B(d-1,D)$ or $K(d-1,D)$, respectively. In order to compute the cardinality of $UF_G(d)$ we compute the cardinality of $F_G(d)$, since neither $F_G(d)$ has loops nor $UF_G(d)$ has ex-digons. Finally, the map $b : F_G(d) \to G'$ defined by $b(x_0 \dots x_{D-1} d) = x_0 \dots x_{D-1}$ is a bijection and we have that

$$|F_{B(d,D)}(d)| = |B(d-1,D)| = (d-1)^D,$$

$$|F_{K(d,D)}(d)| = |K(d-1,D)| = d(d-1)^{D-1}. \qquad \square$$

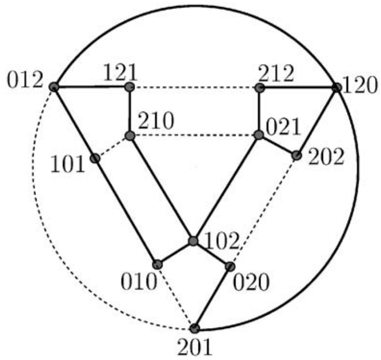

Fig. 2. The set of dashed edges is superfluous in $UK(2,3)$.

Table 1
Lower bounds for der($UK(d,D)$)

		D			D				
d	Δ	2	d	Δ	3	4	5	6	7
2	3	**2**	2	4	**6**	**9**	**19**	**31**	**51**
3	5	**6**	3	6	**20**	**34**	48	96	192
4	7	**14**	4	8	**48**	108	324	972	2 916
5	9	**26**	5	10	**97**	320	1 280	5 120	20 480
6	11	**42**	6	12	**174**	750	3 750	18750	93 750
7	13	**62**	7	14	252	1 512	9 072	54 432	326 592
8	15	**86**	8	16	392	2 744	19 208	134 456	941 192

Corollary 10. *If* $D \geqslant 3$ *and* $d \geqslant 2$, *then*

$$e_D(d^D, 2d) \leqslant d^{D+1} - (d-1)^D - d^2/2 - d/2,$$

$$e_D(d^D + d^{D-1}, 2d) \leqslant d^{D+1} + d^D - d(d-1)^{D-1} - d^2/2 - d/2. \qquad \square$$

The bounds of Corollary 9 are not tight in general. For instance, by Proposition 5, the set of edges $\{\overline{0\,1}, \overline{0\,2}, \overline{1\,2}\}$ is superfluous in $UK(2,3)$ and, even this set, is not of maximum order since the set of edges $SF = \{2012, 0210, 2101, 2010, 1212, 0202\}$ is still a superfluous one, as can be verified by hand (see Fig. 2). The set SF was obtained by computer search in such a way that it is a maximal superfluous set of edges. In Table 1, we present lower bounds for der($UK(d,D)$) derived either from Corollary 9 or by computer search (indicated in bold). The latter arise from maximal superfluous set of edges. We split the table into two subtables, depending on whether D is equal to 2 or not. This is carried out because the maximum degree Δ of $UK(d,D)$ is $2d-1$ if $D=2$, but $2d$ if $D \geqslant 3$.

3.3. Some useful superfluous sets of edges

Next, we give superfluous sets of edges which are not as large as $F_G(s)$, but that will enable us to add vertices in different suitable ways. Let G be a De Bruijn or a Kautz digraph, and let $s \in \Sigma = \Sigma(G)$ be an element of its alphabets. Given a tournament $T = (\Sigma \setminus \{s\}, A)$ over $\Sigma \setminus \{s\}$, we associate to any subset $A_T \subset A$ of its arcs a subset, $F_G(A_T) \subset A(G)$ of arcs of G defined by

$$F_G(A_T) = \{ax_1x_2 \dots x_{D-1}b \in A(G) \mid (a,b) \in A_T,$$
$$x_{D-1} = s, x_i \neq s \text{ if } i \neq D-1\} \quad \text{for } D \geqslant 2. \tag{3}$$

The next proposition states that the set of underlying edges of the arcs of $F_G(A_T)$ is a superfluous set of edges of UG.

Proposition 11. *If G is $B(d,D)$ or $K(d,D)$, then the set $UF_G(A_T)$ of underlying edges of the arcs of $F_G(A_T)$ is a superfluous set of edges of UG for $D \geqslant 2$. That is,*

$$UB(d,D) \triangleright UB(d,D) - UF_{B(d,D)}(A_T) \quad \text{in } \mathscr{H}_{\Delta(G),D},$$
$$UK(d,D) \triangleright UK(d,D) - UF_{K(d,D)}(A_T) \quad \text{in } \mathscr{H}_{\Delta(G),D}.$$

Proof. We proceed as in Proposition 8 noticing that the arcs of the set $F_G(A_T)$ do not belong to any digon, since they contain more than two different symbols (i.e., "a","b" and "s"). Next, in order to use Lemma 2 and proceed by way of contradiction, let us suppose that there exists a directed closed walk $W = \overline{x_0 \dots x_{l-1}}$ in G of length $l \leqslant 2D$, and two arcs e, e' of $F_G(A_T)$, such that

$$\begin{array}{lllll} e = x_{(i \bmod l)} & x_{(i+1 \bmod l)} & \dots & x_{(i+D-1 \bmod l)} & x_{(i+D \bmod l)} \\ = a & x_{(i+1 \bmod l)} & \dots & s & b \end{array} \tag{4}$$

and

$$\begin{array}{lllll} e' = x_{(i' \bmod l)} & x_{(i'+1 \bmod l)} & \dots & x_{(i'+D-1 \bmod l)} & x_{(i'+D \bmod l)} \\ = a' & x_{(i'+1 \bmod l)} & \dots & s & b' \end{array}$$

for two different integer $i, i' \in \{0, \dots, l-1\}$. We claim that

$$i' \notin I = \{(i \bmod l), (i+1 \bmod l), \dots, (i+D-1 \bmod l)\}. \tag{5}$$

Indeed, if $i' \equiv i + j \pmod l$, for some j in $\{1, \dots, D-1\}$, then $x_{(i'-j+D-1 \bmod l)} = x_{(i+D-1 \bmod l)}$ which is, by (4), equal to s. Thus, e' has an s at position $D-j-1 \in \{0, \dots D-2\}$, but since $e' \in F_G(A_T)$ it has an s only in position $D-1$. By symmetry, we conclude that $i \notin I' = \{(i' \bmod l), \dots, (i'+D-1 \bmod l)\}$, which together with (5) implies that I and I' are disjoint sets of cardinality D included in the set $\mathbb{Z}_l$ of cardinality $l \leqslant 2D$. Thus, $l = 2D$ and $i \equiv i' + D \pmod{2D}$. Therefore, $(a,b) = (b', a')$, contradicting the fact that, by definition of $F_G(A_T)$, both (a,b) and (a',b') are arcs of the tournament T. $\square$

4. Expansions for De Bruijn and Kautz graphs

The deletion of superfluous set of edges enables us to add new vertices to the new graphs. For example, Fig. 3 shows a $(4,3)$-admissible expansion of $UK(2,3)$ in 10 vertices. In the next subsections, we describe general methods.

4.1. (Δ, D)-admissible expansions of Kautz graphs in $\Delta + 2$ and $2\Delta + 2$ vertices

In this subsection, we show that we can double the number of vertices added in [4] by means of the removal of the digons. First, we prove the existence of an expansion in $\Delta + 2$ vertices.

Proposition 12. *If $D \geqslant 3$ and $G_0 = UK(d,D)$ there exist graphs $G_i \in \mathscr{H}_{2d,D}$ for $i = 1,\ldots,d+3$ such that*:

(1) $G_0 \sqsubset G_1 \sqsubset \cdots \sqsubset G_{d+1} \triangleright G_{d+2} \sqsubset G_{d+3}$ *in* $\mathscr{H}_{2d,D}$,
(2) $|G_{i+1}| = 1 + |G_i|$ *for* $i = 0,\ldots,d$,
(3) $|G_{d+3}| = |G_{d+2}| + d + 1$.

Proof. For $i = 0,\ldots,d$ we obtain the graph G_{i+1} by adding a new vertex v_i to the graph G_i and joining v_i to the previously added vertices (i.e., v_j with $j < i$) and to any vertex $\overline{x\,i}$ with first symbol $x \neq i$; formally,

$$v_i \sim v_j, \quad \forall j < i \tag{6}$$

and

$$v_i \sim \overline{x\,i}, \quad \forall x \in \mathbb{Z}_{d+1} \setminus \{i\}. \tag{7}$$

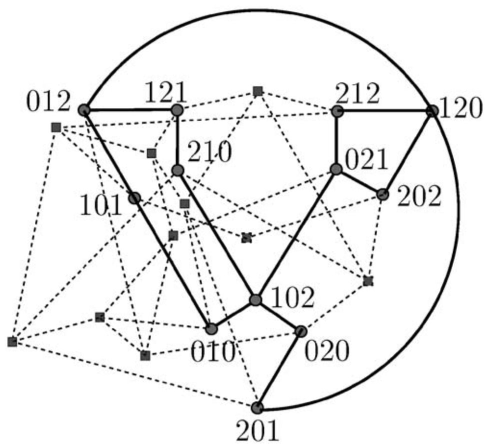

Fig. 3. A $(4,3)$-admissible expansion of $UK(2,3)$ in 10 vertices.

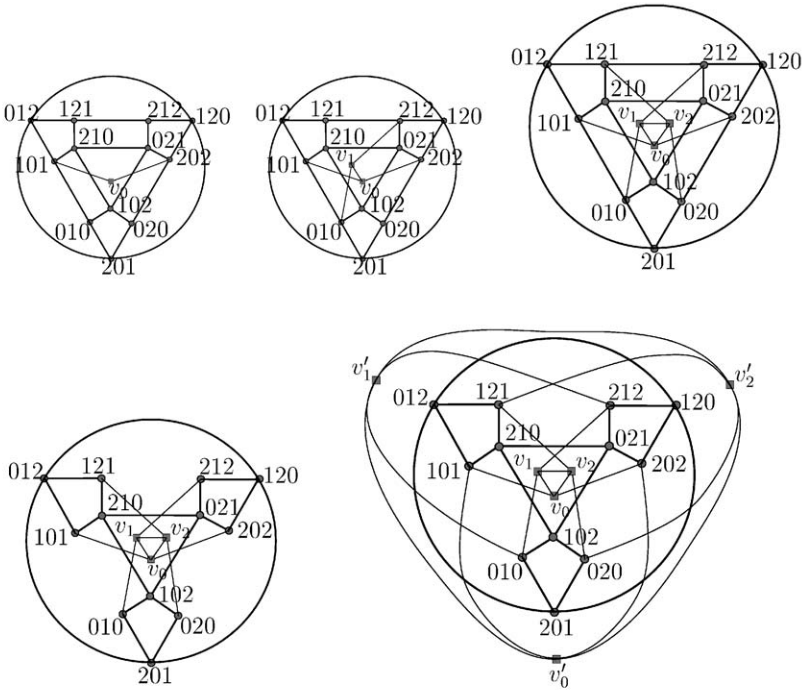

Fig. 4. Graphs G_1, G_2, G_3, G_4 and G_5 of Proposition 12 for $UK(2,3)$.

We construct G_{d+2} by deleting from G_{d+1} all the ex-digons of G_0. Finally, we obtain G_{d+3} by adding a complete graph on $d+1$ vertices $\{v'_0,\dots,v'_d\}$ and joining each vertex v'_i to the same vertices of G_0 that v_i is adjacent with. Thus, the map ϕ that exchange v_i and v'_i for all i is a graph automorphism. Fig. 4 shows some graphs of the sequence for $(d,D)=(2,3)$.

In order to compute the maximum degree of the graphs G_i, we observe that any vertex v_i is adjacent, by (6), with i vertices v_j and, by (7) with d vertices of G_0. Thus, the degree of these vertices is at most $2d$ and the same holds for the vertices v'_i, since the degree is invariant under automorphisms. On the other hand, any vertex $\overline{x\,i}$ is adjacent, in G_j with:

- $2d-1$ vertices of G_0 if $j \leqslant d+1$,
- $2d-2$ vertices of G_0 if $j \geqslant d+2$,
- no vertex v_k if $j \leqslant i$,
- 1 vertex (i.e., v_i) if $i < j \leqslant d+2$,
- 2 vertices (i.e., v_i and v'_i) if $j = d+3$.

Thus, their degrees are upper bounded by $2d$.

Next, in order to verify that the diameters of the graphs of the expansion are upper bounded by D, we claim that it suffices to find a walk of length at most D between any

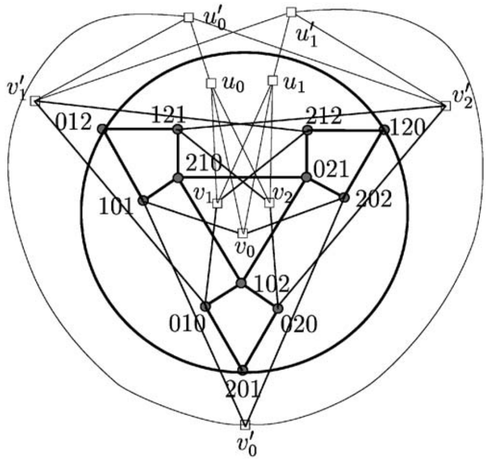

Fig. 5. The G_{10} graph of Proposition 13 for $UK(2,3)$.

vertex in $V(G_0)$ and any vertex v_i. Indeed, we do not need to find such walks between two vertices of G_0 because the set of edges deleted is superfluous by Proposition 5. Also, we do not need to find such walks between vertices v_i and v_i' since they are either adjacent or have an adjacent vertex in common. Besides, any walk of G_{d+3} joining a vertex v of G_0 with another vertex v_i gives rise, via ϕ, to a $v-v_i'$ walk of the same length.

Therefore, let $v=x_1\ldots x_D$ be a vertex of G_0 and v_i one of the added vertices. First, if $v=\overline{a\,b}$ then v and v_i have v_b as an adjacent vertex in common. Otherwise, if $v\neq\overline{a\,b}$, then, by Remark 4, the distance between v and $w=\overline{x_{D-1}\,i}$ or $w=\overline{x_D\,i}$ is at most $D-1$ depending upon x_D be equal to i or not. In both cases, since v_i and w are adjacent in G_j, the distance between v and v_i is at most D. □

Refining the above argument we obtain a similar construction that duplicates the number of added vertices in [4]. The idea is illustrated in Fig. 5. We enunciate the result without proof, but give a possible definition of the graphs in the expansion.

Proposition 13. *If* $D\geqslant 3$ *and* $G_0=UK(d,D)$ *there exist graphs* $G_i\in\mathscr{H}_{2d,D}$ *for* $i=1,\ldots,2d+3$ *such that*:

(1) $G_0\sqsubset G_1\sqsubset\cdots\sqsubset G_{2d+1}\triangleright G_{2d+2}\sqsubset G_{2d+3}$ *in* $\mathscr{H}_{2d,D}$,
(2) $|G_{i+1}|=1+|G_i|$ *for* $i=0,\ldots,2d$
(3) $|G_{2d+3}|=|G_{2d+2}|+2d+1$. □

We first define G_{2d+3} in terms of $G=K(d,D)$ as follows:

$$G_{2d+3}=(U(G-DN(G))\cup K_{d+1,d}\cup K'_{d+1,d})+E,$$

where $K_{d+1,d}$ and $K'_{d+1,d}$ are complete bipartite graphs with stable sets $\{v_0,\ldots,v_d\}\cup\{u_0,\ldots,u_{d-1}\}$ and $\{v'_0,\ldots,v'_d\}\cup\{u'_0,\ldots,u'_{d-1}\}$, respectively, and E is a set of edges defined by

$$\forall i,j,x : i\in\mathbb{Z}_d,\ j\in\mathbb{Z}_{d+1},\ x\in\mathbb{Z}_{d+1}\setminus\{j\},\quad \overline{xj}\sim v_j\sim u_i\sim u'_i\sim v'_j\sim\overline{xj}.$$

The graph G_{2d+2} is the subgraph of G_{2d+3} induced by the vertices of G and the vertices without primes, i.e., v_i and u_i. The graph G_{2d+1} is G_{2d+2} plus the ex-digons of UG, and the graph G_i for $d+2\leqslant i\leqslant 2d$, is the subgraph of G_{2d+1} induced by the vertices of G, the vertices v_j and the vertices u_0 to u_{i-d-2}. Similarly, graphs G_i for $1\leqslant i\leqslant d+1$, are the ones induced in G_{2d+1} by the vertices of G and the vertices v_0 to v_{i-1}.

4.2. (Δ,D)-admissible expansions for diameter 2

Kautz graphs $UK(d,2)$ with diameter 2 are the only regular members of the family (of Kautz graphs), and then the only ones without deficiency vertices. Thus, it is not possible to add new vertices without increasing the maximum degree. Nevertheless, the next result shows that $(\Delta,2)$-admissible expansions of $UK(d,2)$ up to d vertices do, in fact, exist for any d.

Proposition 14. *Let $m\in\mathbb{N}$ and $1\leqslant m\leqslant d$. Then there exists a (Δ,D)-admissible expansion in m vertices of Kautz graph of maximum degree $2d-1$ and diameter 2.*

Proof. By Proposition 8, $UK(d,2)\triangleright G_0=UK(d,2)-UF_{K(d,2)}(d)$ in $\mathscr{H}_{2d-1,2}$. Hence, if we construct a graph G_m such that

$$G_0\sqsubset G_m \quad \text{in } \mathscr{H}_{2d-1,2} \tag{8}$$

and $|G_m|=|UK(d,2)|+m$, the proof will be completed. Let K_m be the complete graph on m vertices with $V(K_m)=\{P_0,\ldots,P_{m-1}\}$. In order to define G_m, we take the union of G_0 with K_m and then, we add a set E of edges defined in the following way:

$$P_i\sim id \quad \text{for } i=0,\ldots,(m-1),$$

$$P_{m-1}\sim id \quad \text{for } i=m,\ldots,(d-1),$$

$$ji\sim P_i \quad \text{for } i=0,\ldots(m-1) \text{ and } j\in\{0,\ldots,(d-1)\}\setminus\{i\}.$$

In other words, $G_m=(G_0\cup K_m)+E$. Hence, G_0 is a subgraph of G_m and then, the only thing we need to verify in order to prove (8) is that $G_m\in\mathscr{H}_{2d-1,2}$ (Fig. 6). Indeed, the degree of the vertices P_i is

$$\deg(P_i)=\begin{cases}(m-1)+1+(d-1)\leqslant 2d-1 & \text{if } i\neq m-1,\\ (m-1)+1+(d-m)+(d-1)=2d-1 & \text{if } i=m-1.\end{cases}$$

The vertices of the form id have degree $\deg(id)=2d-1-(d-1)+1=d+1$ which is at most $2d-1$ if $d\geqslant 2$. The vertices of the form ji with $m<i<d$ have degree

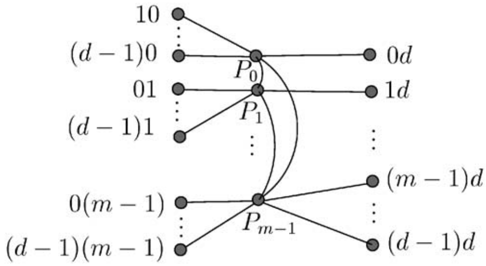

Fig. 6. Addition of vertex to $UK(d,2)$ in Proposition 14.

$2d-2$. The remaining vertices have degree $2d-1$. Finally, in order to verify that G_m has diameter two, we define a compatible routing ρ in the following way:

$$\rho(u,v)=\begin{cases} u,v, & u\sim v,\\ P_i,ji,kj, & u\in K_m\not\ni v=kj \text{ and } d\notin\{k,j\},\\ P_i,id,dj, & u\in K_m\not\ni v=dj,\\ P_i,P_j,jd, & u\in K_m\not\ni v=jd \text{ and } 1\leqslant j<m,\\ P_i,P_m,jd, & u\in K_m\not\ni v=jd \text{ and } j\geqslant m,\\ \rho(v,u)^t, & u\notin K_m\ni v,\\ \text{a minimal walk in } G_0, & u,v\notin K_m, \end{cases}$$

where $\rho(v,u)^t$ is the walk $\rho(v,u)$ travelled in the opposite direction. Proposition 8 guarantees the existence of the minimal walk mentioned in the last case. □

A similar result holds for De Bruijn graphs, even though in this case the result is not so unexpected since the graphs are not regular.

Proposition 15. *Let $m\in\mathbb{N}$ and $1\leqslant m\leqslant d$. Then, there exists a (Δ,D)-admissible expansion in m vertices of $UB(d,2)$.*

Proof. Similarly to the Kautz graphs case,

$$UB(d,2)\triangleright G_0=UB(d,2)-UF_{B(d,2)}(d-1)\sqsubset G_0\cup K_m+E \quad \text{in } \mathscr{H}_{2d-1,2},$$

where K_m is the complete graph on m vertices $P_0,\dots,P_{m-1}$, and E is the set of edges defined as follows:

$$P_i\sim i(d-2) \quad \text{for } i=0,\dots,(m-1),$$

$$P_{m-1}\sim i(d-1) \quad \text{for } i=m,\dots,(d-1),$$

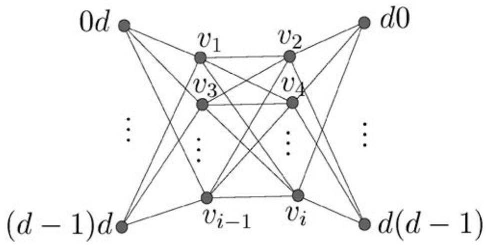

Fig. 7. Atomic vertex additions to $UK(d,2)$. Proposition 16.

$$ji \sim P_i \quad \text{for } i = 0,\dots,(m-2) \text{ and } j = 0,\dots,(d-2),$$

$$jj \sim P_{m-1} \quad \text{for } j = 0,\dots,(d-1).$$

The verification of the conditions on the degrees and the diameter is almost the same as in Proposition 14. □

4.3. (Δ, D)-*admissible extensions*

As we said in the introduction, we are also interested in finding (Δ, D)-admissible extensions, that is, expansions such that two consecutive graphs of the sequence have orders that differ in at most one unit. In order to make these "atomic" steps, we should remove edges of the form $UF_G(A_T)$ for a suitable choice of the tournament T. We choose T to be any tournament T_d over $\mathbb{Z}_{d+1} \setminus \{d\}$ containing the arcs

$$A_{T_d} = \left\{ (a, (a+i) \bmod d) \mid 1 \leqslant i \leqslant \left\lfloor \frac{d-1}{2} \right\rfloor \right\}.$$

By Proposition 11, if G is $B(d+1, D)$ or $K(d, D)$, then the set of edges $UF_G(A_{T_d})$ is superfluous in UG. These sets enable us to make (Δ, D)-admissible extensions for $D = 2$ and 3.

4.3.1. *Diameter two*

Proposition 16. *For any $d \geqslant 3$ there exist graphs $G_i \in \mathscr{H}_{2d-1,2}$ for $i = 0,\dots,l$ with $l = 2\lfloor (d-1)/2 \rfloor$ such that*

(1) $UK(d,2) \triangleright G_0 \sqsubset G_1 \sqsubset \cdots \sqsubset G_l$ *in* $\mathscr{H}_{2d-1,2}$,
(2) $|G_{i+1}| = 1 + |G_i|$ *for* $i = 0,\dots,l-1$.

Proof. Let G_0 be the graph $UK(d,2) - UF_{K(d,2)}(A_{T_d})$ and for $i \geqslant 1$, let us construct graph G_i by adding a new vertex v_i to graph G_{i-1} and joining v_i as in Fig. 7, i.e., vertex v_i is adjacent with

- any previous added vertex v_j with $j \not\equiv i \pmod 2$ (i.e., with the vertices $v_{i-1}, v_{i-3}, \dots, v_{(i+1 \bmod 2)}$),

- vertex dk for $k=0,\dots,d-1$ if i is even,
- vertex kd for $k=0,\dots,d-1$ if i is odd.

Since condition (2) is verified by construction, it remains to prove that $G_i \in \mathscr{H}_{2d-1,2}$. Indeed, the degree of the vertices v_i in the graph G_j is upper bounded by $\lfloor (d-1)/2 \rfloor + d$ which is less than $2d-1$, while the degree of the remaining vertices is at most $2d-1$. Finally, the diameter of each graph G_i is 2 since the following routing:

$$\rho_i(u,v) = \begin{cases} v_i, v & \text{if } u = v_i \sim v, \\ v_i, v_{j-1}, v_j & \text{if } u = v_i,\ v \notin G_0 \text{ and } i \equiv j \,(\text{mod}\, 2), \\ v_i, dj, jk & \text{if } u = v_i,\ v = jk \text{ and } i \text{ even}, \\ v_i, kd, jk & \text{if } u = v_i,\ v = jk \text{ and } i \text{ odd}, \\ \rho_i(v,u)^t & \text{if } v = v_i \\ \rho_{i-1}(u,v) & \text{otherwise} \end{cases}$$

is a well-defined compatible routing for them if ρ_0 is one for G_0. □

A similar result holds for De Bruijn graphs and we shall use the above result to prove it.

Proposition 17. *For any $d \geqslant 3$ there exist graphs $G_i \in \mathscr{H}_{2d+1,2}$ for $i=0,\dots,l$ with $l = 2\lfloor (d-1)/2 \rfloor$ such that*:

(1) $UB(d+1,2) \rhd G_0 \sqsubset G_1 \sqsubset \cdots \sqsubset G_l$ *in* $\mathscr{H}_{2d+1,2}$,
(2) $|G_{i+1}| = 1 + |G_i|$ *for* $i = 0,\dots,l-1$.

Proof. It suffices to bear in mind that $K(d,D)$ and $UK(d,D)$ are subdigraphs and subgraphs of $B(d+1,D)$ and $UB(d+1,D)$, respectively. Applying the previous proposition to the copy of $UK(d,2)$ included in $UB(d+1,2)$ and extending the routing ρ_i to those vertices not in the copy will complete the proof. These vertices are of the form xx and we make the extension by adding the following rules to the definition of ρ_i:

$$\rho_i(v_i, xx) = \begin{cases} v_i, xd, xx & \text{if } x \neq d, \text{ and } i \text{ is odd}, \\ v_i, dx, xx & \text{if } x \neq d, \text{ and } i \text{ is even}, \\ v_i, 0d, dd & \text{if } x = d, \text{ and } i \text{ is odd}, \\ v_i, d0, dd & \text{if } x = d, \text{ and } i \text{ is even}, \end{cases}$$

the fact that $F_{K(d,2)}(A_{T_{d+1}}) \subset F_{B(d+1,2)}(A_{T_d})$, guarantees the correctness of this extension. □

4.3.2. Diameter three

In order to work with diameters greater than two, let us say that two arcs $e = a\,x_1 \dots x_{D-2}\,d\,b$ and $e' = a'\,x'_1 \dots x'_{D-2}\,d\,b'$ of $F_G(A_{T_d})$ are $\simeq$-equivalent, writing $e \simeq e'$, if

$$a' - a \equiv b' - b \equiv x_1' - x_1 \equiv \cdots \equiv x_{D-2}' - x_{D-2} \pmod d.$$

Notice that, from the first equality, we deduce that $b-a \equiv b'-a' \pmod d$. As usual, $[\![e]\!]$ stands for the class of arcs equivalent to e and $F_G(A_{T_d})/\simeq$ for the quotient of $F_G(A_{T_d})$ by $\simeq$. As an example, we show the equivalence class of the arc $e = 3\,1\,4\,1\,5\,9\,2$ of $F_{K(8,6)}(A_{T_8})$:

$$[\![3\,1\,4\,1\,5\,9\,2]\!] = \{3\,1\,4\,1\,5\,9\,2,\ 2\,0\,3\,0\,4\,9\,1,\ 1\,8\,2\,8\,3\,9\,0,\ 0\,7\,1\,7\,2\,9\,8,$$
$$8\,6\,0\,6\,1\,9\,7,\ 7\,5\,8\,5\,0\,9\,6,\ 6\,4\,7\,4\,8\,9\,5,\ 5\,3\,6\,3\,7\,9\,4,$$
$$4\,2\,5\,2\,6\,9\,3\}.$$

The cardinality, $\mathrm{nb}(d,D)$, of $F_{K(d,D)}(A_{T_d})/\simeq$ is

$$\mathrm{nb}(d,D) = |F_{K(d,D)}(A_{T_d})/\simeq| = \left\lfloor \frac{d-1}{2} \right\rfloor (d-1)^{D-2},$$

since the cardinality of $[\![e]\!]$ is d for any arc e and the cardinality of $F_{K(d,D)}(A_{T_d})$ is

$$|F_{K(d,D)}(A_{T_d})| = |A_{T_d}|(d-1)^{D-2} = d \left\lfloor \frac{d-1}{2} \right\rfloor (d-1)^{D-2}.$$

A similar computation gives the cardinality $\mathrm{nbb}(d+1,D)$ of $F_{B(d+1,D)}(A_{T_d})/\simeq$:

$$\mathrm{nbb}(d+1,D) = |F_{B(d+1,D)}(A_{T_d})/\simeq| = \left\lfloor \frac{d-1}{2} \right\rfloor d^{D-2}.$$

In what follows we shall identify $\mathbb{Z}_{d+1} \setminus \{d\}$ with $\mathbb{Z}_d$.

Remark 18. For any arc e, its $\simeq$-class $[\![e]\!]$ verifies:

$$\{b \mid \exists f \in [\![e]\!] : f = a\,x_1 \ldots x_{D-2}\,d\,b\} = \mathbb{Z}_d. \tag{9}$$

This property of $\simeq$ will play a key role in the proceeding constructions. In fact, it seems to be the only property that an equivalence relation over $F_G(A_{T_d})$ must verify in order to prove the next results. This means that different relations satisfying (9) would yield nonisomorphic constructions.

Theorem 19. *For any $d \geqslant 3$ there exist graphs $G_i \in \mathscr{H}_{2d,3}$ for $i = 0,\ldots,l$ with $l = \mathrm{nb}(d,3)$ such that*:

- $UK(d,3) \triangleright G_0 \sqsubset G_1 \sqsubset \cdots \sqsubset G_l$ *in* $\mathscr{H}_{2d,3}$,
- $|G_{i+1}| = 1 + |G_i|$ *for* $i = 0,\ldots,l-1$.

Proof. Let G_0 be the graph $UK(d,3) - UF_{K(d,3)}(A_{T_d})$ and for $i \geqslant 1$ let G_i be the graph obtained by adding a new vertex v_i to G_{i-1} and joining v_i as we next describe. First, we order the set $F_{K(d,3)}(A_{T_d})/\simeq$ saying that $[\![0\,x\,d\,c]\!]$ is previous to any $[\![0\,x'\,d\,c']\!]$ such that either $c < c'$ or $c = c'$ and $x < x'$. Then, we can label each vertex v_i with the ith element of $F_{K(d,3)}(A_{T_d})/\simeq$, since $l = |F_{K(d,3)}(A_{T_d})/\simeq|$. If v_i is labelled with $[\![0\,x\,d\,c]\!]$, we join it to:

- any previous added vertices (labelled with) $[\![0\,x'\,d\,c]\!]$ for any $x' < x$,
- vertex $\langle j,x,c\rangle\,d\,j \in G_0$ for any $j \in \mathbb{Z}_d$, where $\langle j,x,c\rangle = (j + x - c) \bmod d$.

Notice that the vertices of the form $x\,d\,b$ are adjacent to vertices (v_i labelled with) $[\![0\,\langle c,x,b\,\rangle d\,c]\!]$ for $c = 1,\dots,\lfloor (d-1)/2 \rfloor$.

In order to verify the condition on the maximum degree, it suffices to verify it for G_l, because the rest of graphs are subgraphs of it. Let u be a vertex of G_l, then its degree is

$$\deg_{G_l}(u) \leqslant \begin{cases} \deg_{UK(d,D)}(u) - \left\lfloor \dfrac{d-1}{2} \right\rfloor & \text{if } G_0 \ni u = a\,x\,d, \\ \deg_{UK(d,D)}(u) - \left\lfloor \dfrac{d-1}{2} \right\rfloor + \left\lfloor \dfrac{d-1}{2} \right\rfloor & \text{if } G_0 \ni u = x\,d\,b, \\ (d-2) + d & \text{if } G_0 \not\ni u = v_i \text{ for some } i, \\ \deg_{UK(d,D)}(u) & \text{otherwise,} \end{cases}$$

which is always less than $2d$. Next, we verify that the condition on the diameter holds by giving the following compatible routing ρ in G_l (Fig. 8).

$$\rho(u,v) = \begin{cases} u, v & \text{if } u \sim v, \\ \text{a minimal } u - v \text{ walk in } G_0 & \text{if } u, v \in G_0, \\ u, \rho^*(u,v), v & \text{if } u \notin G_0, \\ \rho(v,u)^t & \text{if } v \notin G_0, \end{cases}$$

where $\rho(v,u)^t$ is the walk $\rho(v,u)$ travelled in the opposite direction, and

$$\rho^*([\![0\,x\,d\,c]\!], v) = \begin{cases} [\![0\;y\,d\,c]\!], y\,d\,c & \text{if } G_0 \not\ni v = [\![0\,x'\,d\,c']\!] \\ & \quad \text{and } y = \langle c, x', c'\rangle \neq x, \\ x\,d\,c, & \text{if } G_0 \not\ni v = [\![0\,x'\,d\langle x', c, x\rangle]\!], \\ \langle z_1, x, c\rangle\,d\,z_1,\ d\,z_1 z_2 & \text{if } G_0 \ni v = z_1 z_2 z_3 \\ & \quad \text{and } z_1 \neq d, \\ \langle z_2, x, c\rangle\,d\,z_2 & \text{if } G_0 \ni v = d\,z_2 z_3. \end{cases}$$

The restriction $\rho|_{G_i}$ of ρ to any G_i gives rise to a compatible routing in G_i. □

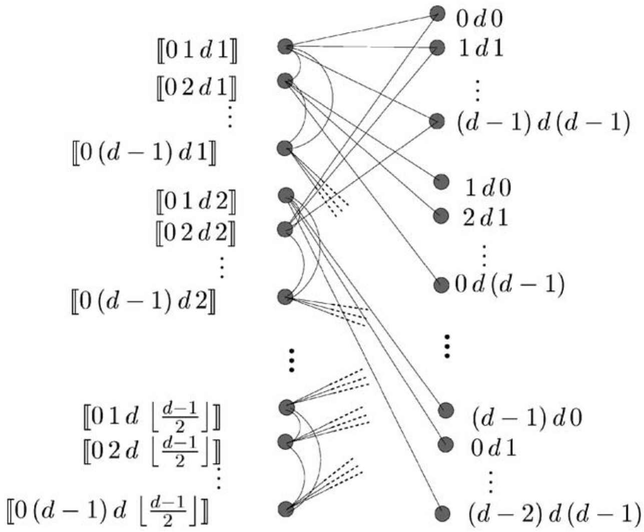

Fig. 8. Graph construction for Proposition 19.

We can derive a similar result for De Bruijn graphs.

Proposition 20. *For any* $d \geqslant 3$ *there exist graphs* $G_i \in \mathscr{H}_{2d+2,3}$ *for* $i = 0, \dots, l$ *with* $l = \lfloor (d-1)/2 \rfloor (d-1)$ *such that*

- $UB(d+1,3) \rhd G_0 \sqsubset G_1 \rhd G_1 \sqsubset \cdots \sqsubset G_l$,
- $|G_{i+1}| = 1 + |G_i|$ *for* $i = 0, \dots, l-1$.

Proof. As in Proposition 17, we apply the previous proposition to the copy of $UK(d,2)$ included in $UB(d+1,2)$ and extend the routing ρ from the new added vertices to those vertices, $u = x_1x_2x_3$ not in the copy of $UK(d,2)$, adding the follow rule:

$$\rho(v,u) = [\![0\,x\,d\,c]\!], \langle x_1, x, c\rangle\, d\, x_1,\; d\, x_1\, x_2,\; x_1\, x_2\, x_3 \quad \text{if } G_0 \not\ni v = [\![0\,x\,d\,c]\!],$$

where the fact that $F_{K(d,3)}(A_{T_d}) \subset F_{B(d+1,3)}(A_{T_{d+1}})$ guarantees the correctness of the extension. □

4.4. The ϕ-union and expansions for diameters greater than 3

In previous constructions, we were able to add an amount of vertices close to $\text{nb}(d,D)$. It seems to be very difficult to obtain similar results (at least with our approach) for larger diameters. Nevertheless, we are still able to add an exponential amount of vertices, as shown below.

Let $\phi : A(\subset A(G)) \to V(H)$ be a map from a subset A of the arcs of a digraph G to the vertices of a graph H. Then, we call the *ϕ-union of H to G* the graph

$$G \cup_\phi H = U(G - A) \cup H + UA_\phi,$$

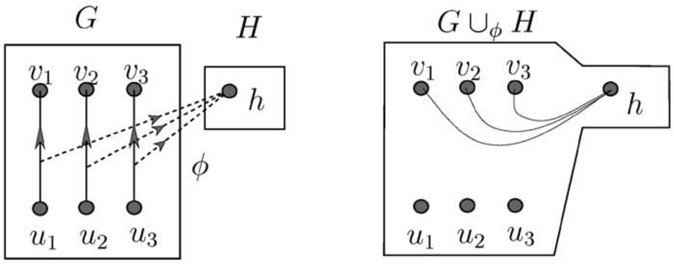

Fig. 9. The ϕ-union of G and H.

where

$$A_\phi = \{(v,h) \in V(G) \times V(H) \colon v \in \Gamma^+(\phi^{-1}(h))\}.$$

Intuitively, the function ϕ selects arcs of the digraph and vertices of the graph and joins them as in Fig. 9.

Now we establish some directs bounds for the diameter and the maximum degree of a ϕ-union.

Lemma 21. *Let $G \cup_\phi H$ be the ϕ-union of a graph H in $\mathscr{H}_{\Delta_1,D_1}$ to a digraph G such that $U(G-A)$ is in $\mathscr{H}_{\Delta_2,D_2}$, where A is the domain of ϕ. Then,*

$$D(G \cup_\phi H) \leqslant \max\left(D_1, D_2, 1 + \max_{h \in H} l_h\right),$$

$$\Delta(G \cup_\phi H) \leqslant \max\left(\Delta_2, \Delta_1 + \max_{h \in H} |\phi^{-1}(h)|\right),$$

where

$$l_h = \max_{u \in G} d(u, \Gamma^+[\phi^{-1}(h)]) = \max_{u \in G} \min_{v \in \Gamma^+[\phi^{-1}(h)]} d(u,v)$$

and $d(u,v)$ denotes the distance between u and v in $U(G-A)$.

Proof. We first bound the diameter. Given two vertices in $G \cup_\phi H$, if both belong to the copy of H or $U(G-A)$, then they are at distance less than D_1 and D_2, respectively. If they are in different copies, then the one in the copy of H, say h, is adjacent to the vertices of $\Gamma^+[\phi^{-1}(h)]$. Among them, there is at least one at distance l_h from the vertex in the copy of $U(G-A)$, say u. Hence, h is at distance at most $1 + l_h$ from u.

In order to verify the upper bound for the maximum degree, notice that the degree of each vertex h of the copy of H in $G \cup_\phi H$ is $|\phi^{-1}(h)| + \deg_H(h)$. On the other hand, the degree of a vertex in $U(G-A)$ does not exceed Δ_2, because either it belongs to $\Gamma^-(A)$ in which case its degree has decreased or its degree has not changed at all. □

Proposition 22. *Let $D \geqslant 4$ and $H \in \mathscr{H}_{d,D}$ a graph with order at most the cardinality of $F_{K(d,D)}(A_{T_d})/\simeq$; i.e.,*

$$|H| \leqslant |F_{K(d,D)}(A_{T_d})/\simeq| = \mathrm{nb}(d,D) = \left\lfloor \frac{d-1}{2} \right\rfloor (d-1)^{D-2}. \qquad (10)$$

Then there exists a subset A of the arcs of $K(d,D)$ and a map $\phi : A \to V(H)$, such that the ϕ-union of H to $K(d,D)$ is a graph in $\mathscr{H}_{2d,D}$; i.e.,

$$K(d,D) \cup_\phi H \in \mathscr{H}_{2d,D}.$$

Proof. By inequality (10), there exists a one-to-one map $b : V(H) \to F_{K(d,D)}(A_{T_d})/\simeq$. We define A to be the set "covered" by the image of b; more precisely

$$A = \cup_{h \in H} b(h)$$

and thus we claim that the map $\phi : A \to V(H)$ defined by $\phi(e) = b^{-1}([\![e]\!])$ verifies the assertion of the proposition.

In order to check that the diameter of $K(d,D) \cup_\phi H$ is at most D, let us apply Lemma 21. With the notation of this lemma, both D_1 and D_2 are upper bounded by D. Indeed, $D_1 \leqslant D$ by hypothesis and $D_2 = D$ because the set of edges $UF_{K(d,D)}(A_{T_d})$ is superfluous. In order to bound l_h, let $u = x_1 \ldots x_D$ be a vertex of $K(d,D)$. Given an $h \in H$, its preimage by ϕ is a $\simeq$-equivalence class $\phi^{-1}(h) = [\![e]\!]$ for some $e \in A(G)$, and, as we observed in Remark 18:

$$\mathbb{Z}_d = \{b \mid \exists f \in [\![e]\!] : f = a\, y_1 \ldots y_{D-2}\, d\, b\}.$$

Then, we can distinguish two cases depending whether x_1 is equal to d or not. If $x_1 = d$ [$x_1 \neq d$], there exists an $f \in [\![e]\!]$ such that its last symbol is x_2 [x_1] and then, if $f = vw$, the vertex w will be at distance at most $D-2$ [$D-1$] to u. In both cases $d(u, \Gamma^+[\phi^{-1}(h)]) \leqslant D-1$ and, therefore, each l_h is upper bounded by $D-1$ and the diameter of $K(d,D) \cup_\phi H$ is at most D.

Finally, the maximum degree of $K(d,D) \cup_\phi H$ is $2d$, since $|\phi^{-1}(h)| = |\mathbb{Z}_d| = d$, $\Delta(H) \leqslant d$ and $\Delta(UK(d,D) - UA) = 2d$. □

The next result says that the ϕ-union operates as a "$\sqsubset$-order homomorphism" in the sense that given an expansion in k vertices

$$H = H_0 \rhd H_1 \sqsubset H_2 \rhd \cdots \rhd H_{2n-1} \sqsubset H_{2n} = H' \quad \text{in } \mathscr{H}_{\Delta(H),D(H)}$$

from H to H' such that H' verifies the hypothesis of previous proposition, we can construct an expansion in k vertices

$$UG \rhd G_0 \sqsubset G \cup_{\phi_0} H_0 \rhd \cdots \rhd G \cup_{\phi_{2n-1}} H_{2n-1} \sqsubset G \cup_{\phi_{2n}} H_{2n} \quad \text{in } \mathscr{H}_{\Delta(G),D(G)}$$

for a suitable Kautz digraph G. Notice that if a step in the first expansion is "atomic", so it is the corresponding step in the second one.

Corollary 23. *If $H' \in \mathscr{H}_{d,D}$ verifies inequality (10), then there exists a map ϕ such that $K(d,D) \cup_\phi H'$ is a (Δ, D)-admissible expansion of $UK(d,D)$ in $|H'|$ vertices.*

Table 2
Number of vertices added to Kautz graphs

		D							
d	Δ	3	4	5	6	7	8	9	10
2	4	*P*13 **10**	*P*13 **10**	*P*13 **10**	*P*13 **10**	*P*13 **10**	*P*13 **10**	*P*13 **10**	*P*13 **10**
3	6	*P*13 **14**	*P*13 **14**	*P*13 **14**	Q_4 **16**	*T* **24**	*IS* **32**	*T* **48**	*IS* **64**
4	8	*P*13 **18**	*P*13 **18**	*UII* **27**	*UB* **64**	*UK* **192**	*UK* **384**	*D* **1 024**	*UK* **1 536**
5	10	*P*13 **22**	*UK* **24**	*D* **64**	*IS* **324**	*T* **1 080**	*IS* **1 944**	*T* **6 480**	*IS* **11 664**
6	12	*P*13 **26**	*UII* **50**	*UB* **243**	*UK* **972**	*D* **4 374**	*UK* **8 748**	*D* **39 366**	*UK* **78 732**
7	14	*P*13 **30**	*UK* **108**	*D* **486**	*IS* **2 304**	*T* **12 096**	*IS* **27 648**	*T* **145 152**	*IS* **331 776**
8	16	*P*13 **34**	*UII* **147**	*UB* **1 024**	*UK* **5 120**	*D* **32 768**	*UK* **81 920**	*D* **524 288**	*Ha* **13 964 808**

UB, *De Bruijn* graphs; *UK*, *Kautz* graphs; *UII*, *Imase-Itoh* graphs [18]; *D*, *Delorme* graphs [5]; *T*, graphs defined by Gómez and Fiol in [16]; *IS* graphs defined by Bond and Delorme in [5]; Q_4 the 4-cube; *Ha* found by Hafner by computer search [17].

Furthermore, if

$$H \triangleright H_1 \sqsubset H' \quad in\ \mathscr{H}_{d,D},$$

then, there exist maps ϕ_0 *and* ϕ_1 *such that*

$$UG \triangleright G_0 \sqsubset G \cup_{\phi_0} H \triangleright G \cup_{\phi_1} H_1 \sqsubset G \cup_{\phi} H' \quad in\ \mathscr{H}_{2d,D},$$

where $G = K(d,D)$ *and* $G_0 = U(G - A)$ *being* A *the domain of* ϕ_0.

Proof. Follows by taking ϕ to be the map of Proposition 22, and the map ϕ_1 [map ϕ_0] the restriction of ϕ [ϕ_1] to the set $\phi^{-1}(H_1)$ [set $\phi^{-1}(H)$]. □

Notice that even though $\mathrm{nb}(\Delta,D)(=\lfloor(\Delta-1)/2\rfloor(\Delta-1)^{D-2})$ is smaller than the Moore bound, $\Delta(\Delta-1)^D/(\Delta-2)$, it is larger than the largest known graphs for many values of the parameters Δ and D. For instance, the largest known $(8,10)$-graph H was found by Hafner [17] and has $|H| = 13\,964\,808$ vertices while $\mathrm{nb}(8,10) = 17\,294\,403$. Hence, there exists a ϕ such that $K(8,10) \cup_\phi H$ is a (Δ,D)-admissible expansion of $UK(8,10)$ in $|H|$ vertices. Table 2 presents this expansion together with other possible ones for $4 \leqslant \Delta \leqslant 16$ and $3 \leqslant D \leqslant 10$. The number of vertices that is possible to add to $UK(d,D)$ is indicated in bold. The name over the numbers are the one of the corresponding graph with which it is made the ϕ-union. The name "P 13" stands for the (Δ,D)-admissible expansions given by Proposition 13. The other names stand for a graph verifying the hypothesis of Corollary 23.

The two previous results of this section have corresponding ones in the case of De Bruijn graphs.

4.4.1. Asymptotic results

The direct application of the ϕ-union to families of asymptotically large graphs gives rise to new asymptotically large ones when the degree grows to infinity. Indeed, one of the best-known general families of large (Δ, D)-graphs are the graphs on alphabets (see [5,16]). For even values of Δ and D, the extensions of the graphs $S(d_1, d_2, q)$ defined in [4] are larger than the corresponding graphs on alphabets. This extensions S' have the following orders:

$$|S'| = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1} + k\left(\frac{\Delta}{2}\right)^{D/2} - (k-3)\left(\frac{\Delta}{2}\right)^{D/2-1},$$

where

$$k = \left\lfloor \frac{D-t+2}{2} \right\rfloor$$

and

$$t = \lfloor (D/2-1)\log_{\Delta/4}(\Delta/2) + \log_{\Delta/4}(\Delta/2-1) \rfloor.$$

However, the next result states that there exists graphs G, which are (Δ, D)-admissible expansions of Kautz graphs, with orders:

$$|G| = \left(\frac{\Delta}{2}\right)^D + \left(\frac{\Delta}{2}\right)^{D-1} + \left(\frac{\Delta}{4}\right)^{D-1}$$

for $\Delta \geqslant 8$ multiple of 4 and $D \geqslant 6$.

Proposition 24. *For $d \geqslant 4$ even and $D \geqslant 6$, there exists an exponential function $f: \mathbb{N} \to \mathbb{N}$ such that, the Kautz graph $UK(d,D)$ has a (Δ, D)-admissible expansion in*

- *$(d/2)^D$ vertices if $d < f(D)$,*
- *$(d/2)^{D-1}$ vertices if $d \geqslant f(D)$.*

Furthermore, $2^{D-3} < f(D) < 2^{D-2}$ for $D \geqslant 6$.

Proof. Let f be the function defined by

$$f(D) = 2\max\{k > 1 \mid (k-1)(2k-1)^{D-2} > k^D\}.$$

By means of Calculus (Appendix A), it is possible to prove that f is well defined and that $2^{D-3} < f(D) < 2^{D-2}$ for $D \geqslant 6$. Hence, if $d = 2k \leqslant f(D)$, we have that $(d/2)^D \leqslant \mathrm{nb}(d,D)$ and, for $D \geqslant 6$, $UB(k,D)$ verifies the hypothesis of Corollary 23. Therefore, there exists a map ϕ, such that $K(d,D) \cup_\phi UB(d/2,D)$ is a (Δ, D)-admissible expansion of $UK(d,D)$ in $(d/2)^D$ vertices. Otherwise, if $d > f(D)$, the graph $UB(k, D-1)$ verifies the hypothesis of Corollary 23, since it has order $N = (d/2)^{D-1} < \mathrm{nb}(d,D)$, and then, $K(d,D) \cup_\phi UB(d/2,D)$ is a (Δ, D)-admissible expansion of $UK(d,D)$ in $(d/2)^{D-1}$ vertices. □

Corollary 25. *For d even and $D \geqslant 4$, the Kautz graph $UK(d,D)$ has an $(2d,D)$-admissible expansion in $(d/2)^{D-1}$ vertices.* □

Asymptotically, many families of (Δ, D)-graphs have orders of the form $k(\Delta/2)^D$ for $k = 1, 2, 3, 5$ depending upon Δ and D (see [9,16]) exceeding bound $\mathrm{nb}(\Delta, D)$ only for $\Delta > \Omega(2^D)$. Thus, using the ideas of Proposition 24, it is possible to prove a similar result without restriction on the parity of d.

Theorem 26. *There exist integers d_0 and D_0, and a function $f : \mathbb{N} \to \mathbb{N}$ such that, $f(D) > c2^D$ for a positive constant c; and $UK(d, D)$ has an (Δ, D)-admissible expansion in*

- $k\lfloor d/2\rfloor^D$ *vertices if $d_0 < d < f(D)$,*
- $k\lfloor d/2\rfloor^{D-1}$ *vertices if $d \geqslant f(D)$,*

where $D > D_0$ and $k = 2, 3, 5$ depending upon the values of d and D. □

5. Conclusions

For all $\Delta \neq 4$, we have expanded Kautz and De Bruijn graphs adding an exponential amount of new nodes in such a way that the routing algorithms remain simple and invariant on the existent vertices. However, we wonder if it is possible to do the same for $\Delta = 4$. Other questions arise:

- How can we extend our results to general graphs or digraphs? For instance, which conditions need a digraph G satisfy in order to expand its underlying graph or the underlying graph of its iterated line digraphs?
- What can we say about the recursive structure obtained by making ϕ-union of graphs of the same family? For example, which are the automorphism groups of the ϕ-union of a Kautz graph with another Kautz graph?

Even though our techniques seem not to be applicable to all families of graphs, we believe that it is possible to use them with the Generalized Compound Graphs [15] which are among the largest known graphs with given diameter and maximum degree.

Acknowledgements

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Appendix A.

Proposition A.1. (1) *For each $D \geqslant 6$ there exists a unique integer $k_0 \geqslant 2$ such that for each integer k:*

$$(k-1)(2k-1)^{D-2} > k^D \tag{A.1}$$

for $2 \leqslant k \leqslant k_0$ *and*

$$(k-1)(2k-1)^{D-2} < k^D \tag{A.2}$$

for $k > k_0$.

(2) $2^{D-3} \leqslant k_0 < 2^{D-2}$.

Proof. (1) Since both sides of inequality (A.1) are positive, we may take logarithms:

$$(k-1)(2k-1)^{D-2} > k^D \quad \Leftrightarrow \quad \log((k-1)(2k-1)^{D-2}) > \log(k^D).$$

Now, we consider the function $g(x) = \log((x-1)(2x-1)^{D-2}) - \log(x^D)$, and prove that exists an x', such that $g(x)$ is monotonic increasing for $2 \leqslant x \leqslant x'$ and monotonic decreasing for $x \geqslant x'$. Indeed, since

$$g'(x) = \frac{1}{(x-1)} + 2\,\frac{D-2}{2x-1} - \frac{D}{x} = \frac{-2x^2 + (D+3)x - D}{(x-1)(2x-1)x},$$

which is positive from $x = 2$ to some value $x' > 2$ and negative for $x > x'$, as we claimed. But $g(x)$ is positive at $x = 2$, since $3^{D-2} - 2^D$ is greater than zero for $D \geqslant 6$, and negative for large enough values of x. Thus, by continuity, k_0 is the integer part of the unique $x_0 > x'$ for which $g(x_0) = 0$. Finally, by parity, $(k-1)(2k-1)^{D-2} \neq k^D$ for any integer k.

(2) Since $(k-1)(2k-1)^{D-2} < k(2k)^{D-2} = 2^{D-2}k^{D-1}$ and $2^{D-2}k^{D-1} \leqslant k^D$ iff $k \geqslant 2^{D-2}$, then $k_0 < 2^{D-2}$. On the other hand, in order to prove that $k_0 \geqslant 2^{D-3}$, let us verify that $D < k \leqslant 2^{D-3}$ implies inequality (A.1). Indeed, inequality (A.1) is verified for $k > 0$ iff

$$2^{D-2}\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{2k}\right)^{D-2} > k.$$

Therefore, it will suffices to prove that $1/2 < (1 - 1/k)(1 - 1/(2k))^{D-2}$. But when $k > D \geqslant 6$ we have that

$$\left(1 - \frac{1}{k}\right)\left(1 - \frac{1}{2k}\right)^{D-2} > \left(1 - \frac{1}{D}\right)\left(1 - \frac{1}{2D}\right)^{D-2} \geqslant \left(1 - \frac{1}{6}\right)\left(1 - \frac{1}{12}\right)^{4},$$

where the last inequality holds because the function $(1-1/D)(1-1/2D)^{D-2}$ is (not obviously) monotonic increasing for $D \geqslant 6$. Finally $(1-1/6)(1-1/12)^4 > 0.5884 > 1/2$. □

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