UNILATERALLY CONNECTED LARGE DIGRAPHS AND GENERALIZED CYCLES

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ABSTRACT. Lower and upper bounds on the order of digraphs and generalized p-cycles with specified maximum degree and unilateral diameter are given for generic values of the parameters. Infinite families of digraphs attaining the bounds asymptotically or even exactly are presented. In particular, optimal results are proved for bipartite digraphs ($p = 2$) and digraphs with unilateral diameter 3.

Keywords: The (Δ, D)–problem; unilaterally connected digraph; Moore digraphs.

1. INTRODUCTION

The construction of the largest [directed] graphs with diameter at most D and maximum [out-] degree Δ has attracted considerable attention both from the graph-theoretical point of view and from the network-designer community; it is known as the (Δ, D)–problem (see [2]).

Besides, as we noted in [6], routing strategies may have different purposes, such as minimizing the length of the paths, minimizing congestion, simplifying routing tables or avoiding deadlock, among others. One of the strategies

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for preventing deadlock is to assign directions (up/down) to the links in the network in such a way that the resulting directed graph is acyclic. If routing tables only allow the use of a sequence of up links followed by a sequence of down links, it is easy to prove that deadlock is not possible. This is the case of the up/down routing in Autonet ([8]). This kind of routing suggests the study of directed graphs with some relaxation in the usual definition of distance between vertices.

In this paper, as a first approach, we deal with unilaterally connected digraphs. A digraph is said to be unilaterally connected with unilateral diameter D^* if for any pair of vertices u, v there exists a directed walk of length at most D^* from u to v or from v to u. According to this definition, the (Δ, D)–problem becomes what we call the (Δ, D^*)–problem, which consists of determining the largest order n^*_{Δ,D^*} of a digraph with unilateral diameter D^* and maximum in- and out-degree Δ.

In Section 3, we derive Moore–like upper bounds for n^*_{Δ,D^*} by counting the possible number of vertices at distance j from or to a fixed vertex. We do this for digraphs without restrictions, as well as for bipartite digraphs and generalized p-cycles (digraphs in which the length of any cycle is a multiple of p).

In Section 4, we study the existence of Moore digraphs, i.e. those attaining such Moore–like bounds. We show that the problem of classifying Moore bipartite digraphs with unilateral diameter 2 and arbitrary in- and out-degrees is equivalent to a difficult open problem in hypergraphs. However, for diameters 2 or 3, we prove that unilateral Moore bipartite digraphs do exist. Finally, we prove that there are no unilateral Moore generalized p-cycles with odd unilateral diameter p.

In Section 5, we study the (Δ, D^*)–problem for asymptotic values of the parameters, finding nontrivial constructions for unilateral diameters 2 and 3, and for p–cycles with p odd and unilateral diameter D^* with $(D^* \mod p) < \lfloor p/2 \rfloor$. With respect to the former, in the last section, we summarize some advantages of the techniques used to construct large unilateral digraphs.

2. Notation

A *directed graph* or *digraph* $G = (V, A)$ consists of a non empty set $V = V(G)$ of elements called *vertices* and a set $A = E(G)$ of ordered pairs of elements of V called *arcs*. The number of vertices $|G| = |V|$ is the *order* of the digraph. If (u, v) is an arc, we say that u *is adjacent to* v and that v *is adjacent from* u, and we write $u \rightsquigarrow v$. We also write uv instead of (u, v) whenever this does not lead to confusion, and we say that uv is *incident from* u and *to* v. An arc uu is called a *loop*.

Usually, for any subset U of vertices of $V(G)$, $\Gamma^+(U)$ denotes the set of vertices of G adjacent from a vertex of U. We define recursively:

$$\Gamma^k(U) = \begin{cases} \Gamma^+(U) & k = 1, \\ \Gamma^+(\Gamma^{k-1}(U)) & k \geqslant 2. \end{cases}$$

If $\Gamma^-(U)$ denotes the set of vertices adjacent to a vertex of U, we can define analogously $\Gamma^k(U)$ for $k < 0$. The cardinality of $\Gamma^+(u) = \Gamma^+(\{u\})$ $[\Gamma^-(u) = \Gamma^-(\{u\})]$ is the *out-degree* [*in-degree*] of u. We write $\Delta(G)$ for the maximum among the in- and out-degrees of the vertices of G. A digraph is *regular* of degree Δ or Δ-*regular* if its vertices have in- and out-degree Δ.

A sequence of vertices $u = u_0, u_1, \ldots, u_{n-1}, u_n = v$ of G, such that $u_{i-1}u_i$ is an arc of G, is called a *u-v walk* or a *directed walk from u to v* of *length* n. A u-v walk is *closed* when $u = v$ and it is a *cycle* when there is no other repetition. A cycle of length 2 is called a *digon*. We say that a directed walk *joins* u and v if it is a u-v or a v-u walk. In this work all walks will be directed, thus the word "walk" will stand for "directed walk".

A digraph is *strongly connected* if there is a walk from any vertex to any other, and it is *unilaterally connected* if there is a walk joining any pair of vertices. The *distance*, $\operatorname{dist}(u, v)$, between two vertices u and v is the number of arcs of a shortest u-v walk, and the *unilateral distance*, $\operatorname{dist}^*(u, v)$, between them is the minimum of $\operatorname{dist}(u, v)$ and $\operatorname{dist}(u, v)$. The *[unilateral] diameter* $[D^*(G)]$ $D(G)$ of a digraph G is the maximum of the [unilateral] distance function, that is:

$$D(G) = \max_{u,v \in V} \{\operatorname{dist}(u, v)\}$$

and

$$D^*(G) = \max_{u,v \in V} \{\operatorname{dist}^*(u, v)\} = \max_{u,v \in V} \{\min\{\operatorname{dist}(u, v), \operatorname{dist}(v, u)\}\ \}.$$

If G has vertex set $\{v_1, \ldots, v_n\}$ and *adjacent matrix* M, i.e. that whose entry i, j is 1 or 0 depending upon the vertex v_i is adjacent to the vertex v_j or not, respectively, then $D^*(G)$ is the minimum k such that

$$I + M + M^\top + \cdots + M^k + (M^\top)^k \geqslant J_n,$$

where J_n is the $n \times n$ matrix with all its entries equal to 1.

A generalized p–cycle is a digraph whose cycles have lengths that are a multiple of p. The vertex set of a generalized p–cycle can be partitioned in p disjoint sets, $V_0, \ldots, V_{p-1}$ in such a way that the vertices in the partite set V_i are adjacent only to the vertices in V_{i+1} where the last sum is reduced modulo p. In particular, a *bipartite digraph* is a generalized 2-cycle.

Finally, in the interests of completeness, we give the definition of an *r–uniform k–regular simple hypergraph*, which is a pair $(X, \{E_i\}_{i\in I})$ where $\{E_i\}_{i\in I}$ is a family of non empty subsets of X whose union is X and such that

- $E_i \subset E_j \Rightarrow i = j$ *(simple)*,
- $r = |E_i| \quad \forall i \in I$ *(r–uniform)*.
- $\forall x \in X,\ |\{i : x \in E_i\}| = k$ *(k–regular)*.

3. Moore–like upper bounds

In this section we state bounds to the largest order n^*_{Δ,D^*} of a digraph G with unilateral diameter D^* and maximum in- and out-degrees Δ. We split the section into three subsections: in the first, we consider digraphs having no restrictions, while in the last two, we treat bipartite digraphs and generalized p-cycle. Although bipartite digraphs are particular cases of generalized p-cycles, we deal with them in a deeper way, considering different in- and out-degrees.

3.1. **General case.** When no additional restrictions are assumed, n^*_{Δ,D^*} is upper-bounded as follows:

(3.1)
$$n^*_{\Delta,D^*} \leqslant \mathrm{M^*}_{\Delta,D^*} = 1+2\Delta+2\Delta^2+\dots 2\Delta^{D^*} = \begin{cases} 1+2\Delta\frac{\Delta^{D^*}-1}{\Delta-1} & \text{for } \Delta > 1, \\ 1+2D^* & \text{for } \Delta = 1. \end{cases}.$$

Indeed, for any digraph G with maximum degree Δ, the number of vertices that can be reached from a fixed vertex v in j steps is Δ^j; thus

$$\left|\Gamma^j(v)\right| \leqslant \Delta^{|j|} \qquad \forall j \in \mathbb{Z}, v \in G. \tag{3.2}$$

Therefore, if G has unilateral diameter D^*, then we have that

$$|G| = \left|\bigcup_{|j|\leqslant D^*} \Gamma^j(v)\right| \leqslant \sum_{|j|\leqslant D^*} \left|\Gamma^j(v)\right| \leqslant \sum_{|j|\leqslant D^*} \Delta^{|j|} = \sum_{j=-D^*}^{D^*} \Delta^{|j|} = \mathrm{M^*}_{\Delta,D^*}.$$

We call $\mathrm{M^*}_{\Delta,D^*}$ the *Moore bound for unilateral digraphs* and *unilateral Moore digraph* any digraph attaining that bound.

3.2. **Bipartite Case.** Let $G = (V_0 \bigcup V_1, A)$ be a bipartite digraph with partite sets V_0 and V_1. Suppose that vertices in the set V_0 [set V_1] have in- and out-degrees Δ_0^- and Δ_0^+ [Δ_1^- and Δ_1^+] respectively. A two-way count argument gives rise to the following equality:

$$\Delta_0^+\Delta_1^+ = \Delta_0^-\Delta_1^-. \tag{3.3}$$

Let us call $\hat{\Delta}$ the square root of that product, i.e. $\hat{\Delta} = \sqrt{\Delta_0^+\Delta_1^+}$. If G is unilaterally connected with odd unilateral diameter $D^* = 2m+1$, and v is a vertex of V_0, then the number of vertices of V_0 at unilateral distance at most D^* from v is upper-bounded by

(3.4)

$$1+\left(\Delta_0^+\Delta_1^+ +\cdots+(\Delta_0^+\Delta_1^+)^m\right)+\left(\Delta_0^-\Delta_1^- +\cdots+(\Delta_0^-\Delta_1^-)^m\right)=1+2\sum_{k=1}^{m}\hat{\Delta}^{2k}.$$

Symmetrically, if $v \in V_1$, the number of vertices of V_1 at unilateral distance at most D^* is not greater than

(3.5)

$$1+\left(\Delta_1^+\Delta_0^+ +\cdots+(\Delta_1^+\Delta_0^+)^m\right)+\left(\Delta_1^-\Delta_0^- +\cdots+(\Delta_1^-\Delta_0^-)^m\right)=1+2\sum_{k=1}^{m}\hat{\Delta}^{2k}.$$

On the other hand, when D^* is even, say $2m$, and v is a vertex of V_0 [of V_1], the number of vertices of V_1 [resp. V_0] at unilateral distance at most D^* from v is not greater than

$$\Delta_0^+ + \Delta_0^- + \cdots + \Delta_0^+(\hat{\Delta}^2)^{m-1} + \Delta_0^-(\hat{\Delta}^2)^{m-1} = (\Delta_0^+ + \Delta_0^-)\sum_{k=0}^{m-1}\hat{\Delta}^{2k}.$$

$$\left[\Delta_1^+ + \Delta_1^- + \cdots + \Delta_1^+(\hat{\Delta}^2)^{m-1} + \Delta_1^-(\hat{\Delta}^2)^{m-1} = (\Delta_1^+ + \Delta_1^-)\sum_{k=0}^{m-1}\hat{\Delta}^{2k}.\right]$$

Therefore, if $\vec{\Delta} = (\Delta_0^-, \Delta_1^-, \Delta_0^+, \Delta_1^+)$ and $\Delta_0^+\Delta_1^+ = \Delta_0^-\Delta_1^-$, then the largest order $n^*_{\vec{\Delta},D^*}$ of a bipartite digraph with unilateral diameter D^* and degrees $\Delta_0^-, \Delta_1^-, \Delta_0^+, \Delta_1^+$ is upper-bounded by

$$\text{(3.6)}\quad \mathrm{M}^*_{\vec{\Delta},D^*} = \begin{cases} 2D^*+1 & \text{if } D^* \text{ is odd and } \hat{\Delta}=1,\\ 2D^* & \text{if } D^* \text{ is even and } \hat{\Delta}=1.\\ 4(\hat{\Delta}^{D^*+1}-1)/(\hat{\Delta}^2-1)-2 & \text{if } D^* \text{ is odd and } \hat{\Delta}>1,\\ 4\bar{\Delta}(\hat{\Delta}^{D^*}-1)/(\hat{\Delta}^2-1) & \text{if } D^* \text{ is even and } \hat{\Delta}>1, \end{cases}$$

where $\bar{\Delta} = (\Delta_0^- + \Delta_0^+ + \Delta_1^- + \Delta_1^+)/4$. We call *unilateral Moore bipartite digraphs* the digraphs attaining that bound.

3.3. **Generalized p-cycles.** Let G be a generalized p-cycle with partite sets V_0, V_1, $\ldots$, V_{p-1} and unilateral diameter D^*. Note that $D^* \geqslant \lfloor p/2 \rfloor$, since the length of any walk joining a vertex in V_0 with another in $V_{\lfloor p/2 \rfloor}$ is at least $\lfloor p/2 \rfloor$. In order to simplify the formulae, let us only consider Δ–regular digraphs. Then, the cardinality of each partite set V_α is upper-bounded as follows:

$$|V_\alpha| \leqslant \min_{v \in V(G)} \sum_{|j| \leqslant D^*} \left|\Gamma^j(v) \cap V_\alpha\right| \leqslant \min_{i \in \mathbb{Z}_p} \min_{v \in V_i} \sum_{\substack{|j| \leqslant D^* \\ i+j \equiv \alpha \pmod p}} \left|\Gamma^j(v)\right| \leqslant$$

$$\leqslant \min_{i \in \mathbb{Z}_p} \min_{v \in V_i} \sum_{\substack{|j| \leqslant D^* \\ j \equiv \alpha - i \pmod p}} \Delta^{|j|} = \min_{k \in \mathbb{Z}_p} \sum_{\substack{|j| \leqslant D^* \\ j \equiv k \pmod p}} \Delta^{|j|}.$$

Where the first inequality holds because each vertex must be reached by any other in at most D^* steps forward or backward, while the second inequality holds because, for any two integers i and j, and any $v \in V_i$,

$$\Gamma^j(v) \subset V_{(i+j) \bmod p}. \tag{3.7}$$

Therefore, the largest order n^*_{p,Δ,D^*} of a generalized p-cycle with maximum in- and out-degrees Δ and diameter D^* is upper-bounded as follows:

$$n^*_{p,\Delta,D^*} \leqslant \mathrm{M}^*{}_{p,\Delta,D^*} = p \min_{k \in \mathbb{Z}_p} \sum_{\substack{|j| \leqslant D^* \\ j \equiv k \pmod p}} \Delta^{|j|}. \tag{3.8}$$

We call this upper bound the *Moore bound for unilateral generalized p-cycles* and *unilateral Moore generalized p-cycle* any digraph attaining it. For $\Delta = 1$ the unilateral Moore generalized p-cycles are directed cycles, thus we will assume $\Delta \geqslant 2$. We end this section by giving a closed form to the summation in (3.8)

Proposition 3.1. *If* $D^* = mp + r$ *with* $r = D^* \bmod p$ *and* $\Delta > 1$*, then*

$$\mathrm{M}^*{}_{p,\Delta,D^*} = \begin{cases} 2p\Delta^{\frac{p}{2}}\frac{\Delta^{mp}-1}{\Delta^p-1} & p \text{ even}\ ,\ r < \lfloor \frac{p}{2} \rfloor , \\ p(\Delta^{\frac{p-1}{2}} + \Delta^{\frac{p+1}{2}})\frac{\Delta^{mp}-1}{\Delta^p-1} & p \text{ odd}\ ,\ r < \lfloor \frac{p}{2} \rfloor , \\ p + 2p\Delta^p\frac{\Delta^{mp}-1}{\Delta^p-1} & r \geqslant \lfloor \frac{p}{2} \rfloor . \end{cases} \tag{3.9}$$

Proof. For each k we set

$$s_k = \sum_{\substack{|j| \leqslant D^* \\ j \equiv k \pmod p}} \Delta^{|j|}.$$

If $m = 0$, then $D^* = r$ and $s_k = 1$ or $2\Delta^k$ depending on whether or not $k = 0$. Thus $\min s_k = 1$ as stated in (3.9) since $r = D^* \geqslant \lfloor p/2 \rfloor$. For $m > 0$, we split the above sum in two, depending on whether j is positive or not.

$$s_k = \sum_{\substack{j=hp+k \leqslant D^* \\ h \geqslant 0}} \Delta^j + \sum_{\substack{-j=hp-k \leqslant D^* \\ h \geqslant 1}} \Delta^{-j} = \sum_{h=0}^{m-1+a} \Delta^{hp+k} + \sum_{h=1}^{m+b} \Delta^{hp-k},$$

where $a = 1$ if $k \leqslant r$ and 0 otherwise, and $b = 1$ if $k + r \geqslant p$ and 0 otherwise. Thus we can distinguish four cases:

(1) If $k \leqslant r$ and $k + r \geqslant p$, then $s_k = (\Delta^k + \Delta^{p-k})(1 + \cdots + \Delta^{mp})$ and reaches its minimum at $k = \lfloor p/2 \rfloor$ which is $s_{(1)} = (\Delta^{\lfloor p/2 \rfloor} + \Delta^{\lceil p/2 \rceil})(1 + \cdots + \Delta^{mp})$.

(2) If $k \leqslant r$ and $k + r < p$, then $s_k = \Delta^k + (\Delta^k + \Delta^{-k})(\Delta^p + \cdots + \Delta^{mp})$ and reaches its minimum at $k = 0$ which is $s_{(2)} = 1 + 2(\Delta^p + \cdots + \Delta^{mp})$.

(3) If $k > r$ and $k + r \geqslant p$, then s_k coincides with s_{p-k} of case 2, thus the minimum $s_{(3)}$ of s_k in this case is greater than or equal to $s_{(2)}$.

(4) If $k > r$ and $k + r < p$, then $s_k = (\Delta^k + \Delta^{p-k})(1 + \cdots + \Delta^{(m-1)p})$, which reaches its minimum at $k = \lfloor p/2 \rfloor$ which is $s_{(4)} = (\Delta^{\lfloor p/2 \rfloor} + \Delta^{\lceil p/2 \rceil})(1 + \cdots + \Delta^{(m-1)p})$.

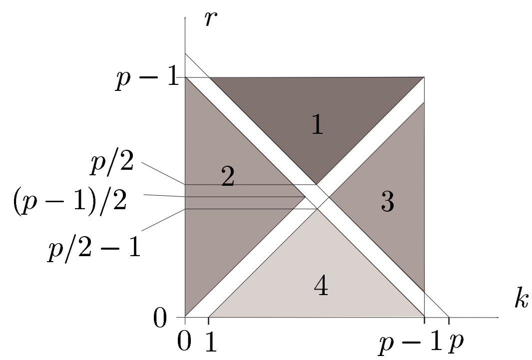

FIGURE 1.

Thus $s_{(4)} \leqslant s_{(2)} \leqslant s_{(3)}, s_{(1)}$ and (see Figure 1) s_k reaches its minimum at $k = \lfloor p/2 \rfloor$ if $r < \lfloor p/2 \rfloor$, (case 4), or at $k = 0$ if $r \geqslant \lfloor p/2 \rfloor$ (case 2), and we have that

$$\min_{k\in\mathbb{Z}_p} s_k = \begin{cases} (\Delta^{\lfloor p/2 \rfloor} + \Delta^{\lceil p/2 \rceil})(1 + \cdots + \Delta^{(m-1)p}) & \text{if } r < \lfloor p/2 \rfloor, \\ 1 + 2(\Delta^p + \cdots + \Delta^{mp}) & \text{otherwise.} \end{cases}$$

From which (3.9) follows readily. □

Note that for $p = 2$ and $\vec{\Delta} = (\Delta, \Delta, \Delta, \Delta)$ we obtain (3.6).

4. Unilateral Moore digraphs

Unlike what happens in the directed and undirected contexts for which the characterizations of Moore graphs are well known (except for $(\Delta, D) = (57, 2)$ in the undirected context), the classification of such digraphs in the unilateral context seems to be difficult even for small values of the parameters. Indeed, for $D^* = 1$, unilateral Moore digraphs are the regular tournaments on $2\Delta + 1$ vertices, but these digraphs are not completely known and are still a subject of research ([4]). The only really trivial case is for $\Delta = 1$, for which unilateral Moore digraphs are the cycles on $2D^* + 1$ vertices. In [6], we prove that there is no unilateral Moore digraph for $\Delta = 2, D^* = 2$, and we present a digraph with 12 vertices (thus, optimal for those values of the parameters). Apart from this no other results are known.

4.1. **Bipartite case.** If a unilateral Moore bipartite digraph with unilateral diameter 2 exists, then its underlying graph must be a bipartite complete graph, because each vertex in one of the partite sets reaches any vertex in the other partite set in 1 or "-1" step. However, the converse is not true, and we need an additional condition besides the non–existence of digons.

Theorem 4.1. *A bipartite digraph G with degrees Δ_0^-, Δ_0^+, Δ_1^- and Δ_1^+ is a unilateral bipartite Moore digraph with unilateral diameter 2 if and only if it has no digons, its underlying graph is a complete bipartite graph and there is no pair of vertices u and v such that $\Gamma^+(u) = \Gamma^+(v)$.*

Proof. Suppose that G is a unilateral Moore bipartite digraph with unilateral diameter 2 and let u and v be two different vertices. If u and v belong to different partite sets, then $\mathrm{dist}^*(u, v) < 2$, and thus $\mathrm{dist}(u, v) = 1$ or $\mathrm{dist}(v, u) = 1$, that is $u \rightsquigarrow v$ or $v \rightsquigarrow u$, which implies that UG is a complete bipartite graph. Besides, if u, v, u is a digon, then the order of G is at most $\Delta_0^- + \Delta_0^+ + \Delta_1^- + \Delta_1^+ - 2$, contradicting the assumption that G is a Moore bipartite digraph. If u and v belong to the same partite set, then $\mathrm{dist}^*(u, v) = 2$, and there exists a vertex w such that either u, w, v or v, w, u is a walk. Since G has no digons, then $w \in \Gamma^+(u) \triangle \Gamma^+(v)$ and $\Gamma^+(u) \neq \Gamma^+(v)$ (where $\triangle$ stands for the symmetric difference between sets, i.e. $A \triangle B = (A \setminus B) \cup (B \setminus A)$).

Conversely, if the underlying graph of a bipartite digraph G is complete, then the order of G is $|\Gamma^+(u) \cup \Gamma^-(u)| + |\Gamma^+(v) \cup \Gamma^-(v)|$ for any two vertices u and v taken from different partite sets. Now, if G has no digons, the previous unions are disjoint and the order of G becomes equal to $|\Gamma^+(u)| + |\Gamma^-(u)| + |\Gamma^+(v)| + |\Gamma^-(v)| = \Delta_0^- + \Delta_0^+ + \Delta_1^- + \Delta_1^+$ which is exactly $\mathrm{M}^*_{\vec{\Delta},D^*}$ for $D^* = 2$. The completeness of UG also implies that the unilateral distance between vertices in different partite sets is 1. Finally, if two different vertices u and v are in the same partite set, since $\Gamma^+(u) \neq \Gamma^+(v)$, there exists $w \in \Gamma^+(u) \triangle \Gamma^+(v)$. Therefore, either u, w, v or v, w, u is a walk, which implies that the unilateral distance between u and v is 2. □

This result allows us to relate the existence of unilateral bipartite Moore digraphs with $D^* = 2$ with the existence of a special kind of hypergraph, as shown in the following corollary.

Corollary 4.2. *A bipartite digraph G is a unilateral Moore bipartite digraph with diameter 2 and degrees Δ_0^-, Δ_0^+, Δ_1^-, Δ_1^+ if and only if the hypergraph defined by*

$$H^{0+} = (V_0, \{\Gamma^+(u)\}_{u \in V_1})$$

is a Δ_1^+–uniform Δ_0^-–regular simple hypergraph.

The existence of such hypergraphs is a particular case of a known (and, to the best of our knowledge open) problem in the theory (see the *open problem* in [1, p. 5]). However, we can construct such digraphs for special values of the parameters. To this end, let us introduce some basic facts about circulant matrices.

An $n\times n$ matrix M is said to be *circulant* if its entries satisfy the equality $m_{i,j} = m_{1,j-i+1}$, where the subscripts are reduced modulo n and lie in the set $\{1,\dots,n\}$. Let W_n denote the $n \times n$ circulant matrix whose first row is $[0,1,0,\dots,0]$. Thus the following basic equalities hold: $W_n^n = I$, $W_n^\top = W_n^{-1} = W_n^{n-1}$ and

$$\mathcal{G}_n(W_n) = J_n$$

where $\mathcal{G}_n(x)$ denotes the polynomial $1+x+\cdots+x^{n-1}$. Furthermore, if j_n denotes any column of J_n, then j_n and $j_n^\top$ are the only right and left eigenvector of W_n with eigenvalue 1. Thus, for any polynomial $q(x)$, $q(W_n)j_n = q(1)j_n$ and $j_n^\top q(W_n) = q(1)j_n^\top$.

In what follows, if $A = (a_{i,j})$ and $B = (b_{i,j})$ are two matrices we will say that $A \leqslant B$ if $a_{i,j} \leqslant b_{i,j}$ for all i,j. We also say that if $p(x) = a_0+\cdots+a_nx^n$ and $q(x) = b_0 + \cdots + b_nx^n$ are two polynomials, then $p(x) \leqslant q(x)$ if $a_i \leqslant b_i$ for all i. Note that if $p(x) \leqslant q(x)$, then $p(A) \leqslant q(A)$ for any given matrix A.

Proposition 4.3. *For any positive integer Δ there exists a unilateral Moore bipartite Δ–regular digraph with unilateral diameter 2.*

Proof. If $W = W_{2\Delta}$, let G be a digraph with adjacency matrix

$$M = \begin{pmatrix} 0 & \mathcal{G}_\Delta(W) \\ W^\Delta \mathcal{G}_\Delta(W^{-1}) & 0 \end{pmatrix}.$$

First, we check that G is Δ–regular. Indeed, both $\mathcal{G}_\Delta(W)$ and $W^\Delta\mathcal{G}_\Delta(W^{-1})$ are circulant matrices whose rows have exactly Δ entries equal to $+1$, since the polynomials $\mathcal{G}_\Delta(x)$ and $x^\Delta\mathcal{G}_\Delta(x^{2\Delta-1})$ have exactly Δ not null terms and all of them have coefficient 1. Besides, if $X = \mathcal{G}_\Delta(W) + (W^\Delta\mathcal{G}_\Delta(W^{-1}))^\top$, then

$$M + M^\top = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}.$$

But $(W^\Delta)^\top = (W^\top)^\Delta = (W^{-1})^\Delta = W^{-\Delta} = W^\Delta$, thus

$$X = \mathcal{G}_\Delta(W) + W^\Delta \mathcal{G}_\Delta(W) = \mathcal{G}_{2\Delta}(W) = J_{2\Delta}, \tag{4.1}$$

which implies the regularity.

Next, in order to prove that the unilateral diameter is 2, we will verify that $A = I + M + M^\top + M^2 + (M^\top)^2 \geqslant J_{4\Delta}$. First note that, by (4.1)

$$A = \begin{pmatrix} r(W) & J_{2\Delta} \\ J_{2\Delta} & r(W) \end{pmatrix},$$

for some polynomial $r(x)$. Thus it suffices to prove that $r(x) \geqslant \mathcal{G}_{2\Delta}(x)$. Now, the corresponding matrix and polynomial computations results in:

$$r(x) = 1 + 2x^\Delta \mathcal{G}_\Delta(x)\mathcal{G}_\Delta(x^{-1}).$$

But, $\mathcal{G}_\Delta(x^{-1}) = x^{1-\Delta}\mathcal{G}_\Delta(x)$ and $\mathcal{G}_\Delta(x)^2 \geqslant \mathcal{G}_{2\Delta-1}(x)$, thus

$$r(x) = 1 + 2x\mathcal{G}_\Delta(x)^2 \geqslant 1 + 2x\mathcal{G}_{2\Delta-1}(x) \geqslant 1 + x\mathcal{G}_{2\Delta-1}(x) = \mathcal{G}_{2\Delta}(x) = \mathcal{G}_n(x).$$

□

For unilateral diameter 3, a similar result can be obtained. In this case we will consider any possible values for the degrees Δ_0^+, Δ_0^-, Δ_1^+, Δ_1^-. However, since the partite sets of a unilateral Moore bipartite digraph with odd unilateral diameter have the same cardinality (equations (3.4) and (3.4)), we have that $\Delta_0^+ = \Delta_1^-$ and $\Delta_1^+ = \Delta_0^-$, and we can consider only two parameters.

Proposition 4.4. *Given two positive integers Δ_0 and Δ_1, there exists a unilateral Moore bipartite digraph with unilateral diameter 3 and maximum out–degrees Δ_0 and Δ_1.*

Proof. Let us call $p(x)$ and $q(x)$ the polynomials $\mathcal{G}_{\Delta_0}(x)$ and $x^{\Delta_1+1}\mathcal{G}_{\Delta_1}(x^{2\Delta_0})$ respectively. Then, if $n = 2\Delta_0\Delta_1 + 1$ and $W = W_n$, let G be a digraph with adjacency matrix

$$M = \begin{pmatrix} 0 & p(W) \\ q(W) & 0 \end{pmatrix}.$$

In order to check the condition on the degrees, we proceed as in Proposition 4.3, noticing that both $p(W)$ and $q(W)$ are circulant matrices whose rows have exactly Δ_0 and Δ_1 entries equal to 1 respectively, since $p(x)$ and $q(x)$ have, respectively, Δ_0 and Δ_1 no null terms and all of them have coefficient 1. Besides, $j_n^\top p(W) = p(1)j_n^\top = \Delta_0 j_n^\top$ and $j_n^\top q(W) = q(1)j_n^\top = \Delta_1 j_n^\top$ which implies the regularity.

Next, in order to prove that the unilateral diameter is 3, we will verify that $A = I + M + M^\top + M^2 + (M^\top)^2 + M^3 + (M^\top)^3 \geqslant J_{2n}$. A matrix computation results in:

$$A = \begin{pmatrix} I + pq + p^\top q^\top & p + q^\top + p^2 q + p^\top (q^\top)^2 \\ p^\top + q + (p^\top)^2 q^\top + pq^2 & I + pq + p^\top q^\top \end{pmatrix},$$

where p and q stand for $p(W)$ and $q(W)$ respectively. Thus it suffices to prove that

(1) $I + pq + p^\top q^\top = J_n$ (since G is a Moore digraph),
(2) $p + q^\top + p^2 q + p^\top (q^\top)^2 \geqslant J_n$.

In order to prove (1), we will restrict the matrices to the orthogonal subspace $j_n^\perp$ of j_n. Note that since the eigenvalues of W are the n-th root of the unity, and since $2\Delta_0$ does not divide $n = 2\Delta_0\Delta_1 + 1$, the eigenvalues of W^{Δ_0} are all different to -1. Then, the following equalities are valid in $j_n^\perp$:

$$pq = \frac{W^{\Delta_0} - I}{W - I} W^{\Delta_0+1} \frac{W^{2\Delta_0\Delta_1} - I}{W^{2\Delta_0} - I} = \frac{I}{W - I} W^{\Delta_0+1} \frac{W^{-1} - I}{W^{\Delta_0} + I} = \frac{-W^{\Delta_0}}{W^{\Delta_0} + I}.$$

And therefore

$$p(W)^\top q(W)^\top = p(W^{-1})q(W^{-1}) = \frac{-W^{-\Delta_0}}{W^{-\Delta_0}+I} = \frac{-I}{W^{\Delta_0}+I}.$$

Thus $I + pq + p^\top q^\top = 0$ in $j_n^\top$, but $(I + pq + p^\top q^\top)j_n = [1 + p(1)q(1) + p(1)q(1)]j_n = (1 + 2\Delta_0\Delta_1)j_n$, which implies (1), since J_n is the only matrix verifying these two properties, i.e., having j_n as a eigenvector with eigenvalue 1 and having $j_n^\perp$ as its kernel.

In order to prove (2), we note that $q(x^{-1}) = x^{\Delta_0 - n}\mathcal{G}_{\Delta_1}(x^{2\Delta_0})$, and that, $p(x)^2 \geqslant \mathcal{G}_{2\Delta_0-1}(x)$ thus if

$$r(x) = \mathcal{G}_{\Delta_0}(x) + x^{\Delta_0}\mathcal{G}_{\Delta_1}(x^{2\Delta_0}) + \mathcal{G}_{2\Delta_0-1}(x)x^{\Delta_0+1}\mathcal{G}_{\Delta_1}(x^{2\Delta_0}),$$

then $p + q^\top + p^2 q \geqslant r(W)$. But

$$\begin{aligned} r(x) &= \mathcal{G}_{\Delta_0}(x) + x^{\Delta_0}\mathcal{G}_{\Delta_1}(x^{2\Delta_0})(1 + x\mathcal{G}_{2\Delta_0-1}(x)) \\ &= \mathcal{G}_{\Delta_0}(x) + x^{\Delta_0}\mathcal{G}_{\Delta_1}(x^{2\Delta_0})\mathcal{G}_{2\Delta_0}(x) \\ &= \mathcal{G}_{\Delta_0}(x) + x^{\Delta_0}\mathcal{G}_{2\Delta_0\Delta_1}(x) = \mathcal{G}_{2\Delta_0\Delta_1+\Delta_0}(x) \end{aligned}$$

and $\mathcal{G}_{2\Delta_0\Delta_1+\Delta_0}(x) \geqslant \mathcal{G}_{2\Delta_0\Delta_1+1}(x)$. Hence, $p + q^\top + p^2 q \geqslant \mathcal{G}_{2\Delta_0\Delta_1+1}(W) = \mathcal{G}_n(W) = J_n$. □

Figure 2 shows the digraph of Proposition 4.4 with $\Delta_0 = \Delta_1 = 2$.

4.2. **Generalized p–cycles.** For $p > 2$, we could not find unilateral Moore p–generalized cycles. Instead, we can prove that they do not exist for odd p and $D^* = p$.

Proposition 4.5. *There is no unilateral Moore generalized p–cycle digraph with odd unilateral diameter p (and maximum degree $\Delta > 1$).*

Proof. Suppose that G is a unilateral Moore generalized p–cycle with odd unilateral diameter p and partite sets $V_0, \ldots, V_{p-1}$. Let u be a vertex in V_0 and v a vertex in $\Gamma^+(u)$. Then, set $A = \Gamma^+(u) \setminus \{v\}$, $B = \Gamma^-(v) \setminus \{u\}$

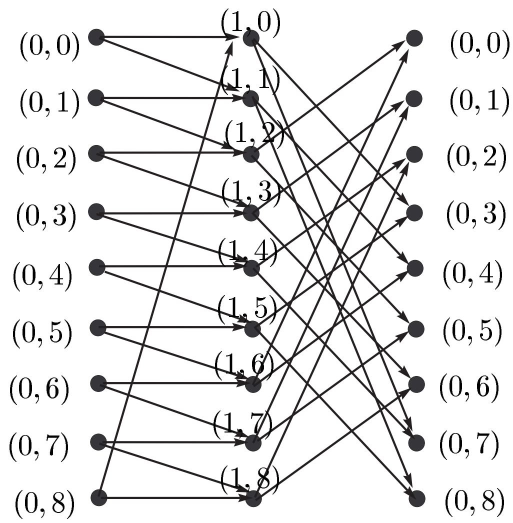

FIGURE 2. A unilateral Moore (2,3)–bipartite digraph.

and $M = \lfloor p/2 \rfloor$. With these choices, and since G is a unilateral Moore generalized p–cycle with odd unilateral diameter p, the following equations hold,

$$V_{M+1} = \Gamma^{M+1}(u) \cup \Gamma^{-M}(u) = \Gamma^{M}(v) \cup \Gamma^{M}(A) \cup \Gamma^{-M}(u),$$

$$V_{M+1} = \Gamma^{M}(v) \cup \Gamma^{-(M+1)}(v) = \Gamma^{M}(v) \cup \Gamma^{-M}(B) \cup \Gamma^{-M}(u),$$

where all the unions are disjoint. Thus we have that,

$$\Gamma^{M}(A) = \Gamma^{-M}(B).$$

Hence, for any $c \in \Gamma^{-M+1}(B)$, all the vertices adjacent to c will be in $\Gamma^{M}(A)$, i.e.

$$\Gamma^{-1}(c) \subset \Gamma^{M}(A) = \bigcup_{a \in A} \Gamma^{M}(a).$$

But $\left|\Gamma^{-1}(c)\right| = \Delta > \Delta - 1 = |A|$, thus, by the pigeonhole principle, there are two vertices $w_1, w_2 \in \Gamma^{-1}(c)$ and a vertex $a \in A$ such that $w_1, w_2 \in \Gamma^{M}(a)$. But this implies that there exist two different walks of length $M + 1 =$

$\lfloor p/2 \rfloor + 1$ from vertex a to vertex c, which is impossible in a unilateral Moore digraph like G. □

Corollary 4.6. *For any positive integers $p = 2k+1$ and Δ,*

$$n^*_{p,\Delta,p} \leqslant p\Delta^{k+1} + p\Delta^k - 1.$$

□

5. Asymptotic lower bounds

In order to study the unilateral (Δ, D^*)–problem when Δ goes to infinity, let us summarize the bounds (3.1) and (3.9) of Section 3:

- General case: $n^*_{\Delta,D^*} \leqslant 2\Delta^{D^*} + \mathcal{O}(\Delta^{D^*-1})$
- Bipartite case: $n^*_{2,\Delta,D^*} \leqslant 4\Delta^{D^*-1} + \mathcal{O}(\Delta^{D^*-2})$
- Generalized p-cycle case: if $D^* = mp + r$ with $r = D^* \bmod p$

$$n^*_{p,\Delta,D^*} \leqslant \begin{cases} 2p\Delta^{mp-p/2} + \mathcal{O}(\Delta^{mp-3p/2}) & p \text{ even },\ r < \left\lfloor \frac{p}{2} \right\rfloor, \\ p\Delta^{mp-(p-1)/2} + \mathcal{O}(\Delta^{mp-3p/2+1/2}) & p \text{ odd },\ r < \left\lfloor \frac{p}{2} \right\rfloor, \\ 2p\Delta^{mp} + \mathcal{O}(\Delta^{mp-p}) & r \geqslant \left\lfloor \frac{p}{2} \right\rfloor. \end{cases}$$

5.1. **General Case.** First of all, let us point out that the Kautz digraphs $\mathrm{K}(\Delta, D)$ have unilateral diameter D, since the unilateral diameter is always at most the diameter, and, with an alphabetic notation (see [6]), the vertices $u = 0101\ldots$ and $v = 2121\ldots$ are at unilateral distance D. Besides, this digraph is Δ-regular and its order is $\Delta^D + \Delta^{D-1}$, thus $n^*_{\Delta,D^*} \geqslant \Delta^{D^*} + \Delta^{D^*-1}$. That is asymptotically half the unilateral Moore bound $\mathrm{M}^*{}_{\Delta,D^*}$.

In [6], we proved that there exist families of Δ-regular digraphs with orders $1.5\Delta^2$ and $1.25\Delta^3$ for $D^* = 2, 3$ respectively, by making the directed product of a suitable digraph with some De Bruijn digraphs. More precisely, if G is a Δ_1–regular digraph which is *k–unilaterally reachable*, i.e., such that for any two vertices u and v (not necessarily different) there exists a walk joining u and v in exactly k steps, then the directed product $G \times \mathrm{B}(\Delta_2, k)$ of G and the De Bruijn digraph $\mathrm{B}(\Delta_2, k)$ of maximum degree Δ_2 and diameter k is a $\Delta_1\Delta_2$–regular digraph with unilateral diameter k and order $|G|\,\Delta_2^k$. Therefore, if G has order larger than Δ_1^k, there is a family of digraphs with unilateral diameter k and orders asymptotically larger than the orders of the corresponding Kautz digraphs. We obtained the previous two bounds by finding 2-regular digraphs which are 2 and 3–unilaterally reachable and have 6 and 10 vertices respectively.

Putting these results together, we have that:

$$n^*_{\Delta,D^*} \geqslant \begin{cases} 1.5\Delta^2 & \text{if, } D^* = 2, \\ 1.25\Delta^3 & \text{if, } D^* = 3, \\ \Delta^{D^*} & \text{if, } D^* \geqslant 4. \end{cases}$$

5.2. **Bipartite and Generalized p-cycles.** The work [7] introduces a family of generalized p-cycles, called $\mathrm{BGC}(p, \Delta, n)$, with vertex set $\mathbb{Z}_p \times \mathbb{Z}_n$ and the following adjacency rules:

$$(\alpha,\, i) \rightsquigarrow (\alpha + 1,\, \Delta i + t) \qquad \text{for } t = 0, \ldots, \Delta - 1.$$

Some instances of these digraphs have asymptotically optimal orders for p odd and $r < \lfloor p/2 \rfloor$, as we stated as a corollary of the following proposition.

Proposition 5.1. *The unilateral diameter of* $\mathrm{BGC}\left(p, \Delta, \Delta^h\right)$ *is at most* $h + \left\lfloor \frac{p}{2} \right\rfloor$ *for* $h \equiv 0, \left\lceil \frac{p}{2} \right\rceil \mod p$.

Proof. First of all, note that for any $n, j \geqslant 0$, the set of vertices at distance at most j from a vertex (α, i) in BGC (p, Δ, n) is

$$\begin{aligned}\Gamma^j(\alpha,i) =& \{(\alpha+j, \Delta^j i + \Delta^{j-1}t_{j-1} + \cdots + \Delta t_1 + t_0) : \quad 0 \leqslant t_i < \Delta\},\\ =& \{(\alpha+j, \Delta^j i + x) : \qquad 0 \leqslant x < \Delta^j\}.\end{aligned}$$

Therefore the cardinality of $\Gamma^j(\alpha,i)$ is $\left|\Gamma^j(\alpha,i)\right| = \min(n, \Delta^j)$ and, if $j = h$ and $n = \Delta^h$, we have that $\left|\Gamma^h(\alpha,i)\right| = n$ and then, $\Gamma^h(\alpha,i) = V_{\alpha+h \mod p}$. Consequently, since any vertex is adjacent from another, we have that:

$$\Gamma^{h+k}(\alpha,i) = V_{\alpha+h+k \mod p} \qquad k = 0, 1, \ldots, \lfloor p/2 \rfloor . \tag{5.1}$$

It remains to be proved that for any $1 \leqslant k' \leqslant \lfloor p/2 \rfloor$ the vertex (α, i) reaches the vertices in $V_{\alpha+h-k'}$ forward or backward in at most $h + \lfloor p/2 \rfloor$ steps. In fact, we will see that the vertex (α, i) reaches the vertices in $V_{\alpha+h-k'}$ backward, or equivalently, that any given vertex $i' \in V_{\alpha+h-k' \mod p}$ reaches (α, i) forward in at most $h + \lfloor p/2 \rfloor$ steps. Indeed, by (5.1) we have that

$$\Gamma^{h+k}(\alpha+h-k', i') = V_{\alpha+2h-k'+k \mod p}$$

for any $0 \leqslant k \leqslant \lfloor p/2 \rfloor$. But, $\alpha + 2h - k' + k \equiv \alpha \mod p$ if we take $k = k' - \epsilon$ and

$$\epsilon = \begin{cases} 1 & \text{if } p \text{ is odd and } h \equiv \lceil p/2 \rceil \mod p\ , \\ 0 & \text{otherwise.} \end{cases}$$

□

Corollary 5.2. *If $l = mp + r$ with $r = l \bmod p$ and*

$$h_{l,p} = \begin{cases} mp - p/2 & p \ even\ ,\ r < \left\lfloor \frac{p}{2} \right\rfloor , \\ mp - (p-1)/2 & p \ odd\ ,\ r < \left\lfloor \frac{p}{2} \right\rfloor , \\ mp & r \geqslant \left\lfloor \frac{p}{2} \right\rfloor , \end{cases}$$

then BGC $\left(p, \Delta, \Delta^{h_{l,p}}\right)$ *has unilateral diameter l and order $p\Delta^{h_{l,p}}$.* □

These digraphs are asymptotically optimal for p odd and $r < \lfloor p/2 \rfloor$. For the other values of the parameters, their orders are asymptotically half the corresponding Moore bound.

6. Conclusions

We have shown that the unilateral (Δ, D^*)–problem is not trivial even for small values of the parameters. As in the undirected case, no general techniques, such as the line digraph in the directed context, seem to exist (see [5] for a discussion of the disadvantages of the line digraph technique in this context). However, for some particular cases, direct products and voltage graphs have been useful. As mentioned in Section 5.1, the direct product of certain classes of digraphs gives rise to families of asymptotically large unilateral digraphs. The method works because the construction of a k-unilaterally reachable Δ-regular digraph leads to a complete infinite family of large digraphs with the same unilateral diameter and maximum degree being a multiple of Δ. Although we only know of two such digraphs, new results may be obtained by means of a computer search.

Voltage graphs give good results [3] in the construction of large graphs with fixed small degree and diameter. In fact we used a simplified version of voltage graphs in [5], obtaining optimal and quasi-optimal results for unilateral diameter 2 and maximum degrees 2, 3 and 4. However, this approach does not seem to work neither for maximum degrees larger than 4 nor for unilateral diameter larger than 2.

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