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Asymptotically large (Δ, D)-graphs

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Abstract

Graphs with maximum degree Δ, diameter D and orders greater than $(\Delta/\alpha)^D$, for a constant $\alpha < 2$, are proved to exist for infinitely many values of Δ and for D larger than a fixed value.
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Keywords: Degree–diameter problem; De Bruijn digraph; Kautz digraph; Exchange graph; Graph–digraph product

1. Introduction

The problem of finding the largest order $n_{\Delta,D}$, among (Δ, D)-graphs, i.e. graphs with maximum degree Δ and diameter D, has attracted considerable attention from the graph-theoretical point of view, as well as from the network-designers community, and it is known as the (Δ, D)-problem (see [2,16]). An upper bound on $n_{\Delta,D}$ derived by counting the maximum possible number of vertices at a fixed distance from a given one is the *Moore bound* $M_{\Delta,D} = 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1}$. Besides the trivial cases ($D = 1$ or $\Delta = 2$), the bound can be attained in two cases ($D = 2$ and $\Delta = 3, 7$) and maybe in a third, which is still open ($D = 2$ and $\Delta = 57$), but for the other values of the parameters the bound cannot be attained (see [6]). Except for a few more cases, and even for small values

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of Δ and D, the largest known graphs (maintained in [8]) have orders far below the Moore bound. One might still wonder whether Δ or D goes to infinity, or if there exist graphs with orders asymptotically equivalent to $M_{\Delta,D}$. The question was answered affirmatively in [4] by means of probabilistic methods, for D going to infinity and fixed Δ. However, for fixed diameter, the question, posed by Bollobás ([3, Ch. IV, p. 8]), remains open, except for $D = 2, 3, 5$ (see [10]). However, the best known large graphs for large values of D have orders of the form

$$k\left(\frac{\Delta}{2}\right)^D + \mathrm{o}(\Delta^D),$$

where $k = 2, 3, 5$ depending on D (see [15]). To the best of our knowledge, only one class of larger graphs, for infinitely many values of the parameters, has been found, namely the generalized compound graphs introduced in [12]. They can be built as a particular case of the construction that we will present in Section 3. This construction enables us to prove (Theorem 7) that there exist constants $\alpha < 2$ and D_0, such that for each $D > D_0$ there exists a sequence of (Δ_n, D)-graphs with $\Delta_n \longrightarrow \infty$ and orders greater than

$$\left(\frac{\Delta_n}{\alpha}\right)^D.$$

Finally, in Section 4, we present a second construction that improves the value of the constant α for diameters congruent with -1, 0 and 1 modulo 6, which is an extension of a work presented in a seminar by one of the authors [13].

2. Notation and basic facts

If $G = (V, E)$ is a graph with vertex set $V = \mathrm{V}(G)$ and edge set $E = \mathrm{E}(G)$, we denote its order by $|G|$. Given two adjacent vertices u and v joined by an edge uv, we write $u \overset{G}{\sim} v$ or simply $u \sim v$. Analogously, if $H = (V, A)$ is a directed graph (or digraph for short) with vertex set $V = \mathrm{V}(H)$ and arc set $E = \mathrm{E}(G)$, we denote its order by $|H|$, and we will write by $u \overset{H}{\rightsquigarrow} v$ or simply $u \rightsquigarrow v$ if u is adjacent to v.

We denote by $\mathrm{U}H$, the underlying graph of H. Conversely, we call the *symmetric looped digraph* of a graph G the digraph obtained from G by replacing each edge by two opposite arcs and adding a loop to each vertex, i.e., the digraph $(\mathrm{V}(G), \{(u, v) : \{u, v\} \in \mathrm{E}(G)\} \cup \{(u, u) : u \in \mathrm{V}(G)\})$. A digraph H is *symmetric* if whenever uv is an arc of H, then vu is also an arc of H.

If $u = u_0, u_1, \ldots, u_{n-1}, u_n = v$ is a u–v walk, then we say that v belongs to $\Gamma^n(u)$ and u belongs to $\Gamma^{-n}(v)$ (we omit the n when $n = 1$). It is said that a directed or undirected graph with vertex set V is *k-reachable* if $\Gamma^k(u) = V$ for any vertex u. Notice that any k-reachable graph (with more than one vertex) is k'-reachable for any $k' > k$ as well. We will denote by $\mathrm{d}_G(u, v)$, or simply $\mathrm{d}(u, v)$, the distance from u to v, and by $\mathrm{D}(G)$ the diameter of G. A related concept is the *unilateral diameter* of a digraph, which is the minimum integer D such that, for any two vertices u and v, $\min(\mathrm{d}(u, v), \mathrm{d}(v, u)) \leqslant D$.

Recall that, if $\delta_H^-(v)$ and $\delta_H^+(v)$ are the in and out-degrees of a vertex v, then

$$\sum_{v\in H}\delta_H^+(v)=\sum_{v\in H}\delta_H^-(v)=|\mathrm{A}(H)|. \tag{1}$$

We denoted by $\Delta(H)$ the maximum among the in and out-degrees, i.e.

$$\Delta(H)=\max_{v\in H}\max(\delta_H^+(v),\delta_H^-(v)).$$

Analogously, the maximum degree of a graph G will be denoted by $\Delta(G)$. A (Δ, D)-*[di]graph* is a [di]graph with maximum degree Δ and diameter D. Notice that if G is a (Δ, D)-graph then its symmetric looped digraph is a D-reachable $(\Delta+1, D)$-digraph. A [di]graph all of whose vertices have the same [in and out-]degree Δ is called *Δ-regular*.

The *line digraph* LH of a digraph H has as vertices the arcs of H and as arcs the pairs of adjacent arcs of H, i.e., $\mathrm{V}(LH)=\mathrm{A}(H)$ and $\mathrm{A}(LH)=\{(uv,vw) : uv, vw\in \mathrm{A}(H)\}$. When H is Δ-regular with $\Delta\geqslant 2$, this operator verifies the following important properties:

- LH is Δ-regular,
- $\mathrm{D}(LH)=\mathrm{D}(H)+1$,
- LH is $(k+1)$-reachable if H is k-reachable.

The first two properties allow us to iterate L to obtain large (Δ, D)-digraphs, since if $\Delta\geqslant 2$ and H is a Δ-regular digraph with diameter D and order n, then L^kH is a Δ-regular digraph with diameter $D+k$ and order $n\Delta^k$. Good examples of this are the two well-known families of iterated line digraphs called *De Bruijn* and *Kautz* digraphs:

- the De Bruijn digraph $\mathrm{B}(\Delta, D)$ is defined as $L^{D-1}K_\Delta^+$, where K_Δ^+ is the complete digraph on Δ vertices with a loop at each vertex (i.e. $|K_\Delta^+|=\Delta$ and $\mathrm{A}(K_\Delta^+)=\mathrm{V}(K_\Delta^+)^2$). Thus, $\mathrm{B}(\Delta, D)$ is a Δ-regular D-reachable digraph with diameter D and order $\Delta\Delta^{D-1}=\Delta^D$, which is the largest possible order for a D-reachable (Δ, D)-digraph.
- The Kautz digraph $\mathrm{K}(\Delta, D)$ is defined as $L^{D-1}K_{\Delta+1}^*$, where $K_{\Delta+1}^*$ is $K_{\Delta+1}^+$ without loops. Thus, $\mathrm{K}(\Delta, D)$ is a Δ-regular digraph with diameter D and order $\Delta^D+\Delta^{D-1}$, which is the largest known order for a (Δ, D)-digraph with $\Delta\geqslant 3$.

Finally, as usual in calculus, by $b_n=\mathrm{o}(a_n)$ we mean that $\lim_{n\to+\infty} b_n/a_n=0$.

3. First construction

Our first construction is based on a graph–digraph product that we call the σ-shuffle c-exchange product, inspired by the ones defined in [9,1]. In order to define this and compute the order, maximum degree and diameter of the graphs obtained from it, let us introduce some previous related concepts.

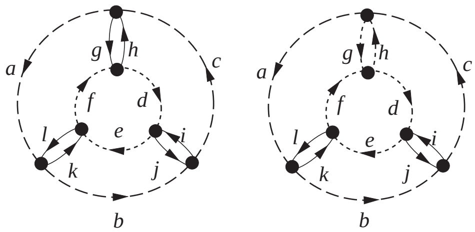

Fig. 1. Two shifts of K(2, 2): $(abc)(efd)(gh)(ij)(lk)$ and $(abc)(efhgd)(ij)(lk)$.

3.1. *Forward arc-colorings of a digraph*

We say that a map $c : \mathrm{A}(H) \longrightarrow C$ from the arcs of a digraph H to a set C is a *forward arc-coloring over C* if the restriction of c to the arcs incident *from* any given vertex is an injection, i.e. $\forall x \in H, \forall y, z \in \Gamma^+(x)$

$$c(xy) = c(xz) \Rightarrow y = z.$$

Since the arcs of a digraph are partitioned according to which vertex they are incident from, any digraph H with maximum degree $\Delta \leqslant |C|$ admits a forward arc-coloring over C. More precisely, for each vertex x, the set $A_x = \{xy : y \in \Gamma^+(x)\}$ has cardinality at most Δ. Thus, there exists an injective function $c_x : A_x \longrightarrow C$. Besides, $A_x \cap A_{x'} = \emptyset$ for $x \neq x'$. Thus, the mapping $xy \mapsto c_x(xy)$ is a well-defined forward arc-coloring.

3.2. *Shifts of a digraph*

We call a *shift* of a digraph, any permutation of its arcs such that the cycles of the permutation are cycles of the digraph. Alternatively, given a digraph H, a permutation σ of $\mathrm{A}(H)$ is a shift if $a \rightsquigarrow \sigma a$ for all a. Notice that any regular digraph has at least one shift since it is Eulerian (see [7, Theorem 2.23]). In Fig. 1, we show two different shifts of the Kautz digraph K(2, 2) by drawing the cycles of the permutation with different dash styles.

3.3. *The σ-shuffle c-exchange product*

Let G_1, H_2 and H_3 be a graph and two digraphs respectively. If $c : \mathrm{A}(H_2) \longrightarrow \mathrm{V}(G_1)$ is a forward arc-coloring of H_2 over the vertex set of G_1, and σ is a shift of H_2, then we call the *σ-shuffle c-exchange product* of H_3 and H_2 *according* to G_1 the graph $G = H_3 \vec{\times} H_2$ whose vertex set is $\mathrm{V}(H_3) \times \mathrm{A}(H_2)$ and such that two pairs (u_1, a_1), (u_2, a_2) are adjacent if $u_1 = u_2$ and $c(a_1)$ is adjacent with $c(a_2)$ in G_1, or if u_1 is adjacent to or from u_2 in H_3 and a_2 is equal to σa_1 or $\sigma^{-1} a_1$ respectively. Equivalently, given a vertex $(u, a) = (u, xy)$ of G, its set $\Gamma((u, a))$ of neighbors consists in three parts: the "exchange" set of neighbors, which is

$$X(u, xy) = \{(u, xz) : x \overset{H_2}{\rightsquigarrow} z,\ c(xy) \overset{G_1}{\sim} c(xz)\},$$

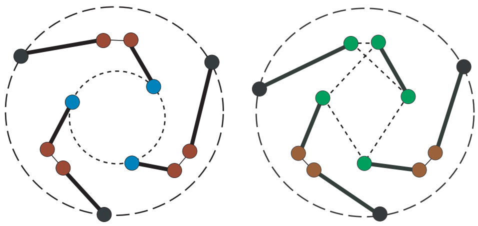

Fig. 2. Two K_2-exchange graphs of Kautz digraph $\mathrm{K}(2,2)$.

and the forward and backward "shuffle" sets of neighbors, which are

$$S^+(u,a) = \{(u', \sigma a) : u \overset{H_3}{\leadsto} u'\},$$
$$S^-(u,a) = \{(u', \sigma^{-1} a) : u' \overset{H_3}{\leadsto} u\}$$

respectively.

In Fig. 2, we give two examples of σ-shuffle c-exchange graphs, using the digraph and shifts of Fig. 1, as G_1 the complete graph K_2 on 2 vertices and as H_3 the digraph with one vertex and a loop on it. We do not specify the arc-forward colorings because in this case ($G_1 = K_2$) any choice gives rise to the same σ-shuffle c-exchange graphs.

Remark 1. Notice that if H_2 is $|G_1|$-regular, then the restriction c_x of c to the arcs A_x incident from a given vertex x, is bijective. Thus, if a and b are two arcs in A_x and $v_0, v_1, \ldots, v_n$ is a $c(a)$–$c(b)$ walk in G_1, then, for each vertex u of H_2, the sequence

$$(u,a) = (u, c_x^{-1}(v_0)), (u, c_x^{-1}(v_1)), \ldots, (u, c_x^{-1}(v_n)) = (u,b)$$

is a (u,a)–(u,b) walk in G.

The order of G and a tight upper bound for its maximum degree follow directly from the definition. We state this as a proposition.

Proposition 2. *Let $G = H_3 \vec{\times} H_2$ be the σ-shuffle c-exchange product of a digraph H_3 with a digraph H_2 according to a graph G_1. If H_2 is $|G_1|$-regular, then*

(1) $|G| = |G_1||H_2||H_3|$.
(2) *If Δ, Δ_1 and Δ_3 are the maximum degrees of G, G_1 and H_3 respectively then*

$$\Delta \leqslant \begin{cases} \Delta_1 + \Delta_3 & \text{if } H_3 \text{ is symmetric and } \sigma \text{ is an involution,} \\ \Delta_1 + 2\Delta_3 & \text{otherwise.} \end{cases}$$

Proof. Since H_2 is $|G_1|$-regular it has $|G_1||H_2|$ arcs, thus

$$|G| = |H_3||\mathrm{A}(H_2)| = |H_3||G_1||H_2|$$

as *item* (1) asserts. In order to bound the degree of a vertex (u, a) of G, we know by definition that

$$\Gamma((u,a)) = X(u,a) \cup S^+(u,a) \cup S^-(u,a).$$

Thus, in general, the cardinality of $\Gamma((u,a))$ is at most $\Delta_1 + 2\Delta_3$, since $|X(u,a)| \leqslant \Delta(G_1)$ and $|S^+(u,a)|, |S^-(u,a)| \leqslant \Delta(H_3)$. However, when H_3 is symmetric, we have that $u \overset{H_3}{\rightsquigarrow} u'$ if and only if $u' \overset{H_3}{\rightsquigarrow} u$. If, in addition, σ is an involution ($\sigma = \sigma^{-1}$), then the forward and backward shuffle sets coincide, i.e., $S^+(u,a) = S^-(u,a)$, and $|\Gamma(u,a)| \leqslant \Delta_1 + \Delta_3$. □

In order to obtain an upper bound on the diameter of a σ-shuffle c-exchange product, we will consider different kinds of diameters for H_2. This requires some detailed analysis, which we develop in the following theorems. We begin with the following lemma.

Lemma 3. *Let G_1 be a graph with diameter D_1 and $G = H_3 \vec{\times} H_2$ the σ-shuffle c-exchange product of a D_3-reachable digraph H_3 and a $|G_1|$-regular digraph H_2 according to G_1. Given any directed walk $u_0, u_1, \ldots, u_l$* in H_3 *and any directed walk $W = x_0, \ldots, x_l$ in H_2, then the distance in G between two vertices (u_0, a_0) and (u_l, a_{l+1}), such that $a_0 = x_0 x_{-1}$ and $a_{l+1} = x_l x_{l+1}$ for some x_{-1} and x_{l+1}, is at most $(D_1 + 1)l + D_1$*

$$\mathrm{d}_G((u_0, a_0), (u_l, a_{l+1})) \leqslant (D_1 + 1)l + D_1. \tag{2}$$

Moreover, if H_3 is symmetric and σ an involution, then W can be taken to be a walk in $\mathrm{U}H_2$.

Proof. Indeed, if $l = 0$, then $u_0 = u_l$ and, from Remark 1, it suffices to find a walk in G_1 joining $c(a_0)$ and $c(a_{l+1})$ in at most D_1 steps. Such a walk does exist since D_1 is the diameter of G_1.

For $l \geqslant 1$, we will bound the distance between (u_0, a_0) and (u_l, a_{l+1}) making use of the triangular inequality. Let us first focus on the digraph P whose vertices are $x_{-1}, x_0, \ldots, x_l, x_{l+1}$ and whose arcs are $a_0, \ldots, a_{l+1}$, where for $1 \leqslant i \leqslant l$, a_i is either $x_{i-1}x_i$ or $x_i x_{i-1}$ depending upon $x_{i-1}x_i$ being an arc of H_2 or not respectively. Next, we define recursively a sequence $w_0, \ldots, w_{l+1}$ of vertices of H_3 beginning with $w_0 = u_0$; and for $i > 0$ we set $w_i = u_{j+\delta}$ if $w_{i-1} = u_j$ and $\delta = \delta_P^-(x_{i-1})$. Solving the recurrence and remembering Eq. (1), we have that $w_{l+1} = u_s$ with

$$s = \sum_{i=1}^{l+1} \delta_P^-(x_{i-1}) = |\mathrm{A}(P)| - \delta_P^-(x_{-1}) - \delta_P^-(x_{l+1}) = l.$$

Finally, by the triangular inequality, we can bound by the above the distance between vertices (u_0, a_0) and (u_l, a_{l+1}) as follows:

$$\mathrm{d}_G((u_0, a_0), (u_l, a_{l+1})) \leqslant \sum_{i=0}^{l} \mathrm{d}_G((w_i, a_i), (w_{i+1}, a_{i+1})). \tag{3}$$

Now, in order to bound each term $d_i = \mathrm{d}_G((w_i, a_i), (w_{i+1}, a_{i+1}))$ of the sum, we distinguish four cases, depending on the directions of a_i and a_{i+1}. First, notice that, when H_3 is not symmetric, then W is a directed walk and the arc a_i must be adjacent to the arc a_{i+1} (forthcoming case 2). Hence, when a_i is not adjacent to a_{i+1} (forthcoming cases 1, 3 and 4) we are under the hypothesis that H_3 is symmetric and σ an involution. Thus, we assume that the forward and backward shuffle sets of neighbors coincide. Let $w_i = u_j$, then

(1) if a_i and a_{i+1} are both incident from the same vertex x_i, then $\delta_P^-(x_i) = 0$ and $w_{i+1} = u_j$. Thus, from Remark 1, the distance in G between (w_i, a_i) and (w_{i+1}, a_{i+1}) is at most D_1, hence $d_i \leqslant D_1$.
(2) If a_i is adjacent to a_{i+1}, then $\delta_P^-(x_i) = 1$, $w_{i+1} = u_{j+1}$ and σa_i is incident in H_2 from the same vertex as a_{i+1} (vertex x_i). Thus, the distance in G between (w_{i+1}, a_{i+1}) and $(u_{j+1}, \sigma a_i)$ is at most D_1. Finally, since $(u_{j+1}, \sigma a_i) \in S^+(u_j, a_i)$, then $d_i \leqslant D_1 + 1$.
(3) If a_i is adjacent from a_{i+1}, then $\delta_P^-(x_i) = 1$, $w_{i+1} = u_{j+1}$ and σa_{i+1} is incident in H_2 from the same vertex as a_i (vertex x_i). Thus, the distance in G between (w_i, a_i) and $(w_i, \sigma a_{i+1})$ is at most D_1. Finally, since H_3 is symmetric and σ an involution, (u_{j+1}, a_{i+1}) is a shuffle neighbor of $(u_j, \sigma a_{i+1})$ in G, and then $d_i \leqslant D_1 + 1$.
(4) Finally, if both a_i and a_{i+1} are incident to the same vertex x_i, then $\delta_P^-(x_i) = 2$, $w_{i+1} = u_{j+2}$ and both σa_i and σa_{i+1} are adjacent in H_2 from the same vertex x_i. Thus, the distance in G between $(u_{j+1}, \sigma a_i)$ and $(u_{j+1}, \sigma a_{i+1})$ is at most D_1. Finally, since H_3 is symmetric, $(u_{j+1}, \sigma a_i)$ and $(u_{j+1}, \sigma a_{i+1})$ are shuffle neighbors of (u_j, a_i) and (u_{j+2}, a_{i+1}) respectively, and then $d_i \leqslant D_1 + 2$.

In any case, it holds that:

$$d_i \leqslant D_1 + \delta_P^-(x_i).$$

Thus, we can upper-bound the sum in inequality (3) by

$$(l+1)D_1 + \sum_{i=0}^{l} \delta_P^-(x_i) = (l+1)D_1 + l,$$

which implies inequality (2), as was claimed. □

With this lemma we are in a position to bound the diameter of a σ-shuffle c-exchange product.

Theorem 4. *Given a graph G_1, let $G = H_3 \vec{\times} H_2$ be the σ-shuffle c-exchange product of a D_3-reachable digraph H_3 with a $|G_1|$-regular digraph H_2 according to G_1. If D, D_1 and D_2 are the diameters of G, G_1 and H_2 respectively, then*

$$D + 1 \leqslant (D_1 + 1)(D_2 + 1) + \max(D_3 - D_2, 0), \tag{4}$$

Moreover, if H_3 is symmetric and σ an involution, D_2 can be taken to be the diameter of UH_2.

Proof. Let us consider two vertices (u, a) and $(\tilde{u}, \tilde{a})$ with $a = xy$ and $\tilde{a} = \tilde{x}\tilde{y}$. We need to find a walk in G joining these vertices in at most the bound in inequality (4). We first treat the case when $D_3 \leqslant D_2$. From the hypothesis, we know that the distance l in H_2 (or in UH_2

if H_3 is symmetric), from x to $\tilde{x}$ is at most D_1, i.e, $l \leqslant D_1$. If there exists a directed walk in H_3 from u to $\tilde{u}$ of length l, we can apply inequality (2) and conclude the proof. Let us then consider the case when there is no directed walk of length l from u to $\tilde{u}$ in H_3. Thus, $l < D_3$ because H_3 is l-reachable for any $l \geqslant D_3$. Next, we set $k = D_3 - l$ and consider a vertex u_0 in $\Gamma^{-k}(u)$ and a directed walk

$$u_0, u_1, \ldots, u_{D_3} = \tilde{u}$$

of length D_3 in H_3, from u_0 to $\tilde{u}$, which exists because H_3 is D_3-reachable. Thus, the sequence $u_k, \ldots, u_{D_3}$ is a directed walk of length l from u_k to $\tilde{u}$, and we can apply inequality (2) to upper bound by $(D_1+1)l + D_1$ the distance in G between (u_k, a) and $(\tilde{u}, \tilde{a})$. Finally, since $u_0 \in \Gamma^{-k}(u)$ there is a directed walk $u_0, u_{-1}, \ldots, u_{-k} = u$ in H_3 from u_0 to u, which gives rise the following walk:

$$\begin{aligned}(u, a) = (u_{-k}, a), (u_{-k+1}, \sigma^{-1}a), \ldots, (u_{-1}, \sigma^{-k+1}a), (u_0, \sigma^{-k}a),\\ (u_1, \sigma^{-k+1}a), \ldots, (u_{k-1}, \sigma^{-1}a), (u_k, a)\end{aligned}$$

in G joining (u, a) and (u_k, a) in $2k$ steps. Therefore,

$$\begin{aligned}\mathrm{d}_G((u, a), (\tilde{u}, \tilde{a})) &\leqslant \mathrm{d}_G((u, a), (u_k, a)) + \mathrm{d}_G((u_k, a), (\tilde{u}, \tilde{a}))\\ &\leqslant 2k + l(D_1 + 1) + D_1 = 2(D_3 - l) + (D_1 + 1)l + D_1,\end{aligned}$$

which is less than or equal to $(D_1 + 1)D_2 + D_1$, since $D_3 \leqslant D_2$ and $D_1 \geqslant 1$. Let us now treat the case when $D_2 < D_3$. We set $h = D_3 - D_2 > 0$ and consider the vertex x' of H_3, incident to the arc $\sigma^h a$. Then, the distance l in H_2 (or in $\mathrm{U}H_2$ if H_3 is symmetric and σ and involution), from x' to $\tilde{x}$ is at most D_2. Let $k = D_2 - l$ and $\tilde{W}$ be a directed walk,

$$u_0, u_1, \ldots, u_{D_3} = \tilde{u},$$

in H_3 from a vertex u_0 in $\Gamma^{-k}(u)$ to $\tilde{u}$. Then, we can apply inequality (2) to the directed walk $u_{D_3-l}, \ldots, u_{D_3}$ of length l, and bound by $(D_1 + 1)l + D_1$ the distance in G between $(u_{D_3-l}, \sigma^h a)$ and $(\tilde{u}, \tilde{a})$. Finally, since $u_0 \in \Gamma^{-k}(u)$, there is a u_0-u directed walk $u_0, u_{-1}, \ldots, u_{-k} = u$ of length k in H_3 which, together with $\tilde{W}$, gives rise to the following walk in G:

$$\begin{aligned}(u, a) = (u_{-k}, a), (u_{-k+1}, \sigma^{-1}a), \ldots, (u_{-1}, \sigma^{-k+1}a),\\ (u_0, \sigma^{-k}a), (u_1, \sigma^{-k+1}a), \ldots, (u_{D_3-l}, \sigma^{-k+D_3-l}a)\end{aligned}$$

joining (u, a) and $(u_{D_3-l}, \sigma^h a)$ in $D_3 - l + k = D_3 + D_2 - 2l$ steps. Thus

$$\begin{aligned}\mathrm{d}_G((u, a), (\tilde{u}, \tilde{a})) &\leqslant \mathrm{d}_G((u, a), (u_{D_3-l}, \sigma^h a)) + \mathrm{d}_G((u_{D_3-l}, \sigma^h a), (\tilde{u}, \tilde{a}))\\ &\leqslant D_3 + D_2 - 2l + (D_1 + 1)l + D_1,\end{aligned}$$

which is at most $(D_1 + 1)D_2 + D_1 + D_3 - D_2$, since $l \leqslant D_2$ and $D_1 \geqslant 1$. □

The first part of the proof of this theorem for the case $D_3 \leqslant D_2$ enables us to prove the same result when D_2 is the unilateral diameter of H_2, as explained in the following proposition.

Proposition 5. *With the same hypothesis of the previous theorem, if D_2 is the unilateral diameter of H_2 and $D_3 \leqslant D_2$ then*

$$D + 1 \leqslant (D_1 + 1)(D_2 + 1). \tag{5}$$

Proof. Let (u, a) and $(\tilde{u}, \tilde{a})$ be two vertices of G with $a = xw$ and $\tilde{a} = \tilde{x}\tilde{w}$. If l is the unilateral distance in H_2 between x and $\tilde{x}$, then there will be a directed walk of length l from x to $\tilde{x}$ or vice-versa. In any case, we can make the same arguments as in the case $D_3 \leqslant D_2$ in Theorem 4 can be made in order to prove that the distance from (u, a) to $(\tilde{u}, \tilde{a})$ verifies (5). □

The above constructions are an extension to other values of the diameter and a unification of the first two constructions defined in [12] (see the Appendix). Indeed, when H_2 and H_3 are the De Bruijn digraphs $\mathrm{B}(k, |G_1|)$ and $\mathrm{B}(k, m)$ respectively, then graph G of Theorem 4 has the same parameters as graph $G_1\{m, k\}$ defined in [12]. If, instead of a De Bruijn digraph, H_2 is the Kautz digraph $\mathrm{K}(k, |G_1|)$, then graph G has the same parameters as graph $G_1(m, k)$ defined in [12]. In fact, a good choice of the forward arc-coloring and shift of the digraph H_2 not only gives rise to graphs with the same parameters, but also to isomorphic ones.

The present construction has a large degree of freedom, since in general a digraph has many different forward arc-colorings, as well as many different shifts. The question of which of these forward arc-colorings and shifts give rise to isomorphic σ-shuffle c-exchange products is not dealt with.

The following corollary, which can be established by means of elementary calculus arguments, will lead us to the main result of the work, namely, Theorem 7.

Corollary 6. *Let D_1, D_2 and $D_3 \geqslant D_2$ be three fixed positive integers, and let $\lambda = D_3/D$ where $D = D_1(D_2 + 1) + D_3$. Suppose that for a given sequence $\Delta_n \longrightarrow +\infty$ there exists a sequence of graphs $G_{1,n}$ and two sequences of digraphs $H_{2,n}$ and $H_{3,n}$ such that for each n:*

- *graph $G_{1,n}$ and digraph $H_{2,n}$ have diameters D_1 and D_2 respectively, and digraph $H_{3,n}$ is D_3-reachable;*
- *the graph and digraphs $G_{1,n}$, $H_{2,n}$ and $H_{3,n}$ have maximum degrees $\Delta_{1,n}$, $|G_{1,n}|$ and $\Delta_{3,n}$ respectively, where $|\Delta_{1,n} - (1 - \lambda)\Delta_n|$ and $|2\Delta_{3,n} - \lambda\Delta_n|$ are upper bounded by a constant;*
- *the previous graph and digraphs have orders $\Delta_{1,n}^{D_1} + \mathrm{o}(\Delta_{1,n}^{D_1})$, $|G_{1,n}|^{D_2} + \mathrm{o}(|G_{1,n}|^{D_2})$ and $\Delta_{3,n}^{D_3} + \mathrm{o}(\Delta_{3,n}^{D_3})$ respectively.*

Then, any σ-shuffle c-exchange product of $H_{3,n}$ and $H_{2,n}$ according with $G_{1,n}$, has maximum degree at most $\Delta_n + \mathrm{o}(\Delta_n)$, diameter at most D and order

$$|G_n| = \left(\frac{\Delta_n}{\alpha}\right)^D + \mathrm{o}(\Delta_n^D),$$

Table 1
First construction, where $k = \lim_{n\to\infty} |G_n|/(\varDelta_n/2)^D$

$D = 11$	15	16	17	18	23	24	25	26	29	30	31	32	33
$D_1 = 5$	7	7	5	5	7	7	7	7	5	5	7	7	7
$D_2 = 1$	1	1	2	2	2	2	2	2	4	4	3	3	3
$D_3 = 1$	1	2	2	3	4	5	6	5	4	5	3	4	5
$k \geqslant 71$	831	39	69	9	2347	248	35	6	297	45	14073	1558	215

where

$$\alpha = f(\lambda) = \frac{2^{\lambda}}{\lambda^{\lambda}(1-\lambda)^{1-\lambda}}.$$

Besides, if $H_{3,n}$ is symmetric, σ_n is an involution, and $|\varDelta_{3,n} - \lambda\varDelta_n|$ is upper-bounded by a constant, then $\alpha = f(\lambda)/2^{\lambda}$.

In order to apply this result, we take into account the existence of large graphs with diameter 5. Indeed, for each odd power q of 3, the quotient graphs $H'(q)$ of the generalized hexagons $H(q)$, described in [9], have maximum degree $(q+1)$, diameter $D_1 = 5$ and orders $(q+1)(q^4+q^2+1)$. Thus, for each integer $D \geqslant 10$ let D_2 and r be, respectively, the quotient and rest of the division of $D-5$ by 6, so $D-5 = 6D_2 + r$. Then, if we set $D_3 = D_2 + r$, $\lambda = D_3/D$ and $\varDelta_n = [(3^{2n+1}+1)/(1-\lambda)]$, we can take the graph $G_{1,n}$ and the digraphs $H_{2,n}$ and $H_{3,n}$ to be, respectively:

- the quotient graph $H'(3^{2n+1})$,
- the Kautz digraph $\mathrm{K}(|G_{1,n}|, D_2)$,
- the De Bruijn digraph $\mathrm{B}([\frac{1}{2}\lambda\varDelta_n], D_3)$.

Therefore, Corollary 6 tells us that there exists a sequence G_n of graphs with diameters at most D, maximum degrees at most $\varDelta_n + \mathrm{o}(\varDelta_n)$ and orders:

$$|G_n| = \left(\frac{\varDelta_n}{f(1/6) + \varepsilon_D}\right)^D + \mathrm{o}(\varDelta_n^D)$$

with $f(1/6) < 1.7614$ and

$$\lim_{D\to+\infty} \varepsilon_D = 0$$

(when $r = 5$, ε_D goes to 0 like $1/D$ and when $r = 0$ it is exactly zero).

In a similar way, if we take as $G_{1,n}$ the graph with diameter $D_1 = 7$, defined in [10] that arise from the incidence graphs of generalized octagons, we will obtain $(\varDelta_n, D)$-graphs with orders $|G_n| = (\varDelta_n/(f(1/8)+\varepsilon_D))^D + \mathrm{o}(\varDelta_n^D)$ with $f(1/8) < 1.5895$ and $\lim_{D\to+\infty} \varepsilon_D = 0$.

Notice that, by the Moore bound, the diameter of each G_n must be exactly D. In Table 1, we give lower bounds to the orders obtained in these two ways, for some small values of the diameter.

The above application of Corollary 6 enables us to state our main theorem:

Table 2
First construction using symmetric looped digraphs of large graphs

$D=15$	16	17	18	19	20	23	24	25	26	31	33	35
$D_1=3$	7	5	5	7	5	7	7	5	7	7	7	5
$D_2=3$	1	2	2	1	2	2	2	3	2	3	3	5
$D_3=3$	2	2	3	5	5	2	3	5	5	3	5	5
$k\geqslant 18$	157	277	78	9	13	9388	1984	123	199	112591	6890	20051

Theorem 7. *There are two constants D_0 and $\alpha<2$ such that for each $D\geqslant D_0$ and infinitely many values of Δ there exists a graph G with maximum degree Δ, diameter D and order*

$$|G|\geqslant\left(\frac{\Delta}{\alpha}\right)^D.$$

Furthermore $\alpha<1.59$.

For particular values of D we can improve some of the values in Table 1 by taking as $H_{2,n}$ and $H_{3,n}$ the symmetric looped digraphs of large graphs and as σ the mapping $uv\mapsto vu$. For instance, the graphs with diameter $D=2$, 3, 5, maximum degrees Δ and orders $\Delta^D+o(\Delta^D)$ described in [9] (for $D=7$ the family described in [10] is useless because its graphs have degrees not dense enough). We illustrate the results in Table 2.

Finally, in [14] it is proved that there exists a family of digraphs with unilateral diameter 2 and orders $1.5\Delta^2$, for even values of Δ. Thus, we can improve the entries of Table 1 for $D=17$ from 69 to 104. Nevertheless, these values are still smaller than the corresponding ones in Table 2.

It is possible to make slight modifications to the σ-shuffle c-exchange product definition, in order to include, as particular cases, the other compound graphs presented in [12]. However, the procedure is similar to the one presented in this section and the graphs obtained do not give better lower bounds for $n_{\Delta,D}$. In next section, a new method is presented which gives rise to families of graphs larger than those defined in [12].

4. Second construction

The next construction provides an improvement on the upper bound of the constant α in Theorem 7, for diameters congruent with 0, 1 or -1 modulo m, with $m=6$ or 8. Specifically, we will prove that α could be taken smaller than 1.57. As in Section 3 we begin by introducing some arc transformations and graph–digraph products.

4.1. Reflections of a digraph

We call a *reflection* of a digraph H any involution $\phi : \mathrm{V}(H)\longrightarrow \mathrm{V}(H)$ of its vertices which is an *antiautomorphism* (i.e. a bijection that reverses the direction of the arcs).

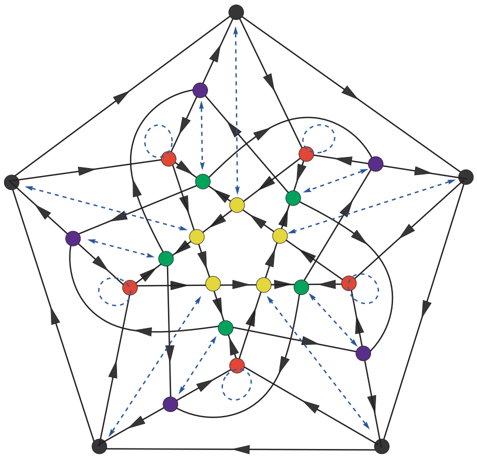

Fig. 3. A reflection of the largest known $(2,4)$-digraph.

Formally, if we write $\bar{u}$ for $\phi(u)$, then $\bar{\bar{u}} = u$ and

$$u \rightsquigarrow v \Rightarrow \bar{v} \rightsquigarrow \bar{u}.$$

Thus, if uv and vw are arcs of H, so are $\bar{v}\bar{u}$ and $\bar{w}\bar{v}$. But, in that case, the arcs uv and $\bar{w}\bar{v}$ are adjacent in LH to the arcs vw and $\bar{v}\bar{u}$ respectively, hence

$$uv \overset{LH}{\rightsquigarrow} vw \Rightarrow \bar{w}\bar{v} \overset{LH}{\rightsquigarrow} \bar{v}\bar{u}.$$

This means that the extension of ϕ to the set of arcs of H given by $uv \mapsto \bar{v}\bar{u}$ is an antiautomorphism of LH as well. Furthermore, it is also an involution since $\bar{v}\bar{u} \mapsto \bar{\bar{u}}\bar{\bar{v}} = uv$; therefore we have proved the following proposition.

Proposition 8. *If the mapping $u \mapsto \bar{u}$ is a reflection of a digraph H, then the mapping $uv \mapsto \bar{v}\bar{u}$ is a reflection of its line digraph LH as well.*

In fact, it can be proved that any reflection of the line digraph of a regular digraph arises in this way. These results enable us to find all the reflections of the De Bruijn and Kautz digraphs by taking reflections in the corresponding complete digraphs which are simply the permutations of order 2.

In Fig. 3 we describe (dashed arrows) a reflection of the largest known digraph with maximum degree 2 and diameter 4 (given in [11]). Therefore, by Proposition 8, the largest known (Δ, D)-digraphs with maximum degree 2 (which are the line digraphs of that in Fig. 3) have reflections.

4.2. The G-antiexchange graph

If $c : \mathrm{A}(H) \longrightarrow \mathrm{V}(G)$ is a forward arc-coloring of a digraph H over the vertex set of a graph G, and $\phi : u \mapsto \bar{u}$ is a reflection of H, then we define the *G-antiexchange graph*

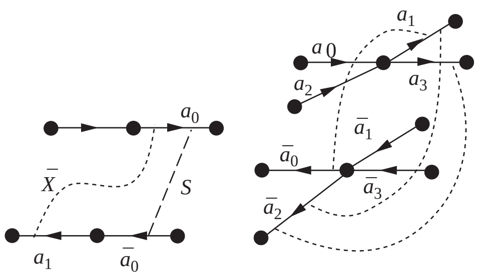

Fig. 4.

$\mathscr{G}=\mathscr{A}(G, H, c, \phi)$ *of* H *according to* c *and* ϕ as the graph whose vertices are the arcs of H, and such that two arcs a_0 and a_1 are adjacent if either $\bar{a}_0$ is adjacent to a_1 in H and $c(a_0)$ and $c(a_1)$ are adjacent in G (the "antiexchange" adjacency) or if $a_1 = \bar{a}_0$ (the "ϕ-adjacency"). Formally,

$$\mathscr{G} = \mathscr{A}(G, H, c, \phi) = (\mathrm{A}(H), \bar{X} \cup S)$$

where

$$\bar{X} = \bar{X}(\mathscr{G}) = \{\{a_0, a_1\} : \bar{a}_0 \overset{LH}{\rightsquigarrow} a_1, c(a_0) \overset{G}{\sim} c(a_1)\}$$

and

$$S = S(\mathscr{G}) = \{\{a, \bar{a}\} : a \in \mathrm{A}(H)\}.$$

Where $\bar{a} = \bar{v}\bar{u}$ if $a = uv$. See Fig. 4, in left.

4.3. *The ϕ-product*

Given two antiexchange graphs A and B, we call the *ϕ-product* of them the graph $G = A \bar{\times} B$ whose vertex set is $\mathrm{V}(A) \times \mathrm{V}(B)$, and where two pairs (a_0, b_0), (a_1, b_1) are adjacent if their first coordinates are antiexchange adjacent and their second are ϕ-adjacent or vice-versa. Formally,

$$(a_0, b_0) \overset{G}{\sim} (a_1, b_1) \Leftrightarrow \{a_0, a_1\} \in \bar{X}(A) \text{ and } \{b_0, b_1\} \in S(B)$$
$$\text{or } \{b_0, b_1\} \in \bar{X}(B) \text{ and } \{a_0, a_1\} \in S(A).$$

Before stating the next theorem, let us give some insights into the local adjacency structure of a ϕ-product.

Remark 9. Suppose that $G = A \bar{\times} B$ with $A = \mathscr{A}(G_1, H_1, c, \phi)$, and let $a_0, a_2, \ldots, a_{2n}$ be arcs adjacent to the arcs $a_1, a_3, \ldots, a_{2n+1}$ in H_1 such that $c(\bar{a}_0), c(a_1), \ldots, c(\bar{a}_{2n})$, $c(a_{2n+1})$ is a walk in G_1. Then, for each vertex b of B, the sequence

$$(\bar{a}_0, b), (a_1, \bar{b}), \ldots, (\bar{a}_{2n}, b), (a_{2n+1}, \bar{b})$$

is a walk in G (see right part in Fig. 4). In particular, if G_1 is D_1-reachable for an odd integer D_1 and H_1 is $|G_1|$-regular, then for any arc a_0 of H_1 adjacent to another arc a_1,

$$\mathrm{d}_G((\bar{a}_0, b), (a_1, \bar{b})) \leqslant D_1$$

for each vertex b of B.

Theorem 10. *Let G_1 and G_2 be two graphs and let H_A and H_B be two $|G_1|$ and $|G_2|$-regular digraphs respectively. Then, the ϕ-product $G = A \bar{\times} B$ between any G_1-antiexchange graph A of H_A and any G_2-antiexchange graph B of H_B verifies*:

(1) $|G| = |A||B| = |G_1||G_2||H_A||H_B|$.
(2) *If Δ, Δ_1 and Δ_2 are the maximum degrees of G, G_1 and G_2 respectively then*

$$\Delta \leqslant \Delta_1 + \Delta_2.$$

(3) *If G_1, G_2 and H_B are D_1, D_2 and D_B-reachable respectively with $D_1 \geqslant 3$, and the diameter D_A of H_A verifies $|D_A - D_B| \leqslant 1$, then*

$$\mathrm{D}(G) \leqslant (D_A + 1)D_1 + (D_B + 1)D_2.$$

Proof. The order of G follows from the fact that the digraphs H_A and H_B have $|G_1||H_A|$ and $|G_2||H_B|$ arcs respectively and that $\mathrm{V}(A) = \mathrm{A}(H_A)$ and $\mathrm{V}(B) = \mathrm{A}(H_B)$. In order to bound the degree of a vertex (a, b) of G, we express its neighbors as follows:

$$\Gamma((a, b)) = \{(a', \bar{b}) : \{a, a'\} \in \bar{X}(A)\} \cup \{(\bar{a}, b') : \{b, b'\} \in \bar{X}(B)\}.$$

Now, since $|\{a' : \{a, a'\} \in \bar{X}(A)\}| \leqslant \Delta_1$ and $|\{b' : \{b, b'\} \in \bar{X}(B)\}| \leqslant \Delta_2$, thus $|\Gamma_G((a, b))| \leqslant \Delta_1 + \Delta_2$, as asserted in *item* (2).

In order to upper bound the diameter let us consider two vertices (a, b) and (a_*, b_*) of G and distinguish three cases depending on the value of $D_A - D_B$.

Case 1: $D_A = D_B - 1$. Let $a = a_0, a_1, \ldots, a_l = \bar{a}_*$ be a shortest directed walk from a to $\bar{a}_*$ in LH_A. Now, we distinguish two cases depending upon l being equal to or smaller than $D_A + 1$. If $l = D_A + 1$. Then we consider a directed walk $\bar{b} = b_0, \ldots, b_{l+1} = b_*$ from $\bar{b}$ to b_* in LH_B of length exactly $D_B + 1$ whose existence is guaranteed by the D_B-reachability of H_B. By the triangular inequality, we have that

$$\mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) \leqslant \mathrm{d}\left(\begin{bmatrix} a_0 \\ \bar{b}_0 \end{bmatrix}, \begin{bmatrix} \bar{a}_0 \\ b_1 \end{bmatrix}\right) + \sum_{i=0}^{l-1} \left[\mathrm{d}\left(\begin{bmatrix} \bar{a}_i \\ b_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ \bar{b}_{i+1} \end{bmatrix}\right) + \mathrm{d}\left(\begin{bmatrix} a_{i+1} \\ \bar{b}_{i+1} \end{bmatrix}, \begin{bmatrix} \bar{a}_{i+1} \\ b_{i+2} \end{bmatrix}\right)\right].$$

(Where we have written the pairs as columns in order to clarify the expression.) Thus, by Remark 9,

$$\mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) \leqslant D_2 + \sum_{i=0}^{l-1} (D_1 + D_2) = (D_A + 1)D_1 + (D_B + 1)D_2.$$

On the other hand, if $k = D_A + 1 - l > 0$, then we consider a walk in G of length $2k$ of the form $(a, b) = (a_0, \bar{\beta}_0)$, $(\bar{a}_{-1}, \beta_0), \ldots, (\bar{a}_{-k}, \beta_{-k})$, $(a_{-k}, \bar{\beta}_{-k-1})$, and a directed walk $\beta_{-k-1} = b_{-k-1}, \ldots, b_l = b_*$ from β_{-k-1} to b_* in LH_B of length $D_B + 1$. As before, by the triangular inequality we have that

$$\begin{aligned}
\mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) &\leqslant \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_{-k} \\ \bar{\beta}_{-k-1} \end{bmatrix}\right) + \mathrm{d}\left(\begin{bmatrix} a_{-k} \\ \bar{\beta}_{-k-1} \end{bmatrix}, \begin{bmatrix} \bar{a}_{-k} \\ b_{-k} \end{bmatrix}\right) \\
&\quad + \mathrm{d}\left(\begin{bmatrix} \bar{a}_{-k} \\ b_{-k} \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) \\
&\leqslant 2k + D_2 + \sum_{i=-k}^{l-1} \left[\mathrm{d}\left(\begin{bmatrix} \bar{a}_i \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}\right) + \mathrm{d}\left(\begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}, \begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}\right)\right] \\
&\leqslant 2k + D_2 + k + lD_1 + (D_A + 1)D_2,
\end{aligned}$$

which is less than or equal to $(D_A + 1)D_1 + (D_B + 1)D_2$ if $D_1 \geqslant 3$. Notice that, in the last inequality, we have used that $\mathrm{d}((\bar{a}_i, b_i), (a_{i+1}, \bar{b}_i)) = 1$ for $i < 0$.

The other cases ($D_A = D_B$ and $D_A = D_B + 1$), which are similar to the previous one, are developed in the Appendix. □

As for the first construction, we can infer the following corollary:

Corollary 11. *Let D_1, D_2, D_A and D_B be four fixed integer such that $D_1 \geqslant 3$ and $|D_A - D_B| \leqslant 1$, and let $\lambda = (D_B + 1)D_2/D$ where $D = (D_A + 1)D_1 + (D_B + 1)D_2$. If for a sequence $\Delta_n \longrightarrow +\infty$ of positive integers there exist two sequences $G_{1,n}$, $G_{2,n}$ of graphs and two sequences $H_{A,n}$, $H_{B,n}$ of digraphs with reflections such that*:

- *for each n, the graphs $G_{1,n}$, $G_{2,n}$ and the digraph $G_{B,n}$ are D_1, D_2 and D_B-reachable respectively, and the graph $G_{A,n}$ has diameter D_A.*
- *The graphs and digraphs $G_{1,n}$, $G_{2,n}$, $H_{A,n}$ and $H_{B,n}$, have maximum degrees $\Delta_{1,n}$, $\Delta_{2,n}$, $|G_{1,n}|$ and $|G_{2,n}|$ respectively, being $|\Delta_{1,n} - (1-\lambda)\Delta_n|$ and $|\Delta_{2,n} - \lambda\Delta_n|$ upper bounded by a constant*;
- *the graphs and digraphs of the previous item have orders $\Delta_{1,n}^{D_1} + \mathrm{o}(\Delta_{1,n}^{D_1})$, $\Delta_{2,n}^{D_2} + \mathrm{o}(\Delta_{2,n}^{D_2})$, $|G_{1,n}|^{D_A} + \mathrm{o}(|G_{1,n}|^{D_A})$ and $|G_{2,n}|^{D_B} + \mathrm{o}(|G_{2,n}|^{D_B})$ respectively.*

Then, for any sequence A_n, of $G_{1,n}$-antiexchange graphs of $H_{A,n}$ and any sequence B_n, of $G_{2,n}$-antiexchange graphs of $H_{B,n}$, the graphs G_n of the sequence of their ϕ-products, i.e.

$$G_n = A_n \bar{\times} B_n,$$

have maximum degrees $\Delta_n + \mathrm{o}(\Delta_n)$, diameters at most D and orders

$$|G_n| = \left(\frac{\Delta_n}{\alpha}\right)^D + \mathrm{o}(\Delta_n^D)$$

Table 3
Second construction, where $k = \lim_{n\longrightarrow\infty}|G_n|/(\Delta_n/2)^D$

$D=12$	13	15	16	17	18	19	23	24	25	29	30	31
$D_1=5$	5	7	7	5	5	5	7	7	7	5	5	7
$D_A=1$	1	1	1	2	2	2	2	2	2	4	4	3
$D_B=1$	2	0	1	1	2	3	1	2	3	3	4	2
$k\geqslant 18$	7	831	157	277	79	29	9388	1984	565	4754	1447	112591

where

$$\alpha = g(\lambda) = \frac{1}{\lambda^{\lambda}(1-\lambda)^{1-\lambda}}.$$

In order to apply this corollary, for each integer $D \geqslant 11$ congruent with -1, 0 or 1 modulo 6, let D_A and r be two integers such that $D-6=6D_A+r$ and $|r|\leqslant 1$. Then, if $D_B=D_A+r$, $\lambda=(D_B+1)/D$ and $\Delta_n=[(3^{2n+1}+1)/(1-\lambda)]$, we can take graphs $G_{1,n}$ and $G_{2,n}$ and digraphs $H_{A,n}$ and $H_{B,n}$ to be, respectively,

- the quotient graph $H'(3^{2n+1})$ of the generalized hexagon $H(3^{2n+1})$,
- the complete graph on $[\lambda\Delta_n]+1$ vertices,
- the Kautz digraph with diameter D_A and maximum degree $|G_{1,n}|$ and
- the De Bruijn digraphs with diameter D_B and maximum degree $[\lambda\Delta_n]+1$,

Consequently, the graphs G_n of Corollary 11 will have diameters at most D, maximum degrees at most $\Delta_n+\mathrm{o}(\Delta_n)$ and orders:

$$|G_n| = \left(\frac{\Delta_n}{g(1/6)+\varepsilon_D}\right)^D + \mathrm{o}(\Delta_n^D)$$

with $g(1/6)\leqslant 1.5692$ and $\varepsilon_D \longrightarrow 0$ when $D\longrightarrow +\infty$.

Similarly, for each integer $D\geqslant 15$ congruent with -1, 0 or 1 modulo 8, we can take as $G_{1,n}$ the graph with diameter $D_1=7$, defined in [10]. The corresponding $(\Delta_n+\mathrm{o}(\Delta_n), D)$-graphs G_n of Corollary 11 will have orders: $|G_n| = (\Delta_n/g(1/8)+\varepsilon_D)^D + \mathrm{o}(\Delta_n^D)$ with $g(1/8)\leqslant 1.4576$ and $\varepsilon_D \longrightarrow 0$ when $D\longrightarrow +\infty$. Table 3 shows the result for some values of the diameter.

As a consequence of the above application, we can state the following final result.

Theorem 12. *For $m=6,8$, there is a constant D_0 such that for each $D\geqslant D_0$ congruent with -1, 0 or 1 modulo m, and for infinitely many values of Δ, there exists a graph G with maximum degree Δ, diameter D and order*

$$|G| \geqslant \left(\frac{\Delta}{\alpha}\right)^D, \quad \text{where } \alpha = \begin{cases} 1.57 & \text{if } m=6,\\ 1.45 & \text{if } m=8.\end{cases}$$

5. Conclusions

As in other works (e.g. [5]), we have illustrated that the implicit parallelism between some constructions, like those of the graphs $G\{m,k\}$ and $G(m,k)$ in [12], and the De Bruijn and Kautz digraphs, can be explicitly given by a special kind of product (in our case the σ-shuffle c-exchange product). As in previous works, we have shown (by means of the ϕ-product) that the existence of a reflection on these digraphs allows the improvement of the constructions based on them by means of a reduction of the maximum degrees.

Finally, we have shown that if there exist graphs with diameter $D_1 \geqslant 5$, maximum degree Δ_1 and orders $\Delta_1^{D_1} + o(\Delta_1^{D_1})$ for infinitely many values of Δ_1, then there exists $\alpha < 2$ such that, for each D greater than a certain constant, graphs with diameter D, maximum degree Δ and order greater than $(\Delta/\alpha)^D$ do exist for infinitely many values of Δ. Furthermore, the larger D_1, the smaller the value of α.

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Appendix A.

In the first two subsections, we give the definition of the first two constructions presented in [12] and the one given in [13]. In the last subsection, we develop the remaining two cases in the proof of Theorem 10.

A.1. Graphs $G_1\{m,k\}$ and $G_1(m,k)$

Given an alphabet A on m symbols and a (Δ_1, D_1)-graph G_1 on N_1 vertices, the graph $G_1\{m,k\}$ has as vertices the words w of the form

$$w = \alpha_1\alpha_2\ldots\alpha_{k-1}, x_1x_2\ldots x_{k-1}x_k, \quad \alpha_j \in A, \ \ x_i \in \mathrm{V}(G_1),$$

and the neighbors of the vertex w are given as follows:

$$w \sim \begin{cases} \alpha_1\alpha_2\ldots\alpha_{k-2}\alpha_{k-1}, x_1x_2\ldots x_{k-1}x'_k & x'_k \sim x_k \text{ in } G_1, \\ \alpha_0\alpha_1\ldots\alpha_{k-3}\alpha_{k-2}, x_0x_1\ldots x_{k-2}x_{k-1} & \alpha_0 \in A, \\ \alpha_2\alpha_3\ldots\alpha_{k-1}\alpha_k, x_2x_3\ldots x_kx_1 & \alpha_k \in A. \end{cases}$$

In order to define $G_1(m,k)$, we take a set of N_1 one to one maps $f_l : C\backslash\{l\} \longrightarrow \mathrm{V}(G_1)$ from an alphabet C on $|C| = N_1 + 1$ symbols to the set of vertices of G_1. Then, a vertex w of $G_1(m,k)$ is a word

$$w = \alpha_1\alpha_2\ldots\alpha_{k-1}, x_1x_2\ldots x_{k-1}x_k \quad \alpha_j \in A, \ \ x_i \in C$$

such that $x_i \neq x_{i+1}$. The neighbors of w are the following:

$$w \sim \begin{cases} \alpha_1\alpha_2 \ldots \alpha_{k-2}\alpha_{k-1}, x_1x_2 \ldots x_{k-1}x'_k & x'_k = f_{x_{k-1}}^{-1}(\Gamma(f_{x_{k-1}}(x_k))), \\ \alpha_0\alpha_1 \ldots \alpha_{k-3}\alpha_{k-2}, x_0x_1 \ldots x_{k-2}x_{k-1} & x_0 = f_{x_1}^{-1}(f_{x_{k-1}}(x_k)), \\ \alpha_2\alpha_3 \ldots \alpha_{k-1}\alpha_k, x_2x_3 \ldots x_kx_1 & x_{k+1} = f_{x_k}^{-1}(f_{x_2}(x_1)). \end{cases}$$

A.2. Definition of the graphs mentioned in Section 4

Given an alphabet A on m symbols and a (Δ_1, D_1)-graph G_1 on N_1 vertices, the graph defined in [13] has as vertices the words w of the form

$$w = \alpha_1\alpha_2 \ldots \alpha_{2k+1}, x_1x_2 \ldots x_{2k+1}, \quad \alpha_j \in A, \quad x_i \in \mathrm{V}(G_1),$$

and the vertex w is adjacent to the following vertices:

$$w \sim \begin{cases} \alpha_{2k+1}\alpha_{2k} \ldots \alpha_2\alpha_1, x_{k-1}x_k \ldots x_1x'_k & x'_k \sim x_k \text{ in } G_1, \\ \alpha_{2k}\alpha_{2k-1} \ldots \alpha_1\alpha'_{2k+1}, x_kx_{k-1} \ldots x_2x_1 & \alpha'_{2k+1} \in A. \end{cases}$$

A.3. Remaining cases in the proof of Theorem 10

Case 2: $D_A = D_B$. Let $\bar{a} = a_0, a_1, \ldots, a_l = \bar{a}_*$ be a shortest directed walk from $\bar{a}$ to $\bar{a}_*$ in LH_A. If $l = D_A + 1$, let $b = b_0, \ldots, b_l = b_*$ be a b-b_* directed walk in LH_B of length $D_B + 1$. Thus,

$$\begin{aligned} \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) &\leqslant \sum_{i=0}^{l-1} \left[\mathrm{d}\left(\begin{bmatrix} \bar{a}_i \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}\right) + \mathrm{d}\left(\begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}, \begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}\right)\right] \\ &\leqslant \sum_{i=0}^{l-1}(D_1 + D_2) = (D_A + 1)D_1 + (D_B + 1)D_2. \end{aligned}$$

On the other hand, if $k=(D_A+1)-l > 0$, we consider a walk $(a, b)=(\bar{a}_0, \beta_0), (a_0, \bar{\beta}_{-1}), \ldots, (\bar{a}_{-k+1}, \beta_{-k+1}), (a_{-k+1}, \bar{\beta}_{-k})$ in G and a β_{-k}-b_* directed walk $\beta_{-k} = b_{-k}, \ldots, b_l = b_*$ in LH_B of length $D_B + 1$. Thus,

$$\begin{aligned} \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) &\leqslant \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_{-k+1} \\ \bar{\beta}_{-k} \end{bmatrix}\right) + \mathrm{d}\left(\begin{bmatrix} a_{-k+1} \\ \bar{\beta}_{-k} \end{bmatrix}, \begin{bmatrix} \bar{a}_{-k+1} \\ b_{-k+1} \end{bmatrix}\right) \\ &\quad + \mathrm{d}\left(\begin{bmatrix} \bar{a}_{-k+1} \\ b_{-k+1} \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) \\ &\leqslant 2k - 3 + D_2 + \sum_{i=-k+1}^{l-1} \left[\mathrm{d}\left(\begin{bmatrix} \bar{a}_i \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}\right)\right. \\ &\quad \left. + \mathrm{d}\left(\begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}, \begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}\right)\right] \\ &\leqslant 2k - 3 + D_2 + (k-1) + lD_1 + (D_A + 1)D_2, \end{aligned}$$

which is less than or equal to $(D_A+1)D_1+(D_B+1)D_2$ if and only if $D_1 \geqslant 3(k-1)/k-1$, but $3(k-1)/k-1 < 3 \leqslant D_1$.

Case 3: $D_A = D_B + 1$. Let $\bar{a} = a_0, a_1, \ldots, a_l = a_*$ be a shortest $\bar{a}$-a_* directed walk in LH_A. If $l = D_A + 1$, let $b = b_0, \ldots, b_{l-1} = \bar{b}_*$ be a b-b_* directed walk in LH_B of length $D_B + 1$. Thus,

$$\begin{aligned} \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) &\leqslant \mathrm{d}\left(\begin{bmatrix} \bar{a}_0 \\ b_0 \end{bmatrix}, \begin{bmatrix} a_1 \\ \bar{b}_0 \end{bmatrix}\right) + \sum_{i=0}^{l-2} \left[\mathrm{d}\left(\begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}, \begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}\right)\right. \\ &\quad \left. + \, \mathrm{d}\left(\begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+2} \\ \bar{b}_{i+1} \end{bmatrix}\right)\right] \\ &\leqslant D_1 + \sum_{i=0}^{l-2} (D_1 + D_2) = (D_A + 1)D_1 + (D_B + 1)D_2. \end{aligned}$$

If $k = (D_A + 1) - l > 0$, we consider a walk $(a, b) = (\bar{a}_0, \beta_0), (a_0, \bar{\beta}_{-1}), \ldots, (\bar{a}_{-k+1}, \beta_{-k+1})$, $(a_{-k+1}, \bar{\beta}_{-k})$ in G and a β_{-k}-$\bar{b}_*$ directed walk $\beta_{-k} = b_{-k}, \ldots, b_{l-1} = \bar{b}_*$ in LH_B of length $D_B + 1$. Thus,

$$\begin{aligned} \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) &\leqslant \mathrm{d}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a_{-k+1} \\ \bar{\beta}_{-k} \end{bmatrix}\right) + \mathrm{d}\left(\begin{bmatrix} a_{-k+1} \\ \bar{\beta}_{-k} \end{bmatrix}, \begin{bmatrix} a_* \\ b_* \end{bmatrix}\right) \\ &\leqslant 2(k-1) + \sum_{i=-k}^{l-2} \left[\mathrm{d}\left(\begin{bmatrix} a_{i+1} \\ \bar{b}_i \end{bmatrix}, \begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}\right)\right. \\ &\quad \left. + \, \mathrm{d}\left(\begin{bmatrix} \bar{a}_{i+1} \\ b_{i+1} \end{bmatrix}, \begin{bmatrix} a_{i+2} \\ \bar{b}_{i+1} \end{bmatrix}\right)\right] \\ &\leqslant 2(k-1) - 1 + D_A D_2 + k + (l - 2 + 1)D_1, \end{aligned}$$

which is less than or equal to $(D_A + 1)D_1 + (D_B + 1)D_2$ if and only if $D_1 \geqslant 3(k-1)/(k+1)$ which is smaller than 3 for any $k > 0$. $\square$

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