

From weighted to unweighted graphs in Synchronizing Graph Theory.

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A way to associate unweighted graphs from weighted ones is presented, such that linear stable equilibria of the Kuramoto homogeneous model associated to both graphs coincide, i.e., equilibria of the system $\dot{\theta}_i = \sum_{j \sim i} \sin(\theta_j - \theta_i)$, where $i \sim j$ means vertices i and j are adjacent in the corresponding graph. As a consequence, the existence of linearly stable equilibrium is proved to be NP-Hard as conjectured by R. Taylor in 2015 and a new lower bound for the minimum degree that ensures synchronization is found.

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Oscillator synchronization is a behavior found in many complex systems, so it is of paramount interest to understand the conditions that ensure it. One fundamental piece of this puzzle is the topology of the interconnection network. The homogeneous Kuramoto model stands out as a good model to answer that question, since its dynamics depends only on that topology. Many concepts from graph theory have been imported in order to shed light on this topic, like biconnectivity, twin vertices and minimum degree. This last parameter has been proved to imply, when large enough, that almost all initial conditions go to an in-phase sync. How large it should be is a limit called *critical connectivity* and it is known to be between 0.6838 and 0.75. The lower bound of 0.6838, has been improved in the last ten years by theoretical and computational effort from an initial value of 0.6809 to 0.6818, then to 0.6828, and finally to 0.6838. All these improvements were achieved by means of the search of circulant graphs with no trivial linearly stable equilibria. However the last of these improvements close the possibility of finding new ones from circulant graphs. In this work, we improve the lower bound to 0.6875 by means of non-circulant graphs. Moreover, we have considered a general Kuramoto model, with different strengths between the oscillators and developed a way to assign a model with equal strengths that share its linearly stable equilibria. This technique also allows us to prove that the computational complexity of classifying equilibria by linear analysis is NP-Hard. Therefore the method has opened a gate between the more general world of weighted graphs to the more restricted world of unweighted ones, which gives hope for a whole set of new results in the field.

I. INTRODUCTION

In 2012 a seminal work by Richard Taylor¹ did a big step toward the classification of synchronizing graphs proving the existence of a critical value $\mu_c < 1$, called *critical connectivity*² such that almost any orbit of the homogeneous Kuramoto model associated with a graph with n vertices and minimum degree at least $\mu(n-1)$, ensure the graph is *synchronizing*, i.e., almost any orbit should go to an equilibrium of the form $(c, c, \dots, c)$. In that work, the author gives an upper and a lower bound for μ_c . Both bounds were later remarkably improved. The upper bound was improved seven years later from 0.9395 to 0.7929 by Ling, Xu and Bandeira³, then from 0.7929 to 0.7889 by Lu and Steinerberger⁴ and finally from 0.7889 to 0.75 by Kassabov, Strogatz and Townsend², in a time-slaps of two years. On the other hand, the lower bound was improved in 2015 from 0.6809 to 0.6818 by Canale and Monzón⁵ by means of Kroenecker product of circulant graphs with complete ones, then from 0.6918 to 0.6828 by Townsend, Stillman and Strogatz⁶ by the exhaustive analysis of graphs with at most 50 vertices and, finally, in an what we consider an amazing work, from 0.6828 to 0.6838 by Yoneda, Tatsukawa, and Teramaer⁷, were they take in count all possible circulant graphs, transform the problem in a family of integer programming problems, solve each of them analytically to find an optimal solution. In the present work we increase the lower bound from 0.6838 to 0.6875 by giving an explicit example of a family of non synchronizing graphs. Our construction comes from a kind of perturbation of the tensor product of C_4 with the complete graph suggested by Kassabov et al.², but going outside the family of circulant graphs. The construction is based on a way to transform weighted graph into unweighted ones. As a collateral result we close an open problem on the complexity of classifying stable equilibria left by Taylor⁸, proving that this decision problem is NP-Hard.

II. PRELIMINARIES

A. Graph theory

A *graph* G consists of a set VG of *vertices*, some of them *joined by edges* in a set EG . If two vertices v and w are joined by an edge e , we say they are *adjacent* and we write $e = vw$ or simply $v \sim w$ if no doubt about G could arise. In this work, all graphs are *simple*, i.e. there are no edge of the form vv and no two different edges join the same vertices. The graph is a *weighted graph* if it comes together with a *weight function* $w : EG \rightarrow \mathbb{N}$. If w is constant equal to 1, then the graph can be considered unweighted.

The *order* $|G|$ of G is the cardinality $|VG|$ of its vertex set while we call *weight* of G to the sum $w(G)$ of all its weights, i.e. $w(G) = \sum_{e \in EG} w_e$. We will denote by G_v the set of vertices adjacent with v in G , thus, $w \in G_v$ iff $v \sim w$. The cardinality $|G_v|$ of G_v is the *degree* of v . The minimum degree amount the vertices of G is denoted δG and called *minimum degree* of G , so $\delta G = \min\{|G(v)| : v \in VG\}$. Let us call *strong density* to the ratio $\mu(G) = \delta G / (|G| - 1)$. In graph theory, the name *density* is reserved for $|EG| / \binom{|G|}{2}$, so, when the graph is regular, the strong and the ordinary density coincide.

Now, we will define a simple graph from a weighted one. Given a weighted graph G and a positive integer $k \geq \max w = \max_v w$ let us define the *k-spinning* $S_k(G)$ of G to be the graph with vertices $VG \times \mathbb{Z}_k$ and edges,

$$ES_k(G) = \{(u, i)(v, j) : uv \in EG, j \in i + \mathbb{N}_{w_{uv}}\} \cup \{(u, i)(u, j) : \forall i \neq j\},$$

where $\mathbb{N}_h = \{0, 1, \dots, h-1\}$, so $i + \mathbb{N}_{w_{uv}} = \{i, i+1, \dots, i+w_{uv}-1\}$. In Figure 1 we show a weighted graph together with its corresponding 2-spinning graph.

Notice that the graph G without weights is the quotient of $S_k(G)$ under the relation $(u, i)R(v, j) \iff u = v$, i.e. $G = S_k(G)/R$. Besides, when all the weights are equal to k we

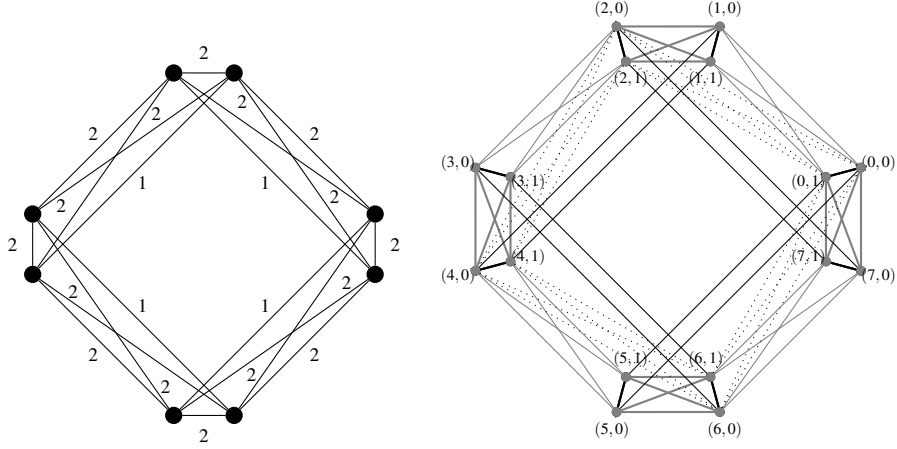


FIG. 1. A weighted graph on the left hand side with its 2-spinning graph on the right hand side..

reobtain the Kronecker product of G/R with the complete graph on k vertices as described in previous works on this topic^{5,6}.

B. Homogeneous Kuramoto Model

Given a weighted graph G with weight function w , the *homogenous Kuramoto model of coupled oscillators coupled through G* , is the system of differential equations given by:

$$\dot{\theta}_v = \sum_{u \in G_v} w_{uv} \sin(\theta_u - \theta_v) \quad v \in V G. \quad (1)$$

The system can be seen as running on the n -dimensional torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$, which is a compact manifold. Besides, the system is an analytic gradient one on $\mathbb{T}^n$, since $\dot{\theta} = -\nabla U$, for U the “*potential energy*” defined as

$$U(\theta_1, \dots, \theta_n) = \frac{1}{2} \sum_{uv \in EG} w_{uv} |e^{i\theta_u} - e^{i\theta_v}|^2 = w(G) - \sum_{uv \in EG} w_{uv} \cos(\theta_u - \theta_v).$$

Therefore, by the theory of analytic gradient systems⁹, every orbit goes to a consensus. A point $\theta \in \mathbb{R}^n$ is an *equilibrium* of G if it is an equilibrium of the system, i.e, if

$$\forall v \in V \quad \sum_{u \in G_v} w_{uv} \sin(\theta_u - \theta_v) = 0. \quad (2)$$

Clearly $\theta = (c, \dots, c)$ are equilibrium and are called *consensus*.

We say that a graph *synchronizes* iff almost every solution tends to a consensus, i.e., if the set of initial conditions that give rise to orbits that do not tend to a consensus has Lebesgue's measure zero. We known since the seminal work of Taylor¹, that there is a limit $\mu_c < 1$ called by Kassabov et al. *critical connectivity*², such that any graph with strong density greater than μ_c should synchronizes. We also known from the work of the latter authors², that the critical connectivity is at most 3/4.

Another important property of the system (1) is the orthogonality of the solutions to the vector $\vec{\mathbf{1}} = (1, 1, \dots, 1)$. Thus, the solutions are always running in a hyperplane orthogonal to $\vec{\mathbf{1}}$, which is in fact a $(n - 1)$ -dimensional torus. So, we only care about the behavior inside these hyperplanes, though, for simplicity, the calculations will be done in $\mathbb{R}^n$.

One way to see that a graph does not synchronizes is to prove that there exists a non consensus equilibrium θ^* where the Hessian matrix U''_{θ^*} of U at θ^* has all its eigenvalue, but one, positive. Another way to express this condition is by considering the smaller eigenvalue $a(U''_{\theta^*})$ in the hyperplane orthogonal to $\vec{\mathbf{1}}$, which is positive iff all the eigenvalues are positive. Let us call $a(U''_{\theta^*})$ the *algebraic connectivity* of θ^* following the notation in spectral graph theory where *algebraic connectivity* or also *Fiedler number* of the graph is reserved to the second smallest eigenvalue of the Laplacian matrix of the graph. The

element uv of U''_{θ^*} is

$$(U''_{\theta^*})_{uv} = \begin{cases} -w_{uv} \cos(\theta_u^* - \theta_v^*) & uv \in EG, \\ \sum_{u \sim v} w_{uv} \cos(\theta_u^* - \theta_v^*) & u = v, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Observe that, since U'' in a consensus coincides with the Laplacian matrix of the graph, then, the algebraic connectivity of a graph is the algebraic connectivity of any consensus.

III. EQUILIBRIA OF k -SPINNINGS

In this section we show a way to obtain equilibria of a spinning graph from equilibria of the original graph, as well as how the former's linear stability can be ensured by the stability of the last one. Given a weighted graph G and its k -spinning $S_k(G)$, if θ is an element of $\mathbb{R}^{|G|}$, let θ^σ be the element of $\mathbb{R}^{|G \times \mathbb{Z}_k|}$ given by

$$\theta_{(u,i)}^\sigma = \theta_u \quad \forall (u,i) \in G \times \mathbb{Z}_k.$$

Proposition 1. *A point θ is an equilibrium of a weighted graph G iff the point θ^σ is an equilibrium of $S_k(G)$.*

Proof. The result is a direct consequence of the definition of equilibrium and the following equalities, where w is the weight of G , given a vertex (u,i) then

$$\sum_{(u,j) \in S_k(G)_{(v,i)}} \sin(\theta_{(u,i)}^\sigma - \theta_{(v,i)}^\sigma) = \sum_{u \in G_v} \sum_{j=1}^{w_{uv}-1} \sin(\theta_u - \theta_v) = \sum_{u \in G_v} w_{uv} \sin(\theta_u - \theta_v) = 0. \quad (4)$$

□

Let us notice that this proposition can be extended to a much more general constructions. For instance, if each vertex v of the graph is substituted by a graph with order greater than the maximum degree of the vertices adjacent with v , the equations 4, will be still valid. However, the stability relationship between both equilibria is weaker.

Proposition 2. *If θ is an equilibrium of a weighted graph G with weight w and Hessian matrix U''_θ , then, the eigenvalues of the Hessian matrix U''_{θ^σ} of $S_k(G)$ in θ^σ are the eigenvalues of U''_θ with the same multiplicity plus a set of $(k-1)|G|$ (linearly independent) eigenvalues. These last eigenvalues are positive for k greater than twice the weight of the graph, i.e. $k > 2w(G)$. In particular, the algebraic connectivity of θ in G and θ^σ in $S_k(G)$ have the same sign for k large enough.*

Proof. From Eq. 3, the matrix U''_{θ^σ} has elements

$$(U''_{\theta^\sigma})_{(u,i)(v,j)} = \begin{cases} k-1 + \sum_{v \in G_u} w_{uv} \cos(\theta_u - \theta_v), & u = v, i = j, \\ -1 & u = v, i \neq j, \\ -\cos(\theta_{(u,i)}^\sigma - \theta_{(v,j)}^\sigma) = -\cos(\theta_u - \theta_v) & u \sim v, j \in i + \mathbb{N}_{w_u v}, \\ 0 & \text{otherwise.} \end{cases}$$

Let x be an eigenvector of U''_{θ^σ} with eigenvalue λ . Therefore,

$$\begin{aligned} \lambda x_{(u,i)} &= (U''_{\theta^\sigma} x)_{(u,i)} = \sum_{(v,j) \in S_k(G)_{(u,i)}} (U''_{\theta^\sigma})_{(u,j),(v,i)} x_{(v,i)} \\ &= \left(k-1 + \sum_{v \in G_u} w_{uv} \cos(\theta_u - \theta_v) \right) x_{(u,i)} - \sum_{j \in \mathbb{Z}_k \setminus \{i\}} x_{(v,j)} - \sum_{(v,i) \in S_k(G)_{(u,j)}} \cos(\theta_u - \theta_v) x_{(v,j)}. \end{aligned}$$

Calling X_v to the sum $\sum_{j \in \mathbb{Z}_k} x_{(v,j)}$, we have

$$\begin{aligned} \lambda x_{(u,i)} &= (k-1 + (U''_{\theta})_{uu}) x_{(u,i)} - (X_u - x_{(u,i)}) - \sum_{v \in G_u} \cos(\theta_u - \theta_v) \sum_{j \in i + \mathbb{N}_{w_{uv}}} x_{(v,j)}, \\ &= (k + (U''_{\theta})_{uu}) x_{(u,i)} - X_u - \sum_{v \in G_u} \cos(\theta_u - \theta_v) \sum_{j \in i + \mathbb{N}_{w_{uv}}} x_{(v,j)}. \end{aligned} \quad (5)$$

If we sum these equations for all $i \in \mathbb{Z}_k$ we obtain

$$\lambda X_u = (k + (U''_{\theta})_{uu}) X_u - kX_u - \sum_{v \in G_u} \cos(\theta_u - \theta_v) w_{uv} X_v$$

which after canceling the $(k-1)$'s, we obtain

$$\lambda X_u = (U''_{\theta})_{uu} X_u - \sum_{v \in G_u} w_{uv} \cos(\theta_u - \theta_v) X_v.$$

Therefore, λ is an eigenvalue of G , with eigenvector $(X_u)_{u \in V_G}$, unless $X_u = 0$ for all u . Conversely, let $(x_u)_{u \in V_G}$ be an eigenvector of U''_{θ} , with eigenvalue λ , then $x_{(u,i)} = x_u$ gives rise an eigenvalue of $U''_{\theta\sigma}$ with eigenvalue λ . Let us return to the case of $X_u = 0$ for all u . Plugin $X_u = 0$ into Eq. 5 we get

$$\lambda x_{(u,i)} = \left(k + \sum_{v:uv \in EG} w_{uv} \cos(\theta_u - \theta_v) \right) x_{(u,i)} - \sum_{v:uv \in EG} \cos(\theta_u - \theta_v) \sum_{j \in i + \mathbb{N}_{w_{uv}}} x_{(v,j)}.$$

Let (u,i) be the greatest component of x in absolute value, that we can suppose w.l.o.g.

that it is 1, i.e. $x_{(u,i)} = 1$ and $|x_{(v,j)}| \leq 1$ for all (v,j) . Then

$$\begin{aligned}\lambda &= k + \sum_{v \in G_u} w_{uv} \cos(\theta_u - \theta_v) - \sum_{v \in G_u} \cos(\theta_u - \theta_v) \sum_{j \in i + \mathbb{N}_{w_{uv}}} x_{(v,j)} \\ &= k + \sum_{v \in G_u} \left(w_{uv} - \sum_{j \in i + \mathbb{N}_{w_{uv}}} x_{(v,j)} \right) \cos(\theta_u - \theta_v) > k - 2w(G),\end{aligned}$$

which is positive if k is large enough, at most twice the weight of G . Since the space of solutions of the system of equations $X_u = 0$ for $u \in G$, has dimension $(k-1)|G|$, we conclude the proof. □

IV. THE COMPLEXITY OF DETERMINING WHETHER A SIMPLE GRAPH HAVE A NON-ZERO STABLE EQUILIBRIUM.

In 2015 R. Taylor proved that *determining whether weighted Kuramoto models have a non-zero linearly stable equilibrium is NP-hard*⁸. However, Taylor left open the un-weighted case, though reducing it to a problem he believed to be NP-Hard. At the light of the previous sections, the k -spinning of the graph built by Taylor with k large enough, will have a linearly stable equilibrium iff the graph has one, so determining if the k -spinning has a stable equilibrium is as hard as determining if the graph does. Therefore we have

Theorem 1. *Determining whether an homogeneous Kuramoto model has a non consensus linearly stable equilibrium is NP-hard.*

V. A NEW LOWER BOUND FOR THE CRITICAL CONNECTIVITY

Let us consider the family of weighted graphs G_p with vertex set $\mathbb{Z}_8$ and edges $EG_p = E_a \cup E_b \cup E_c \cup E_d$ where:

$$E_a = \{(2i)(2i+1) : i \in \mathbb{N}_4\},$$

$$E_b = \{i(i+2) : i \in \mathbb{Z}_8\},$$

$$E_c = \{(2i)(2i+2) : i \in \mathbb{N}_4\},$$

$$E_d = \{(2i)(2i+4) : i \in \mathbb{N}_4\},$$

with weight $w(uv) = t$ if $uv \in E_t$. Figure 1 shows $G_{(2,2,2,1)}$.

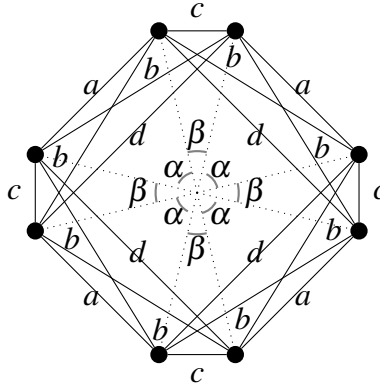


FIG. 2. The weighted graph G_p with $p = (a, b, c, d)$.

Let us find a linearly stable equilibrium θ^* of G_p of the form

$$\theta_{2i+s}^* = \frac{\pi}{2}i + s\frac{\beta}{2} \quad \forall i \in \mathbb{N}_4, \quad s = \pm 1.$$

If $\alpha = \pi/2 - \beta$ the condition for θ^* to be an equilibrium is:

$$a \sin(\alpha) = c \sin(\beta) + d \sin(\alpha + 2\beta),$$

since the edges of type b compensate each other, see Figure 2. Taking in count that $\alpha + \beta = \pi/2$, then $\sin(\beta) = \cos(\alpha)$, $\sin(\alpha + 2\beta) = \sin(\alpha)$ and previous equation is equivalent to $a \sin(\alpha) = c \cos(\alpha) + d \sin(\alpha)$, i.e. $(a - d) \tan \alpha = c$ so $\alpha = \arctan(c/(a - c))$.

Lemma 1. *The equilibria θ of weighted graph G_p with $p = (a, b, c, d)$ is linearly stable equilibrium if $c^2 > 2ad$.*

Proof. The matrix of U''_θ has elements:

$$(U''_\theta)_{uv} = \begin{cases} -w_{uv} \cos(\theta_u - \theta_v) & u \neq v, \\ \sum_{u \sim v} w_{uv} \cos(\theta_u - \theta_v) & u = v, \\ 0 & \text{otherwise,} \end{cases}$$

thus

$$(U''_\theta)_{uv} = \begin{cases} -a \cos(\alpha) & uv \in E_a, \\ -b \cos(\alpha + \beta) & uv \in E_b, \\ -c \cos(\beta) & uv \in E_c, \\ -d \cos(\alpha + 2\beta) & uv \in E_d, \\ a \cos(\alpha) + b \cos(\alpha + \beta) + c \cos(\beta) + d \cos(\alpha + 2\beta) & u = v, \\ 0 & \text{otherwise .} \end{cases}$$

$$(U''_{\theta})_{uv} = \begin{cases} -a \cos \alpha & uv \in E_a, \\ 0 & uv \in E_b, \\ -c \sin \alpha & uv \in E_c, \\ d \cos \alpha & uv \in E_d, \\ a \cos \alpha + c \sin \alpha - d \cos \alpha & u = v, \\ 0 & \text{otherwise .} \end{cases}$$

Extracting $\cos \alpha$ as a factor, we obtain

$$(U''_{\theta})_{uv} = \cos \alpha \begin{cases} -a & uv \in E_a, \\ -c \tan \alpha & uv \in E_c, \\ d & uv \in E_d, \\ a + c \tan \alpha - d & u = v, \\ 0 & \text{otherwise .} \end{cases} = \cos \alpha \begin{cases} -a & uv \in E_a, \\ -\frac{c^2}{a-d} & uv \in E_c, \\ d & uv \in E_d, \\ a + \frac{c^2}{a-d} - d & u = v, \\ 0 & \text{otherwise .} \end{cases}$$

Extracting $1/(a-d)$ as a factor, and setting $f = a-d$, we obtain

$$(U''_{\theta})_{uv} = \frac{\cos \alpha}{f} \begin{cases} -af & uv \in E_a, \\ -c^2 & uv \in E_c, \\ df & uv \in E_d, \\ f^2 + c^2 & u = v, \\ 0 & \text{otherwise .} \end{cases}$$

i.e.

$$\frac{f}{\cos \alpha} U''_{\theta} = \begin{pmatrix} c^2 + f^2 & -af & 0 & 0 & 0 & af - f^2 & 0 & -c^2 \\ -af & c^2 + f^2 & -c^2 & 0 & af - f^2 & 0 & 0 & 0 \\ 0 & -c^2 & c^2 + f^2 & -af & 0 & 0 & 0 & af - f^2 \\ 0 & 0 & -af & c^2 + f^2 & -c^2 & 0 & af - f^2 & 0 \\ 0 & af - f^2 & 0 & -c^2 & c^2 + f^2 & -af & 0 & 0 \\ af - f^2 & 0 & 0 & 0 & -af & c^2 + f^2 & -c^2 & 0 \\ 0 & 0 & 0 & af - f^2 & 0 & -c^2 & c^2 + f^2 & -af \\ -c^2 & 0 & af - f^2 & 0 & 0 & 0 & -af & c^2 + f^2 \end{pmatrix}.$$

The eigenvalues of this matrix are $0, 2f^2, 2c^2, 2c^2 + 2f^2, c^2 + f^2 + \sqrt{4a^2f^2 - 4af^3 + c^4 + f^4}$ with multiplicity two and $c^2 + f^2 - \sqrt{4a^2f^2 - 4af^3 + c^4 + f^4}$ with multiplicity two too (see Appendix A). Besides the null eigenvalue corresponding to the vector $\vec{1}$, the other eigenvalues are positive but the last one, that will be positive iff $2c^2f^2 > 4a^2f^2 - 4af^3$, i.e., if $c^2 > 2a^2 - 2af = 2ad$, as we wanted to prove. $\square$

We are now in conditions to find a lower bound for the critical connectivity by making k -spinnings of the graphs G_p with a special choice of $p = (a, b, c, d)$. Figure 1 shows the 2-spinning of graph G_p for $p = (2, 2, 2, 1)$. We can see that the graphs $S_k(G_p)$ are δ -regular graphs with $\delta = a + 2b + d + c + (k - 1)$, and orders $|S_k(G_p)| = 8k$, so, these graphs have strong density

$$\mu(S_k(G_p)) = \frac{a + 2b + d + c + (k - 1)}{8k}.$$

We will choose parameters a, b, c, d proportional to k , so the strong density will tend to a constant. Unfortunately, that choice does not verify the hypothesis of Proposition 2, but a minor change in the arguments given in the proof of this proposition will be enough to arrive at the same conclusion.

Theorem 2. *The critical connectivity is at least 11/16.*

Proof. Consider the k -spinning graph $S_k(G_{p_k})$ of G_{p_k} with $p_k = (k, k, k, \lceil k/2 \rceil - 1)$, then

$$\mu(S_k(G_{p_k})) = \frac{4k + \lceil k/2 \rceil - 1 + k - 1}{8k} \rightarrow 11/16.$$

Since the choice of $a = b = c = k$ and $d = \lceil k/2 \rceil - 1$ verifies $c^2 = k^2 > 2k(\lceil k/2 \rceil - 1)$, then, by the Lemma 1 the equilibrium θ of the weighted graph G_{p_k} is linearly stable. We would like to apply Proposition 2 to deduce that G_{p_k} does not synchronize, but the hypothesis $k \geq w(G_{p_k})$ is not verified. Nevertheless, we can prove that the $(k-1)|G|$ eigenvalues of $U''_{\theta\sigma}$ that need this condition to be positive in the proof on Proposition 2, are still positive. Indeed, following the last part in Proposition 2's proof, let us consider the formula for the eigenvalue λ :

$$\lambda = k + \sum_{v \in G_u} w_{uv} \cos(\theta_u - \theta_v) - \sum_{v \in G_u} \cos(\theta_u - \theta_v) \sum_{j \in i + \mathbb{N}_{w_{uv}}} x_{(v,j)}.$$

In our case this formula turn out to be

$$\begin{aligned} \lambda &= k + a \cos \alpha + c \sin \alpha - d \cos \alpha \\ &\quad - \cos \alpha \sum_{j \in \mathbb{Z}_k} x_{(v,j)} - \sin \alpha \sum_{j \in \mathbb{Z}_k} x_{(v',j)} + \cos \alpha \sum_{j \in i + \mathbb{N}_d} x_{(v'',j)}, \end{aligned}$$

where $uv \in E_a$, $uv' \in E_c$, $uv'' \in E_d$. Then, since $a = c = k$ and $\sum_{j \in \mathbb{Z}_k} x_{(v,j)} = \sum_{j \in \mathbb{Z}_k} x_{(v',j)} = 0$

$$\begin{aligned} \lambda &= k + k \cos \alpha + k \sin \alpha - d \cos \alpha + \cos \alpha \sum_{j \in i + \mathbb{N}_d} x_{(v'',j)} \\ &= k + k \sin \alpha + \left(k - d + \sum_{j \in i + \mathbb{N}_d} x_{(v'',j)} \right) \cos \alpha, \end{aligned}$$

which is positive since $|x_{(v'',j)}| \leq 1$ and $d = \lceil k/2 \rceil - 1$. □

It is worth to say that $a = b = c = k$ and $d = \lceil k/2 \rceil - 1$ maximize the strong density of the k -spinning graph $G_{(a,b,c,d)}$ subject to the condition $c^2 > 2ad$, although it is not the only choice. We can also have chosen $d = b = c = k$ and $a = \lceil k/2 \rceil - 1$.

The first k such that the strong density μ_k is greater than the best lower bound known so far⁷, i.e., 0.6838, is $k = 34$, that correspond to a graph with 272 vertices, an order far away from exhaustive numerical experiments. The first k such that the strong density μ_k is greater than 0.6874, is $k = 1250$, that corresponds to a graph with 10000 vertices.

Although angles of $\pi/2$ in the equilibrium contribute in other scenarios to nonlinearity behavior⁶, in our case, do not. In fact, the edges corresponding to that angle are those in E_b , which do not give rise to constraints of the equilibrium nor to the positive definition of the quadratic forms, because $\cos \pi/2 = 0$, so $b = k$ is always an optimal choice. However,

VI. CONSEQUENCES

We have develop a way to associate an unweighted graph from a weighted one such that the existence of linearly stable equilibria in the last one implies, under some conditions, the existence of such equilibria in the former. This technique allows us to improve the known lower bound for the critical connectivity from 0.6838 to 0.6875. We hope this technique will allow to improve that limit to at least 0.7495, according to the intuitive arguments given by Townsend et al.⁶. If these arguments are true, the gap from 0.7495 to 0.75 will require a totally different approach. On the other hand we also were able to prove that deciding if a graph has non trivial linearly stable equilibria is an NP-Hard problem based on a result by Taylor⁸ and the weighted-to-unweighted technique. However the complexity of deciding if a graph synchronizes is still open with only partial results¹⁰, moreover, we do not even know if that problem is computable. The last question is related to the absences of a combinatorial characterization of synchronizing graphs. This issue is similar to the characterization of planar graphs, in the sense that both types of graphs, i.e.,

planar and synchronizing are defined in a non combinatorial way. Kazimierz Kuratowski gives a combinatorial characterization of the formers in terms of forbidden induced subgraphs. Unfortunately, this particular way is not possible for synchronizing graphs, since for any fixed graph there are synchronized and unsynchronized ones with that graph as an induced subgraph¹¹. The biggest step into a combinatorial characterization was done by Taylor¹ when he found a connection between the minimum degree and the synchronization property, however we are still far away from a final solution.

VII. ACKNOWLEDGES

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Appendix A: Eigenvector of matrix in proof of Lemma 1

In this section we provide a base of eigenvectors of the scaled Hessian matrix in proof of Lemma 1, i.e.

$$M = \begin{pmatrix} c^2 + f^2 & -af & 0 & 0 & 0 & af - f^2 & 0 & -c^2 \\ -af & c^2 + f^2 & -c^2 & 0 & af - f^2 & 0 & 0 & 0 \\ 0 & -c^2 & c^2 + f^2 & -af & 0 & 0 & 0 & af - f^2 \\ 0 & 0 & -af & c^2 + f^2 & -c^2 & 0 & af - f^2 & 0 \\ 0 & af - f^2 & 0 & -c^2 & c^2 + f^2 & -af & 0 & 0 \\ af - f^2 & 0 & 0 & 0 & -af & c^2 + f^2 & -c^2 & 0 \\ 0 & 0 & 0 & af - f^2 & 0 & -c^2 & c^2 + f^2 & -af \\ -c^2 & 0 & af - f^2 & 0 & 0 & 0 & -af & c^2 + f^2 \end{pmatrix}.$$

A basis of eigenvector of M are the column vectors of the following matrix,

$$V = \begin{pmatrix} -\frac{r}{c^2} & \frac{-2af+f^2}{c^2} & \frac{r}{c^2} & \frac{-2af+f^2}{c^2} & 1 & 1 & -1 & -1 \\ \frac{f(2a-f)}{c^2} & \frac{r}{c^2} & \frac{f(2a-f)}{c^2} & -\frac{r}{c^2} & 1 & -1 & -1 & 1 \\ 0 & -1 & 0 & -1 & 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 0 & 1 & 1 & 1 & 1 \\ \frac{r}{c^2} & \frac{f(2a-f)}{c^2} & -\frac{r}{c^2} & \frac{f(2a-f)}{c^2} & 1 & 1 & -1 & -1 \\ \frac{-2af+f^2}{c^2} & -\frac{r}{c^2} & \frac{-2af+f^2}{c^2} & \frac{r}{c^2} & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

with $r = \sqrt{4a^2f^2 - 4af^3 + c^4 + f^4}$, as can be checked. Therefore $M \cdot V$ is a diagonal matrix with the corresponding eigenvalues of the eigenvector. More concretely,

$$M \cdot V = \begin{pmatrix} c^2 + f^2 + r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 + f^2 + r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 + f^2 - r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c^2 + f^2 - r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2f^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c^2 + 2f^2 \end{pmatrix}$$

REFERENCES

- ¹R. Taylor, “There is no non-zero stable fixed point for dense networks in the homogeneous kuramoto model,” *Journal of Physics A: Mathematical and Theoretical* **45**, 055102 (2012).
- ²M. Kassabov, S. H. Strogatz, and A. Townsend, “Sufficiently dense kuramoto networks are globally synchronizing,” *Chaos* **31**, 5905–5918 (2021).
- ³S. Ling, R. Xu, and A. S. Bandeira, “On the landscape of synchronization networks: A perspective from nonconvex optimization,” *SIAM Journal on Optimization* **29**, 1879–1907 (2019), <https://doi.org/10.1137/18M1217644>.
- ⁴J. Lu and S. Steinerberger, “Synchronization of kuramoto oscillators in dense networks,” *Nonlinearity* **33**, 5905–5918 (2020).
- ⁵E. A. Canale and P. Monzón, “Exotic equilibria of harary graphs and a new minimum degree lower bound for synchronization,” *Chaos: An Interdisciplinary Journal of Nonlinear Science* **25**, 023106 (2015).
- ⁶A. Townsend, M. Stillman, and S. H. Strogatz, “Dense networks that do not synchronize and sparse ones that do,” *Chaos: An Interdisciplinary Journal of Nonlinear Science* **30**, 083142 (2020).
- ⁷R. Yoneda, T. Tatsukawa, and J. Teramae, “The lower bound of the network connectivity guaranteeing in-phase synchronization,” *Chaos: An Interdisciplinary Journal of Nonlinear Science* **31**, 063124 (2021).
- ⁸R. Taylor, “Finding non-zero stable fixed points of the weighted kuramoto model is np-hard,” (2015), 10.48550/ARXIV.1502.06688.
- ⁹S. Lojasiewicz, “Sur les trajectoires du gradient d’une fonction analytique,” *Tech. Rep. (Seminari di Geometria 1982-1983 (Universita di Bologna, Istituto di Geometria, Dipartimento di Matematica), 1984)*.

- ¹⁰E. Canale, P. Monzón, and F. Robledo, “On the complexity of the classification of synchronizing graphs,” in *Grid and Distributed Computing, Control and Automation*, edited by T.-h. Kim, S. S. Yau, O. Gervasi, B.-H. Kang, A. Stoica, and D. Ślęzak (Springer Berlin Heidelberg, Berlin, Heidelberg, 2010) pp. 186–195.
- ¹¹E. Beutler, “Almost global synchronization of symmetric kuramoto coupled oscillators,” in *Systems, Structure and Control*, edited by E. A. Canale and P. Monzón (InTech, 2008) Chap. 4.