A Mixed Formulation For The Fractional Poisson Problem

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Abstract

In this work we explore a mixed formulation of the fractional Poisson problem via the fractional divergence and fractional gradient. Following Hughes and Masud [1] we pursue a stabilized formulation that results in a coercive and well-posed problem. We prove the convergence of this discretization, its order and perfom some numerical experiments.

Fractional Gradient and Fractional Divergence

Definition

Following [2], the fractional gradient and divergence can be expressed in integral form as:

$$\mathbf{grad}^s\varphi(x) = \mu(d,s)\int_{\mathbb{R}^d}\frac{(y-x)(\varphi(y)-\varphi(x))}{|y-x|^{d+s+1}}\,dy,$$

$$\operatorname{div}_s\Phi(x) = \mu(d,s)\int_{\mathbb{R}^d}\frac{(y-x)\cdot(\Phi(y)-\Phi(x))}{|y-x|^{d+s+1}}\,dy,$$

where the normalization constant is given by

$$\mu(d,s) = \frac{2^s\Gamma\left(\frac{d+s+1}{2}\right)}{\pi^{d/2}\Gamma\left(\frac{1-s}{2}\right)}.$$

Problem Formulation

Let $\Omega\subset\mathbb{R}^d$ be a bounded and Lipschitz domain. We consider the following fractional Darcy problem: find $(p,\mathbf{\Phi})\in\tilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)$ such that

$$\begin{cases}\mathbf{\Phi}+\mathbf{grad}^s p = 0 & \text{in } \mathbb{R}^d,\\ \operatorname{div}^s\mathbf{\Phi} = f & \text{in } \Omega,\\ p = 0 & \text{in } \Omega^c,\end{cases}$$

where

$$H(\operatorname{div}^s;\Omega) := \{\mathbf{\Psi}\in L^2(\mathbb{R}^d,\mathbb{R}^d)\colon (\operatorname{div}^s\mathbf{\Psi})\big|_\Omega\in L^2(\Omega)\},$$

furnished with the norm

$$\|\mathbf{\Psi}\|_{H(\operatorname{div}^s;\Omega)} := \left(\|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)} + \|(\operatorname{div}^s\mathbf{\Psi})\big|_\Omega\|^2_{L^2(\Omega)}\right)^{1/2}.$$

We also denote by $\tilde{L}^2(\Omega)$ the space of $L^2(\Omega)$ that are extended by zero to $\mathbb{R}^d$.

The weak formulation of the problem reads: find $(p,\mathbf{\Phi})\in\tilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)$ such that, for all $(q,\mathbf{\Psi})\in\tilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)$,

$$\int_{\mathbb{R}^d}\mathbf{\Phi}\cdot\mathbf{\Psi} - \int_{\mathbb{R}^d}p\operatorname{div}^s\mathbf{\Psi} + \int_{\mathbb{R}^d}q\operatorname{div}^s\mathbf{\Phi} = \int_{\mathbb{R}^d}fq. \tag{1}$$

Note that all but the first of the integrals above need to be effectively computed in Ω.

We are using the integration by parts formula:

$$\int_{\mathbb{R}^d}\mathbf{grad}^s q\cdot\mathbf{\Psi} = -\int_{\mathbb{R}^d}q\operatorname{div}^s\mathbf{\Psi}, \quad \text{for all } q\in H^s(\Omega) \text{ and } \mathbf{\Psi}\in H(\operatorname{div}^s;\Omega).$$

Well-Posedness

Let

$$\begin{aligned} a&\colon H(\operatorname{div}^s;\Omega)\times H(\operatorname{div}^s;\Omega)\to\mathbb{R}, & a(\mathbf{\Phi},\mathbf{\Psi}) &= \int_{\mathbb{R}^d}\mathbf{\Phi}\cdot\mathbf{\Psi},\\ b&\colon \tilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)\to\mathbb{R}, & b(q,\mathbf{\Psi}) &= \int_\Omega q\operatorname{div}^s\mathbf{\Psi},\\ F&\colon \tilde{L}^2(\Omega)\to\mathbb{R}, & F(q) &= \int_\Omega fq. \end{aligned} \tag{2}$$

Problem (1) is well-posed if

I the form a is coercive in $\ker B$, the Riesz representantive of the map $b(\cdot,\mathbf{\Psi})$;

II the form b satisfies an inf-sup condition.

Stabilized Form

To shorten the notation, we define in $\left(\tilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)\right)\times\left(\tilde{L}^2(\Omega)\times H(\operatorname{div}^s;\Omega)\right)$ the form

$$\mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) := a(\mathbf{\Phi},\mathbf{\Psi}) - b(p,\mathbf{\Psi}) + b(q,\mathbf{\Phi}).$$

Let $\mathbb{V} := \tilde{H}^s(\Omega)\times H(\operatorname{div}^s;\Omega)$. We introduce the stabilized form in $\mathbb{V}\times\mathbb{V}$:

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) := \mathcal{L}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) + \frac{1}{2}\int_{\mathbb{R}^d}(\mathbf{\Phi}+\mathbf{grad}^s p)\cdot(-\mathbf{\Psi}+\mathbf{grad}^s q). \tag{3}$$

With this, we consider the stabilized problem: find $(p,\mathbf{\Phi})\in\mathbb{V}$ such that

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) = F(q) \quad \forall (q,\mathbf{\Psi})\in\mathbb{V}.$$

We introduce a norm in $\mathbb{V}$:

$$|||(q,\mathbf{\Psi})||| := \left[\frac{1}{2}\left(\|\mathbf{grad}^s q\|^2_{L^2(\mathbb{R}^d)} + \|\mathbf{\Psi}\|^2_{L^2(\mathbb{R}^d)}\right)\right]^{1/2}.$$

Coercivity/Stability, Continuity and Well-Posedness

We have

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(p,\mathbf{\Phi})) = |||(p,\mathbf{\Phi})|||^2 \quad \forall (p,\mathbf{\Phi})\in\mathbb{V},$$

$$\mathcal{L}_{\text{stab}}((p,\mathbf{\Phi}),(q,\mathbf{\Psi})) \le |||(p,\mathbf{\Phi})|||\,|||(q,\mathbf{\Psi})||| \quad \forall (p,\mathbf{\Phi}),(q,\mathbf{\Psi})\in\mathbb{V}.$$

As usual, the Lax-Milgram theorem gives rise to the well-posedness of our problem.

Finite Element Discretization

We are approximating $\mathbf{\Phi}$, which is not compactly supported, and the form a in (2) and the stabilization term in (3) involve integration in $\mathbb{R}^d$. To tackle this problem, we consider a ball B_H containing Ω and such that $H := d(\Omega, B_H^c) >> 1$.

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of **regular**, simplicial triangulations of $\overline{B_H}$ with mesh size $h>0$. Moreover, we assume that $\{T\in\mathcal{T}_h : T\cap\Omega\neq\emptyset\}$ is a triangulation of Ω for all $h>0$. On the triangulation $\mathcal{T}_h$ we define

$$\mathbb{V}_h = \{(q_h,\mathbf{\Psi}_h)\in\mathcal{P}_1(\mathcal{T}_h)\times\mathcal{P}_1^d(\mathcal{T}_h) : q_h|_{\Omega^c} = 0,\ \mathbf{\Psi}_h|_{B_H^c} = 0\}.$$

We consider the following discrete problem: find $(p_h,\mathbf{\Phi}_h)\in\mathbb{V}_h$ such that

$$\mathcal{L}_{\text{stab}}((p_h,\mathbf{\Phi}_h),(q_h,\mathbf{\Psi}_h)) = F(q_h) \quad \forall (q_h,\mathbf{\Psi}_h)\in\mathbb{V}_h. \tag{4}$$

Galerkin Orthogonality

The fact that $\mathbb{V}_h\subset\mathbb{V}$ implies the existence and uniqueness of solutions to (4). Let $(p,\mathbf{\Phi})\in\mathbb{V}$ and $(p_h,\mathbf{\Phi}_h)\in\mathbb{V}_h$ be the exact solution and the discrete solution, respectively. Then, we have the Galerkin orthogonality:

$$\mathcal{L}_{\text{stab}}((p-p_h,\mathbf{\Phi}-\mathbf{\Phi}_h),(q_h,\mathbf{\Psi}_h)) = 0 \quad \forall (q_h,\mathbf{\Psi}_h)\in\mathbb{V}_h.$$

Therefore,

$$|||(p-p_h,\mathbf{\Phi}-\mathbf{\Phi}_h)||| \le \inf_{(q_h,\mathbf{\Psi}_h)\in\mathbb{V}_h}|||(p-q_h,\mathbf{\Phi}-\mathbf{\Psi}_h)|||.$$

Order of Convergence

To obtain **convergence rates**, we employ regularity estimates up to $\partial\Omega$ for the fractional Poisson problem [3], regularity estimates for the flux given by the mapping properties of the $\mathbf{grad}^s$ operator [4] and quasi-interpolation estimates (cf., [5] and [6]): We have

$$|||(p-p_h,\mathbf{\Phi}-\mathbf{\Phi}_h)||| \le \begin{cases} Ch^{\frac{1}{2}}|\log h|^{\frac{1}{2}}\|f\|_{L^2(\Omega)}, & \text{for } s>1/2,\\ Ch^s|\log h|^{\frac{1}{2}}\|f\|_{L^2(\Omega)}, & \text{for } s<1/2,\\ Ch^{\frac{1}{2}}|\log h|\,\|f\|_{L^2(\Omega)}, & \text{for } s=1/2. \end{cases}$$

Numerical Experiments

We test the convergence rates for different values of s in 1d. We take $f\equiv 1$, $\Omega=(-1,1)$ and the distance $H=d(B_H^c,\Omega)$ is chosen such that $H^{-1-d-2s}\simeq h|\log h|$.

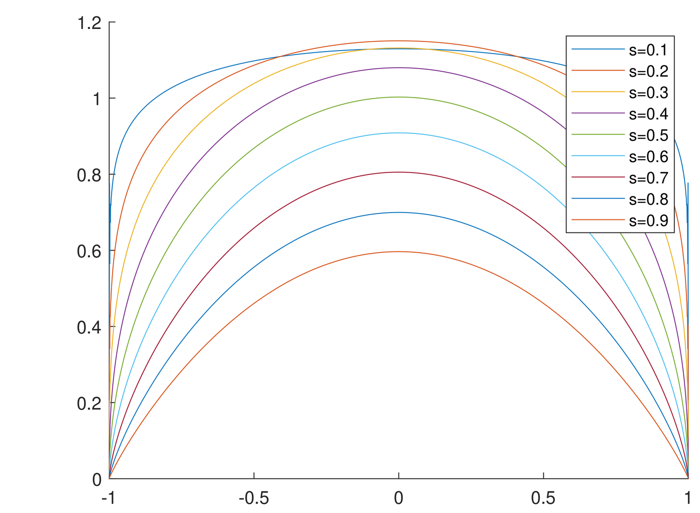

Figure: Computed pressures for different values of s.

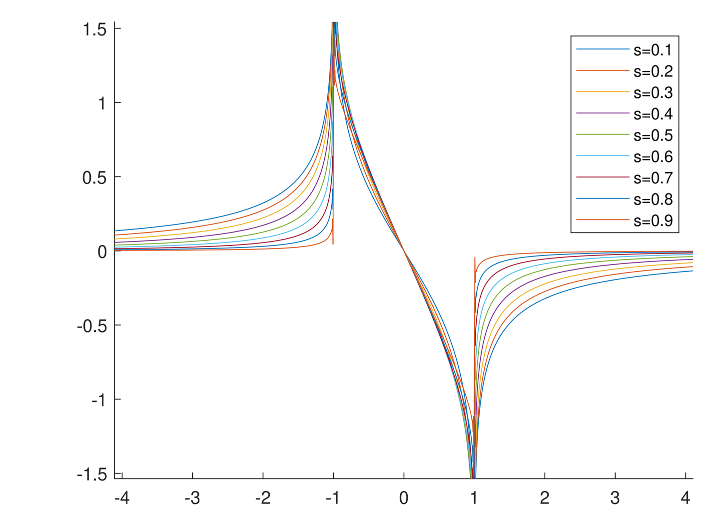

Figure: Computed fluxes for different values of s.

Value of s	H^s order	L^2 order	Value of s	H^s order	L^2 order
0.1	0.4691	0.4949	0.6	0.5005	0.9966
0.2	0.4956	0.6444	0.7	0.5009	0.9928
0.3	0.5000	0.7968	0.8	0.5014	0.9952
0.4	0.5004	0.9236	0.9	0.5017	1.0045
0.5	0.5005	1.0012			

Table: Order of convergence for the pressure p.

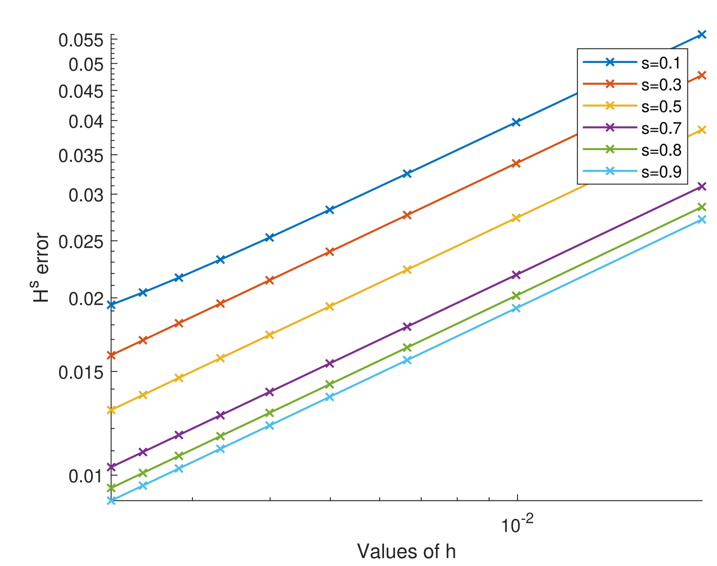

Figure: Error in H^s vs h.

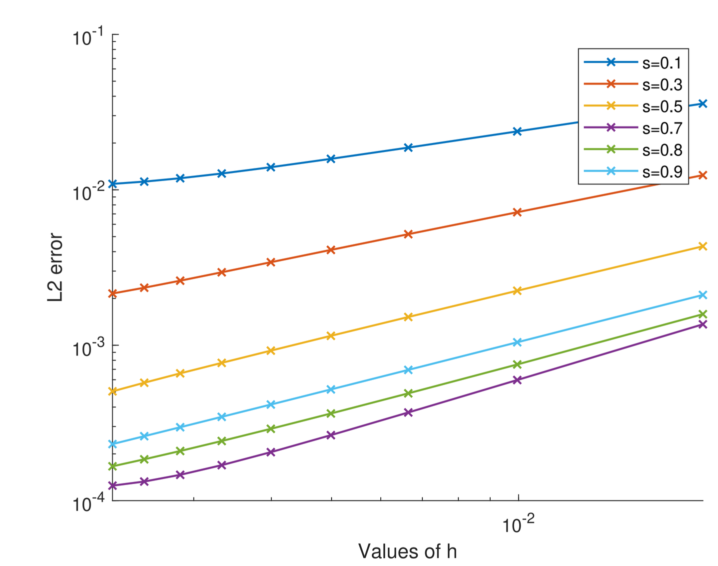

Figure: Error in L^2 vs h.

Work in progress and questions

- 2d implementation.
- Non-zero Dirichlet conditions, and Neumann conditions.
- Convergence rates in non-uniform meshes.
- Test $(\mathcal{P}_0)$ elements for the pressure p.

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