Fractional (p,s)-Laplacian: A Numerical Approach

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Abstract

We study the Dirichlet problem for a fractional p-Laplacian defined through the Riesz fractional gradient.Our approach provides a numerical framework by introducing a discretization scheme and establishing interpolation and error bounds. The study is divided into three main sections: theoretical foundations, numerical methods, and computational results.

Riesz Fractional Gradient and Divergence

Definition

For the definition of the fractional gradient and divergence, following [4], these operators can be expressed in integral form as:

$$\nabla^s\varphi(x) = \mu(N,s)\int_{\mathbb{R}^N}\frac{(y-x)(\varphi(y)-\varphi(x))}{|y-x|^{N+s+1}}\,dy,$$

$$\operatorname{div}_s\Phi(x) = \mu(N,s)\int_{\mathbb{R}^N}\frac{(y-x)\cdot(\Phi(y)-\Phi(x))}{|y-x|^{N+s+1}}\,dy,$$

where $s\in(0,1)$ and the normalization coefficient is given by:

$$\mu(N,s) = \frac{2^s\Gamma\left(\frac{N+s+1}{2}\right)}{\pi^{N/2}\Gamma\left(\frac{1-s}{2}\right)}.$$

Lions-Calderón spaces

From this definition, the *Lions-Calderón spaces* naturally arise as the associated variational spaces, defined by

$$X^{s,p}\left(\mathbb{R}^N\right) := \left\{f\in L^p\left(\mathbb{R}^N\right) : D^s f\in L^p\left(\mathbb{R}^N;\mathbb{R}^N\right)\right\},$$

endowed with the norm

$$\|f\|_{X^{s,p}(\mathbb{R}^N)} := \left(\|f\|^p_{L^p(\mathbb{R}^N)} + \|D^s f\|^p_{L^p(\mathbb{R}^N;\mathbb{R}^N)}\right)^{\frac{1}{p}}.$$

The relationship between these spaces and the classical fractional Sobolev spaces is given by

$$X^{s,p}\left(\mathbb{R}^N\right) = W^{s,p}\left(\mathbb{R}^N\right) \quad \text{if and only if } p=2.$$

Moreover, under the nested inclusion of interpolation spaces, we have

$$X^{s+\varepsilon,p}\left(\mathbb{R}^N\right)\subset W^{s,p}\left(\mathbb{R}^N\right)\subset X^{s-\varepsilon,p}\left(\mathbb{R}^N\right),$$

for all $p\in(1,\infty)$, $s\in(0,1)$, and $\varepsilon>0$. We denote by $\tilde{X}_{s,p}(\Omega)$ the zero extension spaces in Ω.

Problem Statement

Our goal is to study the Dirichlet problem involving the fractional (p,s)-Laplacian with homogeneous boundary conditions in a bounded Lipschitz domain. The continuous problem consists in finding $u\in\tilde{X}^{s,p}(\Omega)$ such that

$$-\Delta_p^s u = f \text{ in } \Omega,\qquad u=0 \text{ in } \Omega^c,$$

where the operator is defined as

$$\Delta_p^s u := \operatorname{div}_s\left(|\nabla^s u|^{p-2}\nabla^s u\right).$$

The weak formulation of this problem consists in finding $u\in\tilde{X}^{s,p}(\Omega)$ such that

$$\int_\Omega |\nabla^s u|^{p-2}\nabla^s u\cdot\nabla^s v\,dx = \int_\Omega f v\,dx,\quad \forall v\in\tilde{X}^{s,p}(\Omega).$$

Spector and Shieh [4] established that, if $g\in L^{p'}(\Omega)$ this problem has a unique solution. Our goal is to develop a numerical implementation based on the Coordinated Decomposition Method by Glowinski-Marocco [2]. Specifically, we reformulate the variational problem as a minimization problem:

$$\min_{v\in X^{s,p}(\Omega)}\left\{\frac{1}{p}\int_{\mathbb{R}^N}|\nabla^s v|^p\,dx - \int_\Omega f v\,dx\right\}. \qquad (1)$$

Minimization Problem Reformulation

Introducing an auxiliary variable $v=\nabla^s u$, the problem (1) can be reformulated as a constrained minimization problem, where the constraint space is defined as:

$$W = \left\{(u,v)\in\tilde{X}^{s,p}(\Omega)\times L^p(\mathbb{R}^N) : \nabla^s u - v = 0 \text{ a.e. in } \mathbb{R}^N\right\}.$$

Using an augmented Lagrangian approach, we transform this saddle-point problem into an unconstrained minimization problem, where the augmented Lagrangian is given by:

$$\mathcal{L}_r(u,w,\lambda) = \int_{\mathbb{R}^N}\frac{|w|^p}{p} - \int_\Omega f v + \langle\lambda,\nabla^s u - w\rangle + \frac{r}{2}\|\nabla^s u - w\|^2_{L^2(\mathbb{R}^N)}.$$

Augmented Lagrangian Iterative Scheme

For the numerical approximation, we employ a finite element discretization. Specifically, the solution u_h^n is approximated using piecewise linear elements (P_1), while w_h^n and λ_h^n are discretized using piecewise constant elements (P_0). Given an initial condition $\{w_h^{n-1},\lambda_h^n\}$, the sequence $\{u_h^n, w_h^n, \lambda_h^{n+1}\}$ is computed through the following iterative process:

Step 1: Update u_h^n

$$r\int_{\mathbb{R}^N}\nabla^s u_h^n\cdot\nabla^s\phi_h\,dx = \int_\Omega f\phi_h\,dx + \langle r w_h^{n-1} - \lambda_h^n, \nabla^s\phi_h\rangle.$$

Step 2: Update w_h^n

$$\int_{\mathbb{R}^N}\left(|w_h^n|^{p-2}+r\right)w_h^n\eta_h\,dx = \langle\lambda_h^n,\eta_h\rangle + r\int_{\mathbb{R}^N}\nabla^s u_h^n\eta_h\,dx.$$

Step 3: Update Lagrange Multiplier λ_h^{n+1}

$$\lambda_h^{n+1} = \lambda_h^n + \rho\left(r\frac{1}{|T_j|}\int_{T_j}\nabla^s u_h^n\,dx - w_h^n\right).$$

Convergence Theorem

If $0<r_0\leqslant\rho\leqslant r_1<\frac{r}{2}$, then for any initial condition λ_h^0, as $n\to\infty$, we have:

$$u_h^n\to u_h,\quad w_h^n\to\nabla^s u_h,\quad \lambda_h^n\to|\nabla^s u_h|^{p-2}\nabla^s u_h.$$

where u_h is the finite-dimensional solution of (1).

Error and Interpolation Estimates

In the interpolation estimate result, we consider $\mathcal{T}_h$ as an admissible triangulation of Ω and denote S_T as the patch associated with the element T, i.e., the set of all elements that share at least one vertex or edge with T. Additionally, these estimates involve quasi-interpolation operators, such as those of Scott-Zhang, Clément, or Chen-Nochetto.

Interpolation Error

For $s\in(0,1)$ and $0<t\leq s$, we have:

$$\forall h\lesssim 1, \forall T\in\mathcal{T}_h, \forall v\in X^{s,p}(S_T),\quad \|v-\Pi_h v\|_{L^p(T)}\lesssim h_T^{s-t}|v|_{X^{s,p}(S_T)}.$$

Error Estimate

Let u be the exact solution of (1) and u_h its finite element approximation. Then, there exists a constant $C>0$, independent of u, such that:

$$\|u-u_h\|\leq\begin{cases}C\inf\limits_{v_h\in V^h}\|u-v_h\|^{p/2}, & \text{if } 1<p\leq 2,\\ C\inf\limits_{v_h\in V^h}\|u-v_h\|^{2/p}, & \text{if } 2\leq p<\infty.\end{cases}$$

Numerical Experiments

We validate our numerical approach by solving the Dirichlet problem involving the fractional (p,s)-Laplacian in both one and two dimensions for $s=0.5$ and varying values of p. The obtained solutions are displayed in the following figure, illustrating the influence of p on the regularity and shape of u.

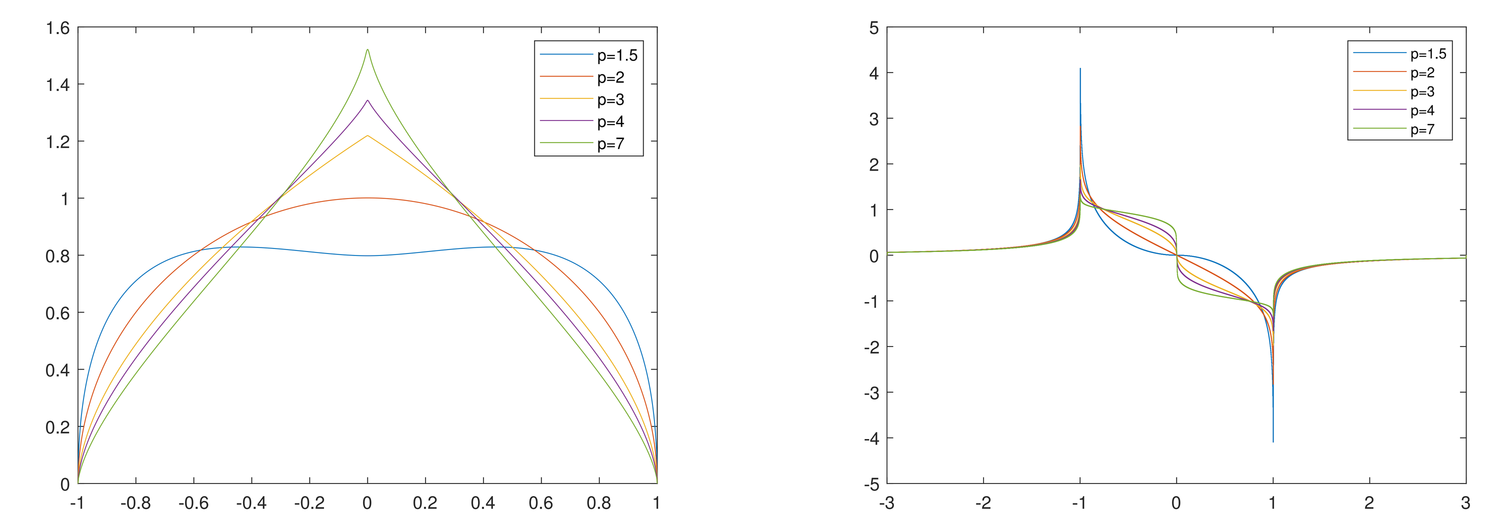

Figure: Computed solutions and their derivatives for $s=0.5$ and multiple values of p in 1D.

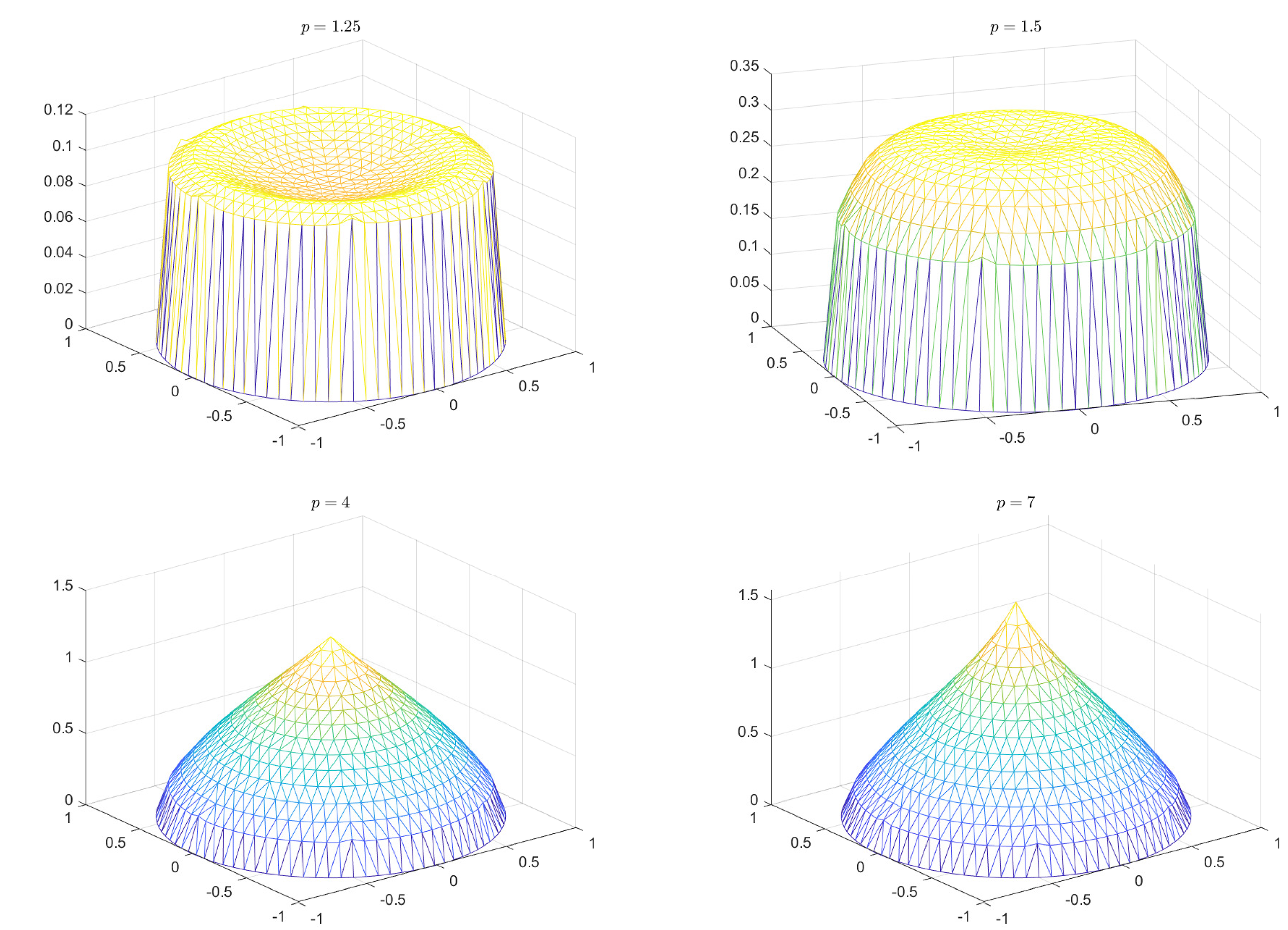

Figure: Computed solutions for $s=0.5$ and multiple values of p in 2D.

Ongoing and Future Works

- We are studying boundary regularity for the Dirichlet problem, aiming to obtain Besov regularity by employing Savaré's [3] approach, which is based on Nirenberg's difference quotient technique.
- We seek to establish new interpolation error estimates using quasi-norms, adapting the methods developed by Ebmeyer and Liu [1] for the p-Laplacian to this class of nonlocal operators.

References

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[4] Shieh, T.-T., Spector, D. E., *On a new class of fractional partial differential equations*, Advances in Calculus of Variations, 8(4), pp. 321–336, 2015.