Robustness of Cucker-Smale flocking model

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October 24, 2018

Abstract

Consider a system of autonomous interacting agents moving in space, adjusting each own velocity as a weighted mean of the relative velocities of the other agents. In order to test the robustness of the model, we assume that each pair of agents, at each time step, can fail to connect with certain probability, the failure rate. This is a modification of the (deterministic) Flocking model introduced by Cucker and Smale in [3]. We prove that, if this random failures are independent in time and space, and have linear or sub-linear distance dependent rate of decay, the characteristic behavior of flocking exhibited by the original deterministic model, also holds true under random failures, for all failure rates.

1 Introduction and main result

The applicability of the results obtained in the mathematical modeling of collective motion, obviously depend on the accuracy of the assumptions of the model. Nevertheless, the complexity of the situation of interest requires simplifications, as very complex models do not admit reasonable mathematical treatment and allow only simulations or other type of experimental verification. In this respect we recommend the recent review by Vicsek and Zafeiris [8] and its complete list of references.

In this direction, the seminal proposal by Cucker and Smale [3] is an important step in the comprehension of collective motion as, under reasonable

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dynamical laws, exhibits cases of flocking, one of the central issues in collective motion and also non-flocking situations, giving the possibility of a mathematical treatment of the question.

One of the characteristics of this proposal is that it lets aside some questions, as the volume of the agents, or its difference in weights, and it also assumes that the communication between agents is perfect. In the present paper we address this third issue. More precisely, departing from the Cucker-Smale model proposed in [3] we analyze the situation in which each interaction is subject to random failure. Our main result states that the convergence type result obtained in [3] holds in our framework, with probability one, in the case of agent interactions with linear or sub-linear distance dependent decay's rate, obtaining then a robustness result for Cucker-Smale model. This asymptotic behavior, called *flocking*, consists in the fact that for large times, all agent velocities become equal, with fixed relative positions. A similar approach in the case of *hierarchical* flocking was considered in [5] (see also [4]).

Consider then a system of k agents with positions and velocities denoted respectively by $X = (X_1, \ldots, X_k)$ and $V = (V_1, \ldots, V_k)$. All individual positions and velocities are vectors in \mathbb{R}^3 , and we refer to X and V as the position and velocity of the system respectively.

Assume that the system evolves following the discrete-time dynamic

$$\begin{cases} X_i(t+h) = X_i(t) + hV_i(t) \\ V_i(t+h) = V_i(t) + h\sum_{j=1}^k a_{ij}(V_j(t) - V_i(t)), \end{cases}$$
(1)

where i = 1, ..., k, h > 0 is the time step, $a_{ij}(t)$ are the weighting coefficients. An equivalent way of writing the second equation in (1) is

$$V_i(t+h) = \left(1 - h\sum_{j=1}^k a_{ij}\right) V_i(t) + h\sum_{j=1}^k a_{ij} V_j(t),$$
(2)

that, under the natural condition

$$0 < h \le \frac{1}{k},\tag{3}$$

(that makes the first coefficient in the r.h.s. of (2) always positive), implies that the velocity at t + h is a linear convex combination of the system velocities at time t. From this follows that the velocity is decreasing in the following sense:

$$\max_{1 \le i \le k} \|V_i(t+h)\| \le \max_{1 \le i \le k} \|V_i(t)\| \le \max_{1 \le i \le k} \|V_i(0)\|,$$

where ||V|| is the Euclidean norm of the vector V in \mathbb{R}^3 . Consequently, in what follows we assume condition (3).

In this framework we introduce the possibility that, at each time step, each pair of agents i, j, can fail to see each other (i.e. they disconnect). These failures are assumed to be random, independent and with a fixed failure rate probability $\lambda \in (0, 1)$. More precisely, the weighting coefficients in (1), for each pair of agents $j \neq i$ are given by

$$a_{ij}(t) = \zeta_t^{ij} \frac{1}{\left(1 + \|X_i(t) - X_j(t)\|\right)^{\alpha}},\tag{4}$$

where ζ_t^{ij} $(t \ge 0)$ are independent and identically distributed Bernoulli random variables with success probability $1 - \lambda$, i.e.

$$\mathbf{P}(\zeta_t^{ij} = 1) = 1 - \lambda, \qquad \mathbf{P}(\zeta_t^{ij} = 0) = \lambda.$$

Through the article **P** and **E** denote probability and expectation respectively, and the (random) event $\zeta_t^{ij} = 0$ (resp. = 1) means that the pair of agents i, j fails to (resp. does) connect (see each other) at time step t.

The factor $(1 + ||X_i(t) - X_j(t)||)^{-\alpha}$ in (4), with $0 \le \alpha \le 1$, is the Cucker-Smale coefficient introduced in [3].

It is interesting to observe (as was done in [2], and [6] for continuous time) that the center of mass of the system travels with constant velocity, that happens to be the initial mean velocity of the flock. Define

$$\bar{X}(t) = \frac{1}{k} \sum_{i=1}^{k} X_i(t), \qquad \bar{V}(t) = \frac{1}{k} \sum_{i=1}^{k} V_i(t).$$

In view of (1) (due to the symmetry of the coefficients $a_{ij} = a_{ji}$), we obtain that $\bar{V}(t+h) = \bar{V}(t)$. We conclude that

$$\bar{X}(th) = \bar{X}(0) + ht\bar{V}(0),, \qquad \bar{V}(th) = \bar{V}(0), \qquad t = 1, 2, \dots$$
 (5)

Theorem 1. Consider the dynamical system governed by equations (1), with coefficients subject to random failure as defined in (4).

(i) Under the condition $\alpha < 1$ all velocities tend to a common velocity that is the mean initial velocity $\overline{V}(0)$, i.e.

$$(V_1(th),\ldots,V_k(th)) \to (\bar{V}(0),\ldots,\bar{V}(0)), \quad as \ t \to \infty.$$
 (6)

Furthermore, there exists a limiting configuration $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k)$ that is the limit of the relative positions of the system with respect to the center of mass defined in (5), more precisely

$$(X_1(th) - \bar{X}(th), \dots, X_k(th) - \bar{X}(th)) \to (\hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_k), \quad as \ t \to \infty.$$
(7)

(ii) Under the condition $\alpha = 1$ there exists a critical initial velocity v^* (that depends only on the failure rate), such that if

$$||(V_1(0),\ldots,V_k(0)) - (\bar{V}(0),\ldots,\bar{V}(0))|| < v^*.$$

the statements of (i) hold.

Remark 1. The critical velocity \mathbf{v}^* in (ii) above is in fact the expectation of the (random) Fiedler number of the non-colored graph associated to the system. We have $\mathbf{v}^* = 0$ when $\lambda = 1$ (complete failure), and $\mathbf{v}^* = k$ when $\lambda = 0$ (non-failure), see the observations following Remark 2.

Remark 2. The case $\lambda = 0$ corresponds to the non-failure system, i.e. to the Cucker-Smale model introduced in [3]. In case $\alpha = 1$ we obtain the condition

$$\|V(0)\| \le \boldsymbol{v}^* = k \le \frac{1}{h}$$

according to (3). This is similar to the condition in Theorem 3, statement ii) in [3]. (Case $\beta = 1/2$ in [3] corresponds to case $\alpha = 1$ in our framework.)

A key feature in order to obtain flocking is the connectedness of the graph induced by the agents of our flock. Within the several ways of quantifying the connectivity of a graph, Cucker and Smale [3] show that a key concept in this situation is the *algebraic connectivity*, also known as the *Fiedler number* (see [1]), that we denote by ϕ . This number is defined as the second smallest eigenvalue of the Laplacian matrix associated to the graph (the first eigenvalue always vanishes). The larger the Fiedler number is, the "more connected" the graph is. A relevant rôle will be played also by the Fiedler number of the noncolored graph induced by the interactions graph, that is defined to have an edge if and only if there is an edge in the original graph. This second Fiedler number is denoted by φ . In particular, when $\varphi = 0$ the graph is not connected, and the complete graph with k vertices has $\varphi = k$.

Observe that, as the edges of our graph are random, this number also becomes a random quantity, and the statistical control of its behavior in time gives us the possibility of establishing our results.

2 Numerical simulations

We perform some simulations of the evolution of the system defined by Equations (1) and (4). The following graphics correspond to that evolution for some different values of the parameters. The graphics show the norm of the system relative velocity, i.e.: the maximum of the 2-norms of the velocity of each agent relative to v_0 , and the logarithm of that relative velocity.

The initial positions and velocities are sampled from standard normal distribution in \mathbb{R}^3 , and we take h = 1/k. It can be observed that the decay seems



Figure 1: Evolution of velocity for $\alpha = 0.5$ and $\lambda = 0.25$.

to be exponential in the sub-linear cases ($\alpha < 1$), as in the hierarchical case (see [5]) although here our technics to not provide this rate of convergence. The simulation results in the linear case ($\alpha = 1$) do not provide clear information about the rate of decay.



Figure 2: Evolution of velocity for $\alpha = 0.5$ and $\lambda = 0.9$.



Figure 3: Evolution of velocity for $\alpha = 1$ and $\lambda = 0.9$.

3 Proof of Theorem 1

As was mentioned, we consider the colored graph induced by the system, with vertices $1, 2, \ldots, k$ corresponding to the agents, and edges $a_{ij}(t)$. We now write the equations that govern the system in matrix form. Denote by Id the $k \times k$ identity matrix, and by L(t) the Laplacian matrix corresponding to the induced graph defined as L(t) = D(t) - A(t) with $A(t) = (a_{ij}(t))$ the incidence matrix of the graph, and D(t) the diagonal matrix with $d_{ii}(t) = \sum_j a_{ij}(t)$. With this notation, the matrix form of the equations (1) are

$$X(t+h) = X(t) + hV(t),$$

$$V(t+h) = (Id - hL(t))V(t),$$
(8)

where the notation AV means that the matrix A is acting on $(\mathbb{R}^3)^k$ by mapping the vector (V_1, \ldots, V_k) into the vector $(a_{i1}V_1 + \cdots + a_{ik}V_k)_{1 \le i \le k}$.

As we have seen in (5), the center of mass of the system has constant velocity. It is useful then to consider coordinates with respect to this point, introducing the relative position and velocity of the flock by

This change of coordinates is equivalent to the projection on the diagonal space in velocities introduced by Cucker and Smale in [3]. A further simplification in the notation is to write $\boldsymbol{x}[t]$ and $\boldsymbol{v}[t]$ instead of $\boldsymbol{x}(th)$ and $\boldsymbol{v}(th)$ respectively (and similarly for other time dependent quantities).

With this change of coordinates and notation, the statements of Theorem 1 are

$$(\boldsymbol{v}_1[t],\ldots,\boldsymbol{v}_k[t]) \to (0,\ldots,0), \text{ as } t \to \infty,$$

and

$$(\boldsymbol{x}_1[t],\ldots,\boldsymbol{x}_k[t]) \to (\hat{\boldsymbol{x}}_1,\ldots,\hat{\boldsymbol{x}}_k), \text{ as } t \to \infty.$$

It is easy to verify that the relative positions $\boldsymbol{x}[t]$ and velocities $\boldsymbol{v}[t]$ just introduced follow the same dynamic than the original ones, i.e. the system (8) holds for \boldsymbol{x} and \boldsymbol{v} instead of X and V respectively:

$$\begin{aligned} \boldsymbol{x}[t+1] &= \boldsymbol{x}[t] + h\boldsymbol{v}[t], \\ \boldsymbol{v}[t+1] &= (\mathrm{Id} - hL[t])\,\boldsymbol{v}[t]. \end{aligned}$$

We introduce the norm of a vector $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)$ in $(\mathbb{R}^3)^k$ by

$$\|\boldsymbol{x}\| = \sum_{i=1}^{k} \sum_{\ell=1}^{3} x_{i\ell}^{2}.$$

With this notation the statements (6) and (7) reduce to

$$\|\boldsymbol{v}[t]\| \to 0 \quad ext{and} \quad \|\boldsymbol{x}[t] - \hat{\boldsymbol{x}}\| \to 0, \quad ext{as } t \to \infty.$$

We further consider the usual operator norm for a matrix A acting as described above by

$$||A|| = \sup\{||Ax|| \colon ||x|| = 1\}$$

As the Fiedler number of A is the second smallest eigenvalue of A, and the smallest is associated to the eigenvector $(1, \ldots, 1)$ in $(\mathbb{R}^3)^k$ (and our vectors are orthogonal with respect to this diagonal vector), from the velocity equation in (8) we obtain that

$$\|\boldsymbol{v}[t+1]\| \le (1 - h\phi[t])\|\boldsymbol{v}[t]\|,\tag{9}$$

where $\phi[t]$ is the Fiedler number of the (random) matrix Id -hL[t].

First, we observe that the Fiedler number of a colored graph satisfies

$$\phi \le \frac{k}{k-1} \min\{d(v) : v \in V(G)\}$$

where d(v) is the degree of the vertex v (see Theorem 2.2 in [7]). From this, as $a_{ij} \leq 1$ for all pairs i, j, we obtain $d(v) \leq k - 1$ for all v, that gives $\phi \leq k$ (the same bound that holds for non-colored graphs). This means, as $h \leq 1/k$ that $0 \leq 1 - h\phi \leq 1$.

Second, we obtain that the norm of the relative velocity of the flock is decreasing, giving a linear bound for the norm of the relative position.

$$\|\boldsymbol{x}[t]\| = \|\boldsymbol{x}[0] + h(\boldsymbol{v}[0] + \dots + \boldsymbol{v}[t-1])\| \le \|\boldsymbol{x}[0]\| + th\|\boldsymbol{v}[0]\|.$$

Furthermore, the iteration of (9) gives

$$\|\boldsymbol{v}[\tau+1]\| \le \prod_{i=0}^{\tau} (1-h\phi[i])\|\boldsymbol{v}[0]\|,$$

that using the equation of the position in (8) give us the bound

$$\begin{aligned} \|\boldsymbol{x}[\tau]\| &\leq \|\boldsymbol{x}[0]\| + h\left(\|\boldsymbol{v}[0]\| + \sum_{j=1}^{\tau-1} \|\boldsymbol{v}[j]\|\right) \\ &\leq \|\boldsymbol{x}[0]\| + h\|\boldsymbol{v}[0]\|\left(1 + \sum_{j=1}^{\tau-1} \prod_{i=0}^{j-1} (1 - h\phi[i])\right). \end{aligned}$$

It is crucial then to study the convergence of the series

$$S[\tau] = \sum_{j=1}^{\tau-1} \prod_{i=0}^{j-1} \left(1 - h\phi[i]\right)$$
(10)

to obtain an upper bound of the position, that, in its turn, will give us a lower bound on the Fiedler number.

An important observation is that the connectedness of the coloured graph with incidence matrix $(a_{ij}[t])$ coincides with the one of the 0-1 graph with incidence matrix (ζ_t^{ij}) (because both have the same zero and non-zero entries). This means that the connectedness of the graph is independent of the position and velocity of the system.

Let $\varphi[t]$ be the Fiedler number of the non colored graph generated by (ζ_t^{ij}) . Then, by Proposition 2 in [3], we have

$$\phi[t] \ge \varphi[t]\mu[t]$$

with

$$\mu[t] = \min\{a_{ij}[t] \colon a_{ij}[t] > 0, i \neq j\}.$$

Note that when the graph is not connected we have $\phi[t] = \varphi[t] = 0$.

We now prove that $\mu[t] \geq \frac{A}{B+t^{\alpha}}$ for some constants A and B. For this, we rely on the inequality $(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha}$ that holds for $a \geq 0$, $b \geq 0$ and $0 \leq \alpha \leq 1$. Applying this inequality with a = 1 + ||x[0]|| and b = h||v[0]||t we obtain

$$\mu[t] \ge \frac{1/(h\|v[0]\|)^{\alpha}}{\left(\frac{1+\|x[0]\|}{h\|v[0]\|}\right)^{\alpha} + t^{\alpha}} =: \frac{A}{B + t^{\alpha}}$$

for all t, that gives

$$\phi[t] \ge \varphi[t] \frac{A}{B + t^{\alpha}} =: \nu[t].$$

With this result we obtain the following bound for our sum in (10). Denote $\bar{\varphi} = \mathbf{E} \varphi[t]$ and note that $\bar{\varphi} > 0$ since φ is a nonnegative and not identically zero random variable. As the quantities { $\nu[t]$ } form a sequence of independent (non identically distributed) random variables, and

$$\mathbf{E}(1-h\nu[t]) = 1 - h\bar{\varphi}\frac{A}{B+t^{\alpha}}$$

Furthermore

$$\mathbf{E} S[\tau] \le \sum_{j=1}^{\tau-1} \prod_{i=1}^{j} \mathbf{E} \left(1 - h\nu[i] \right) = \sum_{j=1}^{\tau-1} \prod_{i=1}^{j} \left(1 - \bar{\varphi}h \frac{A}{B + i^{\alpha}} \right)$$

Assume $\alpha < 1$. For each summand above, as $\log(1-x) \leq -x$, we have

$$\prod_{i=1}^{j} \left(1 - \bar{\varphi}h \frac{A}{B + i^{\alpha}} \right) = \exp\left(\sum_{i=1}^{j} \log\left(1 - \bar{\varphi}h \frac{A}{B + i^{\alpha}} \right) \right) \leq \exp\left(- \sum_{i=1}^{j} \bar{\varphi}h \frac{A}{B + i^{\alpha}} \right) \\
\leq \exp\left(- \bar{\varphi}hA \sum_{i=1}^{j} \frac{1}{i^{\alpha}} \right) \leq \exp\left(-\gamma j^{1-\alpha} \right),$$
(11)

where $\gamma = (\bar{\varphi}hA)/(1-\alpha)$. Finally we observe that a series with general term given by (11) is convergent, and this implies the convergence of $\mathbf{E}(S[\tau])$ as $\tau \to \infty$ to a finite limit. As the series $S[\tau]$ itself is increasing with τ , we obtain that there exists an almost sure finite limit

$$S = \lim_{\tau \to \infty} S[\tau] < \infty.$$
⁽¹²⁾

The case $\alpha = 1$ is treated separately. We have to modify the final bound in (11), in this case we have

$$\prod_{i=1}^{j} \left(1 - \bar{\varphi}h \frac{A}{B+i} \right) = \exp\left(\sum_{i=1}^{j} \log\left(1 - \bar{\varphi}h \frac{A}{B+i} \right) \right) \le \exp\left(-\bar{\varphi}hA \sum_{i=1}^{j} \frac{1}{B+i} \right)$$
$$\le \exp\left(-\bar{\varphi}hA \log\left(\frac{B+j+1}{B+1} \right) \right) = \left(\frac{B+1}{B+j+1} \right)^{\bar{\varphi}hA}.$$

Now, a series with this general term converges when $\bar{\varphi}hA > 1$, that is $||v[0]|| < \bar{\varphi}$. Under this condition we obtain (12).

From the convergence obtained in (12) (in both cases $\alpha < 1$ and $\alpha = 1$) for the series defined in (10), we get that the series of the velocities norm is convergent, i.e.

$$\sum_{j=0}^{\tau-1} \|v[j]\| \le \|v[0]\|(1+S)$$

This implies the convergence stated in (6). It also implies that the series of the velocities converges itself, i.e. there exists

$$\hat{oldsymbol{v}} = \sum_{j=0}^{\infty} oldsymbol{v}[j].$$

This fact gives the convergence for the positions of the system:

$$egin{aligned} m{x}[au] &= m{x}[0] + \sum_{j=0}^{ au-1} (m{x}[j+1] - m{x}[j]) = m{x}[0] + h \sum_{j=0}^{ au-1} m{v}[j] \ & o m{x}[0] + h \hat{m{v}} =: \hat{m{x}}, \end{aligned}$$

concluding the proof of the Theorem.

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