Building Reliability-Improving Network Transformations

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Motivation

Remark

- *Graph with either bridges or cut-points are not good from a telecommunication viewpoint.*
- *The intuition suggests that we can transform them into biconnected graphs, winning in terms of reliability.*
- *Kelmans already provided in 1981 a reliability-improving transformation.*
- *To the best of our knowledge, there is no other reliability-improving transformation in the scientific literature.*

Goal

Here we formalize this intuition, finding reliability-improving transformations. They require the movement of a single link.

Problem

Definition (Unreliability)

The *unreliability* of a simple graph G with independent link failures with probability ρ is:

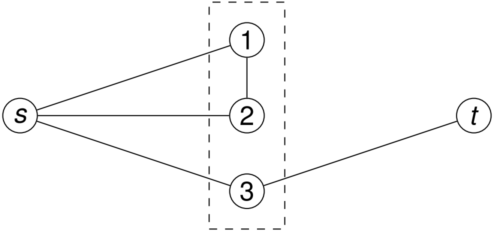
$$U_G(\rho) = \sum_{k=0}^{q} m_k \rho^k (1-\rho)^{q-k},$$

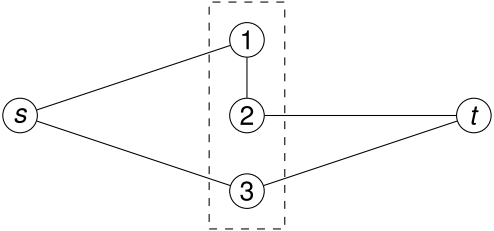
being m_k the number of ways to disconnect G removing k links. A (p,q)-graph is a graph with p nodes and q links.

Definition (Reliability-Improving Transformation)

Given a (p,q)-graph G, a reliability-improving transformation is a mapping $f : G \to H$, where H is another (p,q) graph but $U_H(\rho) < U_G(\rho)$ for all $\rho \in (0,1)$.

Old Transformation (Kelmans, 1981)





New Transformations (Canale et. al., 2019)

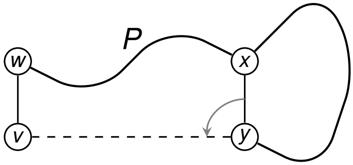


Figure: Building a bridgeless graph.

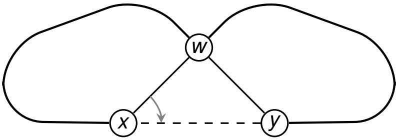


Figure: Building a biconnected graph.

Results 1/3

Theorem

Let $q \geq p \geq 3$. For any connected (p, q)-graph $G = (V, E)$ with a bridge, there is some bridgeless (p, q)-graph G' such that $m_k(G') \leq m_k(G)$ for all k.

Main Idea of the Proof:

Define $G_1 = G - xy + vy$. It has less bridges than G. Find a one-to-one mapping $f_k : M_k(G_1) \to M_k(G)$, to conclude that $m_k(G_1) = |M_k(G_1)| \leq |M_k(G)| = m_k(G)$. The following function works:

$$f_k(S) = \begin{cases} S & vy \notin S, \\ S - vy + xy & vy \in S, \end{cases}$$

Results 2/3

Theorem

Let $q \geq p \geq 3$. For any connected (p, q)-graph G with a cut-point, there is some biconnected (p, q)-graph G' such that $m_k(G') \leq m_k(G)$ for all k.

Main Idea of the Proof:

Analogous. Consider the graph $G_1 = G - wx + xy$, and find a one-to-one mapping from $f_k : M_k(G_1) \to M_k(G)$:

$$f_k(S) = \begin{cases} S & xy \notin S, \\ S - xy + wx & xy \in S. \end{cases}$$

Results 3/3

Theorem

Yutsis graph Y_6 is uniformly most-reliable.

Main Idea of the Proof:

It suffices to prove that $m_k(Y_6) \leq m_k(G)$ for all $(12,18)$-graphs G. Observe that the minimum-degree must be $\delta(G) \leq 3$. The case $\delta(G) = 1$ is not necessary (Theorem 3). There are 85 non-isomorphic cubic graphs with 12 nodes (Bussemake, 1977), so, a simple test finishes these cases. A previous computational test shows that Yutsis is t-optimal (Chen, 2005). Yutsis has superconnectivity $\lambda = 3$, so, $m_3(Y_6)$ is minimum. The remaining cases for $\delta(G) = 2$ are longer.

Yutsis Graph

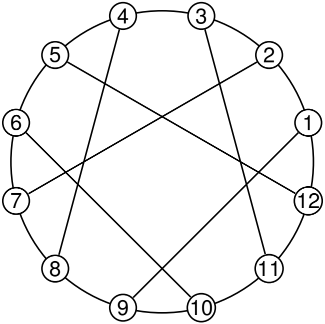


Figure: Yutsis-18jF

Conclusions

1. Two reliability-improving transformations are proposed.
2. Both transformation lead to biconnected graphs.
3. Yutsis is uniformly most-reliable.
4. A formal proof is waiting for Heawood and Kantor-Mobius graphs.
5. The progress in the theory of UMR graphs is slow.