LOOPS, HOLONOMY AND SIGNATURE

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Abstract. We show that there is a topology on certain groups of loops in euclidean space such that these group are embedded in a Banach Lie group which is the structural group of a principal bundle with connection whose holonomy coincides with the Chen signature map. We also give an alternative geometric new proof of Chen signature Theorem and a generalization of this theorem in classes strictly containing the one originally considered by Chen.

1. Introduction

This paper is about the intimate relationship between two theories that developed almost independently of each other. In topology and differential geometry, it traces back to the well known theory of representations of the fundamental group and constructions of local systems over a manifold by the holonomy map of a principal bundle with a flat connection. A natural question is how to extend this construction to arbitrary connections. This question leads directly to the loop space of the manifold, an object that will be defined shortly [1]. In contrast to the fundamental group of a manifold, that is a purely topological object, the loop space we are referring here will be not. Specifically, while the fundamental group of manifold coincides with the one calculated in the smooth category of curves, the loop space of the manifold and therefore the results concerning this space will depend on the category of curves where it is defined.

Consider a point p in a manifold M. Although there are several frameworks of loops based at p that could be chosen a priori, in order to get the group structure the first requirement for a particular class to be *concatenable*; that is the concatenation of two loops in the class belongs to the class. Examples of classes with this property are the class of piecewise smooth loops $\Omega^{ps}(M,p)$ or the proper class of piecewise analytic loops $\Omega^{pa}(M,p)$. Examples not having this property are the smooth and analytic classes.

Identifying loops of a concatenable class by reparameterization, we obtain associativity of the operation but in general $\alpha\,\alpha^{-1} \neq c$ where c denotes the constant loop. A further equivalence relation is needed. Considering the problem before, the natural relation is the one finitely generated by $\alpha a a^{-1}\beta \sim \alpha\beta$. This will be called the *retrace relation*. Considering the original problem instead, the natural relation is the one which identifies a pair of loops if, given a Lie group G, their holonomies coincide for every G-principal bundle with connection. This relation will be called the *G-holonomy relation*.

2020 *Mathematics Subject Classification.* Primary: 53C29, 55P10, 51H25; Secondary: 81Q70.
Key words and phrases. Loop space, Holonomy, Chen signature.

[1]Do not confuse this space with the one defined in algebraic topology. The ambiguity in the names will disappear after we introduce the right notation.

It turns out that in general the resulting groups under the retrace and G-holonomy relation are not isomorphic. However, as it was shown in [Sp], it is quite remarkable that for every connected and non-solvable Lie group G, these groups are isomophic in the class of piecewise analytic loops,

$$\mathcal{L}^{\,pa,ret}(M,p) \;\cong\; \mathcal{L}^{\,pa,G}(M,p), \qquad G \textit{ connected and non} - \textit{solvable}. \tag{1}$$

In [Tl], Tlas proves a similar result in the concatenable class of C^1 loops based at p with zero derivative at p. He proves that if the Lie group G is semisimple then a pair of loops in this class are G-holonomy related if and only if they are related by a rank one homotopy. In the class of piecewise smooth loops, Tlas result is equivalent to the following isomorphism under *thin homotopy relation* (see remark 4 in [Me])

$$\mathcal{L}^{\,ps,thin}(M,p) \;\cong\; \mathcal{L}^{\,ps,G}(M,p), \qquad G \textit{ semisimple}. \tag{2}$$

Along the lines of this result, Meneses in [Me] comments that the analog result of (1) claimed in [Sp] is not valid in the class of piecewise smooth loops and he gives a counterexample of a thin and non-retreacable loop, hence

$$\mathcal{L}^{\,ps,ret}(M,p) \;\ncong\; \mathcal{L}^{\,ps,G}(M,p), \qquad G \textit{ semisimple}. \tag{3}$$

One of the difficulties with piecewise smooth loops, in contrast with piecewise analytic ones, is that they can intersect in very complicated ways. This led Baez and Sawin in [BS] to consider instead the class of piecewise smoothly immersed loops $\Omega^{psi}(M,p)$. In this class they develop the technology of *tassels* and *webs*. A careful inspection into the work in [Sp], which uses the technology developed in [BS], shows that the claimed result holds for the class of piecewise smoothly immersed loops. That is, for a connected and non-solvable Lie group G, it is proved in [Sp] the isomorphism [2]

$$\mathcal{L}^{\,psi,ret}(M,p) \;\cong\; \mathcal{L}^{\,psi,G}(M,p), \qquad G \textit{ connected and non} - \textit{solvable}. \tag{4}$$

All of these loop spaces equivalences have in common some sort of factorization. In [Tl], Tlas associates a transfinite word for every loop in the class described before and identifies a pair of loops if they have the same reduced word. A loop whose reduced word is trivial is called a *whisker* by him. In [Sp], Spallanzani uses the technology of tassels and webs developed by Baez and Sawin [BS] to factorize smoothly immersed loops. The factorization in the piecewise analytic class follows almost directly by definition.

Another natural question is, given a manifold, how to reconstruct the principal bundle with connection from its holonomy map. This led to an abstract formulation of holonomy both in mathematics [Mi, La, Te1, Te2] and theoretical physics as well [Ba, BS, GT, Lo]. We recommend the recent paper [Me] by Meneses for references and historical account of the theory.

Independently, in the fifties Chen defined a map in the space of paths in the euclidean space $\mathbb{R}^n$, the *signature map*

$$\Theta(\gamma) \;=\; 1 + \sum_{p=1}^{\infty} \; \sum_{i_1,\, i_2,\, \dots,\, i_p = 1}^{n} \; \int_\gamma dx_{i_1} dx_{i_2} \dots dx_{i_p} X_{i_1} X_{i_2} \dots X_{i_p} \tag{5}$$

taking values in the ring of the formal power series with the non-commutative variables $X_1, \dots, X_n$. Chen proved that this map is a faithful representation of the

[2]We do not know why Spallanzani did not write the piecewise immersion hypothesis.

group of loops $\mathcal{L}^{\,psi,ret}(\mathbb{R}^n, p)$ [Ch1, Ch2, Ch3]. In recent years, this theory had a resurgence in the context of stochastic processes [Ly], bounded variation paths [HL] and rough paths [Ly, BGLY].

In this paper, we construct a principal bundle with connection whose holonomy coincides with the Chen signature map on the group of loops $\mathcal{L}^{\,psi,ret}(\mathbb{R}^n, p)$ in euclidean space. In other words, we show that the Chen signature map on this loop space is the holonomy map of a principal bundle with connection. In particular, this holonomy map is a monomorphism.

In view of the mentioned counterexample given by Meneses in [Me] of a non-retreacable thin loop in the class of piecewise smoothly loops showing in particular that the group $\mathcal{L}^{\,ps,ret}(M, p)$ is a proper extension of $\mathcal{L}^{\,ps,thin}(M, p)$, we show that no such counterexample is possible in the class of piecewise smoothly immersed loops. Actually, in section 3 we prove the following proposition.

Proposition 1.1. *In the class of piecewise smoothly immersed curves based at a point p in M, the retrace and thin homotopy relations coincide. That is, there is a natural isomorphism*

$$\mathcal{L}^{\,psi,ret}(M, p) \;\cong\; \mathcal{L}^{\,psi,thin}(M, p).$$

From now on, when the manifold M is understood, unless otherwise specified we will denote these groups simply by $\mathcal{L}_p$. The following is the main result of the paper, for $M = \mathbb{R}^n$.

Theorem 1.2. *There is a Banach Lie group $\hat{G}$ and a $\hat{G}$-principal bundle with connection $(E, \mathbb{R}^n, \hat{G}, \pi, \nabla)$ such that for any p in $\mathbb{R}^n$, the holonomy map is a monomorphism*

$$Hol_{\nabla,\,p} : \mathcal{L}_p \hookrightarrow \hat{G}$$

and it coincides with the Chen signature map (6) on $\mathcal{L}_p$. Moreover, the closure of its image is characterized by those elements verifying the shuffle relation (see section 2.2) with vanishing degree one terms.

The structural group $\hat{G}$ is a connected and simply connected infinite dimensional Banach Lie group. It is the completion under the pro-nilpotent topology of the infinite dimensional Lie group G whose Lie algebra is freely generated in n independent generators. These groups have an exponential map and will be constructed in section 4. The problem of finding such a group is not trivial and is described in (section 1.2.1, *Differential equations on matrix groups*, p. 225 in [Ly]), where an heuristic motivational argument is given assuming its existence. In this paper we solve this problem and this is the content of Proposition 4.1. After this work was completed, we noticed the recent result in [Ner] which is similar to Proposition 4.1. On the other hand, the purpose of our main result, Theorem 1.2, is to establish the relationship between the signature and holonomy map.

As an immediate corollary we have an alternative proof of Chen Theorem [Ch2].

Corollary 1.3. *The logarithm of the Chen signature is a Lie element.*

There is a C^∞-topology on the group of based loops $\mathcal{L}_p$ such that it becomes a topological group [Te1, Te2]. However, this group is not a Lie group and Chen developed a theory of differentiable spaces [Ch4, Ch5], a generalization of manifolds, to deal with this problem. Here, we show that there is a topology on the group of loops giving it the structure of a a topological group with the property that it can be embedded in Lie group. Identifying the group of loops with its image under the holonomy map in Theorem 1.2, we have another corollary of the theorem.

Corollary 1.4. *Consider p in $\mathbb{R}^n$. There is a topology τ on $\mathcal{L}_p$ and a proper closed normal Lie subgroup $H \lhd \hat{G}$ verifying the following:*

(1) *H contains an embedded copy of the space $(\mathcal{L}_p, \tau)$ as a topological subgroup; i.e. $\mathcal{L}_p \subset H$.*
(2) *It is simple relative to $\mathcal{L}_p$: there are no proper closed normal subgroups $H' \lhd H$ such that $\mathcal{L}_p \subset H'$.*
(3) *The Lie algebra of H contains an infinitely generated free Lie algebra.*
(4) *The codimension of H in $\hat{G}$ is at least n.*

We do not know if the group H coincides with $[\hat{G}, \hat{G}]$ in corollary 1.4.

In section 5 we give a geometric alternative new proof of the Chen signature theorem as a corollary of either the Tlas result in [Tl] or the holonomy theorem in [Sp]. A verbatim argument gives a generalization of this theorem either in the class of piecewise smooth loops under thin homotopy relation. We state this in the following theorem.

Theorem 1.5. *Consider the formal power series ring $R = \mathbb{R}[[X_1, \dots X_n]]$ with the non-commutative variables $X_1, \dots X_n$. Then, the Chen map (5) is a faithful representation of the group of loops in the class of piecewise smooth loops under thin homotopy relation $\mathcal{L}^{\,ps,thin}(\mathbb{R}^n, p)$.*

As an immediate corollary of the previous theorem, we have the following complement to the main Theorem 1.2.

Corollary 1.6. *Theorem 1.2 and its corollaries hold also for the group of loops in the class of piecewise smooth loops under thin homotopy relation $\mathcal{L}^{\,ps,thin}(\mathbb{R}^n, p)$.*

2. Preliminaries

2.1. **Loops and Holonomy.** Let I be the unit interval and M a smooth manifold. We begin by recalling some standard notations. A *path* in M is a continuous function from I to M, and we say that two paths $a, b : I \to M$ are equivalent modulo reparametrization if there is an orientation preserving homeomorphism $\sigma : I \to I$ such that $a \circ \sigma = b$. Denote by $\mathcal{O}_0(M)$ the quotient set under this equivalence relation. If $a(1) = b(0)$ we define ab and a^{-1} as follows: $ab\,(t) = a\,(2t)$ if $t \in [0, 1/2]$ and $ab\,(t) = b\,(2t-1)$ if $t \in [1/2, 1]$; $a^{-1}(t) = a\,(1-t)$ for every $t \in [0, 1]$. Let $e_p \in \mathcal{O}_0(M)$ be the constant path at the point p, i.e. $e_p(t) = p$ for every $t \in [0, 1]$.

We need to consider another preliminary equivalence, which amounts to collapse constant sub-paths. Let a be a non-constant path in M. We shall define a *minimal form* a_r for a as follows: let $I_i \subset I$ be the family of maximal subintervals in which a is constant, and let $\sigma : I \to I$ be a surjective non-decreasing continuous function, constant in each I_i and strictly increasing in $I - \bigcup_i I_i$. Then there is $a_r : I \to M$ such that $a = a_r \circ \sigma$, which is non-constant on any subinterval of I (this map is obtained by a universal property of quotients). Different choices of the function σ give rise to minimal forms that are equivalent modulo reparametrization, and moreover, if two paths a and b are equivalent, so are any of their minimal forms a_r and b_r. This allows us to define the *minimal class* of an element of $\mathcal{O}_0(M)$ (as the class of any minimal form of any representative), and take a quotient $\mathcal{O}_1(M)$ where we identify two elements of $\mathcal{O}_0(M)$ if they have the same minimal class (extending the definition to constant paths in the trivial way). The product and inverse are well defined on $\mathcal{O}_1(M)$, and the classes of constant paths are units for the product.

Let $\mathcal{O}^{psi}(M) \subset \mathcal{O}_1(M)$ be the set of classes of either constant paths or paths that are *piecewise smoothly immersed*, i.e. a finite concatenation of smooth immersions. Notice that for $\alpha \in \mathcal{O}^{psi}(M)$ there are well defined notions of endpoints $\alpha(0)$ and $\alpha(1)$. Throughout the paper we will abuse of notation and refer to the elements $\alpha \in \mathcal{O}^{psi}(M)$ also as *curves*, and say that α is a *closed curve* if $\alpha(0) = \alpha(1)$.

In the set $\mathcal{O}^{psi}(M)$, consider the equivalence relation finitely generated by the identifications $\alpha a a^{-1}\beta \sim \alpha\beta$. This is called the *retrace relation*. With the formal definition in hand, we recall the concepts from the introduction: Let $\mathcal{E}(M)$ denote the quotient set of $\mathcal{O}^{psi}(M)$ under retrace relation, and let $\mathcal{L}^{\ psi,ret}(M,p) \subset \mathcal{E}(M)$ be the projection under the quotient map of the set of closed curves starting and ending at the point p. Note that $\mathcal{L}^{\ psi,ret}(M,p)$ is a group under concatenation whose neutral element is the equivalence class of e_p, the constant path at p.

Now, consider the class $\Omega^{psi}(M,p)$ of piecewise smoothly immersed closed curves based at the point p. Another equivalence relation in this class is given by the finite composition of thin homotopies. A *thin homotopy* between two curves α and γ is a homotopy $\eta : [0,1]^2 \to M$ such that its image is contained in the union of the images of the curves, that is

$$\eta\left([0,1]^2\right) \subseteq \alpha\left([0,1]\right) \cup \beta\left([0,1]\right).$$

Denote the quotient under this relation by $\mathcal{L}^{\ psi,thin}(M,p)$ and note that, as before, it is a group under concatenation whose neutral element is the equivalence class of e_p, the constant path at p.

As it was mentioned at the introduction, in general the retrace and thin relations are not equivalent. However, in the class of piecewise smoothly immersed closed curves considered in this paper, they are. In section 3, we will prove Proposition 1.1 which states the existence of a natural isomorphism

$$\mathcal{L}^{\ psi,ret}(M,p) \cong \mathcal{L}^{\ psi,thin}(M,p).$$

In view of this isomorphism, we will simply denote these groups by $\mathcal{L}(M,p)$.

Other possible identification is, given a Lie group G, identify two closed curves α and β in $\Omega^{psi}(M,p)$ if they have the same holonomy for any G-principal bundle with connection over M. This relation is the *G-holonomy relation*. The group obtained this way is called the *group of hoops based at p* and will be denoted by $\mathcal{H}^G(M,p)$ [3]. The following is Theorem 1.7 from [Sp] and will be referred to as the *holonomy theorem* throughout the paper.

Theorem 2.1. *If G is a connected and non-solvable Lie group, then there is a natural isomorphism $\mathcal{L}(M,p) \cong \mathcal{H}^G(M,p)$ for every point p in M.*

In the case where the manifold M is the euclidean space $\mathbb{R}^n$, the case considered in this paper, from now on we will simply denote by $\mathcal{L}_p$ the group $\mathcal{L}(\mathbb{R}^n,p)$.

2.2. **Iterated Integrals and Signature.** If $\omega_1, \omega_2, \ldots, \omega_q$ are one forms in a manifold M and $\gamma : I \to M$ is a piecewise regular path, define the iterated integral by the formula

[3] This is the group denoted by $\mathcal{L}^{\ psi,G}(M,p)$ in the introduction.

$$\int_\gamma \omega_1\omega_2\ldots\omega_q \;=\; \int_{t>t_1>\ldots>t_q} \omega_1(\gamma'(t_1))\omega_2(\gamma'(t_1))\ldots\omega_q(\gamma'(t_q))dt_q\ldots dt_1.$$

If γ is a piecewise regular curve γ, Chen [Ch3] defines the formal power series

$$\Theta(\gamma) \;=\; 1 + \sum_{p=1}^{\infty} \sum_{i_1,\, i_2,\, \ldots,\, i_p\, =1}^{n} \int_\gamma dx_{i_1}dx_{i_2}\ldots dx_{i_p} X_{i_1}X_{i_2}\ldots X_{i_p} \tag{6}$$

in the noncommutative indeterminates $X_1,\ldots,X_n$. This will be called the *Chen signature map*. Chen signature theorem can be stated as follows.

Theorem 2.2. *If $\Theta(\gamma)=1$, then γ is trivial in $\mathcal{E}(\mathbb{R}^n)$.*

The reconstruction of the path from its signature is a non trivial task and an active research area [BGLY, HL, LX, WLC]. The *shuffle product* is a commutative associative product on the tensor algebra of coordinates

$$\mathcal{T}^* \;=\; \mathcal{T}(X_1^*,\, X_2^*,\, \ldots,\, X_n^*), \qquad X_i^*\,(X_j) \;=\; \delta_{ij},$$

defined on monomials as the sum of all possible ways of writing two given monomials together while preserving their orders. For example:

$$X_1^* \,\cdot_{sh}\, X_2^*X_3^* \;=\; X_1^*X_2^*X_3^* \,+\, X_2^*X_1^*X_3^* \,+\, X_2^*X_3^*X_1^*.$$

The tensor algebra $\mathcal{T}^*$ acts linearly on the graded completed algebra

$$\hat{\mathcal{T}} \;=\; \mathcal{T}(\widehat{X_1,\, X_2,\, \ldots,}\, X_n)$$

and we say that an element θ in $\hat{\mathcal{T}}$ verifies the *shuffle relation* if

$$f(\theta)\, g(\theta) \;=\; (f \,\cdot_{sh}\, g)\,(\theta), \qquad \forall\, f,\, g \,\in\, \mathcal{T}^*.$$

The Chen signature of every path verifies the shuffle relation and we have the following partial result by Chow [WLC]. Recall that the completed algebra $\hat{\mathcal{T}}$ has a natural ascending filtration by grading. For a natural n let $\hat{\mathcal{T}}_n$ be the ideal generated by the monomials of degree at least n.

Theorem 2.3. *For every θ in $\hat{\mathcal{T}}$ verifying the shuffle relation and every natural n, there is a piecewise linear path γ such that $\Theta(\gamma)=\theta$ modulo $\hat{\mathcal{T}}_n$.*

3. Retrace and thin relation on the piecewise immersed class

Next we prove Proposition 1.1 which states the existence of a natural isomorphism

$$\mathcal{L}^{\,psi,ret}(M,p) \;\cong\; \mathcal{L}^{\,psi,thin}(M,p).$$

That is to say that for piecewise smoothly immersed curves, thin homotopy is the same as retrace equivalence. We will use the concept of *tree-like paths* of Hambly and Lyons [HL], in the equivalent formulation introduced in [BGLY] and [HL2]:

Definition 3.1. *A path $\gamma:[a,b]\to M$ is tree-like if there is an $\mathbb{R}$-tree T and continuous functions $\phi:[a,b]\to T$ and $\psi:T\to M$ such that $\gamma=\psi\circ\phi$ and $\phi(a)=\phi(b)$.*

Namely, a tree-like path is a loop that factors through a loop in an $\mathbb{R}$-tree. In [Le], Lévy shows that for bounded variation curves, being a tree-like path is equivalent to being thin homotopic to the constant path. Also, for C^1 loops with zero derivative at the endpoints, Tlas [Tl] shows directly the equivalence between Definition 3.1 and being thinly homotopic to a constant path. Note that a piecewise smoothly immersed path is of bounded variation, and also can be reparametrized to be C^1 with zero derivative at its endpoints, so these results apply to them.

Proposition 1.1 follows immediately from the previous remarks and the following lemma:

Lemma 3.2. *Let $\gamma : [a, b] \to M$ be a tree-like path that is piecewise smoothly immersed. Then γ is retraceable (i.e. retrace equivalent to a constant path).*

Proof. Let $a = t_0 < t_1 < \cdots < t_k = b$ be a partition so that $\gamma|_{[t_i,t_{i+1}]}$ is a smooth embedding for $i = 0, \ldots, k-1$. Consider the $\mathbb{R}$-tree T and the maps $\phi : [a, b] \to T$ and $\psi : T \to M$ as in Definition 3.1. Notice that $\phi|_{[t_i,t_{i+1}]}$ must be injective, so its image is the unique geodesic in T with endpoints $\phi(t_i)$ and $\phi(t_{i+1})$. Then the image of ϕ is the convex hull of finitely many points, namely $\phi(t_i)$ for $i = 0, \ldots, k$. In an $\mathbb{R}$-tree, the convex hull of finitely many points is (isometric to) a finite simplicial tree (this fact can be obtained by straightforward induction on the number of points). Moreover, each injective path $\phi|_{[t_i,t_{i+1}]}$ can be reparametrized to be the geodesic between $\phi(t_i)$ and $\phi(t_{i+1})$. After such reparametrization, ϕ is a closed edge-path in a simplicial tree, which is retraceable, and then so is $\gamma = \psi \circ \phi$.

□

4. Holonomy and Signature

In this section we construct a principal bundle with connection whose holonomy is a monomorphism and coincides with the Chen signature map on loops. The main step towards this goal is the construction of a Lie group with a free Lie algebra generated on finite elements. As we mentioned in the introduction, this group was treated heuristically in (section 1.2.1, *Differential equations on matrix groups*, p. 225 in [Ly]) for motivational purposes.

Denote by $\mathfrak{g}$ the real Lie algebra freely generated by the elements $X_1, X_2, \ldots, X_n$, that is

$$\mathfrak{g} = \mathrm{Lie}\,(X_1, X_2, \ldots, X_n).$$

Because this Lie algebra is infinite dimensional, the Lie algebra/Lie group correspondence is not assured by Lie's third Theorem. Even so, it may happen that the corresponding exponential map does not cover a neighbourhood of the neutral element as it occurs with the Witt group of vector fields of the circle. Moreover, even covering a neighbourhood of the neutral element, in the case of a non-compact Lie group the exponential may not be surjective as it happens with $SL(2, \mathbb{R})$.

We shall construct a Lie group G whose Lie algebra is $\mathfrak{g}$ in the next Proposition. Before going into details, we will give an illustrative examples of the essence of the idea.

Example 1. *Suppose that we want to understand the group $\mathbb{Z}$ of integers but for some reason we can only understand finite groups. Is there a way to approximate the group of integers by finite groups? The simple answer is just no. However,* *there are bigger groups containing a copy of $\mathbb{Z}$ which have this property. A simple example is the well known Cantor dyadic group $\mathbb{Z}_2$ which is the inverse limit of the finite dyadic quotients*

$$\mathbb{Z}_2 \,=\, \varprojlim_{n\in\mathbb{N}} \mathbb{Z}\,/\,2^n\mathbb{Z}.$$

These quotients are residually finite; i.e. for every non-zero x in $\mathbb{Z}_2$, there is a natural n such that the respective class of x in the quotient $\mathbb{Z}\,/\,2^n\mathbb{Z}$ is non-zero as well. In particular, there is an embedded copy of $\mathbb{Z}$ in $\mathbb{Z}_2$ where the latter group can be approximated by finite groups.

Proposition 4.1. *There is a connected and simply connected Lie group G whose Lie algebra is $\mathfrak{g}$ and the exponential map is bijective and continuous with respect to the pro-nilpotent topology.*

Proof. Consider the Lie algebra $\mathfrak{g}$ and its *nilpotent completion* $\hat{\mathfrak{g}}$; i.e. the one induced by the lower central series $(\,\mathfrak{g}_n\,)_{n\in\mathbb{N}}$ of the algebra

$$\hat{\mathfrak{g}} \,=\, \varprojlim_{n\in\mathbb{N}} \mathfrak{g}\,/\,\mathfrak{g}_n, \qquad \mathfrak{g}_1 \,=\, \mathfrak{g}, \qquad \mathfrak{g}_{m+1} \,=\, [\mathfrak{g},\,\mathfrak{g}_m]\,.$$

We say that the completed Lie algebra $\hat{\mathfrak{g}}$ is a *pro-nilpotent Lie algebra*. Note that since $\mathfrak{g}$ is a rational free Lie algebra, its nilpotent completion coincides with its *graded completion* as well as with its *Malcev completion*.

Since the quotients $\pi_n : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_n$ are residually finite, that is for every non-zero element $x \neq \mathbf{0}$ there is a natural n such that $\pi_n(x) \neq \mathbf{0}$, there is a monomorphism of $\mathfrak{g}$ into its completion

$$\varphi : \mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}. \tag{7}$$

For every natural n, denote the quotient by $\mathfrak{h}_n \,=\, \mathfrak{g}\,/\,\mathfrak{g}_n$. This is a finite dimensional nilpotent Lie algebra hence by Lie's third Theorem, there is a Lie group H_n with Lie algebra $\mathfrak{h}_n$ and exponential map $\exp : \mathfrak{h}_n \to H_n$. Moreover, this correspondence is functorial and we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{h}_n & \xrightarrow{\exp_n} & H_n \\ {\scriptstyle \pi_{n,\,m}}\downarrow & & \downarrow{\scriptstyle \pi'_{n,\,m}} \\ \mathfrak{h}_m & \xrightarrow{\exp_m} & H_m \end{array} \quad , \qquad m \leq n \tag{8}$$

where $\pi_{n,\,m}$ denotes the projection induced by the inclusion $\mathfrak{g}_n \trianglelefteq \mathfrak{g}_m$ and $\pi'_{n,\,m}$ denotes the corresponding one on the Lie groups.

Since $\mathfrak{h}_n$ is a finite dimensional nilpotent Lie algebra, the exponential map $\exp_n$ is a homeomorphism. We will prove this claim by explicitly constructing the exponential map along with the corresponding Lie group. As a set, define $H_n \,=\, \mathfrak{h}_n$ and the exponential simply as the identity map where the group product is given by the *Baker-Campbell-Hausdorff formula*

$$x \cdot y \,=\, x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,[x,y]] + \frac{1}{12}[y,[y,x]] + \dots\ .$$

This is well defined since $\mathfrak{h}_n$ is nilpotent hence there is only a finite amount of terms in the sum. Adding a deformation parameter, the group product can be seen as a non-commutative continuous deformation of the addition in $\mathfrak{h}_n$ [4]. This proves the claim.

Taking the inverse limit of the inverse system (8) gives a Lie group and an exponential map

$$\hat{\mathfrak{g}} \,=\, \varprojlim_{n\in\mathbb{N}} \mathfrak{h}_n \xrightarrow{\widehat{\exp}} \varprojlim_{n\in\mathbb{N}} H_n \,=\, \hat{G}.$$

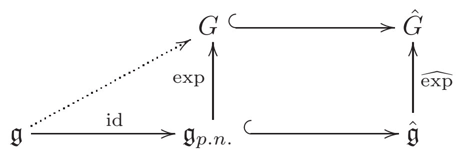
Because $\exp_n$ is a homeomorphism for every natural n, the exponential map $\widehat{\exp}$ is so. We have proved that there is a Lie group $\hat{G}$ whose Lie algebra is $\hat{\mathfrak{g}}$ and the exponential map $\widehat{\exp} : \hat{\mathfrak{g}} \to \hat{G}$ is a homeomorphism.

This construction can be seen in the context of *Malcev Lie algebras* (Definition 2.1, [PS]) since $\hat{\mathfrak{g}}$ is a Malcev Lie algebra hence it has an associated Lie group $\hat{G}$ (Definition 2.2, [PS]). It can also be seen in the category of *pro-nilpotent Lie algebras*, see (Remark V.1.3 (e) in [Nee]).

Now, consider the Lie algebra $\mathfrak{g}$ with the initial topology of the morphism (7). This is a coarser topology and will be called the *pro-nilpotent topology*. The Lie algebra with this new topology will be denoted by $\mathfrak{g}_{p.n.}$ and we have that

$$\mathrm{id} : \mathfrak{g} \to \mathfrak{g}_{p.n.} \tag{9}$$

is continuous. In particular, $\mathfrak{g}_{p.n.}$ is connected and simply connected since $\mathfrak{g}$ is so. Because the exponential $\widehat{\exp}$ is a homeomorphism, the image of its restriction on $\mathfrak{g}_{p.n.}$ gives a subgroup $G < \hat{G}$, that is the following diagram commutes



and it is clear that the exponential map exp is a homeomorphism. In particular, G is connected and simply connected and the composition of the exponential exp with the map (9) gives the claimed bijective continuous exponential map, the pointed arrow in the diagram above. We have the result. □

Remark 1. *We have proved that the exponential map*

$$\exp : \mathfrak{g}_{p.n.} \to G$$

is a homeomorphism with respect to the pro-nilpotent topology. The family of sets

$$U(\varepsilon,\, n) \,=\, B(\mathbf{0};\, \varepsilon) \,+\, \mathfrak{g}_n$$

generate this topology under translations on the Lie algebra and also on the Lie group via the exponential map. This is the topology of the Lie group G.

Now, consider the trivial principal bundle $E = \mathbb{R}^n \times \hat{G}$ with connection ∇ whose associated one-form is

$$\omega_\nabla \,=\, \sum_{i=1}^{n} X_i\, dx^i, \qquad \omega_\nabla \in \Omega^1(\mathbb{R}^n,\, \mathfrak{g}).$$

This is well defined [Va].

[4]This may be a little bit confusing at first. The new product is no longer commutative in general although the neutral element $\mathbf{e}$ of this new group corresponds to the zero element under this correspondence.

Proof of Theorem 1.2. We prove that the holonomy $Hol_{\nabla,x}$ coincides with the Chen signature map (6) on loops. Because $G < \hat{G} = \hat{\mathfrak{g}}$, where the last equality is at the level of sets, it will be enough to show that the universal enveloping algebra of the Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to the graded completion of the tensor algebra in the generators $X_1, X_2, \ldots, X_n$,

$$U(\hat{\mathfrak{g}}) \cong \mathcal{T}(\widehat{X_1, X_2, \ldots, X_n}). \tag{10}$$

Then, the result follows immediately by the path-ordering exponential representation of the holonomy map [Lo].

The universal enveloping algebra functor U is left adjoint hence it preserves inverse limits. Since $\mathfrak{g}$ is the free Lie algebra generated by $X_1, X_2, \ldots, X_n$, its universal enveloping algebra is isomorphic to the tensor algebra of these generators

$$U(\mathfrak{g}) \cong \mathcal{T}(X_1, X_2, \ldots, X_n).$$

Moreover, under the previous isomorphism, $U(\mathfrak{g}_n))$ is isomorphic to T_n, where $\mathfrak{g}_n$ and T_n are the ideals generated by the elements of degree greater than or equal to n in the respective algebras. In particular, the universal enveloping algebra commutes with the graded completion

$$U(\hat{\mathfrak{g}}) = U\left(\varprojlim_{n\in\mathbb{N}} \mathfrak{g}\,/\,\mathfrak{g}_n\right) \cong \varprojlim_{n\in\mathbb{N}} U(\mathfrak{g}\,/\,\mathfrak{g}_n) \cong \varprojlim_{n\in\mathbb{N}} U(\mathfrak{g})\,/\,T_n = \widehat{U(\mathfrak{g})}$$

and we have proved the identity (10).

By Chen signature Theorem 2.2, the holonomy is $Hol_{\nabla,x}$ is a monomorphism and by Chow Theorem 2.3 (see remark 1) the closure of its image consist of those elements verifying the shuffle relation with vanishing degree one terms. This concludes the proof. □

Proof of Corollary 1.4. By Theorem 1.2, the holonomy is a monomorphism and we can take the initial topology on $\mathcal{L}_x$. This gives an embedding $\mathcal{L}_x \subset \hat{G}$ as a topological group.

We can think about the smallest closed normal subgroup $H \trianglelefteq \hat{G}$ such that $\mathcal{L}_x \subset H$. This subgroup exists by a standard Zorn lemma argument. By definition, it is *simple relative to* $\mathcal{L}_x$.

The Lie subgroup $H \trianglelefteq \hat{G}$ is normal hence its Lie subalgebra $\mathfrak{h} \trianglelefteq \hat{\mathfrak{g}}$ is actually an ideal. It is clear that $\mathfrak{h}$ is not trivial since otherwise the Lie group H would be trivial as well but this is absurd since it contains the non-trivial loop group $\mathcal{L}_x$. Since $\mathfrak{g}$ is a free Lie algebra and $\mathfrak{h}$ is a non-trivial ideal of $\hat{\mathfrak{g}}$, $\mathfrak{g} \cap \mathfrak{h}$ is an infinitely generated free Lie algebra [Ek] contained in $\mathfrak{h}$.

Since the holonomy map coincides with the Chen signature map by Theorem 1.2 and the degree one terms of this map vanish on the loop group $\mathcal{L}_x$, it is clear that H must be contained in the closed Lie subgroup generated via the exponential by the Lie subalgebra $\hat{\mathfrak{g}}_2 = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$. In particular $\mathfrak{h} \trianglelefteq \hat{\mathfrak{g}}_2 < \hat{\mathfrak{g}}$ giving

$$[\mathfrak{h} : \hat{\mathfrak{g}}] \geq [\hat{\mathfrak{g}}_2 : \hat{\mathfrak{g}}] = n.$$

Taking the exponential gives the respective codimension result on the Lie groups and this concludes the proof. □

Definition 4.2. *Consider x in $\mathbb{R}^n$. The Lie group H and the topology τ in Corollary 1.4 will be called the Lie group generated by $\mathcal{L}_x$ and the embedded topology on $\mathcal{L}_x$ respectively.*

Question: Is $\mathfrak{h} = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$? or equivalently, is $H = [\hat{G}, \hat{G}]$?

5. Geometric proof of Chen Signature Theorem

5.1. **Main lemma.** The content of the following Lemma is similar to the Fundamental Lemma 3.5 in [Ch3]. We show that it can be derived from Theorem 2.1 giving in the next subsection a geometric proof Theorem 2.2.

Lemma 5.1. *If $\gamma \in \Omega$ is not trivial in $\mathcal{E}(\mathbb{R}^n)$, then there are compactly supported one forms $\omega_1, \ldots, \omega_q$ such that*

$$\int_\gamma \omega_1\omega_2 \ldots \omega_q \neq 0.$$

Proof. Let γ be a non trivial class in $\mathcal{E}$ and set $x = \gamma(0)$. Assume first that γ is closed; i.e. γ is not trivial in $\mathcal{L}_x(\mathbb{R}^n)$. We consider the trivial bundle $\mathbb{R}^n \times SL(2, \mathbb{R})$.

Since $SL(2, \mathbb{R})$ is simple, then Theorem 2.1 applies, that is, there is a connection A in the bundle $\mathbb{R}^n \times SL(2, \mathbb{R})$, such that its holonomy $H_A(\gamma)$ is not trivial. This can also be deduced from Tlas result [Tl] and Proposition 1.1 as follows: since every whisker in the piecewise smooth class is a thin loop (see remark 4 in [Me]), by Proposition 1.1 it is a retreacable loop in the piecewise smoothly immersed class. In particular, γ is not a whisker and by Tlas result we have the same conclusion as before.

The holonomy can be expressed (see [Lo], p. 1416) as

$$H_A(\gamma) = I + \sum_{n>0} (-1)^n \int_{t>t_1>\ldots>t_n} A(\gamma'(t_1)) \,\ldots\, A(\gamma'(t_n))\, dt_n \ldots dt_1.$$

Since $H_A(\gamma)$ is not the identity, there is $q > 0$ such that

$$\int_{t>t_1>\ldots>t_q} A(\gamma'(t_1)) \,\ldots\, A(\gamma'(t_q))\, dt_q \ldots dt_1 \neq 0.$$

Write $A = (a_{ij})$ where $a_{i,j}$ are real valued one-forms. Note that the (i, j)- entry of

$$\int_{t>t_1>\ldots>t_q} A(\gamma'(t_1)) \,\ldots\, A(\gamma'(t_q))\, dt_q \ldots dt_1$$

is given by

$$\int_{t>t_1>\ldots>t_q} \sum_{k_1,\ldots k_{q-1}} a_{ik_1}(\gamma'(t_1))\, a_{k_1k_2}(\gamma'(t_2)) \,\ldots\, a_{k_{q-1}j}(\gamma'(t_q))\, dt_q \ldots dt_1.$$

Then, there are indexes i, j and a $(q-1)$-tuple $k_1, \ldots, k_{q-1}$ such that

$$\int_{t>t_1>\ldots>t_q} a_{ik_1}(\gamma'(t_1))\, a_{k_1k_2}(\gamma'(t_2)) \,\ldots\, a_{k_{q-1}j}(\gamma'(t_q))\, dt_q \ldots dt_1 \neq 0,$$

i.e. there are q real one-forms $\omega_1, \ldots, \omega_q$ such that

$$\int_{t>t_1>\ldots>t_q} \omega_1(\gamma'(t_1))\, \omega_2(\gamma'(t_2)) \,\ldots\, \omega_q(\gamma'(t_q))\, dt_q \ldots dt_1 \neq 0,$$

that is

$$\int_\gamma \omega_1 \omega_2 \dots \omega_q \neq 0.$$

In the proof of Theorem 7.1 of [Sp] it is assumed that the support of A is compact, hence the supports of $\omega_1, \omega_2, \dots \omega_q$ are compact as well. If γ is not closed, i.e. $\gamma(0) \neq \gamma(1)$, we consider a compactly supported real smooth function f such that $f(\gamma(0)) \neq f(\gamma(1))$ and therefore obtain $\int_\gamma df \neq 0$. □

5.2. **Proof of Theorem 2.2.** Now, the kernel of the signature can be obtained as in [Ch3]. Since the supports of $\omega_1 \dots \omega_q$ are compact, there are $\bar{\omega}_1, \dots, \bar{\omega}_q$ with polynomial coefficients such that

$$\int_\gamma \bar{\omega}_1 \bar{\omega}_2 \dots \bar{\omega}_q \neq 0.$$

By additivity, we may assume that $\bar{\omega}_i$ have monomial coefficients, namely, there are monomials $g_1, \dots, g_q$ such that

$$\int_\gamma g_1 dx_{j_1}\, g_2 dx_{j_2} \dots g_q dx_{j_q} \neq 0.$$

We have to show that this expression is a linear combination of elementary iterated integrals, i.e. integrals of the form

$$\int_\gamma dx_{i_1} dx_{i_2} \dots dx_{i_p},$$

and then one of those integrals has to be non zero, finishing the theorem.

The claim follows by induction in q as in Lemma 4.1 of [Ch3]. Assume that $q = 1$. Since g_1 is a monomial we have $g_1 = x_1^{m_1} \dots x_n^{m_n}$. Let γ_t be the restriction of γ to $[0, t]$. Hence we have

$$x_i(\gamma(t)) = \int_{\gamma_t} dx_i + x_i(\gamma(0)),$$

and thus

$$\int_\gamma g_1 dx_{j_1} = \int_0^1 \left(\int_{\gamma_t} dx_1 + x_1(\gamma(0)) \right)^{m_1} \dots \left(\int_{\gamma_t} dx_n + x_n(\gamma(0)) \right)^{m_n} dx_{j_1}(\gamma'(t))dt$$

which gives a linear combination of integrals

$$\int_0^1 \left(\int_{\gamma_t} dx_{i_1} dx_{i_2} \dots dx_{i_k} \right) dx_{j_1}(\gamma'(t))dt = \int_\gamma dx_{i_1} dx_{i_2} \dots dx_{i_k} dx_{j_1}.$$

For the induction step, note that

$$\int_\gamma g_1 dx_{j_1} g_2 dx_{j_2} \dots g_q dx_{j_q} = \int_0^1 \left(\int_{\gamma_t} g_1 dx_{j_1} g_2 dx_{j_2} \dots g_{q-1} dx_{j_{q-1}} \right) g_q(\gamma(t)) dx_{j_q}(\gamma'(t))dt.$$

By the induction hypothesis,

$$\int_{\gamma_t} g_1 dx_{j_1} g_2 dx_{j_2} \dots g_{q-1} dx_{j_{q-1}}$$

is a linear combination of integrals of the form

$$\int_{\gamma_t} dx_{i_1} dx_{i_2} \dots dx_{i_k}.$$

Applying the base step we have that

$$\int_\gamma g_1 dx_{j_1} g_2 dx_{j_2} \dots g_q dx_{j_q}$$

is a linear combination of integrals of the form

$$\int_0^1 \left(\int_{\gamma_t} dx_{i_1} dx_{i_2} \dots dx_{i_k} \int_{\gamma_t} dx_{i_{k+1}} dx_{i_{k+2}} \dots dx_{i_{k+r}} \right) dx_{j_q}(\gamma'(t))dt.$$

which is a linear combination of elementary integrals. This assertion follows from the following definition and lemma from [Ch3] which we include for the sake of completeness.

Definition 5.2.

$$\int_a^b f_1(t)dt \dots f_p(t)dt = \int_a^b \left(\int_a^s f_1(t)dt \dots f_{p-1}(t)dt \right) f_p(s)ds.$$

The following is Lemma 2.1 in [Ch3].

Lemma 5.3. *The product* $\int_a^b f_1(t)dt \dots f_i(t)dt \int_a^b f_{i+1}(t)dt \dots f_p(t)dt$ *is a linear combination of integrals* $\int_a^b f_{i_1}(t)dt \dots f_{i_p}(t)dt$.

Proof. The cases $p = 1$, $p = i$, $i = 0$ are trivial so assume that $p \geq 2$ and $0 < i < p$. Set

$$g(s) = \int_a^s f_1(t)dt \dots f_i(t)dt \int_a^s f_{i+1}(t)dt \dots f_p(t)dt,$$

$$h_1(s) = \int_a^s f_1(t)dt \dots f_{i-1}(t)dt \int_a^s f_{i+1}(t)dt \dots f_p(t)dt,$$

$$h_2(s) = \int_a^s f_1(t)dt \dots f_i(t)dt \int_a^s f_{i+1}(t)dt \dots f_{p-1}(t)dt.$$

Observe that

$$g(b) = \int_a^b g'(t)dt = \int_a^b h_1(t) f_i(t)dt + \int_a^b h_2(t) f_p(t)dt.$$

The Lemma is finished applying the induction hypothesis to h_1 and h_2. □

Acknowledgments

We would like to thank Pablo Lessa for putting our attention on the Chen signature map.

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