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BIG HEEGNER POINTS, GENERALIZED HEEGNER CLASSES AND p-ADIC L-FUNCTIONS IN THE QUATERNIONIC SETTING

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ABSTRACT. The goal of this paper is to study the p-adic variation of Heegner points and generalized Heegner classes for ordinary families of quaternionic modular forms. We compare classical specializations of big Heegner points (introduced in the quaternionic setting in [LV17] by one of the authors in collaboration with S. Vigni) with generalized Heegner classes, extending a result of Castella [Cas20] to the quaternionic setting. We also compare big Heegner points with p-adic families of generalized Heegner classes, introduced in this paper in the quaternionic setting, following works by Jetchev–Loeffler–Zerbes, [JLZ21], Büyükboduk–Lei [BL21] and Ota [Ota20]. These comparison results are obtained by exploiting the relation between p-adic families of generalized Heegner classes and p-families of p-adic L-functions, introduced in this paper following constructions of [HB15] and [BCK21].

1. INTRODUCTION

The goal of this paper is to study the p-adic variation of Heegner points and classes for ordinary families of quaternionic modular forms. The main result is a comparison between, one the one side, quaternionic big Heegner points and their classical specializations introduced in [LV17], and, on the other side, p-adic families of quaternionic generalized Heegner classes and their classical specializations defined in this paper following the works of [JLZ21] and [BL21] in the GL_2 case. The connection between these two objects is made possible by their relations with quaternionic families of p-adic L-functions, defined in this paper. We now explain more carefully the main goals and results of this paper.

1.1. **Higher specializations of Big Heegner points.** Big Heegner points were introduced by Howard [How07] in the case of Hida families for GL_2; soon after, big Heegner points were generalized to Shimura curves over totally real fields by Fouquet [Fou13] and to Shimura curves over $\mathbb{Q}$ by [LV11]. The main idea in these papers is to define a compatible system of Heegner points in a tower of modular or Shimura curves of increasing p-power level and tame level $\Gamma_0(N)$, and form suitable (inverse) limits of these points. While it is clear how specializations of big Heegner points at weight 2 arithmetic primes in the Hida family are related to Heegner points on the modular or Shimura curve of p-power level, their specializations at higher weight arithmetic primes are less clear. However, building on the relation in weight 2, Castella established in [Cas20] and [Cas13] a relation between big Heegner points and families of BDP-B p-adic L-functions (here, following [Kob13] and [KO20], BDP-B refers to the p-adic L-function constructed in [BDP13] and [Bra11]); using the well-known relation between BDP-B p-adic L-functions and generalized Heegner cycles in [BDP13] and [CH18a], one obtains an explicit relation between higher weight specializations of big Heegner points and generalized Heegner classes. As may be observed, this approach is in some sense local, going through the study of specific p-adic L-functions; a global approach is instead used by Ota [Ota20] to obtain the same results.

The *first goal* of this paper is to pursue this program in the quaternionic setting, following especially [Cas13] and [Cas20]. The crucial ingredients are, as remarked above, the study of weight 2 specializations of quaternionic big Heegner points obtained in Theorem 9.11, and the explicit reciprocity law proved in Theorem 9.13 explaining the relation between the p-adic

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family of *analytic* BDP-B p-adic L-functions constructed in Section 5 (see especially §5.3) and the p-adic family of *geometric* p-adic L-functions, obtained by applying a big Perrin-Riou logarithm to quaternionic big Heegner points. The main result, Theorem 9.13, states the equality (up to a simple explicit factor):

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}} \doteq \mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}} \tag{1.1}$$

where:

- $\mathbb{I}$ is the p-adic Hida-Hecke algebra corresponding to a primitive p-adic family of quaternionic modular forms of tame level N^+ and attached to an indefinite quaternion algebra $B/\mathbb{Q}$ of discriminant N^-; here N^+ and N^- are two coprime integers such that $p \nmid N = N^+N^-$, and $N^>1-$ is a square-free product of an even number of primes; (see §3.5, §3.6 for details);
- $\boldsymbol{\xi}\colon K^\times\backslash\widehat{K}^\times \to \mathbb{I}^\times$ is a continuous character of conductor c, where $K/\mathbb{Q}$ is a quadratic imaginary field where p is split; here c and D_K are prime to each other, and prime to Np; moreover, we require that all primes dividing N^- are inert in K, while those primes dividing N^+ are split in K (see §5.2 for details);
- $\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}$ and $\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}$ are continuous functions on $\mathbb{I}[\![\Gamma_\infty]\!]$ where Γ_∞ is a finite index subgroup of the anticyclotomic $\mathbb{Z}_p$-extension of H_c, the ring class field of conductor c of K; here $\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}$ is obtained as the image, via a big Perrin-Riou map, of the big Heegner point of conductor c (see §10.3 for details), while $\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}$ has a more analytic construction and interpolates BDP-B p-adic L-functions at arithmetic points of $\mathbb{I}$ (see §5.3 for the construction, and §5.4 for the relation with BDP-B quaternionic p-adic L-functions for a fixed modular form).

Equation (1.1) is obtained by a continuity argument from a direct comparison (following similar strategies in [DR17], [Cas20], [Cas13], [CL16]) at weight 2 primes, where the specialization of big Heegner points is explicit, since it comes directly from the construction and interpolates Heegner points. The resulting *explicit reciprocity law* at weight 2 specializations is explained in detail in Proposition 9.12. Since the specialization of $\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}$ is known, thanks to a result of Magrone [Mag22], to interpolate generalized Heegner cycles at arithmetic points of $\mathbb{I}$ of trivial character and even weight $k \geq 2$, we see that the higher weight specialization of big Heegner points is explicitly related to generalized Heegner cycles; a more precise statement (see details in Theorem 11.1) says that for all arithmetic points $\nu \in \mathrm{Spec}(\mathbb{I})(\bar{\mathbb{Q}}_p)$ of weight $k \equiv 2 \mod 2(p-1)$, we have

$$(\mathrm{pr}_*)(\mathfrak{z}_c(\nu)) = \boldsymbol{z}_c^\sharp \tag{1.2}$$

where:

- $\mathfrak{z}_c$ is (obtain from) the big Heegner point of conductor c in the quaternionic setting, and $\mathfrak{z}_c(\nu)$ is its specialization at ν (see §8.2);
- $\boldsymbol{z}_c^\sharp$ is a generalized Heegner cycle of conductor c (see §11.1);
- pr_* is a combination of the two degeneracy maps involving curves of level N^+ and N^+p (see §6.1).

1.2. **Generalized Heegner classes.** More recently, other approaches to the p-adic variation of Heegner points have been proposed, and extended to Coleman families, by Jetchev–Loeffler–Zerbes [JLZ21], Büyükboduk–Lei [BL21], Ota [Ota20] and, for Shimura curves over totally real number fields, Disegni [Dis22]. These approaches differ from each other in many ways, but a common feature can be recognized in the fact that the p-adic variation of the geometric objects is obtained by varying the coefficients in the cohomology of the fixed modular or Shimura curve of level $\Gamma_0(Np)$, as opposite to consider towers of modular or Shimura curves. In other words, the strategy of Hida [Hid86] to construct big Galois representations via the tower of modular curves of increasing p-power level is replaced by the approach of Stevens [Ste94] and Pollack–Stevens [PS11] via locally analytic distributions interpolating polynomial representations attached to modular curves. In particular, the paper [JLZ21] takes this perspective and proposes a motivic construction of p-adic families of generalized Heegner classes in the case of Coleman families. The *second goal* of our paper, achieved in Section 10 (see especially §10.9), is to propose an analogue motivic construction of generalized Heegner classes in the quaternionic ordinary case.

1.3. **Comparison.** The *third* (and final) *goal* of this paper is to compare p-adic families of quaternionic Heegner points and p-adic families of quaternionic generalized Heegner classes. This comparison is made possible by the fact that both families specialize in higher weight to quaternionic generalized Heegner classes, and the conclusion follows from a density argument, restricting both families to suitable open affinoids. The main result is Theorem 11.3, which shows that quaternionic big Heegner points and big generalized Heegner classes agree in a suitable connected open affinoid of the weight space.

1.4. **The quaternionic setting.** The quaternionic setting presents some technical difficulties and interesting features with respect to the GL_2 case. We list some of them.

We need to set up a sufficiently explicit integral theory of p-adic quaternionic modular forms, suitable for computations with Serre–Tate coordinates which are crucial for the definition of analytic p-adic L-functions in Section 5. For this, we need to extend in Sections 2 and Section 3 some results of Buzzard [Buz97], and Brasca [Bra13, Bra14, Bra16]; we note that for the results in [LV11] there is no need of such a theory, since the required properties in [LV11] are obtained by identifying the Hida big Galois representation with the representation constructed by means of the inverse limit of p-adic Tate modules of Jacobians of Shimura curves of increasing p-power level. Since Shimura curves are moduli spaces for quaternionic multiplication abelian surfaces, it becomes necessary, as usual in this context, a careful use of certain idempotents to be able to cut the dimension of the relevant cohomology groups.

A crucial ingredient for the explicit reciprocity law is the interpolation in p-adic families of the Eichler–Shimura isomorphism, for which we use the new approach of Chojecki–Hansen–Johansson [CHJ17] via perfectoid techniques; we also make use of some of the results by Barrera–Gao [SG17], generalizing to the quaternionic setting the paper by Andreatta–Iovita–Stevens [AIS15].

An other crucial ingredient for the explicit reciprocity law is the use of Coleman integration to describe Perrin-Riou logarithm, and we extend to the quaternionic setting the relevant results of the theory for p-power level Shimura curves, which, to the best knowledge of the authors, have not been settled properly elsewhere.

Results related to this paper, especially on the construction of quaternionic generalized Heegner classes, have been sketched in a recent preprint [Wan23], based on a work in progress by Jetchev, Skinner and Wan. However, on this side, we would like to remark that the focus on this paper is rather on the comparison between quaternionic generalized Heegner classes and quaternionic big Heegner points.

1.5. **Organization of the paper.** The paper is organized as follows. In Section 2 and Section 3 we review (and extend to the required level of generality) the integral theory of Shimura curves of p-power level and the theory of quaternionic ordinary families of integral modular forms. In Section 4 we review the theory of Serre–Tate coordinates, which allows to define T-expansions of quaternionic families of modular forms. In Section 5 we apply the previous results to the construction of the *analytic* families of BDP-B p-adic L-functions, which is the first result of this paper. After reviewing the construction of p-adic families of Galois representations in Section 6, big Perrin-Riou maps in Section 7, and big Heegner points in Section 8, in Section 9 we prove the second result of this paper, the equality between geometric and analytic families of p-adic L-functions, where the *geometric* p-adic L-function is defined to be the image of quaternionic big Heegner points by the big Perrin-Riou map. The construction of p-adic families of generalized Heegner classes is outlined, in the case of Hida families, in Section 10. Section 11 contains the main results of this paper. In §11.1 we prove the specialization result relating quaternionic big Heegner points and quaternionic generalized Heegner classes, answering a question in [LV11]; this is our third result. Finally, in §11.2 we relate quaternionic big Heegner points and quaternionic families of generalized Heegner classes, showing that these two objects coincide on suitable affinoid subsets of the weight space; this is our forth result. Of course, a similar result should hold in the GL_2-case; we do not state the result in the GL_2-case, but the reader will encounter any problem in stating it.

1.6. **Notation.** We set up some notations which will be used throughout the paper. Fix a square-free integer N^- which is the product of an even number of primes, and integer $N^+ \geq 5$ coprime with N^-, and a prime number $p \nmid N = N^+N^-$; we suppose that $N^- > 1$, since the case $N^- = 1$ is the GL_2-case, where our results have been already established. We also fix an imaginary quadratic field K of discriminant $-D_K$ such that all primes dividing N^- are inert in K and all primes dividing N^+p are split in K. We fix throughout the paper embeddings $\iota_\infty \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We fix a p-stabilized newform f_0 of even weight $k_0 \geq 2$ and tame level $\Gamma_0(N)$, ordinary at p. We assume throughout the paper that the residual Galois representation $\bar{\rho}$ attached to f_0 is absolutely irreducible, p-distinguished and ramified at all primes $\ell \mid N^-$; we also assume the multiplicity one hypothesis [LV11, Assumption 9.2], which ensures that the quaternionic Hida family passing through f_0 has multiplicity one (a generalization of [CKL17, Theorem 3.5] to the present setting would prove that [LV11, Assumption 9.2] holds under the current hypothesis on $\bar{\rho}$, but we will not consider this result in this paper).

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2. Shimura curves

2.1. **Quaternion algebras.** Let B be the (indefinite) quaternion algebra over $\mathbb{Q}$ of discriminant N^-. Since K splits B, we may fix an embedding of $\mathbb{Q}$-algebras $\iota_K \colon K \hookrightarrow B$. Define $\delta = \sqrt{-D_K}$ and $\vartheta = \frac{D'+\delta}{2}$, where $D' = D_K$ if $2 \nmid D_K$ and $D' = D_K/2$ if $2 \mid D_K$. For each place $v \mid N^+p\infty$ of $\mathbb{Q}$, choose an isomorphism $i_v \colon B_v \cong \mathrm{M}_2(\mathbb{Q}_v)$ satisfying

$$i_v(\vartheta) = \begin{pmatrix} \mathrm{T}_{K/\mathbb{Q}}(\vartheta) & -\mathrm{N}_{K/\mathbb{Q}}(\vartheta) \\ 1 & 0 \end{pmatrix}.$$

(Here for any field extension L/F, we denote by $\mathrm{T}_{L/F}$ and $\mathrm{N}_{L/F}$ the trace and norm maps.) For each prime $\ell \nmid Np$, choose isomorphisms $i_\ell \colon B_\ell \cong \mathrm{M}_2(\mathbb{Q}_\ell)$ such that $i_\ell(\mathcal{O}_{K,\ell}) \subseteq \mathrm{M}_2(\mathbb{Z}_\ell)$, where $\mathcal{O}_{K,\ell} = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. In particular, for each divisor M of N^+p^m and each integer $m \geq 0$, we obtain isomorphisms

$$i_M \colon \mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}) \cong \mathrm{M}_2(\mathbb{Z}/M\mathbb{Z}).$$

For each integer $m \geq 0$, let R_m be the Eichler order of B of level N^+p^m with respect to the chosen isomorphisms i_ℓ for all finite places $\ell \nmid N^-$. Let $U_m = \widehat{R}_m^\times = (R_m \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times$ and let $\widetilde{U}_m$ be the subgroup of U_m consisting of those elements g whose p-component is congruent to a matrix of the form $\left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) \mod p^m$.

Remark 2.1. Following Mori [Mor11], other papers (including [HB15], [Bur17], [Lon23]) choose different isomorphisms by fixing a so called *Hashimoto model* for B, for which one needs to chose a quadratic real field M which splits B. This is useful, among other things, to fix a global idempotent element which is fixed by the involution $x \mapsto x^\dagger$ (see §2.3); our choices are more directly comparable with those in [CH18b], [CH15], [CH18a].

2.2. **Idempotents.** Define the following elements in $K \otimes_{\mathbb{Q}} K$:

$$e = \frac{\vartheta \otimes 1 - 1 \otimes \vartheta}{(\vartheta - \bar{\vartheta}) \otimes 1} \quad \text{and} \quad \bar{e} = \frac{1 \otimes \vartheta - \bar{\vartheta} \otimes 1}{(\vartheta - \bar{\vartheta}) \otimes 1}.$$

We will often write simply $\vartheta - \bar{\vartheta} = \delta$ for the denominators of e and $\bar{e}$. A simple computation shows that e and $\bar{e}$ are orthogonal idempotents such that $e + \bar{e} = 1$.

Let $\ell \mid N^+p$ be a prime number. Then ℓ splits in K ad $\ell = \mathfrak{l}\bar{\mathfrak{l}}$, where $\mathfrak{l}$ be the prime corresponding to the chosen embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, so $K_\ell = K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ splits as the direct sum $\mathbb{Q}_\ell e_{\mathfrak{l}} \oplus \mathbb{Q}_\ell e_{\bar{\mathfrak{l}}}$ of two copies of $\mathbb{Q}_\ell$. We have a canonical map

$$j_\ell \colon K \otimes_{\mathbb{Q}} K \longrightarrow K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow B_\ell \xrightarrow{i_\ell} \mathrm{M}_2(\mathbb{Q}_\ell)$$

and one computes that $j_\ell(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $j_\ell(\bar{e}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Denote by $i\colon K \hookrightarrow \mathrm{M}_2(\mathbb{Q})$ the $\mathbb{Q}$-linear map which takes ϑ to $\left(\begin{smallmatrix} \mathrm{T}_{K/\mathbb{Q}}(\vartheta) & -\mathrm{N}_{K/\mathbb{Q}}(\vartheta) \\ 1 & 0 \end{smallmatrix}\right)$. Choose any isomorphism $I_B\colon B \otimes_{\mathbb{Q}} K \xrightarrow{\sim} \mathrm{M}_2(K)$ such that, if we denote $\iota_B\colon B \hookrightarrow \mathrm{M}_2(K)$ the embedding obtained by composition of the canonical map $B \hookrightarrow B \otimes_{\mathbb{Q}} K$ and I_B, then we have $\iota_B \circ \iota_K = i$, where we view $i\colon K \hookrightarrow \mathrm{M}_2(K)$ via the canonical inclusion $\mathrm{M}_2(\mathbb{Q}) \subseteq \mathrm{M}_2(K)$. We thus obtain a further map

$$j\colon K \otimes_{\mathbb{Q}} K \longrightarrow \mathrm{M}_2(K)$$

defined by $j(x \otimes y) = i(x)y$, and one computes again that $j(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $j(\bar{e}) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Remark 2.2. If we work over a sufficiently big extension containing both K and M, where M is a real quadratic field as in Remark 2.1, the choices of the idempotents made above and those obtained from the Hashimoto model are essentially equivalent; see [Mor11, Remark 2.1] for details.

2.3. **Quaternionic multiplication abelian varieties.** We introduce a class of abelian surfaces which play a central role in the theory of Shimura curves.

Definition 2.3. Let S be a scheme. A *quaternionic multiplication (QM) abelian surface* (A, ι) over S is an abelian scheme $A \to S$ of relative dimension 2 equipped with an injective algebra homomorphism $\iota\colon \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A)$.

Remark 2.4. QM abelian surfaces are often called *fake elliptic curves.* We often write $(A, \iota)/S$, or even A/S if the quaternionic action need not to be specified, to denote QM abelian surfaces; if $S = \mathrm{Spec}(R)$, we also write $(A, \iota)/R$ or A/R for $(A, \iota)/S$ and A/S, respectively.

Definition 2.5. An *isogeny* (resp. an *isomorphism*) of QM abelian surfaces is an isogeny (resp. an isomorphism) of abelian schemes commuting with the $\mathcal{O}_B$-action.

Let $t \in \mathcal{O}_B$ be such that $t^2 = -D_K < 0$, which exists because B splits over K, and define the involution $\dagger$ given by $b^\dagger := t^{-1}\bar{b}t$, where $\bar{\cdot}$ denotes the main involution on B. Each QM abelian surface over S can be equipped with a unique principal polarization such that for each geometric point s of S the corresponding Rosati involution of $\mathrm{End}(A_s)$, where A_s is the fiber of $A \to S$ at s, coincides with the involution $x \mapsto x^\dagger$ on $\mathcal{O}_B$ ([Mil79, Lemma 1.1]; see also [DT94, Lemma 5], [Buz97, §1]). If $\pi\colon A \to B$ is an isogeny, taking duals and composing with principal polarizations gives an isogeny $\pi^\vee\colon B \to A$. We say that π has *degree* d if the composition of $\pi^\vee \circ \pi$ is, locally on A, the multiplication by a unique integer d.

2.4. **Shimura curves with naïve level structures.** Given a group G and a scheme S, we write G_S for the constant group scheme of value G over S; when the context is clear, we often simplify the notation and write G for G_S.

Definition 2.6. Let $M \mid N^+p$ be a positive integer and (A,ι) a QM abelian surface over a $\mathbb{Z}[1/M]$-scheme S. A *naïve full level M structure* on A is an isomorphism

$$\alpha\colon \mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\sim} A[M]$$

of S-group schemes locally for the étale topology of S which commutes with the left actions of $\mathcal{O}_B$ given by ι on $A[M]$, and the multiplication from the left of $\mathcal{O}_B$ on the constant S-group scheme $\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z})$.

Remark 2.7. Notice that a full level M-structure is equivalent via i_M to an isomorphism $\mathrm{M}_2(\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\sim} A[M]$ of finite flat group schemes over S, which commutes with the left action of $\mathcal{O}_B$ given by ι on $A[M]$ and by left matrix multiplication on $\mathrm{M}_2(\mathbb{Z}/M\mathbb{Z})$. Also note that, if k is an algebraically closed field, to give a full level M structure on a QM abelian surface defined over $S = \mathrm{Spec}(k)$ is equivalent to fix a $\mathbb{Z}/M\mathbb{Z}$-basis of the group $A[M](k)$.

The group $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ acts from the left on the set of full level M structures on a QM abelian surface (A,ι) as follows. If $g \in (\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^\times$, then right multiplication $r_g(x) = xg$ by g defines an automorphism of the group $(\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^\times$ which commutes with the left action of $(\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^\times$ on itself by left multiplication; for a naïve full level M structure $\alpha\colon (\mathcal{O}_B \otimes (\mathbb{Z}/M\mathbb{Z}))^\times \xrightarrow{\sim} A[M]$ on (A,ι), we see that $\alpha_g = \alpha \circ r_g$ is a naïve full level M structure on (A,ι), and the map $\alpha \mapsto \alpha_g$ gives a left action of $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ on the set of naïve full level M structures on (A,ι). For any subgroup U of $\widehat{\mathcal{O}}_B^\times$ (where $\widehat{\mathcal{O}}_B = \mathcal{O}_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$), we obtain an action of U on full level M structures by composing the action of $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ with the map $U \subseteq \widehat{\mathcal{O}}_B^\times \overset{\hat{\pi}_M}{\twoheadrightarrow} (\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ (where $\hat{\pi}_M$ is the canonical projection).

Definition 2.8. Let (A,ι) be a QM abelian surface over a $\mathbb{Z}[1/M]$-scheme S and U a subgroup of $\widehat{\mathcal{O}}_B^\times$. A *naïve level-$U$ structure* is an equivalence class of full level M structures under the left action of U.

We say that two triples (A,ι,α) and (A',ι',α') consisting of QM abelian surfaces equipped with level-U structures are *isomorphic* if there is an isomorphism of QM abelian surfaces $\varphi\colon A \to A'$ such that $\varphi \circ \alpha = \alpha'$. The functor which takes a $\mathbb{Z}[1/(MN^-)]$-scheme S to the set of isomorphism classes of such triples (A,ι,α) over S is representable by a $\mathbb{Z}[1/(MN^-)]$-scheme $\mathcal{X}_U$, which is projective, smooth, of relative dimension 1 and geometrically connected.

We are especially interested in $V_1(M)$ and $V_0(M)$-level structures, where $V_0(M) \subseteq \widehat{\mathcal{O}}_B^\times$ is the inverse image via $\hat{\pi}_M$ of the subgroup of $(\mathcal{O}_B \otimes_{\mathbb{Z}} (\mathbb{Z}/M\mathbb{Z}))^\times$ consisting of matrices which are upper triangular modulo M, and $V_1(M)$ is the subgroup of $V_0(M)$ consisting of elements g such that $\hat{\pi}_M(g) = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ for some a, b. We also note that the map $g \mapsto g' = \mathrm{norm}(g)g^{-1}$ defines an anti-isomorphism of $V_0(M)$ to itself, and from $V_1(M)$ to $U_1(M)$, the subgroup of $U_0(M) = V_0(M)$ consisting of elements g such that $\hat{\pi}_M(g) = \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right)$ for some b, d. We thus get an induced right action of $U_0(M)$ and $U_1(M)$ on full naïve level M structures, and two such structures are equivalent under the right action of $U_0(M)$ (respectively, of $U_1(M)$) if and only if they are equivalent under the left action of $V_0(M)$ (respectively, of $V_1(M)$). We simply say that two full naïve level structure are $U_1(M)$ or $U_0(M)$-equivalent in this case, understanding the right action.

Remark 2.9. The representability result is due to Morita [Mor81, Main Theorem 1] for naïve full level M structures. A complete proof of the general case can be found in [Buz97, §2] (see especially [Buz97, Corollary 2.3 and Propositions 2.4 and 2.5]) combining the representability result of [BBG+79, Theorem §14, Exposé III] and the proof in [Buz97, Lemma 2.2] that the moduli problem $\mathcal{F}_U$ is rigid. See also [DT94, §4] and [HB15, Theorem 2.2].

2.5. **Shimura curves with Drinfeld level structures.** Let $m \geq 1$ an integer. Recall that we have a left action of $\mathcal{O}_B$ on $A[p^m]$ and therefore we also obtain a left action of $\mathcal{O}_{B,p} = \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on $A[p^m]$; thus, through i_p, we have a left action of $\mathrm{M}_2(\mathbb{Z}_p)$ on $A[p^m]$. Recall the idempotent $e \in \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ such that $i_p(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We have a decomposition

$$A[p^m] = \ker(e) \oplus \ker(1-e).$$

The element $w \in \mathcal{O}_{B,p}$ satisftying $i_p(w) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ induces an isomorphism of group schemes $w\colon \ker(e) \xrightarrow{\sim} \ker(1-e)$, and we have $eA[p^m] = \ker(1-e)$ and $(1-e)A[p^m] = \ker(e)$.

Definition 2.10. Let (A, ι) be a QM abelian surface defined over a scheme S. A $\Gamma_1(p^m)$-*level structure on* A is the datum of a cyclic finite flat S-subgroup scheme H of $eA[p^m]$ which is locally free of rank p^m, equipped with the choice of a generator P of H.

A simple generalization of [Buz97, Lemma 4.4] shows that, for a QM $(A, \iota)/S$ over a $\mathbb{Q}_p$-scheme S, there are canonical bijections between $\Gamma_1(p^m)$-level structures and $V_1(p^m)$-level structures (and therefore also with $U_1(p^m)$-level structures, accordingly with our definitions).

Remark 2.11. Comparing with [Buz97, Lemma 4.4], the reader will notice that in *loc. cit.* is shown the existence of a canonical isomorphism between $V_1(p)$-level structures and the choice of a generator Q of a finite flat subgroup scheme T of $\ker(e) \subseteq A[p]$; after the generalization to the case of higher powers of p of *loc. cit.*, which does not present any difficulty and is left to the reader, in our notation, the generator of $eA[p^m]$ is then $P = wQ$ and the subgroup is $H = wT$.

We denote $(A, \iota, \alpha, (H, P))$ quadruplets consisting of a QM abelian surface $(A, \iota)/S$ over a scheme S equipped with a naïve U-level structure α and a $\Gamma_1(p^m)$-level structure (H, P) on A. Two such quadruplets $(A, \iota, \alpha, (H, P))$ and $(A', \iota', \alpha', (H', P'))$ are said to be *isomorphic* if there is an isomorphism $\varphi\colon A \to A'$ of QM abelian surfaces which takes α to α' and such that $\varphi(H) = H'$ and $\varphi(P) = P'$. The functor which takes a $\mathbb{Z}_{(p)}$-scheme S to the set of isomorphism classes of such quadruplets $(A, \iota, \alpha, (H, P))$ over S is representable by a $\mathbb{Z}_{(p)}$-scheme $\mathcal{X}_{U,\Gamma_1(p^m)}$, which is proper and finite over $\mathcal{X}_U$ (here $\mathcal{X}_U$ is viewed as a $\mathbb{Z}_{(p)}$-scheme). Moreover, there is a canonical isomorphism of $\mathbb{Q}$-schemes between the generic fiber of $\mathcal{X}_{U,\Gamma_1(p^m)}$ and the generic fiber of $\mathcal{X}_{U \cap U_1(p^m)}$. We sometimes understand H and simply write (A, ι, α, P) for $(A, \iota, \alpha, (H, P))$.

Remark 2.12. The proof of this result is similar to the proof of [Buz97, Proposition 4.1] which only considers the case $m = 1$; the extension to the general case does not present difficulties and is left to the reader.

2.6. **CM points on Shimura curves.** Combining [CH18b], [CH15], [CL16], [CKL17] (in the definite setting) and [Cas20], [CH18a], [BCK21] (for the indefinite case), we introduce a more explicit version of the families of Heegner points introduced in [LV11].

Recall the imaginary quadratic field K. Fix an integer $c \geq 1$ with $p \nmid c$ and for each integer $n \geq 0$ let $\mathcal{O}_{cp^n} = \mathbb{Z} + cp^n\mathcal{O}_K$ be the order of K of conductor cp^n. Class field theory gives an isomorphism $\mathrm{Pic}(\mathcal{O}_{cp^n}) \cong \mathrm{Gal}(H_{cp^n}/K)$ for an abelian extension H_{cp^n} of K, called the *ring class field of K of conductor cp^n*. Define the union of these fields $H_{cp^\infty} = \bigcup_{n \geq 1} H_{cp^n}$. Since c is prime to p, $H_c \cap H_{p^\infty} = H$, where $H = H_1$ is the Hilbert class field of K, so we have an isomorphism of groups

$$\mathrm{Gal}(H_{cp^\infty}/K) \cong \mathrm{Gal}(H_c/K) \times \mathrm{Gal}(H_{p^\infty}/H).$$

Since p is split in K, we have $\mathrm{Gal}(H_{p^\infty}/H) \cong \mathbb{Z}_p^\times$; as usual we decompose $\mathbb{Z}_p^\times \cong \Delta \times \Gamma$, with $\Gamma = (1 + p\mathbb{Z}_p)$ and $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$.

Let $\mathcal{H}^\pm = \mathbb{C}\backslash\mathbb{R}$, equipped with the action of $B^\times$ by fractional linear transformations via the embedding $i_\infty\colon B^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R})$. We will often identify $\mathcal{H}^\pm$ with $\mathrm{Hom}_\mathbb{R}(\mathbb{C}, B_\infty)$. Recall, from §2.1 U_m, and $\widetilde{U}_m$, equivalently characterized by $U_m = U_0(N^+p^m)$ and $\widetilde{U}_m = U_0(N^+)\cap U_1(p^m)$. For any integer $m \geq 0$, define the Riemann surfaces (see [LV11, §2] for details)

$$X_m(\mathbb{C}) = B^\times\backslash(\mathcal{H}^\pm \times \widehat{B}^\times)/U_m \cong \Gamma_m\backslash\mathcal{H}, \tag{2.1}$$

$$\widetilde{X}_m(\mathbb{C}) = B^\times\backslash(\mathcal{H}^\pm \times \widehat{B}^\times)/\widetilde{U}_m \cong \widetilde{\Gamma}_m\backslash\mathcal{H}, \tag{2.2}$$

where Γ_m (respectively, $\widetilde{\Gamma}_m$) is the subgroup of norm 1 elements in $B^\times \cap U_m$ (respectively, $B^\times \cap \widetilde{U}_m$). We will write $[(x, g)]$ for the point in any of the two Riemann surfaces $X_m(\mathbb{C})$ and $\widetilde{X}_m(\mathbb{C})$ represented by the class of the pair (x, g) in $\mathcal{H}^\pm \times \widehat{B}^\times$. Then there are algebraic curves X_m and $\widetilde{X}_m$, defined over $\mathbb{Q}$, whose complex points are canonically identified with $X_m(\mathbb{C})$ and $\widetilde{X}_m(\mathbb{C})$, respectively; moreover, the curves X_m and $\widetilde{X}_m$ are the generic fibers of $\mathcal{X}_m = \mathcal{X}_{U_m}$ and $\widetilde{\mathcal{X}}_m = \mathcal{X}_{\widetilde{U}_m}$, respectively.

We refer to [LV11, Definition 3.1] for the definition of Heegner points on X_m and $\widetilde{X}_m$ in terms of optimal embeddings; to fix the notation, for $c \geq 1$ an integer prime to pND_K (and $N = N^+N^-$) the point $x = [(f, g)]$ represents a Heegner point on X_m (respectively, $\widetilde{X}_m$) if $f(\mathcal{O}_{cp^n}) = f(K) \cap gU_mg^{-1}$ (respectively, $f(\mathcal{O}_{cp^n}) = f(K) \cap g\widetilde{U}_mg^{-1}$, plus a natural condition on the images of the elements congruent to 1 modulo cp^n); here $f\colon K \hookrightarrow B$ is viewed as a point in $\mathcal{H}^\pm$ by scalar extension to $\mathbb{R}$. Moreover, for $a \in \widehat{K}^\times$, by Shimura reciprocity law we have $x^\sigma = [(f, \hat{f}(a^{-1})g)]$ where $\hat{f}\colon \widehat{K} \to \widehat{B}$ is the adelization of f, $\mathrm{rec}_K(a) = \sigma$, and rec_K is the geometrically normalized reciprocity map ([Shi71, Theorem 9.6]).

Let $c = c^+c^-$ with c^+ divisible by primes which are split in K and c^- divisible by primes which are inert in K. Choose decompositions $c^+ = \mathfrak{c}^+\bar{\mathfrak{c}}^+$ and $N^+ = \mathfrak{N}^+\bar{\mathfrak{N}}^+$. For each prime number ℓ and each integer $n \geq 0$, define

- $\xi_\ell = 1$ if $\ell \nmid N^+cp$.
- $\xi_p^{(n)} = \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{p}) = \mathrm{GL}_2(\mathbb{Q}_p)$.
- $\xi_\ell = \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell^s & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{l}) = \mathrm{GL}_2(\mathbb{Q}_\ell)$ if $\ell \mid c^+$ and ℓ^s is the exact power of ℓ dividing c^+, where $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ is a factorization into prime ideals in $\mathcal{O}_K$ and $\mathfrak{l} \mid \mathfrak{c}^+$.
- $\xi_\ell = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \ell^s & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q}_\ell)$ if $\ell \mid c^-$ and ℓ^s is the exact power of ℓ dividing c^-.
- $\xi_\ell = \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\mathfrak{l}) = \mathrm{GL}_2(\mathbb{Q}_\ell)$ if $\ell \mid N^+$, where $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ is a factorization into prime ideals in $\mathcal{O}_K$ and $\mathfrak{l} \mid \mathfrak{N}^+$.

We understand these elements $\xi_\bullet^\star$ as elements in $\widehat{B}^\times$ by implicitly using the isomorphisms i_ℓ defined before. With this convention, define $\xi^{(n)} = (\xi_\ell, \xi_p^{(n)})_{\ell\neq p} \in \widehat{B}^\times$. Define a map $x_{cp^n,m}\colon \mathrm{Pic}(\mathcal{O}_{cp^n}) \to X_m(\mathbb{C})$ by $[\mathfrak{a}] \mapsto [(\iota_K, a\xi^{(n)})]$, where if $\mathfrak{a}$ is a representative of the ideal class $[\mathfrak{a}]$, then $a \in \widehat{K}^\times$ satisfies $\mathfrak{a} = a\widehat{\mathcal{O}}_{cp^n} \cap K$; here $a \in \widehat{K}^\times$ acts on $\xi^{(n)} \in \widehat{B}^\times$ via left multiplication by $\hat{\iota}_K(a)$. We often write $x_{cp^n,m}(a)$ or $x_{cp^n,m}(\mathfrak{a})$ for $x_{cp^n,m}([\mathfrak{a}])$. One easily verifies that $x_{cp^n,m}(a)$ are Heegner points of conductor cp^n in $X_m(H_{cp^n})$, for all $a \in \mathrm{Pic}(\mathcal{O}_{cp^n})$, and all integers $n \geq 0$ and $m \geq 0$. More generally, we define a map $\tilde{x}_{cp^n,m}\colon K^\times\backslash\widehat{K}^\times \to \widetilde{X}_m(\mathbb{C})$ by $\tilde{x}_{cp^n,m}(a) = [(\iota_K, a\xi^{(n)})]$, and again verify that $\tilde{x}_{cp^n,m}(a)$ are Heegner points of conductor cp^n in $\widetilde{X}_m(H_{cp^n})$, for all $a \in \widehat{K}^\times$, and all integers $n \geq 0$ and $m \geq 0$.

We consider the pro-$\mathbb{Z}_p$-scheme (viewing $\widetilde{X}_m$ as a $\mathbb{Z}_p$-scheme by scalar extension)

$$\widetilde{X}_\infty = \varprojlim_m \widetilde{X}_m.$$

Then we have a uniformization $\mathcal{H}^\pm \times \widehat{B}^\times \to \widetilde{X}_\infty(\mathbb{C})$ and define the point $x(a) = [(\iota_K, a^{-1}\xi)]$ for each $a \in \widehat{K}^\times$. If $\mathfrak{a}$ is an integral ideal of $\mathcal{O}_c$ such that $(\mathfrak{a}, \mathfrak{N}^+\mathfrak{p}) = 1$, then if $a \in \widehat{K}^\times$ satisfies $\mathfrak{a} = a\widehat{\mathcal{O}}_K \cap \mathcal{O}_c$ and $a_\mathfrak{q} = 1$ for all $\mathfrak{q} \mid \mathfrak{N}^+\mathfrak{p}$, we write $x(\mathfrak{a}) = x(a)$ (one easily checks that this definition does not depend on the choice of a). The points $x(\mathfrak{a})$ are rational over H_{cp^∞} in the following sense: for each m the canonical projection $x_m(\mathfrak{a})$ of $x(\mathfrak{a})$ to $\widetilde{X}_m(\mathbb{C})$ belongs to $\widetilde{X}_m(H_{cp^\infty})$.

2.7. **Igusa towers.** Let (A,ι) be a QM abelian surface over a $\mathbb{Z}_p$-scheme S. For an integer $m\geq 1$, let $A[p^m]^0$ be the connected component of the identity of the p^m-torsion subgroup $A[p^m]$ of A, and let $A[p^\infty]^0$ be the connected component of the identity of the p-divisible group $A[p^\infty]$ of A. Let $\boldsymbol{\mu}_{p^\infty}$ (respectively, $\boldsymbol{\mu}_{p^m}$) denote the S-group scheme of p-power roots of unity (respectively, of p^m-th roots of unity).

Definition 2.13. An *arithmetic trivialization* on $A[p^\infty]$ (respectively, $A[p^m]$) is an isomorphism $\beta\colon \boldsymbol{\mu}_{p^\infty}\times\boldsymbol{\mu}_{p^\infty}\overset{\sim}{\to} A[p^\infty]^0$ (respectively, an isomorphism $\beta\colon \boldsymbol{\mu}_{p^m}\times\boldsymbol{\mu}_{p^m}\overset{\sim}{\to} A[p^m]^0$) of finite flat group schemes over S which is equivariant for the $\mathcal{O}_{B,p}=\mathcal{O}_B\otimes_{\mathbb{Z}}\mathbb{Z}_p$-action, where the action of $\mathcal{O}_{B,p}$ on $\boldsymbol{\mu}_{p^\infty}\times\boldsymbol{\mu}_{p^\infty}$ is by left matrix multiplication via ι_p.

Remark 2.14. The existence of an arithmetic trivialization on $A[p^\infty]$ implies that A is an ordinary abelian scheme over S.

One can easily show that an arithmetic trivialization of $A[p^\infty]$ is equivalent to an isomorphism $\beta\colon \boldsymbol{\mu}_{p^\infty}\overset{\sim}{\to} eA[p^\infty]^0$, and similarly an arithmetic trivialization of $A[p^m]$ is equivalent to an isomorphism $\beta\colon \boldsymbol{\mu}_{p^m}\overset{\sim}{\to} eA[p^m]^0$ of finite flat connected group schemes over S, equivariant for the action of $e\mathcal{O}_{B,p}$ (where the action of $e\mathcal{O}_{B,p}$ on $\boldsymbol{\mu}_{p^\infty}$ is defined through the isomorphism with $e(\boldsymbol{\mu}_{p^\infty}\times\boldsymbol{\mu}_{p^\infty})\cong\boldsymbol{\mu}_{p^\infty}$ given by the first projection). An arithmetic trivialization β of $A[p^\infty]$ induces for each integer $m\geq 1$ an arithmetic trivialization $\bar\beta^{(m)}\colon \boldsymbol{\mu}_{p^m}\overset{\sim}{\to} eA[p^m]^0$. An arithmetic trivialization β_m of $A[p^m]$ is said to be *compatible* with a given arithmetic trivialization β of $A[p^\infty]$ if the composition

$$\boldsymbol{\mu}_{p^m}\xrightarrow{\beta_m} eA[p^m]^0\xrightarrow{(\bar\beta^{(m)})^{-1}}\boldsymbol{\mu}_{p^m}$$

is the identity.

Let us denote $\mathcal{X}_0$ viewed as $\mathbb{Z}_{(p)}$-scheme. Let $\mathbf{Ha}$ be the Hasse invariant of the special fiber $\mathbb{X}_0$ of $\mathcal{X}_0$, and let $\widetilde{\mathbf{Ha}}$ be a lift of $\mathbf{Ha}$ to $\mathcal{X}_0$ ([Kas04, §7]). Then $\mathcal{X}_0^{\mathrm{ord}}=\mathcal{X}_0[1/\widetilde{\mathbf{Ha}}]$ is an affine open $\mathbb{Z}_{(p)}$-subscheme of $\mathcal{X}_0$ representing the moduli problem which associates to any $\mathbb{Z}_{(p)}$-scheme S the isomorphism classes of triplets (A,ι,α) where (A,ι) is an ordinary QM abelian surface over S and α a naïve $U_0(N^+)$-level structure.

Let $\mathcal{A}^{\mathrm{ord}}\to\mathcal{X}_0^{\mathrm{ord}}$ be the universal ordinary abelian variety and for any $\mathbb{Z}_{(p)}$-algebra R define $\mathcal{A}_R^{\mathrm{ord}}=\mathcal{A}^{\mathrm{ord}}\otimes_{\mathbb{Z}_{(p)}}R$; for $R=\mathbb{Z}/p^n\mathbb{Z}$, we set $\mathcal{A}_n^{\mathrm{ord}}=\mathcal{A}^{\mathrm{ord}}_{\mathbb{Z}/p^n\mathbb{Z}}$. Denote $\mathcal{A}_n^{\mathrm{ord}}[p^m]^0$ the connected component of the p^m-torsion subgroup scheme $\mathcal{A}_n^{\mathrm{ord}}[p^m]$ of $\mathcal{A}_n^{\mathrm{ord}}$. For integers $m\geq 1$ and $n\geq 1$, let

$$\mathcal{P}_{m,n}(S)=\mathrm{Isom}_{\mathcal{O}_{B,p}}\left(\boldsymbol{\mu}_{p^m}\times\boldsymbol{\mu}_{p^m},\mathcal{A}_n^{\mathrm{ord}}[p^m]^0\right)$$

be the set of arithmetic trivializations on $\mathcal{A}_n^{\mathrm{ord}}[p^m]$, where for two group schemes G and H equipped with a left $\mathcal{O}_{B,p}$-action, $\mathrm{Isom}_{\mathcal{O}_{B,p}}(G,H)$ denotes the set of isomorphisms of groups schemes $G\to H$ which are equivariant for the action of $\mathcal{O}_{B,p}$. The moduli problem $\mathcal{P}_{m,n}$ is represented by a $\mathbb{Z}/p^n\mathbb{Z}$-scheme $\mathrm{Ig}_{m,n}$, the *p^m-layer of the Igusa tower over $\mathbb{Z}/p^n\mathbb{Z}$*, which is finite étale over $\mathrm{Ig}_{0,n}$ (see [Hid04, Chapter 8], [Hid02, §2.1] or [Bur17, §2.5]). We also note that, by the universality of $\mathcal{A}_n^{\mathrm{ord}}$, the $\mathbb{Z}/p^n\mathbb{Z}$-scheme $\mathrm{Ig}_{m,n}$ represents the moduli problem which associates to any $\mathbb{Z}/p^n\mathbb{Z}$-scheme S the set of isomorphism classes of quadruplets (A,ι,α,β) consisting of a QM abelian surface (A,ι) over S equipped with a $U_1(N^+)$-level structure α and an arithmetic trivialization β of $A[p^m]$. For each $m\geq 0$, each integer $n\geq 1$ and each

$\mathbb{Z}/p^n\mathbb{Z}$-scheme S, the canonical monomorphism $\boldsymbol{\mu}_{p^m} \hookrightarrow \boldsymbol{\mu}_{p^{m+1}}$ of S-group schemes induces a canonical map $\mathrm{Ig}_{m+1,n} \to \mathrm{Ig}_{m,n}$. We can therefore consider the $\mathbb{Z}/p^n\mathbb{Z}$-scheme

$$\widehat{\mathrm{Ig}}_n = \varprojlim_m \mathrm{Ig}_{m,n}.$$

The $\mathbb{Z}/p^n\mathbb{Z}$-scheme $\widehat{\mathrm{Ig}}_n$ represents then over $\mathbb{Z}/p^n\mathbb{Z}$ the moduli problem

$$\mathcal{P}_n(S) = \mathrm{Isom}_{\mathcal{O}_{B,p}}\left(\boldsymbol{\mu}_{p^\infty} \times \boldsymbol{\mu}_{p^\infty}, \mathcal{A}_n^{\mathrm{ord}}[p^\infty]^0\right)$$

classifying the set of arithmetic trivializations of $\mathcal{A}_n^{\mathrm{ord}}[p^\infty]$, or, equivalently, the moduli problem which associates to a $\mathbb{Z}/p^n\mathbb{Z}$-scheme S the set of isomorphism classes of quadruplets $(A, \iota, \alpha, \beta)$ for each integer $m \geq 1$ consisting of a QM abelian surface (A, ι) over S equipped with a $U_1(N^+)$-level structure α and a family of arithmetic trivializations β_m of $\mathcal{A}_n^{\mathrm{ord}}[p^m]$, one for each integer $m \geq 1$, such that there is a trivialization β of $\mathcal{A}_n^{\mathrm{ord}}[p^\infty]$ for which β_m is compatible with β, for all $m \geq 1$. Define finally the $\mathbb{Z}_p$-formal scheme

$$\widehat{\mathrm{Ig}} = \varinjlim_n \widehat{\mathrm{Ig}}_n$$

where the direct limit is computed with respect to the canonical maps induced by the canonical projection maps $\mathbb{Z}/p^{n+1}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ for each $n \geq 1$.

Recall the point $x(\mathfrak{a}) = [(\iota_K, a^{-1}\xi)]$ in the pro-$\mathbb{Q}_p$-scheme $\widetilde{X}_\infty(H_{cp^\infty})$ defined before, which corresponds to the sequence $(x_m(\mathfrak{a}))_{m\geq 0}$ of points in $\widetilde{X}_m(H_{cp^\infty})$. Let $x_m(\mathfrak{a})$ correspond to a quadruplet $(A_\mathfrak{a}, \iota_\mathfrak{a}, \alpha_\mathfrak{a}, \beta_{\mathfrak{a},m})$. The abelian variety $A_\mathfrak{a}$ can be defined over $\mathcal{V} = K^{\mathrm{ab}} \cap \mathbb{Z}_p^{\mathrm{unr}}$, and is p-ordinary because p is split in K, so there exists a unique arithmetic trivialization $\beta_\mathfrak{a}$ compatible with the arithmetic trivializations defined by $\beta_{\mathfrak{a},m}$; therefore, the point $x(\mathfrak{a})$ corresponds to a point, still denoted $x(\mathfrak{a}) = (A_\mathfrak{a}, \iota_\mathfrak{a}, \alpha_\mathfrak{a}, \beta_\mathfrak{a})$, in the Igusa tower $\widehat{\mathrm{Ig}}$.

3. Modular forms

3.1. **Geometric modular forms.** Following [Bra14], we introduce the notion of quaternionic geometric modular forms. Let S be a $\mathbb{Z}_p$-scheme and (A, ι) a QM abelian surface over a S; denote $\pi\colon A \to S$ the structural map. Then $\pi_*\Omega_{A/S}$, where $\Omega_{A/S}$ is the bundle of relative differentials, inherits an action of $\mathcal{O}_B$. For $S = \mathrm{Spec}(R)$, we write $\Omega_{A/R} = \Omega_{A/\mathrm{Spec}(R)}$. Tensoring the action of $\mathcal{O}_B$ on $\pi_*\Omega_{A/R}$ with the scalar action of $\mathbb{Z}_p$ we obtain an action of $\mathcal{O}_{B,p}$ on $\pi_*\Omega_{A/R}$, and therefore the sheaf $\pi_*\Omega_{A/R}$ is equipped with an action of the idempotent element e considered in §2.4. Define the invertible sheaf $\underline{\omega}_{A/R} = e\pi_*\Omega_{A/R}$.

Definition 3.1. Let R be a $\mathbb{Z}_p$-algebra. A *test object* over a R-algebra R_0 is a quintuplet $T = (A, \iota, \alpha, (H, P), \omega)$ consisting of a QM abelian surface (A, ι) over R_0, a $U_0(N^+)$-level structure α on A, a $\Gamma_1(p^m)$-level structure (H, P) on A and a non-vanishing global section of the line bundle $\underline{\omega}_{A/R_0}$ of relative differentials. Two test objects are *isomorphic* if there is an isomorphism of QM abelian surfaces which induces isomorphisms of $V_1(M)$ and $\Gamma_1(p^m)$-level structure and pulls back the generator of the differentials of A' to that of A.

Definition 3.2. Let R be a $\mathbb{Z}_p$-algebra and k an integer. A *R-valued geometric modular form of weight k on $\widetilde{\mathcal{X}}_m$* is a rule $\mathcal{F}$ that assigns to every isomorphism class of test objects $T = (A, \iota, \alpha, (H, P), \omega)$ over an R-algebra R_0 a value $\mathcal{F}(T) \in R_0$ such that the following two conditions are satisfied:

- Compatibility with base change: $\mathcal{F}(A', \iota', \alpha', (H', P'), \omega') = \varphi\big(\mathcal{F}(A, \iota, \alpha, (H, P), \varphi^*(\omega'))\big)$ for any morphism $\varphi\colon R_0 \to R_0'$ of R-algebras, where $A' = A \otimes_{R,\varphi} R_0'$, and ι', α' and (H', P') are obtained by base change from ι, α and (H, P), respectively;
- Weight condition: $\mathcal{F}(A, \iota, \alpha, (H, P), \lambda\omega) = \lambda^{-k}\mathcal{F}(A, \iota, \alpha, (H, P), \omega)$, for any $\lambda \in R_0^\times$.

We denote $M_k(N^+, p^m, R)$ the R-module of R-valued weight k modular forms on $\widetilde{\mathcal{X}}_m$.

We also need to recall an alternative definition of modular forms. First, any test object gives rise to a *test quadruplet* over a R-algebra R_0 is a quadruplet $T = (A, \iota, \alpha, (H, P))$ by forgetting the differential form; we say that two such quadruplet are *isomorphic* if, as before, there is an isomorphism of QM abelian surfaces which induces isomorphisms of $V_1(M)$ and $\Gamma_1(p^m)$-level structure.

Definition 3.3. Let R be a $\mathbb{Z}_p$-algebra and k an integer. A *R-valued geometric modular form of weight k on $\widetilde{\mathcal{X}}_m$* is a rule $\mathcal{F}$ that assigns to every isomorphism class of test quadruplet $T = (A, \iota, \alpha, (H, P))$ over an R-algebra R_0 a section $\mathcal{F}(T)$ of $\omega_{A/R_0}^{\otimes k}$ which is compatibility with base change, *i.e.* $\mathcal{F}(A', \iota', \alpha', (H', P')))\omega' = \varphi\big(\mathcal{F}(A, \iota, \alpha, (H, P))\big)\varphi^*(\omega')$ for any morphism $\varphi\colon R_0 \to R_0'$ of R-algebras, where $A' = A \otimes_{R,\varphi} R_0'$, and ι', α' and (H', P') are obtained by base change from ι, α and (H, P).

The equivalence between Definitions 3.2 and 3.3 is given the map $\mathcal{F} \mapsto \mathcal{G}$ defined by

$$\mathcal{G}(T) = \mathcal{F}(T, \omega)\omega^{\otimes k}$$

for any choice of a section ω. We also note that both definitions are equivalent to the existence of a global section of $\underline{\omega}_{m,R}^{\otimes k}$, where $\underline{\omega}_{m,R} = \underline{\omega}_{\mathscr{A}/R}$ and $\mathscr{A}_m \to \widetilde{\mathcal{X}}_m$ is the universal QM abelian surface (here we view $\widetilde{\mathcal{X}}_m$ and $\mathscr{A}_m$ as defined over R; see [HB15, §3.1] for details). We denote $\omega_{\mathcal{F}}$ the global section of $\underline{\omega}_{m,R}^{\otimes k}$ associated with $\mathcal{F}$ as in Definitions 3.2 and 3.3.

We have an action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ on $\Gamma_1(p^m)$-level structures of a QM abelian surface A over R_0, denoted $(H, P) \mapsto u \cdot (H, P) = (H, u \cdot P)$, and defined by taking the generator section P to the section $u \cdot P$ (multiplication of P by u). We define an action of $(\mathbb{Z}/p^m\mathbb{Z})^\times$ on $\mathcal{F} \in M_k(N^+, p^m, R)$ by setting $(\mathcal{F}|\langle u\rangle)(T) = \mathcal{F}(u \cdot T)$ where for any test object $T = (A, \iota, \alpha, (H, P))$ defined over a R-algebra R_0, we define $u \cdot T = (A, \iota, \alpha, u \cdot (H, P\rangle)$. It is easy to check that $\mathcal{F}|\langle u\rangle$ still belongs to $M_k(N^+, p^m, R)$.

Definition 3.4. Let $\psi\colon (\mathbb{Z}/p^m\mathbb{Z})^\times \to R$ be a finite order character. We say that a modular form $\mathcal{F} \in M_k(N^+, p^m, R)$ has *character ψ* if $\mathcal{F}|\langle u\rangle = \psi(u)\mathcal{F}$ for all $u \in (\mathbb{Z}/p^m\mathbb{Z})^\times$.

We denote $M_k(N^+p^m, R)$ the R-submodule of $M_k(N^+, p^m, R)$ consisting of those modular forms with trivial character; if p is invertible in R, then an element in $M_k(N^+p^m, R)$ is a rule $T \mapsto \mathcal{F}(T)$ defined on test objects $T = (A, \iota, \alpha, H, \omega)$ (obtained from a test object as in Definition 3.1 by forgetting the generator P of H) and satisfying similar compatibilities as before. See [Bra14, Definition 1.3] for more details.

3.2. **p-adic modular forms.** Recall that a *p-adic ring* R is a $\mathbb{Z}_p$-algebra which is complete and Hausdorff with respect to the p-adic topology, so that $R \cong \varprojlim_n R/p^nR$.

Definition 3.5. Let R be a p-adic ring. The space $V_p(N^+, R)$ of *p-adic modular forms* of tame level N^+ on B is a global section of $\widehat{\mathrm{Ig}}_R = \widehat{\mathrm{Ig}} \otimes_{\mathbb{Z}_p} R$.

For a p-adic ring R, define $\mathrm{Ig}_{R,m,n} = \mathrm{Ig}_{m,n} \otimes_{\mathbb{Z}_p} R$. Accordingly with Definition 3.5, for a p-adic ring R we have

$$V_p(N^+, R) \cong \varprojlim_n \varinjlim_m H^0(\mathrm{Ig}_{R,m,n}, \mathcal{O}_{\mathrm{Ig}_{R,m,n}}).$$

An element in $V_p(N^+, R)$ is then a rule $\mathcal{F}$ that for each p-adic ring R_0 with an R-algebra structure and each pair of integers $m \geq 1$ and $n \geq 1$ assigns to each quaternionic multipication abelian surface $(A, \iota, \alpha, \beta)$, with $U_0(N^+)$-level structure α and an arithmetic trivialization β_m of $A[p^m]$, over the R/p^nR-algebra R_0/p^nR_0 a value $\mathcal{F}(A, \iota, \alpha, \beta_m) \in R_0/p^nR_0$ which is compatible with respect to the canonical maps which are used to compute the direct and inverse limit, depends only on the isomorphism class of the quadruplet and is compatible under base change given by continuous morphisms between R-algebras.

The action of $\Gamma = 1 + p\mathbb{Z}_p$ on $\mathcal{A}_n^{\mathrm{ord}}[p^\infty]$ by left multiplication on compatible sequences of p-power torsion sections give rise to an action of Γ on arithmetic trivializations β of $A[p^\infty]$, denoted $\beta \mapsto u \cdot \beta$ for $u \in \Gamma$, obtained by composition

$$u \cdot \beta\colon\ \boldsymbol{\mu}_{p^\infty} \xrightarrow{\ \beta\ } \mathcal{A}_n^{\mathrm{ord}}[p^\infty] \xrightarrow{\ u\ } \mathcal{A}_n^{\mathrm{ord}}[p^\infty].$$

Alternatively, if $u\colon \boldsymbol{\mu}_{p^\infty} \to \boldsymbol{\mu}_{p^\infty}$ denotes the action of $u \in \Gamma$ on the p-power roots of unity given, for any $\mathbb{Z}_p$-algebra R and any $\zeta \in \boldsymbol{\mu}_{p^\infty}(R)$, by $\zeta \mapsto \zeta^u$ (raise to the u-power), then $u \cdot \beta$ is equal to the composition $u \cdot \beta\colon \boldsymbol{\mu}_{p^\infty} \overset{u}{\to} \boldsymbol{\mu}_{p^\infty} \overset{\beta}{\to} \mathcal{A}_n^{\mathrm{ord}}[p^\infty]$. The R-module $V_p(N^+, R)$ is equipped with a structure of $\Lambda_R = R[\![\Gamma]\!]$-module $\mathcal{F} \mapsto \mathcal{F}|\langle\lambda\rangle$ for $\mathcal{F} \in V_p(N^+, R)$ and $\lambda \in \Lambda_R$, defined for $u \in \Gamma$ by

$$(\mathcal{F}|\langle u\rangle)(A, \iota, \alpha, \beta) = \mathcal{F}(A, \iota, \alpha, u \cdot \beta)$$

and then extending by R-linearity.

Definition 3.6. Let $\psi\colon 1 + p\mathbb{Z}_p \to \boldsymbol{\mu}_{p^\infty}(\overline{\mathbb{Q}}_p)$ be a finite order character and $k \in \mathbb{Z}_p$ a p-adic integer. We say that a p-adic modular form $\mathcal{F}$ is of *signature* (k, ψ) if for every $u \in 1 + p\mathbb{Z}_p$, we have $\mathcal{F}|\langle u\rangle = u^k\psi(u)\mathcal{F}$.

3.3. **Geometric modular forms and p-adic modular forms.** Let R be a p-adic ring and $\mathcal{F}$ an R-valued modular form of weight k on $\mathcal{X}_m$. Consider a QM abelian surface $(\mathcal{A}, \iota, \alpha, \beta)$ over R equipped with a U_0-level structure α and an arithmetic trivialization β, and let $A = \mathcal{A} \otimes_R k$, with $k = R/\mathfrak{m}_R$ the residue field of R. The trivialization β determines by Cartier duality a point x_β in $e^\dagger\mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p)$ (see for example [Mag22, §3.1]), where $\mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p)$ is the p-adic Tate module of the dual abelian variety $A^\vee$ of A (use that the abelian S-scheme A is equipped with a unique principal polarization $\theta_A\colon A \overset{\cong}{\to} A^\vee$ such that the associated Rosati involution of $\mathrm{End}(A)$ coincides with the involution $b \mapsto b^\dagger$ on $\mathcal{O}_B$). Now recall from [Kat81, page 150] that $\mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\widehat{\mathcal{A}}, \widehat{\mathbb{G}}_m)$, where $\widehat{\mathcal{A}}$ is the formal group of $\mathcal{A}$ and $\widehat{\mathbb{G}}_m$ is the formal multiplicative group over R, and therefore we have

$$e^\dagger\, \mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(e\widehat{\mathcal{A}}, \widehat{\mathbb{G}}_m).$$

Let $\varphi_\beta \in \mathrm{Hom}_{\mathbb{Z}_p}(e\widehat{\mathcal{A}}, \widehat{\mathbb{G}}_m)$ denote the homomorphism corresponding to x_β. Thus, we can consider the pull-back $\omega_\beta = \varphi_\beta^*(dT/T)$ of the standard differential dT/T of $\widehat{\mathbb{G}}_m$. Then ω_β is a formal differential form on $\widehat{\mathcal{A}}$, which is then identified with a differential form on $\mathcal{A}$, denoted with the same symbol. Moreover, β defines a $\Gamma_1(p^m)$-level structure as follows. Fix a generator of $\boldsymbol{\mu}_{p^\infty}$, which induces a generator of $\boldsymbol{\mu}_{p^m}$, call it ζ_{p^m}; then $\beta(\zeta_{p^m})$ gives a point P in $\mathcal{A}[p^m]^0$ of exact order p^m, which gives a $\Gamma_1(p^m)$-level structure (H, P), where H is the subgroup scheme generated by P. Define

$$\widehat{\mathcal{F}}(\mathcal{A}, \iota, \alpha, \beta) := \mathcal{F}(\mathcal{A}, \iota, \alpha, (H, P), \omega_\beta).$$

The map $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ is compatible with base change, only depends on the isomorphism class of $(A, \iota, \alpha, \beta)$ and it is compatible with the maps occurring in the direct and inverse limit in the definition of p-adic modular forms; thus $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ establishes a map

$$M_k(N^+, p^m, R) \longrightarrow V_p(N^+, R) \tag{3.1}$$

from geometric to p-adic modular forms. If the character of $\mathcal{F}$ is ψ (*cf.* Definition 3.4) then the signature of $\widehat{\mathcal{F}}$ is (k, ψ). We say that a p-adic modular form is *classic* if it belongs to the image of the map (3.1).

3.4. *U* **and** *V* **operators.** Following [Gou88, §II.2, II.3], we recall the definitions for the V and U operators on the space $V_p(N^+, R)$ of p-adic modular forms over a p-adic ring R. Given a triple (A, α, β) over an R-algebra, with $A = (A, \iota)$ ordinary QM abelian surface, α a $U_0(N^+)$-level structure and β an arithmetic trivialization as in Definition 2.13, we can consider the quotient $A_0 = A/C_p$ of A by its canonical subgroup C_p. Write $\phi\colon A \twoheadrightarrow A_0$ for the canonical projection; then (A_0, ι_0) is a QM abelian surface, where ι_0 is the principal polarization induced by ι. Since ϕ has degree prime to N^+, it induces an isomorphism $\phi\colon A[N^+] \to A_0[N^+]$, which defines a $U_0(N^+)$-level structure $\alpha_0 = \phi \circ \alpha$ on A_0. Also, the isogeny ϕ is étale, so it induces an isomorphism on formal completions $\hat{\phi}\colon \widehat{A}_0 \to \widehat{A}$, and therefore also an isomorphism $\phi\colon e\widehat{A}_0 \to e\widehat{A}$. As in §3.3, the trivialization β determines by Cartier duality a point x_β in $e^\dagger \mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p)$ and therefore a morphism $\varphi_\beta\colon e\widehat{A} \to \widehat{\mathbb{G}}_m$. Hence we obtain an isomorphism

$$\varphi_{\beta_0}\colon e\widehat{A}_0 \xrightarrow{\hat{\phi}} e\widehat{A} \xrightarrow{\varphi_\beta} \widehat{\mathbb{G}}_m,$$

which corresponds to an arithmetic trivialization β_0 on A_0.

Definition 3.7. The operator $V\colon V_p(N^+, R) \to V_p(N^+, R)$ is defined for any p-adic modular form f over R by the equation $(Vf)(A, \alpha, \beta) = f(A_0, \alpha_0, \beta_0)$, with α_0 and β_0 as above.

Definition 3.8. The operator $U\colon V_p(N^+, R) \to V_p(N^+, R)$ is defined for any p-adic modular form f over R by the equation $(Uf)(A, \alpha, \beta) = \frac{1}{p}\sum_{(A_i,\alpha_i,\beta_i)} f(A_i, \alpha_i, \beta_i)$, where the sum is over the set of equivalence classes of triples (A_i, α_i, β_i) such that (A, α, β) is obtained from (A_i, α_i, β_i) by quotient by the canonical subgroup.

For any $\mathcal{F} \in V_p(M, R)$, we then have $VU\mathcal{F} = \mathcal{F}$. The limit $\lim_{n\to\infty} U^{n!}$ defines an idempotent acting on $V_p(N^+, R)$, which we denote by e^{ord} (see [Hid00, Lemma 3.2.7] and [Hid00, pag. 238]).

Definition 3.9. The R-module $V_p(N^+, R)^{\mathrm{ord}} := e^{\mathrm{ord}} V_p(N^+, R)$ is the *ordinary* submodule of $V_p(N^+, R)$.

If $\mathcal{F} \in V_p(N^+, R)$ is a p-adic modular forms over the valuation ring R of a finite extension of $\mathbb{Q}_p$, which is an eigenform for the operator U with eigenvalue λ, then, $\mathcal{F}$ is ordinary if and only if $|\lambda|_p = 1$, i.e. if λ is invertible in R, where $|\cdot|_p$ denotes the p-adic absolute value, which we normalize so that $|p|_p = 1/p$. Also in this case we have $e^{\mathrm{ord}} \cdot \mathcal{F} = \mathcal{F}$.

Definition 3.10. We define the *p-depletion* of a p-adic modular form $\mathcal{F}$ in $V_p(N^+, R)$ to be $\mathcal{F}^{[p]} := (\mathrm{Id} - UV)\mathcal{F}$.

3.5. **Ordinary families of modular forms on** GL_2**.** Fix as in Section 1 an embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ (sometimes, we simply write x for $\iota_p(x)$. We begin by recalling some notation and results on $\mathbb{I}$-adic families of modular forms on GL_2. Let $\mathcal{O}$ be the valuation ring of a finite extension L of $\mathbb{Q}_p$ and $N = N^+N^-$ as above with $p \nmid N$. Define $\Gamma = 1 + p\mathbb{Z}_p$ and $\Lambda = \mathcal{O}[\![\Gamma]\!]$. If $\mathbb{I}$ is a finite flat extension of Λ, we say that a $\mathcal{O}$-linear homomorphism $\nu\colon \mathbb{I} \to \overline{\mathbb{Q}}_p$ is *arithmetic* if its restriction to Λ is of the form $\nu(\gamma) = \psi_\nu(\gamma)\gamma^{k_\nu - 2}$ for a finite order character $\psi_\nu\colon \Gamma \to \overline{\mathbb{Q}}_p^\times$ and an integer $\nu \geq 2$. We call k_ν the *weight* of k_ν, ψ_ν the *wild character* of ν and the pair (k_ν, ψ_ν) the *signature* of ν. Let

$$f_\infty = \sum_{n\geq 1} \mathbf{a}_n q^n \in \mathbb{I}[\![q]\!]$$

be a primitive branch of the Hida family of modular forms of tame level N. Then for each arithmetic morphisms $\nu\colon \mathbb{I} \to \overline{\mathbb{Q}}_p$ of signature (k_ν, ψ_ν), the *ν-specialization*

$$f_\nu = \nu(f_\infty) = \sum_{n\geq 1} \nu(\mathbf{a}_n) q^n = \sum_{n\geq 1} a_n(f_\nu) q^n$$

of f_∞ is the q-expansion of a modular form of level $\Gamma_1(Np^r)$ (where $r = \max\{1, \mathrm{cond}(\psi_\nu)\}$), weight k_ν and character $\psi_{\mathbb{I}}\psi_\nu\omega^{-(k_\nu-2)}$, for a character $\psi_{\mathbb{I}}\colon (\mathbb{Z}/Np\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$, independent of ν, called *tame character*. We assume that the following condition is satisfied: There exists an arithmetic morphism ν_0 such that $f_0 = f_{\nu_0} \in S_k(\Gamma_0(Np))$ has even weight $k_0 \geq 2$ with $k_0 \equiv 2 \mod p-1$ and trivial character. When this condition holds, $\psi_{\mathbb{I}}$ is trivial and f_0 is an ordinary p-stabilized newform (either f_0 is a newform of level $\Gamma_0(Np)$, or f_0 is the ordinary p-stabilization of a newform f_0 of level $\Gamma_0(N)$).

Let $\mathcal{X} = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ be the group of continuous $\mathbb{Z}_p$-linear homomorphisms of $\mathbb{Z}_p^\times$ into itself; we view $\mathbb{Z}$ as a dense subset of $\mathcal{X}$ via the map $\mathbb{Z} \hookrightarrow \mathcal{X}$ which takes k to $[x \mapsto x^{k-2}]$. The p-adic Lie group $\mathcal{X}$ is isomorphic to $p-1$ copies of Γ; for a connected open neighborhood $\mathscr{U}$ of $k_0 \in \mathbb{Z}$, let $\kappa_{\mathscr{U}}\colon \Gamma \hookrightarrow \Lambda^\times$ be the universal character of $\mathscr{U}$. Since $\Lambda \hookrightarrow \mathbb{I}$ is étale at k_0, if $\mathscr{U}$ is sufficiently small we can identity $\mathscr{U}$ with a neighborhood in $\mathbb{I}$ of the unique arithmetic morphism lying over $x \mapsto x^{k-2}$; in this case, for any $k \in \mathbb{Z} \cap \mathscr{U}$, we let ν_k the arithmetic morphism lying over k, and put $f_k = f_{\nu_k}$, which is a modular form of weigh k, level $\Gamma_0(Np)$ and trivial character.

3.6. **Quaternionic ordinary families of modular forms.** We now study the quaternionic analogue of the notion of Hida families. For this, we introduce the notion of quaternionic $\mathbb{I}$-adic (ordinary) modular forms, and prove a version of a p-adic Jacquet–Langlands correspondence between these forms and Hida families; the results of this subsection are probably well-known, however we discuss some details for lacking of precise references. We fix $\mathbb{I}$ as before.

Definition 3.11. The $\mathbb{I}$-module $V_p(N^+, \mathbb{I}) = V_p(N^+, \mathcal{O})\widehat{\otimes}_\Lambda \mathbb{I}$ is the $\mathbb{I}$-module of *quaternionic $\mathbb{I}$-adic modular forms*.

An element in $V_p(N^+, \mathbb{I})$ is then a rule $\mathcal{F}$ that for each integer $m \geq 1$, each integer $n \geq 1$, and each integer $r \geq 1$, assigns a value $\mathcal{F}(T) \in \mathbb{I}/\mathfrak{m}_{\mathbb{I}}^r$ to each quadruplet $T = (A, \iota, \alpha, \beta)$ consisting of a QM abelian surface (A, ι) over a $\mathcal{O}/p^n\mathcal{O}$-algebra R, a $U_0(N^+)$ naïve level structure α on A and a trivialization β of $A[p^m]$, where $\mathfrak{m}_{\mathbb{I}}$ is the maximal ideal of $\mathbb{I}$; the rule $(A, \iota, \alpha, \beta) \mapsto \mathcal{F}(A, \iota, \alpha, \beta)$ is compatible with respect to the canonical maps for varying n, m and r, depends only on the isomorphism class of the quadruplet and is compatible under base change. If $\nu\colon \mathbb{I} \to \mathcal{O}_\nu$ is an arithmetic character, where $\mathcal{O}_\nu$ is the valuation ring of the finite extension F_ν of $\mathbb{Q}_p$ containing the Fourier coefficients of f_ν, and $\mathcal{F}$ is a $\mathbb{I}$-adic modular form, we obtain a p-adic modular form $\mathcal{F}_\nu$ via the canonical map $V_p(N^+, \mathbb{I}) \to V_p(N^+, \mathcal{O}_\nu)$.

Definition 3.12. The $\mathbb{I}$-module $V_p(N^+, \mathbb{I}) = V_p^{\mathrm{ord}}(N^+, \mathcal{O}) \otimes_\Lambda \mathbb{I}$ is the $\mathbb{I}$-module of *ordinary* $\mathbb{I}$-adic modular forms.

Recall that $V_p^{\mathrm{ord}}(N^+, \mathcal{O})$ is a finitely generated Λ-module by [Hid02, Theorem 1.1]; also, we have a canonical map $V_p^{\mathrm{ord}}(N^+, \mathbb{I}) \to V_p(N^+, \mathbb{I})$ and, for each arithmetic character $\nu\colon \mathbb{I} \to \mathcal{O}_\nu$, a specialization map $V^{\mathrm{ord}}(N^+, \mathbb{I}) \to V^{\mathrm{ord}}(N^+, \mathcal{O}_\nu)$ denoted $\mathcal{F}_\infty \mapsto \mathcal{F}_\nu$.

Theorem 3.13. *The $\mathbb{I}$-module $V_p^{\mathrm{ord}}(N^+, \mathbb{I})$ is free of rank 1. In particular, there exists an $\mathbb{I}$-adic modular form $\mathcal{F}_\infty$ such that for any arithmetic weight ν, $\mathcal{F}_\nu$ is a classical modular form sharing the same eigenvalues as f_ν.*

Sketch of proof. This result is well-known, and is an instance of p-adic Jacquet–Langlands correspondence, developed in different frameworks in [Che05], [LV17] and [GS16]. In the present context, it can be deduced from the existence of the quaternionic eigencurve $\mathcal{C}^{\mathrm{ord}}$, proved in [Bra16], [Bra13], combined with results of Chenevier [Che05] to identify the connected component of the GL_2-eigenvariety corresponding to the fixed Hida family f_∞ with a connected component of $\mathcal{C}^{\mathrm{ord}}$, which we denote $\mathcal{C}_{\mathbb{I}}$. More general results on the p-adic Jacquet–Langlands correspondence are available in [Han17], [New13]. The reader is in particular referred to [Han17, Theorem 5.3.1] for a general result in this direction. $\square$

Remark 3.14. Since $V^{\mathrm{ord}}(N^+,\mathbb{I})$ is torsion free, the canonical map $V^{\mathrm{ord}}(N^+,\mathbb{I}) \to V(N^+,\mathbb{I})$ of Λ-modules is injective. We will then identify $V^{\mathrm{ord}}(N^+,\mathbb{I})$ as a $\mathbb{I}$-submodule of $V(N^+,\mathbb{I})$ in the following.

For a given Hida family f_∞ for GL_2, we say that $\mathcal{F}_\infty$ in Theorem 3.13 is the $\mathbb{I}$-adic quaternionic modular form associated with f_∞; note that actually $\mathcal{F}_\infty$ is only well defined up to units in $\mathbb{I}$, because q-expansions are not available in the quaternionic setting, so we shall understand that we fix a choice $\mathcal{F}_\infty$, which is unique up to units; note that the specializations $\mathcal{F}_\nu$ are also well defined up to p-adic units. Our results, however, do not depend on those choices.

4. Power series expansion of modular forms

4.1. **Serre–Tate coordinates for Shimura curves.** Following [Kat81], we introduce the Serre–Tate deformation theory for ordinary abelian varieties, which provides a way to attach power series expansions to modular forms on Shimura curves, replacing classical q-expansions for elliptic modular forms that are not available in the quaternionic case.

Consider an *ordinary* abelian variety A over $\overline{\mathbb{F}}_p$, *i.e.* $A[p](\overline{\mathbb{F}}_p) \cong (\mathbb{Z}/p\mathbb{Z})^{\dim(A)}$. Let $A^\vee$ denote the dual abelian variety, which is isogenous to A and hence ordinary too. Denote the p-adic Tate modules of A and $A^\vee$ by $\mathrm{Ta}_p(A)$ and $\mathrm{Ta}_p(A^\vee)$, respectively; then $\mathrm{Ta}_p(A)$ and $\mathrm{Ta}_p(A^\vee)$ are free $\mathbb{Z}_p$-modules of rank $g := \dim A = \dim A^\vee$. Let $\mathscr{C}$ be the category of artinian local rings with residue field $\overline{\mathbb{F}}_p$.

Definition 4.1. If R is an object of $\mathscr{C}$, a *deformation* of an ordinary abelian variety $A/\overline{\mathbb{F}}_p$ to R is an abelian scheme $\mathcal{A}/R$ equipped with an isomorphism $\mathcal{A}\otimes_R \overline{\mathbb{F}}_p \cong A$ over $\overline{\mathbb{F}}_p$. Two deformations $\mathcal{A}/R$ and $\mathcal{A}'/R$ of $A/\overline{\mathbb{F}}_p$ are said to be *isomorphic* if there exists an isomorphism between the abelian R-schemes $\mathcal{A}$ and $\mathcal{A}'$ that induces the identity on $A/\overline{\mathbb{F}}_p$.

We denote by $R \mapsto \mathscr{D}_A(R)$ the functor $\mathscr{D}_A\colon \mathscr{C} \to \mathbf{Sets}$ which takes a ring R in $\mathscr{C}$ to the set of isomorphism classes of deformations of $A/\overline{\mathbb{F}}_p$ to R. The functor $\mathscr{D}_A$ is pro-representable by a complete noetherian local ring $\mathscr{R}_A$ with residue field $\overline{\mathbb{F}}_p$. Let $\mathscr{A}_A/\mathscr{R}_A$ be the universal object. To describe the ring $\mathscr{R}_A$, one uses Serre–Tate coordinates. For this, let $R \in \mathscr{C}$, and write $\mathfrak{m}_R$ for its maximal ideal. Following a construction due to Serre, the Weil pairing induces, for each deformation $\mathcal{A}/R$ of $A/\overline{\mathbb{F}}_p$, a $\mathbb{Z}_p$-bilinear form

$$q_{\mathcal{A}/R}\colon\ \mathrm{Ta}_p(A)\times \mathrm{Ta}_p(A^\vee) \longrightarrow \widehat{\mathbb{G}}_m(R) = 1+\mathfrak{m}_R,$$

where $\widehat{\mathbb{G}}_m$ is the completion of the multiplicative group scheme $\mathbb{G}_m$ over $\overline{\mathbb{F}}_p$. By [Kat81, Theorem 2.1], the construction $\mathcal{A}/R \mapsto q_{\mathcal{A}/R}$ establishes a bijection, functorial in R, between the set $\mathscr{D}_A(R)$ and the set of $\mathbb{Z}_p$-linear homomorphisms $\mathrm{Ta}_p(A)\otimes_{\mathbb{Z}_p}\mathrm{Ta}_p(A^\vee) \to \widehat{\mathbb{G}}_m(R)$. Let now $\mathcal{R}$ be a complete noetherian local ring with maximal ideal $\mathfrak{m}_{\mathcal{R}}$ and $\mathcal{A}/\mathcal{R}$ a deformation of $A/\overline{\mathbb{F}}_p$, *i.e.* as above an abelian scheme $\mathcal{A}/R$ endowed with an isomorphism $\mathcal{A}\otimes_R \overline{\mathbb{F}}_p \cong A$ over $\overline{\mathbb{F}}_p$. Set

$$q_{\mathcal{A}/R} = \varprojlim_n q_{\mathcal{A}/(R/\mathfrak{m}_{\mathcal{R}}^n)}.$$

This gives a $\mathbb{Z}_p$-linear homomorphism $\mathrm{Ta}_p(A)\otimes_{\mathbb{Z}_p}\mathrm{Ta}_p(A^\vee)\to\widehat{\mathbb{G}}_m(\mathcal{R})$, so if we pick $\mathbb{Z}_p$-bases $\{x_1,\dots,x_g\}$ and $\{y_1,\dots,y_g\}$ of $\mathrm{Ta}_p(A)$ and $\mathrm{Ta}_p(A^\vee)$, respectively, then for each deformation $\mathcal{A}/\mathcal{R}$ we have g^2 elements $T_{ij}(\mathcal{A}/\mathcal{R}) = q_{\mathcal{A}/R}(x_i,y_j)-1 \in \mathcal{R}$. The elements $T_{ij}(\mathscr{A}_A/\mathscr{R}_A)$ are called *Serre–Tate coordinates* and the maps defined by $T_{ij}\mapsto T_{ij}(\mathscr{A}_A/\mathscr{R}_A)$ establish a non-canonical ring isomorphism $\mathbb{Z}_p^{\mathrm{unr}}[\![T_{ij}]\!] \xrightarrow{\sim} \mathscr{R}_A$, where $\mathbb{Z}_p^{\mathrm{unr}}$ is the ring of Witt vectors of $\overline{\mathbb{F}}_p$ ([Kat81, §3.1]).

Fix now an *ordinary* QM abelian surface (A,ι) over $\overline{\mathbb{F}}_p$ with a $U_1(N^+)$ naïve level structure α and write $x=(A,\iota,\alpha)$. Let $\mathscr{D}_A = \mathrm{Spf}(\mathscr{R}_A)$ be the deformation functor associated with $A/\overline{\mathbb{F}}_p$ and $\mathscr{A}_A/\mathscr{R}_A$ be the universal object as before. Consider now the subfunctor $\mathscr{D}_x$ of $\mathscr{D}_A$, which sends an artinian local ring R with residue field $\overline{\mathbb{F}}_p$ to the set of deformations of A, as a QM abelian surface with $U_1(N^+)$-level structure, to R, *i.e.* deformations $\mathcal{A}$ of A to R together with an embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}_R(\mathcal{A})$ deforming the given embedding $\mathcal{O}_B \hookrightarrow \mathrm{End}_k(A)$ and a $U_1(N^+)$-level structure on $\mathcal{A}$ deforming the given $U_1(N^+)$-level structure on A. Note that the $U_1(N^+)$-level structure automatically lifts uniquely, as $A[N^+]$ is étale over R.

The Tate module $\mathrm{Ta}_p(A)$ attached to A inherits an action of $\mathcal{O}_B$ and hence of $\mathcal{O}_B \otimes \mathbb{Z}_p$, which is identified with $\mathrm{M}_2(\mathbb{Z}_p)$. Consider the idempotent $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}_p)$ acting on $\mathrm{Ta}_p(A)$. Since $\mathrm{Ta}_p(A) = e\,\mathrm{Ta}_p(A) \oplus (1-e)\,\mathrm{Ta}_p(A)$, we can find a $\mathbb{Z}_p$-basis $\{x_1, x_2\}$ of $\mathrm{Ta}_p(A)$ such that $ex_1 = x_1$ and $ex_2 = 0$. Recall that A is equipped with a unique principal polarization $\theta_A \colon A \xrightarrow{\sim} A^\vee$ satisfying $\theta_A(b(P)) = b^\dagger \theta_A(P)$ for all $b \in \mathcal{O}_B$ and all $P \in A(\overline{\mathbb{F}}_p)$. We thus obtain an isomorphism of $\mathbb{Z}_p$-modules $\theta_{\mathrm{Ta}_p(A)} \colon \mathrm{Ta}_p(A) \xrightarrow{\sim} \mathrm{Ta}_p(A^\vee)$ which satisfies the condition $\theta_{\mathrm{Ta}_p(A)}(b(x)) = b^\vee \theta_{\mathrm{Ta}_p(A)}(x)$ for all $b \in \mathcal{O}_B$ and all $x \in \mathrm{Ta}_p(A)$. Given a point $P \in A(\overline{\mathbb{F}}_p)$ we define $P^\vee = \theta_A(P) \in A^\vee(\overline{\mathbb{F}}_p)$; also, given $x \in \mathrm{Ta}_p(A)$, we set $x^\vee = \theta_{\mathrm{Ta}_p(A)}(x)$. By [Kat81, Theorem 2.1, 4)] (see also [HB15, Proposition 4.5] and [Mor11, Proposition 3.3]), the subfunctor $\mathscr{D}_x$ of $\mathscr{D}_A$ is pro-representable by a ring $\mathscr{R}_x$ that is the quotient of $\mathscr{R}_A$ by the closed ideal generated by the relations $q_{\mathscr{A}_A/\mathscr{R}_A}(bx, y) = q_{\mathscr{A}_A/\mathscr{R}_A}(x, b^\dagger y)$ for all $b \in B$, $x \in \mathrm{Ta}_p(A)$ and $y \in \mathrm{Ta}_p(A^\vee)$. Furthermore, there is an isomorphism $\mathscr{R}_x \cong \mathbb{Z}_p^{\mathrm{unr}}[\![T]\!]$, where $T = T_{11} = t_{11} - 1$ and $t_{11} = q_{\mathscr{A}_A/\mathscr{R}_A}(x_1, x_1^\vee)$. In particular, if we denote by $\widehat{\mathcal{O}}_x$ the completion of the local ring of $\mathcal{X}_0^{\mathrm{ord}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p^{\mathrm{unr}}$ at x, then $\mathscr{D}_x \cong \mathrm{Spf}(\widehat{\mathcal{O}}_x)$ as $\mathbb{Z}_p$-formal schemes.

4.2. *T*-**expansions of modular forms.** Fix a point $x = (\bar{x}, \beta) \in \widehat{\mathrm{Ig}}(\overline{\mathbb{F}}_p)$ in the Igusa tower, *i.e.*, the isomorphism class of a quadruple $(A, \iota, \alpha, \beta)$, lying above a point $\bar{x} = (A, \iota, \alpha)$ in $\mathbb{X}^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$ and equipped with an arithmetic trivialization β of $A[p^\infty]$. Then β determines a point $x_\beta^\vee \in e^\dagger\,\mathrm{Ta}_p(A^\vee)(\overline{\mathbb{F}}_p)$ as in §3.3. Take $x_\beta = \theta_{\mathrm{Ta}_p(A^\vee)}(x_\beta^\vee) \in e\,\mathrm{Ta}_p(A)(\overline{\mathbb{F}}_p)$, where $\theta_{A^\vee}$ is the dual of θ_A (again a principal polarization). We fix the Serre–Tate coordinate $t_x = t_{(\bar{x},\beta)}$ around $\bar{x}$ to be $t_x(\mathcal{A}/R) = q_{\mathcal{A}/R}(x_\beta, x_\beta^\vee)$, for each deformation of QM abelian surface $\mathcal{A}$ of A. As before, denote by $\left(\mathscr{A}_{\bar{x}}, \iota_{\bar{x}}, \alpha_{\bar{x}}\right)$ the universal object of $\mathscr{D}_{\bar{x}}$; in particular, recall that $\mathscr{A}_{\bar{x}}$ is a QM abelian variety over $\mathscr{R}_{\bar{x}} \cong \mathbb{Z}_p^{\mathrm{unr}}[\![T_x]\!]$, with $T_x = t_x - 1$.

Definition 4.2. Let $\mathcal{F} \in V_p(M, \mathbb{Z}_p^{\mathrm{unr}})$ be a p-adic modular form and $x = (\bar{x}, \beta) \in \widehat{\mathrm{Ig}}(\overline{\mathbb{F}}_p)$ with $\bar{x} = (A, \iota, \alpha) \in \mathbb{X}^{\mathrm{ord}}(\overline{\mathbb{F}}_p)$. The formal series $\boldsymbol{\mathcal{F}}(T_x) = \mathcal{F}\left(\mathscr{A}_{\bar{x}}, \iota_{\bar{x}}, \alpha_{\bar{x}}\right)$ in $\mathbb{Z}_p^{\mathrm{unr}}[\![T_x]\!]$, where $T_x = t_x - 1$, is the *T_x-expansion of $\mathcal{F}$.*

For the case of $\mathbb{I}$-adic modular forms (with $\mathbb{I}$ as before a primitive branch of the Hida family passing through a p-stabilized newform f of trivial character and even weight), let $\widetilde{\mathbb{I}} = \mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathbb{Z}_p^{\mathrm{unr}}$ and $V_p(N^+, \widetilde{\mathbb{I}}) = V_p(N^+, \mathbb{I})\widehat{\otimes}_{\mathbb{I}}\widetilde{\mathbb{I}}$. For any $\mathcal{F}_\infty \in V_p(N^+, \widetilde{\mathbb{I}})$ and $x \in \widehat{\mathrm{Ig}}(\overline{\mathbb{F}}_p)$, we can then form the T_x-expansion of $\mathcal{F}_\infty$

$$\boldsymbol{\mathcal{F}}_\infty(T_x) = \mathcal{F}_\infty\left(\mathscr{A}_{\bar{x}}, \iota_{\bar{x}}, \alpha_{\bar{x}}\right) \in \widetilde{\mathbb{I}}[\![T_x]\!].$$

The canonical map $\mathcal{O} \hookrightarrow \mathbb{Z}_p^{\mathrm{unr}}$ induces an injective map $\mathbb{I} \hookrightarrow \widetilde{\mathbb{I}}$ and we may define the T_x-expansion of $\mathcal{F}_\infty \in V_p(N^+, \mathbb{I})$ to be the T_x-expansion of the image of $\mathcal{F}_\infty$ in $V_p(N^+, \widetilde{\mathbb{I}})$.

Lemma 4.3. *Let $\mathcal{F}_\infty$ be as in Theorem* 3.13. *Then for each arithmetic morphism ν we have an equality of formal power series $\boldsymbol{\mathcal{F}}_\nu(T_x) = \nu(\boldsymbol{\mathcal{F}}_\infty(T_x))$.*

Proof. This is clear from the functoriality properties of modular forms and T-expansions. □

4.3. **Serre–Tate coordinates of CM points.** Recall the point $x(\mathfrak{a}) = [(\iota_K, a^{-1}\xi)]$ defined in §2.6, which corresponds to the sequence $(x_m(\mathfrak{a}))_{m \geq 0}$ of points, each in $\widetilde{X}_m(H_{cp^\infty})$. Fix an integer $n \geq 1$. For any x in $\mathbb{Q}_p$, define the $\star$-*action* of $\mathbf{n}(x)$ on the point $x(\mathfrak{a})$ by the formula

$$x(\mathfrak{a}) \star \mathbf{n}(x) = [(\iota_K, a^{-1}\xi\mathbf{n}(x)]$$

where $\mathbf{n}(x)$ denotes the element in $\widehat{B}^\times$ whose p-component has image equal to $\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ via the map i_p and whose components at other primes are trivial. A simple computation (see also [CH18a, page 587]) shows that for any $u\in\mathbb{Z}_p^\times$ we have

$$\xi\cdot\mathbf{n}(u/p^n)=i_{\mathfrak{p}}(u/p^n)\xi^{(n)}\cdot\left(\begin{smallmatrix}u^{-1}&u^{-1}\\0&1\end{smallmatrix}\right),$$

where $i_{\mathfrak{p}}(u/p^n)$ is the element of $\widehat{K}^\times$ having all components equal to 1 except the $\mathfrak{p}$-component, equal to u/p^n. We thus obtain

$$x(\mathfrak{a})\star\mathbf{n}(u/p^n)=[(\iota_K,a^{-1}\xi\mathbf{n}(u/p^n))]=\left[\left(\iota_K,a^{-1}i_{\mathfrak{p}}(u/p^n)\xi^{(n)}\cdot\left(\begin{smallmatrix}u^{-1}&u^{-1}\\0&1\end{smallmatrix}\right)\right)\right].\tag{4.1}$$

By [BCK21, Proposition 4.1], for any $u\in\mathbb{Z}_p^\times$, $(x(\mathfrak{a})\star\mathbf{n}(u/p^n))$ is still a CM point defined over $\mathcal{V}$. Moreover, we have $(x(\mathfrak{a})\star\mathbf{n}(u/p^n))\otimes_{\mathcal{V}}\overline{\mathbb{F}}_p=\bar{x}(\mathfrak{a})$ and

$$t_{x(\mathfrak{a})}(x(\mathfrak{a})\star\mathbf{n}(u/p^n))=\zeta_{p^n}^{-u\mathrm{N}(\mathfrak{a}^{-1})\sqrt{-D_K}^{-1}},$$

where for an ideal $\mathfrak{c}\subseteq\mathcal{O}_c$, we define $\mathrm{N}(\mathfrak{c})=c^{-1}\cdot\sharp(\mathcal{O}_c/\mathfrak{c})$.

5. The analytic p-adic L-function

5.1. p-adic L-functions of modular forms. Recall that p is split in K and fix embeddings $\iota_\infty\colon\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{C}}$ and $\iota_p\colon\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}_p$ as before. Write $p=\mathfrak{p}\bar{\mathfrak{p}}$ in $\mathcal{O}_K$ and let $\mathfrak{p}$ be the prime ideal corresponding to the fixed embedding ι_p. For an element $x\in\mathbb{A}_K^\times$, let $x_{\mathfrak{p}}\in K_{\mathfrak{p}}^\times$ and $x_{\bar{\mathfrak{p}}}\in K_{\bar{\mathfrak{p}}}^\times$ denote the components of x at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, respectively. For an algebraic Hecke character $\xi\colon K^\times\backslash\mathbb{A}_K^\times\to\mathbb{C}^\times$, write $\xi=\xi_{\mathrm{fin}}\xi_\infty$ with $\xi_{\mathrm{fin}}\colon\widehat{K}^\times\to\mathbb{C}^\times$ and $\xi_\infty\colon K_\infty^\times\to\mathbb{C}^\times$ the finite and infinite restrictions of ξ respectively, where $\widehat{K}^\times$ and $K_\infty^\times$ are respectively the groups of finite and infinite ideles. We say that $\xi\colon K^\times\backslash\mathbb{A}_K^\times\to\mathbb{C}^\times$ has infinity type (m,n) if $\xi_\infty(x)=x^m\bar{x}^n$, and in this case denote by $\hat{\xi}\colon K^\times\backslash\widehat{K}^\times\to\overline{\mathbb{Q}}_p^\times$ its p-adic avatar, defined by

$$\hat{\xi}(x)=(\iota_p\circ\iota_\infty^{-1})(\xi_{\mathrm{fin}}(x))x_{\mathfrak{p}}^m x_{\bar{\mathfrak{p}}}^n.$$

To simplify the notation, we sometimes write $\hat{\xi}(x)=\xi(x)x_{\mathfrak{p}}^m x_{\bar{\mathfrak{p}}}^n$ for $x\in\widehat{K}^\times$, understanding that $\xi(x)$ denotes $(\iota_p\circ\iota_\infty^{-1})(\xi_{\mathrm{fin}}(x))$. If ξ is a Hecke character of conductor $\mathfrak{c}\subseteq\mathcal{O}_K$ and $\mathfrak{b}$ is an ideal prime to $\mathfrak{c}$, we write $\xi(\mathfrak{b})$ for $\xi(b)$, where $b\in\widehat{K}^\times$ is a finite idele with trivial components at the primes dividing $\mathfrak{c}$ and such that $b\widehat{\mathcal{O}}_K\cap K=\mathfrak{b}$.

Fix $\mathcal{F}\in M_k(M,R)$ of signature $(k,\mathbf{1})$, where $\mathbf{1}$ denotes the trivial character, $M=N^+$ or $M=N^+p$, and R a p-adic ring obtained as the ring of integers of a finite extension of $\mathbb{Q}_p$, and an anticyclotomic Hecke character ξ of infinity type $(k/2,-k/2)$. We recall the construction of the p-adic L-function associated with $\mathcal{F}$ and ξ in [Mag22, §4.6] (see also [BCK21, §4]). Denote $\mathbb{Z}_p^{\mathrm{unr}}(\xi)$ the extension of $\mathbb{Z}_p^{\mathrm{unr}}$ obtained by adjoining the values of ξ and R. Let $c\mathcal{O}_K$ be the prime to p part of the conductor of ξ and recall the CM points $x(\mathfrak{a})$, defined in §2.6, for each $\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)$. As before, since $x(\mathfrak{a})$ has a model over $\mathcal{V}=\mathbb{Z}_p^{\mathrm{unr}}\cap K^{\mathrm{ab}}$, we can consider the reduction $\bar{x}(\mathfrak{a})=x(\mathfrak{a})\otimes_W\overline{\mathbb{F}}_p$ of $x(\mathfrak{a})$ modulo p. For each ideal class $[\mathfrak{a}]\in\mathrm{Pic}(\mathcal{O}_c)$ with $\mathfrak{a}\subseteq\mathcal{O}_c$, define the $\mathbb{Z}_p^{\mathrm{unr}}(\xi)$-valued measure $\mu_{\mathcal{F},\mathfrak{a}}$ over $\mathbb{Z}_p$ by

$$\int_{\mathbb{Z}_p}t_{\mathfrak{a}}^x\,d\mu_{\mathcal{F},\mathfrak{a}}(x)=\boldsymbol{\mathcal{F}}^{[p]}\left(t_{\mathfrak{a}}^{\mathrm{N}(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}}\right)\in\mathbb{Z}_p^{\mathrm{unr}}(\xi)[\![T_{\mathfrak{a}}]\!],\tag{5.1}$$

where $\mathcal{F}^{[p]}=\mathcal{F}_{VU-UV}$ is the *p-depletion* of f defined *e.g.* in [Bur17, (5.2)] or [Mag22, §4.4], with U and V defined in [HB15, §3.6], and $\boldsymbol{\mathcal{F}}^{[p]}$ denotes as in Definition 4.2 the $T_{\mathfrak{a}}$-espansion of $\mathcal{F}^{[p]}$ seen as a p-adic modular form in $V_p(M,R)$ via the map in (3.1) (also, recall that $T_{\mathfrak{a}}=t_{\mathfrak{a}}-1$ and $t_{\mathfrak{a}}=t_{\bar{x}(\mathfrak{a})}$ is the Serre–Tate coordinate at $\bar{x}(\mathfrak{a})$, and $\mathrm{N}(\mathfrak{a})=c^{-1}\sharp(\mathcal{O}_c/\mathfrak{a})$). The measures $\mu_{\mathcal{F},\mathfrak{a}}$ are supported on $\mathbb{Z}_p^\times$ ([Mag22, Remark 4.2]).

Recall that $\mathrm{rec}_K\colon K^\times\backslash\widehat{K}^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ denotes the geometrically normalized reciprocity map, where K^{ab} is the maximal abelian extension of K; let $\mathrm{rec}_{K,\mathfrak{p}}\colon K_\mathfrak{p}^\times \to \mathrm{Gal}(K_\mathfrak{p}^{\mathrm{ab}}/K_\mathfrak{p})$ be the local reciprocity map, and view $\mathrm{Gal}(K_\mathfrak{p}^{\mathrm{ab}}/K_\mathfrak{p})$ as a subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ by the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Define $\widetilde{\Gamma}_\infty = \mathrm{Gal}(H_{cp^\infty}/K)$ and let $\mathcal{O}_{\mathbb{C}_p}$ be the valuation ring of the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$. For each continuous function $\varphi\colon \widetilde{\Gamma}_\infty \to \mathcal{O}_{\mathbb{C}_p}$, and each ideal class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)$, define the continuous function $\varphi\big|[\mathfrak{a}]\colon \mathbb{Z}_p^\times \to \mathcal{O}_{\mathbb{C}_p}$ by the formula

$$\varphi|[\mathfrak{a}](u) = \varphi\left(\mathrm{rec}_K(a)\mathrm{rec}_{K,\mathfrak{p}}(u)\right),$$

where $a \in \widehat{K}^\times$ is a finite idele with trivial components at the primes lying over cp such that $a\widehat{\mathcal{O}}_c \cap K = \mathfrak{a}$ and we view an element $u \in \mathbb{Z}_p^\times$ as an element in $K_\mathfrak{p}^\times$ via the canonical inclusion $\mathbb{Z}_p^\times \subseteq K_\mathfrak{p}^\times$.

Define a $\mathbb{Z}_p^{\mathrm{unr}}(\xi)$-valued measure $\mathscr{L}_{\mathcal{F},\xi}$ on $\widetilde{\Gamma}_\infty$ by the formula

$$\mathscr{L}_{\mathcal{F},\xi}(\varphi) = \sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)} \xi(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-k/2}\int_{\mathbb{Z}_p^\times} \xi_\mathfrak{p}(u)(\varphi\big|[\mathfrak{a}])(u)d\mu_{\mathcal{F},\mathfrak{a}}(u)$$

for any continuous function $\varphi\colon \widetilde{\Gamma}_\infty \to \mathcal{O}_{\mathbb{C}_p}$, where, as before, $\xi(\mathfrak{a})$ denotes $\xi(a)$ for $a \in \widehat{K}^\times$ finite idele with trivial components at the primes dividing the conductor of ξ and such that $a\widehat{\mathcal{O}}_c \cap K = \mathfrak{a}$.

We recall some results from [Mag22] and [BCK21]. For any ideal $\mathfrak{a} \subseteq \mathcal{O}_c$, any continuous function $\phi\colon \mathbb{Z}_p^\times \to \mathcal{O}_{\mathbb{C}_p}$ and any power series $G(T_\mathfrak{a}) \in W[\![T_\mathfrak{a}]\!]$, define the formal power series $([\phi]G)(T_\mathfrak{a}) \in \mathbb{Z}_p^{\mathrm{unr}}(\phi)[\![T_\mathfrak{a}]\!]$, where $\mathbb{Z}_p^{\mathrm{unr}}(\phi)$ is the extension of $\mathbb{Z}_p^{\mathrm{unr}}$ generated by the values of ϕ, by the formula $([\phi]G)(T_\mathfrak{a}) = \int_{\mathbb{Z}_p^\times}\phi(t)t_\mathfrak{a}^x d\mu_{f,\mathfrak{a}}(x)$. Define

$$\boldsymbol{\mathcal{F}}_\mathfrak{a}^{[p]}(T_\mathfrak{a}) := \boldsymbol{\mathcal{F}}^{[p]}\left((T_\mathfrak{a}+1)^{\mathrm{N}(\mathfrak{a})^{-1}\sqrt{-D_K}^{-1}}\right).$$

By [Mag22, Proposition 4.5] (see also [BCK21, Proposition 4.1]), if $\phi\colon (\mathbb{Z}/p^n\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ is a primitive Dirichlet character, and $[\mathfrak{a}]$ is an ideal class in $\mathrm{Pic}(\mathcal{O}_c)$ with $p \nmid c$ as before, we have

$$([\phi]\boldsymbol{\mathcal{F}}_\mathfrak{a}^{[p]})(0) = p^{-n}\mathfrak{g}(\phi)\sum_{u\in(\mathbb{Z}/p^n\mathbb{Z})^\times}\phi^{-1}(u)\mathcal{F}(x(\mathfrak{a}) \star \mathbf{n}(u/p^n)), \tag{5.2}$$

where $\mathfrak{g}(\phi)$ is the Gauss sum of ϕ.

We finally recall the following result which will be useful later. Suppose that $\mathcal{F}$ is a modular form of level N^+p which is the ordinary p-stabilization of a newform $\mathcal{F}^\sharp$ of level N^+; in other words, we have $\mathcal{F} = \mathcal{F}^\sharp - a_p\mathcal{F}^\sharp|W_p$ where W_p is the quaternionic Atkin–Lehner involution, and a_p is the eigenvalue of the T_p-operator acting on $\mathcal{F}^\sharp$ (see [LV14a, Eq. (28)] for details).

Lemma 5.1. $\mathscr{L}_{\mathcal{F},\xi} = \mathscr{L}_{\mathcal{F}^\sharp,\xi}$.

Proof. In light of Definition 5.1, it is enough to observe that $\boldsymbol{\mathcal{F}}_\mathfrak{a}^{[p]} = \boldsymbol{\mathcal{F}}^{\sharp}{}_\mathfrak{a}^{[p]}$ for each ideal class $[\mathfrak{a}]$ in $\mathrm{Pic}(\mathcal{O}_c)$. This follows immediately from [Bur17, Lemma 5.2] and the references therein; see [Mag22, §4.4] for details. □

5.2. **Families of Hecke characters.** Let $G_\mathbb{Q} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\chi_{\mathrm{cyc}}\colon G_\mathbb{Q} \to \mathbb{Z}_p^\times$ be the cyclotomic character. Factor χ_{cyc} as $\chi_{\mathrm{cyc}} = \chi_{\mathrm{tame}}\cdot\chi_{\mathrm{wild}}$, where $\chi_{\mathrm{tame}}\colon G_\mathbb{Q} \to \mu_{p-1}$ takes values in the group of $p-1$ roots of unity in $\mathbb{Z}_p^\times$ and $\chi_{\mathrm{wild}}\colon G_\mathbb{Q} \to \Gamma$ takes values in the group of principal units $\Gamma = 1 + p\mathbb{Z}_p$; in other words, if we write an element $x \in \mathbb{Z}_p^\times$ as $x = \zeta_x \cdot \langle x\rangle$, where $\zeta_x = \omega(x) \in \mu_{p-1}$ and $x \mapsto \langle x\rangle$ is the projection $\mathbb{Z}_p^\times \to \Gamma$ from $\mathbb{Z}_p^\times$ to the group Γ of principal units, then $\chi_{\mathrm{tame}}(\sigma) = \zeta_{\chi_{\mathrm{cyc}}(\sigma)}$ and $\chi_{\mathrm{wild}}(\sigma) = \langle\chi_{\mathrm{cyc}}(\sigma)\rangle$. We also denote with $\boldsymbol{\chi}_{\mathrm{cyc}}\colon \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mathbb{Z}_p^\times$, $\boldsymbol{\chi}_{\mathrm{tame}}\colon \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mu_{p-1}$ and $\boldsymbol{\chi}_{\mathrm{wild}}\colon \mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \Gamma$ the composition of χ_{cyc}, χ_{tame} and χ_{wild}, respectively, with the reciprocity map $\mathrm{rec}_{\mathbb{Q}}$; we then have $\boldsymbol{\chi}_{\mathrm{tame}}(x) = \zeta_{\boldsymbol{\chi}_{\mathrm{cyc}}(x)}$ and $\boldsymbol{\chi}_{\mathrm{wild}}(x) = \langle \boldsymbol{\chi}_{\mathrm{cyc}}(x)\rangle$.

Fix a finite flat extension $\mathbb{I}$ of Λ. Let $z \mapsto [z]$ be the inclusion of group-like elements $\mathbb{Z}_p^\times \hookrightarrow \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]^\times$ and $\Gamma \hookrightarrow \Lambda^\times$. Recall the critical character $\Theta\colon G_{\mathbb{Q}} \to \Lambda^\times$ defined in [How07, Definition 2.1.3] by

$$\Theta(\sigma) = \chi_{\mathrm{tame}}^{\frac{k_0-2}{2}}(\sigma)\cdot[\chi_{\mathrm{wild}}^{1/2}(\sigma)],$$

where $x \mapsto x^{1/2}$ is the unique square root of $x \in \Gamma$. We still write $\Theta\colon G_{\mathbb{Q}} \to \mathbb{I}^\times$ for the composition of Θ with the canonical inclusion $\Lambda \hookrightarrow \mathbb{I}$. Write $\boldsymbol{\theta}\colon \mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{I}^\times$ for the composition of Θ with the geometrically normalized reciprocity map $\mathrm{rec}_{\mathbb{Q}}$. We denote $\mathbb{Q}_p^{\mathrm{cyc}} = \mathbb{Q}(\zeta_{p^\infty}) = \bigcup_{n\geq 1}\mathbb{Q}(\zeta_{p^n})$ the p-cyclotomic extension of $\mathbb{Q}$, where, for all integers $n \geq 1$, ζ_{p^n} is a primitive p^n-root of unity, and define $G_\infty^{\mathrm{cyc}} = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})$. The cyclotomic character induces an isomorphism $\chi_{\mathrm{cyc}}\colon G_\infty^{\mathrm{cyc}} \overset{\sim}{\to} \mathbb{Z}_p^\times$. Since Θ factors through G_∞^{cyc}, precomposing it with the inverse of the cyclotomic character, we obtain a character of $\mathbb{Z}_p^\times$ which we denote with $\boldsymbol{\vartheta}\colon \mathbb{Z}_p^\times \to \mathbb{I}^\times$. If $\nu\colon \mathbb{I} \to \overline{\mathbb{Q}}_p$ is an arithmetic morphism of signature (k_ν, ψ_ν) we put $\theta_\nu = \nu\circ\boldsymbol{\theta}$ and $\vartheta_\nu = \nu\circ\boldsymbol{\vartheta}$. For any $x \in \mathbb{Z}_p^\times$, if $k_\nu \equiv k_0 \mod 2(p-1)$, then we have (also recall that $k_0 \equiv 2 \mod 2(p-1)$)

$$\vartheta_\nu(x) = \psi_\nu^{1/2}(\langle x\rangle)\cdot x^{\frac{k_\nu-2}{2}}.$$

Denote by $\mathbf{N}_{K/\mathbb{Q}}\colon \mathbb{A}_K^\times \to \mathbb{A}_{\mathbb{Q}}^\times$ the adelic norm map, by $\mathbf{N}_{\mathbb{Q}}\colon \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{Q}^\times$ the adelic absolute value and let $\mathbf{N}_K\colon \mathbb{A}_K^\times \to \mathbb{Q}^\times$ denote the composition $\mathbf{N}_K = \mathbf{N}_{\mathbb{Q}}\circ\mathbf{N}_{K/\mathbb{Q}}$. Define the character $\boldsymbol{\chi}\colon K^\times\backslash\widehat{K}^\times \to \mathbb{I}^\times$ by $\boldsymbol{\chi} = \boldsymbol{\theta}\circ\mathbf{N}_{K/\mathbb{Q}}^{-1}$. For an arithmetic morphism ν, define $\hat{\chi}_\nu = \nu\circ\boldsymbol{\chi}$. Since $\chi_{\mathrm{cyc}}\circ\mathrm{rec}_{\mathbb{Q}}$ is the p-adic avatar of the adelic absolute value $\mathbf{N}_{\mathbb{Q}}\colon \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{Q}^\times$, we obtain, for $x \in \widehat{K}^\times$ and $k_\nu \equiv k_0 \mod 2(p-1)$, $k \equiv 2 \mod 2(p-1)$,

$$\hat{\chi}_\nu(x) = \psi_\nu^{-1/2}(\langle \mathbf{N}_K(x)x_{\mathfrak{p}}x_{\bar{\mathfrak{p}}}\rangle)\cdot(\mathbf{N}_K(x)x_{\mathfrak{p}}x_{\bar{\mathfrak{p}}})^{-\frac{k_\nu-2}{2}}. \tag{5.3}$$

Let $\lambda\colon K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ be a Hecke character of infinity type $(1,0)$, unramified at p and whose p-adic avatar $\hat{\lambda}\colon K^\times\backslash\widehat{K}^\times \to \overline{\mathbb{Q}}_p^\times$ takes values in $\mathcal{O}^\times$ (possibly after enlarging $\mathcal{O}$ if necessary). Denote now by $\bar{\lambda}$ the complex conjugate character of λ, defined by $x \mapsto \lambda(\bar{x})$, where $x \mapsto \bar{x}$ is given by the complex conjugation on K. Then $\bar{\lambda}$ has infinity type $(0,1)$ and the p-adic avatar of $\lambda\bar{\lambda}$ is equal to the product $\chi_\lambda\cdot\chi_{\mathrm{cyc},K}$ where $\chi_{\mathrm{cyc},K} = \chi_{\mathrm{cyc}}\circ\mathrm{rec}_{\mathbb{Q}}\circ\mathbf{N}_{K/\mathbb{Q}}$ and χ_λ is a finite order character unramified at p. We define a character $\boldsymbol{\lambda}\colon K^\times\backslash\widehat{K}^\times \to \mathcal{O}[\![W]\!]^\times$ by the formula $\boldsymbol{\lambda}(x) = \hat{\lambda}(x)[\langle\hat{\lambda}(x)\rangle^{1/2}]$, where we view $\hat{\lambda}(x) \in \mathcal{O}^\times \hookrightarrow \mathcal{O}[\![W]\!]^\times$ via the map $a \mapsto a\cdot 1_W$, with 1_W the identity element of W, $\langle\hat{\lambda}(x)\rangle$ denote the projection of $\hat{\lambda}(x)$ in W and $z \mapsto [z]$ is the inclusion of group-like elements $W \hookrightarrow \mathcal{O}[\![W]\!]^\times$.

To define the specializations of $\boldsymbol{\lambda}$, we need to extend arithmetic morphisms from Γ to W. We briefly explain this point. Recall that F is the field of fractions of $\mathcal{O}$ and note that $\mathcal{O}^\times \cong \boldsymbol{\mu}(F)\times\mathbb{Z}_p^d$, as topological groups, where $d = [F:\mathbb{Q}_p]$. Therefore each element $x \in \mathcal{O}^\times$ can be written uniquely as a product $\zeta_x\langle x\rangle$, where ζ_x is the projection of x in $\boldsymbol{\mu}(F)$ and $\langle x\rangle$ is the projection on $\mathbb{Z}_p^d$. Let $\mathcal{O}_{\mathrm{free}}^\times \cong \mathbb{Z}_p^d$ be the maximal $\mathbb{Z}_p$-free quotient of $\mathcal{O}^\times$ and denote by $\langle - \rangle$ the projection $\mathcal{O}^\times \twoheadrightarrow \mathcal{O}_{\mathrm{free}}^\times$. Let $W = \langle \mathrm{im}\,\hat{\lambda}\rangle$ be the projection of the image of $\hat{\lambda}$ in $\mathcal{O}_{\mathrm{free}}^\times$. If λ has conductor $\mathfrak{c}$ prime to p, then $\hat{\lambda}$ factors through $\mathrm{Gal}(K(\mathfrak{p}^\infty\mathfrak{c})/K)$, where $K(\mathfrak{p}^\infty\mathfrak{c}) = \bigcup_{n\geq 1}K(\mathfrak{p}^n\mathfrak{c})$ and $K(\mathfrak{p}^n\mathfrak{c})$ is the ray class field of K of conductor $\mathfrak{p}^n\mathfrak{c}$. Since $\mathcal{O}_{\mathrm{free}}^\times$ is a free pro-p group, the composition $\langle\hat{\lambda}\rangle$ of $\hat{\lambda}$ with the projection $\langle - \rangle$ factorizes through the maximal free pro-p quotient of $\mathrm{Gal}(K(\mathfrak{p}^\infty\mathfrak{c})/K)$ which is a cyclic pro-p group isomorphic to $\mathbb{Z}_p$; hence W is isomorphic to $\mathbb{Z}_p$ and we can see Γ as a subgroup of W of finite index, *cf.* [Hid11, pp. 64–65]. Write $p^m = [W:\Gamma]$. Let $w \in W$ be a topological generator of W such that $w^{p^m} = \gamma \in \Gamma$ is a topological generator. Consider an arithmetic morphism

$\nu\colon \mathbb{I} \to \mathcal{O}_\nu$ with signature (ψ_ν, k_ν), where $\psi_\nu\colon \Gamma \to \mathcal{O}_\nu^\times$ is a continuous character of finite order and $\mathcal{O}_\nu$ is the ring of integer of a finite extension L_ν of $\mathbb{Q}_p$. Fix a p^m-th root $u \in \overline{\mathbb{Q}}_p$ of $\psi_\nu(\gamma)$ and consider a finite extension M_ν of L_ν containing u. We can extend ψ_ν to a continuous morphism $W \to \mathcal{O}_{M_\nu}^\times$, with $\mathcal{O}_{M_\nu}$ the ring of integers of M_ν, sending w to u. We will denote this morphism again by ψ_ν. Then the restriction $\nu_{|\Gamma}\colon \Gamma \to \mathcal{O}_\nu^\times$ of ν to Γ takes x to $\psi_\nu(x)x^{k_\nu-2}$ and can be extended to a continuous morphism $\nu_W\colon W \to \mathcal{O}_{M_\nu}^\times$ by $w \to \psi_\nu(w)w^{k_\nu-2}$, so that $\nu_W(v) = \psi_\nu(v)v^{k_\nu-2}$ for any $v \in W$. This induces a unique continuous morphism $\nu_{\mathcal{O}[\![W]\!]}\colon \mathcal{O}[\![W]\!] \to \mathcal{O}_{M_\nu}$. Indeed, if $\mathcal{O}[\![W]\!]$ is contained in $\mathbb{I}$, then we can find $u \in \mathcal{O}_\nu^\times$ such that $\psi_\nu(w) = u$ and $u^{p^m} = \psi_\nu(\gamma)$ (possibly enlarging $\mathcal{O}_\nu$ by adding a p^m-root of $\psi_\nu(1+p)$ as above), so that $\nu(w) = uw^{k_\nu-2}$ and, extending ψ_ν to W by $\psi_\nu(w) = u$, we can say that $\nu(v) = \psi_\nu(v)v^{k_\nu-2}$ for any $v \in W$. Otherwise, if $\mathcal{O}[\![W]\!]$ is not contained in $\mathbb{I}$, we can replace $\mathbb{I}$ by $\mathbb{I}\widehat{\otimes}_{\mathcal{O}[\![\Gamma]\!]}\mathcal{O}[\![W]\!]$ and extend ν to $\nu'\colon \mathbb{I}\widehat{\otimes}_{\mathcal{O}}\mathcal{O}[\![W]\!] \to \mathcal{O}_{M_\nu}$, fixing a p^m-th root $u \in \overline{\mathbb{Q}}_p$ of $\psi_\nu(\gamma)$ as above, which is the map obtained by universal property of (completed) tensor product from the $\mathcal{O}$-bilinear map $\mathbb{I} \times \mathcal{O}[\![W]\!] \to \mathcal{O}_{M_\nu}$ given by $(x,y) \to \nu(x)\nu_{\mathcal{O}[\![W]\!]}(y)$. Note that $\psi_\nu\colon W \to \mathcal{O}_{M_\nu}^\times$ is still a finite order character; indeed, since ψ_ν is of finite order as a character of Γ, there exists $n \geq 0$ such that ψ_ν^n is trivial on Γ, so $\psi_\nu^{np^m}$ is trivial on W. By the above argument, we can assume in the following that $\mathcal{O}[\![W]\!]$ is contained in $\mathbb{I}$ and that ν restricted to W is given by $\nu(v) = \psi_\nu(v)v^{k_\nu-2}$ for any $v \in W$.

We can see $\boldsymbol{\lambda}\colon K^\times\backslash\widehat{K}^\times \to \mathbb{I}^\times$ assuming that $\mathcal{O}[\![W]\!]$ is contained in $\mathbb{I}$. Let $\nu\colon \mathbb{I} \to \overline{\mathbb{Q}}_p$ be an arithmetic morphism of signature (k_ν, ψ_ν) and write $\hat{\lambda}_\nu = \nu \circ \boldsymbol{\lambda}$. Then, for $x \in \widehat{K}^\times$ and $k_\nu \equiv k \mod 2(p-1)$, $k \equiv 2 \mod 2(p-1)$, we have

$$\hat{\lambda}_\nu(x) = \psi_\nu^{1/2}(\langle\hat{\lambda}(x)\rangle) \cdot \lambda(x)^{k_\nu/2}x_{\mathfrak{p}}^{k_\nu/2}.$$

Hence $\hat{\lambda}_\nu$ is the p-adic avatar of an algebraic Hecke character λ_ν of infinity type $(k_\nu/2, 0)$. Denote by $x \mapsto \hat{\lambda}(\bar{x})^{-1} = \lambda(\bar{x})^{-1}x_{\bar{\mathfrak{p}}}^{-1}$ (for $x \in \widehat{K}^\times$) the p-adic avatar of the Hecke character given by $x \mapsto \lambda(\bar{x})^{-1}$ (for $x \in \mathbb{A}_K^\times$) of infinity type $(0,-1)$. Then define the character

$$\boldsymbol{\lambda}^{-1}(\bar{x}) = \hat{\lambda}(\bar{x})^{-1}[\langle\hat{\lambda}(\bar{x})^{-1}\rangle^{1/2}]$$

which we see as taking values in $\mathbb{I}^\times$. Consider now the character $\boldsymbol{\xi}\colon K^\times\backslash\widehat{K}^\times \to \mathbb{I}^\times$ given by $\boldsymbol{\xi}(x) = \boldsymbol{\lambda}(x)\cdot\boldsymbol{\lambda}^{-1}(\bar{x})$. Note that $\boldsymbol{\xi}_{|\widehat{\mathbb{Q}}^\times}$ is trivial. For any arithmetic morphisms ν, set as above $\hat{\xi}_\nu = \nu \circ \boldsymbol{\xi}$. For any $x \in \widehat{K}^\times$ and $k_\nu \equiv k \mod 2(p-1)$, $k \equiv 2 \mod 2(p-1)$, we have

$$\hat{\xi}_\nu(x) = \psi_\nu^{1/2}(\langle\lambda(x\bar{x}^{-1})x_{\mathfrak{p}}x_{\bar{\mathfrak{p}}}^{-1}\rangle) \cdot \lambda(x\bar{x}^{-1})^{k_\nu/2} \cdot x_{\mathfrak{p}}^{k_\nu/2}x_{\bar{\mathfrak{p}}}^{-k_\nu/2}. \tag{5.4}$$

Therefore, $\hat{\xi}_\nu$ is the p-adic avatar of an anticyclotomic Hecke character ξ_ν of infinity type $(k_\nu/2, -k_\nu/2)$. If we want to emphasise the dependence of $\boldsymbol{\xi}$, $\hat{\xi}_\nu$ and ξ_ν from λ, we write $\boldsymbol{\xi}^{(\lambda)}$, $\hat{\xi}_\nu^{(\lambda)}$ and $\xi_\nu^{(\lambda)}$.

5.3. **p-adic L-functions for families of modular forms.** Fix a primitive branch $\mathbb{I}$ of the Hida family passing through a p-stabilized newform $f \in S_k(\Gamma_0(Np))$ of trivial character and even weight $k \equiv 2 \mod 2(p-1)$. Let $\mathcal{F}_\infty \in V_p(N^+, \widetilde{\mathbb{I}})$ be the quaternionic form associated with $\mathbb{I}$ as in Theorem 3.13, where recall that $\widetilde{\mathbb{I}} = \mathbb{I} \otimes_{\mathcal{O}} \mathbb{Z}_p^{\mathrm{unr}}$. Let $c\mathcal{O}_K$, with $c \geq 1$ and $p \nmid c$, be the conductor of $x \mapsto \lambda(x)\lambda^{-1}(\bar{x})$. Consider the CM points $x(\mathfrak{a})$ with $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)$, defined in §2.6. Recall that $x(\mathfrak{a})$ has a model defined over $\mathbb{Z}_p^{\mathrm{unr}}$, and define the fiber product $x(\mathfrak{a})_{\widetilde{\mathbb{I}}} := x(\mathfrak{a}) \otimes_W \widetilde{\mathbb{I}}$. Define now a $\widetilde{\mathbb{I}}$-valued measure $\mu_{\mathcal{F}_\infty,\mathfrak{a}}$ on $\mathbb{Z}_p$ by

$$\int_{\mathbb{Z}_p} t_{x(\mathfrak{a})}^x d\mu_{\mathcal{F}_\infty,\mathfrak{a}}(x) = \boldsymbol{\mathcal{F}}_\infty^{[p]}\left(t_{x(\mathfrak{a})}^{\mathrm{N}(\mathfrak{a}^{-1})\sqrt{(-D_K)}^{-1}}\right) \in \widetilde{\mathbb{I}}[\![T_{\mathfrak{a}}]\!],$$

where as before $T_{x(\mathfrak{a})}=t_{x(\mathfrak{a})}-1$ and $t_{x(\mathfrak{a})}$ is the Serre–Tate coordinate around $x(\mathfrak{a})_{\mathbb{I}_W}\otimes\overline{\mathbb{F}}_p$, $\mathcal{F}_\infty^{[p]}$ is the p-depletion of $\mathcal{F}_\infty$ and $\boldsymbol{\mathcal{F}}_\infty^{[p]}$ is the $T_{x(\mathfrak{a})}$-expansion of $\mathcal{F}_\infty^{[p]}$ (see Definition 4.2). The measures $\mu_{\mathcal{F}_\infty,\mathfrak{a}}$ are supported on $\mathbb{Z}_p^\times$.

Definition 5.2. Let $\mathcal{F}_\infty\in V_p(N^+,\widetilde{\mathbb{I}})$ and let $\boldsymbol{\xi}\colon K^\times\backslash\widehat{K}^\times\to\mathbb{I}^\times$ be a continuous character as in §5.2. The measure $\mathscr{L}_{\mathcal{F}_\infty,\boldsymbol{\xi}}$ associated with $\mathcal{F}_\infty$ and $\boldsymbol{\xi}$ is the $\widetilde{\mathbb{I}}$-valued measure on the Galois group $\widetilde{\Gamma}_\infty=\mathrm{Gal}(H_{cp^\infty}/K)$ given for any continuous function $\varphi\colon\widetilde{\Gamma}_\infty\to\widetilde{\mathbb{I}}$ by

$$\mathscr{L}_{\mathcal{F}_\infty,\boldsymbol{\xi}}(\varphi)=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\boldsymbol{\chi}^{-1}\boldsymbol{\xi}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-1}\int_{\mathbb{Z}_p^\times}(\varphi\big|[\mathfrak{a}])(u)d\mu_{\mathcal{F}_\infty,\mathfrak{a}}(u).$$

Definition 5.3. Define the *analytic anticyclotomic p-adic L-function* $\mathscr{L}_{\mathbb{I},\boldsymbol{\xi}}^{\mathrm{an}}\in\widetilde{\mathbb{I}}[\![\widetilde{\Gamma}_\infty]\!]$ to be the power series corresponding to the measure $\mathscr{L}_{\mathcal{F}_\infty,\boldsymbol{\xi}}$ in Definition 5.2.

For any continuous character $\varphi\colon\widetilde{\Gamma}_\infty\to\overline{\mathbb{Q}}_p^\times$ and any arithmetic morphism $\nu\colon\mathbb{I}\to\mathcal{O}_\nu$ we adopt the common notations $\mathscr{L}_{\mathbb{I},\boldsymbol{\xi}}^{\mathrm{an}}(\nu)=\nu(\mathscr{L}_{\mathbb{I},\boldsymbol{\xi}}^{\mathrm{an}})$ and $\mathscr{L}_{\mathbb{I},\boldsymbol{\xi}}^{\mathrm{an}}(\nu,\varphi)=\varphi(\nu(\mathscr{L}_{\mathbb{I},\boldsymbol{\xi}}^{\mathrm{an}}))$.

5.4. **Interpolation.** Let $\mathbb{I}$ be the fixed Hida family passing though $f\in S_k(\Gamma_0(Np))$ with $k\equiv 2\mod p-1$, and let $\mathcal{F}_\infty\in V_p(N^+,\widetilde{\mathbb{I}})$ be as in the previous subsection. The following result generalizes [Cas20, Theorem 2.11] to the quaternionic setting. Fix $\lambda\colon K^\times\backslash\mathbb{A}_K^\times\to\overline{\mathbb{Q}}^\times$ be an algebraic Hecke character of infinity type $(1,0)$ of conductor $\mathfrak{c}\subseteq\mathcal{O}_K$ prime to p whose p-adic avatar takes values in $\mathcal{O}^\times$ and let $\boldsymbol{\xi}=\boldsymbol{\xi}^{(\lambda)}$.

Theorem 5.4. *Let $\nu\colon\mathbb{I}\to\overline{\mathbb{Q}}_p$ be an arithmetic morphism of weight $k_\nu\equiv k\mod 2(p-1)$, and recall that $k\equiv 2\mod 2(p-1)$. Then $\mathscr{L}_{\mathbb{I},\boldsymbol{\xi}}^{\mathrm{an}}(\nu)=\vartheta_\nu^{-1}(c)\mathscr{L}_{\mathcal{F}_\nu,\xi_\nu}$.*

Proof. By Lemma 4.3 we have that, for any point $x=x(\mathfrak{a})$, the T-expansion of $\mathcal{F}_\nu$ at x and the specialization at ν of the T-expansion of $\mathcal{F}$ at x are equal, that is $\mathcal{F}_\nu(T_x)=\nu(\mathcal{F}_\infty(T_x))$. Denote again by $\nu\colon\widetilde{\mathbb{I}}\to\mathbb{Z}_p^{\mathrm{unr}}(\nu)$ the natural extension of ν, where $\mathbb{Z}_p^{\mathrm{unr}}(\nu)$ denotes the finite extension of $\mathbb{Z}_p^{\mathrm{unr}}$ obtained by adjoining the image of ν. Then for any continuous $\varphi\colon\mathbb{Z}_p\to\widetilde{\mathbb{I}}$, using its Mahler expansion, we obtain for $\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)$:

$$\nu\left(\int_{\mathbb{Z}_p^\times}\varphi(u)d_{\mu_{\mathcal{F}_\infty,\mathfrak{a}}}(u)\right)=\int_{\mathbb{Z}_p^\times}(\nu\circ\varphi)(u)d_{\mu_{\mathcal{F}_\nu,\mathfrak{a}}}(u).$$

Recall that, for an ideal $\mathfrak{a}\subseteq\mathcal{O}_c$, we write $\mathrm{N}(\mathfrak{a})=c^{-1}\sharp(\mathcal{O}_c/\mathfrak{a})$; if $\mathfrak{a}=a\widehat{\mathcal{O}}_c\cap K$ for an element $a\in\widehat{\mathcal{O}}_K$, we have $\mathrm{N}(\mathfrak{a})=c^{-1}\cdot\mathbf{N}_K^{-1}(a)$. Choose representatives $\mathfrak{a}$ of $\mathrm{Pic}(\mathcal{O}_c)$ such that $\mathfrak{p}\nmid\mathfrak{a}$ and $\bar{\mathfrak{p}}\nmid\mathfrak{a}$; by (5.3) we then have

$$\hat{\chi}_\nu^{-1}(\mathfrak{a})=\hat{\chi}_\nu^{-1}(a)=\mathbf{N}_K(a)^{k_\nu/2-1}=\mathrm{N}(\mathfrak{a})^{-k_\nu/2+1}c^{-k_\nu/2+1}$$

and $\xi_\nu(\mathfrak{a})=\hat{\xi}_\nu(\mathfrak{a})$. Therefore, since ξ_ν is unramified at p, we have

$$\begin{aligned}\mathscr{L}_{\mathcal{F}_\infty,\boldsymbol{\xi}}(\nu,\varphi)&=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}(\hat{\chi}_\nu^{-1}\hat{\xi}_\nu)(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-1}\int_{\mathbb{Z}_p^\times}(\nu\circ\varphi\big|[\mathfrak{a}])(u)d\mu_{\mathcal{F}_\infty,\mathfrak{a}}(u)\\&=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}c^{-k_\nu/2+1}\mathrm{N}(\mathfrak{a})^{-\frac{k_\nu}{2}}\xi_\nu(\mathfrak{a})\int_{\mathbb{Z}_p^\times}\xi_{\nu,\mathfrak{p}}(u)(\nu\circ\varphi\big|[\mathfrak{a}])(u)d\mu_{\mathcal{F}_\nu,\mathfrak{a}}(u)\\&=c^{-k_\nu/2+1}\mathscr{L}_{\mathcal{F}_\nu,\xi_\nu}(\nu\circ\varphi).\end{aligned}$$

In particular we have $\mathscr{L}_{\mathcal{F}_\infty,\boldsymbol{\xi}}(\nu,\mathbf{1}_U)=c^{-k_\nu/2+1}\mathscr{L}_{\mathcal{F}_\nu,\xi_\nu}(\mathbf{1}_U)$ for any open compact subset U of the Galois group $\widetilde{\Gamma}_\infty$, where $\mathbf{1}_U$ is the characteristic function of U. We conclude by the equivalence between $\mathbb{Z}_p^{\mathrm{unr}}(\nu)$-valued measures on $\widetilde{\Gamma}_\infty$ and additive functions on the set of open compact subsets of $\widetilde{\Gamma}_\infty$ with values in $\mathbb{Z}_p^{\mathrm{unr}}(\nu)$. $\square$

We recall the interpolation properties satisfied by this p-adic L-function. Let f_ν be the specialization at ν of the Hida family f_∞, and suppose that $f_\nu \in S_k(\Gamma_0(Np))$ has trivial character. Then, since f_ν is ordinary, either f_ν has weight 2 and is a newform of level Np, or there is a newform $f_\nu^\sharp \in S_k(\Gamma_0(N))$ whose ordinary p-stabilization is f_ν. Suppose we are in the second case. If $\hat{\varphi}\colon \widetilde{\Gamma}_\infty \to \overline{\mathbb{Q}}_p^\times$ corresponds via rec_K to the p-adic avatar of an anticyclotomic Hecke character φ of infinity type $(n,-n)$ for some integer $n \geq 0$, then

$$\left(\mathscr{L}^{\mathrm{an}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\varphi})\right)^2 = C \cdot L(f_\nu^\sharp, \xi_\nu\varphi, k_\nu/2)$$

where C is a non-zero constant depending on ν, ξ_ν, φ and K, and, up to a p-adic unit, on the choice of $\mathcal{F}_\infty$ (this follows immediately from Theorem 5.4 and Lemma 5.1 combined with [Mag22, Theorem 4.6]).

6. Galois representations

We first recall a standard notation for the symmetric tensors (see [KLZ17, §2.2] for details). For a free abelian group of finite rank H and an integer $m \geq 0$, denote $\mathfrak{S}_m$ the symmetric group on m elements, acting over $H^{\otimes m}$ by permutation of the factors. Let $\mathrm{Sym}^m(H)$ denote the quotient of $H^{\otimes m}$ by the action of $\mathfrak{S}_m$, *i.e.* the group of $\mathfrak{S}_m$-coinvariant elements of $H^{\otimes m}$. We also define the submodule of $H^{\otimes m}$ consisting of the elements which are $\mathfrak{S}_m$-invariant, which is denoted by $\mathrm{TSym}^m(H)$; the ring $\mathrm{TSym}^\bullet(H) = \bigoplus_{m\geq 0} \mathrm{TSym}^m(H)$ is equipped with a ring structure, obtained by symmetrization of the tensor product $(x,y) \mapsto x \cdot y$; for $x \in \mathrm{TSym}^m(H)$ and $y \in \mathrm{TSym}^n(H)$ we thus have $x \cdot y \in \mathrm{TSym}^{m+n}(H)$ (see [KLZ17, (2.2.1)]). There is a natural homomorphism $\mathrm{Sym}^m(H) \to \mathrm{TSym}^m(H)$, which is an isomorphism if H is a module over a ring where $m!$ is invertible, and a canonical duality isomorphism $\mathrm{Sym}^m(H^\vee) \cong (\mathrm{TSym}^m(H))^\vee$ (for a R-module M, we denote $M^\vee$ the R-linear dual of M). The module TSym^m sheafifies in the étale cohomology (see [KLZ17, §3.1] and [Kin15] for more details).

If F is a field, let $\mathrm{Sym}^m(F^2)$ denote the left representation of $\mathrm{GL}(F^2)$ afforded by the space of homogeneous polynomials P in two variables with coefficients in F and degree m, given by $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot P\right)(X,Y) = P(aX+cY, bX+dY)$. The vector space $\mathrm{TSym}^m(F^2)$ is equipped with the dual left action of $\mathrm{GL}_2(F)$ as follows: if we fix an isomorphism $F^2 \cong (F^2)^\vee$, then for $\varphi \in \mathrm{TSym}^m(F^2)$ and $P \in \mathrm{Sym}^m(F^2)$, we set $(\gamma\cdot\varphi)(P) = \varphi(\gamma^{-1}\cdot P)$.

6.1. **Galois representations.** Recall the discrete valuation ring $\mathcal{O}$ fixed in §3.5, and its fraction field $L = \mathrm{Frac}(\mathcal{O})$. Let $m \geq 0$ be an integer and let $M_k = M_k(N^+, p^m, L)$ or $M_k = M_k(N^+p^m, L)$; in the first case, let $\mathcal{C} = \widetilde{X}_m$ and in second case let $\mathcal{C} = X_m$. In both cases, let $\pi\colon \mathcal{A} \to \mathcal{C}$ be the universal abelian surface. Fix $\mathcal{F} \in M_k$ which is an eigenform for all Hecke operators. We consider the motive $h^1(\mathcal{A})$, the degree 1 part of the relative Chow motive of $\mathcal{A}$ over $\mathcal{C}$, whose étale realization is $R^1\pi_*\mathbb{Q}_p$ (see [Anc15, §2, Exemple 3.3(ii), Proposition 3.5]). The idempotents e and $\bar{e}$ defined in §2.2 induce a decomposition into isomorphic factors $R^1\pi_*\mathbb{Q}_p = eR^1\pi_*\mathbb{Q}_p \oplus \bar{e}R^1\pi_*\mathbb{Q}_p$, which lifts to a decompositon of motives $h^1(\mathcal{A}) = eh^1(\mathcal{A}) \oplus \bar{e}h^1(\mathcal{A})$ ([Anc15, Théorème 6.1]). Form the motives

$$\mathscr{V}^k := \mathrm{TSym}^{k-2}(eh^1(\mathcal{A})(1)) \quad \text{and} \quad (\mathscr{V}^k)^\vee = \mathrm{Sym}^{k-2}(eh^1(\mathcal{A})),$$

where $\vee$ denotes the dual motive, notice that $eh^1(\mathcal{A})(1) \cong eh^1(\mathcal{A})$, by [Anc15, Corollaire 2.6]. Denote by $\mathscr{V}^k_{\text{ét}} = \mathrm{TSym}^{k-2}(eR^1\pi_*\mathbb{Q}_p(1))$ and $(\mathscr{V}^k_{\text{ét}})^\vee = \mathrm{Sym}^{k-2}(eR^1\pi_*\mathbb{Q}_p)$ their étale realizations. Put $\mathcal{C}_{\overline{\mathbb{Q}}} = \mathcal{C} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. The subspace

$$V_{\mathcal{F}} \subseteq H^1_{\text{ét}}\left(\mathcal{C}_{\overline{\mathbb{Q}}}, (\mathscr{V}^k_{\text{ét}})^\vee\right) \otimes_{\mathbb{Q}_p} F_{\mathcal{F}}$$

(here $F_{\mathcal{F}} \subseteq \overline{\mathbb{Q}}_p$ is the Hecke field of $\mathcal{F}$) corresponding to the Hecke eigenspace for $\mathcal{F}$ is the Deligne p-adic Galois representation attached to $\mathcal{F}$, characterized by requiring that the trace of the geometric Frobenius element at a prime $\ell \nmid Np$ is equal to $a_\ell(\mathcal{F})$, the Hecke eigenvalue of T_ℓ. The dual, or contragredient, Galois representation $V_{\mathcal{F}}^*$ of $V_{\mathcal{F}}$ is then the maximal quotient

$$H^1_{\text{ét}}\left(\mathcal{C}_{\overline{\mathbb{Q}}}, \mathscr{V}^k_{\text{ét}}(1)\right) \otimes_{\mathbb{Q}_p} F_{\mathcal{F}} \longrightarrow\!\!\!\!\!\rightarrow V_{\mathcal{F}}^*,$$

where the dual Hecke operators T'_ℓ act by $a_\ell(\mathcal{F})$, for all $\ell \nmid Np$, and U' acts by $a_p(\mathcal{F})$, the Hecke eigenvalue of U; see [KLZ17, §2.8].

If $\mathcal{F} \in M_k(N^+p, L)$ is the ordinary p-stabilization of a newform $\mathcal{F}^\sharp \in M_k(N^+, L)$, then we have two isomorphic Galois representations $V_{\mathcal{F}}$ and $V_{\mathcal{F}^\sharp}$, and an explicit isomorphism is

$$\text{(6.1)} \qquad \mathrm{pr}_* = (\mathrm{pr}_1)_* - \frac{(\mathrm{pr}_2)_*}{\alpha} \colon V_{\mathcal{F}} \longrightarrow V_{\mathcal{F}^\sharp},$$

where $\mathrm{pr}_1, \mathrm{pr}_2 \colon X_1 \to X_0$ are the canonical degeneration maps and α is the p-adic unit-root of the Hecke polynomial of T_p acting on $\mathcal{F}^\sharp$ (this follows from a standard generalization of the argument in [KLZ17, Proposition 7.3.1]). By [Ota20, Corollary 5.8] (whose proof works in the quaternionic case also), the maps pr_1 and pr_2 are related to Hida ordinary projector e^{ord} ([LV11, S 6.2]) by the formula

$$(\mathrm{pr}_1)_* \circ e^{\mathrm{ord}} = \frac{\alpha(\pi_1)_* - (\pi_2)_*}{\alpha - \alpha^{-1}p^{k-1}}.$$

6.2. **Big Galois representations.** Define $\widetilde{J}_m = \mathrm{Jac}(\widetilde{X}_m)$ and

$$\mathrm{Ta}_p = \varprojlim_m \left(\mathrm{Ta}_p(\widetilde{J}_m) \otimes_{\mathbb{Z}_p} \mathcal{O}\right).$$

The $\mathcal{O}$-module $\mathrm{Ta}_p(\widetilde{J}_m) \otimes_{\mathbb{Z}_p} \mathcal{O}$ is equipped with an action of Hecke operators T_ℓ for primes $\ell \nmid Np$ and U_ℓ for primes $\ell \mid Np$ attached to the indefinite quaternion algebra B ([LV11, §6.2]). Taking the projective limit of these Hecke algebras one defines a big Hecke algebra acting on Ta_p. Consider the ordinary submodule $\mathrm{Ta}_p^{\mathrm{ord}} = e^{\mathrm{ord}}\,\mathrm{Ta}_p$ of Ta_p. Since $\mathbb{I}$ is a primitive branch of the Hida ordinary Hecke algebra $\mathfrak{h}_N^{\mathrm{ord}}$, again as a consequence of the Jacquet-Langlands correspondence for p-adic families of modular forms ([LV11, Proposition 6.4]; see also [Che05]), one has that $\mathbf{T} = \mathrm{Ta}_p^{\mathrm{ord}} \otimes_{\mathfrak{h}_N^{\mathrm{ord}}} \mathbb{I}$ is a free $\mathbb{I}$-module of rank 2 equipped with a $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, having the following property: $\mathbf{T}$ is unramified outside Np and the characteristic polynomial of the *arithmetic* Frobenius element Frob_ℓ at a prime ideal $\ell \nmid Np$ is equal to

$$P_\ell(X) = X^2 - T_\ell X + (\chi_{\mathrm{cyc}}\Theta^2)(\ell).$$

For each arithmetic character $\nu \colon \mathbb{I} \to F_\nu$, where F_ν is a finite extension of $\mathbb{Q}_p$, $\mathbf{T}_\nu = \mathbf{T} \otimes_{\mathbb{I},\nu} F_\nu$ is isomorphic to $V^*_{\mathcal{F}_\nu}$.

Let v be the place of $\overline{\mathbb{Q}}$ over p corresponding to the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and let $D_v \cong G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ denote the decomposition group of $G_{\mathbb{Q}}$ at v and $I_v \subseteq D_v$ the inertia subgroup, isomorphic to the inertia subgroup $I_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ via the isomorphism $D_v \cong G_{\mathbb{Q}_p}$. Let $\eta_v \colon D_v/I_v \to \mathbb{I}$ be the unramified character defined by $\eta_v(\mathrm{Frob}_v) = U_p$, where Frob_v is an arithmetic Frobenius element of D_v/I_v; we identify η_v with a character of $G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$. There is a short exact sequence of $G_{\mathbb{Q}_p}$-modules (depending on the choice of v, and thus on $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$)

$$\text{(6.2)} \qquad 0 \longrightarrow \mathbf{T}^+ \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}^- \longrightarrow 0$$

such that both $\mathrm{Fil}^+(\mathbf{T})$ and $\mathrm{Fil}^-(\mathbf{T})$ are free $\mathbb{I}$-modules of rank 1, and $G_{\mathbb{Q}_p}$ acts on $\mathrm{Fil}^+(\mathbf{T})$ via $\eta_v^{-1}\chi_{\mathrm{cyc}}\Theta^2$ and acts on the unramified quotient $\mathrm{Fil}^-(\mathbf{T})$ via η_v; see [LV11, §5.5] and [LV11, Corollary 6.5] for details. As $G_{\mathbb{Q}_p}$-representations we have then an isomorphism

$$\mathbf{T} \cong \begin{pmatrix} \eta_v^{-1}\chi_{\mathrm{cyc}}\Theta^2 & * \\ 0 & \eta_v \end{pmatrix}.$$

Define the *critical twist* of $\mathbf{T}$ to be the twist $\mathbf{T}^\dagger = \mathbf{T} \otimes \Theta^{-1}$ of $\mathbf{T}$ by the Galois action of Θ ([LV11, §6.4]). Then $\mathbf{T}^\dagger_\nu = \mathbf{T}^\dagger \otimes_{\mathbb{I},\nu} F_\nu$ is isomorphic to the self-dual twist $V^\dagger_{\mathcal{F}_\nu} = V_{\mathcal{F}_\nu}(k/2)$ of the Deligne representation $V_{\mathcal{F}_\nu}$. Fix a continuous character $\boldsymbol{\xi}\colon K^\times \backslash \widehat{K}^\times \to \mathbb{I}^\times$ as in §5.2, and denote by the same symbol the associated Galois character $\boldsymbol{\xi}\colon G_K \to \mathbb{I}^\times$. Let $\mathbf{T}^\dagger_{|G_K}$ denote the G_K-representation $\mathbf{T}^\dagger$ obtained by restriction to the subgroup $G_K \subseteq G_{\mathbb{Q}}$. Define the G_K-representation $\mathbf{T}^\dagger_{\boldsymbol{\xi}} = \mathbf{T}^\dagger_{|G_K} \otimes \boldsymbol{\xi}^{-1}$. From (6.2) we obtain a filtration of $D_v \cong G_{\mathbb{Q}_p}$-modules (recall that p is split in K)

$$0 \longrightarrow \mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}} \longrightarrow \mathbf{T}^\dagger_{\boldsymbol{\xi}} \longrightarrow \mathbf{T}^{\dagger,-}_{\boldsymbol{\xi}} \longrightarrow 0$$

and as $G_{\mathbb{Q}_p}$-representations we have an isomorphism

$$\mathbf{T}^\dagger_{\boldsymbol{\xi}} \cong \begin{pmatrix} \eta_v^{-1}\chi_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1} & * \\ 0 & \eta_v\Theta^{-1}\boldsymbol{\xi}^{-1} \end{pmatrix}.$$

Define the Galois character $\Psi\colon G_K \to \mathbb{I}^\times$ by $\Psi = \eta_v^{-1}\chi_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1}$.

Lemma 6.1. $\Psi\colon G_{K_{\mathfrak{p}}} \to \mathbb{I}^\times$ *is unramified.*

Proof. Since λ has infinity type $(1,0)$ and it is unramified at p, we have $\hat{\lambda}\hat{\bar{\lambda}} = \chi_{\mathrm{cyc}}\beta$, with $\hat{\bar{\lambda}}\colon x \mapsto \hat{\lambda}(\bar{x})$ and β a character of finite order and unramified at $\mathfrak{p}$. Since $k \equiv 2 \mod 2(p-1)$, a simple computation shows that $\eta_v^{-1}\chi_{\mathrm{cyc}}\boldsymbol{\vartheta}\boldsymbol{\xi}^{-1} = \eta_v^{-1}\beta^{-1}\hat{\bar{\lambda}}^2[\langle\beta\rangle^{-1/2}][\langle\hat{\bar{\lambda}}\rangle]$, and the result follows because η_v^{-1} is unramified as a $G_{K_{\mathfrak{p}}}$-character and $\beta^{-1}\hat{\bar{\lambda}}^2[\langle\beta\rangle^{-1/2}][\langle\hat{\bar{\lambda}}\rangle]$ is unramified at $\mathfrak{p}$ seen as the p-adic avatar of a Hecke character. $\square$

For each arithmetic morphism $\nu\colon \mathbb{I} \to \mathcal{O}_\nu$, where $\mathcal{O}_\nu$ is the valuation ring of the finite extension F_ν of $\overline{\mathbb{Q}}_p$, define $\mathbf{T}^\dagger_{\xi_\nu} = \mathbf{T}^\dagger_{\boldsymbol{\xi}} \otimes_{\mathbb{I},\nu} F_\nu$, where the tensor product is taken with respect to ν, composed with the inclusion $\mathcal{O}_\nu \subseteq F_\nu$ as indicated. We have then an exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \longrightarrow \mathbf{T}^{\dagger,+}_{\xi_\nu} \longrightarrow \mathbf{T}^\dagger_{\xi_\nu} \longrightarrow \mathbf{T}^{\dagger,-}_{\xi_\nu} \longrightarrow 0$$

where $\mathbf{T}^{\dagger,\pm}_{\xi_\nu} = \mathbf{T}^{\dagger,\pm}_{\boldsymbol{\xi}} \otimes_{\mathbb{I},\nu} F_\nu$ are F_ν-vector spaces of dimension one.

6.3. **Specializations.** Fix a prime $\mathfrak{P}$ of $\overline{\mathbb{Q}}$ over $\mathfrak{p}$. Denote $F = H_{c,\mathfrak{P}}$ the completion of H_c at $\mathfrak{P}$, $F_\infty = \mathbb{Q}_p^{\mathrm{unr}}$ the maximal unramified extension of $\mathbb{Q}_p$ (which contains F and is also the maximal unramified extension of F because $p \nmid c$) and $L_\infty = H_{cp^\infty,\mathfrak{P}}$, the completion of H_{cp^∞} at $\mathfrak{P}$. Recall that $L_\infty = F(\mathfrak{F})$ is obtained by adjoining the torsion points of the relative Lubin–Tate formal group $\mathfrak{F}$ of parameter $\pi/\bar{\pi}$, where if δ is the order of $\mathfrak{p}$ in $\mathrm{Pic}(\mathcal{O}_c)$, then $\mathfrak{p}^\delta = (\pi)$ with $\pi \in \mathcal{O}_c$ (see [Shn16, Proposition 8.3] for the proof; see also [CH18a, page 604]). Let $K_\infty = L_\infty(\mu_{p^\infty})$ and define $\mathcal{G} = \mathrm{Gal}(K_\infty/F)$, $\Gamma_\infty = \mathrm{Gal}(L_\infty/F)$, $\Gamma_{\mathrm{cyc}} = \mathrm{Gal}(F(\mu_{p^\infty})/F)$. We also notice that if we let $\widetilde{H}_{cp^n} = H_{cp^n}(\mu_{p^n})$ and $\widetilde{H}_{cp^\infty} = \bigcup_{n\geq 1}\widetilde{H}_{cp^n}$, then $K_\infty = \widetilde{H}_{cp^\infty,\mathfrak{P}}$ is the completion of $\widetilde{H}_{cp^\infty}$ at $\mathfrak{P}$. We thus have the following diagram of local fields:

(6.3)
$$\begin{tikzcd}
 & K_\infty \arrow[dl, dash] \arrow[d, dash] \arrow[dd, dash, bend left, "\mathcal{G}"] \\
F(\mu_{p^\infty}) \arrow[dr, dash, "\Gamma_{\mathrm{cyc}}"'] & L_\infty \arrow[d, dash, bend right, "\Gamma_\infty"'] \\
 & F
\end{tikzcd}$$

For any finite extension L of F in L_∞, and any $\mathcal{G}$-stable subquotient $\mathbf{M}$ of $\mathbf{T}^\dagger_{\boldsymbol{\xi}}$, define

$$H^1_{\mathrm{Iw}}(L_\infty/L,\mathbf{M}) = H^1_{\mathrm{Iw}}\left(\mathrm{Gal}(L_\infty/L),\mathbf{M}\right) = \varprojlim_{L'} H^1(L',\mathbf{M})$$

where L' runs over the finite extensions of L contained in L_∞.

Let (ν,ϕ) be a pair consisting of an arithmetic morphism $\nu\colon \mathbb{I}\to\mathcal{O}_\nu$ and a Hodge–Tate character $\phi\colon\mathcal{G}\to\overline{\mathbb{Q}}_p^\times$ of Hodge–Tate weight $m\in\mathbb{Z}$; we adopt the convention that the Hodge–Tate weight of the cyclotomic character $\chi_{\mathrm{cyc}}\colon G_{\overline{\mathbb{Q}}_p}\to\mathbb{Z}_p^\times$ is $+1$, so $\phi=\chi_{\mathrm{cyc}}^m\psi$ for some unramified character $\psi\colon\mathcal{G}\to\overline{\mathbb{Q}}_p^\times$. For $\bullet$ being $+$, $-$ or no symbol, let $\mathbf{T}^{\dagger,\bullet}_{\xi_\nu}(\phi)$ denote the twist of the representation $\mathbf{T}^{\dagger,\bullet}_{\xi_\nu}$ by ϕ. We then have specialization maps

$$\mathrm{sp}_{\nu,\phi}\colon H^1_{\mathrm{Iw}}(\Gamma_\infty,\mathbf{T}^{\dagger,\bullet}_{\boldsymbol{\xi}}) \xrightarrow{\sim} H^1(F,\mathbf{T}^{\dagger,\bullet}_{\boldsymbol{\xi}}\widehat{\otimes}_{\mathbb{I}}\mathbb{I}[\![\mathcal{G}]\!]) \longrightarrow H^1(F,\mathbf{T}^{\dagger,\bullet}_{\xi_\nu}(\phi))$$

where the first map is induced by Shapiro's isomorphism.

7. Big Perrin-Riou logarithm map

7.1. **Big Eichler–Shimura isomorphisms.** Faltings Eichler-Shimura isomorphism is constructed in the quaternionic case by [SG17]. Faltings Eichler-Shimura isomorphism in the quaternionic setting is constructed in [SG17, Section 2], following the approach of Faltings [Fal87]. Recall the Shimura curve $\widetilde{X}_m$ in (2.2), which we view as defined over $\mathbb{Q}_p$. Let $\widetilde{\mathcal{A}}_m\to\widetilde{X}_m$ be the universal QM abelian surface and set $\mathcal{G}_m=\mathrm{Ta}_p((e\widetilde{\mathcal{A}}_m[p^\infty]))$ and, for any integer $k\geq 2$ and any finite extension of $\mathbb{Q}_p$, define $\mathcal{V}_{k-2}(L)=\mathrm{Sym}^{k-2}(\mathcal{G})\otimes L$ (tensor product as $\mathbb{Z}_p$-modules). Put $\underline{\omega}_m=\underline{\omega}_{\widetilde{\mathcal{A}}_m^\vee/\widetilde{X}_m}=e\pi_*(\Omega_{\widetilde{\mathcal{A}}_m^\vee/\widetilde{X}_m})$. Then by [SG17, Proposition 2.1] there is an isomorphism (where the tensor product is over L)

$$H^1(\widetilde{X}_m,\mathcal{V}_{k-2}(L))\otimes\mathbb{C}_p(1)\simeq\left(H^0(\widetilde{X}_m,\underline{\omega}_m^k)\otimes\mathbb{C}_p\right)\oplus\left(H^1(\widetilde{X}_m,\underline{\omega}_m^{-k-2})\otimes\mathbb{C}_p(k-1)\right) \tag{7.1}$$

of $\mathbb{C}_p$-vector spaces, which is $G_L=\mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$ and Hecke equivariant. We use the results by [CHJ17] to interpolate this isomorphism along the Hida family $\mathbb{I}$.

Let $\mathbf{V}$ be the sheaf of distributions on $\mathbb{Z}_p$ defined in [CHJ17, §4.2], and $\mathcal{M}^\dagger$ the sheaf of perfectoid modular forms defined in [CHJ17, §4.3]; also, let $\mathcal{C}=\mathcal{C}_{\mathbf{V}}=\mathcal{C}_{\mathcal{M}^\dagger}$ be the eigencurve constructed by the eigenmachine in [CHJ17, §5.1] (see especially [CHJ17, Proposition 5.2]). The Hida family $\mathbb{I}$ corresponds to an irreducible component $\mathcal{C}_{\mathbb{I}}$ of $\mathcal{C}$, which coincides with the base-change of the eigencurve in the proof of Theorem 3.13; let $\mathbf{V}_{\mathbb{I}}$ and $\mathcal{M}_{\mathbb{I}}$ be the restrictions of $\mathbf{V}$ and $\mathcal{M}^\dagger$ to $\mathcal{C}_{\mathbb{I}}$. By [CHJ17, Theorem 5.12], the $G_{\mathbb{Q}}$-representation $\mathbf{V}_{\mathbb{I}}(1)$ is isomorphic to the base change of the big Galois representation $\mathbf{T}$ considered before. Moreover, the sheaf of perfectoid modular forms $\mathcal{M}_{\mathbb{I}}$ is canonically isomorphic to the base change of the sheaf of p-adic modular forms constructed from the Igusa tower (see [BHW22, §3.4] for the argument over modular curves, which extends to the quaternionic setting; also, note that [BHW22] work over the anticanonical tower while [CHJ17] work over the canonical tower, which are isomorphic by the Arkin-Lehner involution, as explained in [BHW22, §3.2]). By [CHJ17, Theorem 5.12], the sheaves $\mathbf{V}_{\mathbb{I}}$ and $\mathcal{M}_{\mathbb{I}}$ are locally free of rank 2 and 1 respectively. Denote

$$\mathcal{ES}:\mathbf{V}_{\mathbb{I}}(1)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p\longrightarrow\mathcal{M}_{\mathbb{I}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p$$

the restriction of the Eichler-Shimura map constructed in [CHJ17, Theorem 5.3]. The kernel $\ker(\mathcal{ES})$ and the image $\mathrm{im}(\mathcal{ES})$ are both locally free sheaves of rank 1. By [CHJ17, Theorem 5.14] we have, outside a Zariski closed subset of dimension 0 disjoint from the subset of classical points, a splitting

$$\mathbf{V}_{\mathbb{I}}(1)\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p\longrightarrow(\mathcal{M}_{\mathbb{I}}\widehat{\otimes}_{\mathbb{Q}_p}\mathbb{C}_p)\oplus\ker(\mathcal{ES}). \tag{7.2}$$

Proposition 7.1. *Zariski generically, the map* (7.2) *interpolates* (7.1).

Proof. This is one of the main results of the paper [CHJ17] (see especially [CHJ17, §5.2]). However, the paper [CHJ17] does not explicitly mention specializations at arithmetic morphisms with arbitrary wild ramification. For this reason, we sketch the construction of this specialization morphisms by modifying the relevant maps in [CHJ17]; we advise the reader to keep her/his copy of the paper [CHJ17] for the notation which is not fully introduced here. A similar result can be obtained by adapting in the same way the techniques of [SG17], which relies on a quaternionic analogue of [AIS15]. However, we remark that in the ordinary case one can probably obtain finer results using measures instead of distributions, and only working directly over the ordinary locus instead of its neighborhoods of overconvergence. To keep the proof at a reasonable length, we prefer to explain how to modify the more general result of [CHJ17].

We follow closely, in this proof only, the notation in [CHJ17], which we briefly recall and adapt to our setting. For any $m \geq 0$, we denote $\Delta_0(p^m)$ (respectively, $K_0(p^m)$ or $K_1(p^m)$) the semigroup (respectively, the group) of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{M}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Z})$ (respectively, in $\mathrm{GL}_2(\mathbb{Z}_p)$) with $c \equiv 0 \mod p^m$ and $d \in \mathbb{Z}_p^\times$ (respectively, $c \equiv 0$ or $c \equiv 0$ and $d \equiv 1$) modulo p^m. For $K = K_0(p^m)$ or $K = K_1(p^m)$, denote $\mathcal{X}_K$ the adic Shimura curve with $U_0(N^+)$-level structure, and K-structure at p, and let $\mathcal{X} = \mathcal{X}_{K_0(p)}$ and $\mathcal{X}_m = \mathcal{X}_{K_1(p^m)}$ (*cf.* [CHJ17, §2.2]; all these curves are viewed as adic spaces over $\mathrm{Spa}(\mathbb{Q}_p^{\mathrm{cyc}}, \mathbb{Z}_p^{\mathrm{cyc}})$; also, $\mathcal{X}_{K_0(p^m)}$ and $\mathcal{X}_{K_1(p^m)}$ are the analytifications of the curves previously denoted X_m and $\widetilde{X}_m$). The perfectoid Shimura curve considered in this proof is $\mathcal{X}_\infty \sim \varprojlim_m \mathcal{X}_m$; this is not exactly the same object considered in [CHJ17, Theorem 2.2], where $\mathcal{X}_m$ is replaced by Shimura curves with full level structure; however, the arguments used in [CHJ17] work as well in our setting, since $\mathcal{X}_\infty$ is a perfectoid space as well ([Sch15, Proposition 3.2.34]). The perfectoid space $\mathcal{X}_\infty$ is equipped with the Hodge-Tate map $\pi_{\mathrm{HT}} : \mathcal{X}_\infty \to \mathbb{P}^1$ and the variable $\mathfrak{z} \in H^0(\mathcal{V}_1, \mathcal{O}^+_{\mathcal{X}_\infty})$ is $\mathfrak{z} = z \circ \pi_{\mathrm{HT}}$ with $\mathcal{V}_1 = \pi_{\mathrm{HT}}^{-1}(V_1)$ and $z = -y/x$ the standard variable in $V_1 = \{[x : y] : x \neq 0\}$, so that we have $\gamma^*(\mathfrak{z}) = \frac{a\mathfrak{z}+c}{b\mathfrak{z}+d}$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_0(p)$ (see [CHJ17, §2.2] for details; alternatively, one can use a different affine subset of $\mathbb{P}^1$ and obtain a new variable $\mathfrak{z}'$ as in [BHW22] satisfying $\gamma^*\mathfrak{z}' = \frac{a\mathfrak{z}'+b}{c\mathfrak{z}'+d}$, but we prefer to follow more closely the computations in [CHJ17]). Let $\mathcal{X}_m(0)_c$ denote the canonical component of the ordinary locus $\mathcal{X}_m(0)$ of $\mathcal{X}_m$, and $\mathcal{X}_\infty(0)_c = q_m^{-1}(\mathcal{X}_m(0)_c)$, where $q_m : \mathcal{X}_\infty \to \mathcal{X}_m$ is the canonical projection map. Let $\omega = \pi_{\mathrm{HT}}^*(\mathcal{O}(1))$ and let $\mathfrak{s} \in H^0(\mathcal{X}_\infty(0)_c, \omega)$ be the (fake) Hasse invariant defined in [CHJ17, §2.4], satisfying the transformation formula $\gamma^*(\mathfrak{s}) = \mathfrak{s}(b\mathfrak{z}+d)$ (alternatively, as in [BHW22], one could work over the anticanonical ordinary tower instead, which is isomorphic to the canonical tower by an Atkin-Lehner operator, and has the advantage of being directly related to Katz convergent modular forms as noticed before).

Step 1. Twist distributions and polynomials by finite order characters. In [CHJ17, §3.1], replace the $\mathbb{Q}_p$-vector space of powerbounded p^s-locally analytic functions $\mathbf{A}_k^{s,\circ}$ on $\mathbb{Z}_p$ with the $\mathbb{Q}_p$-vector space $\mathbf{A}_\nu^{s,\circ}$, which coincides with $\mathbf{A}_k^{s,\circ}$ as $\mathbb{Q}_p$-vector space, but the right $\Delta_0(p)$-actions is defined by replacing the automorphic factor $x \mapsto (cx+d)^{k-2}$ with $x \mapsto \psi_\nu(cx+d)(cx+d)^{k_\nu-2}$. Equip then the linear dual $\mathbf{D}_\nu^{s,\circ}$ with the dual left action as in [CHJ17, §3.1] (here, working in the ordinary setting, one could even use measures instead of locally analytic distributions, but the modifications required to adapt the notation in [CHJ17] would led to a too long proof). Similarly, for a ring A, replace the A-module of polynomials $\mathscr{L}_k(A)$ with coefficients in A and degree at most $k-2$ with the A-module $\mathscr{L}_\nu(A)$, which coincides with $\mathscr{L}_k(A)$ as A-module, but the left $\mathrm{M}_2(A)$-action is twisted by ψ_ν as follows: for $f \in \mathscr{L}_\nu(A)$, put

$$(f \cdot_\nu \gamma)(X) = \psi_\nu(d)(d+bX)^{k_\nu-2} f\left(\frac{c+aX}{d+bX}\right). \tag{7.3}$$

Replace then $\mathscr{L}_k^\circ$ with $\mathscr{L}_\nu^\circ = \mathscr{L}_\nu(\mathbb{Z}_p)$ and $\mathscr{L}_k$ with $\mathscr{L}_\nu = \mathscr{L}_\nu(\mathbb{Q}_p)$. Change the map ρ_k appearing before [CHJ17, Definition 3.2] with the map $\rho_\nu : \mathbf{D}_\nu^{s,\circ} \to \mathscr{L}_\nu^\circ$ defined by the integration formula

$$\rho_\nu(\mu) = \int_{\mathbb{Z}_p} \psi_\nu(x)(1+Xx)^{k_\nu-2} d\mu(x)$$

with the convention that if ψ_ν is non-trivial, then $\psi_\nu(x) = 0$ for $x \in p\mathbb{Z}_p$, and $\psi_\nu(x) = 1$ otherwise. A standard computation shows that the map ρ_ν is then $\Delta_0(p^m)$-equivariant for $m = \max\{1, m_\nu\}$:

$$\begin{aligned}\int (1+xX)^{k_\nu-2} d(\gamma\cdot\mu) &= \int_{\mathbb{Z}_p} \psi_\nu(cx+d)(cx+d)^{k_\nu-2}\left(1+\frac{ax+b}{cx+d}X\right)^{k_\nu-2} d\mu(x)\\ &= \int_{\mathbb{Z}_p} \psi_\nu(d)(cx+d+(ax+b)X)^{k_\nu-2} d\mu(x)\\ &= \psi_\nu(d)\int_{\mathbb{Z}_p} (bX+d+x(aX+c))^{k_\nu-2} d\mu(x)\\ &= \psi_\nu(d)\int_{\mathbb{Z}_p} (bX+d)^{k-2}\left(1+x\frac{aX+c}{bX+d}\right)^{k_\nu-2} d\mu(x)\\ &= \left(\int (1+xX)^{k-2} d\mu\right)|_\nu\gamma\end{aligned}$$

where we use that $\psi_\nu(cx+d) = \psi_\nu(d)$ because $\gamma \in \Delta_0(p^m)$. We also need to consider twists at the level of overconvergent distributions as follows. Replace $\mathcal{O}\mathbf{D}^s_k(V)$ in [CHJ17, §4.5] with

$$\mathcal{O}\mathbf{D}^s_\nu(V) = (\mathbf{D}^s_\nu \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}_m}(V_\infty))^{K_1(p^m)}$$

where $V \in (\mathcal{X}_m)_{\text{proét}}$ and $V_\infty = V \times_{\mathcal{X}_m} \mathcal{X}_\infty$. The integration map ρ_ν gives then a map

$$\rho_\nu : \mathcal{O}\mathbf{D}^s_\nu(V) \longrightarrow (\mathscr{L}_\nu \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}m}(V_\infty))^{K_1(p^m)},$$

which is equivariant for the actions of $\Delta_0(p^m)$ on both sides.

Step 2. Let $\mathcal{A}_m \to \mathcal{X}_m$ be the universal QM abelian surface. Define the $\mathbb{Z}_p$-modules $\mathcal{G}_m = e\mathcal{A}_m[p^\infty]$ and $\mathcal{T}_m = \mathrm{Ta}_p(\mathcal{G}_m)$, and the sheaf $\widehat{\mathcal{T}}_m = \mathcal{T}_m \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}_{\mathcal{X}_m}$. Form the sheaf $\widehat{\mathcal{V}}_\nu = \mathrm{Sym}^{k_\nu-2}(\widehat{\mathcal{T}}_m)$. As in [CHJ17, Lemma 4.13], by [CHJ17, Lemma 4.1] we obtain a canonical isomorphism

$$\widehat{\mathcal{V}}_\nu(V) \simeq (\mathscr{L}_\nu \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}_m}(V_\infty))^{K_1(p^m)}$$

for each qcqs $V \in (\mathcal{X}_m)_{\text{proét}}$; we equip $\widehat{\mathcal{V}}_\nu$ with the left $\Delta_0(p^m)$-action by $\gamma \cdot_\nu x = \psi_\nu(d)(\gamma\cdot x)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, where $\gamma \cdot x$ denotes the untwisted action considered in [CHJ17, §4.5]. We thus get a map still denoted $\rho_\nu : \mathcal{O}\mathbf{D}^s_\nu \to \widehat{\mathcal{V}}_\nu$, which is equivariant for the action of $\Delta_0(p^m)$.

Step 2. Let $\omega_m = e\mathrm{Lie}(\mathcal{A}_m/X_m)$. Then $\omega = q_m^*(\omega_m) = \pi^*_{\mathrm{HT}}(\mathcal{O}(1))$ (see [CHJ17, Theorem 2.8]). The Hodge-Tate map gives a map $\mathcal{T}_m \to \omega_m$ and we thus obtain a map $\widehat{\mathcal{V}}_\nu \to \omega_m^{\otimes(k_\nu-2)}$; we twist the usual action of $\Delta_0(p^m)$ on the target by ψ_ν to obtain the sheaf ω_ν so that there is an equivariant map $j_\nu : \widehat{\mathcal{V}}_\nu \to \omega_\nu$. This map can be described as in [CHJ17, Lemma 4.15] as follows. For any qcqs open subset V of $\mathcal{X}_m(0)_c$, define

$$\hat{\omega}_m = \left(\widehat{\mathcal{O}}_{\mathcal{X}_m}(V_\infty)_{|X_\infty(0)_c}\right)^{K_1(p^m)}.$$

Let $\hat{\omega}_\nu$ be the sheaf sheaf $\omega_m^{\otimes(k_\nu-2)}$ with $\Delta_0(p^m)$ action twisted by ψ_ν. By [CHJ17, Lemma 4.14], the map $f \mapsto f \cdot \mathfrak{s}^{\otimes(k_\nu-2)}$ defines then an isomorphism $\hat{\omega}_\nu \cong \omega_\nu$ which is equivariant for the action of $\Delta_0(p^m)$. Define a map $\mathscr{L}_\nu \widehat{\otimes}_{\mathbb{Q}_p} \widehat{\mathcal{O}}_{\mathcal{X}_m}(V_\infty) \to \widehat{\mathcal{O}}_{\mathcal{X}_m}(V_\infty)$ by $X^i \mapsto \mathfrak{z}^i$. Restricting to $\mathcal{X}_m(0)_c$ and taking $K_1(p^m)$-invariants, we thus obtain the explicit description of the map

$j_\nu : \widehat{\mathcal{V}}_\nu \to \omega_\nu$. By the argument in [CHJ17, Lemma 4.15], we thus obtain a commutative diagram on which $\Delta_0(p^m)$ acts equivariantly:

$$\begin{tikzcd}
\mathcal{O}\mathbf{D}^s_\nu \arrow[r, "j_\nu"] \arrow[d, "\rho_\nu"'] & \hat{\omega}_\nu \\
\widehat{\mathcal{V}}_\nu . \arrow[ur, "X^i \mapsto \mathfrak{z}^i \mathfrak{s}^{\otimes(k_\nu - 2)}"'] &
\end{tikzcd}$$

Step 3. The Eichler-Shimura map $\mathcal{ES}$ in [CHJ17, Definition 4.12] can be factorized for $\nu \in \mathcal{U}$ as

$$\begin{tikzcd}
\mathbf{V}_{\mathcal{U}} \arrow[r, "\cong"] \arrow[ddrr, "\mathcal{ES}_\nu"] & H^1_{\text{proét}}(\mathcal{X}_{\mathbb{C}_p}, \mathcal{O}\mathbf{D}^s_{\mathcal{U}}) \arrow[r, "\text{res}"] & H^1_{\text{proét}}(\mathcal{X}_m(0)_c, \mathcal{O}\mathbf{D}^s_\nu) \arrow[d, "j_\nu"] \\
 & & H^1_{\text{proét}}(\mathcal{X}_m(0)_c, \hat{\omega}_\nu) \arrow[d] \\
 & & H^0(\mathcal{X}_m, \omega_\nu \otimes \Omega^1_{\mathcal{X}_{K(p^m)}})(-1)
\end{tikzcd}$$

where the unlabeled vertical arrow is the map constructed in [CHJ17, Proposition 5.7] and res is the canonical restriction composed with the canonical specialization map $\mathcal{O}\mathbf{D}^s_{\mathcal{U}} \to \mathcal{O}\mathbf{D}^s_\nu$. The composite map $\mathcal{ES}_\nu$ is surjective by the argument in [CHJ17, Proposition 5.8], and the result follows by restriction to $\mathbf{V}_{\mathbb{I}}$ (and multiplicity one) after taking into consideration the following observation. Recall that the action of $\Delta_0(p^m)$ on $\mathcal{O}\mathbf{D}_{\mathcal{U}}$ is the dual of the action on locally analytic distributions $\mathbf{A}^s_{\mathcal{U}}$ given in [CHJ17, §3.1] by the formula

$$(f \cdot_{\mathcal{U}} \gamma)(x) = \chi_{\mathcal{U}}(cx+d) f\left(\frac{ax+b}{cx+d}\right)$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For each $\nu \in \mathcal{U}$, this action is equal to the action defined in (7.3) once $\chi_{\mathcal{U}}$ is defined to be the square of (any choice of) the critical character $\Theta : \mathbb{Z}_p^\times \hookrightarrow \mathbb{I}^\times$. $\square$

We now recall an argument in [LZ16, §6.1] to interpret these results in terms of p-adic Hodge theory. Taking $G_{\mathbb{Q}_p}$-invariants, and looking at Hodge-Tate weights we obtain, Zariski generically, an isomorphism

$$\mathcal{ES} : \left(\mathbf{V}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)^{G_{\mathbb{Q}_p}} \simeq \mathcal{M}_{\mathbb{I}} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p.$$

There is an isomorphism

$$\left(\mathbf{V}_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)^{G_{\mathbb{Q}}} \simeq \left(\mathbf{D}_{\text{Sen}}(\mathbf{V}_{\mathbb{I}}(1))\right)^{\Gamma}$$

(recall that $\mathbf{D}_{\text{Sen}}(M)$ is defined by means of the functor $M \mapsto \mathbf{D}^\dagger_{\text{rig}}(M)$ from p-adic $G_{\mathbb{Q}_p}$-representations to (φ, Γ)-modules). We may fix an isomorphism $\mathbf{V}_{\mathbb{I}}(1) \simeq \mathbf{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and let $\mathbf{V}^+_{\mathbb{I}}(1)$ and $\mathbf{V}^-_{\mathbb{I}}(1)$ denote the inverse images of $\mathbf{T}^+ \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathbf{T}^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, respectively, under this isomorphism. Now, $\left(\mathbf{D}_{\text{Sen}}\left(\mathbf{V}^+_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)\right)^{\Gamma} = 0$ and $\left(\mathbf{D}_{\text{Sen}}\left(\mathbf{V}^-_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)\right)^{\Gamma}$ is equal to $\left(\mathbf{V}^-_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)^{\Gamma}$ because $\mathbf{V}^-_{\mathbb{I}}(1)$ is unramified, and therefore we obtain an isomorphism

$$\left(\mathbf{V}_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)^{G_{\mathbb{Q}_p}} \simeq \left(\mathbf{V}^-_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)^{\Gamma}.$$

We thus obtain, Zariski generically, an isomorphisms:

$$\mathcal{ES} : \left(\mathbf{V}^-_{\mathbb{I}}(1) \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p\right)^{\Gamma} \simeq \mathcal{M}_{\mathbb{I}} \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p. \tag{7.4}$$

Define the $\mathbb{I}$-module $\mathbf{D}(\mathbf{T}^-) = (\mathbf{T}^- \widehat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}}_p^{\mathrm{unr}})^{G_{\mathbb{Q}_p}}$. Since $\mathbf{T}_\kappa^-$ is unramified, we have

$$\mathbf{D}_{\mathrm{cris}}\left(\mathbf{T}_\kappa^-\right) = \left(\mathbf{T}_\kappa^- \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{unr}}\right)^{G_{\mathbb{Q}_p}}$$

and therefore we have a canonical map $\mathrm{sp}_\kappa : \mathbf{D}(\mathbf{T}^-) \to \mathbf{D}_{\mathrm{cris}}(\mathbf{T}_\kappa^-)$.

Theorem 7.2. *There exists an isomorphism $\eta_{\mathcal{F}_\infty} : \mathbf{D}(\mathbf{T}^-) \cong \mathbb{I}$ of $\mathbb{I}$-modules such that if $\omega_{\mathcal{F}_\infty}$ is a generator of the $\mathbb{I}$-modules $\mathbf{D}(\mathbf{T}^-)$, then $\mathrm{sp}_\kappa(\omega_{\mathcal{F}_\infty}) = \omega_{\mathcal{F}_\kappa}$ where $\omega_{\mathcal{F}_\kappa}$ is a differential form attached to $\mathcal{F}_\kappa$.*

Proof. We know that the $\mathbb{I}$-module $\mathbf{D}(\mathbf{T}^-)$ is free of rank 1 because the analogue result is known for the isomorphic Galois representation obtained from elliptic modular forms: see [LV11, Proposition 6.4] and [Cas20, Lemma 5.1]. Fix a $\mathbb{I}$-basis $\omega_{\mathcal{F}_\infty}$ of the free $\mathbb{I}$-module $\mathbf{D}(\mathbf{T}^-)$-module of rank 1. For each ν, $\omega_{\mathcal{F}_\nu}$ gives a (canonical, up to the choice of $\omega_{\mathcal{F}_\infty}$) choice of an element in $\mathbf{D}_{\mathrm{dR}}(\mathbf{T}_\nu^-)$. If we extend the coefficients to $\mathbb{C}_p$ and identify $\mathbf{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathbf{V}_{\mathbb{I}}$, this is exactly the image of the ν-specialization of the map (7.4) by Theorem 7.2. ☐

Recall that $\mathbf{T}$ is equipped with a perfect $\mathbb{I}(1) \otimes \Theta^2$-valued alternating pairing which induces a perfect and Galois equivariant pairing of free $\mathbb{I}$-modules of rank 1:

$$\mathbf{T}^+ \times \mathbf{T}^- \longrightarrow \mathbb{I}(1) \otimes \Theta^2. \tag{7.5}$$

Define as before the G_K-representation $\mathbf{T}_{\boldsymbol{\xi}}^\dagger = \mathbf{T}_{|G_K}^\dagger \otimes \boldsymbol{\xi}^{-1}$, which we restrict at the decomposition group at $\mathfrak{p}$, obtaining a $G_{\mathbb{Q}_p}$-representation, still denoted $\mathbf{T}_{\boldsymbol{\xi}}^\dagger$. Note that the $G_{\mathbb{Q}_p}$-representation $\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+} \cong \mathbb{I}(\eta_v^{-1}\chi_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1})$ is unramified. We also define $\mathbf{T}_{\boldsymbol{\xi}^{-1}}^\dagger = \mathbf{T}_{|G_K}^\dagger \otimes \boldsymbol{\xi}$. From the Galois equivariant pairing (7.5) we construct a map

$$\mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}^{-1}}^{\dagger,-}) \longrightarrow \mathrm{Hom}_{\mathbb{I}}\left(\mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+}), \mathbf{D}\left(\mathbb{I}(\chi_{\mathrm{cyc}})\right)\right),$$

where we put $\mathbf{D}\left(\mathbb{I}(\chi_{\mathrm{cyc}})\right) = \left((\mathbb{I} \otimes \chi_{\mathrm{cyc}}) \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{unr}}\right)^{G_{\mathbb{Q}_p}}$. In particular, the element $\omega_{\mathcal{F}_\infty}$ of $\mathbf{D}(\mathbf{T}^-)$ gives rise to a map

$$\boldsymbol{\omega}_\infty : (-, \omega_{\mathcal{F}_\infty} \otimes \Theta^{-1}\boldsymbol{\xi}) \colon \mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+}) \longrightarrow \mathbf{D}\left(\mathbb{I}(\chi_{\mathrm{cyc}})\right)$$

which is given, as indicated, by pairing with the class $\omega_{\mathcal{F}_\infty}$ and twisting by the Galois character $\Theta^{-1}\boldsymbol{\xi}$.

We now compare with the de Rham pairing for the representations $\mathbf{T}_{\xi_\nu}^{\dagger,+}$. Since $\mathbf{T}_{\xi_\nu}^{\dagger,+}$ and $\mathbf{T}_\nu^-$ are unramified, $\mathbf{D}(\mathbf{T}_{\xi_\nu}^{\dagger,+}) \cong \mathbf{D}_{F_\nu,\star}(\mathbf{T}_{\xi_\nu}^{\dagger,+})$ and $\mathbf{D}(\mathbf{T}_\nu^-) \cong \mathbf{D}_{F_\nu,\star}(\mathbf{T}_\nu^-)$ for $\star = \mathrm{dR}$ or $\star = \mathrm{cris}$. We thus get a commutative diagram:

$$\begin{tikzcd}
\mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+}) \arrow[r, "\boldsymbol{\omega}_\infty"] \arrow[d, "\mathrm{sp}_\nu"] & \mathbf{D}\left(\mathbb{I}(\chi_{\mathrm{cyc}})\right) \arrow[d] \\
\mathbf{D}_{\mathrm{dR}}(\mathbf{T}_{\xi_\nu}^{\dagger,+}) \arrow[r, "\boldsymbol{\omega}_\nu"] & \mathbf{D}_{\mathrm{dR}}\left(F_\nu(\chi_{\mathrm{cyc}})\right),
\end{tikzcd}$$

where the map $\boldsymbol{\omega}_\nu$ is given by pairing agains the class $\omega_{\mathcal{F}_\nu} \otimes (\Theta_\nu^{-1}\xi_\nu)$. When ν has weight 2, define the twisted modular form $\mathcal{F}_\nu^* = \mathcal{F}_\nu \otimes \vartheta_\nu^{-1}$ (recall that ϑ_ν is the Dirichlet character of $\mathcal{F}_\nu$ in this case). Then the map $\boldsymbol{\omega}_\nu$ is actually given by pairing with the differential form $\omega_{\mathcal{F}_\nu^*} \otimes \xi_\nu$ associated with $\mathcal{F}_\nu^*$, further twisted by ξ_ν. Fix a compatible sequence $(\zeta_{p^n})_{n\geq 1}$ of p-power roots of unity; so for each integer $n \geq 1$, ζ_{p^n} is a primitive p^n-th root of unity such that $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. This choice defines a generator of $\mathbb{Q}_p(j)$, denoted e_j. Let t denote Fontaine's p-adic analogue of $2\pi i$, defined, *e.g.* in [Kat91, Ch. II, §1.1.15]. Then $\delta_r = t^{-r} \otimes e_r$ is a generator of the 1-dimensional $F_{\nu,\phi}$-vector space $\mathbf{D}_{\mathrm{dR}}\left(F_{\nu,\phi}(\chi_{\mathrm{cyc}}^r)\right)$. Having fixed this basis, the above diagram becomes:

$$\begin{tikzcd} \mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+}) \arrow[r, "\boldsymbol{\omega}_\infty"] \arrow[d, "\mathrm{sp}_\nu"] & \mathbf{D}\left(\mathbb{I}(\chi_{\mathrm{cyc}})\right) \arrow[d] \\ \mathbf{D}_{\mathrm{dR}}(\mathbf{T}_{\xi_\nu}^{\dagger,+}) \arrow[r, "\boldsymbol{\omega}_\nu"] & F_\nu. \end{tikzcd} \tag{7.6}$$

Consider now the Galois group $\mathcal{G}=\mathrm{Gal}(K_\infty/F)$. Let $\phi\colon \mathcal{G}\to\overline{\mathbb{Q}}_p^\times$ be the p-adic avatar of a Hecke character of Archimedean type $(r,-r)$ for an integer $r\in\mathbb{Z}$. Then $\phi\chi_{\mathrm{cyc}}^{-r}$ is unramified at $\mathfrak{p}$. For any $G_{\mathbb{Q}_p}$-representation V, denote $V(\phi)$ the twist of V by ϕ. Fix a basis $\omega_{\phi\chi_{\mathrm{cyc}}^{-r}}$ of the 1-dimensional $F_{\nu,\phi}$-vector space $\mathbf{D}_{\mathrm{dR}}\left(F_{\nu,\phi}(\phi\chi_{\mathrm{cyc}}^{-r})\right)$. One defines (see for example [Cas20, page 2144]) a map $\psi_\phi\colon \mathbb{Z}_p[\![\mathcal{G}]\!]\to\mathbf{D}_{\mathrm{dR}}\left(F_{\nu,\phi}(\phi\chi_{\mathrm{cyc}}^{-r})\right)$ setting $\psi_\phi(\sigma)=(\phi\chi_{\mathrm{cyc}}^{-r})(\sigma)\omega_{\phi\chi_{\mathrm{cyc}}^{-r}}$ on group-like elements. From this we construct a map

$$\mathrm{sp}_{\nu,\phi}\colon \mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![\mathcal{G}]\!]\longrightarrow\mathbf{D}_{\mathrm{dR}}(\mathbf{T}_{\xi_\nu}^{\dagger,+}(\phi))$$

setting $\mathrm{sp}_{\nu,\phi}=\mathrm{sp}_\nu\otimes\psi_\phi(x)\otimes\delta_r$, where we use the description if the right hand side in terms of $\mathbf{D}_{\mathrm{dR}}(\mathbf{T}_{\xi_\nu}^{\dagger,+}\otimes_{F_\nu}F_{\nu,\phi})$, $\mathbf{D}_{\mathrm{dR}}(F_{\nu,\phi}(\phi\chi_{\mathrm{cyc}}^{-r}))$ and $\mathbf{D}_{\mathrm{dR}}(F_{\nu,\phi}(\chi_{\mathrm{cyc}}^{r}))$. We thus get from (7.6) a commutative diagram:

$$\begin{tikzcd} \mathbf{D}(\mathbf{T}_{\boldsymbol{\xi}}^{\dagger,+})\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![\mathcal{G}]\!] \arrow[r, "\boldsymbol{\omega}_\infty"] \arrow[d, "\mathrm{sp}_{\nu,\phi}"] & \mathbf{D}\left(\mathbb{I}(\chi_{\mathrm{cyc}})\right)\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p[\![\mathcal{G}]\!] \arrow[d, "\mathrm{sp}_{\nu,\phi}"] \\ \mathbf{D}_{\mathrm{dR}}(\mathbf{T}_{\xi_\nu}^{\dagger,+}(\phi)) \arrow[r, "\boldsymbol{\omega}_\nu\otimes\phi^{-1}"] & F_{\nu,\phi}. \end{tikzcd} \tag{7.7}$$

7.2. **The big Perrin-Riou map.** Recall that we still denote $\boldsymbol{\xi}\colon G_K\to\mathbb{I}^\times$ the Galois character associated with $\boldsymbol{\xi}\colon K^\times\backslash\widehat{K}^\times\to\mathbb{I}^\times$. Define as before the Galois character $\Psi\colon G_{K_\mathfrak{p}}\to\mathbb{I}^\times$ by $\Psi=\eta_v^{-1}\chi_{\mathrm{cyc}}\Theta\boldsymbol{\xi}^{-1}$. Let $\mathrm{Frob}_\mathfrak{p}\in G_K$ be an arithmetic Frobenius at $\mathfrak{p}$; for each arithmetic character $\nu\colon\mathbb{I}\to\overline{\mathbb{Q}}_p$, let $\Psi_\nu=\nu\circ\Psi$. Since $k\equiv 2 \mod 2(p-1)$ and, identifying Galois and adelic character, accordingly with convenience, $\chi_{\mathrm{cyc}}(i_\mathfrak{p}(p))=\mathbf{N}_K(i_\mathfrak{p}(p)^{-1})(i_\mathfrak{p}(p)^{-1})$, a simple computation shows that $\Psi(\mathrm{Frob}_\mathfrak{p})=\mathbf{a}_p^{-1}\boldsymbol{\xi}(i_\mathfrak{p}(p))$, and therefore $\Psi_\nu(\mathrm{Frob}_\mathfrak{p})=\nu(\mathbf{a}_p)^{-1}\xi_{\nu,\mathfrak{p}}(p)p$.

Set $\mathfrak{j}=\Psi(\mathrm{Frob}_\mathfrak{p})-1\in\mathbb{I}$. Define $\mathcal{J}=(\mathfrak{j},\gamma_{\mathrm{cyc}}-1)$ to be the ideal of $\mathcal{I}$ generated by $\mathfrak{j}$ and $\gamma_{\mathrm{cyc}}-1$, where γ_{cyc} is a fixed topological generator of Γ_{cyc}. Say that an arithmetic morphism $\nu\colon\mathbb{I}\to\overline{\mathbb{Q}}_p$ is *exceptional* if its signature is $(2,\mathbf{1})$, where $\mathbf{1}$ is the trivial character, and $\Psi_\nu(\mathrm{Frob}_\mathfrak{p})=1$, so that $\mathfrak{j}=0$.

Let $\phi\colon\mathcal{G}\to\overline{\mathbb{Q}}_p^\times$ be a character of Hodge–Tate type of Hodge–Tate weight w and conductor p^n for some integer $n\geq 0$. Write $\phi=\chi_{\mathrm{cyc}}^w\phi'$ for some unramified character ϕ'. For each ν we may consider the 1-dimensional (over F_ν) representation $V(\Psi_\nu)=F_\nu(\Psi_\nu)$ and its crystalline Dieudonné module $\mathbf{D}_{\mathrm{cris}}(V(\Psi_\nu))$. The crystalline Frobenius acts then on $\mathbf{D}_{\mathrm{cris}}(V(\Psi_\nu))$ by $\Phi_\nu=\Psi_\nu^{-1}(\mathrm{Frob}_\mathfrak{p})$ ([BC09, Lemma 8.3.3]). Define $\mathcal{E}_p(\phi,\nu)$ by

$$\mathcal{E}_p(\phi,\nu)=\begin{cases}\frac{1-p^w\phi'(\mathrm{Frob}_\mathfrak{p})\Phi_\nu}{1-(p^{w+1}\phi'(\mathrm{Frob}_\mathfrak{p})\Phi_\nu)^{-1}}, \text{ if } n=0,\\ \epsilon(\phi^{-1})\cdot\Phi_\nu^n, \text{ if } n\geq 1,\end{cases}$$

where, for any Hodge–Tate character $\psi\colon \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)\to\overline{\mathbb{Q}}_p^\times$, $\epsilon(\psi)$ is the ϵ-factor of the Weil–Deligne representation $\mathbf{D}_{\mathrm{pst}}(\psi)$; we adopt the convention in [LZ14, §2.8] for ϵ-factors, and we refer to *loc. cit.* for a careful discussion.

If $\phi\colon\mathcal{G}\to\overline{\mathbb{Q}}_p$ is a character of Hodge–Tate type with Hodge–Tate weight $w\leq -1$, then the finite subspace $H^1_f(F,\mathbf{T}_{\xi_\nu}^{\dagger,+}(\phi^{-1}))$ of $H^1(F,\mathbf{T}_{\xi_\nu}^{\dagger,+}(\phi^{-1}))$ coincides with $H^1(F,\mathbf{T}_{\xi_\nu}^{\dagger,+}(\phi^{-1}))$; the

Bloch–Kato logarithm for $V^{\dagger}_{\nu,\phi}$ gives rise to a map

$$\log\colon H^1(F,\mathbf{T}^{\dagger,+}_{\xi_\nu}(\phi^{-1}))\longrightarrow \mathbf{D}_{\mathrm{dR}}(\mathbf{T}^{\dagger,+}_{\xi_\nu}(\phi^{-1})).$$

By [Cas20, Theorem 3.7], there exists an injective $\mathbb{I}[\![\mathcal{G}]\!]$-linear map

$$\mathrm{Log}\colon H^1_{\mathrm{Iw}}(\Gamma_\infty,\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}})\longrightarrow \lambda^{-1}\cdot\mathcal{J}\cdot(\mathbf{D}(\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[\![\mathcal{G}]\!])$$

where $\widehat{\mathcal{O}}_{F_\infty}$ is the completion of the valuation ring $\mathcal{O}_{F_\infty}$ of F_∞, such that for each non-exceptional $\nu\colon\mathbb{I}\to\mathcal{O}_\nu$ and each non-trivial Hodge–Tate character $\phi\colon\mathcal{G}\to L^\times$ of conductor p^n and Hodge–Tate weight $w\leq -1$ as above, the following diagram commutes:

$$\begin{tikzcd} H^1_{\mathrm{Iw}}(\Gamma_\infty,\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}}) \arrow[r, "\mathrm{Log}"] \arrow[d, "\mathrm{sp}_{\nu,\phi^{-1}}"] & \lambda^{-1}\cdot\mathcal{J}\cdot(\mathbf{D}(\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}})\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathcal{O}}_{F_\infty}[\![\mathcal{G}]\!]) \arrow[d, "\mathrm{sp}_{\nu,\phi^{-1}}"] \\ H^1(F,\mathbf{T}^{\dagger,+}_{\xi_\nu}(\phi^{-1}))) \arrow[r, "\frac{(-1)^{-w-1}}{(-w-1)!}\log\mathcal{E}_p(\phi,\nu)"] & \mathbf{D}_{\mathrm{dR}}(\mathbf{T}^{\dagger,+}_{\xi_\nu}(\phi^{-1}))). \end{tikzcd} \tag{7.8}$$

Combining Diagrams (7.8) and (7.7), the argument in [Cas20, Proposition 5.2] shows that exists an injective $\widetilde{\mathbb{I}}_W$-linear map

$$\mathcal{L}^{\Gamma_\infty}_{\omega_{\mathcal{F}_\infty}}\colon H^1_{\mathrm{Iw}}(\Gamma_\infty,\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}})\longrightarrow\widetilde{\mathbb{I}}[\![\Gamma_\infty]\!] \tag{7.9}$$

with pseudo-null kernel and cokernel, such that for all characters $\phi\colon\Gamma_\infty\to\overline{\mathbb{Q}}_p$ of Hodge–Tate type, with Hodge–Tate weight $w\leq -1$ and conductor p^n, all $\mathfrak{Y}\in H^1_{\mathrm{Iw}}(\Gamma_\infty,\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}})$ and all non-exceptional ν we have

$$\mathrm{sp}_{\nu,\phi^{-1}}\left(\mathcal{L}^{\Gamma_\infty}_{\omega_{\mathcal{F}_\infty}}(\mathfrak{Y})\right)=\frac{(-1)^{-w-1}}{(-w-1)!}\cdot\mathcal{E}(\phi,\nu)\cdot(\boldsymbol{\omega}_\nu\otimes\phi)\left(\log(\mathrm{sp}_{\nu,\phi^{-1}}(\mathfrak{Y}))\right). \tag{7.10}$$

7.3. **p-stabilizations.** Let ν_k be an arithmetic morphism corresponding to a p-stabilized form $\mathcal{F}_k=\mathcal{F}_{\nu_k}$ in $M_k(N^+p,\mathcal{O})$, and let $\mathcal{F}^\sharp_k$ be the form in $M_k(N^+,\mathcal{O})$ whose ordinary p-stabilization is $\mathcal{F}_k$. Let $\hat{\xi}_k=\boldsymbol{\xi}_{\nu_k}$ and define the G_K-representation $V^\dagger_{\mathcal{F}^\sharp_k,\xi_k}=V^\dagger_{\mathcal{F}^\sharp_k}\otimes\hat{\xi}_k^{-1}$; let L be the field of definition of $V^\dagger_{\mathcal{F}^\sharp_k,\xi_k}$, $\mathcal{O}_L$ its valuation ring and $\widehat{\mathcal{O}}^{\mathrm{unr}}_L$ the completion of its maximal unramified extension. We consider the Perrin-Riou logarithm map

$$\mathrm{Log}_{\mathcal{F}^\sharp_k,\xi_k}\colon H^1_{\mathrm{Iw}}(\Gamma_\infty,V^{\dagger,+}_{\mathcal{F}^\sharp_k,\xi_k})\longrightarrow\widehat{\mathcal{O}}^{\mathrm{unr}}_L[\![\Gamma_\infty]\!][1/p]\widehat{\otimes}\mathbf{D}_{\mathrm{dR}}(V^{\dagger,+}_{\mathcal{F}^\sharp_k,\xi_k})$$

constructed in this setting in [Mag22, §7.1.1] (see also [CH18a, §5.3, Theorem 5.1]). Let $\mathrm{pr}^*=\mathrm{pr}_1^*-\frac{\mathrm{pr}_2^*}{\alpha}$ and let $\omega_{\mathcal{F}^\sharp_\nu,\phi}$ be such that $\mathrm{pr}^*(\omega_{\mathcal{F}^\sharp_\nu,\phi})=\omega_{\mathcal{F}\nu,\phi}$. Pairing against the class $\omega_{\mathcal{F}^\sharp_\nu,\phi}:=\omega_{\mathcal{F}^\sharp_\nu}\otimes\omega_{\phi^{-1}\chi^r_{\mathrm{cyc}}}\otimes\delta_r$ gives then a map

$$\mathcal{L}^{\Gamma_\infty}_{\mathcal{F}^\sharp_k,\xi_k}\colon H^1_{\mathrm{Iw}}(H_{cp^\infty,\mathfrak{P}},V^{\dagger,+}_{\mathcal{F}^\sharp_k,\xi_k})\longrightarrow\widehat{\mathcal{O}}^{\mathrm{unr}}_L[\![\Gamma_\infty]\!][1/p]. \tag{7.11}$$

Composing the map $\mathcal{L}^{\Gamma_\infty}_{\omega_{\mathcal{F}_\infty}}$ in (7.9) with the specialization map we obtain a second map

$$\mathcal{L}^{\Gamma_\infty}_{\mathcal{F}_k,\xi_k}\colon H^1_{\mathrm{Iw}}(H_{cp^\infty,\mathfrak{P}},\mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}_k})\longrightarrow\widehat{\mathcal{O}}^{\mathrm{unr}}_L[\![\Gamma_\infty]\!][1/p].$$

Lemma 7.3. $\mathcal{L}^{\Gamma_\infty}_{\mathcal{F}_k,\xi_k}=\mathcal{L}^{\Gamma_\infty}_{\mathcal{F}^\sharp_k,\xi_k}\circ\mathrm{pr}_*$.

Proof. Put $\mathrm{Log} = \mathrm{Log}_{\mathcal{F}_k,\xi_k}$, $\mathrm{Log}^\sharp = \mathrm{Log}_{\mathcal{F}_k^\sharp,\xi_k}$, $\omega = \omega_{\mathcal{F}_\nu,\phi}$, $\omega^\sharp = \omega_{\mathcal{F}_\nu^\sharp,\phi}$ to simplify the notation. We have $\langle \mathrm{Log}(x), \omega\rangle_{\mathrm{dR}} = \langle \mathrm{Log}(x), \mathrm{pr}^*(\omega^\sharp)\rangle_{\mathrm{dR}}$ and since pr^* is adjoint to pr_* we also have $\langle \mathrm{Log}(x), \mathrm{pr}^*(\omega^\sharp)\rangle_{\mathrm{dR}} = \langle \mathrm{pr}_*\mathrm{Log}(x), \omega\rangle$. By [Ota20, Corollary 5.8] we have

$$\begin{pmatrix}(\mathrm{pr}_1)_*\\(\mathrm{pr}_2)_*\end{pmatrix} e^{\mathrm{ord}} = \frac{1}{\alpha-\beta}\begin{pmatrix}\alpha(\mathrm{pr}_1)_* - (\mathrm{pr}_2)_*\\ \alpha\beta(\mathrm{pr}_1)_* - \beta(\mathrm{pr}_2)_*\end{pmatrix}.$$

So $\mathrm{pr}_* \circ e^{\mathrm{ord}} = \mathrm{pr}_*$. It follows that $\mathrm{pr}_*\mathrm{Log}(x) = \mathrm{Log}^\sharp(\mathrm{pr}_*(x))$ and the result follows. □

8. p-adic families of Heegner points

8.1. **Families of Heegner points.** Denote $\chi_{\mathrm{cyc}} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ the cyclotomic character, and let $\vartheta \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{p^*})) \to \mathbb{Z}_p^\times/\{\pm 1\}$ be the unique character which satisfies $\vartheta^2 = \chi_{\mathrm{cyc}}$, where $p^* = (-1)^{\frac{p-1}{2}}p$ (see [LV11, §4.4] for details). For integers $n \geq 0$ and $m \geq 1$, define $L_{cp^n,m} = H_{cp^n}(\mu_{p^m})$. Recalling the notation in §2.6, define $\widetilde{P}_{cp^n,m} = \tilde{x}_{cp^n,m}(1)$. These points are known to satisfy the following properties:

1. $\widetilde{P}_{cp^n,m} \in \widetilde{X}_m(L_{cp^n,m})$;
2. $\widetilde{P}^\sigma_{cp^n,m} = \langle\vartheta(\sigma)\rangle \cdot \widetilde{P}_{cp^n,m}$ for all $\sigma \in \mathrm{Gal}(L_{cp^n,m}/H_{cp^{n+m}})$;
3. *Vertical compatibility*: if $m > 1$, then $\sum_\sigma \widetilde{\alpha}_m(\widetilde{P}^\sigma_{cp^n,m}) = U_p \cdot \widetilde{P}_{cp^n,m-1}$, where the sum is over all $\sigma \in \mathrm{Gal}(L_{cp^n,m}/L_{cp^{n-1},m})$ and $\widetilde{\alpha}_m \colon \widetilde{X}_m \to \widetilde{X}_{m-1}$ is the canonical projection map;
4. *Horizontal compatibility*: if $n > 0$, then $\sum_\sigma \widetilde{P}^\sigma_{cp^n,m} = U_p \cdot \widetilde{P}_{cp^{n-1},m}$, where the sum is over all $\sigma \in \mathrm{Gal}(L_{cp^n,m}/L_{cp^{n-1},m})$.

Remark 8.1. See [CL16, Theorem 1.2] for a proof of the above properties; in *loc.cit* only the case of definite quaternion algebras and $c = 1$ is treated, but it is easy to see that the proof, which combines results in [LV11] and the description of optimal embeddings in [CH15], works in this generality as well.

8.2. **Big Heegner points.** Fix an integer $c \geq 1$ prime to $D_K N p$. Recall from §2.6 the family of points $\widetilde{P}_{cp^{n+m},m}$ in $\widetilde{X}_m\left(L_{cp^{n+m},m}\right)$ and from §6.2 the Jacobian variety $\widetilde{J}_m$ of $\widetilde{X}_m$. Write $\mathbb{Z}_p^\times = \Delta \times (1 + p\mathbb{Z}_p)$ with $\Delta \cong (\mathbb{Z}/p\mathbb{Z})^\times$ and let e_{k-2} denote the projector

$$e_{k-2} = \frac{1}{p-1}\sum_{\delta\in\Delta} \omega^{-(k-2)}(\delta)[\delta] \in \mathbb{Z}_p[\mathbb{Z}_p^\times].$$

By [LV11, (42)], $\Theta(\sigma) = \langle\vartheta(\sigma)\rangle$ for all $\sigma \in \mathrm{Gal}(L_{cp^{n+m},m}/H_{cp^{n+m}})$, as endomorphisms of $(e_{k-2} \cdot e^{\mathrm{ord}}) \cdot \widetilde{J}_m(L_{cp^{n+m},m})$, and therefore, using that U_p has degree p (*cf.* [LV11, §6.2]), projecting to the ordinary submodule gives points

$$\mathbf{P}_{cp^{n+m},m} = (e_{k-2} \cdot e^{\mathrm{ord}}) \cdot \widetilde{P}_{cp^{n+m,m}} \in H^0\left(H_{cp^{n+m}}, \widetilde{J}^{\mathrm{ord}}_m(L_{cp^{n+m},m})^\dagger\right),$$

where $\widetilde{J}^{\mathrm{ord}}_m(L) = e^{\mathrm{ord}} \cdot \widetilde{J}_m(L)$ for any extension $L/\mathbb{Q}$, and for any $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module M, we denote $M^\dagger$ the Galois module $M \otimes \Theta^{-1}$. Corestricting from $H_{cp^{n+m}}$ to H_{cp^n}, we obtain classes

$$\mathcal{P}_{cp^n,m} \in H^0\left(H_{cp^n}, \widetilde{J}^{\mathrm{ord}}_m(L_{cp^{n+m},m})^\dagger\right).$$

Composing the (twisted) Kummer map we obtain classes $\mathfrak{X}_{cp^n,m}$ in $H^1(H_{cp^n}, \mathrm{Ta}^{\mathrm{ord}}_p(\tilde{J}_m)^\dagger)$ (where $\mathrm{Ta}^{\mathrm{ord}}_p(\tilde{J}_m) = e^{\mathrm{ord}}\,\mathrm{Ta}_p(\tilde{J}_m)$) and then, using the trace-compatibility properties enjoyed by the collection of points $\widetilde{P}_{cp^{n+m},m}$ recalled in §2.6, we may define a class

$$\mathfrak{X}_{cp^n} = \varprojlim_m U_p^{-m}\mathfrak{X}_{cp^n,m} \in H^1(H_{cp^n}, \mathbf{T}^\dagger).$$

Under the assumption that p does not divide the class number of K, using the properties of the points $\widetilde{P}_{cp^{n+m},m}$ once again, we also may define Iwasawa classes

$$\mathfrak{X}_{cp^\infty} = \varprojlim_n U_p^{-n}\mathfrak{X}_{cp^n} \in H^1_{\mathrm{Iw}}(H_{cp^\infty}/H_c, \mathbf{T}^\dagger) := \varprojlim_{n\geq 0} H^1(H_{cp^n}, \mathbf{T}^\dagger),$$

where the inverse limit is taken with respect to the corestriction maps. Since $\mathfrak{P}$ in totally ramified in the extension H_{cp^∞}/H_c, we have $\mathrm{Gal}(H_{cp^\infty}/H_c) \cong \Gamma_\infty$, so we can write

$$\mathfrak{X}_{cp^\infty} \in H^1_{\mathrm{Iw}}(\Gamma_\infty, \mathbf{T}^\dagger).$$

Recall now the notation fixed in §6.3, and let $\boldsymbol{\xi}$ be the character of conductor c constructed in *loc. cit.* We may thus consider the class

$$\mathfrak{X}_{\boldsymbol{\xi}} := \mathfrak{X}_{cp^\infty} \otimes \boldsymbol{\xi}^{-1} \in H^1_{\mathrm{Iw}}(\Gamma_\infty, \mathbf{T}^\dagger_{\boldsymbol{\xi}}).$$

Let $\widetilde{\Gamma}_\infty = \mathrm{Gal}(H_{cp^\infty}/K)$. Taking corestriction we get a class

$$\mathfrak{Z}_{\boldsymbol{\xi}} = \mathrm{cor}_{H_c/K}(\mathfrak{X}_{\boldsymbol{\xi}}) \in H^1_{\mathrm{Iw}}(\widetilde{\Gamma}_\infty, \mathbf{T}^\dagger_{\boldsymbol{\xi}}) := \varprojlim_{n\geq -1} H^1(H_{cp^n}, \mathbf{T}^\dagger_{\boldsymbol{\xi}}) \tag{8.1}$$

where $H_{cp^{-1}} := K$. Under the condition that the residual Galois representation $\bar{\rho}$ attached to the Hida family f_∞ is ramified at all primes dividing N^-, one can prove that $\mathfrak{X}_{cp^n}$ belongs to the Greenberg Selmer group (see [CW22, Proposition 4.5]).

8.3. **Geometric p-adic L-function attached to big Heegner points.** Recall the big Perrin-Riou map $\mathcal{L}^{\Gamma_\infty}_{\omega_{\mathcal{F}_\infty}}$ in (7.9) and define $\mathcal{L}^{\widetilde{\Gamma}_\infty}_{\omega_{\mathcal{F}_\infty}} = \varrho \circ \mathcal{L}^{\Gamma_\infty}_{\omega_{\mathcal{F}_\infty}}$, where $\varrho\colon \widetilde{\mathbb{I}}[\![\Gamma_\infty]\!] \to \widetilde{\mathbb{I}}[\![\widetilde{\Gamma}_\infty]\!]$ is the map arising from the canonical map $\Gamma_\infty \hookrightarrow \widetilde{\Gamma}_\infty$. Since p is split in K, $\mathrm{res}_{\mathfrak{P}}(\mathfrak{X}_{\boldsymbol{\xi}})$ belongs to $H^1_{\mathrm{Iw}}(\Gamma_\infty, \mathbf{T}^{\dagger,+}_{\boldsymbol{\xi}})$ by [How07, Proposition 2.4.5], so the following definitions make sense.

Definition 8.2. $\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}} = \mathcal{L}^{\widetilde{\Gamma}_\infty}_{\omega_{\mathcal{F}_\infty}}(\mathrm{res}_{\mathfrak{P}}(\mathfrak{Z}_{\boldsymbol{\xi}}))$ is the *geometric* anticyclotomic p-adic L-function attached to the family $\mathcal{F}_\infty$.

9. Reciprocity law for big Heegner points

The goal of this section is to derive an explicit reciprocity law for higher weight specialization of big Heegner points using a reciprocity law for weight 2 specializations; this strategy has been successively used in a series of paper ([Cas20], [Cas13], [CL16]).

9.1. **Coleman integration on Shimura curves.**

9.1.1. *Rigid analytic Shimura curves.* Recall the Shimura curves $\widetilde{\mathcal{X}}_m$ for integers $m \geq 0$, viewed as $\mathbb{Z}_p$-schemes, and denote $\widetilde{X}^{\mathrm{rig}}_m$ the rigid analytic space over $\mathbb{Q}_p$ associated with $\widetilde{\mathcal{X}}_m$; if $m = 0$ we simply write X^{rig}_0 for $\widetilde{X}^{\mathrm{rig}}_0$. Also recall that we denote $\mathcal{X}^{\mathrm{ord}}_0 = \mathcal{X}_0[1/\widetilde{\mathbf{Ha}}]$ the ordinary locus of $\mathcal{X}_0$: the rigid analytic space $X^{\mathrm{ord,rig}}_0$ associated with $\mathcal{X}^{\mathrm{ord}}_0$ is the complement in X^{rig}_0 of residue disks D_x corresponding to supersingular points x in the special fiber of $\mathcal{X}^{\mathrm{ord}}_0$ (we refer *e.g.* to [Buz97, §3] for the notion of *supersingular* abelian surface with quaternionic multiplication).

For any real number $0 \leq \varepsilon < 1$, denote $\mathcal{X}^-_0(\varepsilon)$ the open rigid analytic subspace of X^{rig}_0 defined by the condition $|\widetilde{\mathbf{Ha}}| > |p|^\varepsilon$; we view $\mathcal{X}^-_0(\varepsilon)$ as defined over any field extension $L/\mathbb{Q}_p$ in which there exists an element $x \in L$ with $|x| = |p|^\varepsilon$. For any integer $m \geq 1$, let $\varepsilon_m = \frac{1}{p^{m-2}(p-1)}$; then $\mathcal{X}^-_0(\varepsilon_m)$ is defined over $\mathbb{Q}_p(\zeta_{p^m})$, and later we will adopt the same symbol for their base change to finite field extensions L of the cyclotomic field $\mathbb{Q}_p(\zeta_{p^m})$. By [Bra13, Proposition 6.30], any point $x = (A, \iota, \alpha)$ in $\mathcal{X}^-_0(\varepsilon_m)$ admits a *canonical subgroup* $C_{p^m} \subseteq A[p^m]$ of order p^{2m} (see [Bra13, §3] for the notion of canonical subgroup in this setting; see also [Kas04, §10] and [Sch15, §3.2] for related results).

Let $\alpha_m : \widetilde{\mathcal{X}}_m \to \mathcal{X}_0$ denote the forgetful map. Define $\mathcal{W}_1(p^m)$ (respectively, $\mathcal{W}_2(p^m)$) to be the open rigid analytic subspace of $\widetilde{X}_m^{\mathrm{rig}}$ whose closed points corresponds to QM abelian surfaces with level structure $x = (A, \iota, \alpha, \beta)$ where:

- (A, ι) is a QM abelian surface equipped with a $V_0(N^+)$-structure α;
- $\beta : \boldsymbol{\mu}_{p^m} \to eC_{p^m}$ is an isomorphism, where as before, we indicate $C_{p^m} \subseteq A[p^m]$ the canonical subgroup of A of order p^{2m}; thus, $\beta(\zeta_{p^m})$ is a generator of eC_{p^m};
- $\alpha_m(x)$ belongs to $\mathcal{X}_0^-(\varepsilon_m)$ (respectively, $\mathcal{X}_0^-(\varepsilon_{m+1})$).

We thus have a chain of inclusions of rigid analytic spaces $\mathcal{W}_1(p^m) \subseteq \mathcal{W}_2(p^m) \subseteq X_m^{\mathrm{rig}}$.

The *Deligne-Tate map* $\phi : \mathcal{X}_0^-(\varepsilon_{m+1}) \to \mathcal{X}_0^-(\varepsilon_m)$ is defined by taking quotients by the canonical subgroup, *i.e.* we put $\phi(A, \iota, \alpha) = (A_0, \iota_0, \alpha_0)$ where $A_0 = A/C_p$, and $C_p \subseteq A[p]$ denotes as before the canonical subgroup of A of order p^2, and if $\phi : A \to A/C_p$ is the canonical isogeny, ι_0 is the polarization induced by ι and ϕ, and α_0 is the $V_1(N^+)$-level structure induced by α and ϕ. The map $\bar{\phi}$ induced by ϕ on the special fibers of $\mathcal{X}_0^-(\varepsilon_{m+1})$ and $\mathcal{X}_0^-(\varepsilon_m)$ coincides with the Frobenius map Frob_p, and so $\phi : \mathcal{X}_0^-(\varepsilon_{m+1}) \to \mathcal{X}_0^-(\varepsilon_m)$ is also called *Frobenius map.* The map ϕ thus obtained can be lifted to a map (denoted with the same symbol and also called *Frobenius map*)

$$\phi : \mathcal{W}_2(p^m) \longrightarrow \mathcal{W}_1(p^m) \tag{9.1}$$

setting $\phi(A, \iota, \alpha, \beta) = (A_0, \iota_0, \alpha_0, \beta_0)$ where $\beta_0 : \mu_{p^m} \to A/C_p$ sends ζ_{p^m} to $\phi(P_{m+1})$ where $P_{m+1} \in C_{p^{m+1}}$ satisfies $pP_{m+1} = P_m = \beta(\zeta_{p^m})$.

9.1.2. *Semistable models and rigid de Rham cohomology.* We denote by $\mathcal{Y}_m$ the proper, flat, regular balanced model of $\widetilde{\mathcal{X}}_m$ over $\mathbb{Z}[\zeta_{p^m}]$. The special fiber of $\mathcal{Y}_m$ is the union of a finite number of reduced Igusa curves over $\mathbb{F}_p$, meeting at their supersingular points, and two of these components, denoted Ig_∞ and Ig_0, are isomorphic to the Igusa curve $\mathrm{Ig}_{m,1}$ over $\mathbb{F}_p$; we let Ig_∞ denote the connected component corresponding to the canonical inclusion of $\mathrm{Ig}_{m,1}$ into $\mathcal{Y}_m \otimes_{\mathbb{Z}[\mu_{p^m}]} \mathbb{F}_p$. We have an involution $w_{\zeta_{p^m}}$ attached to the chosen p^n-root of unity ζ_{p^m} which interchanges the two components Ig_∞ and Ig_0 (see [Mor81] and its generalization to totally real fields in [Car86]).

Remark 9.1. The results of Carayol [Car86] formally exclude the case under consideration when the fixed totally real number field F is equal to $\mathbb{Q}$, but refers to the paper of Morita [Mor81] for this case. A proof of these facts can also be obtained by a direct generalization of the arguments in [Buz97, Theorem 4.10] which considers the case $m = 1$.

Let L be a finite extension of $\mathbb{Q}_p(\zeta_{p^m})$ where $\mathcal{Y}_m$ acquires semistable reduction. Let $\mathcal{O}_L$ be the valuation ring of M and k_L its residue field. We denote $\varpi : \mathscr{Y}_m \to \mathcal{Y}_m \otimes_{\mathbb{Z}_p[\zeta_{p^m}]} \mathcal{O}_L$ a semistable model of $\mathcal{Y}_m$ over $\mathcal{O}_L$. Let $\mathcal{G}_m$ denote the dual graph of the special fiber $\mathbb{Y}_m$ of $\mathscr{Y}_m$; the set $\mathcal{V}(\mathcal{G}_m)$ of vertices of $\mathcal{G}_m$ is in bijection with the irreducible components of the special fiber $\mathbb{Y}_m$ of $\mathscr{Y}_m$, and the set $\mathcal{E}(\mathcal{G}_m)$ of oriented edges of $\mathcal{G}$ is in bijection with the singular points of $\mathbb{Y}_m$, together with an ordering of the two components which intersect at that point. Given $v \in \mathcal{V}(\mathcal{G}_m)$, let $\mathbb{Y}_v$ denote the associated component in $\mathbb{Y}_m$, and let $\mathbb{Y}_v^{\mathrm{sm}}$ denote the smooth locus of $\mathbb{Y}_v$. Let $\mathrm{red} : \mathscr{Y}_m(\mathbb{C}_p) \to \mathbb{Y}_m(\overline{\mathbb{F}}_p)$ be the canonical reduction map. For any $v \in \mathcal{V}(\mathcal{G}_m)$, let $\mathcal{W}_v = \mathrm{red}^{-1}(\mathbb{Y}_m(\overline{\mathbb{F}}_p)$ denote the wide open space associated with the connected component $\mathbb{Y}_v$, and let $\mathcal{A}_v = \mathrm{red}^{-1}(\mathbb{Y}_m^{\mathrm{sm}}(\overline{\mathbb{F}}_p)$ denote the underlying affinoid $\mathcal{A}_v \subseteq \mathcal{W}_v$. If $e = (s(e), t(e)) \in \mathcal{E}(\mathcal{G}_m)$ is a edge, then $\mathcal{W}_e = \mathcal{W}_{s(e)} \cap \mathcal{W}_{t(e)}$ is equal to $\mathrm{red}^{-1}(\{x_e\})$, where $\{x_e\} = \mathbb{Y}_{s(e)} \cap \mathbb{Y}_{t(e)}$. The set $\{\mathcal{W}_v : v \in \mathcal{V}(\mathcal{G}_m)\}$ form an *admissible cover* of the rigid analytic space $\mathscr{Y}_m(\mathbb{C}_p) = \widetilde{X}_m(\mathbb{C}_p)$ by wide open subsets. Let $d : \mathcal{O}(\mathcal{V}) \to \Omega^1_{\mathrm{rig}}(\mathcal{V})$ be the differential map for any wide open $\mathcal{V}$, where $\mathcal{O} = \mathcal{O}^{\mathrm{rig}}_{\mathscr{Y}_m}$ is the sheaf of rigid analytic functions on $\mathscr{Y}_m$ and Ω^1_{rig} the sheaf of rigid 1-forms; the de Rham cohomology group can be described as the set of hyper-cocycles

$$(\omega = \{\omega_v\}_{v\in\mathcal{V}(\mathcal{G}_m)}, \{f_e\}_{e\in\mathcal{E}(\mathcal{G}_m)}) \in \prod_{v\in\mathcal{V}(\mathcal{G}_m)} \Omega^1(\mathcal{W}_v) \times \prod_{e\in\mathcal{E}(\mathcal{G}_m)} \mathcal{O}_{\mathcal{W}_e}$$

such that $df_e = \omega_{t(e)} - \omega_{s(e)}$ and $f_{\bar{e}} = -f_e$ for each $e = (s(e), t(e) \in \mathcal{E}(\mathcal{G}_m)$ (where for each $e = (s(e), t(e))$, we let $\bar{e} = (t(e), s(e)))$ modulo hyper-coboundaries, which are elements of the form $(df_v, f_{t(e)} - f_{s(e)})$ for a set $\{f_v\}_{v\in\mathcal{V}(\mathcal{G}_m)}$ of functions $f_v \in \mathcal{O}_{\mathcal{W}_v}$. For each edge $e = (s(e), t(e))$, we have an *annular residue map* $\mathrm{res}_{\mathcal{W}_e} : \Omega^1_{\mathscr{Y}_m}(\mathcal{W}_e) \to \mathbb{C}_p$ defined by expanding a differential form $\omega \in \Omega^1_{\mathscr{Y}_m}(\mathcal{V})$ as $\omega = \sum_{n\in\mathbb{Z}} a_n t^n dt$ for a fixed uniformizing parameter t on $\mathcal{W}_e$ and setting $\mathrm{res}_{\mathcal{W}_e}(\omega) = a_{-1}$. We say that a class $\omega \in H^1_{\mathrm{dR}}(\mathscr{Y}_m)$ is *pure* if it has vanishing annular residues for all $v \in \mathcal{V}(\mathcal{G}_m)$. For pure classes $\omega = (\omega_v, f_g)$, $\eta = (\eta_v, g_v)$ the de Rham pairing $\langle\omega, \eta\rangle_{\mathrm{dR}}$ is computed by the formula

$$\langle\omega, \eta\rangle_{\mathrm{dR}} = \sum_{e=(s(e),t(e))\in\mathcal{E}(\mathcal{G}_m)} \mathrm{res}_{\mathcal{W}_e}(F_e \eta_{s(e)}) \tag{9.2}$$

where F_e is an analytic primitive of the restriction to $\mathcal{W}_e$ of $\omega_{s(e)}$, which exists because ω_v has vanishing annular residues for all $v \in \mathcal{V}(\mathcal{G}_m)$, and is well defined up to a constant (and since η_v has also vanishing annular residue at v, the value of the pairing is independent of this choice). See [CI10, §3.5] (or [DR17, §3.1]) for more details.

The birational map $\varpi : \mathscr{Y}_m \to \mathcal{Y}_m \otimes_{\mathbb{Z}_p[\zeta_{p^m}]} \mathcal{O}_L$ induces an isomorphism between the generic fibers; it also induces an isomorphism between two of the components of the special fiber $\mathbb{Y}_m$ of $\mathscr{Y}_m$ with $\mathrm{Ig}_\infty \otimes_{\mathbb{F}_p} k_L$ and $\mathrm{Ig}_0 \otimes_{\mathbb{F}_p} k_L$: we denote $\mathtt{Ig}_\infty$ and $\mathtt{Ig}_0$ these two components of $\mathbb{Y}_m$. Let $\mathcal{W}_\infty(p^m) = \mathrm{red}^{-1}(\mathtt{Ig}_\infty)$ and $\mathcal{W}_0(p^m) = \mathrm{red}^{-1}(\mathtt{Ig}_0)$ be the corresponding wide open subsets with associated underlying affinoids $\mathcal{A}_\infty(p^m)$ and $\mathcal{A}_0(p^m)$, respectively. The L-valued points of the rigid anaytic space $\mathcal{A}_\infty(p^m)$ are in bijection with quadruplets $(A, \iota, \alpha, \beta)$ where (A, ι) is a QM abelian surface, α is a $V_0(N^+)$-structure and $\beta : \boldsymbol{\mu}_{p^m} \to eC_{p^m}$ is an isomorphism (as before, we indicate $C_m \subseteq A[p^m]$ the canonical subgroup of A of order p^{2m}). The L-vector spaces

$$H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^m)) = \frac{\Omega^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^m))}{d\mathcal{O}_{\mathcal{W}_\infty(p^m)}},$$

$$H^1_{\mathrm{rig}}(\mathcal{W}_0(p^m)) = \frac{\Omega^1_{\mathrm{rig}}(\mathcal{W}_0(p^m))}{d\mathcal{O}_{\mathcal{W}_0(p^m)}}$$

are equipped with a canonical action of Hecke operators T_ℓ for primes $\ell \nmid Np$, and with canonical L-linear Frobenius endomorphisms defined by choosing characteristic zero lifts Φ_∞ and Φ_0 of the Frobenius endomorphism in characteristic p to a system of wide open neighborhoods of the affinoids $\mathcal{A}_\infty(p^m)$ in $\mathcal{W}_\infty(p^m)$ and $\mathcal{A}_0(p^m)$ in $\mathcal{W}_0(p^m)$, respectively. In the case of Shimura curves, we take $\Phi_\infty = \phi$ and $\Phi_0 = w_{\zeta_p^m} \circ \Phi_\infty \circ w_{\zeta_{p^m}}$, where $w_{\zeta_{p^m}}$ is the Atkin-Lehner involution associated with the choice of ζ_{p^m} which interchanges the two wide opens $\mathcal{W}_\infty(p^m)$ and $\mathcal{W}_0(p^m)$.

Let

$$\mathrm{res}_{\mathcal{W}} : H^1_{\mathrm{dR}}(\mathscr{Y}_m) \longrightarrow H^1_{\mathrm{rig}}(\mathcal{W}) = \frac{\Omega^1_{\mathrm{rig}}(\mathcal{W})}{d\mathcal{O}_{\mathcal{W}}}$$

be the restriction map, where $\mathcal{W}$ is an admissible wide open space obtained as inverse image via the reduction map of an irreducible component of the special fiber of $\mathcal{Y}_m$; in particular we have the two maps $\mathrm{res}_\infty = \mathrm{res}_{\mathcal{W}_\infty(p^m)}$ and $\mathrm{res}_0 = \mathrm{res}_{\mathcal{W}_0(p^m)}$. Let $H^1_{\mathrm{dR}}(\mathscr{Y}_m)^{\mathrm{prim}}$ be the subspace of the de Rham cohomology of $\mathscr{Y}_m$ associated with the primitive subspace of the L-vector space of modular forms of weight 2 and level N^+p^m, and $H^1_{\mathrm{rig}}(\mathcal{W})^{\mathrm{pure}}$ is the subspace generated by pure classes of rigid differentials (*i.e.* those classes with vanishing annular residues, as before), for $\mathcal{W} = \mathcal{W}_\infty(p^m)$ and $\mathcal{W} = \mathcal{W}_0(p^m)$.

Proposition 9.2. *The restriction maps* res_∞ *and* res_0 *induce an isomorphism of* L*-vector spaces*

$$\mathrm{res} = \mathrm{res}_\infty \oplus \mathrm{res}_0 : H^1_{\mathrm{dR}}(\mathscr{Y}_m)^{\mathrm{prim}} \simeq H^1_{\mathrm{rig}}(\mathcal{W}_\infty(p^m))^{\mathrm{pure}} \oplus H^1_{\mathrm{rig}}(\mathcal{W}_0(p^m))^{\mathrm{pure}}$$

which is equivariant with respect to the action of Hecke operators T_ℓ *for* $\ell \nmid Np$ *on both sides, the Frobenius endomorphism* Φ *acting on the LHS and the Frobenius endomorphism* (Φ_∞, Φ_0) *acting on the RHS.*

Proof. The proof of these results can be obtained as in [BE10, §4.4] using a generalization of [Col97, Theorem 2.1] to the case of Shimura curves. This generalization does not present technical difficulties and is left to the interested reader. □

Fix a finite set of points S of $\mathscr{Y}_m(\mathbb{C}_p)$ which reduce to smooth points in $\mathbb{Y}_m(\overline{\mathbb{F}}_p)$. The residue disk D_Q of each $Q \in S$ (defined as the set of points of $\mathscr{Y}_m(\mathbb{C}_p)$ whose reduction is equal to the reduction of Q) is conformal to the open unit disk $D \subseteq \mathbb{C}_p$ because $\mathrm{red}(Q)$ is smooth, and we may fix an isomorphism $\varphi_Q : D_Q \xrightarrow{\sim} D$ of rigid analytic space which takes Q to 0. For each $Q \in S$, fix a real number $r_Q < 1$ which belongs to the set $\{|p|^m : m \in \mathbb{Q}\}$. Let $\mathcal{V}_Q \subseteq D_Q$ be the annulus consisting of points $x \in D_Q$ such that $r_Q < |\varphi_Q(x)|_p < 1$; define the *orientation* of $\mathcal{V}_Q$ by choosing the subset $\{x \in D_Q : |\varphi(x)|_p \leq r_Q\}$ of the set $D_Q - \mathcal{V}_Q$, which consists in two connected components. We may then consider the affinoid

$$\mathcal{A}_S = \mathscr{Y}_m(\mathbb{C}_p) - \bigcup_{Q \in S} D_Q$$

and the wide open neighborhood

$$\mathcal{W}_S = \mathcal{A}_S \cup \bigcup_{Q \in S} \mathcal{V}_Q$$

of $\mathcal{A}_S$, so that $\mathcal{A}_S$ is the underlying affinoid of $\mathcal{W}_S$. We also put

$$\widetilde{\mathcal{W}}_\infty = \mathcal{W}_\infty(p^m) - \bigcup_{Q \in S} (D_Q - \mathcal{V}_Q),$$

$$\widetilde{\mathcal{W}}_0 = \mathcal{W}_0(p^m) - \bigcup_{Q \in S} (D_Q - \mathcal{V}_Q).$$

For a Hecke module M, denote $M[\mathcal{F}]$ the eigencompnent corresponding to an eigenform $\mathcal{F}$. Let $Y_S = \mathscr{Y}_m - S$ and let $\mathcal{F}$ be a weight 2 newform on $\widetilde{X}_m$. An excision argument from Proposition 9.2 shows that the canonical restriction map $\mathrm{res} = (\mathrm{res}_0, \mathrm{res}_\infty)$ induces an isomorphism

$$\mathrm{res} : H^1_{\mathrm{dR}}(Y_S/L)[\mathcal{F}] \xrightarrow{\simeq} H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)[\mathcal{F}] \oplus H^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)[\mathcal{F}]. \tag{9.3}$$

Moreover, again from Proposition 9.2, a class in $H^1_{\mathrm{dR}}(Y_S/L)[\mathcal{F}]$ is the restriction of a class of $H^1_{\mathrm{dR}}(\mathscr{Y}_m)$ if and only if it can be represented by a pair of differentials $\tilde{\omega}_\infty \in \Omega^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)$ and $\tilde{\omega}_0 \in \Omega^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)$ with vanishing annular residues. If ω and η are classes in $H^1_{\mathrm{dR}}(\mathscr{Y}_m)^{\mathrm{prim}}$, denote $\omega_\infty = \mathrm{res}_\infty(\omega)$, $\omega_0 = \mathrm{res}_0(\omega)$, $\eta_\infty = \mathrm{res}_\infty(\eta)$, $\eta_0 = \mathrm{res}_0(\eta)$. Let $F_{\infty|\mathcal{V}_Q}$ be any solution of the differential equation $dF = \omega_\infty$ on $\mathcal{V}_Q$, and let $F_{0|\mathcal{V}_Q}$ be any solution of the differential equation $dF = \omega_0$ on $\mathcal{V}_Q$. It follows from (9.2) that for each $\omega, \eta \in H^1_{\mathrm{dR}}(\mathscr{Y}_m)[\mathcal{F}]$ we have

$$\langle \eta, \omega \rangle_{\mathrm{dR}} = \sum_{\mathcal{V} \subseteq \widetilde{\mathcal{W}}_\infty} \mathrm{res}_{\mathcal{V}}(F_{\infty|\mathcal{V}} \cdot \eta_{\infty|\mathcal{V}}) + \sum_{\mathcal{V} \subseteq \widetilde{\mathcal{W}}_0} \mathrm{res}_{\mathcal{V}}(F_{0|\mathcal{V}} \cdot \eta_{0|\mathcal{V}}) \tag{9.4}$$

where the sum is over all annuli $\mathcal{V}$.

9.1.3. *Coleman primitives.* Fix a $\mathbb{Z}_p$-algebra R and an ordinary modular form $\mathcal{F}$ of weight 2 in $M_2(N^+, p^m, R)$, and recall that we denote $\omega_\mathcal{F} \in H^0(\widetilde{X}_m(R), \underline{\omega}_{m,R}^{\otimes 2})$ the global section corresponding to $\mathcal{F}$. Let $x = (\mathcal{A}, \iota, \alpha, P)$ be a point of $\widetilde{\mathcal{X}}_m$ which reduces to a smooth point $\bar{x} = (A, \bar{\iota}, \bar{\alpha})$ in the special fiber of $\mathcal{X}_0$, where $A = \mathcal{A} \otimes_R k$ and the polarization $\bar{\iota}$ and the level structure $\bar{\alpha}$ and induced by ι and α, respectively (here $\mathcal{A}$ is defined over R and k is the residue filed; we write simply P for (H, P)). Fix a $\mathbb{Z}_p$-basis $\{x_A, x'_A\}$ of $\mathrm{Ta}_p(A)$ such that x_A is a $\mathbb{Z}_p$-basis of $e\,\mathrm{Ta}_p(A)$ and $ex'_A = 0$. As in §3.3, we consider the formal differential form $\hat{\omega}_x := \omega_{x_A}$ obtained by pull-back of dT/T along the map $\widehat{\mathscr{A}_{\bar{x}}} \to \mathbb{G}_m$, where $\widehat{\mathscr{A}_{\bar{x}}}$ is the formal group associated with the universal object $\mathscr{A}_{\bar{x}}$ (see §4.2). Let D_x be the residue disk of $\bar{x}$ in $\widetilde{\mathcal{X}}_m$, defined to be the set of points of the associated rigid analytic space whose reduction is equal to $\bar{x}$. Using the Serre-Tate coordinates around A associated with the choice of the basis $\{x_A, x'_A\}$, we may write on D_x

$$\omega_\mathcal{F} = \boldsymbol{\mathcal{F}}(T_x)\hat{\omega}_x. \tag{9.5}$$

We further simplify the notation and write (B, t) with $t = (\iota, \alpha)$ for a test object where B is an ordinary QM abelian surface over a $\mathbb{Z}_p$-algebra R which reduces to A, equipped with a principal polarization ι and la $U_0(N^+)$-level structure α; we also let (B, t, P) be a test object in which (B, t) is as before and defined over a p-adic ring R, and P is a p^m-torsion point in $C_{p^m} \subseteq B[p^m]$. Let finally $D_x^\phi = \phi(D_x)$ be the residue disk in $\widetilde{\mathcal{X}}_m$ of $\bar{\phi}(x) = (A^{\mathrm{Frob}}, t')$ with $t' = (\iota', \alpha')$, where the polarization ι' and the level structure α' are induced by ι and α and the Frobenius map $\mathrm{Frob} = \bar{\phi}$.

Lemma 9.3. $\phi^*(\omega_\mathcal{F}) = p\omega_{V\mathcal{F}}$.

Proof. The operator V is described by the formula (using the previous notation)

$$V\mathcal{F}(B, t, P) = \mathcal{F}(B_0, t_0, P_0)$$

where:

- $B_0 = B/C_p$ is the quotient by the canonical subgroup;
- $t_0 = (\iota_0, \alpha_0, P_0)$ where ι_0 and α_0 are induced by the quotient map $\phi : B \to B_0$ from ι ad α respectively, and P_0 is the p-th root of P in $C_{p^{m+1}}$.

From this and (9.5) we thus have

$$\phi^*(\omega_\mathcal{F}) = (V\boldsymbol{\mathcal{F}})(T_x)\phi^*(\hat{\omega}_{\bar{\phi}(x)}).$$

On the other hand, $\phi^*(\hat{\omega}_{\bar{\phi}(x)}) = p\omega_x$ by [Kat81, Lemma 3.5.1] (see also [HB15, Lemmas 4.4, 4.11]), concluding the proof. ☐

Let a_p denote its U_p-eigenvalue of $\mathcal{F}$ and define the polynomial $L(X) = 1 - \frac{a_p}{p}X$.

Proposition 9.4. (1) *There exists a locally analytic function F_∞ on $\mathcal{W}_\infty(p^m)$, unique up to a constant, such that $dF_\infty = \omega_\mathcal{F}$ on $\mathcal{W}_\infty(p^m)$ and $L(\phi^*)F_\infty$ is a rigid analytic function on a wide open neighborhood $\mathcal{W}_\infty$ of $\mathcal{A}_\infty(p^m)$ contained in $\mathcal{W}_\infty(p^m)$.*

(2) *Let $\tilde{\phi} = w_{\zeta_{p^m}} \circ \phi \circ w_{\zeta_{p^m}}$. There exists a locally analytic function F_0 on $\mathcal{W}_0(p^m)$, unique up to a constant, such that $dF_0 = \omega_\mathcal{F}$ on $\mathcal{W}_0(p^m)$ and $L(\tilde{\phi}^*)F_0$ is a rigid analytic function on a wide-open neighborhood $\mathcal{W}_0$ of $w_{\zeta_{p^m}}\widetilde{X}_m(0)$ in $\mathcal{W}_0$.*

Proof. (1) In $\mathcal{W}_\infty = \phi^{-1}(\mathcal{W}_\infty(p^m) \cap \mathcal{W}_1(p^m))$ we have $L(\phi^*)\omega_\mathcal{F} = 0$ by Lemma 9.3; moreover, $L(\phi^*)$ induces an isomorphism of the sheaf of locally analytic functions on $\mathcal{W}_\infty(p^m)$ because the (complex) absolute value of a_p is $p^{1/2}$. Then (1) follows from [Col94, Theorem 8.1], using [Kat73, Proposition 3.1.2] (see also [CI10, Lemma 5.1]) to check the condition on regular singular annuli. For (2), apply (1) to $w_{\zeta_p^m}\omega_\mathcal{F}$. ☐

Definition 9.5. The functions F_∞ and F_0 in Proposition 9.4 are the *Coleman primitives* of $\mathcal{F}$ on $W_\infty(p^m)$ and $W_0(p^m)$, respectively.

Note that (1) of Proposition 9.4 says that $L(\phi^*)F_\infty$ is overconvergent. More precisely, for any integer $m \geq 1$ and any real number $0 \leq \varepsilon < \varepsilon_m$, let $\mathcal{X}_m(\varepsilon)$ denote the affinoid subdomain of $\mathcal{W}_1(p^m)$ consisting of those points x such that $|\widetilde{\mathbf{Ha}}(\alpha_m(x))| \geq |p|^\varepsilon$; to complete the notation, when $m = 0$ and $0 \leq \varepsilon < 1$, we also denote $\mathcal{X}_0(\varepsilon)$ the affinoid subdomain of X_0^{rig} defined by the condition $|\widetilde{\mathbf{Ha}}| \geq |p|^\varepsilon$, so that $\mathcal{X}_0^-(\varepsilon) \subseteq \mathcal{X}_0(\varepsilon)$. For any integer k and any integer $m \geq 0$, define the $\mathbb{C}_p$-vector space of *overconvergent modular forms* of weight k pm $\widetilde{X}_m$ to be

$$M_k^{\mathrm{oc}}(\widetilde{X}_m) = \varprojlim_{\varepsilon} H^0\left(\mathcal{X}_m(\varepsilon), \underline{\omega}_{m,\mathbb{C}_p}^{\otimes k}\right)$$

where $0 \leq \varepsilon < \varepsilon_m$ with ε approaching ε_m. Then we have $L(\phi^*)F_\infty \in M_k^{\mathrm{oc}}(\widetilde{X}_m)$.

The proof of [Col94, Theorem 10.1] shows that $d(L(\phi^*)(F_\infty) = L(\phi^*)\omega_{\mathcal{F}}$; on the other hand, $L(\phi^*)\omega_{\mathcal{F}} = \omega_{\mathcal{F}^{[p]}}$, where recall that $\mathcal{F}^{[p]} = \mathcal{F} - a_p V\mathcal{F}$. Define the overconvergent modular form

$$d^{-1}\omega_{\mathcal{F}^{[p]}} = L(\phi^*)(F_\infty).$$

Then $L(\phi^*)^{-1}d^{-1}\omega_{\mathcal{F}^{[p]}} = F_\infty$. Note that the definition of $d^{-1}\omega_{\mathcal{F}^{[p]}}$ depends on the choice of a constant defining F_∞, which we fix as follows.

Pick a point x_∞ in the wide open neighborhood $\mathcal{W}_\infty$ of $\mathcal{A}_\infty(p^m)$ appearing Proposition 9.4; accordingly with our definitions, $\mathrm{red}(x_\infty) = (\bar{x}_\infty, \beta)$ belongs to $\mathtt{Ig}_\infty(\overline{\mathbb{F}}_p)$, so we may consider the T_{x_∞}-expansion $\boldsymbol{\mathcal{F}}(T_{x_\infty})$ of $\mathcal{F}$ at x_∞ associated with the choice of a basis $\{x_A, x'_A\}$ of $\mathrm{Ta}_p(A)$ coming from β as described in §4.2. The T_{x_∞}-expansion of $\mathcal{F}^{[p]}$ is then

$$\boldsymbol{\mathcal{F}}^{[p]}(T_{x_\infty}) = \sum_{p \nmid n} \alpha_n T_{x_\infty}^n$$

for suitable elements $\alpha_n \in \mathbb{Z}_p^{\mathrm{unr}}$ and $n \geq 0$ ([HB15, Proposition 4.17] and [Bur17, Lemma 5.2]). Define

$$d^{-1}\boldsymbol{\mathcal{F}}^{[p]}(T_{x_\infty}) = \sum_{p \nmid n} \frac{\alpha_n}{n+1} T_{x_\infty}^{n+1}. \tag{9.6}$$

We may then normalize the choice of F_∞ by imposing that the T_{x_∞}-expansion of $d^{-1}\omega_{\mathcal{F}^{[p]}}$ is that in (9.6); more precisely, we introduce the following:

Definition 9.6. Let $d^{-1}\mathcal{F}_{x_\infty}^{[p]}$ denote the unique overconvergent modular form such that:

- $d(d^{-1}\mathcal{F}_{x_\infty}^{[p]}) = \mathcal{F}^{[p]}$;
- The T_{x_∞}-expansion of $d^{-1}\mathcal{F}_\infty^{[p]}$ is equal to $d^{-1}\boldsymbol{\mathcal{F}}^{[p]}(T_{x_\infty})$.

The previous definition fixes the choice of $d^{-1}\omega_{\mathcal{F}^{[p]}}$ and, consequently, of F_∞, to be $d^{-1}\mathcal{F}_{x_\infty}^{[p]}$. Note that in the residue disk of x_∞ we have $d^{-1}\mathcal{F}_{x_\infty}^{[p]} = d^{-1}\boldsymbol{\mathcal{F}}^{[p]}(T_{x_\infty})\hat{\omega}_{x_\infty}$.

Definition 9.7. We say that the Coleman primitive F_∞ in $W_\infty(p^m)$ appearing in Definition 9.5 *vanishes* at x_∞ if the choice of the constant is normalized as in (9.6).

With these definitions, if F_∞ vanishes at x_∞, we have

$$d^{-1}\mathcal{F}_{x_\infty}^{[p]} = L(\phi^*)F_\infty. \tag{9.7}$$

9.1.4. *Logarithmic de Rham cohomology.* Let L_0 be the maximal unramified extension of L. The work of Hyodo-Kato [HK94] equips the L-vector space $H^1_{\mathrm{dR}}(\widetilde{X}_m/L)$ with a canonical L_0-subvector-space

$$H^1_{\text{log-cris}}(\mathscr{Y}_m) \hookrightarrow H^1_{\mathrm{dR}}(\widetilde{X}_m/L)$$

equipped with a semi-linear Frobenius operator φ; by the results of Tsuji [Tsu99], there is a canonical comparison isomorphism $\mathbf{D}_{\mathrm{dR}}(V_m)\otimes_{\mathbb{Q}_p}L \simeq H^1_{\mathrm{dR}}(X_m/L)$ of filtered φ-modules, where $V_m = H^1_{\text{ét}}(\widetilde{X}_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$. For a Hecke module M, let us denote $M[\mathcal{F}]$ the eigencomponent corresponding to the eigenform $\mathcal{F}$; we also denote $F_{\mathcal{F}} \subseteq \overline{\mathbb{Q}}_p$ the Hecke field of $\mathcal{F}$ inside the algebraic closure of $\mathbb{Q}_p$. We then have a canonical isomorphism of $L_0 \otimes_{\mathbb{Q}_p} F_{\mathcal{F}}$-modules $\mathbf{D}_{\mathrm{cris}}(V_{\mathcal{F}}) \simeq H^1_{\text{log-cris}}(\mathscr{Y}_m)[\mathcal{F}]$ compatible with the φ-action which induces after extending scalars an isomorphism of $L \otimes_{\mathbb{Q}_p} F_{\mathcal{F}}$-modules

$$\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}) \simeq H^1_{\mathrm{dR}}(\widetilde{X}_m/L)[\mathcal{F}].$$

9.1.5. *Abel-Jacobi map.* Let $\widetilde{J}_m = \mathrm{Jac}(\widetilde{X}_m \otimes_{\mathbb{Q}_p} L)$ and consider the map

$$\delta_m : \widetilde{J}_m(L) \xrightarrow{\mathrm{Kum}} H^1_f(L, \mathrm{Ta}_p(\widetilde{J}_m)) \xrightarrow{\mathrm{proj}} H^1_f(L, V_{\mathcal{F}}(1)) \xrightarrow{\log} \frac{\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))}{\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))} \xrightarrow{\sim} (\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V^*_{\mathcal{F}})))^\vee$$

where:

- Kum is the Kummer map;
- proj is induced by the projection map $\mathrm{Ta}_p(\widetilde{J}_m) \to V^*_{\mathcal{F}}$ and the isomorphism $V^*_{\mathcal{F}} \simeq V_{\mathcal{F}}(1)$ induced by Kummer duality;
- log is the inverse of the Bloch-Kato exponential map
$$\exp : \frac{\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))}{\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))} \xrightarrow{\sim} H^1_f(L, V_{\mathcal{F}}(1))$$
which is an isomorphism in our setting;
- The isomorphism
$$\frac{\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))}{\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))} \simeq (\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V^*_{\mathcal{F}})))^\vee$$
is induced by the de Rham pairing.

Following [BDP13, §3.4] and [Cas13, §2.2], the map δ_m can be described as follows. First, recall that the Bloch-Kato Selmer group can be identified with the group of cristalline extensions

$$0 \longrightarrow V_{\mathcal{F}}(1) \longrightarrow W \xrightarrow{\rho} \mathbb{Q}_p \longrightarrow 0$$

and since $\mathbf{D}_{\mathrm{cris}}(V_{\mathcal{F}}(1))^{\varphi=1}$ is trivial, the resulting extension of φ-modules

$$0 \longrightarrow \mathbf{D}_{\mathrm{cris}}(V_{\mathcal{F}}(1)) \longrightarrow \mathbf{D}_{\mathrm{cris}}(W) \longrightarrow L_0 \longrightarrow 0 \tag{9.8}$$

(where L_0 is the maximal unramified subextension of L) admits a unique section

$$s^{\mathrm{Frob}}_W : L_0 \longrightarrow \mathbf{D}_{\mathrm{cris}}(W)$$

with $\eta^{\mathrm{Frob}}_W = s^{\mathrm{Frob}}_W(1) \in \mathbf{D}_{\mathrm{cris}}(W)^{\varphi=1}$. We also fix a section

$$s^{\mathrm{Fil}}_W : L \longrightarrow \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(W))$$

of the exact sequence of L-vector spaces

$$0 \longrightarrow \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))) \longrightarrow \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(W)) \longrightarrow L \longrightarrow 0 \tag{9.9}$$

obtained by extending scalars from L_0 to L in (9.8), using the canonical isomorphism with de Rham cohomology, and taking the Fil^0-parts of the resulting sequence. Define $\eta^{\mathrm{Fil}}_W = s^{\mathrm{Fil}}_W(1)$ and consider the difference

$$\eta_W = \eta^{\mathrm{Frob}}_W - \eta^{\mathrm{Fil}}_W$$

viewed as an element in $\mathbf{D}_{\mathrm{dR}}(W)$; this difference comes from an element in $\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))$, denoted with the same symbol η_W, and its image modulo $\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1)))$ is well defined. Then we have (see [Cas13, Lemma 2.4] and the references therein)

$$\log(W) = \eta_W \mod \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))).$$

Let $\Delta \in J_s(L)$ be the class of a degree zero divisor in $\widetilde{X}_m$, with support contained in the finite set of points $S \subseteq \widetilde{X}_m(L)$. Define the map

$$\kappa_m : \widetilde{J}_m(L) \xrightarrow{\mathrm{Kum}} H^1_f(L, \mathrm{Ta}_p(\widetilde{J}_m)) \xrightarrow{\mathrm{proj}} H^1_f(L, V_{\mathcal{F}}(1)) \tag{9.10}$$

and consider the class $\kappa_m(\Delta) \in H^1_f(L, V_{\mathcal{F}}(1))$. Denote W_Δ the extension class associated with $\kappa_m(\Delta)$. Attached to W_Δ we then have the class η_{W_Δ} in $\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1))$ constructed before, and we may consider the (weight 2) newform $\mathcal{F}^*$ associated with the twisted form $\mathcal{F} \otimes \chi_{\mathcal{F}}^{-1}$, where $\chi_{\mathcal{F}}$ denotes the character of $\mathcal{F}$. Let as before $\omega_{\mathcal{F}^*}$ denote the differential form attached to $\mathcal{F}^*$; denote with the same symbol $\omega_{\mathcal{F}^*}$ the corresponding element in $\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}^*})$ via the isomorphism $\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}^*}) \simeq H^1_{\mathrm{dR}}(\widetilde{X}_m/L)[\mathcal{F}^*]$ (here for a Hecke module M, we denote $M[\mathcal{F}^*]$ the eigencomponent corresponding to the eigenform $\mathcal{F}^*$, and we also denote $F_{\mathcal{F}^*} \subseteq \overline{\mathbb{Q}}_p$ the Hecke field of $\mathcal{F}^*$ inside the algebraic closure of $\mathbb{Q}_p$). Note that $\omega_{\mathcal{F}^*}$ belongs to $\mathrm{Fil}^1(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}^*}))$, which is equal to $\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}^*))$; we therefore obtain a class $\omega_{\mathcal{F}^*} \in \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}^*))$.

Lemma 9.8. $\delta_m(\Delta)(\omega_{\mathcal{F}^*}) = \langle \eta_{W_\Delta}, \omega_{\mathcal{F}^*}\rangle_{\mathrm{dR}}$.

Proof. Follow the argument in the case of modular curves in [BDP13, §4.1] (the good reduction case) and [Cas13, §2.2] (the bad reduction case). ☐

Pick as before a point x_∞ in the wide open space $\mathcal{W}_\infty$. Let F_∞^* be the Coleman primitive of $\omega_{\mathcal{F}^*}$ on $W_\infty(p^m)$ which vanishes at x_∞ (*cf.* Definition 9.7). We may then consider the map $j_m^{(x_\infty)} : \widetilde{X}_m(\mathbb{C}_p) \to \widetilde{J}_m(\mathbb{C}_p)$ which associates to P the divisor $(P) - (x_\infty)$. We simply write j_m for this map when x_∞ is understood.

Lemma 9.9. *Let $\Delta = j_m(P)$ and F_∞^* the Coleman primitive of $\omega_{\mathcal{F}^*}$ on $W_\infty(p^m)$ which vanishes at ∞. Assume that $m > 1$. Then $\langle \eta_{W_\Delta}, \omega_{\mathcal{F}^*}\rangle_{\mathrm{dR}} = F_{\omega_{\mathcal{F}^*}}(P)$.*

Proof. The proof follows [DR17, §4.2] and [Cas13, Proposition 2.9], which adapts the proof of [BDP13, Proposition 3.21] to the semistable setting. We proceed with the computations using (9.4).

Step 1. We first describe the classes $\eta_{W_\Delta}^{\mathrm{Fil}}$ and $\eta_{W_\Delta}^{\mathrm{Frob}}$. Let $S = \{P, x_\infty\}$ and $Y_S = \mathscr{Y}_m(\mathbb{C}_p) - S$ as before.

The class $\eta_{W_\Delta}^{\mathrm{Fil}}$ is an element in $\mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(W_\Delta))$ with $\rho_{\mathrm{dR}}(\eta_{W_\Delta}^{\mathrm{Fil}}) = 1$, where ρ_{dR} is the top right arrow map in the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(V_{\mathcal{F}}(1)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Fil}^0(\mathbf{D}_{\mathrm{dR}}(W_\Delta)) \arrow[r, "\rho_{\mathrm{dR}}"] \arrow[d] & L \otimes_{\mathbb{Q}_p} F_{\mathcal{F}} \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathrm{Fil}^1\left(H^1_{\mathrm{dR}}(\widetilde{X}_m/L)\right)[\mathcal{F}] \arrow[r] & \mathrm{Fil}^1\left(H^1_{\mathrm{dR}}(Y_S/L)\right)[\mathcal{F}] \arrow[r, "\oplus \mathrm{res}_Q"] & (L \otimes_{\mathbb{Q}_p} F_{\mathcal{F}})_0^S \arrow[r] & 0
\end{tikzcd}$$

which realizes the exact sequence in the top horizontal line (which is (9.9)) as the pull-back of the bottom horizontal line with respect to the rightmost $L \otimes_{\mathbb{Q}_p} F_{\mathcal{F}}$-linear vertical map Δ taking 1 to $(P, -x_\infty)$; in the bottom horizontal arrow, $\mathrm{res}_Q(\omega)$ is the residue at $Q \in S$ of the differential form ω, and the subscript 0 denotes the degree zero elements, *i.e.* those $(x_Q)_{Q\in S}$ in $L \otimes_{\mathbb{Q}_p} F_{\mathcal{F}}$ with $\sum_{Q\in S} n_Q = 0$. Therefore, we have $\mathrm{res}_P(\eta_{W_\Delta}^{\mathrm{Fil}}) = 1$ and $\mathrm{res}_{x_\infty}(\eta_{W_\Delta}^{\mathrm{Fil}}) = -1$.

Similarly, the class $\eta_{W_\Delta}^{\mathrm{Frob}}$ is an element in $\mathbf{D}_{\mathrm{cris}}(W_\Delta)^{\varphi=1}$ with $\rho_{\mathrm{cris}}(\eta_{W_\Delta}^{\mathrm{Frob}})=1$, where ρ_{dR} is the top right arrow map in the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{D}_{\mathrm{cris}}(V_{\mathcal{F}}(1)) \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}_{\mathrm{cris}}(W_\Delta) \arrow[r, "\rho_{\mathrm{cris}}"] \arrow[d] & L_0\otimes_{\mathbb{Q}_p}F_{\mathcal{F}} \arrow[r] \arrow[d, "\Delta"] & 0 \\ 0 \arrow[r] & H^1_{\text{log-cris}}(\widetilde{X}_m/L_0)[\mathcal{F}](1) \arrow[r] & H^1_{\text{log-cris}}(Y_S/L_0)[\mathcal{F}](1) \arrow[r, "\oplus\mathrm{res}_Q"] & (L_0\otimes_{\mathbb{Q}_p}F_{\mathcal{F}})_0^S \arrow[r] & 0 \end{tikzcd}$$

which realizes the exact sequence in the top horizontal line (which is (9.8)) as the pull-back of the bottom horizontal line with respect to the rightmost $L_0\otimes_{\mathbb{Q}_p}F_{\mathcal{F}}$-linear vertical map Δ taking 1 to $(P,-x_\infty)$; as before in the bottom horizontal arrow, $\mathrm{res}_Q(\omega)$ is the residue at $Q\in S$ of the differential form ω, and the subscript 0 denotes the degree zero elements. By the discussion closing §9.1.2 (see especially (9.3)), $\eta_{W_\Delta}^{\mathrm{Frob}}$ is represented by a pair of sections $(\eta_\infty^{\mathrm{Frob}},\eta_0^{\mathrm{Frob}})$ of $\Omega^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_\infty)\times\Omega^1_{\mathrm{rig}}(\widetilde{\mathcal{W}}_0)$. Since $\eta_{W_\Delta}^{\mathrm{Frob}}$ is fixed by φ, we have $\eta_\infty^{\mathrm{Frob}}=\phi\eta_\infty^{\mathrm{Frob}}+dG_\infty$ for a rigid analytic function G_∞ on $\widetilde{\mathcal{W}}_\infty$, and $(\eta_0^{\mathrm{Frob}})=(\phi')\eta_0^{\mathrm{Frob}}+dG_0$ for a rigid analytic function G_0 on $\widetilde{\mathcal{W}}_0$. Moreover, we also have $\mathrm{res}_Q(\eta_{W_\Delta}^{\mathrm{Frob}})=\mathrm{res}_Q(\eta_{W_\Delta}^{\mathrm{Fil}})$ for all $Q\in S$, and since $\mathrm{res}_Q(\eta_{W_\Delta}^{\mathrm{Frob}})=\mathrm{res}_{\mathcal{V}_Q}(\eta_{W_\Delta}^{\mathrm{Frob}})$ for all $Q\in S$, we may rewrite the last condition in the form $\mathrm{res}_{\mathcal{V}_Q}(\eta_{W_\Delta}^{\mathrm{Frob}})=\mathrm{res}_Q(\eta_{W_\Delta}^{\mathrm{Fil}})$ for all $Q\in S$.

Step 2. (*Cf.* [BDP13, Lemma 3.20].) We now show that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle F_\infty^*,\eta_\infty^{\mathrm{Frob}}\rangle_{\mathrm{dR}})+\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_0}\mathrm{res}_{\mathcal{V}}(\langle F_0^*,\eta_0^{\mathrm{Frob}}\rangle_{\mathrm{dR}})=0. \tag{9.11}$$

We begin by showing that the first summand in (9.11) is zero. Recall that $\eta_\infty^{\mathrm{Frob}}=\phi\eta_\infty^{\mathrm{Frob}}+dG_\infty$. By the Leibeniz rule we then have

$$d(\langle\phi F_\infty^*,G_\infty\rangle_{\mathrm{dR}})=\langle\phi F_\infty^*,dG_\infty\rangle_{\mathrm{dR}}+\langle\phi\omega_{\mathcal{F}^*},G_\infty\rangle_{\mathrm{dR}}$$

where we use that $d(\phi F_\infty^*)=\phi dF_\infty^*$ because ϕ is horizontal for d. Therefore, the RHS is exact on each $\mathcal{V}$, so we have $\mathrm{res}_{\mathcal{V}}(\langle\phi F_\infty^*,dG_\infty\rangle_{\mathrm{dR}})=-\mathrm{res}_{\mathcal{V}}(\langle\phi\omega_{\mathcal{F}^*},G_\infty\rangle_{\mathrm{dR}})$; on the other hand, $\langle\phi\omega_{\mathcal{F}^*},G_\infty\rangle_{\mathrm{dR}}$ is a rigid analytic differential form on $\widetilde{\mathcal{W}}_\infty$, so the sum of its residues is zero for all $\mathcal{V}$. We conclude that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle\phi F_\infty^*,dG_\infty\rangle_{\mathrm{dR}})=0. \tag{9.12}$$

We then observe that $\mathrm{res}_{\mathcal{V}}(\langle F_\infty^*,\eta_\infty^{\mathrm{Frob}}\rangle_{\mathrm{dR}})=\mathrm{res}_{\mathcal{V}}(\langle\phi F_\infty^*,\phi\eta_\infty^{\mathrm{Frob}}\rangle_{\mathrm{dR}})$; combing this with the equation $\eta_\infty^{\mathrm{Frob}}=\phi\eta_\infty^{\mathrm{Frob}}+dG_\infty$ and the equation (9.12) we conclude that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle F_\infty^*,\eta_\infty^{\mathrm{Frob}}\rangle_{\mathrm{dR}})=\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle\phi F_\infty^*,\eta_\infty^{\mathrm{Frob}}\rangle_{\mathrm{dR}}).$$

It follows that

$$L(1)\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle F_\infty^*,\eta_{W_\Delta}^{\mathrm{Frob}}\rangle_{\mathrm{dR}})=\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle L(\phi)F_\infty^*,\eta_\infty^{\mathrm{Frob}}\rangle_{\mathrm{dR}}).$$

Now $L(\phi)F_\infty^*$ is rigid analytic, and therefore the RHS is zero; since $L(1)\neq0$, we conclude that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(\langle F_\infty^*,\eta_{W_\Delta}^{\mathrm{Frob}}\rangle_{\mathrm{dR}})=0.$$

A similar argument, replacing $\widetilde{\mathcal{W}}_\infty$ with $\widetilde{\mathcal{W}}_0$, η_∞ with η_0, G_∞ with G_0, F_∞^* with F_0^* and ϕ by ϕ' shows that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_0}\mathrm{res}_{\mathcal{V}}(\langle F_0^*,\eta_0^{\mathrm{Frob}}\rangle_{\mathrm{dR}})=0$$

and (9.11) follows.

Step 3. (*Cf.* [BDP13, Lemma 3.19].) We now show that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(F^*_\infty\eta^{\mathrm{Fil}}_\infty)+\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_0}\mathrm{res}_{\mathcal{V}}(F^*_0\eta^{\mathrm{Fil}}_0)=F^*_\infty(P). \tag{9.13}$$

Since F^*_∞ vanishes at x_∞, $F^*_\infty\eta^{\mathrm{Fil}}_\infty$ is locally analytic in a neighborhood of x_∞, and it follows that $\mathrm{res}_{x_\infty}(F^*_\infty\eta^{\mathrm{Fil}}_\infty)=0$. On the other hand, since $\mathrm{res}_P(\eta^{\mathrm{Fil}}_{W_\Delta})=1$, we have $\mathrm{res}_P(F^*_\infty\eta^{\mathrm{Fil}}_{W_\Delta})=F^*_\infty(P)$, so we conclude that

$$\sum_{\mathcal{V}\subseteq\widetilde{\mathcal{W}}_\infty}\mathrm{res}_{\mathcal{V}}(F^*_\infty\eta^{\mathrm{Fil}}_\infty)=F^*_\infty(P).$$

On the other hand, $F^*_0\eta^{\mathrm{Fil}}_{W_\Delta}$ is analytic on $\mathcal{W}_0$, so the second summand in the LHS of (9.13) is zero, and (9.13) follows.

Step 4. The result now follows combining (9.11) and (9.13) with (9.4) and using that, since $m>1$, the wide opens $\widetilde{\mathcal{W}}_\infty$ and $\widetilde{\mathcal{W}}_0$ are disjoint. □

Corollary 9.10. *Let* $\Delta=(P)-(x_\infty)$ *and* F^*_∞ *the Coleman primitive of* $\omega_{\mathcal{F}^*}$ *on* $W_\infty(p^m)$ *which vanishes at* ∞. *Assume that* $m>1$. *Then* $\delta_m(\Delta)(\omega_{\mathcal{F}^*})=F_{\omega_{\mathcal{F}^*}}(P)$.

Proof. This follows immediately from Lemma 9.8 and Lemma 9.9. □

9.2. **Weight 2 specializations.** Let ν an arithmetic homomorphisms of signature $(2,\psi)$ and let the conductor of ψ be p^m for some integer $m\geq 1$. Let $\hat{\phi}\colon K^\times\backslash\widehat{K}^\times\to F^\times$ be the p-adic avatar of a Hecke character $\phi\colon K^\times\backslash\mathbb{A}_K^\times\to\overline{\mathbb{Q}}^\times$ of infinity type $(1,-1)$ and conductor p^n for some integer $n\geq m$ such that the Galois character $\tilde{\phi}\colon\mathrm{Gal}(K^{\mathrm{ab}}/K)\to F^\times$ factors through $\widetilde{\Gamma}_\infty$. The next task consists in computing the $(\nu,\hat{\phi}^{-1})$-specialization of $\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}$. We put

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1})=\mathrm{sp}_{\nu,\phi}\left(\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}\right).$$

For a number field L and the ring of algebraic integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}$ there is a canonical exact sequence

$$0\longrightarrow\widetilde{J}_m(L)\otimes_{\mathbb{Z}}\mathcal{O}\longrightarrow\mathrm{Pic}(\widetilde{X}_m/L)\otimes_{\mathbb{Z}}\mathcal{O}\overset{\deg}{\longrightarrow}\mathcal{O}\longrightarrow 0$$

and taking ordinary parts, since the degree of U_p is p, we obtain a canonical isomorphism

$$\widetilde{J}_m(L)^{\mathrm{ord}}\otimes_{\mathbb{Z}}\mathcal{O}\longrightarrow\mathrm{Pic}(\widetilde{X}_m/L)^{\mathrm{ord}}\otimes_{\mathbb{Z}}\mathcal{O}. \tag{9.14}$$

We denote ϱ_m the inverse of this canonical isomorphism. Consider the divisor

$$Q_{cp^n,m}=\sum_{\sigma\in\mathrm{Gal}(H_{cp^{n+m}}/H_{cp^n})}\widetilde{P}^{\tilde{\sigma}}_{cp^{n+m},m}\otimes\chi_\nu(\tilde{\sigma})$$

where $\tilde{\sigma}\in\mathrm{Gal}(L_{cp^{n+m},m}/H_{cp^n})$ is any lift of σ (the independence of the lift follows the results recalled in 2.6). We define a canonical class $\varrho_m(Q_{cp^n,m})$ in $\widetilde{J}_m(\overline{\mathbb{Q}})\otimes_{\mathbb{Z}}\mathcal{O}_\nu(\chi_\nu)$, which is fixed by the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/L_{cp^{n+m},m})$. Tracing through the definition of big Heegner points, we see (*cf.* [LV14b, §3.4], see especially [LV14b, (3.6)]) that when $n\geq m\geq 2$

$$\varrho_m(Q_{cp^n,m})=\left(\frac{\nu(\mathbf{a}_p)}{p}\right)^m\cdot\mathrm{sp}_\nu(\mathfrak{X}_{cp^n}). \tag{9.15}$$

For the next theorem, let ν an arithmetic homomorphisms of signature $(2,\psi)$ such that $\psi\colon\Gamma\to\overline{\mathbb{Q}}_p^\times$ has conductor $\mathrm{cond}(\psi)=p^m$ for some integer $m\geq 2$. Let $\hat{\phi}\colon K^\times\backslash\widehat{K}^\times\to F^\times$ be the p-adic avatar of a Hecke character $\phi\colon K^\times\backslash\mathbb{A}_K^\times\to\overline{\mathbb{Q}}^\times$ of infinity type $(1,-1)$ and conductor $\mathrm{cond}(\psi)=p^n$ for some integer $n\geq m$ such that the Galois character $\tilde{\phi}\colon\mathrm{Gal}(K^{\mathrm{ab}}/K)\to F^\times$

factors through $\widetilde{\Gamma}_\infty$. Finally, recall that $\tilde{x}_{cp^n,m} = \tilde{x}_{cp^n,m}(1) = \widetilde{P}_{cp^n,m}$. We finally write $d^{-1}\mathcal{F}^{[p]}_{\nu,x_\infty}$ to denote the overconvergent modular form in Definition 9.6 for $\mathcal{F} = \mathcal{F}_\nu$.

Theorem 9.11. *Let ν and ψ be as before, so ν has signature $(2,\psi)$ with $\mathrm{cond}(\psi) = p^m$ with $m \geq 2$ and $\phi\colon K^\times\backslash\mathbb{A}_K^\times \to \overline{\mathbb{Q}}^\times$ of infinity type $(1,-1)$ and $\mathrm{cond}(\phi) = p^n$ with $n \geq m$. Then*

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1}) = \frac{\epsilon(\phi)}{\xi_{\nu,\mathfrak{p}}(p^n)\cdot p^n}\cdot \sum_{a\in\mathrm{Pic}(\mathcal{O}_{cp^n})} (\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi})(a)d^{-1}\mathcal{F}^{[p]}_{\nu,x_\infty}\left(x_{cp^n,m}(a^{-1})\right).$$

Proof. We first relate $\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1})$ to the Coleman primitive. Since $\hat{\phi}\colon \Gamma_\infty \to \overline{\mathbb{Q}}_p$ has Hodge–Tate weight $w = 1$ and conductor $n > 1$, from (7.10) we have

$$\begin{aligned}\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1}) &= \mathrm{sp}_{\nu,\hat{\phi}}\left(\mathcal{L}^{\widetilde{\Gamma}_\infty}_{\omega_{\mathcal{F}}}(\mathrm{res}_{\mathfrak{P}}(\mathfrak{Z}_{\boldsymbol{\xi}}))\right)\\ &= \mathcal{E}(\hat{\phi}^{-1},\nu)\cdot(\boldsymbol{\omega}_\nu\otimes\phi^{-1})\left(\log(\mathrm{sp}_{\nu,\hat{\phi}}(\mathrm{res}_{\mathfrak{P}}(\mathfrak{Z}_{\boldsymbol{\xi}}))\right).\end{aligned}\tag{9.16}$$

Using that the characters ξ_ν and ϕ has conductors p^m and p^n respectively, and $n \geq m$, by (9.15) we have

$$\begin{aligned}\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1}) &= \mathcal{E}(\hat{\phi}^{-1},\nu)\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/H_c)}(\hat{\xi}_\nu^{-1}\hat{\phi}^{-1})(\sigma)\log(\mathrm{sp}_\nu(\mathrm{res}_{\mathfrak{p}}(\mathrm{cor}_{H_c/K}(\mathfrak{X}^\sigma_{cp^\infty}))))(\omega_{\mathcal{F}^*_\nu})\\ &= \mathcal{E}(\hat{\phi}^{-1},\nu)\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}(\hat{\xi}_\nu^{-1}\hat{\phi}^{-1})(\sigma)\log(\mathrm{sp}_\nu(\mathrm{res}_{\mathfrak{p}}(\mathfrak{X}^\sigma_{cp^\infty})))(\omega_{\mathcal{F}^*_\nu})\\ &= \mathcal{E}(\hat{\phi}^{-1},\nu)\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}\nu(\boldsymbol{a}_p)^{-n}(\hat{\xi}_\nu^{-1}\hat{\phi}^{-1})(\sigma)\log(\mathrm{sp}_\nu(\mathrm{res}_{\mathfrak{p}}(\mathfrak{X}^\sigma_{cp^n})))(\omega_{\mathcal{F}^*_\nu})\\ &= \mathcal{E}(\hat{\phi}^{-1},\nu)\left(\frac{p}{\nu(\mathbf{a}_p)}\right)^m\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}\nu(\boldsymbol{a}_p)^{-n}(\hat{\xi}_\nu^{-1}\hat{\phi}^{-1})(\sigma)\log(\varrho_m(Q^\sigma_{cp^n,m}))(\omega_{\mathcal{F}^*_\nu})\\ &= \mathcal{E}(\hat{\phi}^{-1},\nu)\left(\frac{p}{\nu(\mathbf{a}_p)}\right)^m\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}\nu(\boldsymbol{a}_p)^{-n}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi}^{-1})(\sigma)\log(\varrho_m(\widetilde{P}^\sigma_{cp^{n+m},m}))(\omega_{\mathcal{F}^*_\nu}).\end{aligned}\tag{9.17}$$

Let F^*_∞ be the Coleman primitive of $\omega_{\mathcal{F}^*_\nu}$ on $W_\infty(p^m)$ which vanishes at x_∞. It follows from (9.14) that

$$\log(\varrho_m(\widetilde{P}^\sigma_{cp^{n+m},m})) = \log(j_m(\widetilde{P}^\sigma_{cp^{n+m},m})).$$

Applying Corollary 9.10 (and using linearity) we thus obtain

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1}) = \mathcal{E}(\hat{\phi}^{-1},\nu)\left(\frac{p}{\nu(\mathbf{a}_p)}\right)^m\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}\nu(\boldsymbol{a}_p)^{-n}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi}^{-1})(\sigma)F^*_\infty(\widetilde{P}^\sigma_{cp^{n+m},m}).$$

On the other hand, since $\widetilde{P}_{cp^{n+m},m}$ is defined over the subfield $H_{cp^{n-1}}(\zeta_{p^n})$ of L_{cp^n}, and χ_ν is a primitive character modulo p^n, we see that, after setting $\psi = \nu(\boldsymbol{a}_p)^{-n}\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi}^{-1}$ to simplify the notation,

$$\begin{aligned}\sum_\sigma\psi(\sigma)F^*_\infty(\widetilde{P}^\sigma_{cp^{n+m},m}) &= \sum_\sigma\psi(\sigma)F^*_\infty(\widetilde{P}^\sigma_{cp^{n+m},m}) - \frac{\nu(\mathbf{a}_p)}{p}\sum_\sigma\psi(\sigma)F^*_\infty(\phi(\widetilde{P}^\sigma_{cp^{n+m},m}))\\ &= \sum_\sigma\psi(\sigma)L(\phi^*)F^*_\infty(\widetilde{P}^\sigma_{cp^{n+m},m})\\ &= \sum_\sigma\psi(\sigma)d^{-1}\mathcal{F}^{[p]}_{\nu,x_\infty}(\widetilde{P}^\sigma_{cp^{n+m},m})\end{aligned}\tag{9.18}$$

where the sum is over all $\sigma \in \mathrm{Gal}(H_{cp^n}/K)$, and the last equation follows from (9.7) and the fact that $d^{-1}\omega_{\mathcal{F}^{[p]}} = d^{-1}\omega_{\mathcal{F}^{*[p]}}$. Therefore,

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1}) = \mathcal{E}(\hat{\phi}^{-1},\nu)\left(\frac{p}{\nu(\mathbf{a}_p)}\right)^m\cdot\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}\nu(\boldsymbol{a}_p)^{-n}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi}^{-1})(\sigma)d^{-1}\mathcal{F}^{[p]}_{\nu,x_\infty}(\widetilde{P}^\sigma_{cp^{n+m},m}).\tag{9.19}$$

We now observe that

$$U_pF^*_\infty=\left(\frac{\nu(\mathbf{a}_p)}{p}\right)F^*_\infty. \tag{9.20}$$

Since $\widetilde{P}_{cp^{n+m},m}=U_p^m\widetilde{P}_{cp^n,m}=U_p^m\tilde{x}_{cp^n,m}$; it follows from (9.20) and (9.19) that (use Shimura's reciprocity law to keep trace of the Galois action)

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1})=\mathcal{E}(\hat{\phi}^{-1},\nu)\sum_{\sigma\in\mathrm{Gal}(H_{cp^n}/K)}\nu(\boldsymbol{a}_p)^{-n}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi}^{-1})(\sigma)d^{-1}\mathcal{F}^{[p]}_{\nu,x_\infty}(\tilde{x}^\sigma_{cp^n,m}). \tag{9.21}$$

Since $\Phi_\nu=\nu(\mathbf{a}_p)(\xi_{\nu,\mathfrak{p}}(p)p)^{-1}$, we have

$$\mathcal{E}(\hat{\phi}^{-1},\nu)=\frac{\epsilon(\phi)\nu(\mathbf{a}_p)^n}{\xi_{\nu,\mathfrak{p}}(p^n)\cdot p^n}$$

and the result follows. □

9.3. **Reciprocity Laws.** Fix an algebraic Hecke character $\lambda\colon K^\times\backslash\mathbb{A}_K^\times\to\overline{\mathbb{Q}}^\times$ as in §5.2 and set $\boldsymbol{\xi}=\boldsymbol{\xi}^{(\lambda)}$. We fix ν and ϕ as in the proof of Theorem 9.11; so ν is an arithmetic homomorphisms of signature $(2,\psi)$ with $\mathrm{cond}(\psi)=p^m$ for some integer $m\geq 2$, and $\hat{\phi}$ is the p-adic avatar of a Hecke character $\phi\colon K^\times\backslash\mathbb{A}_K^\times\to\overline{\mathbb{Q}}^\times$ of infinity type $(1,-1)$ and conductor p^n for some integer $n\geq m$, so the associated Galois character $\tilde{\phi}$ factors through $\widetilde{\Gamma}_\infty$.

Proposition 9.12. *Let ν and ϕ be as before. Then*

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1})=\left(\frac{\phi_{\mathfrak{p}}(-1)}{\sqrt{-D_K}}\right)\mathscr{L}^{\mathrm{an}}_{\mathbb{I},\boldsymbol{\xi}}(\nu,\hat{\phi}^{-1}).$$

Proof. The character $\hat{\xi}_\nu$ has infinity type $(1,-1)$, so the character $\varphi=\hat{\xi}_\nu\hat{\phi}^{-1}$ has infinity type $(0,0)$, thus finite order. Recall that, by definition,

$$\nu(\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}(\tilde{\phi}^{-1}))=\sum_{\mathfrak{a}\in\mathrm{Pic}\,\mathcal{O}_c}\hat{\xi}_\nu\hat{\chi}_\nu^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-1}\int_{\mathbb{Z}_p^\times}\tilde{\phi}^{-1}|[\mathfrak{a}](z)d\mu_{\mathcal{F}_\nu,\mathfrak{a}}(z).$$

Since ϕ^{-1} has infinity type $(-1,1)$ and we chose the representatives $\mathfrak{a}$ such that $((p),\mathfrak{a})=1$, then

$$\tilde{\phi}^{-1}|[\mathfrak{a}](z)=\tilde{\phi}^{-1}(\mathrm{rec}_K(a)\mathrm{rec}_{K,\mathfrak{p}}(z))=\hat{\phi}^{-1}(ai_{\mathfrak{p}}(z))=\phi^{-1}(\mathfrak{a})\phi_{\mathfrak{p}}^{-1}(z)z^{-1},$$

where recall that $\mathfrak{a}=a\widehat{\mathcal{O}}_c\cap K$ and $i_{\mathfrak{p}}\colon\mathbb{Z}_p^\times\to\widehat{K}^\times$ denotes the map which takes $z\in\mathbb{Z}_p^\times\cong\mathcal{O}_{K,\mathfrak{p}}^\times$ to the element $i_{\mathfrak{p}}(z)$ with $\mathfrak{p}$-component equal to z and trivial components at all the other places. Hence,

$$\nu(\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}(\hat{\phi}^{-1}))=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\hat{\xi}_\nu\hat{\chi}_\nu^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-1}\phi^{-1}(\mathfrak{a})\int_{\mathbb{Z}_p^\times}\phi_{\mathfrak{p}}^{-1}(z)z^{-1}d\mu_{\mathcal{F}_{\nu,\mathfrak{a}}}(z).$$

By [Hid93, §3.5, (5)] ([Mag22, (6.7)] for negative exponents), we have

$$\nu(\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}(\hat{\phi}^{-1}))=\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\hat{\xi}_\nu\hat{\chi}_\nu^{-1}(\mathfrak{a})\mathrm{N}(\mathfrak{a})^{-1}\phi^{-1}(\mathfrak{a})\cdot([\phi_{\mathfrak{p}}^{-1}]d^{-1}\boldsymbol{\mathcal{F}}^{[p]}_{\nu,\mathfrak{a}}(T_{x(\mathfrak{a})}))|_{T_{x(\mathfrak{a})}=0},$$

where $d=t_{x(\mathfrak{a})}\frac{d}{dt_{x(\mathfrak{a})}}$ is as before the Katz operator. Set $C_0(\xi_\nu,\chi_\nu,\phi)=\sqrt{-D_K}p^{-n}\mathfrak{g}(\phi_{\mathfrak{p}}^{-1})$. Applying (5.2), and using the equality $\mathrm{N}(\mathfrak{a})\sqrt{-D_K}(d^{-1}\boldsymbol{\mathcal{F}}^{[p]}_\nu)_{\mathfrak{a}}=d^{-1}\boldsymbol{\mathcal{F}}^{[p]}_{\nu,\mathfrak{a}}$, we see that

$$\nu(\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}(\hat{\phi}^{-1}))=C_0(\xi_\nu,\chi_\nu,\phi)\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\sum_{u\in(\mathbb{Z}/p^n\mathbb{Z})^\times}(\hat{\xi}_\nu\hat{\chi}_\nu^{-1}\phi^{-1})(\mathfrak{a})\phi_{\mathfrak{p}}(u)d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}(x(\mathfrak{a})\star\mathbf{n}(u/p^n)).$$

Here $d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}$ denotes the overconvergent modular form in Definition 9.6 for $\mathcal{F}=\mathcal{F}_\nu$ where the basis point is taken to be $x(\mathfrak{a})$ instead of the point x_∞ fixed before. Since $d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}$ has weight 0 and character ψ, using (4.1) we obtain (recall that $\mathfrak{a}=a\mathcal{O}_c^\times\cap K$)

$$d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}(x(\mathfrak{a})\star\mathbf{n}(u/p^n))=\psi^{-1}(\langle u\rangle)d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}\left(\left[\left(\iota_K,a^{-1}i_\mathfrak{p}(u/p^n)\xi^{(n)}\right)\right]\right).$$

To simplify the notation, we temporarily write $z_n(a)=\left(\left[\left(\iota_K,a^{-1}i_\mathfrak{p}(u/p^n)\xi^{(n)}\right)\right]\right)$. We have

$$\hat{\xi}_\nu^{-1}\hat{\chi}_\nu(a^{-1}i_\mathfrak{p}(u/p^n))=\hat{\xi}_\nu\hat{\chi}_\nu^{-1}(\mathfrak{a})\hat{\xi}_\nu^{-1}\hat{\chi}_\nu(i_\mathfrak{p}(u/p^n)),$$
$$\hat{\phi}(a^{-1}i_\mathfrak{p}(u/p^n))=\phi^{-1}(\mathfrak{a})\phi_\mathfrak{p}(u)\phi_\mathfrak{p}(p^{-n})up^{-n}.$$

By (5.3), $\chi_{\nu,\mathfrak{p}}^{-1}(z)=\psi^{1/2}(\langle z\rangle)$ for $z\in\mathbb{Z}_p^\times\cong\mathcal{O}_{K,\mathfrak{p}}^\times$. Also, $\chi_{\nu,\mathfrak{p}}^{-1}(p^{-n})=\psi^{1/2}(\langle p^np^{-n}\rangle)=1$ and by (5.4), $\xi_{\nu,\mathfrak{p}}(z)=\psi^{1/2}(\langle z\rangle)$ for $z\in\mathbb{Z}_p^\times\cong\mathcal{O}_{K,\mathfrak{p}}^\times$. Therefore, after setting

$$C(\xi_\nu,\chi_\nu,\phi)=C_0(\xi_\nu,\chi_\nu,\phi)\xi_{\nu,\mathfrak{p}}(p^{-n})\phi_\mathfrak{p}(p^n)=\frac{\sqrt{-D_K}\cdot\mathfrak{g}(\phi_\mathfrak{p}^{-1})p^{-n}\phi_\mathfrak{p}(p^n)}{\xi_{\nu,\mathfrak{p}}(p^n)}$$

we have

$$\begin{aligned}\nu(\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}(\hat{\phi}^{-1}))&=C(\xi_\nu,\chi_\nu,\phi)\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_c)}\sum_{u\in(\mathbb{Z}/p^n\mathbb{Z})^\times}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi})(a^{-1}i_\mathfrak{p}(u/p^n))d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}(z_n(a))\\&=C(\xi_\nu,\chi_\nu,\phi)\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{cp^n})}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi})(a)d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}\left([(\iota_K,a^{-1}\xi^{(n)})]\right)\\&=C(\xi_\nu,\chi_\nu,\phi)\sum_{\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{cp^n})}(\hat{\xi}_\nu^{-1}\hat{\chi}_\nu\hat{\phi})(a)\cdot d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}\left(y_{cp^n,m}(a^{-1})\right)\end{aligned}$$

where for each $\mathfrak{a}\in\mathrm{Pic}(\mathcal{O}_{cp^n})$ we let $\mathfrak{a}=a\widehat{\mathcal{O}}_{cp^n}\cap K$. We now observe that $d^{-1}\mathcal{F}^{[p]}_{\nu,x_\infty}$ and $d^{-1}\mathcal{F}^{[p]}_{\nu,x(\mathfrak{a})}$ differ by a constant; however, since the character $\hat{\chi}_\nu$ is primitive, we can replace the first with the second in the previous formula. Comparing with Theorem 9.11, the result follows from the equality $\epsilon(\phi_\mathfrak{p})=\mathfrak{g}(\phi_\mathfrak{p}^{-1})\phi_\mathfrak{p}(-p^n)$. □

Theorem 9.13. *Let $\sigma_{-1,\mathfrak{p}}:=\mathrm{rec}_\mathfrak{p}(-1)$. Then in $\widetilde{\mathbb{I}}[\![\widetilde{\Gamma}_\infty]\!]$ we have:*

$$\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}=\left(\frac{\sigma_{-1,\mathfrak{p}}}{\sqrt{-D_K}}\right)\cdot\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}.$$

Proof. The equality holds when specialized at arithmetic primes of weight 2 by Proposition 9.12, and the result follows because these primes are dense. □

Corollary 9.14. *$\mathfrak{Z}_c$ is not $\mathbb{I}$-torsion.*

Proof. Since $\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}}$ is not zero, the same is true for $\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}}$; any specializations at $\nu\colon\mathbb{I}\to\overline{\mathbb{Q}}_p$ has therefore only a finitely many zeroes. If $\mathfrak{Z}_c$ is torsion, then there are specializations having infinitely many zeroes, which is a contradiction. □

10. Big Generalized Heegner classes

We recall some general results on representations of algebraic groups obtained in [Anc15]. Let G be an algebraic group, (G,D) be a PEL Shimura datum, U a compact open subset of $G(\mathbb{A}_f)$, S the canonical model of the Shimura variety $\mathrm{Sh}_U(G,D)$ of level U over the reflex field F and $\pi\colon\mathcal{A}\to S$ be the universal PEL abelian variety. Then there is a functor

$$\mathrm{Hod}_S\colon\ \mathrm{CHM}_F(S)\longrightarrow\mathrm{VHS}_F(S(\mathbb{C})),$$

called the *Hodge realization* functor, from the category $\mathrm{CHM}_F(S)$ of relative Chow motives (X,p,n) (where $X\to S$ is a smooth projective scheme, $p\in\mathrm{CH}^{\dim(X)}(X\times_S X)_F$ satisfies

$p^2 = p$ and $n \in \mathbb{Z}$), to the category $\mathrm{VHS}_F(S(\mathbb{C}))$ of variations of F-Hodge structures over $S(\mathbb{C})$ (see *ibid.* Example 3.3(i) and Proposition 3.5). Under the functor Hod_S, $R^1\pi_*F$ corresponds to $h^1(\mathcal{A}) \in \mathrm{CHM}_F(X)$, the degree 1 part of the relative Chow motive of $\mathcal{A}$ over S/F, and decompositions into direct summands in $\mathrm{VHS}_F(S(\mathbb{C}))$ lift to decompositions in $\mathrm{CHM}_F(S)$ (*cf. ibid.* Thérorème 6.1). By *ibid.* Théorème 8.6, the canonical construction functor ([Pin90, §1.18]) lifts through Hod_S to a functor

$$\mathrm{Anc}_G\colon \mathrm{Rep}_F(G) \longrightarrow \mathrm{CHM}_F(S),$$

where $\mathrm{Rep}_F(G)$ is the category of F-representations of G, with the following properties:

- Anc_G is F-linear, preserves duals and tensor products;
- If V_G is the standard algebraic representation of G, $\mathrm{Anc}_G(V_G(F)) = h^1(\mathcal{A})$ (following the normalization in [LSZ22, Remark 6.2.3], see also [Tor20, §8]).

See [Anc15, §2] for properties of the decomposition of Chow motives.

10.1. **Generalized Heegner cycles.** Let $m \in \mathbb{Z}_{\geq 0}$. We present the definition of generalized Heegner cycles over the curves X_m adopting, as discussed before, a motivic approach as in [JLZ21]; in this paper we are primarily interested in the cases $m \in \{0,1\}$.

Let $k \geq 2$ be an even integer and fix, as in [Mag22, §2.4.2], the QM abelian surface with CM by $\mathcal{O}_K$ given by $A = E \times E$, where $E = \mathbb{C}/\mathcal{O}_K$. Let $W_{k,m} := \mathcal{A}_m^{k/2-1} \times_{X_m} A^{k/2-1}$ be the generalized Kuga–Sato variety introduced in [HB15, §2.6], where $\mathcal{A}_m^{k/2-1}$ is the $(k/2-1)$-fold fiber product over the Shimura curve X_m of the universal QM abelian surface $\mathcal{A}_m \to X_m$. Since $\mathcal{A}_m$ can be defined over $\mathbb{Q}$, as well as X_m, and A can be defined over the Hilbert class field H of K, the variety $W_{k,m}$ has a model, which we fix, defined over H.

Fix an integer $n \geq m$ and, as before, an integer $c \geq 1$ prime to ND_Kp. Let $F_{\mathfrak{N}^+}$ be the ray class field of K modulo $\mathfrak{N}^+$ and $F_{\mathfrak{N}^+,cp^n}$ the smallest abelian extension of K containing both $F_{\mathfrak{N}^+}$ and H_{cp^n}; note that $F_{\mathfrak{N}^+,cp^n}$ corresponds by class field theory to the group $U_{\mathfrak{N}^+,cp^n}$ of elements $x \in \widehat{\mathcal{O}}_{cp^n}^\times$ such that $x \equiv 1 \mod \mathfrak{N}^+$. To simplify the notation, we forget the dependence on $\mathfrak{N}^+$ and set $F_{cp^n} := F_{\mathfrak{N}^+,cp^n}$.

Let $\epsilon_{\mathcal{A}} \in \mathrm{Corr}_{X_m}(\mathcal{A}_m^{k/2-1})$ be the projector defined in [Bes95, Theorem 5.8], and define the projector $\epsilon_A \in \mathrm{Corr}^{k-2}(A,A)$ to be the image of $\epsilon_{\mathcal{A}}$ when specializing all the factors of $\mathcal{A}_m^{k/2-1}$ to A. Finally, define $\epsilon_W = \epsilon_{\mathcal{A}}\epsilon_A \in \mathrm{Corr}_{X_m}^{k/2-1}(W_{k,m}, W_{k,m})$. Let

$$\Delta_{cp^n,m}^{[k]} = \epsilon_W(\mathrm{graph}(\phi_{cp^n}))^{k/2-1} \in \epsilon_W \mathrm{CH}^{k-1}(W_{k,m} \otimes_H F_{cp^n})_{\mathbb{Q}}$$

denote the generalized Heegner cycle, constructed in [HB15, §6.2], associated with the canonical cyclic cp^n-isogeny

$$\phi_{cp^n}\colon A = \mathbb{C}/\mathcal{O}_K \times \mathbb{C}/\mathcal{O}_K \longrightarrow A_{cp^n} = \mathbb{C}/\mathcal{O}_{cp^n} \times \mathbb{C}/\mathcal{O}_{cp^n},$$

which is defined over F_{cp^n}.

The Chow group of codimension $k-1$ cycles in $W_{k,m}$ can be interpreted as a motivic cohomology group [MVW06, Corollary 19.2] and the generalized Heegner cycle is in its ϵ_W-component:

$$\Delta_{cp^n,m}^{[k]} \in \epsilon_W \,\mathrm{CH}^{k-1}(W_{k,m} \otimes_H F_{cp^n})_{\mathbb{Q}} \cong \epsilon_W H^{2k-2}_{\mathrm{mot}}(W_{k,m} \otimes_H F_{cp^n}, \mathbb{Q}(k-1)). \tag{10.1}$$

Denote

$$r_{\text{ét}}\colon \epsilon_W H^{2k-2}_{\mathrm{mot}}(W_{k,m} \otimes_H F_{cp^n}, \mathbb{Q}(k-1)) \longrightarrow \epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m} \otimes_H F_{cp^n}, \mathbb{Q}_p(k-1)) \tag{10.2}$$

the étale realization. We can use Lieberman's trick on the étale cohomology groups to replace the base scheme $W_{k,m}$ with the simpler Shimura variety X_m, to the cost of having a slightly more complicated coefficient system. We now describe the trick in our context. From Künneth theorem ([Mil80, Theorem 8.21]),

$$H^{2k-2}_{\text{ét}}(W_{k,m},\mathbb{Q}_p)=\bigoplus_{i+j=2k-2} H^i_{\text{ét}}(\mathcal{A}_m^{k/2-1},\mathbb{Q}_p)\otimes H^j_{\text{ét}}(A^{k/2-1},\mathbb{Q}_p).$$

Let $\pi_{\mathcal{A}_m}\colon \mathcal{A}_m\to X_m$ and $\pi_A\colon A\to X_m$ be the canonical projections. Since the Leray spectral sequence degenerates at page 2 (*cf.* [Del68, §2.4]), each of the groups in the right-hand side decomposes as

$$H^i_{\text{ét}}(\mathcal{A}_m^{k/2-1},\mathbb{Q}_p)=\bigoplus_{a+b=i} H^a_{\text{ét}}(X_m,R^b\pi_{\mathcal{A}_m,*}(\mathbb{Q}_p)),$$
$$H^j_{\text{ét}}(A^{k/2-1},\mathbb{Q}_p)=\bigoplus_{a+b=j} H^a_{\text{ét}}(X_m,R^b\pi_{A,*}(\mathbb{Q}_p))$$

and the image of the projectors $\epsilon_{\mathcal{A}}$ and ϵ_A are motives whose Betti realizations are of type $((k-1,0),(0,k-1))$, as in [Bes95] (see the proof of Theorem 5.8 and the paragraph after the proof of Proposition 5.9 in *op. cit.*). Therefore, the only summand remaining after applying the projectors corresponds to the indexes $i=j=k-1$, so

$$\begin{array}{ccc}\epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m},\mathbb{Q}_p) & = & H^1_{\text{ét}}\left(X_m,\mathrm{TSym}^{k-2}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)\right)\otimes H^1_{\text{ét}}\left(X_m,\mathrm{TSym}^{k-2}(eR^1\pi_{A,*}\mathbb{Q}_p)\right)\\ & & \Big\downarrow{\scriptstyle \mathrm{PD}}\\ & & H^2_{\text{ét}}\left(X_m,\mathrm{TSym}^{k-2}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)\otimes \mathrm{TSym}^{k-2}(eR^1\pi_{A,*}\mathbb{Q}_p)\right),\end{array}$$

where we write $R^1\pi_*\mathbb{Q}_p=eR^1\pi_*\mathbb{Q}_p\oplus\bar{e}R^1\pi_*\mathbb{Q}_p$ for the decomposition in isomorphic factors induced by the idempotents e and $\bar{e}$ from §2.2; note that the action of e is built in the definition of the projector ϵ_W (*ibid.*, Theorem 5.8); the map PD is Poincaré duality (see for example [Mil80, Corollary 11.2]). Finally, one can twist by $k-1$ to achieve

(10.3)
$$\epsilon_W H^{2k-2}_{\text{ét}}(W_{k,m},\mathbb{Q}_p(k-1))\longrightarrow H^2_{\text{ét}}\left(X_m,\mathrm{TSym}^{k-2}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)\otimes\mathrm{TSym}^{k-2}(eR^1\pi_{A,*}\mathbb{Q}_p)(k-1)\right).$$

It will be convenient to have the twists distributed in the following way: denoting

$$\mathscr{M}_{\text{ét}}:=\mathrm{TSym}^{k-2}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p(1))\otimes\mathrm{TSym}^{k-2}(eR^1\pi_{A,*}\mathbb{Q}_p),$$

we have that

$$\mathrm{TSym}^{k-2}(eR^1\pi_{\mathcal{A},*}\mathbb{Q}_p)\otimes\mathrm{TSym}^{k-2}(eR^1\pi_{A,*}\mathbb{Q}_p)(k-1)\cong\mathscr{M}_{\text{ét}}(1). \tag{10.4}$$

The composition of the étale realization $r_{\text{ét}}$ with the Lieberman's trick map (10.3) and the map induced by the isomorphism above gives a map

$$\epsilon_W H^{2k-2}_{\text{mot}}(W_{k,m}\otimes_H F_{cp^n},\mathbb{Q}(k-1))\longrightarrow H^2_{\text{ét}}\left(X_m\otimes_H F_{cp^n},\mathscr{M}_{\text{ét}}(1)\right). \tag{10.5}$$

Definition 10.1. The image of $\Delta^{[k]}_{cp^n,m}$ by (10.5) is the *generalized Heegner class* $z^{[k]}_{cp^n,m}$.

10.2. **Representations associated to motives.** Consider the motive

$$\mathscr{M}:=\mathrm{TSym}^{k-2}(eh^1(\mathcal{A}_m)(1))\otimes\mathrm{TSym}^{k-2}(eh^1(A)),$$

whose étale realization is $\mathscr{M}_{\text{ét}}$. Under Ancona's functor, each factor comes respectively from K-representations of the algebraic groups $\mathbf{G}:=\mathrm{Res}_{K/\mathbb{Q}}(B^\times)$ and $\mathbf{H}:=\mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_m^\times)$ as follows.

We first study the representations of $\mathbf{G}$. The algebraic group $\mathbf{G}$ is associated to the PEL Shimura curve X_m. Since $\mathbf{G}(K)=\mathrm{GL}(B\otimes_{\mathbb{Q}}K)$ and $\dim_K(B\otimes_{\mathbb{Q}}K)=4$, the standard representation of $\mathbf{G}$ over K is K^4. By [Anc15, Corollaire 2.6], $h^1(\mathcal{A}_m)^\vee\cong h^1(\mathcal{A}_m)(1)$ and, since $e(h^1(\mathcal{A}_m))^\vee\cong(e^\dagger h^1(\mathcal{A}_m))^\vee$, we have

$$eh^1(\mathcal{A}_m)(1)\cong(e^\dagger h^1(\mathcal{A}_m))^\vee.$$

The idempotents $e^\dagger$ and $\bar{e}^\dagger$ split K^4 into $K^2 \oplus K^2 = e^\dagger K^4 \oplus \bar{e}^\dagger K^4$. Therefore,

$$\mathscr{V}^{k-2} \cong \mathrm{TSym}^{k-2}\left((e^\dagger h^1(\mathcal{A}_m))^\vee\right) \cong \mathrm{Anc}_{\mathbf{G}}\left(\mathrm{TSym}^{k-2}((K^2)^\vee)\right) \cong \mathrm{Anc}_{\mathbf{G}}\left((\mathrm{Sym}^{k-2}(K^2))^\vee\right).$$

We next study the representations of $\mathbf{H}$. We fix a field F of characteristic 0 equipped with an embedding $\sigma_F\colon K \hookrightarrow F$; assume that the image of $\bar{\sigma}_F(x) := \sigma_F(\bar{x})$ (a priori defined in a Galois closure of F, where $x \mapsto \bar{x}$ is the action of the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$) is still contained in F. For all pairs of integers (ℓ_1, ℓ_2), we define the F-representation $\sigma_F^{\ell_1} \otimes \bar{\sigma}_F^{\ell_2}$ of $\mathbf{H}$ by $(\sigma_F^{\ell_1} \otimes \bar{\sigma}_F^{\ell_2})(x \otimes a) = a\sigma_F^{\ell_1}(x)\bar{\sigma}_F^{\ell_2}(x)$ for all $x \in K$ and all elements $a \in A^\times$ for a $\mathbb{Q}$-algebra A. For $F = K$ and σ_K the identity map, we simplify the notation and write $\sigma^{\ell_1} \otimes \bar{\sigma}^{\ell_2}$ for $\sigma_K^{\ell_1} \otimes \bar{\sigma}_K^{\ell_2}$. Recall the group $U_{\mathfrak{N}^+,cp^n}$ corresponding to $F_{cp^n} = F_{\mathfrak{N}^+,cp^n}$ by class field theory. The PEL Shimura variety of level $U_{\mathfrak{N}^+,cp^n}$ associated to the torus $\mathbf{H}$ is zero-dimensional, admits a canonical model S_{cp^n} defined over K, and is identified with the $\mathrm{Gal}(F_{cp^n}/K)$-orbit of elliptic curves defined over F_{cp^n} with CM by $\mathcal{O}_K$; see [Mil05, Proposition 12.11] and [AGHMP17, §1.3]. Recall that $A \cong E \times E$, where E is an elliptic curve with CM by $\mathcal{O}_K$ (thus defining a point in S_{cp^n}), and that this isomorphism is equivariant respect to the action of $\mathrm{M}_2(\mathcal{O}_K)$. Since $j(e) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we have $eh^1(A) = h^1(E)$, and $h^1(E)$ corresponds to a relative Chow motive in $\mathrm{CHM}_K(S_{cp^n})$. Let $V \in \mathrm{Rep}_K(\mathbf{H})$ be such that $\mathrm{Anc}_{\mathbf{H}}(V) = h^1(E)$. For an integer $0 \le j \le k-2$, one can consider the $(k-2-j, j)$-component $V^{k-2-j,j}$ of V, where the complex multiplication by $x \in \mathcal{O}_K$ acts as the multiplication by $x^{k-2-j}\bar{x}^j$. Then $V^{k-2-j,j} = \sigma^{k-2-j} \otimes \bar{\sigma}^j$. We define

$$h^{(k-2-j,j)}(A) = \mathrm{Anc}_{\mathbf{H}}(\sigma^{k-2-j} \otimes \bar{\sigma}^j).$$

We finally study the representations of $\mathbf{G} \times \mathbf{H}$. Piecing all of the above together, we get a K-representation

$$V^{(k-2-j,j)} = (\mathrm{TSym}^{k-2}(K^2)^\vee) \boxtimes (\sigma^{k-2-j} \otimes \bar{\sigma}^j) \in \mathrm{Rep}_K(\mathbf{G} \times \mathbf{H})$$

(here, as usual, $\boxtimes$ denotes the external tensor product). Therefore, defining

$$\mathscr{M}^{(k-2-j,j)} := \mathscr{V}^{k-2} \otimes h^{(k-2-j,j)}(A), \tag{10.6}$$

we find that $\mathrm{Anc}_{\mathbf{G}\times\mathbf{H}}(V^{(k-2-j,j)}) = \mathscr{M}^{(k-2-j,j)}$ in $\mathrm{CHM}_K(X_m \times S_{cp^n})$.

10.3. **Vectors from CM points.** We now use CM points to construct basis elements of the representation $V^{(k-2-j,j)}$ base-changed to suitable p-adic fields. We begin by observing that the fixed embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and the canonical inclusion $K \subseteq \overline{\mathbb{Q}}$ induce an embedding $\sigma_{\overline{\mathbb{Q}}_p}\colon K \hookrightarrow \overline{\mathbb{Q}}_p$ whose image is contained in $\mathbb{Q}_p$ because p is split in K. Fix a subfield $L \subseteq \overline{\mathbb{Q}}_p$; therefore, we obtain two embeddings $\sigma_L\colon K \hookrightarrow L$ and $\bar{\sigma}_L\colon K \hookrightarrow L$ satisfying $\bar{\sigma}_L(x) = \sigma_L(\bar{x})$. The map $\iota_K\colon K \hookrightarrow B$ induces an embedding of algebraic groups $i\colon \mathbf{H} \hookrightarrow \mathbf{G}$ and, as before, for each pair of integers (ℓ_1, ℓ_2), we have a one-dimensional L-representation $\sigma_L^{\ell_1} \otimes \bar{\sigma}_L^{\ell_2}$ of $\mathbf{H}$.

Recall that the choice of ϑ is normalized with respect to ι_K so that $\iota_K(x)\left(\begin{smallmatrix} \vartheta \\ 1 \end{smallmatrix}\right) = \sigma_L(x)\left(\begin{smallmatrix} \vartheta \\ 1 \end{smallmatrix}\right)$. More generally, we can similarly define vectors $e_{cp^n} \in L^2$ attached to CM points $[(\iota_K, \xi^{(n)})]$ as follows. First, write $\xi^{(n)} = b^{(n)}u^{(u)}$ for some $b^{(n)} \in B^\times$ and $u^{(n)} \in U_m$; then $\vartheta_{cp^n} := (b^{(n)})^{-1}(\vartheta)$ is an eigenvector for $i_{cp^n} = (b^{(n)})^{-1}\iota_K b^{(n)}$, and $i_{cp^n}(\mathbf{H})$ acts on $v_{cp^n} := \left(\begin{smallmatrix} \sigma_L(\vartheta_{cp^n}) \\ 1 \end{smallmatrix}\right) \in L^2$ as the representation σ_L; note that, under the isomorphism (of compact Riemann surfaces) $X_m(\mathbb{C}) \cong \Gamma_m \backslash \mathcal{H}$ in §2.6 the point $[(\iota_K, \xi^{(n)})]$ is sent to the class of the point ϑ_{cp^n}. Dually, setting $\vartheta^*_{cp^n} = -1/\bar{\vartheta}_{cp^n}$, a simple computation shows that $e_{cp^n} := \left(\begin{smallmatrix} \sigma_L(\bar{\vartheta}^*_{cp^n}) \\ 1 \end{smallmatrix}\right)$ an eigenvector for the dual $(L^2)^\vee$ of the standard representation L^2 of $\mathbf{H}$, *i.e* for each $x \in K$ we have $(i_{cp^n}(x)^{-1})^{\mathrm{T}}(e_{cp^n}) = \sigma_L^{-1}(x)e_{cp^n}$ (where A^{T} denotes the transposed matrix of a matrix A). Define $e_{cp^n}^{[k,j]} := (e_{cp^n})^{\otimes(k-2-j)} \cdot (\bar{e}_{cp^n})^{\otimes j}$ in $\mathrm{TSym}^{k-2}((L^2)^\vee)$ where $\bar{e}_{cp^n} := \left(\begin{smallmatrix} \bar{\sigma}_L(\vartheta^*_{cp^n}) \\ 1 \end{smallmatrix}\right)$. Then

$e_{cp^n}^{[k,j]}$ defines an element in the L-representation

$$V_L^{(k-2-j,j)} := \mathrm{TSym}((L^2)^\vee) \boxtimes \sigma_L^{(k-2-j)} \otimes \bar{\sigma}_L^j$$

of $\mathbf{H}$, which is invariant under the diagonal action of K by $i_{cp^n} \otimes \mathrm{id}$.

10.4. **The j-component of generalized Heegner classes.** Each of the embeddings i_{cp^n} induces an embedding of $\delta_{cp^n} \colon \mathbf{H} \hookrightarrow \mathbf{G} \times \mathbf{H}$, defined by $\delta_{cp^n} = (i_{cp^n}, \mathrm{id})$, where id is the identity map. By [Tor20, Theorem 9.7], we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_K(\mathbf{G} \times \mathbf{H}) \arrow[r, "\delta_{cp^n}^*"] \arrow[d, "\mathrm{Anc}_{\mathbf{G}\times\mathbf{H}}"] & \mathrm{Rep}_K(\mathbf{H}) \arrow[d, "\mathrm{Anc}_{\mathbf{H}}"] \\
\mathrm{CHM}_K(X_m \times S_{cp^n}) \arrow[r, "\delta_{cp^n}^*"] & \mathrm{CHM}_K(S_{cp^n})
\end{tikzcd}$$

where the top map $\delta_{cp^n}^* \colon \rho \mapsto \delta_{cp^n} \circ \rho$ is the restriction via δ_{cp^n} of representations, and the bottom map $\delta_{cp^n}^*$ is the pullback of motives via the map $\delta_{cp^n}^* \colon X_m \times S_{cp^n} \to S_{cp^n}$ induced by δ_{cp^n}. Via the functoriality of the étale realization, the maps described above descend to maps of lisse étale shaves over L. To simplify the notation, denote $X_m \times S_{cp^n}$ the K-scheme $(X_m \otimes_{\mathbb{Q}} K) \times S_{cp^n}$ (*i.e.* we simply view X_m as a K-scheme and take the product as K-schemes).

Let $\mathscr{M}_{\text{ét}}^{(k-2-j,j)}$ be the étale realization of the motive $\mathscr{M}^{(k-2-j,j)} = \mathrm{Anc}_{\mathbf{G}\times\mathbf{H}}(V^{(k-2-j,j)})$ in $\mathrm{CHM}_K(X_m \times S_{cp^n})$ introduced in (10.6); consider the motive $\delta_{cp^n}^*(\mathscr{M}_{\text{ét}}^{(k-2-j,j)})$ in $\mathrm{CHM}_K(S_{cp^n})$. Composing the Gysin map (see [KLZ20, Definition 3.1.2, §5.2])

$$H^0_{\text{ét}}\left(S_{cp^n}, \delta_{cp^n}^*(\mathscr{M}_{\text{ét}}^{(k-2-j,j)})\right) \longrightarrow H^2_{\text{ét}}\left(X_m \times S_{cp^n}, \mathscr{M}_{\text{ét}}^{(k-2-j,j)}(1)\right)$$

with the isomorphism

$$H^2_{\text{ét}}\left(X_m \times S_{cp^n}, \mathscr{M}_{\text{ét}}^{(k-2-j,j)}(1)\right) \xrightarrow{\sim} H^2_{\text{ét}}\left(X_m \otimes_K F_{cp^n}, \mathscr{M}_{\text{ét}}^{(k-2-j,j)}(1)\right)$$

that comes from the identification of S_{cp^n} as a K-variety with the $\mathrm{Gal}(F_{cp^n}/K)$-orbit of ϑ_{cp^n}, we obtain a map

$$\delta_{cp^n,*} \colon H^0_{\text{ét}}\left(S_{cp^n}, \delta_{cp^n}^*(\mathscr{M}_{\text{ét}}^{(k-2-j,j)})\right) \longrightarrow H^2_{\text{ét}}\left(X_m \otimes_K F_{cp^n}, \mathscr{M}_{\text{ét}}^{(k-2-j,j)}(1)\right). \tag{10.7}$$

Definition 10.2. The image of $e_{cp^n}^{[k,j]}$ under the map $\delta_{cp^n,*}$ in (10.7) is the *j-component* $z_{cp^n,m}^{[k,j]}$ of the generalized Heegner class $z_{cp^n,m}^{[k]}$.

The above construction is useful for the p-adic interpolation of the vectors $e_{cp^n}^{[k,j]}$ as way to interpolate generalized Heegner classes. However, there is an equivalent and simpler construction of the classes $z_{cp^n,m}^{[k,j]}$. The projection of $\mathrm{TSym}^{k-2}(h^1(A))$ onto the direct summand $h^{(k-2-j,j)}(A)$ is a correspondence $\mathscr{M} \to \mathscr{M}^{(k-2-j,j)}$ in $\mathrm{Corr}^0_{X_m}(W_{k,m})$, which induces, under the étale realization of motives, a projection $\mathscr{M}_{\text{ét}} \to \mathscr{M}_{\text{ét}}^{(k-2-j,j)}$ and therefore a pushforward map in the étale cohomology

$$H^2_{\text{ét}}\left(X_m \otimes_{\mathbb{Q}} F_{cp^n}, \mathscr{M}_{\text{ét}}(1)\right) \longrightarrow H^2_{\text{ét}}\left(X_m \otimes_{\mathbb{Q}} F_{cp^n}, \mathscr{M}_{\text{ét}}^{(k-2-j,j)}(1)\right), \tag{10.8}$$

under which $z_{cp^n,m}^{[k]}$ maps to the class $z_{cp^n,m}^{[k,j]}$.

10.5. **The Abel–Jacobi map.** The degeneration at page 2 of the Hochschild–Lyndon–Serre spectral sequence [Nek00, §1.2] yields an isomorphism

$$H^2_{\text{ét}}\left(X_m \otimes_{\mathbb{Q}} F_{cp^n}, \mathscr{M}^{(k-2-j,j)}_{\text{ét}}(1)\right) \xrightarrow{\sim} H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{M}^{(k-2-j,j)}_{\text{ét}}(1))\right). \tag{10.9}$$

Recall the motive $\mathrm{TSym}^{k-2}(eh^1(\mathcal{A}_m)(1)) = \mathscr{V}^{k-2}$ introduced in §6.1 and its étale realization $\mathscr{V}^{k-2}_{\text{ét}}$, an étale lisse sheaf over L. The right-hand side of (10.9) can be further rewritten as

(10.10)
$$H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{M}^{(k-2-j,j)}_{\text{ét}}(1))\right) \cong H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}^{k-2}_{\text{ét}}(1) \otimes \sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}})\right).$$

Here $\sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}$ denotes the étale realization of the $\mathbf{H}$-representation $\sigma^{k-2-j} \otimes \bar{\sigma}^j$; concretelly, the reciprocity map induces an isomorphism $(1+\mathfrak{N}^+\widehat{\mathcal{O}}_K)^\times \cong \mathrm{Gal}(K^{\mathrm{ab}}/F_{\mathfrak{N}^+})$, and the characters $\sigma_{\text{ét}}, \bar{\sigma}_{\text{ét}} \colon \mathrm{Gal}(K^{\mathrm{ab}}/F_{\mathfrak{N}^+}) \to \mathbb{Q}_p^\times$ are given by $x \mapsto \sigma^{-1}(x_p)$ and $x \mapsto \bar{\sigma}^{-1}(x_p)$ on $(1+\mathfrak{N}^+\widehat{\mathcal{O}}_K)^\times$, respectively (recall that the reciprocity map is geometrically normalized). The composition of the maps (10.1), (10.5), (10.8), (10.9) and (10.10) gives the *p-adic Abel–Jacobi map*

(10.11)
$$\Phi^{[k,j]}_m \colon \epsilon_W \mathrm{CH}^{k-1}(W_{k,m} \otimes_H F_{cp^n})_{\mathbb{Q}} \longrightarrow H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}^{k-2}_{\text{ét}}(1) \otimes \sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}})\right).$$

Note that the image of $\Delta^{[k]}_{cp^m,n}$ under (10.11) is the j-component $z^{[k,j]}_{cp^n,m}$ introduced above, *i.e* $z^{[k,j]}_{cp^n,m} = \Phi^{[k,j]}_m(\Delta^{[k]}_{cp^m,n})$.

10.6. **Classes associated to quaternionic modular forms.** Let $\mathcal{F} \in M_k(N^+p^m, L)$ be a quaternionic newform of weight k over X_m. Since $V^\dagger_{\mathcal{F}} \cong V^*_{\mathcal{F}}(1-k/2) \cong V_{\mathcal{F}}(k/2)$, as $G_{F_{cp^n}}$-representations we have $V^*_{\mathcal{F}} \otimes \sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}} \cong V^\dagger_{\mathcal{F}} \otimes (\sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}\chi^{k/2-1}_{\text{cyc}})$, so there is a projection map (*cf.* [KLZ17, §2.8])

(10.12)
$$\mathrm{pr}_{\mathcal{F}} \colon H^1\left(F_{cp^n}, H^1_{\text{ét}}(X_m \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathscr{V}^{k-2}_{\text{ét}}(1) \otimes \sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}\right) \longrightarrow H^1\left(F_{cp^n}, V^\dagger_{\mathcal{F}} \otimes (\sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}\chi^{k/2-1}_{\text{cyc}})\right).$$

Definition 10.3. The class $z^{[\mathcal{F},j]}_{cp^n,m} := \mathrm{pr}_{\mathcal{F}}\left(z^{[k,j]}_{cp^n,m}\right)$ is the *generalized Heegner class* associated to $\mathcal{F}$ and j with $0 \le j \le k-2$.

Lemma 10.4. *Let ξ be a Hecke character of infinity type $(k-2-j, j)$ of conductor c prime to Np. Then the generalized Heegner class $z^{[\mathcal{F},j]}_{cp^n,m}$ belongs to the* $\mathrm{Gal}(F_{cp^n}/H_{cp^n})$*-invariant subspace of $H^1(F_{cp^n}, V^\dagger_{\mathcal{F}} \otimes \xi\chi^{k/2-1}_{\text{cyc}})$.*

Proof. The argument is taken from [JLZ21, Proposition 3.5.2]. The 0-dimensional variety S_{cp^n} has an action of $\widehat{\mathcal{O}}^\times_{cp^n}/U_{\mathfrak{N}^+,cp^n} \cong (\mathbb{Z}/N^+\mathbb{Z})^\times$, and the embedding δ_{cp^n} intertwines this action with the action of $(\mathbb{Z}/N^+\mathbb{Z})^\times$ on X_m given by diamond operators. Now $\mathcal{F}$ has trivial character, and ξ has conductor prime to N^+, so ξ restricts to the character $\sigma^{k-2-j}_{\text{ét}}\bar{\sigma}^j_{\text{ét}}$ on $\mathrm{Gal}(F_{cp^n}/H_{cp^n})$, thus extending $\sigma^{k/2-1-j}\bar{\sigma}^{j-(k/2-1)}$ to $\mathrm{Gal}(K^{\mathrm{ab}}/H_{cp^n})$. This proves the result, in light of Shimura reciprocity law and the fact that $z^{[\mathcal{F},j]}_{cp^n,m}$ lies in the finite dimensional subspace of classes which are unramified outside Np. $\square$

The map $\xi \mapsto \xi\chi^{k/2-1}_{\text{cyc}}$ is a bijection between Hecke characters of infinity type (ℓ_1, ℓ_2) and those of infinity type $(\ell_1-(k/2-1), \ell_2-(k/2-1))$. The inflation-restriction exact sequence and the irreducibility of $V_{\mathcal{F}}$ induce for all such ξ an isomorphism

$$\left(H^1(F_{cp^n}, V^\dagger_{\mathcal{F}} \otimes \xi)\right)^{\mathrm{Gal}(F_{cp^n}/H_{cp^n})} \cong H^1(H_{cp^n}, V^\dagger_{\mathcal{F}} \otimes \xi). \tag{10.13}$$

Definition 10.5. Suppose that ξ is an algebraic Hecke character of infinity type $(\ell, -\ell)$ with $-(k/2-1) \leq \ell \leq (k/2-1)$, and let and $j = k/2-1-\ell$. The *ξ-component* of the generalized Heegner class $z^{[\mathcal{F},j]}_{cp^n,m}$ is its image of via the isomorphism (10.13) "untwisted" by ξ, that is, the class $z^{[\mathcal{F},j]}_{cp^n,m} \otimes \xi^{-1} \in H^1(H_{cp^n}, V^{\dagger}_{\mathcal{F}})$.

The ξ-component of a generalized Heegner class can naturally be restricted to F_{cp^n}, giving a class $z^{[\mathcal{F},j]}_{cp^n,m} \otimes \xi^{-1} \in H^1(F_{cp^n}, V^{\dagger}_{\mathcal{F}})$.

10.7. **p-adic families of basis vectors.** As in §3.5, let $\mathscr{U} \subseteq \mathcal{X} = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ be a connected open neighborhood of an integer $k_0 \in \mathbb{Z}$ and $\kappa\colon \Gamma \hookrightarrow \Lambda^\times$ be its universal character. Let $\Lambda_{\mathscr{U}}$ be the Iwasawa algebra of $\mathscr{U}$. For a character $\sigma\colon R^\times \to \Gamma$, where R is a p-adic ring, we write $\sigma^{\kappa_{\mathscr{U}}} := \kappa_{\mathscr{U}} \circ \sigma$, which naturally extends the exponentiation by an integer power. For any pair of integers (ℓ_1, ℓ_2) we also write $\sigma^{\pm\kappa_{\mathscr{U}}+\ell_1}\bar{\sigma}^{\ell_2} := (\sigma^{\kappa_{\mathscr{U}}})^{\pm 1}\sigma^{\ell_1}\bar{\sigma}^{\ell_2}$. Let $\mathscr{C}(\mathbb{Z}_p, \Lambda[1/p])$ be the $\Lambda[1/p]$-module of continuous $\Lambda[1/p]$-valued functions on $\mathbb{Z}_p$ equipped with the left action of the monoid $\Sigma = \mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{M}_2(\mathbb{Z}_p)$ given by $\gamma \cdot f(Z) = \kappa_{\mathscr{U}}(bZ+d) f(Z \cdot \gamma)$, where $Z \cdot \gamma = \frac{aZ+c}{bZ+d}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The canonical embedding $\mathrm{Sym}^{k-2}(L^2) \hookrightarrow \mathscr{C}(\mathbb{Z}_p, \Lambda[1/p])$ defined by $P(X,Y) \mapsto P(Z,1)$ with $Z = X/Y$ is equivariant with respect to this action, which reduces to the weight k action from the beginning of Section 6. The dual of this embedding gives the *moment map*

$$\mathrm{mom}^{k-2}\colon D_{\mathscr{U}} := \mathrm{Hom}_{\Lambda_{\mathscr{U}}[1/p]}(\mathscr{C}(\mathbb{Z}_p, \Lambda[1/p]), \Lambda_{\mathscr{U}}[1/p]) \longrightarrow \mathrm{TSym}^{k-2}((L^2)^\vee)$$

defined by the integration formula

$$\left(\mathrm{mom}^{k-2}(\mu)\right)(\varphi) = \int_{\mathbb{Z}_p} \varphi(x,1) d\mu(x), \tag{10.14}$$

for each $\varphi \in \mathrm{Sym}^{k-2}(L^2)$.

Lemma 10.6. *$\sigma(\vartheta^*_{cp^n}) = \alpha_{cp^n} p^n$ for some $\alpha_{cp^n} \in \mathbb{Z}_p^\times$.*

Proof. The question is local. Write $b^{(n)} = i_p(b^{(n)})$ to simplify the notation. Recall that $\xi^{(n)}_p = b^{(n)} u^{(n)}$ and $\xi^{(n)}_p = \delta^{-1}\left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right)$; since $\vartheta \in \mathbb{Z}_p$ and $\delta \in \mathbb{Z}_p^\times$, then, locally at p, we have $b^{(n)} = \left(\begin{smallmatrix} \vartheta & \bar{\vartheta} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^n & 1 \\ 0 & 1 \end{smallmatrix}\right) u_0$ for some element u_0 in $\mathrm{GL}_2(\mathbb{Z}_p)$. Therefore, using again that $\delta \in \mathbb{Z}_p^\times$, we see that $(b^{(n)})^{-1} = u_1 \left(\begin{smallmatrix} p^{-n} & -p^{-n} \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\bar{\vartheta} \\ -1 & \vartheta \end{smallmatrix}\right)$ for some $u_1 = u_0^{-1} \in \mathrm{GL}_2(\mathbb{Z}_p)$. It follows that $\vartheta_{cp^n} = p^{-n} u_1 \left(\begin{smallmatrix} \vartheta - \bar{\vartheta} \\ 0 \end{smallmatrix}\right)$. Now recall that the pair $[(\iota_K, \xi^{(n)})]$ is an Heegner point on X_m *for all* $m \geq 0$, therefore $u^{(n)}$ satisfies the congruence $u^{(n)} \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod p^m$ for all $m \geq 0$, from which we conclude that $u_1 = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $a \in \mathbb{Z}_p^\times$, $d \in \mathbb{Z}_p^\times$ and $b \in \mathbb{Z}_p$. Therefore we have $\vartheta_{cp^n} = p^{-n} a(\vartheta - \bar{\vartheta})$ for a suitable $a \in \mathbb{Z}_p^\times$, and the result follows. ☐

Thanks to Lemma 10.6, we can define $\mathbf{e}_{\mathscr{U},cp^n} \in D_{\mathscr{U}}$ by the integration formula

$$\mathbf{e}_{\mathscr{U},cp^n}(\varphi) = \int_{\mathbb{Z}_p} \varphi(x) \mathbf{e}_{\mathscr{U},cp^n}(x) := \varphi(\sigma(\vartheta^*_{cp^n}))$$

for any continuous function $\varphi \in \mathscr{C}(\mathbb{Z}_p, \Lambda[1/p])$.

Lemma 10.7. *The distribution $\mathbf{e}_{\mathscr{U},cp^n}$ has the following properties:*

- *The action of $i_{cp^n}\left((\mathcal{O}_{cp^n} \otimes \mathbb{Z}_p)^\times\right)$ on $\mathbf{e}_{\mathscr{U},cp^n}$ is via $\sigma^{-\kappa_{\mathscr{U}}}$.*
- *For all integers $k \geq 0$ we have $\mathrm{mom}^{k-2}(\mathbf{e}_{\mathscr{U},cp^n}) = e^{[k,0]}_{cp^n}$.*

Proof. For the first statement, recall that i_{cp^n} is an optimal embedding of $\mathcal{O}_{cp^n}$ into the Eichler order $R_m = B^\times \cap U_m$. For u in $(\mathcal{O}_{cp^n} \otimes \mathbb{Z}_p)^\times$, we have $(i_{cp^n}(u)^{-1})^{\mathrm{T}} \left(\begin{smallmatrix} \vartheta^*_{cp^n} \\ 1 \end{smallmatrix}\right) = \sigma^{-1}(u) \left(\begin{smallmatrix} \vartheta^*_{cp^n} \\ 1 \end{smallmatrix}\right)$ (see

§10.3). Let u act by $\left(\begin{smallmatrix} a & b \\ cp^n & d\end{smallmatrix}\right)$. For each $\varphi \in \mathscr{C}(\mathbb{Z}_p, \Lambda[1/p])$ we then have

$$\begin{aligned} i_{cp^n}(u)^{-1}\mathbf{e}_{\mathscr{U},cp^n}(\varphi) &= \int_{\mathbb{Z}_p} \varphi(x) d(i_{cp^n}(u)^{-1}\mathbf{e}_{\mathscr{U},cp^n})(x) \\ &= \int_{\mathbb{Z}_p} \kappa_{\mathscr{U}}(bx+d)\varphi\left(\frac{ax+cp^n}{bx+d}\right)\mathbf{e}_{\mathscr{U},cp^n}(x) \\ &= \sigma^{\kappa_{\mathscr{U}}}(u)\varphi\left(\frac{a\sigma(\vartheta^*_{cp^n})+cp^n}{b\sigma(\vartheta^*_{cp^n})+d}\right) \\ &= \sigma^{\kappa_{\mathscr{U}}}(u)\varphi(\sigma(\vartheta^*_{cp^n})) \\ &= \sigma^{\kappa_{\mathscr{U}}}(u)\mathbf{e}_{\mathscr{U},cp^n}(\varphi). \end{aligned}$$

Therefore, we conclude that the action of $i_{cp^n}(u)$ on the measure $\mathbf{e}_{\mathscr{U},cp^n}$ is just the product $\sigma^{-\kappa_{\mathscr{U}}(u)}\mathbf{e}_{\mathscr{U},cp^n}$, which is the first statement. For the second, take $P \in \mathrm{Sym}^{k-2}(L^2)$; then we have

$$\left(\mathrm{mom}^{k-2}(\mathbf{e}_{\mathscr{U},cp^n})\right)(P) = \int_{\mathbb{Z}_p} P(x,1)d\mathbf{e}_{\mathscr{U},cp^n}(x) = P\left(\sigma(\vartheta^*_{cp^n}),1\right) = e^{[k,0]}_{cp^n}(P),$$

concluding the proof of the second equality. □

We now define $\mathbf{e}^{[j]}_{\mathscr{U},cp^n} = \mathbf{e}_{\mathscr{U},cp^n}\cdot(\sigma^{-j}\otimes\bar{\sigma}^j)$, where $\cdot$ is the symmetrized tensor product.

Lemma 10.8. *The distribution $\mathbf{e}^{[j]}_{\mathscr{U},cp^n}$ has the following properties:*

- *The group $i_{cp^n}((\mathcal{O}_{cp^n}\otimes\mathbb{Z}_p)^\times)$ acts on $\mathbf{e}^{[j]}_{\mathscr{U},cp^n}$ via the representation $\sigma^{-(\kappa_{\mathscr{U}}-j)}\bar{\sigma}^j$*
- *For all integers $k \geq 0$ we have $\mathrm{mom}^{k-2}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n}) = e^{[k,j]}_{cp^n}$.*

Proof. This follows immediately from Lemma 10.7. □

10.8. **p-adic interpolation of generalized Heegner classes.** Let $\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\sigma^j_{\text{ét}}$ the étale realization of $\sigma^{\kappa_{\mathscr{U}}-j}\sigma^j$. We then have a map

$$H^0_{\text{ét}}\left(S_{cp^n}, \delta^*_{cp^n}(D_{\mathscr{U}}\otimes\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}^j_{\text{ét}})\right) \xrightarrow{\delta_{cp^n,*}} H^2_{\text{ét}}\left(X_m\otimes_K F_{cp^n}, D_{\mathscr{U}}(1)\otimes\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}^j_{\text{ét}}\right) \tag{10.15}$$

Definition 10.9. Let $j \geq 0$ and $n \geq m \geq 1$ be integers. Define the *j-component of the big generalized Heegner class* to be $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m} := \delta_{cp^n,*}(\mathbf{e}^{[j]}_{\mathscr{U},cp^n})$.

The Gysin map for the interpolated coefficient systems is compatible with the one in (10.7) via the moment maps; more precisely, for each $k \in \mathbb{Z}\cap\mathscr{U}$ with $k \geq j$ we have, writing $\overline{X}_m = X_m\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ and $D_{\mathscr{U}}\otimes\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}^j_{\text{ét}} = \mathscr{D}^{(k-2-j,j)}_{\text{ét}}$ to simplify the notation,

(10.16)

$$\begin{array}{ccccc} H^0_{\text{ét}}\left(S_{cp^n},\delta^*_{cp^n}(\mathscr{D}^{(k-2-j,j)}_{\text{ét}})\right) & \xrightarrow{\delta_{cp^n,*}} & H^2_{\text{ét}}\left(X_m\otimes F_{cp^n},\mathscr{D}^{(k-2-j,j)}_{\text{ét}}\right) & \xrightarrow{\sim} & H^1(F_{cp^n},H^1_{\text{ét}}(\overline{X}_m,\mathscr{D}^{(k-2-j,j)}_{\text{ét}})) \\ \big\downarrow{\scriptstyle \mathrm{mom}^{k-2}} & & \big\downarrow{\scriptstyle \mathrm{mom}^{k-2}} & & \big\downarrow{\scriptstyle \mathrm{mom}^{k-2}} \\ H^0_{\text{ét}}\left(S_{cp^n},\delta^*_{cp^n}(\mathscr{M}^{(k-2-j,j)}_{\text{ét}})\right) & \xrightarrow{\delta_{cp^n,*}} & H^2_{\text{ét}}\left(X_m\otimes F_{cp^n},\mathscr{M}^{(k-2-j,j)}_{\text{ét}}(1)\right) & \xrightarrow{\sim} & H^1(F_{cp^n},H^1_{\text{ét}}(\overline{X}_m,\mathscr{M}^{(k-2-j,j)}_{\text{ét}}(1))). \end{array}$$

The class $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}$ gives an element of $H^1(F_{cp^n},H^1_{\text{ét}}(X_m\otimes_{\mathbb{Q}}\overline{\mathbb{Q}},\mathscr{D}^{(k-2-j,j)}_{\text{ét}}))$ via Diagram (10.16), and it follows from the definitions and Lemma 10.8 that for each $k\in\mathbb{Z}\cap\mathscr{U}$ with $k\geq j$ we have $\mathrm{mom}^{k-2}(\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}) = z^{[k,j]}_{cp^n,m}$.

Lemma 10.10. $\mathrm{cores}_{F_{cp^{n+1}}/F_{cp^n}}(\mathbf{z}^{[j]}_{\mathscr{U},cp^{n+1},m}) = U_p'\cdot\mathbf{z}^{[j]}_{\mathscr{U},cp^n,m}$, *for all $n\geq m\geq 0$ and all $j\geq 0$.*

Proof. The generalization of the proof of [JLZ21, Proposition 5.1.2] presents no difficulty and is left to the reader. □

10.9. **Big Heegner classes associated to quaternionic Hida families.** We now consider the special case $m=1$ in the previous constructions. Let $\mathcal{F}_\infty$ be the quaternionic Hida family fixed before, passing through the p-stabilized modular form $\mathcal{F}_0$ of weight $k_0 \equiv 2 \pmod{2(p-1)}$ and trivial character. The universal character $\kappa_{\mathscr{U}}$ is the restriction of $\boldsymbol{\vartheta}^2 \colon \mathbb{Z}_p^\times \to \mathbb{I}^\times$ to $\mathscr{U}$ (recall that the extension $\mathbb{I}/\Lambda$ is locally étale at the point $z \mapsto x^{k-2}$ in $\mathcal{X}$). We still denote $\mathcal{F}_\infty$ the restriction of of $\mathcal{F}_\infty$ to $\mathscr{U}$; we also denote in this section by k an integer in $\mathbb{Z}\cap\mathscr{U}$ and by $\mathcal{F}_k$ the specialization of $\mathcal{F}_\infty$ at k (note that for all such k, the modular form $\mathcal{F}_k$ has trivial character, and $k \equiv 2 \mod p-1$). Let $a_p(\mathcal{F}_k) = a_p(f_{\nu_k})$ be the U_p-eigenvalue (as before, ν_k is the arithmetic morphism corresponding to k). We consider the Galois representation

$$\mathbf{V}_{\mathscr{U}} = H^1_{\text{ét}}(X_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \Lambda[1/p]) \widehat{\otimes}_{\Lambda[1/p]} \Lambda_{\mathscr{U}}[1/p].$$

Then the restriction of $\mathbf{T}[1/p]$ to $\mathscr{U}$ is isomorphic to $\mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) = e^{\text{ord}}\mathbf{V}_{\mathscr{U}}(1)$. For each $k \in \mathbb{Z}\cap\mathscr{U}$ the canonical specialization map $\Lambda_{\mathscr{U}}[1/p] \to F_k$ induces a map $\mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \to V^*_{\mathcal{F}_k}$ (here F_k is the image of the Hecke field of $\mathcal{F}_k$ in $\overline{\mathbb{Q}}_p$). By projection, we therefore obtain a map

$$H^1\left(F_{cp^n}, \Lambda[1/p](1) \otimes \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j}\right) \longrightarrow H^1\left(F_{cp^n}, \mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \otimes \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j}\right). \tag{10.17}$$

Definition 10.11. The $(\mathcal{F}_\infty, j)$-*th component* of the generalized big Heegner class is the image, denoted $\mathbf{z}^{[\mathcal{F}_\infty,j]}_{\mathscr{U},cp^n,1}$ of the generalized big Heegner class $\mathbf{z}^{[j]}_{\mathscr{U},cp^n,1}$ in Definition 10.9 under the map (10.17).

As in [JLZ21, §3.5], observe that the groups appearing above may be infinite dimensional over $\Lambda_{\mathscr{U}}[1/p]$. For a G_K-module M, we shall denote $H^1_\Sigma(K,M) = H^1(\mathrm{Gal}(K_\Sigma/K), M)$, where K_Σ is the maximal extension of K unramified outside the set Σ of all places dividing Np. Then, accordingly with this notation, $\mathbf{z}^{[\mathcal{F}_\infty,j]}_{\mathscr{U},cp^n,1}$ belongs to $H^1_\Sigma(F_{cp^n}, \mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \otimes \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j})$.

We now interpolate the characters $\sigma^{-j}\bar{\sigma}^j$ for $j \geq 0$. Let $F_{cp^\infty} = \bigcup_{n=1}^\infty F_{cp^n}$; the Galois group $\mathrm{Gal}(F_{cp^\infty}/F_c)$ is isomorphic to the group $\mathrm{Gal}(H_{cp^\infty}/H_c)$ (because, since $(cp,N)=1$, $F_{\mathfrak{N}^+}$ and H_{cp^n} are linearly disjoint over K), and this Galois group is isomorphic to the group Γ_∞ in §6.3 (because $\mathfrak{P}$ is totally ramified in H_{cp^∞}). For any integer $n \geq 1$, let Γ_n denote the subgroup $\Gamma_n = \mathrm{Gal}(F_{cp^\infty}/F_{cp^n})$ of Γ_∞. By class field theory, the reciprocity map induces an isomorphism $\Gamma_1 \cong (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times/\mathbb{Z}_p^\times$, so the character $\sigma/\bar{\sigma} \colon (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times/\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ induces a Galois character $\sigma_{\text{ét}}/\bar{\sigma}_{\text{ét}} \colon \Gamma_1 \to \mathbb{Z}_p^\times$. We view $\Lambda_1 = \mathbb{Z}_p[\![\Gamma_1]\!]$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{cp})$-module via the canonical projection and the canonical embedding of group-like elements $\Gamma_1 \hookrightarrow \Lambda_1$; also, denote by $\Lambda_1(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})$ the Galois module Λ_1 equipped with the twisted action by the inverse of $\sigma_{\text{ét}}/\bar{\sigma}_{\text{ét}}$. For each integer j and each integer $n \geq 1$, there is a canonical specialization map $\Lambda_1(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}}) \to \sigma_{\text{ét}}^{-j}\bar{\sigma}_{\text{ét}}^{j}$ from the category of $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{cp})$-representations to the category of $\mathrm{Gal}(\overline{\mathbb{Q}}/F_{cp^n})$-representations which takes $\mu \in \Lambda_1(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})$ to $(\int_{\Gamma_{cp^n}} d\mu)(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})^j$. Finally, define $\sigma^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}^{\mathbf{j}} := \sigma^{\kappa_{\mathscr{U}}}\widehat{\otimes}_{\mathbb{Z}_p}\Lambda_1(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})$. We therefore obtain a map

$$\mathrm{mom}_j \colon H^1\left(F_{cp}, \mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \widehat{\otimes}_{\mathbb{Z}_p} \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right) \longrightarrow H^1(F_{cp^n}, \mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \otimes \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-j}\bar{\sigma}_{\text{ét}}^{j}). \tag{10.18}$$

Proposition 10.12. *There exists an element* $\mathbf{z}^{[\mathcal{F}_\infty,\mathbf{j}]}_{\mathscr{U},cp} \in H^1\left(F_{cp}, \mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \widehat{\otimes}_{\mathbb{Z}_p} \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right)$ *such that* $\mathrm{mom}_j(\mathbf{z}^{[\mathcal{F}_\infty,\mathbf{j}]}_{\mathscr{U},cp}) = \mathbf{a}_p^{-n}\mathbf{z}^{[\mathcal{F}_\infty,j]}_{\mathscr{U},cp^n,1}$*, for all* $n \geq 1$ *and all* $j \geq 0$.

Proof. This follows from [LZ16, Proposition 2.3.3] with $h = \lambda = 0$ and Lemma 10.10. □

For each $k \in \mathbb{Z}\cap\mathscr{U}$, composing with the weight k specialization, we also have a map

$$\mathrm{mom}_{k-2,j} \colon H^1\left(F_{cp}, \mathbf{V}^{\text{ord}}_{\mathscr{U}}(1) \widehat{\otimes}_{\mathbb{Z}_p} \sigma_{\text{ét}}^{\kappa_{\mathscr{U}}-\mathbf{j}}\bar{\sigma}_{\text{ét}}^{\mathbf{j}}\right) \longrightarrow H^1(F_{cp^n}, V^*_{\mathcal{F}_k} \otimes \sigma_{\text{ét}}^{k-2-j}\bar{\sigma}_{\text{ét}}^{j}).$$

Theorem 10.13. *For each* $k \in \mathbb{Z}\cap\mathscr{U}$ *with* $k \geq j$*, we have* $\mathrm{mom}_{k-2,j}(\mathbf{z}^{[\mathcal{F}_\infty,\mathbf{j}]}_{\mathscr{U},cp}) = a_p^{-n}(\mathcal{F}_k) z^{[\mathcal{F}_k,j]}_{cp^n,1}$.

Proof. By construction, for each $k \in \mathbb{Z} \cap \mathscr{U}$ with $k \geq j$, the image (which we denoted $z_{cp^n,1}^{[\mathcal{F}_k,j]}$) of $z_{cp^n,1}^{[k,j]}$ under the map (10.12) is the image of $\mathbf{z}_{\mathscr{U},cp^n,1}^{[\mathcal{F}_\infty,j]}$ under the weight k specialization map. Therefore, the result follows from Proposition 10.12. □

10.10. **Specialization of big Heegner classes.** Let $(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})^{\kappa_{\mathscr{U}}/2}\colon \Gamma_1 \to \Lambda^\times$ be the composition of $\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}}\colon \Gamma_1 \to \mathbb{Z}_p^\times$ with the character $\boldsymbol{\vartheta}\colon \mathbb{Z}_p^\times \to \Lambda^\times$. Using the isomorphism $\sigma(\mathcal{O}_K \otimes \mathbb{Z}_p)^\times \cong \mathbb{Z}_p^\times$ we see that $\sigma^{\kappa_{\mathscr{U}}}\bar{\sigma}^{\kappa_{\mathscr{U}}} = \boldsymbol{\vartheta}_K^2$, where $\boldsymbol{\vartheta}_K$ is the composition of $\boldsymbol{\vartheta}$ with the norm map. Therefore, $\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}}(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})^{\kappa_{\mathscr{U}}/2} = \boldsymbol{\vartheta}_K$. Define $\mathbf{V}_{\mathscr{U}}^\dagger := \mathbf{V}_{\mathscr{U}}^{\text{ord}}(1) \otimes \Theta^{-1}$. As G_K-representations, we then have

$$\mathbf{V}_{\mathscr{U}}^{\text{ord}}(1)\widehat{\otimes}_{\mathbb{Z}_p}\sigma_{\text{ét}}^{\kappa_{\mathscr{U}}}(\bar{\sigma}_{\text{ét}}/\sigma_{\text{ét}})^{\kappa_{\mathscr{U}}/2} \cong \mathbf{V}_{\mathscr{U}}^\dagger. \tag{10.19}$$

The element in Proposition 10.12 corresponds to an element $\mathbf{z}_{\mathscr{U},cp}$ in $H^1_\Sigma(F_{cp}, \mathbf{V}_{\mathscr{U}}^\dagger)$. As before, using that $\mathcal{F}_\infty$ has trivial character and that $H^1_\Sigma(F_{cp}, \mathbf{V}_{\mathscr{U}}^\dagger)$ is finite dimensional, we see that $\mathbf{z}_{\mathscr{U},cp}$ is invariant under the action of $\mathrm{Gal}(F_{cp}/H_{cp})$, and therefore may be regarded as an element in $H^1_\Sigma(H_{cp}, \mathbf{V}_{\mathscr{U}}^\dagger)$. For each finite order character χ and each $k \in \mathbb{Z} \cap \mathscr{U}$, we thus have specialization maps

$$\mathrm{sp}_{\nu_k,\chi}\colon H^1_\Sigma(H_{cp}, \mathbf{V}_{\mathscr{U}}^\dagger) \to H^1(H_{cp}, \mathbf{T}_{\nu_k}^\dagger(\chi)),$$

where $\mathbf{T}_{\nu_k}^\dagger(\chi)$ is the twist of $\mathbf{T}_{\nu_k}^\dagger$ by χ, where ν_k is the arithmetic morphism associated with k. The following result is then an immediate consequence of our constructions:

Theorem 10.14. *For each algebraic finite order character χ of conductor cp^n for some $n \geq 1$, we have* $\mathrm{sp}_{k,\chi}(\mathbf{z}_{\mathscr{U},cp}) = z_{cp^n,1}^{[\mathcal{F}_k,k/2-1]} \otimes \chi$.

11. Results

11.1. **Specializations of big Heegner points.** We consider the construction of generalized Heegner classes in §10 for the case $m = 0$ and $j = k/2 - 1$. Let $\mathcal{F}_k^\sharp$ be a p-ordinary newform on X_0 of weight $k \equiv 2 \mod 2(p-1)$ and trivial character, and consider the self-dual twist $V_{\mathcal{F}^\sharp}^\dagger = V_{\mathcal{F}_k^\sharp}(k/2)$ of the Deligne Galois representation associated with $\mathcal{F}_k^\sharp$. Let $W_k = W_{k,0}$ and

$$\Phi_{\mathcal{F}_k^\sharp}^{[k,j]} = \mathrm{pr}_{\mathcal{F}_k^\sharp} \circ \Phi_m^{[k,j]}\colon \epsilon_W \, \mathrm{CH}^{k-1}(W_k \otimes_H F_{cp^n})_{\mathbb{Q}} \longrightarrow H^1\left(F_{cp^n}, V_{\mathcal{F}_k^\sharp}^\dagger \otimes (\sigma_{\text{ét}}\bar{\sigma}_{\text{ét}}^{-1})^{k/2-1}\chi_{\text{cyc}}^{k/2-1}\right)$$

the p-adic Abel–Jacobi map. Taking $\xi = \chi_{\text{cyc}}^{k/2-1}$ in Lemma 10.4 we obtain a map

$$\Phi_{\mathcal{F}_k^\sharp}^{\text{ét}}\colon \epsilon_W \, \mathrm{CH}^{k-1}(W_k \otimes_H F_{cp^n})_{\mathbb{Q}} \longrightarrow H^1(F_{cp^n}, V_{\mathcal{F}_k^\sharp}^\dagger).$$

Set $\Delta_{cp^n} = \Delta_{cp^n,0}$; then we have generalized Heegner classes $\Phi_{\mathcal{F}_k^\sharp}^{\text{ét}}(\Delta_{cp^n})$ for $n \geq 0$ as in Definition 10.5. Let $u_c = \sharp(\mathcal{O}_c^\times)/2$ and α the unit root of the Hecke polynomial at p acting on $\mathcal{F}_k^\sharp$. We normalize these points to obtain a non-compatible family of quaternionic generalized Heegner classes by setting (here recall that $\mathrm{Frob}_{\mathfrak{p}}$ is the Frobenius element at $\mathfrak{p}$ and similarly denote $\mathrm{Frob}_{\bar{\mathfrak{p}}}$ the Frobenius element at $\bar{\mathfrak{p}}$)

- $z_{\mathcal{F}_k^\sharp,c} = \frac{1}{u_c}\left(1 - \frac{p^{k/2-1}}{\alpha}\mathrm{Frob}_{\mathfrak{p}}\right)\left(1 - \frac{p^{k/2-1}}{\alpha}\mathrm{Frob}_{\bar{\mathfrak{p}}}\right) \cdot \Phi_{\mathcal{F}_k^\sharp}^{\text{ét}}(\Delta_c)$;
- $z_{\mathcal{F}_k^\sharp,cp^n} = \left(1 - \frac{p^{k-2}}{\alpha}\right) \cdot \Phi_{\mathcal{F}_k^\sharp}^{\text{ét}}(\Delta_{cp^n})$ for $n \geq 1$.

Then $\mathrm{cores}_{H_{cp^n}/H_{cp^{n-1}}}(z_{\mathcal{F}_k^\sharp,cp^n}) = \alpha \cdot z_{\mathcal{F}_k^\sharp,cp^{n-1}}$ for all $n \geq 1$ ([Mag22, §7.1.2]) and we can define (using Shapiro's lemma for the isomorphisms)

- $\boldsymbol{x}_c^\sharp = \varprojlim_n \alpha^{-n} z_{\mathcal{F}_k^\sharp,cp^n} \in H^1_{\mathrm{Iw}}(\Gamma_\infty, V_{\mathcal{F}_k^\sharp}^\dagger) \cong H^1(H_c, V_{\mathcal{F}_k^\sharp}^\dagger \otimes \mathcal{O}[\![\Gamma_\infty]\!])$;
- $\boldsymbol{z}_c^\sharp = \mathrm{cores}_{H_c/K}(\boldsymbol{x}_c^\sharp) \in H^1_{\mathrm{Iw}}(\widetilde{\Gamma}_\infty, V_{\mathcal{F}_k^\sharp}^\dagger) \cong H^1(K, V_{\mathcal{F}_k^\sharp}^\dagger \otimes \mathcal{O}[\![\widetilde{\Gamma}_\infty]\!])$.

For any character $\chi\colon \widetilde{\Gamma}_\infty \to \overline{\mathbb{Q}}_p^\times$, we can then consider the specialization map, and obtain an element $\boldsymbol{z}_\chi^\sharp \in H^1(H_c, V^\dagger_{\mathcal{F}_k^\sharp,\chi})$; here $V^\dagger_{\mathcal{F}_k^\sharp,\chi} = V^\dagger_{\mathcal{F}_k^\sharp}\otimes\chi$.

Let $\mathcal{F}_\infty$ be the quaternionic Hida family passing through the modular form $\mathcal{F}_k$. We also assume that the residual p-adic representation $\bar\rho_f$ is irreducible, p-ordinary and p-distinguished. Define $\mathfrak{z}_c = \boldsymbol{\vartheta}^{-1}\left(\frac{-\sqrt{-D_K}}{c^2}\right)\mathfrak{Z}_c$ and write as before $\mathfrak{z}_c(\nu)$ for $\nu(\mathfrak{z}_c)$.

Theorem 11.1. *For all ν of weight $k\equiv 2 \mod 2(p-1)$, we have* $(\mathrm{pr}_*)(\mathfrak{z}_c(\nu)) = \boldsymbol{z}_c^\sharp$.

Proof. Let ν be as in the statement, let $\mathcal{F}_k=\mathcal{F}_\nu$ and $\mathcal{F}_k^\sharp$ the form whose ordinary p-stabilization is $\mathcal{F}_k$. Let $\hat\phi\colon \widetilde{\Gamma}_\infty \to \overline{\mathbb{Q}}_p^\times$ be the p-adic avatar of a Hecke character ϕ of infinity type $(k/2,-k/2)$; then $\chi = \hat\phi\hat\xi_k^{-1}$ is a finite order character. Consider the map $\mathcal{L}^{\widetilde{\Gamma}_\infty}_{\mathcal{F}_k^\sharp,\xi_k}$ obtained by composing the map $\mathcal{L}^{\Gamma_\infty}_{\mathcal{F}_k^\sharp,\xi_k}$ in (7.11) for $\hat\xi_k=\boldsymbol{\xi}_\nu$ with the canonical map arising from the inclusion $\Gamma_\infty\hookrightarrow\widetilde{\Gamma}_\infty$. Combining Theorem 9.13, Theorem 5.4 and [Mag22, Theorem 7.2], we have:

$$\begin{aligned}
\nu(\mathscr{L}^{\mathrm{geo}}_{\mathbb{I},\boldsymbol{\xi}})(\hat\phi^{-1}) &= \nu\left(\frac{\sigma_{-1,\mathfrak{p}}}{\sqrt{-D_K}}\right)\cdot\nu(\mathscr{L}^{\mathrm{alg}}_{\mathbb{I},\boldsymbol{\xi}})(\hat\phi^{-1}) \text{ (by Theorem 9.13)}\\
&= \left(\frac{\sigma_{-1,\mathfrak{p}}}{\sqrt{-D_K}}\right)\left(c^{-k_\nu/2+1}\mathscr{L}_{\mathcal{F}_\nu,\xi_\nu}(\hat\phi^{-1})\right) \text{ (by Theorem 5.4)}\\
&= \left(\frac{\sigma_{-1,\mathfrak{p}}}{\sqrt{-D_K}}\right)\left(c^{-k_\nu/2+1}\mathscr{L}_{\mathcal{F}_\nu^\sharp,\xi_\nu}(\hat\phi^{-1})\right) \text{ (by Lemma 5.1)}\\
&= (-1)^{k_\nu/2-1}\cdot\frac{\sqrt{-D_K}^{k_\nu/2-1}}{c^{k_\nu-2}}\mathcal{L}^{\widetilde{\Gamma}_\infty}_{\mathcal{F}_k^\sharp,\xi_k}(\mathrm{res}_\mathfrak{p}(\boldsymbol{z}^\sharp_{\xi_k}))(\hat\phi^{-1}) \text{ (by [Mag22, Theorem 7.2])}.
\end{aligned}$$

Because of the injectivity of $\mathcal{L}^{\widetilde{\Gamma}_\infty}_{\mathcal{F}_k^\sharp,\xi_k}\circ\mathrm{res}_\mathfrak{p}$ (*cf.* [Cas20, Lemma 6.4] and the proof of [Cas20, Theorem 6.5]), it follows from Lemma 7.3 that $\mathrm{pr}_*\left(\nu(\mathfrak{Z}_{\boldsymbol{\xi}})\right) = \left(\frac{-\sqrt{-D_K}}{c^2}\right)^{k/2-1}\cdot\boldsymbol{z}^\sharp_{\xi_k}$. Now we have $\vartheta_\nu(u)=u^{k/2-1}$ for $u\in\mathbb{Z}_p^\times$, and the result follows. $\square$

Remark 11.2. We shall compare Theorem 11.1 with analogues results by Castella [Cas20] and Ota [Ota20]. We first observe the difference in the Euler factors are implicit in the use of pr_*, while [Ota20] uses the map $(\mathrm{pr}_1)_*$ (this formulation is similar to the one in [JLZ21]). Comparing Theorem 11.1 with the analogue result in the GL_2 case obtained in [Cas20, Theorem 6.5] (especially [Cas20, Equation (6.9)]), the reader should notice the correction factor $\nu\left(\frac{-\sqrt{-D_K}}{c^2}\right)$ which is present in our paper and does not appear in [Cas20]. This difference is due to two minor corrections: the first one is in [CH18a, Theorem 4.9] (noticed by Kobayashi [Kob23] and fixed in [Mag22, Theorem 7.2]) and explains the contribution of $\sqrt{-D_K}$; the second one, explaining the contribution of c, arises from the comparison result in Theorem 5.4, which is not considered in [Cas20, Equation (2.5)]. Accordingly with [Ota20, Theorem 1.2], the correction factor $\nu\left(\frac{-\sqrt{-D_K}}{c^2}\right)$ (in the GL_2-case) should *not* appear in [Cas20]. The reason for the discrepancy between [Cas20, Theorem 6.5] (in the corrected form of Theorem 11.1) and Ota's result [Ota20, Theorem 1.2] is the following. In the proof of [Cas20, Proposition 4.4], the author claims that the Heegner points considered in *loc. cit.* and [CH18a] coincide with those considered in [How07]; comparing the two set of points explains this discrepancy. For simplicity, we only treat the case of the elliptic curve $E_{c,n}$ whose complex points are $E_{c,n}(\mathbb{C})=\mathbb{C}/\mathcal{O}_{cp^n}$. Recall that the p-power level structure in [How07] is (in the current notation) $c\vartheta\in\mathcal{O}_K$, which gives rise to a p^n-torsion point in $\mathbb{C}/\mathcal{O}_{cp^n}$. On the other hand, the p^n-level structure in [Cas20] is defined in [CH18a] and amounts to consider the level structure $\mu_{p^n}\overset{\sim}{\to}E_{c,n}[\mathfrak{p}^n]\hookrightarrow E_{c,n}[p^n]$

which takes ζ_{p^s} to the idempotent $e_{\mathfrak{p}}$ (in [CH18a] the idempotents $e_{\mathfrak{p}}$ and $e_{\bar{\mathfrak{p}}}$ coincide with those chosen in §2.2); however, note that $c\vartheta = c\vartheta(e_{\mathfrak{p}} + e_{\bar{\mathfrak{p}}}) = c\vartheta e_{\mathfrak{p}} + c\vartheta e_{\bar{\mathfrak{p}}}$ as elements in $\mathcal{O}_K \otimes \mathbb{Z}_p$. This implies that the Heegner points considered in [How07] and [CH18a, Cas20] are Galois conjugate to each other, explaining the correction factors appearing in Theorem 11.1.

11.2. **Big Heegner points and big generalized Heegner classes.** Let $\mathscr{U}$ be a sufficiently small fixed neighborhood of $k \in \mathbb{Z}$ with $k \equiv 2 \mod 2(p-1)$. Since $k-2$ is even for all $k \in \mathbb{Z} \cap \mathscr{U}$, we can define $\mathbf{j} = \mathbf{k}/2 = \kappa_{\mathscr{U}}/2$ and define

$$\mathbf{z}_c := \mathrm{cores}_{H_{cp}/K}\left(\mathbf{z}_{\mathscr{U},cp}\right) \in H^1_\Sigma(K, \mathbf{V}^\dagger_{\mathscr{U}}).$$

Theorem 11.3. *There exists a sufficiently small neighborhood $\mathscr{U}$ of k where $\mathfrak{z}_c = \mathbf{z}_c$.*

Proof. By Theorem 10.14, for each finite order character χ of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ of conductor p^n for some integer $n \geq 1$ and each ν_k corresponding to $k \in \mathbb{Z} \cap \mathscr{U}$, we see that the image $\mathrm{sp}_{\nu_k,\chi}(\mathbf{z}_c)$ of $\mathbf{z}_c$ in $H^1(K, \mathbf{T}^\dagger_{\nu_k}(\chi))$ is $z^{[\mathcal{F}_k, k/2-1]}_{cp^n,1} \otimes \chi$. Therefore, $\mathrm{pr}_*\left(\mathrm{sp}_{\nu_k,\chi}(\mathbf{z}_c)\right) = \mathrm{pr}_*(\mathrm{sp}_{\nu_k,\chi}(\mathfrak{z}_c))$ by Theorem 11.1. Since pr_* is an isomorphism, we have $\mathrm{sp}_{\nu_k,\chi}(\mathbf{z}_c) = \mathrm{sp}_{\nu_k,\chi}(\mathfrak{z}_c)$ for all such ν_k and χ. Since $H^1_\Sigma(K, \mathbf{V}^\dagger_{\mathscr{U}})$ is a finitely generated $\Lambda_{\mathscr{U}}$-module, the result follows. □

Remark 11.4. Theorem 11.3 only works over neighborhoods of integers points in $\mathcal{X}$. Even if the construction of big generalized Heegner classes can probably (at least in the ordinary case) be extended to the whole Hida family $\mathbb{I}$, the strategy to prove Theorem 11.3 involves a comparison result at integer points in $\mathcal{X}$, which does not immediately extends to bigger sets in $\mathbb{I}$. We hope to come to this problem (at least in the ordinary case) in a future work.

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