ON POLYNOMIAL AUTOMORPHISMS COMMUTING WITH A SIMPLE DERIVATION

Pierre-Louis Montagard[1], Iván Pan[2], Alvaro Rittatore[2]

Abstract. Let D be a simple derivation of the polynomial ring $\Bbbk[x_1,\dots,x_n]$, where $\Bbbk$ is an algebraically closed field of characteristic zero, and denote by $\operatorname{Aut}(D)\subset\operatorname{Aut}\big(\Bbbk[x_1,\dots,x_n]\big)$ the subgroup of $\Bbbk$-automorphisms commuting with D. We show that the connected component of $\operatorname{Aut}(D)$ passing through the identity is a unipotent algebraic group of dimension at most $n-2$, this bound being sharp. Moreover, $\operatorname{Aut}(D)$ is an algebraic group if and only if it is a connected ind-group. Given a simple derivation D, we characterize when $\operatorname{Aut}(D)$ contains a normal subgroup of translations. As an application of our techniques we show that if $n=3$, then either $\operatorname{Aut}(D)$ is a discrete group or it is isomorphic to the additive group acting by translations, and give some insight on the case $n=4$.

1. Introduction

Let $\Bbbk[x_1,\dots,x_n]$ be the polynomial ring over an algebraically closed field $\Bbbk$ of characteristic 0. Recall that a $\Bbbk$-derivation of $\Bbbk[x_1,\dots,x_n]$ is a linear endomorphism $D\in\operatorname{End}\big(\Bbbk[x_1,\dots,x_n]\big)$ such that $D(fg)=D(f)g+fD(g)$ for all $f,g\in\Bbbk[x_1,\dots,x_n]$; we denote $D\in\operatorname{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$.

A central and, of course, very difficult problem is to classify derivations up conjugation by an automorphism of $\Bbbk[x_1,\dots,x_n]$. If $n=2$ there is a partial classification ([BaPa2019]), but essentially nothing is known in higher dimension. In order to provide contributions to the solution of this problem, one may try to classify derivations whose interest has been proven in relation to various areas of mathematics. This is the case, for example, of the *locally nilpotent derivations* and the *simple derivations* (see Definition 2.1 below). In the first case, significant progress has been made in recent decades (see [Fr2017] and references therein), while little is known about the second one.

Recall that if Δ is a locally nilpotent derivation then the formal series $e^{t\Delta}:=\sum_{i=0}^{\infty}\frac{t^i\Delta^i}{i!}$ acts as a finite sum on every polynomial and defines an element in $\operatorname{Aut}\big(\Bbbk[x_1,\dots,x_n]\big)$ (the group of $\Bbbk$-automorphisms of $\Bbbk[x_1,\dots,x_n]$) for all $t\in\Bbbk$; in this way, Δ induces an action of the additive group $\mathbb{G}_a=(\Bbbk,+)$ on $\Bbbk[x_1,\dots,x_n]$. Conversely, every action of $\mathbb{G}_a$ on $\Bbbk[x_1,\dots,x_n]$ is of this form (see for example [Fr2017, §1.5]). When $\Bbbk=\mathbb{C}$, the field of complex numbers, if one associates to a derivation D the polynomial vector field $\big(D(x_1),\dots,D(x_n)\big)$, then D is a locally nilpotent derivation if and only if the general solution of the differential equation of the corresponding vector field is polynomial ([FiWa1997]).

[1]IMAG, Univ Montpellier, CNRS, Montpellier, France
[2]CMAT, Universidad de la Republica, Montevideo, Uruguay, supported by CSIC (Udelar), ANII and PEDECIBA, of Uruguay
This work was partially supported by the CNRS International Research Laboratory IFUMI

On the other hand, given a simple derivation $D \in \operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ — that is the only D-stable ideals are the trivial ones —, one may extend the polynomial ring $\Bbbk[x_1,\dots,x_n]$ to a "skew" polynomial ring R by adjoining a new indeterminate t with the rule: $t\cdot f - f\cdot t = D(f)$ for any $f \in \Bbbk[x_1,\dots,x_n]$. It is well known that D is simple if and only if R has no non trivial bilateral ideals (see for example [GoWa1989, Chap. 2]). Finally, if $\Bbbk = \mathbb{C}$ and we consider the singular foliation $\mathcal{F}$ associated with the vector field $\bigl(D(x_1),\dots,D(x_n)\bigr)$ on $\mathbb{C}^n$, then D being simple is equivalent to saying that $\mathcal{F}$ is not singular and any of its leaves is Zariski dense. For a more general field, the previous equivalence may be described in terms of algebraic independence of power series in one indeterminate ([Le2008]).

Thus, if we compare the two kind of derivations previously mentioned by focusing on their corresponding flows (when $\Bbbk = \mathbb{C}$), then locally nilpotent and simple derivations appear as opposite in a certain sense. More generally, if $\Bbbk$ is an arbitrary field and Δ is a locally nilpotent derivation of $\Bbbk[x_1,\dots,x_n]$, then $\ker\Delta = \bigl\{f \in \Bbbk[x_1,\dots,x_n]; \Delta(f) = 0\bigr\}$ is a $\Bbbk$-subalgebra of $\Bbbk[x_1,\dots,x_n]$ with transcendence degree over $\Bbbk$ equal to $n-1$ (see for example [vdE, Pro.1.3.32(i)]), whereas clearly $\ker D = \Bbbk$ when D is a simple derivation. One finds further evidence for this intuition on the opposite behavior of simple and locally nilpotent derivations in dimension 2 ([MePa2016], [Pa2022]), looking at the automorphisms group of the derivations:

Definition 1.1. Let $D \in \operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$. The *automorphisms group of* D is the isotropy group of D for the action of $\operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ on $\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ by conjugations:

$$\operatorname{Aut}(D) = \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)_D \subset \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr).$$

Theorem 1.2. *If $D \in \operatorname{Der}\bigl(\Bbbk[x_1,x_2]\bigr)$, then*

(*a*) *D is locally nilpotent if and only is $\operatorname{Aut}(D)$ is not an algebraic group.*

(*b*) *If D is simple, then $\operatorname{Aut}(D) = 1$.* □

Recall that if $n \geq 2$, then $\operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ does not admit a compatible structure of group scheme, but it is an ind-group with filtration induced by total degree (see for example [Sha1981] and [Ku2002, Chapter 4]). More precisely, if we consider the set theoretical inclusion $\operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr) \hookrightarrow \Bbbk[x_1,\dots,x_n]^n$, given by $\varphi \mapsto \bigl(\varphi(x_1),\dots,\varphi(x_n)\bigr)$, then the *degree of* φ induces a filtration $\operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr) = \bigcup_p \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)_p$, where

$$\operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)_p = \bigl\{\varphi = (f_1,\dots,f_n) \in \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr) : \forall i \ \deg(f_i) \leq p\ \bigr\}.$$

It follows that a closed group $G \subset \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ is algebraic if and only if there exists p such that $G \subset \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)_p$. Clearly, $\operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ identifies in a canonical way with $\operatorname{Aut}(\mathbb{A}^n)$, the automorphisms group of the n-dimensional affine space; under this identification, if $G \subset \operatorname{Aut}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ is algebraic, then the induced action $G \times \mathbb{A}^n \to \mathbb{A}^n$ is regular.

If $n > 2$, it is not difficult to show that $\operatorname{Aut}(D)$ is never algebraic when D is locally nilpotent; however, very few is known about the simple case. In fact, $\operatorname{Aut}(D)$ is expected to be algebraic: in [Ya2022], after exhibiting examples of simple derivations whose isotropy is $\mathbb{G}_a$ acting by translations, the author conjectures that in general, up to conjugation, if D is simple then $\operatorname{Aut}(D)$ acts by translations.

The main objective of this work is to give evidence in the direction of proving that $\operatorname{Aut}(D)$ is in fact an algebraic group. The structure of this work is as follows:

In Section 2 we collect some basic definitions and results on $\mathrm{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$ and the ind-group structure of $\mathrm{Aut}\big(\Bbbk[x_1,\dots,x_n]\big)$.

In Section 3 we prove that if D is a simple derivation, then $\mathrm{Aut}(D)^0$, the connected component of $\mathrm{Aut}(D)$ passing through the identity, is a unipotent algebraic group (Theorem 3.2), and that any algebraic element of $\mathrm{Aut}(D)$ is unipotent (Theorem 3.4). In particular, $\mathrm{Aut}(D)$ is algebraic if and only if it is a connected group.

In Section 4 we show that if $D \in \mathrm{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$ is a simple derivation, then $\dim \mathrm{Aut}(D)^0 \leq n-2$ (Theorem 4.7), and that this bound is sharp (Corollary 4.13).

When a simple derivation is such that $\mathrm{Aut}(D)$ contains a non trivial translation, then one can give more insight on the structure of $\mathrm{Aut}(D)^0$; this is done in Section 5. More precisely, if D is a simple derivation, we characterize in terms of a coordinate system when $\mathrm{Aut}(D)$ contains a non trivial normal subgroup of translations and we use this characterization to somewhat describe $\mathrm{Aut}(D)^0$ (Theorem 5.4). We apply our point of view in order to prove that the isotropy group of a simple derivation of $\Bbbk[x_1,x_2,x_3]$ is either trivial, either the additive group acting by translations (upon conjugation) or it is an infinite countable discrete group — however, we remark that, up to our knowledge, there is no known example of this last possibility.

Most examples of explicit simple derivations of $\Bbbk[x_1,\dots,x_n]$ are produced by extending, recursively, a simple derivation in $\Bbbk[x_1]$, using Shamsuddin's criterion (see Proposition 4.8), or some of its variants, and are therefore of the form $D = \partial/\partial x_1 + \sum_{i=2}^{n} a_i \partial/\partial x_i$, $a_i \in \Bbbk[x_1,\dots,x_n]$. In this case — we say that D admits x_1 as a *linear coordinate* —, one can describe quite well the action of an element of $\mathrm{Aut}(D)^0$ on the coordinate x_1 (see Lemma 5.8). If $n=4$, we use this description in order to give more insight on the algebraicity of $\mathrm{Aut}(D)$ (Theorem 5.10).

2. Preliminaries

In this section we recall some basic definitions and results on simple and locally nilpotent derivations.

Let V be a $\Bbbk$-vector space. A linear endomorphism $\varphi : V \to V$ is *locally finite* if for all $v \in V$, there exists a finite dimensional φ-stable vector space W containing v. If moreover $\varphi|_W$ is nilpotent (resp. semisimple) for all finite dimensional φ-stable vector space W, then φ is said to be *locally nilpotent* (resp. *locally semisimple*). We say that φ is *locally unipotent* if $\varphi - \mathrm{Id}$ is locally nilpotent.

If $\varphi \in \mathrm{Aut}\big(\Bbbk[x_1,\dots,x_n]\big)$, we denote by $\langle\varphi\rangle$ the group generated by φ; we say that φ is *algebraic* if the closure $\overline{\langle\varphi\rangle} \subset \mathrm{Aut}(\mathbb{A}^n)$ is an algebraic group — along this work, closures are taken with respect to the inductive topology of the ind-variety structure. It is well known that φ is algebraic if and only if the total degree of the positive powers of φ is bounded: $\max\big\{\deg(\varphi^\ell) : \ell \geq 0\big\} < \infty$ (see for example [FuKr2018, Lemma 9.1.4]).

Finally, it is easy to prove that φ is algebraic if and only if φ is locally finite (see [FuKr2018, Lemma 9.1.4]); in particular, φ is an algebraic unipotent automorphism if and only if φ is locally unipotent.

Definition 2.1. A *locally nilpotent derivation* is a derivation D that is locally nilpotent as a linear endomorphism of $\Bbbk[x_1,\dots,x_n]$.

Let D be a derivation of $\Bbbk[x_1,\dots,x_n]$. An ideal I of $\Bbbk[x_1,\dots,x_n]$ is *D-stable* if $D(I)\subset I$. If the only D-stable ideals are the trivial ones, we say that D is a *simple derivation.*

Remark 2.2. (1) It is clear that if D is a derivation of $\Bbbk[x_1,\dots,x_n]$, then $D=\sum_i a_i\frac{\partial}{\partial x_i}$, with $a_i\in\Bbbk[x_1,\dots,x_n]$.
(2) Let $D=\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}$ be a derivation of $\Bbbk[x_1,\dots,x_n]$. Then the ideal $\langle a_1,\dots,a_n\rangle_{\Bbbk[x_1,\dots,x_n]}\subset\Bbbk[x_1,\dots,x_n]$ is D-stable.

If D is a simple derivation then $\langle a_1,\dots,a_n\rangle=\Bbbk[x_1,\dots,x_n]$.
(3) If $D=\sum_{i=1}^n a_i\frac{\partial}{\partial x_i}\in\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ and s is such that $a_i\in\Bbbk[x_{s+1},\dots,x_n]$ for $i>s$, then the restriction $\overline{D}=D|_{\Bbbk[x_{s+1},\dots,x_n]}:\Bbbk[x_{s+1},\dots,x_n]\to\Bbbk[x_{s+1},\dots,x_n]$ is also a derivation. If moreover D is simple, then $\overline{D}$ is also a simple derivation.

Definition 2.3. Let $D\in\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$. We say that $f\in\Bbbk[x_1,\dots,x_n]\setminus\{0\}$ is a *Darboux polynomial for D of eigenvalue* $\lambda\in\Bbbk[x_1,\dots,x_n]$ if $D(f)=\lambda f$. In the literature f is also called an *eigenvector of D.*

Remark 2.4. (1) If $f\in\Bbbk[x_1,\dots,x_n]$ is a Darboux polynomial for D, then the ideal $\langle f\rangle\subset\Bbbk[x_1,\dots,x_n]$ is D-stable.

(2) The kernel of a derivation D, denoted as $\ker(D)$, is the subspace of Darboux polynomials of eigenvalue equal to 0.

(3) In particular, if D is a simple derivation, the only Darboux polynomials are the constant polynomials, and therefore $\ker(D)=\Bbbk$.

Remark 2.5. Let $\Delta\in\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ be a locally nilpotent derivation and consider $\mathbb{G}_a=\bigl\{e^{t\Delta}:t\in\Bbbk\bigr\}$. Recall that the linear span $\langle\Delta\rangle_\Bbbk\subset\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ is naturally identified with the Lie algebra of $\mathbb{G}_a$ and $\ker(\Delta)=\Bbbk[x_1,\dots,x_n]^{\mathbb{G}_a}$, the subalgebra of $\mathbb{G}_a$-invariants.

Definition 2.6. If $\Delta\in\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ is a locally nilpotent derivation, we say that $s\in\Bbbk[x_1,\dots,x_n]$ is a *slice* for Δ if $\Delta(s)=1$.

Let $D\in\operatorname{Der}\bigl(\Bbbk[x_1,\dots,x_n]\bigr)$ be an arbitrary derivation. We say that $s\in\Bbbk[x_1,\dots,x_n]$ is a *linear coordinate for D* if $D(s)=1$ and there exist polynomials $s_2,\dots,s_n$ such that $\Bbbk[s,s_2,\dots,s_n]=\Bbbk[x_1,\dots,x_n]$:

$$D=\frac{\partial}{\partial s}+\sum_{i=2}^n a_i\frac{\partial}{\partial s_i}\ ,\ a_i\in\Bbbk[s,s_2,\dots,s_n].$$

3. Automorphisms of simple derivations: first results

As noted in the introduction, the main object of study of this work is the isotropy subgroup of a simple derivation (see Definition 1.1).

3.1. On the ind-group structure of the isotropy group of a simple derivation.

Recall that $\operatorname{Aut}(\mathbb{A}^n)$ is an ind-group, therefore, $\operatorname{Aut}(D)$ being an isotropy group, it is also an ind-group. We begin this section by showing that if D is simple, then $\operatorname{Aut}(D)^0\subset\operatorname{Aut}(D)$, the unique connected component containing the identity, is an algebraic group.

First, we recall the following result of Baltazar (see [B2016, Proposition 7]):

Proposition 3.1. *Let D be a simple derivation of $\mathbb{k}[x_1, \ldots, x_n]$. Then every non trivial element of $\mathrm{Aut}(D)$ admits no fixed point.* □

As a consequence of Proposition 3.1 above and some general results about ind-groups, we obtain the following description of $\mathrm{Aut}(D)$ as an ind-group.

Theorem 3.2. *If $D \in \mathrm{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ is simple, then $\mathrm{Aut}(D)^0$ is an affine algebraic group of dimension at most n which acts freely on $\mathbb{A}^n$, and there exists a family of automorphisms $(\varphi_i)_{i\in\mathbb{N}}$, with $\varphi_i \in \mathrm{Aut}(D)$, such that $\varphi_0 = \mathrm{Id}$ and*

$$\mathrm{Aut}(D) = \bigcup_{i\in\mathbb{N}} \varphi_i \mathrm{Aut}(D)^0. \tag{1}$$

In particular $X_n = \bigcup_{i=0}^{n} \varphi_i \mathrm{Aut}(D)^0$, $n \in \mathbb{N}$, is a filtration of $\mathrm{Aut}(D)$ by (affine) algebraic varieties.

Proof. The first assertion follows from Proposition 3.1 above and [FuKr2018, Propositions 1.8.3 and 7.1.2]. By [FuKr2018, proposition 1.7.1 and 2.2.1] it follows that $\mathrm{Aut}(D)$ is a countable disjoint union of connected algebraic varieties: $\mathrm{Aut}(D) = \bigcup_{i\in\mathbb{N}} Y_i$, such that $Y_0 = \mathrm{Aut}(D)^0$, where $\mathrm{Aut}(D)^0$ is a connected and normal algebraic subgroup of $\mathrm{Aut}(D)$.

If we choose $\varphi_i \in Y_i$, $i \neq 0$, then $\varphi_i \mathrm{Aut}(D)^0 = Y_i$, since left multiplication by φ_i is an isomorphism of ind-varieties. It follows that $\mathrm{Aut}(D) = \bigcup_{i\in\mathbb{N}} \varphi_i \mathrm{Aut}(D)^0$, where $\varphi_0 = \mathrm{Id}$. □

Question 3.3. To our knowledge, there are no known examples of a simple derivation of $\mathbb{k}[x_1, \ldots, x_n]$ with non-connected automorphisms group.

Is it true that if D is a simple derivation, then $\mathrm{Aut}(D)$ is connected?

In view of Theorem 3.2, this would imply that any simple derivation has algebraic automorphisms group.

Theorem 3.4. *Let $D \in \mathrm{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ be a simple derivation and $\varphi \in \mathrm{Aut}(D)$. If φ is algebraic, then φ is unipotent. In particular, if $G \subset \mathrm{Aut}(D)$ is an algebraic subgroup, then G is a unipotent subgroup and therefore G is a closed connected subgroup of $\mathrm{Aut}(D)^0$.*

Proof. If φ is algebraic, then there exists a Jordan decomposition $\varphi = \varphi_s \varphi_u$, with φ_s semi-simple and φ_u unipotent and such that $\varphi_s, \varphi_u \in \overline{\langle \varphi \rangle} \subset \mathrm{Aut}(D)$. By a result of Furter and Kraft ([FuKr2018, Proposition 15.9.3.]), φ_s acts in $\mathbb{A}^n$ with a fixed point, and it follows from Proposition 3.1 that φ_s is trivial. Hence, $\varphi = \varphi_u$ — that is, φ is unipotent.

If $G \subset \mathrm{Aut}(D)$ is algebraic, then we deduce that G is unipotent, so it is connected, proving the second assertion. □

Remark 3.5. If D is a simple derivation and $\varphi \in \mathrm{Aut}(D)$ is a locally finite automorphism, then applying Theorem 3.4 to $G = \overline{\langle \varphi \rangle} \subset \mathrm{Aut}(\mathbb{A}^n)$ we deduce that $\varphi \in \mathrm{Aut}(D)^0$.

As a direct follow up of theorems 3.4 and 3.2, we have the following characterization of the algebraicity of the automorphisms group of a simple derivation.

Corollary 3.6. *If $D \in \mathrm{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ is simple then the following are equivalent:*

(1) The ind-group $\mathrm{Aut}(D)$ *is an algebraic group.*

(2) The ind-group $\mathrm{Aut}(D)$ *is connected.*

(3) Every element $\varphi \in \mathrm{Aut}(D)$ *is algebraic.*

(4) Every element $\varphi \in \mathrm{Aut}(D)$ *is a locally finite automorphism.*

(5) Every element $\varphi \in \mathrm{Aut}(D)$ *is a unipotent locally finite automorphism.* □

3.2. **Additive subgroups of** $\mathrm{Aut}(D)$**.**

Definition 3.7. Let X be an affine algebraic variety and $\varphi : \mathbb{G}_a^s \times X \to X$ be a regular action. We say that φ is *globally (equivariantly) trivial*, if there exists an affine algebraic variety V and an action $\phi : \mathbb{G}_a^s \times (\mathbb{A}^s \times V)$, $\phi\big(a, (t, v)\big) = (t + a, v)$ for all $a \in \mathbb{G}_a^s$, $t \in \mathbb{A}^s$, $v \in V$, together with an equivariant isomorphism $X \cong \mathbb{A}^s \times V$.

We say that φ is *locally trivial* if there exists a cover $X = \bigcup_{i=1}^{\ell} U_i$ by affine open $\mathbb{G}_a^s$-stable subsets such that the restrictions $\varphi|_{\mathbb{G}_a^s \times U_i} : \mathbb{G}_a^s \times U_i \to U_i$ are globally trivial actions.

Let $\mathbb{G}_a^s \hookrightarrow \mathrm{Aut}(\mathbb{A}^n)$ be a closed immersion. We say that (the image of) $\mathbb{G}_a^s$ *acts by translations* or that $\mathbb{G}_a$ is a *group of translations* if the induced action $\varphi : \mathbb{G}_a^s \times \mathbb{A}^n \to \mathbb{A}^n$ is globally trivial where $\mathbb{A}^n \cong \mathbb{A}^s \times \mathbb{A}^{n-s}$ and $\mathbb{G}_a^s$ acts by translations in the first s coordinates.

A subgroup of automorphisms $\mathbb{G}_a^s \subset \mathrm{Aut}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ is said to act in a locally trivial (resp. globally trivial, resp. by translations) way if the induced action $\varphi : \mathbb{G}_a^s \times \mathbb{A}^n \to \mathbb{A}^n$ is so.

Remark 3.8. (1) Notice that $\mathbb{G}_a^s$ acts by translations on $\mathbb{A}^n$ if and only if $\mathbb{G}_a^s$ is conjugated to a subgroup of the group of translations of $\mathbb{A}^n$ (as subgroups of $\mathrm{Aut}(\mathbb{A}^n)$).

(2) If $\Delta \in \mathrm{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ is a locally nilpotent derivation then $s \in \mathbb{k}[x_1, \ldots, x_n]$ is a slice (see Definition 2.6) if and only if the canonical action of $H = \big\{e^{t\Delta} : t \in \mathbb{k}\big\} \cong \mathbb{G}_a$ over $\mathbb{A}^n$ is globally trivial, where $V \cong \mathrm{Spec}\big(\ker(\Delta)\big)$ — this is the content of the slice theorem, see for example [Fr2017, Corollary 1.26].

If moreover s is a linear coordinate for Δ, then the action of H is by translations

(3) Let $\varphi : \mathbb{G}_a^s \times X \to X$ be a globally trivial action, with $X \cong \mathbb{A}^s \times V$. Then it is easy to show that the projection $p_2 : X \cong \mathbb{A}^s \times V \to V$ is the geometric quotient. In particular, $\mathbb{k}[V] = \mathbb{k}[X]^{\mathbb{G}_a^s}$, the subalgebra of invariants of $\mathbb{G}_a^s$.

In [Fr2017], the following characterization of the local triviality of a free action of $\mathbb{G}_a$ on $\mathbb{A}^n$ is given.

Definition 3.9. Let $\Delta : \mathbb{k}[x_1, \ldots, x_n] \to \mathbb{k}[x_1, \ldots, x_n]$ be a locally nilpotent derivation with kernel A. We call $\mathrm{pl}(\Delta) = A \cap \mathrm{Im}(\Delta)$ the *plinth ideal of* Δ.

Lemma 3.10. *Let* Δ *be a locally nilpotent derivation of* $\mathbb{k}[x_1, \ldots, x_n]$ *and consider* $\mathbb{G}_a = \big\{e^{t\Delta} : t \in \mathbb{k}\big\} \subset \mathrm{Aut}(D)$ *(see Remark 2.5). Then the restricted action of* $\mathbb{G}_a$ *is locally trivial if and only if* $\big\langle\, \mathrm{pl}(\Delta)\big\rangle_{\mathbb{k}[x_1, \ldots, x_n]} = \mathbb{k}[x_1, \ldots, x_n]$.

Proof. See [Fr2017, p. 34]. □

If Δ is a locally nilpotent derivation that commutes with a simple derivation, the characterization of the local triviality of the action of the additive group $\{e^{t\Delta} : t \in \mathbb{k}\}$ takes the following form:

Proposition 3.11. *Let $D \in \operatorname{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ be a simple derivation and assume that there exists a closed immersion $\mathbb{G}_a \hookrightarrow \operatorname{Aut}(D)$. Then the induced action of $\mathbb{G}_a$ on $\mathbb{A}^n$ is locally trivial. In particular, the action is proper — that is, the orbit map $\varphi : \mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$, $\varphi(g, x) = \big(x, g(x)\big)$ is a proper morphism.*

Proof. Let $\Delta \in \operatorname{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ be such that e^{Δ} is a generator of (the image of) $\mathbb{G}_a$ and write $A = \ker(\Delta)$. Then $e^{t\Delta}D = De^{t\Delta}$ for all $t \in \mathbb{k}$, and therefore $[D, \Delta] = 0$. Hence, $D(A) \subset A$ and it follows that the plinth ideal $\operatorname{pl}(\Delta) \subset A$ is D-stable so $\big\langle \operatorname{pl}(\Delta)\big\rangle_{\mathbb{k}[x_1,\ldots,x_n]}$ is also D-stable. We conclude by Lemma 3.10 and the simplicity of D.

Finally, the properness of the action follows from [DeFiGe1994] (see also [Fr2017, Theorem 3.37]). ☐

We will use the previous characterization later in order to study the subgroup of translations of the isotropy group of a simple derivation, see Lemma 5.9.

4. Automorphisms of simple derivations : dimension of $\operatorname{Aut}(D)^0$

We begin this section by dealing with the special case where $D \in \operatorname{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ is simple and $\operatorname{Aut}(D)$ contains a subgroup acting in a globally trivial way or by translations.

Proposition 4.1. *Let $D \in \operatorname{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ be a simple derivation and $H \subset \operatorname{Aut}(D)$ be a (closed) s-dimensional subgroup, $H \cong \mathbb{G}_a^s$, acting in a globally trivial way. If $B = \mathbb{k}[x_1, \ldots, x_n]^H$, then $\operatorname{Im} D \subset B$. Moreover, the restriction $\overline{D} = D|_B : B \to B$ is a simple derivation, and if $\{f_1, \ldots, f_\ell\}$ is a generating set of the $\mathbb{k}$-algebra B, then $\big\langle D(f_1), \ldots, D(f_\ell)\big\rangle_B = B$ and $\big\langle D(f_1), \ldots, D(f_\ell)\big\rangle_{\mathbb{k}[x_1,\ldots,x_n]} = \mathbb{k}[x_1, \ldots, x_n]$ unless $B = \mathbb{k}$ — that is $s = n$.*

Proof. By the global triviality of the $\mathbb{G}_a^s$-action, there exists a $\mathbb{G}_a^s$-equivariant isomorphism $\mathbb{A}^n \cong \mathbb{A}^s \times \operatorname{Spec}(B)$ and therefore $\mathbb{k}[x_1, \ldots, x_n] = B[y_1, \ldots, y_s]$, with $y_1, \ldots, y_s$ algebraically independent over B. If $p \in \mathbb{k}[x_1, \ldots, x_n]$ and $\Delta \in \operatorname{Lie}(H) \subset \operatorname{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$, then $\Delta\big(D(p)\big) = D\big(\Delta(p)\big) = 0$. We deduce that

$$D(p) \in \bigcap_{\Delta \in \operatorname{Lie}(H)} \ker(\Delta) = \mathbb{k}[x_1, \ldots, x_n]^H = B.$$

Finally, it is clear that the restriction $\overline{D} : B \to B$ must be a simple derivation and, since $I = \big\langle D(f_1), \ldots, D(f_\ell)\big\rangle_B$ is a $\overline{D}$-stable ideal, it is either 0 or B. But $D(f_i) = 0$ implies that f_i is a constant so the result follows. ☐

Remark 4.2. We will show in Theorem 4.7 that, in the notations of Proposition 4.1, $s = \dim H \leq n - 2$, so $\mathbb{k}[x_1, \ldots, x_n]^H \neq \mathbb{k}$.

Corollary 4.3. *Let $D \in \operatorname{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ be a simple derivation and $H \subset \operatorname{Aut}(D)$ be a (closed) s-dimensional subgroup of translations. Then there exist coordinates such that $D = \sum_i a_i \frac{\partial}{\partial x_i}$,* *with* $a_i \in \mathbb{k}[x_{s+1}, \ldots, x_n]$. *Moreover,* $\big\langle a_{s+1}, \ldots, a_n \big\rangle_{\mathbb{k}[x_{s+1},\ldots,x_n]} = \mathbb{k}[x_{s+1}, \ldots, x_n]$ *and therefore* $\big\langle a_{s+1}, \ldots, a_n \big\rangle_{\mathbb{k}[x_1,\ldots,x_n]} = \mathbb{k}[x_1, \ldots, x_n]$.

Proof. In this case, $\mathbb{k}[x_1, \ldots, x_n]^H$ is a polynomial ring and result follows directly from Proposition 4.1. □

In [DEFM2011, theorem 2], the authors describe the structure of the action of an unipotent group of dimension $n-1$ on an affine variety of dimension n. We present here an adaptation to our special case where X is an affine space:

Theorem 4.4. *Let* U *be a unipotent group of dimension* $n-1$ *acting freely on* $\mathbb{A}^n$*. Then* $\mathbb{A}^n$ *is* U*-isomorphic to* $U \times \mathbb{k}$. □

Using Lie-Kolchin Theorem and the well known identification of $\mathrm{Lie}(U)$ as a sub-Lie algebra of $\mathrm{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ (see Remark 2.5), in [DEFM2011, proposition 1] the authors give the following nice description:

Proposition 4.5. *Let* U *be a unipotent group of dimension* n *acting on* $\mathbb{A}^n$ *such that there exists* $x \in \mathbb{A}^n$ *with* $U_x = \{\mathrm{Id}\}$*. Then, upon conjugation with an automorphism, there exists a basis* $\Delta_1, \ldots, \Delta_n$ *of the Lie algebra* $\mathrm{Lie}(U)$ *such that* $\Delta_i(x_j) = \delta_{ij}$ *for* $1 \le i \le j \le n$. □

Corollary 4.6. *Let* U *be a unipotent group of dimension* $d = n-1$ *or* $d = n$ *which acts freely on* $\mathbb{A}^n$*. Then, upon conjugation with an automorphism, there exist* U*-invariant locally nilpotent derivations* $\{\Delta_1, \ldots, \Delta_d\} \subset \mathrm{Der}\big(\mathbb{k}[x_1, \ldots, x_n]\big)$ *such that* $\Delta_i(x_j) = \delta_{ij}$ *for* $i = 1, \ldots, d$ *and* $j = i, \ldots, n$.

Proof. If $\dim U = n$, the result is a direct consequence of Proposition 4.5, and if $\dim U = n-1$, then by Theorem 4.4, the affine space $\mathbb{A}^n$ is U-isomorphic to $U \times \mathbb{k}$ and we conclude by applying again Proposition 4.5 on U acting on itself. □

Now we can state the main result of this section.

Theorem 4.7. *Let* D *be a simple derivation of* $\mathbb{k}[x_1, \ldots, x_n]$*, with* $n \ge 2$*. Then* $\dim \mathrm{Aut}(D)^0 \le n-2$.

Proof. By Theorem 3.2, $\mathrm{Aut}(D)^0$ is an algebraic unipotent group of dimension at most n. Let $d = \dim \mathrm{Aut}(D)^0$ and suppose that $d = n$ or $d = n-1$. It follows from Corollary 4.6 that, upon a choice of coordinates, there exist derivations $\{\Delta_1, \ldots, \Delta_d\}$ such that $\Delta_i(x_j) = \delta_{ij}$ for $i = 1, \ldots, d$ and $j = i, \ldots, n$ and such that the subgroups $\{e^{t\Delta_i}, t \in \mathbb{k}\}$ are contained in $\mathrm{Aut}(D)^0$.

Let $D = \sum a_i \frac{\partial}{\partial x_i}$. By Corollary 4.3, since $e^{t\Delta_1}$ is a translation for all t, we deduce that $a_j \in \mathbb{k}[x_2, \ldots, x_n]$ for all $j = 1, \ldots, n$.

Let $\overline{D}$ and $\overline{\Delta}_2$ be the restrictions of D and Δ_2 to $\mathbb{k}[x_2, \ldots, x_n]$ respectively. Then for $t \in \mathbb{k}$, $e^{t\overline{\Delta_2}}$ is a translation along x_2 that belongs to $\mathrm{Aut}(\overline{D})$. We conclude that $a_j \in \mathbb{k}[x_3, \ldots, x_n]$ for $j = 2, \ldots, n$.

By recurrence, we deduce that $a_{n-1} \in \mathbb{k}[x_n]$; moreover $a_n \in \mathbb{k}$ if $d = n$ or $a_n \in \mathbb{k}[x_n]$ if $d = n-1$. In both cases, we can consider the restriction $\hat{D}$ of D to $\mathbb{k}[x_{n-1}, x_n]$ which is simple by Remark 2.2. Write $\hat{D} = a_{n-1}\frac{\partial}{\partial x_{n-1}} + a_n \frac{\partial}{\partial x_n}$ with $a_{n-1}, a_n \in \mathbb{k}[x_n]$; restricting to the last coordinate, we deduce that $a_n \in \Bbbk$, and therefore $\hat{D}$ is not simple (see Remark 4.9), which is a contradiction. □

Our next goal is to provide examples that show the bound given by Theorem 4.7 is optimal for all $n \geq 2$, see Example 4.12 below. In order to do so, we will use the following criterion, attributed to Shamsuddin.

Proposition 4.8. *Let A be a $\Bbbk$-algebra and consider a simple derivation $\delta \in \mathrm{Der}(A)$. Let $D_{a,b} \in \mathrm{Der}\big(A[y]\big)$ the derivation obtained by extending δ by $D_{a,b}(y) = ay + b$ with $a, b \in A$. Then $D_{a,b}$ is simple if and only if*

$$\delta h \neq ah + b \text{ for all } h \in A.$$

Proof. See [No1994, Theorem 13.2.1]. □

Remark 4.9. (1) Notice that in particular there exist infinitely many simple derivations of the form $\frac{\partial}{\partial u} + c(u,v)\frac{\partial}{\partial v}$, $c \in \Bbbk[u,v]$.

(2) If $c \in \Bbbk[u]$ and $\alpha \in \Bbbk^*$, then the derivation $\delta = \alpha\frac{\partial}{\partial u} + c'(u)\frac{\partial}{\partial v}$, is not simple, since $\delta(\alpha^{-1}c) = c'(u)$.

Proposition 4.10. *Let $D \in \mathrm{Der}\big(\Bbbk[u,v,x_1,\ldots,x_n]\big)$ be a simple derivation of the form $D = \frac{\partial}{\partial u} + c(u,v)\frac{\partial}{\partial v} + \sum_{j=1}^{n} b_j(u,v,x)\frac{\partial}{\partial x_j}$. Then $\varphi|_{\Bbbk[u,v]} = \mathrm{Id}_{\Bbbk[u,v]}$ for all $\varphi \in \mathrm{Aut}(D)$.*

Proof. Let $\varphi = (f_1, f_2, g_1, \ldots, g_n) \in \mathrm{Aut}(D)$. Since $1 = \varphi D(u) = D\big(\varphi(u)\big) = D(f_1)$ we have that $D(f_1 - u) = 0$ so $f_1 = u + t$ for some $t \in \Bbbk$.

We write $f_2 = \sum_r \alpha_r x^r$, where $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n_{\geq 0}$ and $x^r = x_1^{r_1} \ldots x_n^{r_n}$ and $\alpha_r \in \Bbbk[u,v]$. Let d be the multidegree of f_2 for the lexicographic order, with $x_1 \geq \cdots \geq x_n$. First, we show that $d = 0$; for this, assume that $d \neq 0$ and write $c(u,v) = \sum_{k=0}^{\ell} c_k(u)v^k$. Then $\ell > 0$ (see Remark 4.9). Let us calculate in a explicit way the equality $\varphi(D)(v) = D\varphi(v)$:

$$\varphi D(v) = \sum_{k=0}^{\ell} c_k(u+t)f_2^k = c_\ell(u+t)\alpha_r^\ell x^{\ell d} + \text{strictly lower degree terms},$$

and, since $c_\ell(u+t)\alpha_r^\ell \neq 0$, it follows that $\varphi D(v)$ is of multidegree ℓd. On the other hand,

$$\begin{aligned} D\big(\varphi(v)\big) = D(f_2) &= \sum_r \delta(\alpha_r)x^r + \sum_r \sum_{j=1}^{n} i_j b_j(u,v,x)\alpha_r x^{r-e_j} \\ &= \delta(\alpha_d)x^d + \text{strictly lower degree terms}, \end{aligned}$$

where $\{e_j\}$ is the canonical basis of the lattice $\mathbb{Z}^n$ and $\delta \in \mathrm{Der}\big(\Bbbk[u,v]\big)$ is restriction of D to $\Bbbk[u,v]$. It follows that $D\big(\varphi(v)\big)$ is at most of multidegree d.

From the equation $\varphi D(v) = D\varphi(v)$ we deduce $\ell \leq 1$ and hence $\ell = 1$. It follows that $c_1(u+t)\alpha_d = \delta(\alpha_d)$ and α_d is a Darboux polynomial for δ, and therefore it is a constant. But $c_1(u+t) \neq 0$, so $\alpha_d = 0$ which is a contradiction.

Applying the same reasoning to $\varphi^{-1} \in \mathrm{Aut}(D)$ we deduce that $(f_1, f_2) \in \mathrm{Aut}(\delta)$, and it follows from Theorem 1.2 that $(f_1, f_2) = \varphi|_{\Bbbk[u,v]} = \mathrm{Id}_{\Bbbk[u,v]}$. □

Proposition 4.11. *Let $\delta \in \operatorname{Der}(\Bbbk[u,v])$ be a simple derivation and let $I = \{a_1, \ldots, a_n\} \subset \Bbbk[u,v]$ be a linearly independent subset. If the linear span of I is such that $\langle I\rangle_{\Bbbk} \cap \operatorname{Im}(\delta) = \{0\}$, then the derivation $D_I \in \operatorname{Der}([u,v,x_1,\ldots,x_n])$ obtained by extending δ as $D_I(x_j) = a_j$ for $j = 1, \ldots, n$ is simple.*

Proof. We proceed by induction on $n = \#I$; denote $D_n = D_I$. If $n = 0$, then $D_0 = \delta$ is simple by hypothesis. Suppose now that D_I is a simple derivation of $\Bbbk[u,v,x_1,\ldots,x_n]$ for $\#I \leq n$, and consider $I = \{a_1, \ldots, a_{n+1}\}$ as in the hypothesis. Then D_I restricts to a simple derivation $D_n \in \operatorname{Der}(\Bbbk[u,v,x_1,\ldots,x_n])$ by hypothesis.

By Shamsuddin's criterion (see Proposition 4.8) D_{n+1} is not simple if and only if there exists $f \in \Bbbk[u,v,x_1,\ldots,x_n]$ such that $D_n(f) = a_{n+1}$. If $f \in \Bbbk[u,v,x_1,\ldots,x_n]$ is such that $D_n(f) = a_{n+1}$, write $f = \sum_{r\in\mathbb{Z}^n_{\geq 0}} \alpha_r x^r$, with $\alpha_r \in \Bbbk[u,v]$, and let d be the multidegree of $f \in \Bbbk[u,v][x_1,\ldots,x_n]$ for the lexicographic order. Then

$$\begin{aligned} a_{n+1} = D_n(f) &= \sum_r \delta(\alpha_r)x^r + \sum_r \alpha_r \sum_{j=1}^{n} i_j a_j x^{r-e_j} \\ &= \delta(\alpha_d)x^d + \text{strictly lower degree terms.} \end{aligned} \tag{2}$$

By considering the term of degree d in Equation (2), we deduce that $\delta(\alpha_d) = a_{n+1}$ if $d = 0$ or $\delta(\alpha_d) = 0$ otherwise. In the first case, we deduce that a_{n+1} belongs to $\langle I\rangle_{\Bbbk} \cap \operatorname{Im}(\delta)$ and $a_{n+1} = 0$ which is a contradiction.

If $\delta(\alpha_d) = 0$, then $\alpha_d \in \Bbbk$ because δ is simple. Consider $j_0 = \max\{j : (d)_j \neq 0\}$, and let $d' = d - e_{j_0}$. By definition of j_0, for all $j = 1, 2, \ldots, n$ and for all multi-indexes r such that $0 \leq r < d$ we have $d' \neq r - e_j$. We deduce that the term of degree d' in Equation (2) is

$$\delta(\alpha_{d'}) + \alpha_d i_{j_0} a_{j_0}$$

and this term is equal to a_{n+1} if $d' = 0$ or 0 otherwise. In both cases, we have a contradiction with the hypothesis $\langle I\rangle_{\Bbbk} \cap \operatorname{Im}(\delta) = \{0\}$. □

The following example exhibits a derivation $\delta \in \operatorname{Der}(\Bbbk[u,v])$ such that δ admits linearly independent subsets I as in the hypothesis of Proposition 4.11, with arbitrary cardinal. Moreover, for the family of simple derivations that we produce the bound given in Theorem 4.7 is reached.

Example 4.12. Consider the derivation $\delta = \frac{\partial}{\partial u} + (1+uv)\frac{\partial}{\partial v} \in \operatorname{Der}(\Bbbk[u,v])$ — notice that δ it is simple by Shamsuddin's Criterion.

Let us show that $\operatorname{Im}(\delta) \cap \Bbbk[v] = \Bbbk$. Assume that there exists $f = \sum_{i,j} \alpha_{i,j} u^i v^j \in \Bbbk[u,v]$ such that $\delta(f) \in \Bbbk[v]$. By a direct computation, we have that:

$$\delta(f) = \sum_{i,j} \alpha_{i,j}(iu^{i-1}v^j + ju^i v^{j-1} + ju^{i+1}v^j).$$

Let (i_0, j_0) be the multidegree of f for lexicographic order with $v \geq u$. If $j_0 \neq 0$, then $\delta(p)$ is of multidegree (i_0+1, j_0), and we cannot have $\delta(p) \in \Bbbk[v]$. If $j_0 = 0$ then $f \in \Bbbk[u]$ and $\delta(f) \in \Bbbk[v]$ implies $\delta(f) \in \Bbbk$ and our assertion follows.

Notice in particular that there exist linearly independent subsets I as in the hypothesis of Proposition 4.11, of arbitrary finite cardinal.

Corollary 4.13. *For every $n \geq 2$, there exists a simple derivation $D \in \operatorname{Der}(\mathbb{k}[x_1, \dots, x_n])$ such that* $\dim \operatorname{Aut}(D)^0 = n - 2$.

Proof. The case $n = 2$ is the content of Theorem 1.2. If $n > 2$, consider δ as in Example 4.12 and let $I = \{b_1, \dots, b_n\} \subset \mathbb{k}[v]$ a linearly independent subset of cardinal n, such that $\langle I \rangle_{\mathbb{k}} \cap \operatorname{Im}(\delta) = \{0\}$. Then the derivation $D_I = \delta + \sum_i b_i(v) \frac{\partial}{\partial x_i}$. is simple by Proposition 4.11. Moreover, since D_I commutes with $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ for $n \geq 1$, it follows that $\operatorname{Aut}(D_I)^0$ contains the subgroup $\mathbb{G}_a^n$ of translations on the x coordinates. Hence $\operatorname{Aut}(D_I)^0 = \mathbb{G}_a^s$ by Theorem 4.7, and the result follows. □

5. Derivations invariant under the action of a group of translations

Let D be a simple derivation such that $\operatorname{Aut}(D)$ contains a non trivial subgroup of translations. In this section we give some insight on how to exploit this fact in order to describe $\operatorname{Aut}(D)$.

5.1. Isotropy groups with non trivial subgroups of translations.

Lemma 5.1. *Let $H \subset \operatorname{Aut}(\mathbb{A}^n)$, $H \cong G_a^s$, $1 \leq s \leq n-1$, be a subgroup of automorphisms acting in a globally trivial way, and consider an equivariant isomorphism $\mathbb{A}^n \cong \mathbb{A}^s \times V$ as in Definition 3.7. Then the normalizer of H in $\operatorname{Aut}(\mathbb{A}^n)$ has the form*

$$N_{\operatorname{Aut}(\mathbb{A}^n)}(H) = \{(x, v) \mapsto (Ax + g_1(v), g_2(v)) : A \in \operatorname{GL}_s(\mathbb{k}), g_1 : V \to \mathbb{A}^s, g_2 \in \operatorname{Aut}(V)\}.$$

Proof. We describe an automorphism $f \in \operatorname{Aut}(\mathbb{A}^n) \cong \operatorname{Aut}(\mathbb{A}^s \times V)$ as a pair of morphisms (f_1, f_2), with $f_1 : \mathbb{A}^s \times V \to \mathbb{A}^s$, $f_2 : \mathbb{A}^s \times V \to V$. Then $H = \{(t_a, \operatorname{Id}_V) : a \in \mathbb{k}^s\}$, and if $f = (f_1, f_2) \in N_{\operatorname{Aut}(\mathbb{A}^n)}(H)$, we have that $\sigma_f : H \to H$, $\sigma_f(t_a, \operatorname{Id}_V) = f(t_a, \operatorname{Id}_V) f^{-1}$ is a morphism of algebraic groups. It follows that there exists $A_f \in GL_s(\mathbb{k})$ such that $\sigma_f(t_a, \operatorname{Id}_V) = (t_{A_f a}, \operatorname{Id}_V)$ for all $a \in \mathbb{k}^s$ — recall that $\operatorname{char}(\mathbb{k}) = 0$.

From the equality

$$f(t_a, \operatorname{Id}_V) = (t_{A_f a}, \operatorname{Id}_V) f : \mathbb{A}^s \times V \to \mathbb{A}^s \times V$$

we deduce that

$$f_1(x + a, v) = f_1(x, v) + A_f a \quad , \quad f_2(x + a, v) = f_2(x, v) \text{ for all } a \in \mathbb{k}^s, v \in V.$$

Let $f_1(x, v) = (f_{11}(x, v), \dots f_{1s}(x, v))$ and consider the maps f_{1j} as polynomials in $\mathbb{k}[V][x]$. Then $f_{ij}(x + a, v) = f_{ij}(x, v) + (A_f a)_j$ and an easy calculation on the coefficients shows that

$$f_1(x, v) = A_f x + g_1(v), g_1 : V \to \mathbb{k}^s$$

On the other hand, since $f_2 : \mathbb{A}^s \times V \to V$ is a $\mathbb{G}_a^s$-invariant morphism and that $p_2 : \mathbb{A}^s \times V \to V$ is the geometric quotient, it follows that there exists $g_2 : V \to V$ such that $f_2(x, v) = g_2 \circ p_2(x, v) = g_2(v)$.

Finally, applying the same reasoning to the inverse $f^{-1} \in N_{\operatorname{Aut}(\mathbb{A}^n)}(H)$, we deduce that $g_2 \in \operatorname{Aut}(V)$. □

Proposition 5.2. *Let D be a simple derivation and $H \subset \operatorname{Aut}(D)$ a normal subgroup, $H \cong \mathbb{G}_a^s$, $s \geq 1$, that acts in a globally trivial way. Consider an equivariant isomorphism $\mathbb{A}^n \cong \mathbb{A}^s \times V$ as in Definition 3.7 and let $\overline{D} = D|_{\Bbbk[V]} : \Bbbk[V] \to \Bbbk[V]$ (see Proposition 4.1). If we identify $\operatorname{Aut}(D)$ as a subgroup of $\operatorname{Aut}(\mathbb{A}^s \times V)$ and $\operatorname{Aut}(\overline{D})$ as a subgroup of $\operatorname{Aut}(V)$, then*

$$\operatorname{Aut}(D) \subset \big\{ \big(Ax + g_1(v), g_2(v)\big) : A \in \operatorname{GL}_s(\Bbbk)\,,\ g_1 : V \to \mathbb{A}^s\,,\ g_2 \in \operatorname{Aut}(\overline{D}) \big\}.$$

Proof. By Lemma 5.1, follows that

$$\operatorname{Aut}(D) \subset \big\{ \big(Ax + g_1(v), g_2(v)\big) : A \in \operatorname{GL}_s(\Bbbk)\,,\ g_1 : V \to \mathbb{A}^s\,,\ g_2 \in \operatorname{Aut}(V) \big\},$$

so it remains to prove that if $f = \big(Ax + g_1(v), g_2(v)\big) \in \operatorname{Aut}(D)$, then $g_2 \in \operatorname{Aut}(\overline{D})$ — that is, after identification of $\operatorname{Aut}(\mathbb{A}^n)$ with $\operatorname{Aut}\big(\Bbbk[x_1, \ldots, x_n]\big)$, that $Dg_2(p) = g_2 D(p)$ for all $p \in \Bbbk[V]$. But by definition, we have that

$$\begin{aligned} \overline{D} g_2(p) =& D\big(p\big(g_2(v)\big)\big) = D\big(p\big(f_2(x, v)\big)\big) = D\big(p\big(f(x, v)\big)\big) = Df(p) \\ g_2 \overline{D}(p) =& fD(p), \end{aligned}$$

where we consider $p \in \Bbbk[V] \subset \Bbbk[V][x]$, and the result follows. □

Remark 5.3. If X, Y are affine algebraic varieties, then $\operatorname{Hom}(X, Y)$ inherits a structure of ind-variety (see for example [FuKr2018, Lemma 3.1.4]). In the notations of Proposition 5.2, we identify $\operatorname{Aut}(\mathbb{A}^n) = \operatorname{Aut}(\mathbb{A}^s \times V)$ as a subset of $\operatorname{Hom}(\mathbb{A}^s \times V, \mathbb{A}^s) \times \operatorname{Hom}(\mathbb{A}^s \times V, V)$, and restrict the projection over the second coordinate to $\operatorname{Aut}(D)$. Then $p_2\big(\operatorname{Aut}(D)\big)$ identifies with a subgroup of $\operatorname{Aut}(\overline{D})$, in such a way that the corresponding map $\varphi : \operatorname{Aut}(D) \to \operatorname{Aut}(\overline{D})$, is a morphism of ind-groups. It follows that

$$\operatorname{Aut}(D)^0 \subset \big\{ \big(Ax + g_1(v), g_2(v)\big) : A \in \operatorname{GL}_s(\Bbbk)\,,\ g_1 = V \to \mathbb{A}^s\,,\ g_2 \in \operatorname{Aut}(\overline{D})^0 \big\}.$$

Theorem 5.4. *Let D be a simple derivation such that $\operatorname{Aut}(D)$ contains a non trivial normal subgroup $H \cong \mathbb{A}^s$, $s \geq 1$, such that H acts on a globally trivial way. Consider an equivariant isomorphism $\mathbb{A}^n \cong \mathbb{A}^s \times V$ as in Proposition 5.2 and let $\overline{D} = D|_{\Bbbk[V]}$. Assume moreover that the restriction $\overline{D} : \Bbbk[V] \to \Bbbk[V]$ is such that $\operatorname{Aut}(\overline{D})$ is algebraic.*

Then $\operatorname{Aut}(D)$ is algebraic; in particular, $\operatorname{Aut}(D) = \operatorname{Aut}(D)^0$.

Proof. By Proposition 5.2, if $f \in \operatorname{Aut}(D)$ then $f(x, v) = \big(Ax + g_1(v), g_2(v)\big)$, with $A \in \operatorname{GL}_s(\Bbbk)$ and $g_2 \in \operatorname{Aut}(\overline{D})$. By induction on the number of compositions, we deduce that for $\ell \in \mathbb{N}$

$$f^\ell(x, v) = \Big(A^\ell x + \sum_{i=0}^{\ell-1} A^{\ell-i} g_1\big(g_2^i(v)\big)\,,\ g_2^\ell(v) \Big).$$

Consider the equivariant isomorphism $\psi : \mathbb{A}^n \to \mathbb{A}^s \times V$ and the isomorphism of ind-groups $\operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}(\mathbb{A}^s \times V)$ given by conjugation by ψ. Then $\psi^{-1}(\operatorname{Id}_{\mathbb{A}^s}, g_2)\psi : \mathbb{A}^n \to \mathbb{A}^n$ is an algebraic automorphism, and it follows that, under identification by ψ, the family $\big\{\big(A^\ell x + \sum_{i=0}^{\ell-1} A^{\ell-i} g_1(g_2^i(v)), g_2(v)\big) : \mathbb{A}^n \to \mathbb{A}^n : \ell \geq 0\big\}$ has bounded degree, and therefore f is algebraic. □

Corollary 5.5. *Let $D \in \operatorname{Der}\big(\Bbbk[x_1, \ldots, x_n]\big)$ such that $\operatorname{Aut}(D)^0 = \mathbb{G}_a^{n-2}$, $n \geq 2$, and assume moreover that $\operatorname{Aut}(D)^0$ acts by translations. Then $\operatorname{Aut}(D) = \operatorname{Aut}(D)^0$.*

Proof. This is a direct application of Theorem 5.4, together with the fact that a simple derivation of $\Bbbk[u,v]$ has trivial isotropy (Theorem 1.2). □

Example 5.6. Notice that in particular, Corollary 5.5 shows that the derivations D_I exhibited in Example 4.12 are such that $\mathrm{Aut}(D_I)$ is algebraic (see also Corollary 4.12).

5.2. Simple derivations of $\Bbbk[x_1,\dots,x_n]$ in small dimension and their automorphisms.

In this section we study the automorphisms group of a simple derivation of $\Bbbk[x_1,\dots,x_n]$, with $n=3,4$, the cases $n=1,2$ being well known:

(1) A derivation $D \in \mathrm{Der}\big(\Bbbk[x]\big)$ is simple if and only if it is locally nilpotent, and in this case $\mathrm{Aut}(D)=\{e^{tD} : t\in\Bbbk\}$.

(2) Any simple derivation of $\Bbbk[x,y]$ has trivial isotropy (see Theorem 1.2).

Simple derivations of $\Bbbk[x_1,x_2,x_3]$.

Let D be a simple derivation of $\Bbbk[x_1,x_2,x_3]$. Our objective is to show that $\mathrm{Aut}(D)$ is either isomorphic to $\mathbb{G}_a$ or possible a countable discrete group: More precisely, $\mathrm{Aut}(D)^0$ is unipotent and of dimension 0 or 1 by Theorem 4.7; the following theorem shows that in this last case, $\mathrm{Aut}(D)=\mathrm{Aut}(D)^0$, its action being by translations.

Theorem 5.7. *Let $D\in\mathrm{Der}\big(\Bbbk[x_1,x_2,x_3]\big)$ be a simple derivation such that $\dim \mathrm{Aut}(D)^0=1$. Then $\mathrm{Aut}(D)=\mathrm{Aut}(D)^0$. Moreover, there exist coordinates such that*

$$\begin{aligned}\mathrm{Aut}(D) &= \big\{(x_1+a,x_2,x_3) : a\in\Bbbk\big\}\\ D &= \sum a_i(x_2,x_3)\frac{\partial}{\partial x_i}\end{aligned}\tag{3}$$

Proof. If $u\in\mathrm{Aut}(D)^0$ is a non trivial automorphism, then by Proposition 3.1, u has no fixed point but, by a result of Kaliman (see [Kal2004]), a non trivial unipotent automorphism $u\in\mathrm{Aut}(\mathbb{A}^3)$ without fixed point is conjugated to a translation. It follows from Proposition 5.2 that there exist coordinates as in Equation (3) but for $\mathrm{Aut}(D)^0$. Hence, it remains to prove that $\mathrm{Aut}(D)=\mathrm{Aut}(D)^0$; this is the content of Corollary 5.5. □

A family of simple derivations of $\Bbbk[x_1,x_2,x_3,x_4]$.

Next, we consider the four dimensional affine space and describe the automorphisms group of a simple derivation $D\in\mathrm{Der}\big(\Bbbk[x_1,x_2,x_3,x_4]\big)$ that admits a linear coordinate.

Lemma 5.8. *Let $D\in\mathrm{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$ be a simple derivation that admits a linear coordinate s. Let $\phi\in\mathrm{Aut}(D)^0$ and Δ be a locally nilpotent derivation such that $\phi=e^{\Delta}$ (see Remark 2.5). Then $\Delta(s)\in\Bbbk$. In particular, then either $\phi(s)=s$ or ϕ is a translation along the direction of s.*

Proof. Since $\phi\in\mathrm{Aut}(D)$, then $[D,\Delta]=0$ and therefore $D\big(\Delta(s)\big)=0$, so $\Delta(s)\in\Bbbk$.

If $\Delta(s)=0$, then $\phi(s)=s$. On the other hand, if $\Delta(s)=c\neq 0$, then $\Delta(c^{-1}s)=1$, i.e. $c^{-1}s$ is a slice for Δ. Hence, $\Bbbk[x_1,\dots,x_n]=\ker(\Delta)[c^{-1}s]$. Since $c^{-1}s$ is a coordinate we may, up to conjugation, assume $x_1=c^{-1}s$. Thus $\ker(\Delta)=\Bbbk[x_2,\dots,x_n]$, i.e. ϕ is translation along the s coordinate. □

Now we specialize Lemma 5.8 to the case where $n = 4$. We begin by a useful result.

Lemma 5.9. *Let $D \in \operatorname{Der}\big(\Bbbk[x_1, x_2, x_3, x_4]\big)$ be a simple derivation that admits a linear coordinate. Then every nontrivial element of $\operatorname{Aut}(D)^0$ is conjugate to a translation.*

Proof. Let $\phi \in \operatorname{Aut}(D)^0$ and let Δ be a locally nilpotent derivation such that $\phi = e^{\Delta}$. By Lemma 5.8, if suffices to prove that if s is a linear coordinate for D such that $\Delta(s) = 0$, then ϕ is a translation. By Proposition 3.11 we know that $\mathbb{G}_a = \{e^{t\Delta}; t \in \Bbbk\}$ acts over $\mathbb{A}^4$ in a locally trivial, and in particular a proper, way. Then, the result follows from [Kal2018, Thm. 01]. ☐

Theorem 5.10. *Let $D \in \operatorname{Der}\big(\Bbbk[x_1, x_2, x_3, x_4]\big)$ be a simple derivation that admits a linear coordinate. If $\dim \operatorname{Aut}(D)^0 > 0$ then $\operatorname{Aut}(D)^0$ acts by translations. Moreover, if $\dim \operatorname{Aut}(D)^0 = 2$, then $\operatorname{Aut}(D)$ is algebraic.*

Proof. Recall that, by Theorem 4.7, $\operatorname{Aut}(D)^0$ is a unipotent group of dimension 1 or 2. In the case of dimension 1 the theorem is a direct consequence of Lemma 5.9.

Now, we suppose that $\dim \operatorname{Aut}(D)^0 = 2$. It is well know that in this case $\operatorname{Aut}(D)^0 = U_1 \times U_2$, $U_i \cong \mathbb{G}_a$, — recall that $\operatorname{char} \Bbbk = 0$. Consider two generators u_1, u_2 of U_1 and U_2 respectively. Again by Lemma 5.9 we may suppose that u_1 is a translation with respect to x_1; therefore, we deduce from Lemma 5.1 that $u_2 = \big(ax_1 + g_1(x_2, x_3, x_3), g_2(x_2, x_3, x_4), g_3(x_2, x_3, x_4), g_4(x_2, x_3, x_4)\big)$; moreover, since u_1 and u_2 commute, we deduce that $a = 1$. Then the geometric quotient $q_1 : \mathbb{A}^4 \to X_1 := \mathbb{A}^4/\overline{\langle U_1 \rangle}$ exists and is isomorphic to $\mathbb{A}^3$, and D induces a simple derivation $\overline{D} \in \operatorname{Der}\big(\Bbbk[X_1]\big) = \operatorname{Der}\big(\Bbbk[x_2, x_2, x_4]\big)$. As u_1 and u_2 commute, we deduce that U_2 acts on X_1, in such a way that $U_2 \subset \operatorname{Aut}(\overline{D})$. It follows from Theorem 5.7 that either the action of U_2 over X_1 is trivial or given by translations.

If U_2 acts trivially, it follows that $u_2 = \big(x_1 + g_1(x_2, x_3, x_3), x_2, x_3, x_4\big)$. But if $(p_2, p_3, p_4) \in \mathcal{V}(g_1) \subset \mathbb{A}^3$, it follows that $(0, p_2, p_3, p_4)$ is a fixed point of u_2, and we obtain a contradiction. Hence U_2 acts by translations over X_1. Changing coordinates in $X_1 = \mathbb{A}^3$, we can assume that $u_2 = \big(x_1 + w(x_2, x_3, x_4), x_2 + 1, x_3, x_4\big)$. Let $W \in \Bbbk[x_2, x_3, x_4]$ be such that $\partial/\partial x_2(W) = w$, and consider the coordinates (z, x_2, x_3, x_4), where $z = x_1 - W + x_2$. Then $u_1 = (z + 1, x_2, x_3, x_4)$ and $u_2 = (z + 1, x_2 + 1, x_3, x_4)$, and it follows that in these new coordinates, $\operatorname{Aut}(G)^0$ is included in the group of translations.

In order to finish the proof, we apply Corollary 5.5. ☐

We finish by showing that the absence of linear coordinates is not an obstruction for a simple derivation to have algebraic automorphisms group.

Example 5.11. In [Jor1984], the author shows that the derivations $D_n \in \operatorname{Der}\big(\Bbbk[x_1, \dots, x_n]\big)$ given by

$$D_n = (1 - x_1 x_2)\frac{\partial}{\partial x_1} + x_1^3 \frac{\partial}{\partial x_2} + \sum_{i=3}^{n} x_{i-1} \frac{\partial}{\partial x_i}$$

are simple for all $n \geq 2$. In [Ya2022], the author shows that if $n \geq 3$, then $\operatorname{Aut}(D_n) \cong \mathbb{G}_a$, acting by translations in the last coordinate — recall that $\operatorname{Aut}(D_2) = \{\mathrm{Id}\}$. We affirm that $\operatorname{Im}(D_n) \cap \Bbbk = \{0\}$ — in particular, the family D_n, $n \geq 2$, gives examples of simple derivations without linear coordinates such that their automorphism group is algebraic.

Indeed, $\mathrm{Im}(D_n)$ is the linear span of $G = \{D_n(x^d) : d \in \mathbb{N}^n\}$.

Let $(e_i)_{1\leq i\leq n}$ be the canonical basis of $\mathbb{Z}^n$, a direct computation shows that: $D_n(x^{e_1}) = 1 - x_1x_2$ is the only polynomial which contains monomials 1 and x_1, x_2 with non zero coefficient in G and so $1 \notin \mathrm{Im}(D_n)$.

References

[B2016] R.Baltazar, *On simple Shamsuddin derivations in two variables*, Annals of the Brazilian Academy of Sciences, 88(4) (2016), 2031-2038.

[BaPa2019] R. Baltazar and I. Pan, *On the Automorphism Group of a polynomial differential ring in two variables*, J. of Algebra, v. 576 (2021), 197-227.

[DEFM2011] Derksen, H., van den Essen, A., Finston, D. R., Maubach, S., *Unipotent group actions on affine varieties*, J. Algebra 336 (2011), 200–208.

[DeFiGe1994] Deveney J.K., Finston, D.R.,Gehrke M., $\mathbb{G}_a$ *actions on* $\mathbb{C}^n$, Comm. Algebra 22 (1994), 4977–4988.

[Fr2017] G. Freudenburg, *Algebraic theory of locally nilpotent derivations*, Encyclopaedia of Mathematical Sciences, vol. 136, Second edition, Springer-Verlag, Berlin, (2017).

[FiWa1997] D. R Finston, S. Walcher, *Centralizers of locally nilpotent derivations*, Journal of Pure and Applied Algebra 120 (1997), 39-49.

[FuKr2018] J.P. Furter, H. Kraft, *On the geometry of the automorphism groups of affine varieties*, 2018, 179 pages, available at https://arxiv.org/abs/1809.04175

[Jor1984] Jordan, D. A.,*Differentially simple rings with no invertible derivatives*, Quart. J. Math. Oxford Ser. (2) 32 (1981) 417–424.

[GoWa1989] K.R. Goodearl, R.B. Warfield Jr., *An Introduction to Noncommutative Noetherian Rings*, in: London Mathematical Society Student Text, vol. 16, Cambridge University Press, 1989.

[Kal2004] S. Kaliman, *Free* $\mathbb{C}^+$*-actions on* $\mathbb{C}^3$ *are translations*, Invent. Math. 156 (2004), 163–173.

[Kal2018] S. Kaliman, *Proper* $\mathbb{G}_a$*-actions on* $\mathbb{C}^4$ *preserving a coordinate*, Algebra Number Theory 12:2(2018), 227-258.

[Ku2002] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, 204, Birkhaüser Boston, Inc., Boston, MA, 2002.

[Le2008] Y. Lequain, *Simple Shamsuddin derivations of* $K[X; Y_1, \ldots, Y_n]$*:An algorithmic characterization*, Journal of Pure and Applied Algebra 212 (2008) 801–807.

[MePa2016] L.G. Mendes and I. Pan, *On plane polynomial automorphisms commuting with simple derivations*, Journal of Pure and Applied Algebra Volume 221, Issue 4 (2017), pp. 875-882.

[No1994] A. Nowicki, *Polynomial derivations and their rings of constants*, 1994, available at http://www-users.mat.umk.pl/ anow/ps-dvi/pol-der.pdf

[Pa2022] I. Pan, *A characterization of local nilpotence for dimension two polynomial derivations*, Comm. in Algebra Volume 50 (2022), Issue 5, pp. 1884-1888

[Sha1981] I. R. Shafarevich *On some infinite dimensional algebraic groups. II*, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1) (1981), pp. 214-226, 240.

[vdE] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture*, Progress in Mathematics, Vol. 190, Birkhäuser (2000).

[Ya2022] D. Yan, *A note on isotropy groups and simple derivations*, Comm. in Algebra Volume 50, (2022) Issue 7, pp. 2831-2839.