Dynkin Games for Lévy Processes

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Abstract

We obtain a verification theorem for solving a Dynkin game driven by a Lévy process. The result requires finding two averaging functions that, composed respectively with the supremum and the infimum of the process, summed, and taked the expectation, provide the value function of the game. The optimal stopping rules are the respective hitting times of the support sets of the averaging functions. The proof relies on fluctuation identities of the underlying Lévy process. We illustrate our result with three new simple examples, where the smooth pasting property of the solutions is not always present.

Keywords: Dynkin Game, Lévy processes, Wiener-Hopf Factorization

1 Introduction

By a Dynkin game we understand an optimal stopping game involving two players, called the *max* and *min* players. The max player chooses the moment to stop a stochastic process, and receives the value of the stopped process from the min player; the min player chooses the moment to stop another stochastic process, and has to pay the value of this second stopped process to the max player. The game ends once the first of these two moments has been chosen. In the present paper, these moments are modeled by stopping times under a common filtration, and the payoffs processes are compositions with different functions of a unique stochastic processes, an underlying Lévy process. As a two player game, when a minimax theorem holds, we say that the game has a value, and if there exists a pair of strategies that realizes this value, this pair of stopping times is called a Nash equilibrium.

A variant of this game was first formulated by Dynkin in 1969 (see [10]), as an extension of the classical optimal stopping problem for sequences of random variables, proving the existence of a value for the game, and finding a pair of approximate optimal stopping times. The methodology to prove these results is nowadays called the *martingale approach*, initiated by Snell [34] for the optimal stopping problem (see the presentation of the martingale and markovian approaches to optimal stopping in [30]). In the very same year, two related publications appeared. First, a game formulated for Markov chains by Frid in [13], and second, one formulated for the Wiener process by Gusein-Zade in [15]. In these last two papers, each player has a set where he can stop the process, being these two sets disjoint.

Our formulation is slightly different, and follows the stochastic game proposed by Neveu (see [29, VI-6]), our main reference being the work by Ekström and Peskir [11]. See also the related work by Peskir [30] and the references in these two papers.

As in the case of optimal stopping problems, there is a strong motivation coming from mathematical finance. In the case of a game, the max player is the holder of an American type contract (like in the optimal stopping situation), meanwhile the min player is the issuer of the contract, that, paying a penalty, can also decide the time of execution of the American contract. These contracts receive

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then the name of *callable* American contracts. In this direction, we can mention the work of Kifer [18], who studies the non-arbitrage pricing of such a contract, for both the Cox-Ross-Rubinstein binomial model and the Black-Scholes model; Kyprianou [23], where explicit solutions for put callable perpetual options are found; and Emmerling [12], where explicit solutions of callable call options are provided. These two latter works consider the Black-Scholes model.

Studies regarding Dynkin games have been carried out using martingale and Markovian methods, and also analytic approaches in different situations (see the discussion in [11]). In particular, in [11], Ekström and Peskir prove the existence of a value function and a Nash equilibrium for Dynkin games for a strong Markov process, with right continuous and quasi-left-continuous trajectories. This fine result is achieved in several steps, based first on the martingale approach, using the Wald-Bellman equations and afterwards the Markovian approach. The Nash equilibrium strategies for each player are stopping times, respectively obtained as hitting times of certain Borel sets. Also in a general framework, Peskir [30] proves that the Nash equilibrium for Dynkin games involving strong Markov processes can be characterized in terms of superharmonic and subharmonic functions, thereby providing a comprehensive framework for understanding the value function in these games.

When considering particular classes of processes, situation in which one expects to obtain more explicit solutions to particular problems, there are important contributions when the underlying process is a diffusion. Besides the after-mentioned results [23] and [12], Alvarez [2] relies on the fundamental solutions of the second-order differential equation associated with the generator of the diffusion process in order to find the value of a game and characterize equilibrium points in some particular situations. In this respect, it is well known that in order to find explicit solutions to *boundary* problems for a diffusion process, like computing hitting barrier probabilities, finding a stationary distribution of a reflected diffusion within an interval, solving an optimal stopping problem, or solving a stopping game as above, the fundamental solutions often contains all the necessary information to solve the aforementioned problems.

However, the generalization of these ideas or procedures to Lévy processes encounter significant difficulties. The key problem is the overshoot: boundary conditions in Lévy processes are not limited to one or two points (the extremes of the interval where the process evolves) but extend over the entire line, as the process, when stopped, can potentially reach any point. In analytic terms, many boundary problems for diffusions can be approached by solving an ordinary differential equation on an interval or a half-line, whereas the same type of problem for a Lévy process requires to solve an integro-differential equation, which, in general, does not have explicit solutions.

At least two strategies have been proposed to face this difficulty. One approach is to consider Lévy processes with one-sided jumps, called *spectrally* (say negative) Lévy processes. This fact simplifies the boundary conditions when the process does not jump over a boundary. Even in the case when the process does overshoot, the exponential nature of the distribution of one of the Wiener-Hopf factors (the maximum in the case of negative jumps) can help to obtain useful information regarding the solution of the boundary problem, as the optimal threshold for the American put option for spectrally negative Lévy process found by Chan [7]. In many of these cases, the scale function –which is computable for large classes of these processes (see Hubalek and Kyprianou [16] and Kuznetsov [22])– provides valuable information for solving some problems. Examples of this situation are the Dynkin games for spectrally Lévy processes considered by Bardoux and Kyprianou in [4] (the Mc Kean game) and [5] (the Shepp-Shiryaev game).

A second approach leverages the memoryless property of exponential random variables: when the jump is exponential, the overshoot is independent of the departure point, potentially allowing for closed-form solutions to various boundary problems. Based on this principle, it is possible to consider boundary problems for Lévy processes with jump distributions to include phase-type distributions [3], or those with rational transforms [25], or even when the the density of the Lévy measure is given by an infinite series of exponential functions with positive coefficients, as considered in [20]. Examples of closed solutions for optimal stopping of Lévy processes with exponential-like

jumps can be found, for instance in [3], [19], or [26]. Closed solution for Dynkin games in the context of mathematical finance (i.e. the pricing of certian callable perpetual contracts) for processes with positive exponential jumps have been found by Gapeev and Kühn [14].

At the same time, there have been efforts to solve optimal stopping problems for general Lévy processes using the distributions of the supremum and infimum of the process. The first work addressing one-sided optimal stopping problems through the distribution of the supremum was performed for general random walks by Darling et al. [9], and generalized to Lévy processes by Mordecki [26] (see also [1]). Posterior developments in optimal stopping and the supremum can be found in the monograph by Kyprianou [24]. For two-sided problems, both the infimum and the supremum are expected to participate in the solution. A verification theorem for Lévy processes (as a particular case of Hunt processes) is presented in [28]. Another verification theorem for the two sided case was presented in [8]. A third verification theorem for Lévy processes, where also examples with closed solutions are included, was provided by Mordecki and Oliú [27]. In a parallelism with diffusions, one can think that the distributions of the infimum and the supremum of a Lévy process play the role of the fundamental solutions of a diffusion, i.e., they contain all the information necessary to solve boundary problems, and, if known, allow for an explicit solution of the problem.

As we mentioned, in the present paper we analyze a Dynkin game in the formulation of Neveu [29], where the underlying is a Lévy process. Although the two-sided optimal stopping (with only one max-player) is an optimization problem, and a Dynkin game with two players requires to find an equilibrium, i.e. it is a minimax-problem, we found that some ideas of [27] are applicable, and a (somewhat similar) verification theorem is obtained. More precisely, the verification theorem proposes to construct the value function in terms of the infimum and supremum of the Lévy process, with the help of two *averaging* functions, that should be found. The optimal stopping rules are the support of the respective averaging functions. Naturally, as the distribution of the infimum and the supremum of the process are involved, the Wiener-Hopf factorization obtained by Rogozin [32] for Lévy process (see also [6] or [24]), appear as a natural tool to find explicit formulas for these distributions. We afterwards apply this verification result to three examples with linear payoff functions and Lévy processes with known Wiener-Hopf factors, finding explicit solutions. Although it is not the object of the present work, we hope that these results can find applications in mathematical finance.

The rest of the paper is organized as follows. Section 2 includes some preliminaries, necessary to introduce in Section 3 the main verification theorem. This section also includes the proof of this main result, and the computation of the lateral derivatives of the value function at the contact points, in terms of the averaging functions and the possible mass of the supremum (or infimum) of the process at the origin. Section 4 presents an application. The Dynkin game is driven by a process with known Wiener-Hopf factors, and the functions corresponding to the players are parallel lines. For three processes (Brownian motion with drift, Cramér-Lundberg and Compound Poisson) we obtain the explicit solutions of the considered Dynkin game, and illustrate each situation with numerical examples. The smooth pasting property is analyzed in each example, as in some cases it does not hold. In Section 5 some conclusions are presented.

2 Preliminaries

Let $X = \{X_t : t \ge 0\}$ be a Lévy process defined on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\ge 0}, \mathbf{P}_x)$ departing from $X_0 = x$. Assume that the filtration \mathbf{F} satisfies the usual conditions (see [17]). The corresponding expectation is denoted by \mathbf{E}_x , and for short we denote $\mathbf{E} = \mathbf{E}_0$ and $\mathbf{P} = \mathbf{P}_0$. The Lévy-Khintchine formula characterizes the law of the process, stating, for $z \in i\mathbb{R}$, that $\mathbf{E} e^{zX_t} = e^{t\Psi(z)}$ with

$$\Psi(z) = cz + \frac{\sigma^2}{2}z^2 + \int_{\mathbb{R}} \left(e^{zy} - 1 - zy \mathbf{1}_{\{|y| < 1\}} \right) \Pi(dy),$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ and $\Pi(dy)$ is a non-negative measure (the jump measure) that satisfies the integrability condition $\int_{\mathbb{R}} (1 \wedge y^2) \Pi(dy) < \infty$. For general references on Lévy processes see [6] or [24]. Given the stochastic basis \mathcal{B} the set of stopping times is the set of random variables

$$\mathcal{M} = \{\tau \colon \Omega \to [0, \infty] \text{ such that } \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Observe that we allow the possibility $\tau = \infty$, as several stopping rules that take part in the solution of the considered problems are within this class.

The Wiener-Hopf factorization relates the characteristic functions of the overall infimum and supremum of a Lévy process, stopped at an independent exponential time, with the characteristic exponent given by the Lévy Khintchine formula [6, Chapter VI]. These two random variables play a key role in the solution of two-sided problems, and are defined respectively by

$$I = \inf\{X_t : 0 \le t \le e_r\}, \quad S = \sup\{X_t : 0 \le t \le e_r\},$$
(1)

where e_r is an exponential random variable of parameter r > 0, independent of X. Observe that as r > 0, both random variables S and I are proper. It is also worth saying that the parameter r plays the role of a discount in the definition 1 of the Dynkin game. With these definitions, the Wiener-Hopf factorization (see [32] or [6, Theorem VII.5]) states

$$\frac{r}{r-\Psi(z)} = \mathbf{E}(e^{zS}) \, \mathbf{E}(e^{zI})$$

This equation allows, in some cases, to compute the distribution of the supremum and/or the infimum of a Lévy process (see for instance [21] or [25]).

Definition 1 (Dynkin game). Let $G_1(x) \leq G_2(x)$ be two real continuous payoff functions. Consider a Lévy process X and a discount factor r > 0, such that conditions (3) and (4) below hold. Given two stopping times σ and τ , define the expected payoff

$$\mathbf{M}_x(\sigma,\tau) = \mathbf{E}_x \left(e^{-r\tau} G_1(X_\tau) \mathbf{1}_{\{\tau \le \sigma\}} + e^{-r\sigma} G_2(X_\sigma) \mathbf{1}_{\{\sigma < \tau\}} \right).$$

The Dynkin game (DG) problem consists in finding the value function V(x) and two optimal stopping rules σ^* and τ^* such that

$$V(x) = \inf_{\sigma} \sup_{\tau} \mathbf{M}_x(\sigma, \tau) = \sup_{\tau} \inf_{\sigma} \mathbf{M}_x(\sigma, \tau) = \mathbf{M}_x(\sigma^*, \tau^*).$$
(2)

It is important to mention that, in terms of Game Theory, there are two players, the first one is the max player, that chooses a strategy τ ; the second one is the min player, and chooses a strategy σ . As a result of these elections, the min player pays

$$e^{-r\tau}G_1(X_{\tau})\mathbf{1}_{\{\tau \leq \sigma\}} + e^{-r\sigma}G_2(X_{\sigma})\mathbf{1}_{\{\sigma < \tau\}},$$

to the max player. In this context the pair of optimal strategies (σ^*, τ^*) is called a Nash equilibrium.

The above definition requires two technical conditions. The first one is an integrability condition, following [11, eq. (2.1)], that involves both the payoff functions G_i (i = 1, 2) and the Lévy process X. Assume, given a positive discount factor r, that

$$\mathbf{E}_x\left(\sup_{t\ge 0} \left|e^{-rt}G_i(X_t)\right|\right) < \infty, \quad \text{for } i = 1, 2.$$
(3)

The second condition involves the behaviour at infinity. We require that

$$\lim_{t \to \infty} e^{-rt} G_i(X_t) = 0, \quad i = 1, 2.$$
(4)

Then, for an arbitrary stopping time $\tau \in \mathcal{M}$, we define

$$e^{-r\tau}G_i(X_{\tau})\mathbf{1}_{\{\tau=\infty\}} = 0, \quad i = 1, 2,$$

that is consistent in the sense that

$$\lim_{t \to \infty} \mathbf{E}_x \left(e^{-r(\tau \wedge t)} G_i(X_{\tau \wedge t}) \right) = \mathbf{E}_x \left(e^{-r\tau} G_i(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right), \quad i = 1, 2,$$
(5)

where we pass to the limit by dominated convergence using (3) and (4).

The following simple result presents inequalities useful for solving a Dynkin game (compare with [4, Lemma 5]).

Proposition 1. Consider a pair (σ^*, τ^*) of stopping times for the problem in Definition 1 and a real valued function V(x), defined by

$$V(x) = \mathbf{M}_x(\sigma^*, \tau^*). \tag{MG}$$

Assume that the two following conditions hold

$$V(x) \ge \mathbf{M}_x(\sigma^*, \tau), \quad \text{for all } \tau \in \mathcal{M},$$
 (SPMG)

$$V(x) \le \mathbf{M}_x(\sigma, \tau^*), \quad \text{for all } \sigma \in \mathcal{M}.$$
 (SBMG)

Then, the DG in Definition 1 has value function V(x), and (σ^*, τ^*) is a pair of optimal stopping times for the problem, meaning that (2) holds.

Proof. For an arbitrary pair σ, τ , it holds that

$$\inf_{\sigma} \mathbf{M}_x(\sigma, \tau) \le \mathbf{M}_x(\sigma, \tau) \le \sup_{\tau} \mathbf{M}_x(\sigma, \tau).$$

We take \sup_{τ} in the first inequality, to obtain

$$\sup_{\tau} \inf_{\sigma} \mathbf{M}_x(\sigma, \tau) \le \sup_{\tau} \mathbf{M}_x(\sigma, \tau).$$

If we now take \inf_{σ} above, we obtain

$$\sup_{\tau} \inf_{\sigma} \mathbf{M}_x(\sigma, \tau) \le \inf_{\sigma} \sup_{\tau} \mathbf{M}_x(\sigma, \tau), \tag{6}$$

the first inequality to verify (2). Now, by our hypothesis

$$\mathbf{M}_x(\sigma^*,\tau) \le \mathbf{M}_x(\sigma^*,\tau^*) \le \mathbf{M}_x(\sigma,\tau^*),$$

following

$$\sup_{\tau} \mathbf{M}_x(\sigma^*, \tau) \le \mathbf{M}_x(\sigma^*, \tau^*) \le \inf_{\sigma} \mathbf{M}_x(\sigma, \tau^*),$$

that gives

$$\inf_{\sigma} \sup_{\tau} \mathbf{M}_x(\sigma, \tau) \le \sup_{\tau} \mathbf{M}_x(\sigma^*, \tau) \le \mathbf{M}_x(\sigma^*, \tau^*) = V(x)$$
$$\le \inf_{\sigma} \mathbf{M}_x(\sigma, \tau^*) \le \sup_{\tau} \inf_{\sigma} \mathbf{M}_x(\sigma, \tau),$$

that, in view of (6), concludes the proof.

3 Main result

In this section we present a verification theorem that constructs the solution of a Dynkin game in terms of the supremum and infimum of the process, combined with two (unknown) averaging functions.

Theorem 1. Consider a Lévy process X, a discount rate r > 0, and a two continuous reward functions $G_1 \leq G_2 \colon \mathbb{R} \to \mathbb{R}$. Assume that there exist two points $x_I < x_S$ and two continuous non-decreasing functions: Q_I s.t. $Q_I(x) = 0$ for $x_I \leq x$ and Q_S s.t. $Q_S(x) = 0$ for $x \leq x_S$ (named averaging functions); and such that

$$\mathbf{E}_{x} Q_{I}(I) + \mathbf{E}_{x} Q_{S}(S) = \begin{cases} G_{1}(x), & \text{when } x_{S} \leq x, \\ \\ G_{2}(x), & \text{when } x \leq x_{I}, \end{cases}$$
(7)

where I and S are defined in (1). Define the function

$$V(x) = \mathbf{E}_x Q_I(I) + \mathbf{E}_x Q_S(S), \text{ for all } x \in \mathbb{R}.$$
(8)

Then, if the condition

$$G_1(x) \le V(x) \le G_2(x),\tag{9}$$

holds for all $x \in [x_I, x_S]$, the DG of Definition 1 has value function V(x) in (8), and optimal stopping rules given by

$$\sigma^* = \inf\{t \ge 0 \colon X_t \le x_I\}, \qquad \tau^* = \inf\{t \ge 0 \colon X_t \ge x_S\}.$$
 (10)

Remark 1. Observe that, having into account the asymptotic behavior of a Lévy process (see Thm. VI.12 in [6]) the stopping times in (10) satisfy $\mathbf{P}(\sigma^* \wedge \tau^* < \infty) = 1$. Furthermore, in view of (5) and condition (9), for an arbitrary stopping time in \mathcal{M} , we have

$$\mathbf{E}_x \left(e^{-r\tau} V(X_\tau) \mathbf{1}_{\{\tau=\infty\}} \right) = 0. \tag{11}$$

3.1 Proof

The proof of the main result is carried out in the following two Lemmas.

Lemma 1 (MG). Consider a DG as in Definition 1. Assume that there exist points x_S , x_I and averaging functions Q_S, Q_I that satisfy the hypothesis of Theorem 1. Then, for V defined by (8), the equality (MG) holds.

Proof. Denote $U = \mathbb{R} \setminus (x_I, x_S)$. Define $G: U \to \mathbb{R}$ by

$$G(x) = \begin{cases} G_1(x), & \text{when } x_S \leq x, \\ \\ G_2(x), & \text{when } x \leq x_I. \end{cases}$$

Denote $\rho^* = \sigma^* \wedge \tau^*$. Observe that as X is a Lévy process, $\mathbf{P}_x(\rho^* < \infty) = 1$. As $X_{\rho^*} \in U$ and V = G on U, we have

$$\mathbf{E}_x(e^{-r\rho^*}G(X_{\rho^*})) = \mathbf{E}_x(e^{-r\rho^*}V(X_{\rho^*})).$$
(12)

On the other side, by the monotonicity of the functions Q_S and Q_I ,

$$V(x) = \mathbf{E}_x Q_I(I) + \mathbf{E}_x Q_S(S) = \mathbf{E}_x \left(Q_I \left(\inf_{0 \le t \le e_r} X_t \right) + Q_S \left(\sup_{0 \le t \le e_r} X_t \right) \right)$$
$$= \mathbf{E}_x \left(\inf_{0 \le t \le e_r} Q_I(X_t) + \sup_{0 \le t \le e_r} Q_S(X_t) \right).$$

Observe that if $e_r < \rho^*$ we have $x_I < I \le S < x_S$ (i.e. the process still lives within $[x_I, x_S]$), and then

$$\inf_{0 \le t \le e_r} Q_I\left(X_t\right) = \sup_{0 \le t \le e_r} Q_S\left(X_t\right) = 0,$$

because $Q_I(x) = Q_S(x) = 0$ in $[x_I, x_S]$. Also note, that

$$\inf_{0 \le t < \rho^*} Q_I(X_t) = \sup_{0 \le t < \rho^*} Q_S(X_t) = 0.$$

In conclusion,

$$\mathbf{E}_x Q_I(I) = \mathbf{E}_x \left(\inf_{0 \le t \le e_r} Q_I(X_t) \right) = \mathbf{E}_x \left(\inf_{\rho^* \le t \le e_r} Q_I(X_t) \right) =: q_I(x),$$

$$\mathbf{E}_x Q_S(S) = \mathbf{E}_x \left(\sup_{0 \le t \le e_r} Q_S(X_t) \right) = \mathbf{E}_x \left(\sup_{\rho^* \le t \le e_r} Q_S(X_t) \right) =: q_S(x),$$

and we have

$$V(x) = q_I(x) + q_S(x).$$

Consider now $\tilde{X} = {\tilde{X}_s = X_{\rho^*+s} - X_{\rho^*} : s \ge 0}$ that, by the strong Markov property, is independent of \mathcal{F}_{ρ^*} and has the same distribution as X, and denote by $\tilde{\mathbf{E}}_x$ the expectation w.r.t. \tilde{X} . Based on these considerations, we have

$$q_I(x) = \mathbf{E}_x Q_I(I) = \mathbf{E}_x \left(\sup_{\rho^* \le t \le e_r} Q_I(X_t) \right)$$
$$= \mathbf{E}_x \left(\int_{\rho^*}^{\infty} \sup_{\rho^* \le t \le u} Q_I(X_t) r e^{-ru} du \right)$$
(13)

$$= \mathbf{E}_{x} \left(e^{-r\rho^{*}} \int_{0}^{\infty} \sup_{\rho^{*} \le t \le \rho^{*} + v} Q_{I}(X_{t}) r e^{-rv} dv \right)$$
(14)

$$= \mathbf{E}_{x} \left(e^{-r\rho^{*}} \int_{0}^{\infty} \sup_{\rho^{*} \le t \le \rho^{*} + v} Q_{I} (X_{\rho^{*}} + X_{t} - X_{\rho^{*}}) r e^{-rv} dv \right)$$
(15)

$$= \mathbf{E}_{x} \left(e^{-r\rho^{*}} \int_{0}^{\infty} \sup_{0 \le s \le v} Q_{I} (X_{\rho^{*}} + X_{\rho^{*}+s} - X_{\rho^{*}}) r e^{-rv} dv \right)$$
(16)

$$= \mathbf{E}_{x} \left(e^{-r\rho^{*}} \int_{0}^{\infty} \sup_{0 \le s \le v} Q_{I}(X_{\rho^{*}} + \tilde{X}_{s})re^{-rv}dv \right)$$

$$= \mathbf{E}_{x} \left(e^{-r\rho^{*}} \tilde{\mathbf{E}}_{X_{\rho^{*}}} \left[\int_{0}^{\infty} \sup_{0 \le s \le v} Q_{I}(\tilde{X}_{s})re^{-rv}dv \right] \right)$$

$$= \mathbf{E}_{x} \left(e^{-r\rho^{*}} \tilde{\mathbf{E}}_{X_{\rho^{*}}} \left[\sup_{0 \le s \le e_{r}} Q_{I}(\tilde{X}_{s}) \right] \right) = \mathbf{E}_{x} \left(e^{-r\rho^{*}} q_{I}(X_{\rho^{*}}) \right),$$

where in (13) we take expectation w.r.t. the independent variable e_r , we change variables according to $v = u - \rho^*$ to pass from (13) to (14), and denote $s = t - \rho^*$ to pass from (15) to (16). The same relation holds with q_S and Q_S , i.e.:

$$q_S(x) = \mathbf{E}_x Q_S(S) = \mathbf{E}_x \left(e^{-r\rho^*} q_S(X_{\rho^*}) \right).$$

Summing up these two relations, in view of (12), we obtain

$$V(x) = \mathbf{E}_x \left(e^{-r\sigma^*} V(X_{\rho^*}) \right) = \mathbf{E}_x \left(e^{-r\sigma^*} G(X_{\rho^*}) \right)$$
$$= \mathbf{E}_x \left(e^{-r\tau^*} G_1(X_{\tau^*}) \mathbf{1}_{\{\tau^* \le \sigma^*\}} \right) + \mathbf{E}_x \left(e^{-r\sigma^*} G_2(X_{\sigma^*}) \mathbf{1}_{\{\sigma^* < \tau^*\}} \right),$$

concluding the proof of the Lemma.

Lemma 2 (SPMG-SBMG). Consider the DG of Definition 1. Assume that there exist points x_S , x_I and averaging functions Q_I, Q_S that satisfy the hypothesis of Theorem 1. Then, for V defined by (8),

- (i) the supermartingale inequality (SPMG) holds,
- (ii) the submartingale inequality (SBMG) holds.

Proof. Let us see (i). As $Q_I(x) = 0$ for $x \ge x_S$, in view of the definition (10) of σ^* , for an arbitrary stopping time $\tau \in \mathcal{M}$, we have the following a.s. identity

$$\inf_{0 \le t \le e_r} Q_I(X_t) = \inf_{\tau \land \sigma^* \le t \le e_r} Q_I(X_t) \mathbf{1}_{\{\tau \land \sigma^* \le e_r\}}.$$
(17)

On the other side, as $Q_S(x) \ge 0$, we have

$$\sup_{0 \le t \le e_r} Q_S(X_t) \ge \sup_{\tau \land \sigma^* \le t \le e_r} Q_I(X_t) \mathbf{1}_{\{\tau \land \sigma^* \le e_r\}}.$$
(18)

With these relations in view, we obtain

$$V(x) = \mathbf{E}_{x} Q_{I}(I) + \mathbf{E}_{x} Q_{I}(I)$$

=
$$\mathbf{E}_{x} \left(\inf_{0 \le t \le e_{r}} Q_{I}(X_{t}) + \sup_{0 < t \le e_{r}} Q_{S}(X_{t}) \right)$$
(19)

$$\geq \mathbf{E}_{x} \left[\left(\inf_{\tau \wedge \sigma^{*} \leq t \leq e_{r}} Q_{I}(X_{t}) + \sup_{\tau \wedge \sigma^{*} \leq t \leq e_{r}} Q_{S}(X_{t}) \right) \mathbf{1}_{\{\tau \wedge \sigma^{*} < e_{r}\}} \right]$$
(20)

$$= \mathbf{E}_{x} \left[\int_{\tau \wedge \sigma^{*}}^{\infty} \left(\inf_{\tau \wedge \sigma^{*} \leq t \leq v} Q_{I}(X_{t}) + \sup_{\tau \wedge \sigma^{*} \leq t \leq v} Q_{S}(X_{t}) \right) r e^{-rv} dv \right]$$
(21)

$$= \mathbf{E}_{x} \left[e^{-r(\tau \wedge \sigma^{*})} V(X_{\tau \wedge \sigma^{*}}) \mathbf{1}_{\{\tau < \infty\}} \right] = \mathbf{E}_{x} \left[e^{-r(\tau \wedge \sigma^{*})} V(X_{\tau \wedge \sigma^{*}}) \right],$$
(22)

where we pass from (19) to (20) in view of (17) and (18), from line (20) to (21) integrating with respect to the exponential density of the independent random variable e_r , and the passage from (21) to (22) is the same procedures as in (13) to (16); and, the last equality uses (11). Finally, taking into account our hypothesis (9), and the fact that $V = G_2$ for $x \leq x_I$, we obtain

$$e^{-r(\tau \wedge \sigma^*)}V(X_{\tau \wedge \sigma^*}) \ge e^{-r\tau}G_1(X_{\tau})\mathbf{1}_{\{\tau \le \sigma^*\}} + e^{-r\sigma^*}G_2(X_{\sigma^*})\mathbf{1}_{\{\sigma^* < \tau\}}.$$

Taking expectations, the inequality (SPMG) is proved, concluding the proof of (i). The proof of (ii) is analogous. $\hfill \Box$

Proof of Theorem 1. The proof is an application of Proposition 1, where Lemmas 1 gives (MG), and 2 gives the conditions (SPMG) and (SBMG). \Box

3.2 Smooth pasting

One of the interesting features of Theorem 1 is that, although it is a verification result in the sense that it requires a conjecture to identify a solution (a common characteristic in many optimal stopping problems), it does not rely on the smooth pasting condition. This condition is typically used in stopping problems within the diffusion framework [31]. This distinction is significant because, as illustrated in our examples and well documented in the context of Lévy processes, the smooth pasting condition does not consistently hold (see for instance the discussion in [1] and the examples in [24]).

Due to its similarity, we adopt the approach of [27], where the following result is obtained.

Proposition 2 (Theorem 3.1 in [27]). Consider a function of the form

$$W(x) = \mathbf{E}_x Q(S). \tag{23}$$

Assume that the three following conditions hold:

(i) Q(x) = 0, for $x \le x_0$ and Q(x) is non-decreasing and in class $C^2[x_0, \infty)$,

(ii) there exist A > 0 and $\alpha > 0$ s.t. $|Q''(x)| \le Ae^{\alpha x}$,

(iii) for $\alpha > 0$ above, it holds $\mathbf{E} e^{\alpha X_1} < e^r$.

Then

$$W'(x_0+) - W'(x_0-) = Q'(x_0+) \mathbf{P}(S=0),$$
(24)

$$W'(x+) - W'(x-) = 0, \text{ for } x > x_0.$$
(25)

To apply this result in our situation, observe that the function V(x) is the sum of two functions of the form (23), because $\mathbf{E}_x Q_I(I) = \mathbf{E}_{-x} Q_I(-\hat{S})$ where \hat{S} is the supremum of the dual process $\hat{X} = -X$. In conclusion, if the averaging functions Q_I , Q_S meet conditions (i) to (iii), and the payoffs G_1, G_2 are differentiable, we will have smooth pasting at x_I if and only if the condition $\mathbf{P}(I=0) = 0$ holds, and smooth pasting at x_S if and only if the condition $\mathbf{P}(S=0) = 0$ holds. This will be discussed more in detail in the examples of next section.

4 Applications

In this section, we consider three examples that are similar from an analytic point of view: Brownian motion with drift, Cramér-Lundberg process with exponential claims and Compound Poisson process with double sided exponential jumps. We assume that in all case, the infimum I and the supremum S in (1) have characteristic functions given respectively by

$$\mathbf{E} e^{zI} = \pi_I + (1 - \pi_I) \frac{r_I}{r_I + z},$$
(26)

$$\mathbf{E} e^{zS} = \pi_S + (1 - \pi_S) \frac{r_S}{r_S - z},$$
(27)

where $0 \le \pi_I, \pi_S \le 1$ are probabilities, and $-r_I < 0 < r_S$ are the two roots of the equation

$$\Psi(z) = r.$$

It is important to note that the Brownian motion with drift is obtained when $\pi_I = \pi_S = 0$; the Cramér-Lundberg process is obtained when $\pi_I > 0$ and $\pi_S = 0$, and the Compound Poisson process is obtained when $\pi_I > 0$ and $\pi_S > 0$. Moreover, other processes with rational transform distributed jumps can be hopefully treated with a similar methodology, at the cost or more involved computations (see [25]).

For simplicity to obtain tractable solutions, we assume

$$G_1(x) = x - \delta, \quad G_2(x) = x + \delta, \tag{28}$$

for some $\delta > 0$. In order to check conditions (3) and (4) we provide the following result.

Proposition 3. Assume that there exist positive constants A, B, α such that

$$|G(x)| \le A + B\cosh(\alpha x),\tag{29}$$

and

$$\mathbf{E} e^{\alpha X_1} < e^r, \qquad \mathbf{E} e^{-\alpha X_1} < e^r. \tag{30}$$

Then, conditions (3) and (4) hold.

Proof. In view of (30), the characteristic exponent $\Psi(z)$ has an analytic extension to the strip $\Re(z) \in [-\alpha, \alpha]$ in the complex plane (see [33, Thm. 25.17]). From this follows that $\mathbf{E} |X_1| < \infty$ and that $\mathbf{E} X_1 = \Psi'(0)$. Consider the Lévy process $\{\bar{X}_t = \alpha X_t - rt : t \ge 0\}$. Its characteristic exponent $\bar{\Psi}(z)$ satisfies $\bar{\Psi}(1) = \Psi(\alpha) - r < 0$ (the first statement in (30)). From this follows $\mathbf{E} \bar{X}_1 < 0$, meaning that $\lim_{t\to\infty} (\alpha X_t - rt) = -\infty$. A similar reasoning based in the second statement in (30) implies $\lim_{t\to\infty} (-\alpha X_t - rt) = -\infty$. In view of (29), condition (4) holds.

In order to see now condition (3) we use [26, Lemma 1] that can be stated for the Lévy process $\{\alpha X_t - rt : t \ge 0\}$ without discount, as the equivalence of the following two statements:

$$\mathbf{E}(e^{\sup_{t\geq 0}(\alpha X_t - rt)}) < \infty,\tag{31}$$

$$\mathbf{E}(e^{\alpha X_1 - r}) < 1. \tag{32}$$

Furthermore

$$\sup_{t \ge 0} e^{-rt} |G(X_t)| \le A + \frac{B}{2} e^{\sup_{t \ge 0} (\alpha X_t - rt)} + \frac{B}{2} e^{\sup_{t \ge 0} (-\alpha X_t - rt)}.$$
(33)

As condition (32) above is equivalent to the first statement in (30) (the same for $-\alpha$) in view of condition (31) and (33), the statement (3) follows.

Observe now that the three process considered above (Brownian motion with drift, Cramér-Lundberg process with exponential claims and Compound Poisson process with double sided exponential jumps) have a characteristic exponent that is analytic in a certain strip $-\alpha \leq \Re(z) \leq \alpha$, and, for that $\alpha > 0$ there exist positive constants A, B such that $|G_i(x)| \leq A + B \cosh(\alpha x)$, (i = 1, 2). This allows to apply Theorem 1 in the considered examples.

4.1 General resolution

We first develop some necessary procedures to find the functions Q_I and Q_S and the critical thresholds x_I and x_S under this solely assumptions, in order to meet the conditions of Theorem 1.

In view of (26) and (27), the random variables -I and S have (possibly) defective exponential distributions with respective parameters r_I and r_S , and atoms of size π_I , π_S . Then, with a slight abuse of notation, the respective densities can be denoted by

$$f_I(x) = \pi_I \delta_0(x) + (1 - \pi_I) r_I e^{r_I x}, \quad x \le 0,$$
(34)

$$f_S(x) = \pi_S \delta_0(x) + (1 - \pi_S) r_S e^{-r_S x}, \quad x \ge 0,$$
(35)

where $\delta_0(x)$ denotes the Dirac mass measure at x = 0.

In order to find functions $Q_I(x)$ and $Q_S(x)$ s.t. (7) holds, we can compute $\mathbf{E}_x Q_I(I)$ for $x \leq x_I$ and $\mathbf{E}_x Q_S(S)$ for $x \geq x_S$ (using (34) and (35))

$$q_{I}(x) = \mathbf{E}_{x} Q_{I}(I) = \int_{(-\infty,0]} Q_{I}(x+y) f_{I}(y) dy$$

=
$$\int_{(-\infty,0]} (\pi_{I} \delta_{0}(y) + (1-\pi_{I}) r_{I} e^{r_{I}y}) Q_{I}(x+y) dy$$

=
$$\int_{-\infty}^{x \wedge x_{I}} \left(\pi_{I} \delta_{0}(z-x) + (1-\pi_{I}) r_{I} e^{r_{I}(z-x)} \right) Q_{I}(z) dz$$

=
$$\pi_{I} Q_{I}(x) + (1-\pi_{I}) e^{-r_{I}x} \int_{-\infty}^{x \wedge x_{I}} r_{I} e^{r_{I}z} Q_{I}(z) dz.$$
(36)

Analogously

$$q_S(x) = \mathbf{E}_x Q_S(S) = \pi_S Q_S(x) + (1 - \pi_S) e^{r_S x} \int_{x \lor x_S}^{+\infty} r_S e^{-r_S z} Q_S(z) dz.$$
(37)

For convenience, we introduce the notation

$$A_I = A_I(x_I, x_S) = (1 - \pi_I)e^{-r_I x_S} \int_{-\infty}^{x_I} Q_I(z)r_I e^{r_I z} dz,$$
(38)

$$A_S = A_S(x_I, x_S) = (1 - \pi_S)e^{r_S x_I} \int_{x_S}^{+\infty} Q_S(z)r_S e^{-r_S z} dz.$$
 (39)

Moreover, we denote the constants (that depend only on the parameters of the problem)

$$E_I = -\mathbf{E}I = \frac{1 - \pi_I}{r_I} > 0, \qquad E_S = \mathbf{E}S = \frac{1 - \pi_S}{r_S} > 0, \qquad (40)$$

$$F_I = \frac{1}{\mathbf{E} e^{-r_I S}} = \frac{r_I + r_S}{r_I + \pi_I r_S} > 1, \qquad F_S = \frac{1}{\mathbf{E} e^{r_S I}} = \frac{r_I + r_S}{\pi_S r_I + r_S} > 1, \qquad (41)$$

$$G_I = F_I - 1 = \frac{(1 - \pi_I)r_S}{r_I + \pi_I r_S} > 0, \qquad \qquad G_S = F_S - 1 = \frac{(1 - \pi_S)r_I}{\pi_S r_I + r_S} > 0.$$
(42)

For the solution, we impose the condition (7) with the payoff functions of the problem. Therefore

$$\mathbf{E}_{x} Q_{I}(I) + \mathbf{E}_{x} Q_{S}(S) = \begin{cases} x + \delta, & \text{when } x \le x_{I}, \\ x - \delta, & \text{when } x \ge x_{S}. \end{cases}$$
(43)

We first impose the continuity conditions at the optimal thresholds x_I and x_S . We obtain the system

$$\begin{cases} e^{r_I u} A_I + A_S = x_I + \delta, \\ A_I + e^{r_S u} A_S = x_S - \delta, \end{cases}$$

$$\tag{44}$$

where we denoted

$$u = x_S - x_I$$

As we have jumps, the boundary conditions necessarily comprises the whole half lines $x \leq x_I$ and $x \geq x_S$, instead of only the boundary point. This corresponds to the fact that the process can jump outside the interval $[x_I, x_S]$. Accordingly we impose equation (43) for $x \leq x_I$, that states

$$\pi_I Q_I(x) + (1 - \pi_I) e^{-r_I x} \int_{-\infty}^x r_I e^{r_I z} Q_I(z) dz + e^{r_S(x - x_I)} A_S = x + \delta.$$
(45)

The relationship (45) is an ordinary differential equation to find Q_I . If we multiply by $e^{r_I x}$ both sides, assume enough regularity, and take derivatives, we have

$$\pi_I \left(r_I e^{r_I x} Q_I(x) + e^{r_I x} Q'_I(x) \right) + (1 - \pi_I) r_I e^{r_I x} Q_I(x) + (r_I + r_S) e^{r_I x} e^{r_S(x - x_I)} A_S = r_I(x + \delta) e^{r_I x} + e^{r_I x}.$$

Therefore, the function $Q_S(x)$ satisfies the differential equation

$$\frac{\pi_I}{r_I}Q_I'(x) + Q_I(x) = x + \delta + \frac{1}{r_I} - \left(1 + \frac{r_S}{r_I}\right)A_S e^{r_S(x - x_I)}.$$

In the case that $\pi_I = 0$ we directly obtain the candidate Q_I to use in the verification Theorem 1. When $\pi_I > 0$, we assume that $Q_I(x) \sim x$ when $x \to -\infty$, in order to find a candidate. Solving the differential equation according to the condition, with the introduced above notation, we find the function

$$Q_I(x) = x + \delta + E_I - F_I A_S e^{r_S(x - x_I)}, \quad x \le x_I.$$

$$\tag{46}$$

Similarly, for $x \ge x_S$, we obtain the differential equation

$$\pi_S Q_S(x) + (1 - \pi_S) e^{r_S x} \int_x^{+\infty} r_S e^{-r_S z} Q_S(z) dz + e^{-r_I (x - x_S)} A_I = x - \delta_S$$

with solution

$$Q_S(x) = x - \delta - E_S - F_S A_I e^{-r_I (x - x_I)} \quad x \ge x_S.$$
(47)

Using now the functions (46) and (47), we impose conditions $Q_I(x_I) = 0$ and $Q_S(x_S) = 0$ required in Theorem 1, to get

$$\begin{cases} x_I + \delta + E_I - F_I A_S = 0, \\ x_S - \delta - E_S - F_S A_I = 0. \end{cases}$$
(48)

To solve the system above, we substitute $x_I - \delta$ and $x_S + \delta$ from (44) obtaining the linear system

$$\begin{cases} e^{r_I u} A_I - G_I A_S = -E_I, \\ -G_S A_I + e^{r_S u} A_S = E_S, \end{cases}$$

that has solutions

$$A_{I} = \frac{-E_{I}e^{r_{S}u} + E_{S}G_{I}}{e^{(r_{I} + r_{S})u} - G_{I}G_{S}}, \quad A_{S} = \frac{E_{S}e^{r_{I}u} - E_{I}G_{S}}{e^{(r_{I} + r_{S})u} - G_{I}G_{S}}.$$
(49)

We replace A_I and A_S in the system (48), we subtract both equations and we obtain

$$u - 2\delta - E_I - E_S = \frac{-E_I F_S e^{r_S u} - E_S F_I e^{r_I u} + 2E_S F_S G_I}{e^{(r_I + r_S)u} - G_I G_S}.$$
(50)

where we used $E_S F_S G_I = E_I F_I G_S$. The left-hand side in (50) is a linear increasing function in u, with value at u = 0 strictly less than $-(E_I + E_S)$, as $\delta > 0$. The numerator of the right-hand side (using the constants (40), (41) and (42)) results

$$-E_I F_S e^{r_S u} - F_I E_S e^{r_I u} + 2E_S F_S G_I \le 0,$$

which is a decreasing function in u and the denominator of the right-hand side is an increasing function in u. These properties show that the right-hand side of (50) is a negative increasing function to zero and when u = 0 satisfies that it is equals $-(E_I + E_S)$. Therefore, we conclude that (50) has a unique positive solution for all $\delta > 0$.

Finally, with this value of the root u, from (48) and (49) we find the optimal strategies

$$x_{I} = -\delta - E_{I} + F_{I} \frac{E_{S} e^{r_{I} u} - E_{I} G_{S}}{e^{(r_{I} + r_{S})u} - G_{I} G_{S}}.$$
(51)

$$x_{S} = \delta + E_{S} + F_{S} \frac{-E_{I} e^{r_{S} u} + E_{S} G_{I}}{e^{(r_{I} + r_{S})u} - G_{I} G_{S}},$$
(52)

In order to compute V(x) for $x \in [x_I, x_S]$, we refer to (8). Observe that $Q_I(x) = Q_S(x) = 0$ within this interval, so, taking into account (36)-(37) and (38)-(39), we obtain

$$V(x) = \begin{cases} x + \delta, & \text{if } x \le x_I, \\ A_I e^{-r_I (x - x_S)} + A_S e^{r_S (x - x_I)}, & \text{if } x_I \le x \le x_S, \\ x - \delta, & \text{if } x \ge x_S. \end{cases}$$
(53)

In conclusion, we obtained the following result.

Theorem 2. Under the assumptions (34) and (35) for the densities of the respective infimum and supremum of the process X, and for the linear payoffs in (28), the DG of Definition 1 has optimal thresholds given by (51) and (52), where u is the unique solution of (50), and the value function is given in (53).

4.2 Brownian motion with drift

We consider that $X = \{X_t : t \ge 0\}$ is a Brownian motion with drift,

$$X_t = ct + \sigma W_t, \quad t \ge 0, \tag{54}$$

where $\{W_t: t \ge 0\}$ is a standard Brownian motion. The characteristic exponent of the process is

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + cz,$$

therefore, the denominator in the Wiener-Hopf Factorization is zero when $\Psi(z) = r$ that has real roots $-r_I < 0 < r_S$, with

$$r_I = \frac{\sqrt{c^2 + 2r\sigma^2} + c}{\sigma^2}, \quad r_S = \frac{\sqrt{c^2 + 2r\sigma^2} - c}{\sigma^2}.$$
 (55)

As $r_I r_S = 2r/\sigma^2$, Wiener-Hopf factorization reads

$$\frac{r}{r-\Psi(z)} = \frac{2r}{\sigma^2(r_I + z)(r_S - z)} = \frac{r_I}{r_I + z} \frac{r_S}{r_S - z},$$

giving densities

$$f_I(x) = r_I e^{r_I x}, \quad x \le 0,$$

$$f_S(x) = r_S e^{-r_S x}, \quad x \ge 0.$$

i.e. the formulas in (34) and (35) with $\pi_I = \pi_S = 0$. In conclusion, applying the results of subsection 4.1, we obtain the following results.

Corollary 1. The DG of Definition 1 for the Brownian motion with drift in (54) and linear payoff functions in (28) and optimal stopping rules given by hitting times of the levels

$$x_{I} = -\delta - \frac{1}{r_{I}} - \frac{\sqrt{c^{2} + 2r\sigma^{2}}}{r} \frac{e^{r_{I}u} - 1}{e^{(r_{I} + r_{S})u} - 1}$$
$$x_{S} = \delta + \frac{1}{r_{S}} + \frac{\sqrt{c^{2} + 2r\sigma^{2}}}{r} \frac{e^{r_{S}u} - 1}{e^{(r_{I} + r_{S})u} - 1},$$

where r_I, r_S are the roots in (55), and u is the unique positive solution of the equation

$$u - 2\delta = \frac{\sqrt{c^2 + 2r\sigma^2}}{r} \frac{(e^{r_S u} - 1)(e^{r_I u} - 1)}{e^{(r_I + r_S)u} - 1},$$

and value function (53) with

$$A_{I} = -\frac{1}{r_{I}} \frac{e^{r_{S}u} - 1}{e^{(r_{I} + r_{S})u} - 1}, \quad A_{S} = \frac{1}{r_{S}} \frac{e^{r_{I}u} - 1}{e^{(r_{I} + r_{S})u} - 1}.$$

Moreover, as a consequence of the discussion in Subsection 3.2, in view of the fact that the infimum and supremum of the process have no atoms, the value function is everywhere differentiable, i.e. we have smooth pasting.

Remark 2. The symmetric case is obtained when c = 0, with roots $-r_I = r_S = \sqrt{2r}/\sigma$. Furthermore, in this symmetric case, $A_I = -A_S$, $-x_I = x_S$, and the function V(x) in (53) presents central symmetry around the origin.

Numerical examples

To illustrate our results in this case we consider two examples. In the first one we choose the standard Brownian motion, with c = 0 and $\sigma = 1$, and the second example is the Brownian motion with drift the parameters c = 1 and $\sigma = 1$. We consider r = 1 and $\delta = 1$. The results obtained are $x_I = -x_S = -1.6955$ and $x_I = -1.2426$ and $x_S = 2.3659$ respectively. In Figure 1, we show the Lévy-Khintchine characteristic exponent and in Figure 2 the corresponding value functions.



Figure 1: $\Psi(z)$ (solid) in Brownian Motion and r = 1 (dashed). Left: Standard Brownian Motion Right: Brownian Motion with drift.



Figure 2: Value functions V(x) (solid), payoff functions $G_1(x)$, $G_2(x)$ (dashed). Left: Standard Brownian Motion Right: Brownian Motion with drift.

4.3 Cramér-Lundberg process

We consider the Cramér-Lundberg process $X = \{X_t : t \ge 0\}$ with exponential jumps, given by

$$X_t = x + ct - \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)},$$
(56)

where $N^{(1)} = \{N_t^{(1)} : t \ge 0\}$ is a Poisson process with intensity λ_1 and $Y^{(1)} = \{Y_i^{(1)} : i \ge 1\}$ is a sequence of independent identically distributed exponential random variables with parameter α_1 . The two processes $N^{(1)}$, and $Y^{(1)}$ are independent.

The characteristic exponent of the process is

$$\Psi(z) = cz - \lambda_1 \frac{z}{\alpha_1 + z},$$

therefore, the denominator in the Wiener-Hopf Factorization is zero when $\Psi(z) = r$ that has two

roots $-r_I < 0 < r_S$, with

$$r_{I} = \frac{\sqrt{(c\alpha_{1} - \lambda_{1} - r)^{2} + 4cr\alpha_{1}} + c\alpha_{1} - \lambda_{1} - r}{2c},$$

$$r_{S} = \frac{\sqrt{(c\alpha_{1} - \lambda_{1} - r)^{2} + 4cr\alpha_{1}} - (c\alpha_{1} - \lambda_{1} - r)}{2c}.$$

The Wiener-Hopf factorization to determine the law of S and I is

$$\frac{r}{r-\Psi(z)} = \left(\pi_I + (1-\pi_I)\frac{r_I}{r_I+z}\right) \left(\frac{r_S}{r_S-z}\right),\,$$

where $\pi_I = r/(r_S c)$, and we used that $r_I r_S = r \alpha_1/c$. We observe that the random variables S has exponential distributions with parameter r_S and -I has defective exponential distributions with parameter r_I and an atom at zero of size π_I . The respective densities are

$$f_I(x) = \pi_I \delta_0(x) + (1 - \pi_I) r_I e^{r_I x}, \quad x \le 0,$$

$$f_S(x) = r_S e^{-r_S x}, \quad x \ge 0,$$

where $\delta_0(x)dx$ denotes the Dirac mass measure at x = 0. Consequently, we obtain (34) and (35) with $\pi_I > 0$ and $\pi_S = 0$. We obtain then the following result.

Corollary 2. The DG of Definition 1 for the Cramér Lundberg process in (56) and linear payoff functions in (28) has and optimal stopping rules given by hitting times of the levels given by (51) and (52) with $\pi_S = 0$, where u is the unique solution of (50), and the value function is given in (53).

Moreover, as a consequence of the discussion in Subsection (3.2), in view of the fact that only the infimum has an atom, we have smooth pasting at x_S but the value function at x_I is not differentiable. The jump of the derivative can be computed by (24).

Numerical examples

To illustrate our results in this process we consider the Cramér-Lundberg with c = 1, $\lambda = 1$ and $\alpha_1 = 1$. We consider r = 1 and $\delta = 1$. The critical thresholds are $x_I = -1.6127$ and $x_S = 1.4931$. In Figure 3, we show the Lévy-Khintchine formula (left) and the corresponding value functions (right).



Figure 3: Cramér -Lundberg process. Left: $\Psi(z)$ (solid) and r = 1 (dashed). Right: Value function V(x).

4.4 Compound Poisson process

We consider the compound Poisson process $X = \{X_t : t \ge 0\}$ with double-sided exponential jumps, given by

$$X_t = x - \sum_{i=1}^{N_t^{(1)}} Y_i^{(1)} + \sum_{i=1}^{N_t^{(2)}} Y_i^{(2)}$$
(57)

where $N^{(1)} = \{N_t^{(1)}: t \ge 0\}$ and $N^{(2)} = \{N_t^{(2)}: t \ge 0\}$ are two Poisson process with respective positive intensities λ_1 , λ_2 , the two sequences $Y^{(1)} = \{Y_i^{(1)}: i \ge 1\}$ and $Y^{(2)} = \{Y_i^{(2)}: i \ge 1\}$ are of independent identically distributed exponential random variables with respective positive parameters α_1 , α_2 . The four processes $N^{(1)}$, $N^{(2)}$, $Y^{(1)}$, $Y^{(2)}$ are independent.

The characteristic exponent of the process X is given by

$$\Psi(z) = -\lambda_1 \frac{z}{\alpha_1 + z} + \lambda_2 \frac{z}{\alpha_2 - z},$$

therefore the denominator in the Wiener-Hopf Factorization $\Psi(z) = r$ that has two roots $-r_I$, r_S are the solutions of the equation

$$(r + \lambda_1 + \lambda_2)z^2 + (\alpha_1(\lambda_2 + r) - \alpha_2(\lambda_1 + r))z - r\alpha_1\alpha_2 = 0,$$

that satisfy

$$-\alpha_1 < -r_I < 0 < r_S < \alpha_2.$$

The Wiener-Hopf factorization to determine the law of S and I is

$$\frac{r}{r-\psi(z)} = \frac{r(z-\alpha_2)(z-\alpha_1)}{(r+\lambda+\mu)(z-r_S)(z-r_I)} = \frac{r_I r_S(\alpha_1+z)(\alpha_2-z)}{\alpha_1 \alpha_2 (r_I+z)(r_S-z)} \\ = \left(\frac{r_I}{\alpha_1} + \frac{\alpha_1 - r_I}{\alpha_1} \frac{r_I}{r_I+z}\right) \left(\frac{r_S}{\alpha_2} + \frac{\alpha_2 - r_S}{\alpha_2} \frac{r_S}{r_S-z}\right).$$

In conclusion, due to the uniqueness of the factorization (see Thm. 5(ii) Ch. VI of [6]), we obtain that the random variables -I and S have (strictly) defective exponential distributions with parameters r_I and r_S , and atoms at zero of respective size $\pi_I = r_I/\alpha_1$ and $\pi_S = r_S/\alpha_2$, corresponding to (26) and (27), giving then the following result.

Corollary 3. The DG of Definition 1 for the compound Poisson process with double sided exponential jumps in (57) and linear payoff functions in (28) has optimal stopping rules given by hitting times of the levels given by (51) and (52) with $\pi_I = r_I/\alpha_1$ and $\pi_S = r_S/\alpha_2$, where u is the unique solution of (50), and the value function is given in (53).

Moreover, as a consequence of the discussion in Subsection (3.2), in view of the fact that both the infimum and the supremum have atoms, the value function at x_S and x_I is not differentiable. The jump of the derivative at these points can be computed by (24).

Numerical examples

To illustrate our results we consider two examples. In the first one we choose a symmetric case with $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (1, 1, 1, 1)$ and in the second example an asymmetric case with $(\alpha_1, \lambda_1, \alpha_2, \lambda_2) = (1, 3, 3, 1)$. We consider r = 1 and $\delta = 1$. The results obtained are $x_I = -1.5901$, $x_S = 1.5901$ and $x_I = -3.7750$, $x_S = 0.0834$ respectively. In Figure 4, we show the Lévy-Khintchine formula and in Figure 5 the corresponding value functions V(x). In both cases we observe that in x_I and x_S the value function is not differentiable.



Figure 4: Compound Poisson process. $\Psi(z)$ (solid) and r = 1 (dashed). Left: symmetric case. Right: asymmetric case.



Figure 5: Compound Poisson process. Value function V(x) (solid) and $G_1(x)$, $G_2(x)$ (dashed). Left: symmetric case. Right: asymmetric case.

5 Conclusions

In the present paper we present a verification theorem to solve a Dynkin game driven by a Lévy processes. The theorem requires finding two averaging functions, that give an explicit form of the value function in terms of the infimum and the supremum of the process. The optimal stopping times result to be the entry times of the support sets of these averaging functions. The main challenge in this problem is managing the characteristic overshoot caused by the jumps of the Lévy processes, rendering traditional solution techniques (as the smooth pasting principle) ineffective. Notably, in some instances, when we can explicitly compute Wiener–Hopf factors, we are able to identify the averaging functions and solve the problem completely. This is the case in three situations: Brownian motion with drift, the Cramér-Lundberg process and the compound Poisson process. In these examples we discuss the smooth pasting property, that does not always hold.

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