LOCAL PRODUCT STRUCTURE FOR EXPANSIVE HOMEOMORPHISMS

ALFONSO ARTIGUE, JOAQUIN BRUM, AND RAFAEL POTRIE

Abstract. Let $f\colon M \to M$ be an expansive homeomorphism with dense topologically hyperbolic periodic points, M a closed manifold. We prove that there is a local product structure in an open and dense subset of M. Moreover, if some topologically hyperbolic periodic point has codimension one, then this local product structure is uniform. In particular, we conclude that the homeomorphism is conjugated to a linear Anosov diffeomorphism of a torus.

1. Introduction

Let M be a compact connected boundaryless manifold of dimension n and $f : M \to M$ an expansive homeomorphism, that is, there exists $\alpha > 0$ such that every two points have iterates which are separated at least α from each other (the existence of α is independent of the metric, furthermore, the notion can be defined independently of the metric).

A paradigm of expansive homeomorphisms are Anosov diffeomorphisms. Other class of expansive homeomorphisms are pseudoAnosov maps in surfaces of genus $g > 2$. They satisfy that $\Omega(f) = M$ and they have dense topologically hyperbolic periodic points. In surfaces, pseudoAnosov maps and linear Anosov homeomorphisms (that is, conjugated to a linear Anosov diffeomorphism) describe completely expansive dynamics as was proved in [L2],[H2] obtaining a global classification of expansive homeomorphisms. For this classification, the key step is to prove that in a reduced neighborhood of every point there is a local product structure. To do this, in [L2] it is proved that every point in an expansive homeomorphism has a uniformly big connected stable and unstable set. On surfaces, in some way this is enough to find the local product structure since proving that the connected sets intersect is enough (using Invariance of Domain Theorem, see [Sp]) to find local product structure (these connected sets contain arcs and so a map from $[0, 1]^2$ to a neighborhood of the point can be constructed). In higher dimensions, the existence of connected stable and unstable sets is not enough to find a local product structure, as shown in the example from [FR].

A surprising result is the one of [V2], since it proves that in dimension 3 expansive homeomorphisms whose topologically hyperbolic periodic points are dense, are conjugated to linear Anosov diffeomorphisms in the torus $\mathbb{T}^3$. For doing this it is also very important to find a local product structure in an open and dense subset of the manifold (see [V1]). Again, the technique is to obtain intersections between stable and unstable sets of topologically hyperbolic periodic points which are near and use Invariance of Domain Theorem. This is not completely direct since, a priori, the size of the stable and unstable sets of the periodic points is not controlled, and must study separation properties of these sets to ensure the intersection. The hypothesis of having dense topologically hyperbolic periodic points was weakened in [V4] changing it for having $\Omega(f) = M$

(a necessary condition as can be seen with the example in [FR]) and some smooth hypothesis (f must be a $C^{1+\theta}$ diffeomorphism) to use Pesin theory.

In this work we obtain local product structure in an open dense subset of M when topologically hyperbolic periodic points are dense in M; in fact, we obtain local product structure in neighborhoods of every periodic point. When the codimension of topologically hyperbolic periodic points is arbitrary, this result is optimal, since in the case of a product of two pseudoAnosov maps the local product structure can not be defined in all the manifold.

The somewhat strange aspect of the result from [V2] is that it proves that in dimension 3 no singularities can appear, not as in the surface case where pseudoAnosov maps are expansive with dense topologically hyperbolic periodic points. However, this result has a nice counterpart in the theory of Anosov diffeomorphisms where it is known that codimension one Anosov diffeomorphisms can only exist in torus and be conjugated to a linear one (see [F1],[N]).

Maybe this connection is not a priori obvious, but we give in this work more evidence of it, proving that if the topologically hyperbolic periodic points are dense in M (with dimension higher than 2) and one of them has codimension one, then, the homeomorphism is conjugated to a linear Anosov diffeomorphism of $\mathbb{T}^n$. The reason why this does not work in dimension 2 is that we can disconnect an arc by removing from it one point and not a disc of dimension > 1. The proof in this case is based on proving first that singularities are finite, and then discarding their existence.

1.1. **Definitions and presentation of results.**

In this section we define the concepts that we use in the course of this paper and give precise statements of the results in it.

Definition 1.1. *We say an homeomorphism $f\colon M \to M$ is* expansive *if $\alpha > 0$ exists satisfying that if $x, y \in M$ are different points, then, there exists $n \in \mathbb{Z}$ such that $dist(f^n(x), f^n(y)) > \alpha$.*

Definition 1.2. *We say that a periodic point $p \in M$ of period l is* topologically hyperbolic *($p \in Per_H$) if f^l is locally conjugated to the linear map $L\colon \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^r \times \mathbb{R}^{n-r}$ given by $L(x, y) = (x/2, 2y)$. In this case we say that $p \in Per^r_H \subset Per_H$, we say that r is the index of p.*

In our case f is expansive, so, due to results in [L1] (Lemma 2.7) it is true that $Per^0_H = Per^n_H = \emptyset$ since no stable points exist.

We denote as $H_k(A)$ ($H^k_c(A)$) the k dimensional reduced homology (cohomology with compact support) of A with coefficients in $\mathbb{R}$. As usual, we define the stable and unstable sets of a point $x \in M$ as $W^s(x) = \{y \in M : \text{dist}(f^n(x), f^n(y)) \to 0, n \to +\infty\}$ and $W^u(x) = \{y \in M : \text{dist}(f^n(x), f^n(y)) \to 0, n \to -\infty\}$. The local stable and unstable sets (ε-local) are defined as follows $W^s_\varepsilon(x) = \{y \in M : \text{dist}(f^n(x), f^n(y)) \leq \varepsilon, \forall n \geq 0\}$ and $W^u_\varepsilon(x) = \{y \in M : \text{dist}(f^n(x), f^n(y)) \leq \varepsilon, \forall n \leq 0\}$. We denote as $cc_p(X)$ the connected component of $X \subset M$ containing p.

We prove a separation property verified by the stable and unstable set of a point $p \in Per^r_H$. The proof of this Proposition follows the ideas in [V1],[V2] and it is developed in section 2. The property is the following.

Proposition 1.1. *Let $f : M \to M$ be an expansive homeomorphism. Then, there exists $\varepsilon > 0$ such that for all $x \in M$, $p \in Per^k_H \cap B_\varepsilon(x)$ and $V \subset B_\varepsilon(x)$ homeomorphic to $\mathbb{R}^n$ and containing p, we have $H_{n-k-1}(V \setminus S_p) \cong \mathbb{R}$ with $S_p = cc_p(V \cap W^s(p))$.*

An analogous result is verified for the unstable set.

Remark 1.1. If $f : M \to M$ is an expansive homeomorphism and $z \in M$ then, for all $\varepsilon > 0$ exists $\delta > 0$ such that if $S_z = cc_z(W^s(z) \cap B_\delta(z))$ then $S_z \subset W^s_\varepsilon(z)$. See [L2].

Definition 1.3. *We say that $p \in M$ admits a* local product structure *if there exists a map $h\colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to M$ which is a homeomorphism over its image ($p \in Im(h)$) and if there exists $\varepsilon > 0$ such that for all $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ it is verified that $h(\{x\} \times \mathbb{R}^{n-k}) = W^s_\varepsilon(h(x, y)) \cap Im(h)$ and $h(\mathbb{R}^k \times \{y\}) = W^u_\varepsilon(h(x, y)) \cap Im(h)$. We say that the local product structure is a* uniform local product structure *if in addition to the previous conditions, there exists $r > 0$ such that for all $x \in M$ the points in $B_r(x)$ admit a local product structure.*

We remark that the points admitting a local product structure are an open set. We call the points which do not admit a local product structure singularities.

Theorem 1.1. *Let $f\colon M \to M$ be an expansive homeomorphism such that $\overline{Per_H} = M$. Then, every point in Per_H admits a local product structure. In particular, the set of points with a local product structure is open and dense in M.*

Once this is obtained, in [V2] singularities are studied, discarding their existence by studying the way in which the product structure is glued together in the singularity and proving that this can not happen by discarding the possible dimensions in which that gluing may happen one by one . As was already explained, with the product between the Anosov and the pseudo-Ansov we see that this can not be done in dimension larger than 3, unless we add the hypothesis of having $Per^{n-1}_H = Per_H$. This will be studied in section 4.3.

It is worth observing that the fact of having a local product structure in an open and dense subset does not imply, a priori, that the index of the topologically hyperbolic periodic points should be constant in all the manifold. We shall prove this is true, under the hypothesis of Theorem 1.1, for dimensions 3 and 4. For doing that, in section 4.1 several properties of the points in $\overline{Per^{n-1}_H}$ are studied. The following sharper result is obtained.

Theorem 1.2. *Let $f : M \to M$ be an expansive homeomorphism verifying $\overline{Per_H} = M$. Then, $Per^{n-1}_H = Per_H$ or $Per^{n-1}_H = \emptyset$. Analogously for Per^1_H.*

Corollary 1.1. *Let $f : M \to M$ be an expansive homeomorphism of a manifold of dimension 3 or 4 with $\overline{Per_H(f)} = M$. Then, every topologically hyperbolic point has the same index.*

Proof. In dimension 3 we have $Per_H = Per^1_H \cup Per^2_H$ (see [L1], Lemma 2.7, no stable points can exist); the Theorem 1.2 concludes the proof. In dimension 4, we have $Per_H = Per^1_H \cup Per^2_H \cup Per^3_H$ and since $Per^1_H \cup Per^3_H = \emptyset$ implies $Per_H = Per^2_H$ the proof finishes by using the Theorem 1.2. □

Finally, in section 4.3 we study the singularities in the case of having one topologically hyperbolic point of index $n-1$ (or 1), discarding their existence and concluding that there is a uniform local product structure in all the manifold.

Definition 1.4. *Let $f: M \to M$ be a homeomorphism, we say it verifies the* pseudo orbit tracing property *if for all $K > 0$ exists $\alpha > 0$ such that if $\{x_n\}_{n\in\mathbb{Z}}$ verifies $dist(x_n, f(x_{n-1})) < \alpha$ (i.e. it is an $\alpha-$pseudo-orbit) then there exists $x \in M$ such that $dist(f^n(x), x_n) < K$ for all $n \in \mathbb{Z}$ (i.e. x $K-$shadows the pseudo orbit).*

Theorem 1.3. *Let $f : M \to M$ be an expansive homeomorphism verifying $\overline{Per_H} = M$ and $Per_H^{n-1} \neq \emptyset$ or $Per_H^1 \neq \emptyset$. Then, there is a uniform local product structure in all the manifold. In particular, the pseudo orbit tracing property is verified.*

In dimension 3, in [V3] the uniform local product structure is used for proving that $M = \mathbb{T}^3$ and concluding that f must be conjugated to a linear Anosov diffeomorphism. In higher dimensions, as far as we know, there are no published results which ensure that a manifold with uniform local product structure of codimension one is a torus. However, our results give a codimension one foliation transversal to a dimension one foliation. It is known from the work of Franks that if the foliations are differentiable this implies that the manifold is a torus. This is also the case without the differentiability assumption. The proof is a straightforward adaptation of the work in [V3] and [F1]. However, we shall sketch how to adapt the proof for the sake of completeness. We then have the following Corollary, which is the main result of this paper.

Corollary 1.2. *Let $f : M^n \to M^n$ ($n \geq 3$) be an expansive homeomorphism verifying $\overline{Per_H} = M$. Suppose $Per_H^{n-1} \neq \emptyset$ or $Per_H^1 \neq \emptyset$. Then, $M = \mathbb{T}^n$ and f is conjugated to a linear Anosov diffeomorphism.*

Proof. It is consequence of the Theorem 1.3 and a result of Hiraide ([H1]) which ensures that an expansive homeomorphism in $\mathbb{T}^n$ with the pseudo orbit tracing property is conjugated to a linear Anosov diffeomorphism. The proof that $M = \mathbb{T}^n$ is sketched at the Appendix of this work. □

Acknowledgements: We profoundly thank José Vieitez for his patience and suggestions during the preparation of this work. We want to thank him and Jorge Lewowicz for teaching us all we know about this topics. Also, we thank Alberto Verjovsky for his suggestions and comments on topology (especially for communicating to us the proof sketched in the Appendix) . Finally, we would like to thank the members of the dynamical systems group in Montevideo for useful conversations and advice.

2. Separation properties

In this section, with the help of the ideas in [V1], we prove the Proposition 1.1.

The following Lemma is a general homological property of euclidean spaces.

Lemma 2.1. *Let B be a set homeomorphic to $\mathbb{R}^n$ and $F \subset B$ a closed connected set homeomorphic to an open set of $\mathbb{R}^k$. Then, $H_{n-k-1}(B \setminus F) \cong \mathbb{R}$.*

PROOF. Let $U = B \setminus F$. We then have the following long exact sequence of homology:

$$\ldots \to H_l(B) \to H_l(B,U) \to H_{l-1}(U) \to H_{l-1}(B) \to \ldots$$

We know $H_l(B) = 0$ (recall we work with reduced homology), so, we have $H_l(B,U) \cong H_{l-1}(U)$.

In particular, it is true that $H_{n-k}(B,U) \cong H_{n-k-1}(U)$. Using the duality Theorem of Alexander-Pontryagin we also deduce that $H_{n-k}(B,U) \cong H_c^k(F)$ (see [D],[Sp]). Applying the same Theorem, now to $(F,\emptyset)$, we can conclude that $H_0(F,\emptyset) \cong H_c^k(F)$. Therefore, we can deduce (using that $H_0(F,\emptyset) \cong \mathbb{R}$ since F is connected) that $H_{n-k-1}(U) \cong \mathbb{R}$ as we desired.

□

Lemma 2.2. *Let $f : M \to M$ be an expansive homeomorphism. Then, there exists $\varepsilon > 0$ such that for all $p \in Per_H^r$ exists $\phi : \overline{D^r} \to W^s(p)$ a surjective homeomorphism over its image satisfying that: $\phi(0) = p$ and that for all continuous curve $y : [0,1] \to \overline{D^r}$ such that $y(0) = 0$ and $y(1) \in \partial D^r$ there exists $s \in (0,1]$ such that $\phi \circ y(s) \notin B_\varepsilon(p)$.*

PROOF. Expansivity ensures the existence of $\varepsilon > 0$ such that for every connected set C with diameter smaller than ε satisfying that the diameter of $f^n(C)$ is bigger than the constant α of expansivity for some $n \leq 0$, then, the diameter of $f^m(C)$ is bigger than ε for all $m < n$.

If this affirmation were false there would exist connected sets C_n with diameter smaller than $1/n$ and numbers $k_n > 0$ and $l_n > k_n$ verifying that the diameter of $f^{-k_n}(C_n)$ is bigger than the expansivity constant and the diameter of $f^{-l_n}(C_n)$ smaller than $1/n$. Using the uniform continuity of f we obtain that $k_n \to +\infty$ and $l_n - k_n \to +\infty$. Connectedness of C_n and its iterates allows us to find points x_n and y_n in $f^{-m_n}(C_n)$ (with $0 \leq m_n \leq l_n$ and $l_n - m_n \to \infty$) such that $\alpha/2 \leq d(x_n, y_n) < \alpha$ and $d(f^i(x_n), f^i(y_n)) < \alpha$ for all $-l_n + m_n \leq i \leq m_n$. Considering limit points of the sequences x_n and y_n we contradict the expansivity of f.

Without loss of generality we can suppose that p is a fixed point and we can consider the conjugation $h : \overline{D^r} \to W^s(p)$ between f and the linear hyperbolic map. Also, we know that there exists $N < 0$ such that for all $x \in h(\partial D^r) \subset W^s(p)$ exists $n \in [N, 0]$ satisfying $f^n(x) \notin B(p,\alpha)$ (if not, we can find points in $h(\partial D^r)$ which stay in $B(p,\alpha)$ for an arbitrarily large quantity of iterates of f, taking limit points of that sequences we contradict expansivity). We then define $\phi : \overline{D^r} \to W^s(p)$ by $\phi(x) = f^N \circ h(x)$. Then, for every y connecting p with $\phi(\partial D^r)$ we have that $y([0,1])$ is a connected set of diameter bigger than ε. For this ε the lemma works.

□

PROOF OF PROPOSITION 1.1. After what we have already proved, to conclude the proof, it is enough to prove that if we have a homeomorphism over its image $\phi : \overline{D^k} \to \mathbb{R}^n$ such that $\phi(0) = 0$ and such that for every curve $y : [0,1] \to \overline{D^k}$ verifying $y(0) = 0$ and $y(1) \in \partial D^k$ satisfies that $\phi \circ y([0,1])$ is not contained in $B_\varepsilon(0)$, so, considering X, the connected component of 0 in $\phi^{-1}(B_\varepsilon(0))$ we have $H_{n-k-1}(B_\varepsilon(0) \setminus \phi(X)) = \mathbb{R}$.

In order to do this, let $F = \phi(X)$ and $B = B_\varepsilon(0)$. Since B is open, we have that $\phi^{-1}(B)$ is an open set of $\overline{D^k}$. Since $\overline{D^k}$ is locally arcconnected, X, being a connected component of an open set is open in $\overline{D^k}$ and locally arcconnected. This implies that it is arcconnected.

We have that $X \cap \partial D^k = \emptyset$ since in the other case a curve joining 0 with ∂D^k whose image by ϕ would be included in B would exist. Then, F is homeomorphic to an open set of $\mathbb{R}^k$. Since X is a connected component, X is closed in $\phi^{-1}(B)$ so F is closed in B. Lemma 2.1 implies the thesis.

□

3. Local product structure

The construction of a local product structure is strongly based on proving that stable and unstable sets of the periodic points intersect. This allows us to define a map between $W^s_\varepsilon(p) \times W^u_\varepsilon(p)$ and a neighborhood of p which is a homeomorphism by the invariance of domain theorem and has the desired properties. In this section we prove that this intersection occurs for periodic points close to a given one.

Let $\Delta^m = \{(x_1, \dots x_{m+1}) \in \mathbb{R}^{m+1} \ : \ x_i \geq 0 \ , \ x_1 + \dots x_{m+1} = 1\}$ the canonical simplex of dimension m. We denote $\gamma = \sum_i a_i \sigma_i$ to a $m-$chain, where $\sigma_i \colon \Delta^m \to V$ $(a_i \in \mathbb{R})$. In the course of this section, γ denotes the chain and the union of the images of σ_i indifferently.

Lemma 3.1. *For all $x \in M$, there exists $\varepsilon > 0$ such that if $V \subset B_\varepsilon(x)$ is homeomorphic to $\mathbb{R}^n$ and $p \in V \cap Per^l_H$ then there exists a cycle $\gamma \subset U_p$ which is non trivial in the $n-l-1$ dimensional homology of $V \setminus S_p$ (where $S_p = cc_p(V \cap W^s(p))$ and $U_p = cc_p(V \cap W^u(p))$). Furthermore, given K compact in V we can choose γ so that $\gamma \subset V \setminus K$.*

Proof. Because of Proposition 1.1 we know that $\varepsilon_0 > 0$ exists verifying that $H_{n-l-1}(V \setminus S_p) \neq 0$. Let γ be a cycle such that its $n-l-1$ dimensional homology class $[\gamma]$ is non trivial. Since $H_{n-l-1}(V) = 0$ we can suppose $\gamma = \partial \eta$ where η is a $n-l$ dimensional chain in V.

Say $\eta = \sum_{i=1}^{j} a_i \sigma_i$ with $\sigma_i \colon \Delta^{n-l} \to V$ $(a_i \in \mathbb{R})$.

Besides, we can suppose that σ_i and $\partial\sigma_i$ are topologically transversal to S_p so that the set of points of intersection between every σ_i and S_p is finite and such that $\partial\sigma_i \cap S_p = \emptyset$. Given $\varepsilon_1 > 0$, using barycentric subdivision (see [Sp]), we can also suppose $\operatorname{diam}(\sigma_i) < \varepsilon_1$. We observe that if $\sigma_i \cap S_p = \emptyset$ then $\partial\sigma_i$ is trivial in $H_{n-l-1}(V \setminus S_p)$. So, by choosing ε_1 small enough we can suppose that each σ_i intersects S_p in y_i only for $i = 1, ..., j$.

Let $h \colon U \subset \mathbb{R}^n \to M$ the local conjugation with the hyperbolic map, in a neighborhood of p. Intersecting with V we have that $h(U) \subset V$ and by iteration of f we can suposse that is a neighborhood of S_p.

We can think $U \subset V \subset \mathbb{R}^n$ (with the identification given by h) , $S_p \subset \mathbb{R}^l \times \{p_2\}$ and $U_p \subset \{p_1\} \times \mathbb{R}^{n-l}$ where $p = (p_1, p_2)$.

We can choose ε_1 smaller so that $B_{\varepsilon_1}(y_i) \subset U$

Since $y_i \in \sigma_i$ and $\operatorname{diam}(\sigma_i) < \varepsilon_1$ we have $\sigma_i \subset B_{\varepsilon_1}(y_i)$. Let $h^i_t \colon \mathbb{R}^l \times \mathbb{R}^{n-l} \to \mathbb{R}^l \times \mathbb{R}^{n-l}$ continuous given by $h^i_t(a + y^1_i, b) = (ta + y^1_i, b)$ with $t \in [0,1]$ where $y_i = (y^1_i, y^2_i)$.

Then, for $t \in [0,1]$, $h^i_t \circ \partial\sigma_i$ does not intersect S_p and is contained in V. Also, we have $h^i_1 \circ \partial\sigma_i = \partial\sigma_i$ and $h^i_0 \circ \partial\sigma_i \subset \{y^1_i\} \times \mathbb{R}^{n-l}$. Since $h_0 \circ \partial\sigma_i$ is homotopic to $\partial\sigma_i$ we have they are both homologous in $V \setminus S_p$.

For every $i = 1, ..., j$ let $\beta_i \colon [0,1] \to S_p$ be a continuous curve such that $\beta(0) = y_i$ and $\beta(1) = p$. If we choose a smaller ε_1 again, we have $B_{\varepsilon_1}(\beta_i) \subset U$ for all $i = 1, ..., j$.

Now, we consider $g_t^i\colon \mathbb{R}^n \to \mathbb{R}^n$, another homotopy, given by $g_t^i(z) = z+\beta_i(t)-y_i$. It verifies that $g_t^i(h_0\circ\partial\sigma_i)$ does not intersect S_p for all $t\in[0,1]$, $g_0^i = id_{\mathbb{R}^n}$ and $g_1^i(y_i) = p$ so $\sum_{i=1}^j a_i g_1^i\circ h_0^i\circ\partial\sigma_i \subset U_p$, and since g_t^i is a homotopy, it is homologous to $\gamma = \sum_{i=1}^j a_i\partial\sigma_i$ which is non trivial in the homology of $V\setminus S_p$. We call γ to $\sum_{i=1}^j a_i g_1^i\circ h_0^i\circ\partial\sigma_i$.

To see that there is a cycle homologous to γ outside of every compact set in V, we will use the map of the Lemma 2.2 $\phi\colon \overline{D^{n-l}}\subset\mathbb{R}^{n-l}\to M$ which verifies that $U_p = \phi(X)$ where $X = cc_0(\phi^{-1}(V))$. Consider a subdivision of $\mathbb{R}^{n-l}$ in simplexes of dimension $n-l$ and diameter smaller than ρ. Let us say $\mathbb{R}^{n-l} = \bigcup_{i=1}^\infty\theta_i$ and that $0\in\mathbb{R}^{n-l}$ is in the interior of θ_0.

If we consider a neighborhood $B\subset V$ of p with linear structure as before, we know that $H_{n-l-1}(B\setminus S_p)\cong\mathbb{R}$. So, we have that there exists a non zero $a\in\mathbb{R}$ such that $\gamma = a\partial(\phi\circ\theta_0)$ in $H_{n-l-1}(B\setminus S_p)$ and in particular also in $H_{n-l-1}(V\setminus S_p)$. Let $\eta_1 = \theta_0 - \sum_{\theta_i\subset X}\theta_i$.

We observe that $\partial(\phi\circ\eta_1)$ is a trivial cycle in $V\setminus S_p$. So, $a^{-1}\gamma$ is homologous to $\gamma' = \partial\phi\circ(\sum_{\theta_i\subset X}\theta_i) = \sum_{\theta_i\subset X}\phi\circ\partial\theta_i$.

To conclude the proof is enough to observe that we can suppose $\sum_{\theta_i\subset X}\partial\theta_i\subset B_\rho(\partial X)$ and use the fact that ϕ is uniformly continuous. This is true because every boundary in $B_\rho(\partial X)$ is cancelled for being trivial in homology and we can take θ_i to have arbitrarily small diameter. Given a compact set in V, considering an adequate ρ we conclude the proof.

□

Corollary 3.1. *With the same hypothesis that the previous Lemma, if $p\in Per_H^{n-1}$ then S_p separates V in two connected components V_1 and V_2. Also, p separates U_p in two connected components U_1 and U_2 such that $U_1\subset V_1$ and $U_2\subset V_2$.*

PROOF. Due to the fact that we are working with reduced homology, the previous Lemma implies that $V\setminus S_p$ has two connected components V_1 and V_2. Moreover, U_p is homeomorphic to $\mathbb{R}$, so $U_p\setminus\{p\}$ has two connected components U_1 and U_2. Let us suppose that $U_1, U_2\subset V_1$. Since V_1 is connected, we have that every $\gamma\subset U_1\cup U_2$ would be trivial in the homology of $V\setminus S_p$, contradicting the previous Lemma.

□

We will repeatedly make use of the following Lemma concerning the semicontinuous variation of stable and unstable sets (see [L2]).

Lemma 3.2. *Let $f: M\to M$ be any homeomorphism. Then, given $\varepsilon,\gamma>0$ and $x\in M$, there exists $\delta>0$ verifying that if $dist(x,y)<\delta$ then, $W_\varepsilon^s(y)\in B_\gamma(W_\varepsilon^s(x))$.*

PROOF. Suppose by contradiction that there exists $\gamma,\varepsilon>0$ and $x_n\to x$ such that $y_n\in W_\varepsilon^s(x_n)\cap B_\gamma(W_\varepsilon^s(x))^c$ exist. If we consider z a limit point of y_n we have

$$\operatorname{dist}(f^k(z), f^k(x)) = \lim_{n\to+\infty}\operatorname{dist}(f^k(y_n), f^k(x_n)) \le \varepsilon$$

with $z\neq x$ and $k\geq 0$. Thus, $z\in W_\varepsilon^s(x)$, but this is a contradiction since $z\notin B_\gamma(W_\varepsilon^s(x))$.

□

Another result we will repeatedly make use of refers to the distance between local stable and unstable sets of the points (see [V1] also). We think of it as ensuring "big angles" between the local stable and unstable sets.

Lemma 3.3. *Let $f : M \to M$ be an expansive homeomorphism with expansivity constant $\alpha > 0$. Given $V \subset U$ neighborhoods of x and ρ small enough, there exist a neighborhood $W \subset V$ of x such that if $y, z \in W$ we have $dist(S_y \cap (U \setminus V), U_z \cap (U \setminus V)) > \rho$ (where $S_y = cc_y(W^s(y) \cap U$ and $U_z = cc_z(W^u(z) \cap U)$.*

Proof. For $0 < \varepsilon < \alpha$ let us consider $\delta > 0$ given by Remark 1.1. Then, we can see that for given neighborhoods $V \subset U \subset B_\delta(x)$ of x, there are $\rho > 0$ and and $W \subset V$ ($x \in W$) such that if $y, z \in W$, then

$$dist(S_y \cap (U\backslash V), U_z \cap (U\backslash V)) > \rho$$

Otherwise, there would be points y_n and z_n converging to x and such that $\text{dist}(S_{y_n} \cap (U\backslash V), U_{z_n} \cap (U\backslash V)) < 1/n$. Taking a limit point of $a_n \in S_{y_n} \cap (U\backslash V)$ (choosen to verify $\text{dist}(a_n, U_{z_n} \cap (U\backslash V)) < 1/n$) we find a point $\overline{x} \neq x$ such that $\overline{x} \in S_x \cap S_y \cap \overline{(U\backslash V)}$. Thus, by Remark 1.1

$$\text{dist}(f^k(x), f^k(\overline{x})) \leq \varepsilon < \alpha$$

$\forall k \in \mathbb{Z}$ so, expansivity implies $x = \overline{x}$ which is a contradiction.

□

In the following Proposition we prove that the index of topologically hyperbolic periodic points is locally constant and that if two of them are close enough then their local stable and unstable sets intersect. As was already mentioned, this is the key step for obtaining the local product structure.

Proposition 3.1. *Let $f \colon M \to M$ be an expansive homeomorphism. Then*

1. *for all $k = 1, ..., n-1$, Per_H^k is open in Per_H and*
2. *for all $p \in Per_H$ there exists open neighborhoods of p, V_1 and V_2 such that for all $q \in Per_H \cap V_1$ we have $S_q \cap U_p \neq \emptyset$ and $U_q \cap S_p \neq \emptyset$, where $S_x = cc_x(W^s(x) \cap V_2)$, $U_x = cc_x(W^u(x) \cap V_2)$.*

Proof. Let $p \in Per_H^k$, $\varepsilon > 0$ from Lemma 3.1 applied to p and $h \colon B_\rho(0) \subset \mathbb{R}^n \to h(B_\rho(0)) \subset B_\varepsilon(p)$ the local conjugacy, $h(0) = p$, between f and $L \colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ given by $L(x, y) = (x/2, 2y)$, considering in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ the metric $d((x, y), (u, v)) = \max\{\|x-u\|, \|y-v\|\}$. Fix $\rho_1 \in (0, \rho)$ and let $V_2 = h(B_{\rho_1}(0))$. For $q \in h(B_{\rho_1}(0))$ we denote $S'_q = h^{-1}(S_q)$ and $U'_q = h^{-1}(U_q)$.

Let ρ_2 and ρ_3 given by Lemmas 3.2 and 3.3 such that if $\text{dist}(h^{-1}(q), 0) < \rho_3$ then $U'_q \cap B_{\rho_2}(\overline{S'_p} \cap \partial B_{\rho_1}(0)) = \emptyset$ and $S'_q \subset B_{\rho_2}(S'_p)$. Let $V_1 = h(B_{\rho_3}(0))$. Observe that we can use Lemma 3.2 to S_p and U_p because of the choice of ρ_1.

By applying Lemma 3.1 we know that if $q \in V_1 \cap Per_H^m$ then there exists $h \circ \gamma \subset S_q$ a non trivial cycle of the $m-1$ dimensional homology of $V_2 \setminus U_q$. Because of Lemma 3.1 as well, we can suppose that $\gamma \subset B_{\rho_2}(\overline{S'_p} \cap \partial B_{\rho_1}(0))$.

Let $\pi_t \colon \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ given by $\pi_t(x, y) = (x, ty)$ for $t \in [0, 1]$. It is easy to see that $\pi_t(B_{\rho_2}(\overline{S'_p} \cap \partial B_{\rho_1}(0))) \subset B_{\rho_2}(\overline{S'_p} \cap \partial B_{\rho_1}(0))$ for all $t \in [0, 1]$. Then, $\pi_t \circ \gamma$ is a homotopy between γ and $\pi_0 \circ \gamma \subset S'_p$ contained in $B_{\rho_1}(0) \setminus U'_q$, so they are homologous in $B_{\rho_1}(0) \setminus U'_q$. To conclude:

(1) If Per_H^k is not open in Per_H we can suppose there exists $q \in V_1 \cap Per_H^m$ with $m < k$. Then, γ has dimension $m-1 < k-1$ but $m-1$ dimensional homology of $S'_p \setminus U'_q$ is trivial (remember S'_p is a disk) which is absurd.
(2) If $m = k$ a cycle $\eta \subset S'_p$ such that $\partial\eta = \gamma$ exists. Since $h \circ \gamma$ is non trivial in $V_2 \setminus U_q$ we conclude that $S_p \cap U_q \neq \emptyset$.

□

Proof of Theorem 1.1.

We are going to construct a local product structure in a neighborhood of every $p \in Per_H$. We consider the notation of the statement of Proposition 3.1.

Let $\pi_s \colon \overline{V_1} \to S_p$ be defined in the points $q \in Per_H$ as $\pi_s(q) = U_q \cap S_p$. This map is well defined in a dense subset of V_1 because of Proposition 3.1. Let $x \in \overline{V_1}$ and $q_n \to x$, $q_n \in Per_H$ with $\pi(q_n) \to y$. Observe that $y \in W^u_\varepsilon(x) \cap S_p$ and expansivity imply that the intersection point is unique. This allows us to extend π_s to $\overline{V_1}$. The same reason ensures this extension is continuous. Also we have $\pi_s(x) \in U_x \cap S_p$ and because of expansivity $\pi_s(x) = U_x \cap S_p$ for all $x \in V_1$. Expansivity also implies that $\pi_s|S_x$ is injective.

If $q \in Per_H$ then the Invariance of Domain Theorem (see [Sp]) implies that $\pi_s|S_q$ is open and a homeomorphism over its image. Observe that $\pi_s(r) \in \pi_s(S_q)$ with $r, q \in Per_H$ implies $U_r \cap S_q \neq \emptyset$. Let $W \subset S_p$, $p \in W$, W homeomorphic to the disk $\overline{D}^k$ and W relative neighborhood of p in S_p.

We affirm there exists V_3 neighborhood of p such that for all $q \in V_3 \cap Per_H$, $W \subset \pi_s(S_q)$. Otherwise, $q_n \to p$ would exists, such that $q_n \in Per_H$ and $W \not\subseteq \pi_s(S_{q_n})$. Since W is connected and $\pi_s|S_{q_n}$ open, $y_n \in \partial\pi_s(S_{q_n}) \cap W$ must exist (the frontier is relative to S_p). So there must exist $x_n \in \partial V_1 \cap S_{q_n}$ such that $\pi_s(x_n) = y_n$. We can suppose $x_n \to x$ and $y_n \to y$ points of $S_{q_n} \cap \partial V_1$ and W respectively, the first due to semicontinuity of local stable sets (Lemma 3.2) and the second because W is compact. From the construction of π_s we deduce that x and y are over the same local stable and unstable set contradicting expansivity (observe that dist$(W, S_p \cap \partial V_1) > 0$ so $x \neq y$).

Let $V_4 = \pi_s^{-1}(W) \cap V_3$, we have that for every $q, r \in Per_H \cap V_4$, it is true that $S_q \cap U_r \neq \emptyset$ and $S_r \cap U_q \neq \emptyset$ from its construction.

Let $A_s \subset S_p \cap V_4$ and $B_u \subset U_p \cap V_4$ be relative neighborhoods of p, both homeomorphic to disks. Now, let $x \in A_s$ and $y \in B_u$, then, by taking limit points of the intersection of local stable and unstable sets of periodic points converging to x and y respectively, semicontinuity of local stable and unstable sets (Lemma 3.2) and expansivity easily imply that $U_x \cap S_y$ is a unique point. Let $h \colon A_s \times B_u \to V_1$ given by $h(x, y) = U_x \cap S_y$. It is continuous and injective. Using the Invariance of Domain Theorem again we conclude that it is open. This concludes the proof of the existence of a local product structure in an open and dense set.

□

Remark 3.1. Although this does not ensure the dimension of the decomposition in the local product structure to be constant, it is an inmediate consequence of the obtained results if the hypothesis of

f being transitive is added. We prove in section 4.2 that the splitting is constant when $Per_H^{n-1} \neq \emptyset$ or $Per_H^1 \neq \emptyset$.

4. Codimension one case

4.1. **Periodic point ordering and its properties.** We shall study the structure of Per_H^{n-1} in a neighborhood of a singularity $x \in M$ defining a partial order in Per_H^{n-1}. We consider $B_\nu(x)$ so that Proposition 1.1 holds. Let

$$S_p = cc_p(W^s(p) \cap B_\nu(x)),$$
$$U_p = cc_p(W^u(p) \cap B_\nu(x)).$$

Also, we shall suppose that, because of Remark 1.1, $S_p \subset W^s_\varepsilon(p)$ and $U_p \subset W^u_\varepsilon(p)$ for some $\varepsilon > 0$. For every $p \in Per_H^{n-1} \cap B_\nu(x)$ we define $\hat{p} = B_\nu \setminus cc_x(B_\nu(x) \setminus S_p)$.

Given $\delta > 0$ we define the following order relation in $X_\delta = Per_H^{n-1} \cap B_\delta(x)$. If $p, q \in X_\delta$ we say that $p \leq q$ if $\hat{p} \subset \hat{q}$. Clearly this is a partial order which depends on the singularity $x \in M$, $\nu > 0$ from Proposition 1.1 and $\delta \in (0, \nu)$. We call chain to every totally ordered subset of the relation.

Remark 4.1. Since stable sets of different periodic points have empty intersection, we have that if $\hat{p} \cap \hat{q} \neq \emptyset$ then the points p and q must be related by the ordering. So, if $p \leq q$ and $p \leq r$ then $\hat{p} \subset \hat{q} \cap \hat{r}$, q and r must be related. This implies that if $p \leq q$ and $\mathcal{C}$ is a maximal chain containing p, then $q \in \mathcal{C}$.

This order can be well understood in the case of surfaces where, for the pseudo Anosov maps, singularities have more than 2 maximal chains.

Lemma 4.1. *Given a singularity $x \in M$ and $\nu > 0$ there exists $\delta > 0$ such that there are finitely many maximal chains in X_δ. These are pairwise disjoint and every one of them accumulates in x.*

Proof. Let us suppose there were infinitely many maximal chains different from each other. We shall prove this implies the existence of arbitrarily large sets of points which are not pairwise related by the order relation. We prove this using induction.

Let $p_1, ..., p_l \in X_\delta$ be pairwise not related. Let $\mathcal{C}_i$ be maximal chains such that $p_i \in \mathcal{C}_i$ and take $\mathcal{C} \neq \mathcal{C}_i$ another maximal chain. Since two points in the same maximal chain are related, at most one of the p_i's can belong to $\mathcal{C}$.

If $p_i \notin \mathcal{C}$ for all $i = 1, \ldots, l$ then, we can choose $p_{l+1} \in \mathcal{C} \setminus \left(\bigcup_i \mathcal{C}_i\right)$ and it will not be related to any of the p_i by Remark 4.1.

If $p_i \in \mathcal{C}$ for some $1 \leq i \leq l$ then, we can take $p_i' \in \mathcal{C}_i \setminus \mathcal{C}$ and $p_{l+1} \in \mathcal{C} \setminus \mathcal{C}_i$ not related. So, the points in $\{p_1, \ldots, p_i', \ldots p_l, p_{l+1}\}$ will be pairwise not related again by Remark 4.1.

This leads us to a contradiction since Lemma 3.3 implies the existence of $\delta > 0$ and $\nu' \in (0, \nu)$ such that if $p, q \in X_\delta$ then

$$\operatorname{dist}(S_p \cap \partial B_{\nu'}(x), U_q \cap \partial B_{\nu'}(x)) > \rho$$

Given $p_i \in X_\delta$, Lemma 3.1 ensures the existence of $q_i \in \hat{p} \cap \partial B_{\nu'}(x) \cap U_{p_i}$ so that $\mathrm{dist}(q_i, q_j) > \rho$ if $i \neq j$. So, there exists a bound on the number of pairwise not related points since $\partial B_{\nu'}(x)$ is compact.

Once we know there are finitely many maximal chains, we know that the ones that do not accumulate in x are at a positive distance of x, so if we choose δ to be smaller, we obtain that every maximal chain in $B_\delta(x)$ accumulates in x.

Let C and C' be two maximal chains and $q \in C \cap C'$. If we choose δ smaller in such a way that $\hat{q}$ be disjoint with $B_\delta(x)$, we reduce the number of maximal chains in $B_\delta(x)$. So, we can suppose that the maximal chains are pairwise disjoint.

□

We call $[p]$ to the maximal chain of p in X_δ given by the previous Lemma. Now, we define

$$S_{[p]} = \overline{\bigcup_{q \in [p]} \hat{q}} \subset \overline{B_\nu(x)}$$

where $p \in X_\delta$.

We remark that we can choose ν such that for every $p \in Per_H \cap B_\nu(x)$ we have that $S_p \in W^s_\varepsilon(x)$ where $0 < 2\varepsilon < \alpha$ and $\alpha > 0$ is the constant of expansivity.

Lemma 4.2. *For every maximal chain $[p]$, $\partial \left(\bigcup_{q \in [p]} \hat{q} \right) \cap B_\nu(x) \subset W^s_\varepsilon(x)$ verifies.*

Proof. Because of Lemma 4.1 we know that $[p]$ accumulates in x. Let $q_n \in [p]$ such that $q_n \to x$. Take a point $y \in \partial \left(\bigcup_{q \in [p]} \hat{q} \right)$. Then, a sequence $z_n \in \hat{q}_n$ exists such that $z_n \to y$. Without loss of generality we can suppose $z_n \in S_{q_n}$.

Remark 1.1 ensures the existence of $\varepsilon > 0$ such that $S_{q_n} \subset W^s_\varepsilon(z'_n)$. So we have that for all $m \geq 0$

$$\mathrm{dist}(f^m(y), f^m(x)) = \lim_{n \to \infty} \mathrm{dist}(f^m(z'_n), f^m(p_n)) \leq \varepsilon$$

so $y \in W^s_\varepsilon(x)$. Then, $\partial \left(\bigcup_{q \in [p]} \hat{q} \right) \subset W^s_\varepsilon(x)$.

□

Lemma 4.3. *Suppose $\overline{Per_H} = M$ and let $x \in M$ be a singularity. Then, for all $p \in B_\delta(x) \cap Per^{n-1}_H$, there exists a neighborhood V of S_p such that $Per_H \cap V \cap B_\delta(x) \subset [p]$.*

Proof. By contradiction, let us suppose that there exists $y \in S_p \cap B_\delta(x)$ satisfying that $q_n \to y$ with $q_n \in [q] \neq [p]$ (remember that because of Theorem 1.1, near S_p we have local product structure so every periodic point near y must have the same index as p). Then $y \in \partial S_{[q]}$ (because it belongs both to $int(S_{[p]})$ and $S_{[q]}$, and the interiors of $S_{[p]}$ and $S_{[q]}$ have empty intersection). So, by Lemma 4.2, $y \in W^s_\varepsilon(x)$. Therefore $x \in W^s(p)$ because $y \in S_p \subset W^s_\varepsilon(p)$. But, since $p \in Per_H$ we contradict the fact that x is singular, since Theorem 1.1 gives us local product structure in a neighborhood of x by iteration of the local product structure in p.

□

Lemma 4.4. *If $\overline{Per_H} = M$ and let $x \in M$ be a singularity. Then* $\text{int}(S_{[p]}) \cap B_\delta(x) = \bigcup_{q\in[p]} \hat{q} \cap B_\delta(x)$.

PROOF.

The inclusion $\bigcup_{q\in[p]} \hat{q} \cap B_\delta(x) \subset \text{int}(S_{[p]}) \cap B_\delta(x)$ is immediate because if $q \geq r$ then $\hat{r} \subset int(\hat{q})$.

To obtain the other inclusion we proceed by contradiction supposing there exists a point $y \in B_\delta(x)$ in the interior of $S_{[p]}$ but such that $y \notin \hat{q}$ for all $q \in [p]$.

Then, there exists $y_n \in S_{q_n}$ such that $y_n \to y$ (this implies in particular that $y \in W^s_\varepsilon(x)$ because of Lemma 3.2) where $q_n \in [p]$ satisfies $q_n \to x$.

Using Lemma 4.3 and the fact that $\overline{Per_H} = M$ we know that there exist points $r_n \in [p]$ arbitrarily close to y_n. We can suppose $r_n \to y$ and that this points are not bounded in the ordering in $[p]$. On the other hand, we consider $U_{r_n} = cc_{r_n}(B_\nu(x) \cap W^u(r_n)) \subset W^u_\varepsilon(r_n)$ (see Remark 1.1) which is separated by S_{r_n} in two different connected components (see corollary 3.1).

Pick $\gamma > 0$ and choose $z_n \in \partial B_\gamma(y) \cap U_{r_n}$ such that $z_n \notin \hat{r}_n$. We can suppose that $z_n \to z \in \partial B_\gamma(y)$ and using the semicontinuous variation of local stable and unstable sets (Lemma 3.2) we obtain that $z \in W^u_\varepsilon(y)$.

We shall prove that $z \notin S_{[p]}$ and since γ was arbitrary this will imply that $y \in \partial S_{[p]}$ which contradicts the fact that $y \in int(S_{[p]})$.

We know that $z \notin \hat{q}$ for all $q \in [p]$, so, if $z \in S_{[p]}$ it should be accumulated by points in S_{q_n} and therefore verify $z \in W^s_\varepsilon(x)$. Then, $z \neq y$, $z \in W^u_\varepsilon(y)$ and $y, z \in W^s_\varepsilon(x)$ which contradicts expansivity (remember we chose ε so that $2\varepsilon < \alpha$).

□

Remark 4.2. Clearly $x \in S_{[p]}$ and $x \notin \hat{q}$ for all $q \in [p]$. Since $S_{[p]}$ is a closed set with non empty interior and $x \in \partial S_{[p]}$ we have that its complement in $B_\nu(x)$ which is open is also non empty. This implies that $\partial S_{[p]}$ separates $B_\nu(x)$.

The next lemma shows how the stable sets of periodic points converge uniformly towards $\partial S_{[p]}$.

Lemma 4.5. *Suppose $\overline{Per_H} = M$ and let $z \in \partial S_{[p]} \cap B_\delta(x)$ and $\rho > 0$. Then, there exists V a neighborhood of z such that if $q \in [p] \cap V$ then $S_{[p]} \cap \overline{B_\delta(x)} \subset \hat{q} \cup B_\rho(\partial S_{[p]})$.*

PROOF. Given $\rho > 0$, the set $K = \left(S_{[p]} \setminus B_\rho(\partial S_{[p]})\right) \cap \overline{B_\delta(x)}$ is a compact set contained in $\text{int}(S_{[p]} \cap B_\delta(x))$ so, using Lemma 4.4, $\{\text{int}(\hat{q})\}_{q\in[p]}$ is an open cover of K so $r \in [p]$ exists such that $K \subset \hat{r}$. Let V be a neighborhood of z disjoint from $\hat{r}$. Then, for every $q \in [p] \cap V$ we have that $q \geq r$. Then, $K = \left(S_{[p]} \setminus B_\rho(\partial S_{[p]})\right) \cap \overline{B_\delta(x)} \subset \hat{q}$ and therefore $S_{[p]} \cap \overline{B_\delta(x)} \subset \hat{q} \cup B_\rho(\partial S_{[p]})$.

□

The following Lemma represents the key step for proving the uniformity of the local product structure because it allows us to ensure that the stable and unstable sets intersect in a neighborhood of a singularity. This gives uniformity and is also important to give structure to $\partial S_{[p]}$ and discard singularities.

Lemma 4.6. *Suppose $\overline{Per_H} = M$. For all $z \in \partial S_{[p]} \cap B_\delta(x)$ and for all $\varepsilon > 0$ there exists V neighborhood of z such that if $q, r \in V \cap [p]$ then U_q intersects S_r and $\partial S_{[p]}$ in $B_\varepsilon(z) \cap S_{[p]}$.*

PROOF. Let V be a neighborhood of z such that $z \in V \subset B_\delta(x)$. Corollary 3.1 allows us to associate to each $q \in [p] \cap V$ two points $y_1^q, y_2^q \in U_q \cap \partial B_\delta(x)$ such that $y_1^q \in \hat{q}$ and $y_2^q \notin \hat{q}$. Lemma 3.3 gives us $\rho > 0$ such that (maybe taking V smaller) for $i = 1, 2$ and $q \in [p] \cap V$

$$\text{dist}(y_i^q, \partial S_{[p]}) > \rho \tag{1}$$

At the same time, by Lemma 4.5 we can suppose that for every $q \in [p] \cap V$,

$$S_{[p]} \cap B_\delta(x) \subset \hat{q} \cup B_{\rho/2}(\partial S_{[p]}) \tag{2}$$

Then, since $y_2^q \notin \hat{q}$ and $y_2^q \notin B_\rho(\partial S_{[p]})$, we have that $y_2^q \notin S_{[p]}$. Remark 4.2 together with the fact that U_q is connected implies U_q intersects $\partial S_{[p]}$.

Let us take $r \in V \cap [p]$ such that $q \leq r$, that is to say $\hat{q} \subset \hat{r}$. Consider y_1^r and y_2^r associated to r in the same way we did with q. Then $y_1^q \in \hat{q} \subset \hat{r}$. On the other hand, $y_1^r \in S_{[p]} \cap \partial B_\delta(x)$ and by (2) $y_1^r \in \hat{q} \cup B_{\rho/2}(\partial S_{[p]})$. Because of (1) we have $y_1^r \notin B_{\rho/2}(\partial S_{[p]})$ and so $y_1^r \in \hat{q}$. Then $y_1^r, y_1^q \in \hat{q} \subset \hat{r}$.

Now, since $y_2^r \notin \hat{r}$ and $\hat{q} \subset \hat{r}$ we have that $y_2^r \notin \hat{q}$. Previously we said that $y_2^q \notin S_{[p]}$, so, applying (2) (to r instead of q) we have that $y_2^q \notin \hat{r}$. This implies $y_2^q, y_2^r \notin \hat{r} \supset \hat{q}$.

Finally, since $U_q \supset \{y_1^q, y_2^q\}$ and $U_r \supset \{y_1^r, y_2^r\}$ are connected, and S_q and S_r separate the ball $B_\nu(x)$ we deduce that $S_q \cap U_r$ and $U_q \cap S_r$ are not empty as wanted.

Given $\varepsilon > 0$, expansivity and semicontinuous variation of local stable and unstable sets allow us to prove that by means of considering V small enough we can ensure that the intersections lie in $B_\varepsilon(z)$.

□

To prove Theorem 1.2 we shall also make use of some properties of the frontier of the sets $S_{[p]}$.

Proposition 4.1. *If $\overline{Per_H} = M$, $\partial S_{[p]} \cap B_\delta(x)$ is a topological manifold of dimension $n-1$.*

PROOF. Let $z \in \partial S_{[p]} \cap B_\delta(x)$. We choose $\varepsilon > 0$ such that $B_\varepsilon(z) \subset B_\delta(x)$ and let V a neighborhood of z satisfying that if $q, r \in V \cap [p]$ then $U_q \cap S_r \cap B_\varepsilon(z) \neq \emptyset$ as given in Lemma 4.6. Also, for every $q \in V \cap [p]$ we can have $U_q \cap \partial S_{[p]} \cap B_\varepsilon(z) \neq \emptyset$ again by Lemma 4.6.

Pick $q \in V \cap [p]$ and define $h_q \colon S_q \cap \overline{V} \to \partial S_{[p]} \cap \overline{B_\varepsilon(z)}$ given by

$$h_q(y) = \lim_{q_n \to y} U_{q_n} \cap \partial S_{[p]}$$

which is well defined thanks to expansivity and semicontinuous variation of local stable and unstable sets (Lemma 3.2) together with the fact that $\partial S_{[p]} \subset W^s_\gamma(z)$ because of Lemma 4.5. The fact that there is a sequence $q_n \in [p] \to y$ is a consequence of Lemma 4.3 and the fact that $\overline{Per_H} = M$.

The same argument implies that h_q is continuous and injective. Moreover, since the domain is compact, h_q is a homeomorphism over its image.

Again, by Lemma 4.6 , we can take V' and $\varepsilon' > 0$ such that $\overline{B_{\varepsilon'}(z)} \subset V$ and that for $q, r \in [p] \cap V'$, $U_q \cap S_r \cap B_{\varepsilon'}(z) \neq \emptyset$. Analogously, we have that for every $q \in V' \cap [p]$, $U_q \cap \partial S_{[p]} \cap \overline{B_{\varepsilon'}(z)} \neq \emptyset$ verifies.

If we fix $q \in V' \cap [p]$ (Lemmas 4.3 and 4.5 and ensures the existence of such q) we will be able to prove that for all $w \in \partial S_{[p]} \cap V'$ exists $y \in S_q \cap V$ such that $h_q(y) = w$. This holds since for every $w \in \partial S_{[p]} \cap V'$ we can find $\{q_n\} \subset [p] \cap V'$ such that $q_n \to w$ and so that $\emptyset \neq U_{q_n} \cap S_q \cap B_{\varepsilon'}(z) \subset V \cap S_q$. In particular, every point in $\partial S_{[p]} \cap V'$ has a preimage of the map h_q in $S_q \cap \overline{B_{\varepsilon'}(z)} \subset S_q \cap V$.

Since h_q is a homeomorphism over its image, $\partial S_{[p]} \cap V'$ is homeomorphic to its preimage which is an open subset of $S_q \cap V'$ and the proposition is proved (remember $S_q \cap V$ is homeomorphic to an open set of $\mathbb{R}^{n-1}$).

□

4.2. **Constant splitting.** Proof of Theorem 1.2.

By contradiction, we suppose that $\emptyset \neq \overline{Per_H^{n-1}} \neq M$ and consider a singularity $x \in \partial \overline{Per_H^{n-1}}$. We consider ν and δ as in Lemma 4.1, for which we know there is a finite set of maximal chains of the partial order in X_δ. Let $[p]$ be a maximal chain accumulating in x.

Lemma 4.7. *There exists $\delta > 0$ such that $Per_H \cap S_{[p]} \cap B_\delta(x) \subset Per_H^{n-1}$.*

Proof. Suppose, by contradiction, that there exist $p_n, q_n \to x$ where $q_n \in S_{[p]} \cap Per_H \setminus Per_H^{n-1}$ and $p_n \in [p]$.

We know that $p_n, q_n \notin \partial S_{[p]}$ because it would contradict the fact that x is a singularity.

Since $q_n \notin Per_H^{n-1}$, U_{q_n} is a connected topological manifold (and therefore arcconnected) of dimension at least two. Consequently, if we remove a point from U_{q_n} it remains arcconnected.

Clearly, for every p_n there exists $q_m \notin \hat{p}_n$. Remember that $\partial S_{[p]}$ and S_{p_n} separates the ball $B_\nu(x)$.

We shall prove that $U_{q_m} \subset S_{[p]} \setminus \hat{p}_n$. Otherwise, $y \in U_{q_m} \setminus S_{[p]}$ would exist. Since $\partial S_{[p]}$ separates the ball $B_\nu(x)$ we know that every curve contained in U_{q_m} joining q_m to y must intersect $\partial S_{[p]}$. Expansivity implies that U_{q_m} intersects $\partial S_{[p]}$ in at most one point. Then, since two curves in U_{q_m} connecting q_m to y and coinciding only in the extremes exist (because of the dimension of U_{q_m}) they should intersect $\partial S_{[p]}$ in two different points reaching a contradiction. We proceed analogously if we consider $y \in \hat{p}_n$.

Finally, the fact that for every n_0 there exist $m, n \geq n_0$ such that $U_{q_m} \subset S_{[p]} \setminus \hat{p}_n$ contradicts expansivity (see Lemma 3.3).

□

Let $\mathcal{C}$ be the finite set of maximal chains in $B_\delta(x)$ and let

$$S = \bigcup_{[p] \in \mathcal{C}} S_{[p]}$$

Since every $S_{[p]}$ is closed in $B_\nu(x)$ and $\mathcal{C}$ is finite, we have that S is closed. Lemma 4.7 and the fact that $\overline{Per_H} = M$ implies

$$B_\delta(x) \cap S = B_\delta(x) \cap \overline{Per_H^{n-1}} \tag{3}$$

Since $x \in \partial \overline{Per_H^{n-1}}$ we know that S can not be a neighborhood of x. We shall see how this fact represents a contradiction.

In order to do that, we shall make use of Proposition 4.1 and the following lemma.

Lemma 4.8. *For all $p \in Per_H^{n-1} \cap B_\delta(x)$ exists $A_{[p]} \subset \partial S_{[p]}$ such that $A_{[p]}$ is an open and dense subset relative to $\partial S_{[p]} \cap B_\delta(x)$ and $A_{[p]}$ is in the interior of S.*

Proposition 4.1 ensures that $\partial S_{[p]} \cap B_\delta(x)$ is a topological manifold of dimension $n-1$. Then, Lemma 4.8 and a result in [HW] stating that a closed set with empty interior in a topological manifold has dimension smaller than the manifold (chapter IV, section 4), imply that for every $[p]$, $\text{dimtop}(\partial S_{[p]} \setminus A_{[p]}) \leq n-2$. Moreover, since $\partial S \subset \bigcup \partial S_{[p]}$

$$\partial S \subset \bigcup_{[p]\in\mathcal{C}} \partial S_{[p]} \setminus A_{[p]}$$

And, since the union of a finite set of closed spaces has the dimension of the largest one (see [HW] chapter III, section 3) we know that $\text{dimtop}(\partial S) \leq n-2$. So, ∂S can not separate $B_\delta(x)$ because it should have dimension at least $n-1$ (see [HW] chapter IV, section 5). This leads us to a contradiction.

Proof of Lemma 4.8.

Let $\varepsilon > 0$ and $z \in \partial S_{[p]}$. By Lemma 4.6, there exist $q \in [p]$ such that $\{a\} = U_q \cap \partial S_{[p]}$ is in $B_\varepsilon(z)$. Theorem 1.1 implies that q has a neighborhood with local product structure, by iterating this neighborhood to the past, we obtain local product structure over a neighborhood of a, so, a must belong to $int(S) = int(\overline{Per_H^{n-1}} \cap B_\delta(x))$ and the lemma is proved.

□

4.3. **Uniform local product structure.** We shall prove Theorem 1.3 in this section. By Theorem 1.1 we know that there is an open and dense set whose points admit a local product structure. And by Theorem 1.2 we conclude, since $Per_H^{n-1} \neq \emptyset$, that $Per_H = Per_H^{n-1}$.

Let $\mathcal{S}$ be the set of singularities of f, that is to say, the points which do not admit any local product structure. To prove Theorem 1.3 we must prove that $\mathcal{S}$ is an empty set.

With the results proved in 4.1 we obtain the following consequence which allows us to study the set of singularities in codimension one case. The next proposition gives a sort of local product structure in the sets $S_{[p]}$ which will be defined properly in this statement.

Proposition 4.2. *Let $x \in \mathcal{S}$. Then, for every $z \in \partial S_{[p]} \cap B_\delta(x)$ there exists $h : I \times I^{n-1} \to S_{[p]}$ ($I = [0,1]$) homeomorphism over its image, where $h(\{a\} \times I^n)$ is contained in a local stable set, $h(I \times \{b\})$ is contained in a local unstable set and the image of h is a neighborhood of z relative to $S_{[p]}$.*

Proof. By Lemma 4.6 there exists $V \subset B_\delta(x)$ neighborhood of z in M such that if $q, r \in [p] \cap V$ then $S_q \cap U_r \neq \emptyset$ and $U_q \cap \partial S_{[p]} \neq \emptyset$.

Let $D_z \subset \partial S_{[p]} \cap V$ homeomorphic to I^{n-1} (see Proposition 4.1) such that z belongs to the interior of D_z relative to $\partial S_{[p]}$.

Let $V' \subset V$ neighborhood of z such that if $q \in [p] \cap V'$ then $S_q \cap U_r \cap V \neq \emptyset$ and $U_q \cap D_z \neq \emptyset$.

Let $q \in V' \cap [p]$ and we define $h : U_q \cap \overline{V} \cap S_{[p]} \times D_z \to S_{[p]}$ in such a way that $h(y, w) = W_\varepsilon^s(y) \cap W_\varepsilon^u(w)$ is verified. By the choice of V, approximating with topologically hyperbolic periodic points and making use of expansivity and semicontinuous variation of local stable and unstable sets (Lemma 3.2) we can ensure that the map h is well defined, continuous and injective and, since the domain is compact, a homeomorphism over its image.

We are now interested in proving that the image contains $V' \cap S_{[p]}$ and it is enough to show that it contains $[p] \cap V'$, since Per_H^{n-1} is dense in V''. This holds due to the choice of V'.

Let $U = h^{-1}(V' \cap S_{[p]})$ which is open because h is a homeomorphism over its image. Since $z \in V' \cap S_{[p]}$ a relative open set of the image of h and $S_{[p]}$, $h^{-1}(z)$ is in the interior of U. Since $U_q \cap \overline{V} \cap S_{[p]} \times D_z$ is locally connected in $h^{-1}(z)$ we can find in U a set homeomorphic to $I \times I^{n-1}$ neighborhood of $h^{-1}(z)$ whose image will be a relative neighborhood of z in $S_{[p]}$.

The other properties of h claimed in the statement of the proposition are immediate consequences of the definition of h.

□

Lemma 4.9. *If $\overline{Per_H^{n-1}} = M$ then $\mathcal{S}$ is a finite set.*

Proof.

Since the set of points with local product structure is open and invariant we know that $\mathcal{S}$ is compact and invariant. Therefore, $f\colon \mathcal{S} \to \mathcal{S}$ is an expansive homeomorphism.

We shall prove that there exist a neighborhood of $x \in \mathcal{S}$ satisfying that every singularity in that neighborhood belongs to the local stable set of x. This is a consequence of the existence of $\delta > 0$ small enough (given by Lemma 4.1) such that (since Per_H^{n-1} is dense) we have that $B_\delta(x) \subset \bigcup_{i=1}^k S_{[p_i]}$. Proposition 4.2 implies that in the interior of $S_{[p_i]}$ there is a local product structure (maybe by considering δ smaller) so singularities must lie in $\bigcup_{i=1}^k \partial S_{[p_i]}$. Lemma 4.2 now implies that singularities of $B_\delta(x)$ belong to the local stable set of x.

Expansivity implies that Lyapunov stable points are asymptotically stable. Otherwise, points y, w such that $\mathrm{dist}(f^n(y), f^n(w)) \leq \varepsilon \leq \alpha$ (α expansivity constant) and such that a subsequence $n_j \to +\infty$ with $\mathrm{dist}(f^{n_j}(y), f^{n_j}(w)) \geq \delta$ exist. Taking limit points we contradict expansivity.

Since $\mathcal{S}$ is compact and every point is asymptotically stable for f, we conclude that $\mathcal{S}$ must be finite.

□

In the following Lemma we will show that there are no isolated singularities if $\dim(M) \geq 3$. Observe that in surfaces, pseudoAnosov maps have this kind of singular points. The key fact here is how the semilocal product structures given by Proposition 4.1 are glued around the singularity. The idea is that if $S_{[p]}$ and $S_{[q]}$ have semilocal product structure, then $S_{[p]} \cap S_{[q]} \setminus \{x\}$ is a connected component of $W^s_{loc}(x) \setminus \{x\}$. If $\dim(M) \geq 3$ then the set $W^s_{loc}(x) \setminus \{x\}$ is connected and therefore there is no place for a third semilocal product structure. This will let us prove that x has a local product structure.

Lemma 4.10. *If $\dim(M) \geq 3$ and $\overline{Per_H^{n-1}} = M$ then, no isolated singularities exist.*

Proof. By contradiction, suppose $x \in M$ is an isolated singularity. Let $\nu, \delta > 0$ be as in Proposition 4.1 and such that $B_\nu(x) \cap \mathcal{S} = \{x\}$. Fix $[p]$ a maximal chain accumulating in x and let $T = cc_x(\partial S_{[p]} \cap B_\delta(x))$. We know by Proposition 4.1 that T is a topological manifold that is closed in $B_\delta(x)$.

Let $z \in T \setminus \{x\}$ and $[q] \neq [p]$ such that $z \in \partial S_{[q]}$. Define $T' = cc_z(\partial S_{[q]} \cap B_\delta(x))$. Let $F = T' \setminus \{x\} \cap T \setminus \{x\}$, which is a non empty closed set in both $T \setminus \{x\}$ and $T' \setminus \{x\}$. Since for all $w \in F$ there is a local product structure, F is open in both $T \setminus \{x\}$ and $T' \setminus \{x\}$. Thus $F = T \setminus \{x\} = T' \setminus \{x\}$ because $T \setminus \{x\}$ and $T' \setminus \{x\}$ are connected sets ($\dim(M) \geq 3$). Then, since T' is closed, $x \in T'$ which implies that $T' = cc_x(\partial S_{[q]} \cap B_\delta(x))$.

Since Per_H^{n-1} is dense in $S_{[p]}$ we can apply Proposition 4.2 to x. Let

$$h_p : [0,1) \times (-1,1)^{n-1} \to R_p \subset B_\delta(x)$$

be a homeomorphism such that R_p is a neighborhood of x relative to $S_{[p]}$ and $h_p(0) = x$. Let $F_p = T \cap R_p = h(\{0\} \times (-1,1)^{n-1})$.

Now, from Proposition 4.2 we can consider $h_q : (-1,0] \times (-1,1)^{n-1} \to R_q \subset B_\delta(x)$ a homeomorphism satisfying that R_q is a neighborhood of z relative to $S_{[q]}$ and $h_q(0) = x$. Analogously we define $F_q = \partial S_{[q]} \cap R_q = h_q(\{0\} \times (-1,1)^{n-1}) \subset F$. From the previous, we can suppose $F_q \subset F_p$. Let $\pi_2 \colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ the canonical projection over the second coordinate. Furthermore, if we restrict h_p to the set $[0,1) \times \pi_2(h_q^{-1}(F_q))$ we can suppose $F_p = F_q$.

Let $h \colon (-1,1) \times F_p \to B_\delta(x)$ given by

$$h(t,y) = \begin{cases} h_p(t, \pi_2(h_p^{-1}(y))) & \text{if } t \geq 0 \\ h_q(t, \pi_2(h_q^{-1}(y))) & \text{if } t \leq 0 \end{cases}$$

Clearly $h(0,y) = y$ so h is continuous. Again, using the Invariance of Domain Theorem, this allows us to prove that h gives a local product structure around x. This contradicts the fact that x is a singularity.

□

PROOF OF THEOREM 1.3.

Once we have discarded singularities it is very simple to prove there is a uniform local product structure. Otherwise, there would exist points x_n not admitting local product structure in balls of radius greater than $1/n$. Taking a limit point we could find a singularity, a contradiction.

Uniform local product structure implies the pseudo orbit tracing property from the results of [R] which ensure the existence of a hyperbolic metric in the coordinates given by the local product structure (see [V2]).

□

5. APPENDIX

To conclude, we prove the following proposition and then sketch the proof that M is $\mathbb{T}^n$.

Proposition 5.1. *Let M be a $n-$dimensional manifold ($n \geq 3$) and $f : M \to M$ an expansive homeomorphism such that Per_H is dense in M and $Per_H^1 \neq \emptyset$ or $Per_H^{n-1} \neq \emptyset$. Then, M admits a codimension one foliation with leaves homeomorphic to $\mathbb{R}^{n-1}$.*

PROOF. The uniform local product structure obtained in Theorem 1.3 shows the existence of the foliation.

Let us suppose that $Per_H^{n-1} \neq \emptyset$, then, the leaves of the foliation are the stable sets of the points. Let $x \in M$, we shall prove that $W^s(x)$ is homeomorphic to $\mathbb{R}^{n-1}$. To see this, is enough to see that

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(S_\varepsilon(f^n(x))$$

Where $S_\varepsilon(z)$ is a disc of uniform size in $W^s_\varepsilon(z)$ (which exist because of the uniform local product structure). So, $W^s(x)$ may be written (maybe by taking some subsequence $n_j \to \infty$ so that $f^{-n_j}(S_\varepsilon(f^{n_j}(x))) \subset f^{-n_{j+1}}(S_\varepsilon(f^{-n_{j+1}}(x))))$ as an increasing union of $n-1$ dimensional discs, which implies the thesis.

□

Once we know the leaves are homeomorphic to $\mathbb{R}^{n-1}$ classical arguments allow us to prove that M is $\mathbb{T}^n$. As we said, we shall sketch some steps of the proof for the sake of completeness. The ideas are based on [V3] and [F1] section 5.

The first thing it should be proved is that the universal covering space of M ($\overline{M}$) equals $\mathbb{R}^n$.

To prove that $\overline{M} = \mathbb{R}^n$ it suffices to prove that given two points $\overline{x}, \overline{y} \in \overline{M}$ then, the lifts of their stable and unstable manifolds (which are respectively proper copies of $\mathbb{R}^{n-1}$ and $\mathbb{R}$) intersect at a single point.

To see that the intersection has at most one point, we can see that if the manifolds intersect at more than one point then we can obtain a closed loop transversal to the codimension one foliation, thus, bounding a disc (since we are in the universal covering, the loop is nullhomotopic). By using Solodov's methods (see [So] Lemma 5) we see that the disc may be chosen to be in general position so that we obtain a foliation of the disc $\mathbb{D}^2$, transversal to the frontier and such that its singularities are nondegenerate and have no saddle connections (this is the only step where differentiability is used in [F1]). Now, using Haefliger arguments (see [V3] Lemma 2.11 or [F1] Lemma 5.1) we conclude there is a leaf of the codimension one foliation with non trivial holonomy, hence, the leaf is not simply connected, a contradiction.

Finally, proving that the foliations intersect is a straightforward adaptation of the arguments of [F1] Lemma 5.2 after it is known that the the leaves of the codimension one foliation are dense (which follows from the fact that periodic points are dense and the uniform local product structure).

Once this is obtained, it is not difficult to prove that $\pi_1(M)$ is free abelian by studding the action of $\pi_1(M)$ over $\mathbb{R}$ as it permutes without fixed points the leaves of the codimension one foliation (see [HeHi] Chapter VIII, section 3, remember that the leaves of the foliation are dense).

Now one can follow the proof in [F1], by reading the proofs of Proposition (6.2), Theorem (4.2) and Theorem (3.6) in that order (remember that expansive homeomorphisms with local product structure have hyperbolic canonical coordinates, [R]).

One can take a shortcut in dimensions ≥ 5 thanks to a result of [HiWa]. A space with free abelian fundamental group and which is covered by $\mathbb{R}^n$ is an Elienberg-McLane space of the same type of a torus, hence homotopically equivalent to one (see [Hat], Theorem 1.B.8.). From [HiWa] we deduce that if n, the dimension of M, satisfies $n \geq 5$ then M is homeomorphic to $\mathbb{T}^n$.

This proves that $M = \mathbb{T}^n$.

□

References

[D] A.Dold, *Lectures on Algebraic Topology*, Springer-Verlag (1980)

[FR] J. Franks and C. Robinson, A Quasi-Anosov Diffeomorphism that is not Anosov. *Trans. Am. Math. Soc.*, **233** (1976), 267–278,

[F1] J. Franks, Anosov Diffeomorphisms., *Proc. Sympos. Pure Math.*, vol. 14 (1970), 61–93.

[Hat] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002)

[HeHi] G. Hector and U. Hirsch, *Introduction to the Geometry of Foliations, Part B*, Aspects of Math. (1983)

[H1] K. Hiraide, Expansive homeomorphisms with the pseudo-orbit tracing property of n-tori., *J. Math.Soc. Japan*, **41** (1989), 357–389.

[H2] K. Hiraide, Expansive homeomorphisms of compact surfaces are pseudo-Anosov. *Osaka J. Math* **27**, (1990), 117–162.

[HiWa] W. Hsiang and C.T.C. Wall, On homotopy tori II., *Bull. London Math. Soc.*, **1** (1969), 341–342.

[HW] W. Hurewicz and H. Wallman, *Dimension Theory (Princeton Math. Series 4)*, Princeton University Press (1948).

[L1] J. Lewowicz, Persistence in expansive systems., *Ergodic Theory and Dynamical Systems*, **3** (1983), 567–578.

[L2] J. Lewowicz, Expansive Homemorphisms of Surfaces, *Bol. Soc. Bras. de Mat.*, **20** (1989), 113–133,

[N] S. Newhouse, On codimension one Anosov diffeomorphisms, *Amer. Jour. of Math.*, **92** (1970), 761–770.

[R] W. Reddy, Expansive canonical coordinates are hyperbolic, *Topology and its Appl.*,**13** (1982), 327–334.

[So] V.V. Solodov, Components of topological foliations, *Math. USSR Sbornik*, **47** (1984), 329–343.

[Sp] E.H. Spanier, *Algebraic Topology*, McGraw-Hill, (1966).

[V1] J.L. Vieitez, Three dimensional expansive homeomorphisms. *Pitman Research Notes in Math.* **285**, (1993), 299–323.

[V2] J.L. Vieitez, Expansive homeomorphisms and hyperbolic diffeomorphisms on three manifolds. *Ergodic Theory and Dynamical Systems*, **16** (1996), 591– 622.

[V3] J.L. Vieitez, A $3D-$manifold with a uniform local product structure is $\mathbb{T}^3$., *Publicaciones Matemáticas del Uruguay*, **8** (2000), 47–62.

[V4] J.L. Vieitez, Lyapunov functions and expansive diffeomorphisms on 3D manifolds. *Ergodic Theory and Dynamical Systems*, **22** (2002), 601–632.

Imerl, Facultad de Ingeniería, Universidad de la República, Uruguay
E-mail address: aartigue@fing.edu.uy

Cmat, Facultad de Ciencias, Universidad de la República, Uruguay
E-mail address: joaquin@cmat.edu.uy

Imerl, Facultad de Ingeniería, Universidad de la República, Uruguay
E-mail address: rpotrie@cmat.edu.uy