Constructing nearly Frobenius algebras

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Abstract

In the first part we study nearly Frobenius algebras. The concept of nearly Frobenius algebras is a generalization of the concept of Frobenius algebras. Nearly Frobenius algebras do not have traces, nor they are self-dual. We prove that the known constructions: direct sums, tensor, quotient of nearly Frobenius algebras admit natural nearly Frobenius structures.

In the second part we study algebras associated to some families of quivers and the nearly Frobenius structures that they admit. As a main theorem, we prove that an indecomposable algebra associated to a bound quiver (Q, I) with no monomial relations admits a non trivial nearly Frobenius structure if and only if the quiver is $\overrightarrow{A_n}$ and I = 0. We also present an algorithm that determines the number of independent nearly Frobenius structures for Gentle algebras without oriented cycles.

Introduction

A Frobenius algebra over a field k is a (non-necessarily commutative) associative algebra A, together with a non-degenerate trace $\varepsilon : A \to k$. In other words we have that $\langle a, b \rangle = \varepsilon(ab)$ is a non-degenerate bilinear form. They have been studied since the 1930's, specially in representation theory, for their very nice duality properties. In recent times the surprising connection found to topological quantum field theories has made them subject of renewed interest.

An important example for us, of Frobenius algebra, is the Poincaré algebra associated to every compact closed manifold M, $A = H^*(M)$. In this case we can define the trace as $\varepsilon(w) = \int_M w$, for $w \in H^*(M)$. It is a classical result that Poincaré duality is equivalent to the assertion that this trace is non-degenerate. In topology this fact manifests in many ways, for instance in the existence of an intersection product in homology that becomes a coproduct in cohomology. The coproduct Δ is the composition of the Poincaré duality isomorphism $D: H_*(M) \xrightarrow{\cong} H^*(M)$ with the dual map for the ordinary cup product $\mu: H^*(M) \otimes H^*(M) \to H^*(M)$. Note that if we consider the case of a non-compact manifold M, its cohomology algebra is no longer a Frobenius algebra, but we may ask ourselves what structure remains. In this way we arrive at the following definition.

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A nearly Frobenius algebra A is an algebra together with a coassociative coproduct Δ : $A \rightarrow A \otimes A$ such that Δ is an A-bimodule morphism. To the best of our knowledge this notion was first isolated by R. Cohen and V. Godin, see [CG04]. A good reference for this topic is the book [GLSU13].

The other objects studied in this work are the quivers and the associated algebras. It is a known result that to each finite dimensional basic algebra over an algebraically closed field \Bbbk corresponds a graphical structure, called a *quiver*, and that, conversely, to each quiver corresponds an associative \Bbbk -algebra, which has an identity and is finite dimensional under some conditions. Similarly, using the quiver associated to an algebra A, it will be possible to visualice a (finitely generated) A-module as a family of (finite dimensional) \Bbbk -vector spaces connected by linear maps (see [ASS06]). The idea of such a graphical representation seems to go back to the late forties but it became widespread in the early seventies, mainly due to Gabriel [Gab72], [Gab73]. In an explicit form, the notions of quiver and linear representation of quiver were introduced by Gabriel in [Gab72]. It was the starting point of the modern representation theory of associative algebras.

In section 1 we present known concepts required in the rest of the work. We dedicate section 2 to develop the concept of nearly Frobenius algebras. Studying nearly Frobenius structures over an algebra we prove that this family defines a k-vector space. This result permit us to define the Frobenius dimension of an algebra as the dimension of this vector space. Moreover, in this section we determine the Frobenius dimension of particular algebras as the matrix algebra, the group algebra and the truncated polynomial algebra. All these cases verify that $\operatorname{Frobdim}(A) \leq \dim_{\Bbbk}(A)$. In section 3 we show that known constructions, as opposite algebra, direct sum, tensor product and quotient of nearly Frobenius algebras admit natural nearly Frobenius structures. The last section is divided in three parts. In the first part we prove that an indecomposable algebra associated to a bound quiver (Q, I) with no monomial relations admits a non trivial nearly Frobenius structure if and only if the quiver is $\overrightarrow{\mathbb{A}_n}$ and I = 0. Moreover, in this case the Frobenius dimension is one. In the second part we deal with gentle algebras. If the quiver associated to a gentle algebra A has no oriented cycles we show that the Frobenius dimension of A is finite and we determine this number by an algorithm. In the last part we exhibit a family of algebras $A = \{A_C\}_C$ given by bound quivers for which $\operatorname{Frobdim}(A_C) > \dim_{\Bbbk}(A_C)$.

1 Preliminaries

Definition 1.1. A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 (whose elements are called *points*, or *vertices*) and Q_1 (whose elements are called *arrows*), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its *source* $s(\alpha) \in Q_0$ and its *target* $t(\alpha) \in Q_0$, respectively.

An arrow $\alpha \in Q_1$ of source $a = s(\alpha)$ and target $b = t(\alpha)$ is usually denoted by $\alpha : a \to b$. A quiver $Q = (Q_0, Q_1, s, t)$ is usually denoted briefly by $Q = (Q_0, Q_1)$ or even simply by Q. Thus, a quiver is nothing but an oriented graph without any restriction on the number of arrows between two points, to the existence of loops or oriented cycles.

Definition 1.2. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $a, b \in Q_0$. A path of length $l \ge 1$ with source a and target b (or, more briefly, from a to b) is a sequence

$$(a|\alpha_1,\alpha_2,\ldots,\alpha_l|b),$$

where $\alpha_k \in Q_1$ for all $1 \le k \le l$, $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \le k < l$, and $t(\alpha_l) = b$. Such a path is denoted briefly by $\alpha_1 \alpha_2 \dots \alpha_l$.

Definition 1.3. Let Q be a quiver. The *path algebra* &Q is the k-algebra whose underlying k-vector space has as its basis the set of all paths $(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b)$ of length $l \ge 0$ in Q and such that the product of two basis vectors $(a|\alpha_1, \alpha_2, \ldots, \alpha_l|b)$ and $(c|\beta_1, \beta_2, \ldots, \beta_k|d)$ of &Q is defined by

 $(a|\alpha_1, \alpha_2, \dots, \alpha_l|b)(c|\beta_1, \beta_2, \dots, \beta_k|d) = \delta_{bc}(a|\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_k|d),$

where δ_{bc} denotes the Kronecker delta. In other words, the product of two paths $\alpha_1 \dots \alpha_l$ and $\beta_1 \dots \beta_k$ is equal to zero if $t(\alpha_l) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1 \dots \alpha_l \beta_1 \dots \beta_k$ if $t(\alpha_l) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of kQ by distributivity.

Assume, that Q is a quiver and k is a field. Let kQ be the associated path algebra. Denote by R_Q the two-sided ideal in kQ generated by all paths of length 1, i.e. all arrows. This ideal is known as the arrow ideal.

It is easy to see, that for any $m \ge 1$ we have that \mathbb{R}^m_Q is a two-sided ideal generated by all paths of length m. Note, that we have the following chain of ideals:

$$\mathsf{R}^2_Q \supseteq \mathsf{R}^3_Q \supseteq \mathsf{R}^4_Q \supseteq \cdots$$

Definition 1.4. A two-sided ideal I in $\mathbb{k}Q$ is said to be *admissible* if there exists $\mathfrak{m} \geq 2$ such that

$$\mathsf{R}^{\mathfrak{m}}_{\mathsf{Q}} \subseteq \mathsf{I} \subseteq \mathsf{R}^{2}_{\mathsf{Q}}$$

Definition 1.5. Let \Bbbk be a field, and Q a quiver. We call a finite dimensional \Bbbk -algebra A *gentle* if it is Morita equivalent to an algebra $\frac{\&Q}{I}$ where Q is a quiver and $I \subset \&Q$ an admissible ideal subject to the following conditions:

- (1) at each vertex of Q at most 2 arrows start,
 - at each vertex of Q at most 2 arrows finish;
- (2) for each arrow $\beta \in Q_1$ there is at most one arrow $\gamma \in Q_1$ with $\beta \gamma$ a path not contained in I,
 - for each arrow $\beta \in Q_1$ there is at most one arrow $\alpha \in Q_1$ with $\alpha\beta$ a path not contained in I;

(3) the ideal I is generated by paths of length 2;

- (4) for each arrow $\beta \in Q_1$ there is at most one arrow $\gamma' \in Q_1$ with $\beta \gamma'$ a path contained in I,
 - for each arrow $\beta \in Q_1$ there is at most one arrow $\alpha' \in Q_1$ with $\alpha'\beta$ a path contained in I.

2 Nearly Frobenius algebras

The concept of nearly Frobenius algebras is a generalization of the concept of Frobenius algebras. Nearly Frobenius algebras do not have traces, nor they are self-dual.

Definition 2.1. A k-algebra A is a *nearly-Frobenius algebra* if there exists a linear map $\Delta : A \to A \otimes A$ such that

1. Δ is coassociative



2. Δ is a morphism of A-bimodule

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m} & A & A \otimes A & \xrightarrow{m} & A \\ \Delta \otimes 1 & & & \downarrow_{\Delta} & & 1 \otimes \Delta \\ A \otimes A \otimes A & \xrightarrow{m} & A \otimes A & & A \otimes A \otimes A & \xrightarrow{m \otimes 1} & A \otimes A \end{array}$$

Remark 2.2. Any nearly Frobenius coproduct in a k-algebra is determined by the evaluation in the unit of the algebra structure, that is if A is a k-algebra and $\Delta : A \to A \otimes A$ is a k-linear map such that

$$\Delta(x) = (x \otimes 1)\Delta(1) = \Delta(1)(1 \otimes x)$$

for all $x \in A$.

Theorem 2.3. Let A be a fixed k-algebra and \mathcal{E} the set of nearly Frobenius coproducts of A making it into a nearly Frobenius algebra. Then \mathcal{E} is a k-vector space.

Proof. To prove that \mathcal{E} is a k-vector space we prove that \mathcal{E} is a subspace of $V = \{\Delta : A \to A \otimes A \text{ linear transformation}\}$, which is a k-vector space. We consider the linear map $\Delta = \alpha \Delta_1 + \beta \Delta_2 : A \to A \otimes A$, with $\alpha, \beta \in \mathbb{k}$ where $\Delta_1, \Delta_2 \in \mathcal{E}$. First we prove that this map is an A-bimodule morphism, i.e. $(\mathfrak{m} \otimes 1)(1 \otimes \Delta) = \Delta \circ \mathfrak{m} = (1 \otimes \mathfrak{m})(\Delta \otimes 1)$

$$\begin{aligned} (\mathfrak{m}\otimes 1)\big(1\otimes \Delta\big) &= (\mathfrak{m}\otimes 1)\big(1\otimes (\alpha\Delta_1+\beta\Delta_2)\big) \\ &= \alpha\big(\mathfrak{m}\otimes 1\big)\big(1\otimes \Delta_1\big)+\beta\big(\mathfrak{m}\otimes 1\big)\big(1\otimes \Delta_2\big) \\ &= \alpha\Delta_1\mathfrak{m}+\beta\Delta_2\mathfrak{m}=\Delta\mathfrak{m}. \end{aligned}$$

To prove the coassociativity of Δ : $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, we fix a basis, as k-vector space, of A, $\mathcal{B} = \{e_i\}_{i \in I}$ and we note, by the Remark 4.6, that

$$\Delta(\mathbf{e}_{\mathbf{k}}) = (\mathbf{e}_{\mathbf{k}} \otimes \mathbf{1})\Delta(\mathbf{1}) = \Delta(\mathbf{1})(\mathbf{1} \otimes \mathbf{e}_{\mathbf{k}}). \tag{1}$$

If we represent $e_i e_j = \sum_k a_{ij}^k e_k$, $\Delta_1(1) = \sum_{i,j} b_{ij} e_i \otimes e_j$ and $\Delta_2(1) = \sum_{i,j} c_{ij} e_i \otimes e_j$, then the equation (1) by Δ_1 and Δ_2 is expressed as:

$$egin{aligned} &\sum_{\mathrm{i},\mathrm{j},\mathrm{l}} \mathrm{b}_{\mathrm{i}\mathrm{j}} a_{\mathrm{k}\mathrm{i}}^{\mathrm{l}} e_{\mathrm{l}} \otimes e_{\mathrm{j}} = \sum_{\mathrm{i},\mathrm{j},\mathrm{l}} \mathrm{b}_{\mathrm{l}\mathrm{i}} a_{\mathrm{i}\mathrm{k}}^{\mathrm{j}} e_{\mathrm{l}} \otimes e_{\mathrm{j}}, \ &\sum_{\mathrm{i},\mathrm{j},\mathrm{l}} \mathrm{c}_{\mathrm{i}\mathrm{j}} a_{\mathrm{k}\mathrm{i}}^{\mathrm{l}} e_{\mathrm{l}} \otimes e_{\mathrm{j}} = \sum_{\mathrm{i},\mathrm{j},\mathrm{l}} \mathrm{c}_{\mathrm{l}\mathrm{i}} a_{\mathrm{i}\mathrm{k}}^{\mathrm{j}} e_{\mathrm{l}} \otimes e_{\mathrm{j}}, \end{aligned}$$

therefore

$$\sum_{i} b_{ij} a_{ki}^{l} = \sum_{i} b_{li} a_{ik}^{j}, \tag{2}$$

$$\sum_{i} c_{ij} a_{ki}^{l} = \sum_{i} c_{li} a_{ik}^{j}.$$
(3)

Using the definition of Δ , the coassociativity condition is equivalent to

$$\left(\left(\Delta_2\otimes 1\right)\Delta_1-\left(1\otimes\Delta_1\right)\Delta_2\right)+\left(\left(\Delta_1\otimes 1\right)\Delta_2-\left(1\otimes\Delta_2\right)\Delta_1\right)=0.$$
(4)

We will prove that $(\Delta_1 \otimes 1)\Delta_2 - (1 \otimes \Delta_2)\Delta_1 = 0 = (\Delta_2 \otimes 1)\Delta_1 - (1 \otimes \Delta_1)\Delta_2$. To prove that the map $(\Delta_1 \otimes 1)\Delta_2 - (1 \otimes \Delta_2)\Delta_1$ is zero is enough to observe that the evaluation in 1 is zero:

$$\begin{array}{rcl} \bigl(\Delta_1\otimes 1\bigr)\Delta_2(x)-\bigl(1\otimes \Delta_2\bigr)\Delta_1(x)&=&\bigl(\Delta_1\otimes 1\bigr)\Delta_2(1)\bigl(1\otimes x\bigr)-\bigl(1\otimes \Delta_2\bigr)\Delta_1(1)\bigl(1\otimes x\bigr)\\ &=&\bigl(\bigl(\Delta_1\otimes 1\bigr)\Delta_2(1)-\bigl(1\otimes \Delta_2\bigr)\Delta_1(1)\bigr)\bigl(1\otimes x\bigr)=0. \end{array}$$

And the last equation holds from

$$(\Delta_1 \otimes 1)\Delta_2(1) = \sum_{i,j} c_{ij}\Delta_1(e_i) \otimes e_j = \sum_{j,l,m} \left(\sum_{i,k} c_{ij}b_{kl}a_{ik}^m \right) e_m \otimes e_l \otimes e_j$$

and

$$\begin{split} (1 \otimes \Delta_2) \Delta_1(1) &= \sum_{i,j} b_{ij} e_i \otimes \Delta_2(e_j) = \sum_{i,l,m} \left(\sum_{j,k} b_{ij} c_{kl} a_{jk}^m \right) e_i \otimes e_m \otimes e_l \\ &= \sum_{j,l,m} \left(\sum_{i,k} b_{mk} c_{ij} a_{ki}^l \right) e_m \otimes e_l \otimes e_j \\ &= \sum_{j,l,m} \left(\sum_i c_{ij} \left(\sum_k b_{mk} a_{ki}^l \right) \right) e_m \otimes e_l \otimes e_j \\ &= \sum_{j,l,m} \left(\sum_i c_{ij} \left(\sum_k b_{kl} a_{ik}^m \right) \right) e_m \otimes e_l \otimes e_j \quad \text{using (2), and (3).} \end{split}$$

5.

Definition 2.4. The *Frobenius space* associated to an algebra A is the vector space of all the possible coproducts Δ that make it into a nearly Frobenius algebra (\mathcal{E} from Theorem 3.6). Its dimension over \Bbbk is called the *Frobenius dimension* of A, that is,

Frobdim(A) = dim_k(
$$\mathcal{E}$$
).

Example 2.1. Every Frobenius algebra is also a nearly Frobenius algebra.

It is known that the truncated polynomial algebra is a Frobenius algebra, in the next example we prove that this algebra admits nearly Frobenius structures that do not come from Frobenius structures.

Example 2.2. Let A be the truncated polynomial algebra in one variable $k[x]/x^{n+1}$. We will determine all nearly Frobenius structures on A, even more we will determine a basis of the Frobenius space \mathcal{E} of A.

We consider the canonical basis $B = \{1, x, ..., x^n\}$ of A. Then the general expression of a k-linear map $\Delta : A \to A \otimes A$ in the value 1 is

$$\Delta(1) = \sum_{i,j=1}^n a_{ij} x^i \otimes x^j.$$

This map is an A-bimodule morphism if

$$\Delta(\mathbf{x}^{k}) = (\mathbf{x}^{k} \otimes \mathbf{1})\Delta(\mathbf{1}) = \Delta(\mathbf{1})(\mathbf{1} \otimes \mathbf{x}^{k}), \quad \forall \ k \in \{0, \dots, n\}.$$
(5)

The equation (5) when k = 1 is

$$\sum_{i,j=1}^n \alpha_{ij} x^{i+1} \otimes x^j = \sum_{ij,=1}^n \alpha_{ij} x^i \otimes x^{j+1}.$$

This happens if $a_{0j-1} = 0$, j = 1, ..., n; $a_{i-10} = 0$, i = 1, ..., n and $a_{ij-1} = a_{i-1j}$. Then

$$\Delta(1) = \sum_{k=0}^{n} a_{kn} \left(\sum_{i+j=n+k} x^{i} \otimes x^{j} \right)$$

We denote $a_k = a_{kn}$. Applying the Remark 4.6 we need to prove that $\Delta(x^k) = (x^k \otimes 1)\Delta(1) = \Delta(1)(1 \otimes x^k)$ to conclude that Δ is an A-bimodule morphism.

$$\begin{split} \Delta(1)\big(1\otimes x^{l}\big) &= \sum_{k=0}^{n} a_{k}\left(\sum_{i+j=n+k} x^{i}\otimes x^{j}\right)\big(1\otimes x^{l}\big) = \sum_{k=0}^{n} a_{k}\left(\sum_{i+j=n+k} x^{i}\otimes x^{j+l}\right) \\ &= \sum_{k=0}^{n} a_{k}\left(\sum_{i+m=n+k+l} x^{i}\otimes x^{m}\right) = \sum_{k=0}^{n} a_{k}\left(\sum_{r+m=n+k} x^{r+l}\otimes x^{m}\right) \\ &= (x^{l}\otimes 1)\sum_{k=0}^{n} a_{k}\left(\sum_{r+m=n+k} x^{r}\otimes x^{m}\right) = (x^{l}\otimes 1)\Delta(1). \end{split}$$

Finally, we need to check that this map is coassociative: Let $x^{l} \in A$ with $l \geq 0$.

$$\begin{split} (\Delta \otimes 1)(\Delta(x^{l})) &= (\Delta \otimes 1) \left(\sum_{k=0}^{n} a_{k} \left(\sum_{i+j=n+k+l} x^{i} \otimes x^{j} \right) \right) = \sum_{k=0}^{n} a_{k} \left(\sum_{i+j=n+k+l} \Delta(x^{i}) \otimes x^{j} \right) \\ &= \sum_{k,m=0}^{n} a_{k} a_{m} \left(\sum_{i+j=n+k+l} \sum_{r+s=n+m+i} x^{r} \otimes x^{s} \otimes x^{j} \right) \\ &= \sum_{k,m=0}^{n} a_{k} a_{m} \left(\sum_{r+s+j=2n+m+k+l} x^{r} \otimes x^{s} \otimes x^{j} \right) \\ (1 \otimes \Delta)(\Delta(x^{l})) &= (1 \otimes \Delta) \left(\sum_{k=0}^{n} a_{k} \left(\sum_{i+j=n+k+l} x^{i} \otimes x^{j} \right) \right) = \sum_{k=0}^{n} a_{k} \left(\sum_{i+j=n+k+l} x^{i} \otimes \Delta(x^{j}) \right) \\ &= \sum_{k,m=0}^{n} a_{k} a_{m} \left(\sum_{i+j=n+k+l} \sum_{r+s=n+m+j} x^{i} \otimes x^{r} \otimes x^{s} \right) \\ &= \sum_{k,m=0}^{n} a_{k} a_{m} \left(\sum_{r+s+j=2n+m+k+l} x^{r} \otimes x^{s} \otimes x^{j} \right). \end{split}$$

Then the pair (\mathbf{A}, Δ) is a nearly Frobenius algebra. In particular we have that the coproduct Δ is a linear combination of the coproducts $\Delta_{\mathbf{k}}$ defined by

$$\Delta_k \big(x^l \big) = \sum_{i+j=n+k+l} x^i \otimes x^j, \quad \textit{for } k \in \{0, \dots, n\}$$

that is $\Delta = \sum_{k=0}^{n} a_k \Delta_k$ where $a_k \in k$ for all $k \in \{1, \dots, n\}$. It is clear that the set of coproducts Δ_k is a linearly independent set. Then

$$\mathfrak{C} = \left\{ \Delta_k : A \to A \otimes A, k \in \{0, 1, \dots, n\} \right\}$$

is a basis of \mathcal{E} , and $\operatorname{Frobdim}(A) = n + 1 (= \dim_{\mathbb{K}}(A))$.

Note that Δ_0 is the Frobenius coproduct of A where the trace map $\varepsilon : A \to \Bbbk$ is given by $\varepsilon(x^i) = \delta_{i,n}$ and it is the only coproduct that admits a completion to Frobenius algebra structure. This is because if we have a counit map $\varepsilon : A \to \Bbbk$ then it satisfies the counit axiom: $\mathfrak{m}(\varepsilon \otimes 1)(\Delta_k(x^i)) = x^i, \forall i = 0, 1, ..., n$. But

$$m(\epsilon\otimes 1)\big(\Delta_k\big(x^i\big)\big)=\sum_{j+l=n+k+i}\epsilon\big(x^j\big)x^l$$

with l > i so $m(\epsilon \otimes 1)(\Delta_k(x^i)) \neq x^i$ for $k \in \{1, \ldots, n\}$.

Example 2.3. Let A be the algebra $\mathbb{C}[[x, x^{-1}]]$ of formal Laurent series. Consider the coproducts given by:

$$\Delta_j(x^i) = \sum_{k+l=i+j} x^k \otimes x^l.$$

These coproducts define nearly Frobenius structures that do not come from a Frobenius structure and $\operatorname{Frobdim}(A) = \infty$.

Example 2.4. Let be A the matrix algebra $M_{n \times n}(\mathbb{k})$. We consider the canonical basis of A, $\mathcal{B} = \{E_{ij}: i, j = 1, ..., n\}$.

As in the example 2.2 we can prove that $M_{n \times n}(\mathbb{k})$ admits $n \times n$ independent coproducts, these are $\Delta_{kl}(E_{ij}) = E_{ik} \otimes E_{lj}$, and a general coproduct in A is

$$\Delta(\mathsf{E}_{ij}) = \sum_{k,l=1}^{n} \mathfrak{a}_{kl} \Delta_{kl}(\mathsf{E}_{ij}).$$

Then $\mathbb{C} = \{\Delta_{kl} : k, l \in \{1, ..., n\}\}$ is a basis of \mathcal{E} and $\operatorname{Frobdim}(A) = n^2$. The coproduct in the identity matrix is

$$\Delta(I) = \sum_{i=1}^{n} \Delta(E_{ii}) = \sum_{i=1}^{n} \sum_{k,l=1}^{n} a_{kl} E_{ik} \otimes E_{li}.$$

In the particular case that $a_{kl} = 0$ if $k \neq l$ and $a_{kk} = 1$, for all $k \in \{1, ..., n\}$ we recover the Frobenius coproduct

$$\Delta(I) = \sum_{i,k=1}^{n} \mathsf{E}_{ik} \otimes \mathsf{E}_{ki},$$

where the trace map $\varepsilon : M_{n \times n}(\Bbbk) \to \Bbbk$ is $\varepsilon(A) = \operatorname{tr}(A)$.

Example 2.5. Let G be a cyclic finite group. The group algebra &G is a nearly Frobenius algebra. A basis of &G is $\{g^i : i = 1, ..., n\}$ where |G| = n.

Using the bimodule condition of the coproduct we can prove that a basis of the Frobenius space is

$$\mathfrak{C} = \left\{ \Delta_k : \Bbbk G \to \Bbbk G \otimes \Bbbk G : k \in 2, \dots, n \right\}$$

where $\Delta_k(1) = \sum_{i=1}^{k-1} g^i \otimes g^{k-i} + \sum_{i=k}^n g^i \otimes g^{n+k-i}$. Then we have that

 $\operatorname{Frobdim}(A) = n - 1 < \dim(A) = n.$

The general expression of any nearly Frobenius coproduct is

$$\Delta(1) = \sum_{k=2}^{n} a_k \left(\sum_{i=1}^{k-1} g^i \otimes g^{k-i} + \sum_{i=k}^{n} g^i \otimes g^{n+k-i} \right)$$

In the particular case that $a_i = 0$, for $i \in \{2, ..., n-1\}$ and $a_n = 1$ we have

$$\Delta(1) = \sum_{k=1}^{n} g^k \otimes g^{n-k}$$

the Frobenius coproduct of the group algebra A where the counit is $\epsilon(g^i)=\delta_{ni}.$

To complete the construction of the category of nearly Frobenius algebras we need to define the morphisms of them.

Definition 2.5. Let (A, Δ_A) and (B, Δ_B) be nearly Frobenius algebras. We say that $f : A \to B$ is a *morphism of nearly Frobenius algebras* if it is a morphism of algebras and the next diagram commutes



3 Constructing nearly Frobenius structures

In this section we show that known constructions, as opposite algebra, direct sum, tensor product and quotient of nearly Frobenius algebras admit natural nearly Frobenius structures. A basic but important remark in this section is the following. If A and B are isomorphic k-algebras, such that B is a nearly Frobenius algebra, we can provide to A with a nearly Frobenius structure, where the coproduct is defined as

$$\Delta_{\mathsf{A}}(\mathfrak{a}) = \big(\psi \otimes \psi\big) \Delta_{\mathsf{B}}\big(\varphi(\mathfrak{a})\big),$$

with $\varphi : A \to B$ and $\psi : B \to A$ morphisms of k-algebras such that $\psi \circ \varphi = Id_A$ and $\varphi \circ \psi = Id_B$.

- **Theorem 3.1.** 1. An algebra A is nearly Frobenius if and only if A^{op} is a nearly Frobenius algebra.
 - 2. Let A_1, \ldots, A_n be nearly Frobenius k-algebras then $A = A_1 \oplus \cdots \oplus A_n$ is a nearly Frobenius k-algebra.
 - 3. If A and B are nearly Frobenius $\Bbbk\text{-algebras},$ then $A\otimes_\Bbbk B$ also is.
- *Proof.* 1. We define the coproduct $\Delta^{op} : A^{op} \to A^{op} \otimes A^{op}$ as $\tau \circ \Delta$, where Δ is the coproduct in A and τ is the twist, that is $\tau(a \otimes b) = b \otimes a$. It is clear that Δ^{op} is coassociative because Δ is coassociative. We need to check that Δ^{op} is morphism of A^{op} -bimodule.

$$\Delta^{op}(a*b) = \Delta^{op}(ba) = \tau(\Delta(ba)) = \sum a_2 \otimes ba_1 = \sum a_2 \otimes a_1 * b = (1 \otimes *)(\Delta^{op}(a) \otimes b)$$

$$\Delta^{\operatorname{op}}(\mathfrak{a} \ast \mathfrak{b}) = \Delta^{\operatorname{op}}(\mathfrak{b}\mathfrak{a}) = \tau(\Delta(\mathfrak{b}\mathfrak{a})) = \sum \mathfrak{b}_2 \mathfrak{a} \otimes \mathfrak{b}_1 = \sum \mathfrak{a} \ast \mathfrak{b}_2 \otimes \mathfrak{b}_1 = (\ast \otimes 1)(\mathfrak{a} \otimes \Delta^{\operatorname{op}}(\mathfrak{b}))$$

2. Let A_1, \ldots, A_n be nearly Frobenius algebras with $\Delta_1 \ldots, \Delta_n$ are the associated coproducts. We consider the canonical injections $q_i : A_i \to \bigoplus_{i=1}^n A_i$. By the universal

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property of the direct sum in $Vect_k$, there exists a unique morphism Δ in $Vect_k$ such that the diagram



commutes.

The coassociativity is a consequence of the commutativity of the cube



To prove the Frobenius identities, first we note that the diagram



commutes, where $p_j: \bigoplus_{i=1}^n A_i \to A_j$ is the canonical projection. This implies that the

next cube commutes.



Then (A, Δ) is a nearly Frobenius algebra.

3. We can define the coproduct $\Delta : A \otimes B \to (A \otimes B) \otimes (A \otimes B)$ as

$$\Delta = (1 \otimes \tau \otimes 1) \circ (\Delta_1 \otimes \Delta_2), \quad \text{where } \tau \text{ is the twist.}$$

Using that Δ_1 and Δ_2 are the coproducts of A and B respectively we conclude that $(A \otimes B, \Delta)$ is a nearly Frobenius algebra.

Corollary 3.2. Let G be a finite group. If $char(\mathbb{k})$ does not divide the order of G, then $\mathbb{k}G$ is a nearly Frobenius algebra.

Proof. Applying Maschke's theorem we have that $\Bbbk G$ is semisimple, then it is a direct sum of simple algebras $M_{n_i \times n_i}(\Bbbk)$. Therefore, by the Theorem 3.1, we conclude that $\Bbbk G$ is a nearly Frobenius algebra.

Corollary 3.3. If G is a finite abelian group. Then $\Bbbk G$ is a nearly Frobenius algebra.

Proof. If G is a finite abelian group, then, by the fundamental theorem of finite abelian groups, $G = G_1 \oplus \cdots \oplus G_p$, where G_i is a finite cyclic group for $i \in \{1, \ldots, p\}$. Therefore, the group algebra $\Bbbk G$ of G is isomorphic, as a \Bbbk -algebra, to $\Bbbk G_1 \otimes \ldots \otimes \Bbbk G_p$. Finally, applying the Example 2.5 and the Theorem 3.1 we conclude that $\Bbbk G$ is a nearly Frobenius algebra.

Definition 3.4. Let (A, Δ) be a nearly Frobenius algebra. A linear subspace J in A is called a *nearly Frobenius ideal* if

- (a) J is an ideal of A and
- (b) $\Delta(J) \subset J \otimes A + A \otimes J$.

Note that, if A is a bialgebra, i.e. we have a trace map $\varepsilon : A \to k$, the additional condition $\varepsilon(J) = 0$ implies that J is a bi-ideal of A.

Example 3.1. Go back to Example 2.2. We observe that the ideal $J = \langle x \rangle$ is a nearly Frobenius ideal if we consider the coproduct Δ_1 . Because

$$\Delta_1(x) = \sum_{i+j=n+2} x^i \otimes x^j = x^2 \otimes x^n + x^3 \otimes x^{n-1} + \dots + x^n \otimes x^2 \in J \otimes J \subset J \otimes A + A \otimes J$$

Proposition 3.5. Let (A, Δ) be a nearly Frobenius algebra, J a nearly Frobenius ideal. Then A/J admits a unique nearly Frobenius structure such that $p : A \to A/J$ is a nearly Frobenius morphism.

Proof. Since $(p \otimes p)\Delta(J) \subset (p \otimes p)(J \otimes A + A \otimes J) = 0$, by the universal property of the quotient vector space it follows that there exists a unique morphism of vector spaces

$$\overline{\Delta}: A/J \to A/J \otimes A/J$$

for which the diagram

$$\begin{array}{c} A \xrightarrow{p} A/J \\ \downarrow & \downarrow \overline{\Delta} \\ A \xrightarrow{p \otimes p} A/J \otimes A/J \end{array}$$

is commutative. This map is defined by $\overline{\Delta}(\overline{\mathfrak{a}}) = \sum \overline{\mathfrak{a}_1} \otimes \overline{\mathfrak{a}_2}$ where $\overline{\mathfrak{a}} = \mathfrak{p}(\mathfrak{a})$ and $\Delta(\mathfrak{a}) = \sum \mathfrak{a}_1 \otimes \mathfrak{a}_2$, i.e. $\overline{\Delta} = (\mathfrak{p} \otimes \mathfrak{p}) \circ \Delta$.

The fact that $(\overline{\Delta} \otimes 1)\overline{\Delta}(\overline{a}) = (1 \otimes \overline{\Delta})\overline{\Delta}(\overline{a}) = \sum \overline{a_1} \otimes \overline{a_2} \otimes \overline{a_3}$ follows immediately from the commutativity of the diagram.

The last step is to prove that the coproduct is a bimodule morphism:

$$\begin{array}{cccc} A/J \otimes A/J & \xrightarrow{m} & A/J & A/J \otimes A/J & \xrightarrow{m} & A/J \\ \overline{\Delta} \otimes 1 & & & & & & & & & & & \\ \overline{\Delta} \otimes 1 & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & & \\ A/J \otimes A/J \otimes A/J \otimes A/J \otimes A/J & \xrightarrow{1 \otimes \overline{m}} & & & & & & & & & & & & \\ A/J \otimes A/J$$

Example 3.2. Consider the linear quiver

$$a \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and the associated path algebra

$$A = \langle e_1, e_2, e_3, \alpha, \beta, \alpha\beta \rangle$$

This algebra admits a unique nearly Frobenius coproduct:

$$\Delta(e_1) = \alpha \beta \otimes e_1, \quad \Delta(e_2) = \beta \otimes \alpha, \quad \Delta(e_3) = e_3 \otimes \alpha \beta,$$

$$\Delta(\alpha) = \alpha\beta \otimes \alpha, \quad \Delta(\beta) = \beta \otimes \alpha\beta, \quad \Delta(\alpha\beta) = \alpha\beta \otimes \alpha\beta,$$

in particular Frobdim(A) = 1.

Now, let be $J = \langle \alpha \beta \rangle$. Note that J is a nearly Frobenius ideal:

$$\Delta(\alpha\beta) = \alpha\beta \otimes \alpha\beta \in A \otimes J + J \otimes A.$$

Then, applying the Proposition 3.5, $B = A/J = \langle \overline{e_1}, \overline{e_2}, \overline{e_3}, \overline{\alpha}, \overline{\beta} \rangle$ admits a nearly Frobenius structure defined by

$$\overline{\Delta}: \mathbf{B} \to \mathbf{B} \otimes \mathbf{B}$$

$$\overline{\Delta}(\overline{e_1}) = \overline{\Delta}(\overline{e_3}) = \overline{\Delta}(\overline{\alpha}) = \overline{\Delta}(\overline{\beta}) = 0 \text{ and } \overline{\Delta}(\overline{e_2}) = \overline{\beta} \otimes \overline{\alpha}.$$

Note that the algebra B is associated to the quiver

$$\begin{array}{c} \alpha \\ 1 \\ 2 \\ 3 \end{array}$$

where the dashed line represents the relation $\alpha\beta = 0$, admits three independent coproducts, in fact Frobdim(B) = 3:

$$\Delta_{1}(\overline{e_{1}}) = \overline{\alpha} \otimes \overline{e_{1}}, \quad \Delta_{1}(\overline{e_{2}}) = \overline{e_{2}} \otimes \overline{\alpha}, \quad \Delta_{1}(\overline{e_{3}}) = \Delta_{1}(\overline{\beta}) = 0, \quad \Delta_{1}(\overline{\alpha}) = \overline{\alpha} \otimes \overline{\alpha}$$
$$\Delta_{2}(\overline{e_{1}}) = \Delta_{2}(\overline{\alpha}) = 0, \quad \Delta_{2}(\overline{e_{2}}) = \overline{\beta} \otimes \overline{e_{2}}, \quad \Delta_{2}(\overline{e_{3}}) = \overline{e_{3}} \otimes \overline{\beta}, \quad \Delta_{2}(\overline{\beta}) = \overline{\beta} \otimes \overline{\beta}$$
$$\Delta_{3}(\overline{e_{1}}) = \Delta_{3}(\overline{e_{3}}) = \Delta_{3}(\overline{\alpha}) = \Delta_{3}(\overline{\beta}) = 0, \quad \Delta_{3}(\overline{e_{2}}) = \overline{\beta} \otimes \overline{\alpha}.$$

Observe that $\overline{\Delta}$ coincides with Δ_3 .

Theorem 3.6. Let A, B and C nearly Frobenius algebras. Given two epimorphisms of nearly Frobenius algebras $f_A : A \twoheadrightarrow C$ and $f_B : B \twoheadrightarrow C$ the pullback R of f_A and f_B

$$\mathsf{R} = \big\{ \big(\mathfrak{a}, \mathfrak{b} \big) \in \mathsf{A} \times \mathsf{B} : \mathsf{f}_{\mathsf{A}} \big(\mathfrak{a} \big) = \mathsf{f}_{\mathsf{B}} \big(\mathfrak{b} \big) \big\}$$

is a nearly Frobenius algebra.

Proof. The pullback R is a subalgebra of $A \times B$, then the product is defined by

$$(a,b) \cdot (c,d) = (ac,bd)$$

for all $(a, b), (c, d) \in R$. Note that $(ac, bd) \in R$ because

$$f_A(ac) = f_A(a)f_A(c) = f_B(b)f_B(d) = f_B(bd).$$

As A and B are nearly Frobenius algebras there exist $\Delta_A : A \to A \otimes A$ and $\Delta_B : B \to B \otimes B$ coproducts. Then, we define

$$\Delta_{\mathsf{R}}:\mathsf{R}\to\mathsf{R}\otimes\mathsf{R}$$

as $\Delta_R((a, b)) = \sum (a_1, b_1) \otimes (a_2, b_2)$, where $\Delta_A(a) = \sum a_1 \otimes a_2$ and $\Delta_B(b) = \sum b_1 \otimes b_2$. First, we check that the map Δ_R is well defined, that is $\Delta_R((a, b)) \in R \otimes R$ for $(a, b) \in R$.

As the maps f_A and f_B are morphisms of nearly Frobenius algebras the next diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{\Delta_A} & A \otimes A & & B & \xrightarrow{\Delta_B} & B \otimes B \\ f_A & & & & & & & \\ C & \xrightarrow{\Delta_C} & C \otimes C & & C & \xrightarrow{\Delta_C} & C \otimes C \end{array},$$

 $\mathrm{then}\ \big(f_A\otimes f_A\big)\Delta_A(\mathfrak{a})=\Delta_C\big(f_A(\mathfrak{a})\big)=\Delta_C\big(f_B(\mathfrak{b})\big)=\big(f_B\otimes f_B\big)\Delta_B(\mathfrak{b}). \ \mathrm{Using\ this\ we\ have}$

$$\begin{array}{rcl} \big(f_A\otimes f_A\big)\big(\sum a_1\otimes a_2\big)&=& \big(f_B\otimes f_B\big)\big(\sum b_1\otimes b_2\big)\\ &\Rightarrow\\ &\sum f_A(a_1)\otimes f_A(a_2)&=& \sum f_B(b_1)\otimes f_B(b_2)\\ &\Rightarrow\\ \big(\sum f_A(a_1)\big)\otimes \big(\sum f_A(a_2)\big)&=& \big(\sum f_B(b_1)\big)\otimes \big(\sum f_B(b_2)\big)\,. \end{array}$$

Then $\sum f_A(a_1) = \sum f_B(b_1)$ and $\sum f_A(a_2) = \sum f_B(b_2)$ and

$$f_A\left(\sum a_1\right) = f_B\left(\sum b_1\right)$$
 and $f_A\left(\sum a_2\right) = f_B\left(\sum b_2\right)$

therefore

$$\Delta_{\mathsf{R}}((\mathfrak{a},\mathfrak{b})) = \sum (\mathfrak{a}_1,\mathfrak{b}_1) \otimes \sum (\mathfrak{a}_2,\mathfrak{b}_2) \in \mathsf{R} \otimes \mathsf{R}.$$

1. Coassociativity of Δ_R : $(\Delta_R \otimes 1)\Delta_R((a,b)) = (1 \otimes \Delta_R)\Delta_R((a,b))$.

$$\begin{array}{rcl} \bigl(\Delta_R\otimes 1\bigr)\Delta_R\bigl((a,b)\bigr)&=&\sum\Delta_R\bigl(a_1,b_1\bigr)\otimes\bigl(a_2,b_2\bigr)\\ &=&\sum\sum\bigl(a_{11},b_{11}\bigr)\otimes\bigl(a_{12},b_{12}\bigr)\otimes\bigl(a_2,b_2\bigr)\\ \bigl(1\otimes\Delta_R\bigr)\Delta_R\bigl((a,b)\bigr)&=&\sum\bigl(a_1,b_1\bigr)\otimes\Delta_R\bigl(a_2,b_2\bigr)\\ &=&\sum\sum\bigl(a_1,b_1\bigr)\otimes\bigl(a_{21},b_{21}\bigr)\otimes\bigl(a_{22},b_{22}\bigr). \end{array}$$

As the coproducts Δ_A and Δ_B are coassociatives the expressions $\sum \sum (a_{11}, b_{11}) \otimes (a_{12}, b_{12}) \otimes (a_2, b_2)$ and $\sum \sum (a_1, b_1) \otimes (a_{21}, b_{21}) \otimes (a_{22}, b_{22})$ coincide, then the coproduct Δ_R is coassociative.

2. Δ_R is a morphism of bimodules if



commute. We will prove that the first diagram commutes, the other case is analogous.

We knows that Δ_A and Δ_B are bimodule morphisms, then

$$\sum a_1 \otimes a_2 \mathbf{c} = \sum (\mathbf{a}\mathbf{c})_1 \otimes (\mathbf{a}\mathbf{c})_2, \tag{6}$$

for all $a, c \in A$, and

$$\sum b_1 \otimes b_2 d = \sum (bd)_1 \otimes (bd)_2, \tag{7}$$

for all $b, d \in B$.

Let $(a, b), (c, d) \in R$, then

$$\begin{array}{rcl} \Delta_R \bigl(ac, bd \bigr) &=& \sum \bigl((ac)_1, (bd)_1 \bigr) \otimes \bigl((ac)_2, (bd)_2 \bigr) \\ \bigl(\Delta_R \otimes 1 \bigr) \bigl((a, b) \otimes (c, d) \bigr) &=& \sum \bigl(a_1, b_1 \bigr) \otimes \bigl(a_2, b_2 \bigr) \otimes \bigl(c, d \bigr) \\ &\Rightarrow \\ \bigl(1 \otimes m \bigr) \bigl(\Delta_R \otimes 1 \bigr) \bigl((a, b) \otimes (c, d) \bigr) &=& \sum \bigl(a_1, b_1 \bigr) \otimes \bigl(a_2 c, b_2 d \bigr) \end{array}$$

Using (6) and (7) we have that $\sum ((ac)_1, (bd)_1) \otimes ((ac)_2, (bd)_2) = \sum (a_1, b_1) \otimes (a_2c, b_2d)$. Then the first diagram commutes.

Example 3.3. Let be the quivers Q_A , Q_B and Q_C illustrated in the next picture,



as before the dashed lines represent the relations $\alpha \delta = 0$ and $\alpha \beta \gamma = 0$. The pullback algebra $R = A \times_C B$ where $f_A : A \to C$ and $f_B : B \to C$ are the natural projections, by the previous theorem, admits a nearly Frobenius structure. In the next step we develop the associated coproduct.

First, we describe the nearly Frobenius structures of the algebras A, B and C.

The path algebra C admits only one independent coproduct, this is

The path algebra B admits three independent coproducts, these are

The path algebra A admits only one independent coproduct, this is

The pullback algebra R is defined by the next diagram

Then

$$\begin{split} \mathsf{R} = \langle (e_1, e_1), (e_2, e_2), (e_3, e_3), (\alpha, \alpha,), (\beta, \beta), (\alpha\beta, \alpha\beta), (e_5, 0), (\delta, 0), (0, e_4), (0, \gamma), (0, \beta\gamma) \rangle \\ & \stackrel{\simeq}{\simeq} \\ \langle e_1, e_2, e_3, \alpha, \beta, \alpha\beta, e_5, \delta, e_4, \gamma, \beta\gamma \rangle, \end{split}$$

that is the path algebra associated to the pushout quiver $Q_A \coprod_{Q_C} Q_B$. Finally, the coproduct of R, by the last identification, is

Using the Lemma 2.1.2 of [Lév04] we have that R is the path algebra of the pushout quiver $Q_R = Q_A \coprod_{Q_C} Q_B$. This quiver is represented in the next picture.



The algebra associated to the pushout quiver $Q_A \coprod_{Q_C} Q_B$ is generated by

$$\{e_1, e_2, e_3, e_4, e_5, \alpha, \beta, \gamma, \delta, \alpha\beta, \beta\gamma\}$$

This algebra admits two independent nearly Frobenius coproducts, these are

Note that if b = 0 we have the coproduct detected by the pullback structure defined in the Theorem 3.6 and developed in the previous example.

4 Quivers and nearly Frobenius algebras

This section is divided in three parts. In the first part we prove that an indecomposable algebra associated to a bound quiver (Q, I) with no monomial relations admits a non trivial nearly Frobenius structure if and only if the quiver is $\overrightarrow{A_n}$ and I = 0. Moreover, in this case the Frobenius dimension is one. In the second part we deal with gentle algebras. If the quiver associated to a gentle algebra A has no oriented cycles we show that the Frobenius dimension of A is finite and we determine this number by an algorithm. In the last part we exhibit a family of algebras $A = \{A_C\}_C$ given by bound quivers for which Frobdim $(A_C) > \dim_k(A_C)$.

4.1 Path algebras

Lemma 4.1. If $Q = \overrightarrow{\mathbb{A}_n}$, that is, Q is the following quiver

•	α_1	\	α_2		α_3	\		α_{n-1}	
1		$\frac{2}{2}$		3		4	n-1		'n,

the path algebra $A = \Bbbk Q$,

 $kQ = \langle e_1, e_2, \ldots, e_n, \alpha_i \ldots \alpha_j : i = 1, \ldots, n, i \leq j \leq n \rangle,$

admits only one independent nearly Frobenius structure, where the coproduct is defined as follows

$$\begin{array}{rcl} \Delta(e_1) &=& \alpha_1 \dots \alpha_{n-1} \otimes e_1; \\ \Delta(e_n) &=& e_n \otimes \alpha_1 \dots \alpha_{n-1}; \\ \Delta(e_i) &=& \alpha_i \dots \alpha_{n-1} \otimes \alpha_1 \dots \alpha_{i-1}, \quad i=2,\dots,n-1; \\ \Delta(\alpha_i \dots \alpha_j) &=& \alpha_i \dots \alpha_{n-1} \otimes \alpha_1 \dots \alpha_j, \quad 1 \leq i \leq j \leq n-1. \end{array}$$

Proof. If we have a coproduct Δ the next condition is required

$$\Delta(e_i) = \Delta(e_i)(1 \otimes e_i) = (e_i \otimes 1)\Delta(e_i), \quad \forall i = 1, \dots, n.$$
(8)

This implies that the coproduct in the vertexes e_1 and e_n is

$$\Delta(e_1) = a_0^1 e_1 \otimes e_1 + \sum_{\substack{i=1\\n-1}}^{n-1} a_i^1 \alpha_1 \dots \alpha_i \otimes e_1, \quad a_i^1 \in \mathbb{k}$$

$$\Delta(e_n) = a_0^n e_n \otimes e_n + \sum_{\substack{i=1\\i=1}}^{n-1} a_i^n e_n \otimes \alpha_i \dots \alpha_{n-1}, \quad a_i^n \in \mathbb{k}.$$

As $\Delta(\alpha_1 \dots \alpha_{n-1}) = \Delta(e_1)(\alpha_1 \dots \alpha_{n-1}) = (\alpha_1 \dots \alpha_{n-1})\Delta(e_n)$ we have that

$$a_i^I = a_i^n = 0 \ \forall i = 0, \dots, n-2$$

Then the coproduct in these vertexes is

$$\begin{array}{lll} \Delta(e_1) &=& a\alpha_1 \dots \alpha_{n-1} \otimes e_1, \\ \Delta(e_n) &=& ae_n \otimes \alpha_1 \dots \alpha_{n-1}. \end{array}$$

Using the equation (8) the coproduct in the vertex e_i , i = 2, ..., n - 1 is

$$\Delta(e_i) = a_0^i e_i \otimes e_i + \sum_{j=1}^{i-1} a_j^i e_i \otimes \alpha_j \dots \alpha_{i-1} + \sum_{j=i}^{n-1} a_j^i \alpha_i \dots \alpha_j \otimes e_i + \sum_{j=1}^{i-1} \sum_{k=i}^{n-1} a_{jk}^i \alpha_i \dots \alpha_k \otimes \alpha_j \dots \alpha_{i-1}.$$

The coproduct in the path $\alpha_1 \dots \alpha_{i-1}$ is given by

$$\Delta(\alpha_1 \ldots \alpha_{i-1}) = \Delta(e_1)(1 \otimes \alpha_1 \ldots \alpha_{i-1}) = (\alpha_1 \ldots \alpha_{i-1} \otimes 1)\Delta(e_i),$$

then

$$\begin{array}{lll} \Delta\bigl(\alpha_1\ldots\alpha_{i-1}\bigr) &=& a\alpha_1\ldots\alpha_{n-1}\otimes\alpha_1\ldots\alpha_{i-1}\\ &=& a_0^i\alpha_1\ldots\alpha_{i-1}\otimes e_i+\sum_{j=1}^{i-1}a_j^i\alpha_1\ldots\alpha_{i-1}\otimes\alpha_j\ldots\alpha_{i-1}\\ &+& \sum_{j=i}^{n-1}a_j^i\alpha_1\ldots\alpha_{i-1}\alpha_i\ldots\alpha_j\otimes e_i\\ &+& \sum_{j=1}^{i-1}\sum_{k=i}^{n-1}a_{jk}^i\alpha_1\ldots\alpha_{i-1}\alpha_i\ldots\alpha_k\otimes\alpha_j\ldots\alpha_{i-1} \end{array}$$

therefore $a_0^i = a_j^i = 0$, $\forall j = 1, \dots n - 1$, $a_{jk}^i = 0$, $\forall j = 2, \dots n - 1$, $k = 1, \dots n - 2$, $a_{1n-1}^i = a$ and $\Delta(e_i) = a\alpha_i \dots \alpha_{n-1} \otimes \alpha_1 \dots \alpha_{i-1}$. Also, this determine the coproduct on paths $\alpha_i \dots \alpha_j$:

$$\Delta(\alpha_{i}\ldots\alpha_{j}) = a\alpha_{i}\ldots\alpha_{n-1}\otimes\alpha_{1}\ldots\alpha_{j}.$$

To conclude the construction we need to check that Δ is coassociative.

$$\begin{split} (\Delta \otimes 1)\Delta(\mathbf{e}_{i}) &= (\Delta \otimes 1)\big(a\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{i-1}\big) \\ &= a^{2}\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{i-1}, \\ (1\otimes \Delta)\Delta(\mathbf{e}_{i}) &= (1\otimes \Delta)\big(a\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{i-1}\big) \\ &= a^{2}\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{i-1}. \\ (\Delta \otimes 1)\Delta(\alpha_{i}\dots\alpha_{j}) &= (\Delta \otimes 1)\big(a\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{j}\big) \\ &= a^{2}\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{j}, \\ (1\otimes \Delta)\Delta(\alpha_{i}\dots\alpha_{j}) &= (\Delta \otimes 1)\big(a\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{j}\big) \\ &= a^{2}\alpha_{i}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{n-1}\otimes\alpha_{1}\dots\alpha_{j}, \end{split}$$

Then, a basis of the Frobenius space has only one coproduct and $\operatorname{Frobdim}(A) = 1$. \Box

Lemma 4.2. Let $A = \frac{\Bbbk Q}{I}$ be a finite dimensional algebra. If $\alpha, \mu \in Q_1$ with $s(\alpha) = s(\mu) = p$ ($t(\alpha) = t(\mu) = p$) such that no monomial relation ends (starts) on α or β . Then $\Delta(e_p) = \Delta(\alpha) = \Delta(\mu) = 0$ for all nearly Frobenius structure Δ .

Proof. We prove the first case, the other is analogous. The situation is the following



with p, q_1 and $q_2 \in Q_0$, $\alpha, \mu \in Q_1$, $s(\alpha) = s(\mu) = p$, $t(\alpha) = q_1$ and $t(\mu) = q_2$. Since $\Delta(e_p) = (e_p \otimes 1)\Delta(e_p) = \Delta(e_p)(1 \otimes e_p)$ the elementary tensors appearing in $\Delta(e_p)$ must have the first coordinate starting in e_p and the second coordinate ending in e_p , this means:

$$\sum a_{ij}\alpha_i\otimes\beta_j$$

where $s(\alpha_i) = p$ and $t(\beta_j) = p$. In the same way we have that:

$$\Delta\bigl(e_{\mathfrak{q}_1}\bigr)=\sum b_{\mathfrak{i}\mathfrak{j}}\gamma_{\mathfrak{i}}\otimes\delta_{\mathfrak{j}}$$

with $s(\gamma_i) = q_1$, $t(\delta_j) = q_1$. Since $\Delta(\alpha) = \Delta(e_p)(1 \otimes \alpha) = (\alpha \otimes 1)\Delta(e_{q_1})$ we have that

$$\Delta(\alpha) = \sum b_{ij} \alpha \gamma_i \otimes \delta_j = \sum a_{ij} \alpha_i \otimes \beta_j \alpha,$$

then $\alpha \gamma_i = \alpha_i$ and $\delta_j = \beta_j \alpha$ (some $\alpha \gamma_i$ could be zero but $\beta_j \alpha \neq 0 \forall j$ since there is no relation ending on α). Therefore

$$\Delta(e_p) = \sum a_{ij} \alpha \gamma_i \otimes \beta_j.$$

On the other hand we have that

$$\Delta\bigl(e_{\mathfrak{q}_2}\bigr)=\sum c_{ij}\eta_i\otimes\xi_j,$$

with $s(\eta_i) = q_2$, $t(\xi_j) = q_2$. Then

$$\Delta(\mu) = \sum c_{ij} \mu \eta_i \otimes \xi_j = \sum a_{ij} \alpha_i \otimes \beta_j \mu_i$$

So we conclude that $\Delta(e_p) = \sum a_{ij}\mu\eta_i \otimes \beta_j$. Comparing $\Delta(e_p) = \sum a_{ij}\mu\eta_i \otimes \beta_j$ and $\Delta(e_p) = \sum a_{ij}\alpha\gamma_i \otimes \beta_j$ we deduce that $\Delta(e_p) = 0$ and therefore $\Delta(\alpha) = \Delta(\mu) = 0$.

Theorem 4.3. Let $A = \frac{kQ}{I}$ be a finite dimensional indecomposable algebra such that there are no monomial relations. If A admits a non trivial nearly Frobenius structure then I = 0 and $Q = \overrightarrow{\mathbb{A}_n}$.

Proof. If there exist $p, q_1, q_2 \in Q_0$ and $\alpha, \mu \in Q_1$ such that $s(\alpha) = s(\mu) = p$, $t(\alpha) = q_1$ and $t(\mu) = q_2$, that is



applying the Lema 4.2 we have that $\Delta(e_p) = \Delta(\alpha) = \Delta(\mu) = 0$. Moreover, using that there exist no monomial relations, arguing in the same way that in Lemma 4.2 we conclude that $\Delta(e_{q_1}) = \Delta(e_{q_2}) = 0$.

Since A is a finite dimensional indecomposable algebra Q is finite and connected. Given a point r of the quiver there is a walk $w = p \sim p_1 \sim \cdots \sim p_s \sim r$ from p to r, where \sim means that there is an arrow $p_i \rightarrow p_j$ or $p_j \rightarrow p_i$. Then, since $\Delta(e_p) = 0$ and there exist non monomial relations, we can reproduce again the arguments and prove that $\Delta(e_{p_1}) = \cdots = \Delta(e_{p_s}) = \Delta(e_r) = 0$. Therefore $\Delta(e_r) = 0$ for any point of Q₀. Then the coproduct Δ is trivial.

Corollary 4.4. Let A be the path algebra associated to Q, a finite connected quiver. Then A admits a non trivial nearly Frobenius structure if and only if $Q = \overrightarrow{\mathbb{A}_n}$.

Proof. If $Q = \overrightarrow{A_n}$, by the Lemma 4.1, there exists a unique non trivial nearly Frobenius structure on A.

Suppose now that A is the path algebra associated to Q with a non trivial nearly Frobenius structure, then, by the Theorem 4.3, we have that $Q = \overrightarrow{\mathbb{A}_n}$.

4.2 Gentle algebras

Lemma 4.5. The algebra associated to the quiver

$$Q: \underbrace{\alpha_1}_{1} \xrightarrow{\alpha_2}_{2} \xrightarrow{\alpha_2}_{3} \underbrace{\alpha_m}_{m} \xrightarrow{\beta_1}_{0} \underbrace{\beta_1}_{m+1} \xrightarrow{\beta_n}_{m+n}$$

with the relation $\alpha_m \beta_1 = 0$, admits mn + 2 independent nearly Frobenius structures, these are

$$\begin{split} \Delta(e_{1}) &= \alpha \alpha_{1} \dots \alpha_{m} \otimes e_{1}, & \Delta(e_{m+1}) &= b \beta_{2} \dots \beta_{n} \otimes \beta_{1}, \\ \Delta(e_{i}) &= \alpha \alpha_{i} \dots \alpha_{m} \otimes \alpha_{1} \dots \alpha_{i-1}, & \Delta(e_{m+i}) &= b \beta_{i+1} \dots \beta_{n} \otimes \beta_{1} \dots \beta_{i}, \\ \Delta(e_{m}) &= \alpha \alpha_{m} \otimes \alpha_{1} \dots \alpha_{m-1}, & \Delta(e_{m+n}) &= b e_{m+n} \otimes \beta_{1} \dots \beta_{n}, \\ \Delta(\alpha_{i} \dots \alpha_{j}) &= \alpha \alpha_{i} \dots \alpha_{m} \otimes \alpha_{1} \dots \alpha_{j}, & \Delta(\beta_{i} \dots \beta_{j}) &= b \beta_{i} \dots \beta_{n} \otimes \beta_{1} \dots \beta_{j}, \\ \Delta(e_{0}) &= \alpha e_{0} \otimes \alpha_{1} \dots \alpha_{m} + b \beta_{1} \dots \beta_{n} \otimes e_{0} + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \beta_{1} \dots \beta_{j} \otimes \alpha_{i} \dots \alpha_{m}, \end{split}$$

where $a, b, c_{ij} \in \mathbb{k}$, i = 1, ..., m and j = 1, ..., n. Therefore Frobdim $\left(\frac{\mathbb{k}Q}{I}\right) = mn + 2$.

Proof. Using that the coproduct satisfies

$$\Delta(e_i) = (e_i \otimes 1)\Delta(e_i) = \Delta(e_i)(1 \otimes e_i)$$

we conclude that, on the vertex e_i for i = 1, ..., m, the coproduct is

$$\Delta(e_i) = a_i e_i \otimes e_i + \sum_{j=i}^m a_i^j \alpha_i \dots \alpha_j \otimes e_i + \sum_{j=1}^i a_j^i e_i \otimes \alpha_j \dots \alpha_i + \sum_{j=i}^m \sum_{k=1}^i a_{jk}^i \alpha_i \dots \alpha_j \otimes \alpha_k \dots \alpha_i.$$

and

$$\Delta(e_1) = a_1 e_1 \otimes e_1 + \sum_{j=1}^m a_1^j \alpha_1 \dots \alpha_j \otimes e_1.$$

Similarly, we have that the coproduct on the vertex e_{i+m} , for i = 1, ..., n is

$$\Delta(e_{m+i}) = b_i e_{m+i} \otimes e_{m+i} + \sum_{j=i}^n b_i^j \beta_i \dots \beta_j \otimes e_{m+i} + \sum_{j=1}^i b_j^i e_{m+i} \otimes \beta_j \dots \beta_i + \sum_{j=i}^n \sum_{k=1}^i b_{jk}^i \beta_i \dots \beta_j \otimes \beta_k \dots \beta_i$$

and

$$\Delta(e_{\mathfrak{m}+\mathfrak{n}}) = \mathfrak{b}_{\mathfrak{n}} e_{\mathfrak{m}+\mathfrak{n}} \otimes e_{\mathfrak{m}+\mathfrak{n}} + \sum_{j=1}^{\mathfrak{n}} \mathfrak{b}_{j}^{\mathfrak{n}} e_{\mathfrak{m}+\mathfrak{n}} \otimes \beta_{j} \dots \beta_{\mathfrak{n}}.$$

In the vertex e_0 the situation is different.

$$\Delta(e_0) = c_0 e_0 \otimes e_0 + \sum_{k=1}^m c^k e_0 \otimes \alpha_k \dots \alpha_m + \sum_{j=1}^n c_j \beta_1 \dots \beta_j \otimes e_0 + \sum_{j=1}^n \sum_{k=1}^m c_{jk} \beta_1 \dots \beta_k \otimes \alpha_k \dots \alpha_m.$$

If we consider the path $\alpha_1 \dots \alpha_m$ the coproduct $\Delta(\alpha_1 \dots \alpha_m)$ is

$$\Delta(e_1)(1 \otimes \alpha_1 \dots \alpha_m) = (\alpha_1 \dots \alpha_m \otimes 1)\Delta(e_0).$$

Then $a_1 = a_1^j = 0$ for all $j = 1, \dots m - 1$ and $c_0 = c^k = 0$ for all $k = 2, \dots m$. The only coefficient not zero is $c^1 = a_1^m$.

Therefore

$$\Delta(e_1) = a\alpha_1 \dots \alpha_m \otimes e_1,$$

$$\Delta(e_0) = ae_0 \otimes \alpha_1 \dots \alpha_m + \sum_{j=1}^n c_j \beta_1 \dots \beta_j \otimes e_0 + \sum_{j=1}^n \sum_{k=1}^m c_{jk} \beta_1 \dots \beta_k \otimes \alpha_k \dots \alpha_m.$$

On the other hand, if we consider the path $\beta_1\ldots\beta_n$

$$\Delta(\beta_1\ldots\beta_n)=\Delta(e_0)(1\otimes\beta_1\ldots\beta_n)=(\beta_1\ldots\beta_n\otimes 1)\Delta(e_{m+n}),$$

then $b_n = b_j^n = 0$ for all j = 2, ... n and $c_j = 0$ for all j = 1, ... n - 1. The only coefficient not zero is $c_n = b_1^n$.

Consequently

$$\Delta(e_{m+n}) = b\beta_1 \dots \beta_n \otimes e_{m+n},$$

$$\Delta(e_0) = ae_0 \otimes \alpha_1 \dots \alpha_m + b\beta_1 \dots \beta_n \otimes e_0 + \sum_{j=1}^n \sum_{k=1}^m c_{jk}\beta_1 \dots \beta_j \otimes \alpha_k \dots \alpha_m.$$

To complete the prove we consider the internal paths $\alpha_1 \dots \alpha_{i-1}$, $\beta_1 \dots \beta_i$, $\alpha_i \dots \alpha_j$ and $\beta_i \dots \beta_j$.

To the first family of paths we have that

$$\Delta(\alpha_1 \dots \alpha_{i-1}) = \Delta(e_1)(1 \otimes \alpha_1 \dots \alpha_{i-1}) = (\alpha_1 \dots \alpha_{i-1} \otimes 1)\Delta(e_i),$$

then $a_{jk}^i = 0$ for all $j = i, \dots, m-1, k = 2, \dots, i$, $a_i = a_i^j = a_j^i = 0$ for all $j = 1, \dots, m$ and $a = a_{m1}^i$. Accordingly

$$\Delta(e_i) = \mathfrak{a}\alpha_i \dots \alpha_m \otimes \alpha_1 \dots \alpha_{i-1}, \text{ for all } i = 2, \dots, m.$$

In a very similar way we have that

$$\Delta(e_{m+i}) = b\beta_{i+1} \dots \beta_n \otimes \beta_1 \dots \beta_i, \quad \forall i = 1, \dots, n-1.$$

An immediate consequence of these results is

$$\Delta(\alpha_i \dots \alpha_j) = \mathfrak{a} \alpha_i \dots \alpha_m \otimes \alpha_1 \dots \alpha_j \quad \mathrm{and} \quad \Delta(\beta_i \dots \beta_j) = \mathfrak{b} \beta_i \dots \beta_n \otimes \beta_1 \dots \beta_j.$$

The coassociativity of the coproduct in the vertices e_i , e_{i+m} and in the arrows $\alpha_i \dots \alpha_j$, $\beta_i \dots \beta_j$ is analogous to the example 4.1. In the vertex e_0 is a simple calculus.

Lemma 4.6. The algebra associated to the quiver



with the relation $\beta_r \alpha_{m+1} = 0$, admits only one nearly Frobenius structure, this is

$$\begin{array}{rcl} \Delta(e_1) &=& \alpha_1 \dots \alpha_{m+n} \otimes e_1, & \Delta(\alpha_1) &=& \alpha_1 \dots \alpha_{m+n} \otimes \alpha_1, \\ \Delta(e_i) &=& \alpha_i \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_{i-1}, & \Delta(\alpha_i) &=& \alpha_i \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_i, \\ \Delta(e_{m+n}) &=& e_{m+n} \otimes \alpha_1 \dots \alpha_{m+n}, & \Delta(\alpha_{m+n}) &=& \alpha_{m+n} \otimes \alpha_1 \dots \alpha_{m+n}, \\ & \Delta(e_{m+n+i}) = 0 \quad and \quad \Delta(\beta_i) = 0 \quad \forall \ i = 1, \dots, r. \end{array}$$

Therefore Frobdim(A) = 1.

Proof. In the extreme vertices the coproduct has the general expression

$$\Delta(e_1) = a_1 e_1 \otimes e_1 + \sum_{i=1}^{m+n} a_i^1 \alpha_1 \dots \alpha_i \otimes e_1,$$

$$\Delta(e_{m+n}) = b_n e_{m+n} \otimes e_{m+n} + \sum_{i=1}^{m+n} b_i^n e_{m+n} \otimes \alpha_i \dots \alpha_{m+n}.$$

The coproduct in the path $\alpha_1 \dots \alpha_{m+n}$ is

$$\Delta(\alpha_1 \ldots \alpha_{m+n}) = \Delta(e_1)(1 \otimes \alpha_1 \ldots \alpha_{m+n}) = (\alpha_1 \ldots \alpha_{m+n} \otimes 1)\Delta(e_{m+n}),$$

then

$$a_{1}e_{1} \otimes \alpha_{1} \dots \alpha_{m+n} + \sum_{i=1}^{m+n} a_{i}^{1}\alpha_{1} \dots \alpha_{i} \otimes \alpha_{1} \dots \alpha_{m+n} = b_{n}\alpha_{1} \dots \alpha_{m+n} \otimes e_{m+n}$$

$$+ \sum_{i=1}^{m+n} b_{i}^{n}\alpha_{1} \dots \alpha_{m+n} \otimes \alpha_{i} \dots \alpha_{m+n}.$$
Consequently $a_{i}^{1} = b_{j}^{n} = 0$, for all $i = 1, \dots, m+n-1$, for all $j = 2, \dots, m+n$ and
 $a_{m+n}^{1} = b_{1}^{n},$
and
 $\Delta(e_{1}) = a\alpha_{1} \dots \alpha_{m+n} \otimes e_{1},$

$$\Delta(e_1) = a\alpha_1 \dots \alpha_{m+n} \otimes e_1,$$

$$\Delta(e_{m+n}) = ae_{m+n} \otimes \alpha_1 \dots \alpha_{m+n}.$$

Repeating the procedure of the previous lemma we can prove that

$$\Delta(e_i) = a \alpha_i \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_{i-1}, \quad \forall \ i = \dots, m+n.$$

In the vertex \boldsymbol{e}_0 the situation is different

$$\begin{split} \Delta(e_0) &= \Delta(e_0) (1 \otimes e_0) = (e_0 \otimes 1) \Delta(e_0) \\ &= c_0 e_0 \otimes e_0 + \sum_{j=1}^n c_j \alpha_{m+1} \dots \alpha_{m+j} \otimes e_0 + \sum_{j=1}^m c^j e_0 \otimes \alpha_k \dots \alpha_m \\ &+ \sum_{j=1}^n \sum_{k=1}^m c_{jk} \alpha_{m+1} \dots \alpha_{m+j} \otimes \alpha_k \dots \alpha_m \\ &+ \sum_{j=1}^r d_j e_0 \otimes \beta_j \dots \beta_r + \sum_{j=1}^n \sum_{k=1}^r d_{jk} \alpha_{m+1} \dots \alpha_{m+j} \otimes \beta_j \dots \beta_r. \end{split}$$

By the other hand

$$\Delta(e_{\mathfrak{m}+\mathfrak{n}+1}) = \mathfrak{b}e_{\mathfrak{m}+\mathfrak{n}+1} \otimes e_{\mathfrak{m}+\mathfrak{n}+1} + \sum_{i=1}^{r} \mathfrak{b}_{i}\beta_{1} \dots \beta_{i} \otimes e_{\mathfrak{m}+\mathfrak{n}+1}.$$

Now, consider the path $\beta_1 \dots \beta_r,$ then the coproduct is described by

$$\Delta(\beta_1 \dots \beta_r) = \Delta(e_{m+n+1})(1 \otimes \beta_1 \dots \beta_r) = (\beta_1 \dots \beta_r \otimes 1)\Delta(e_0),$$

then

$$\begin{split} be_{m+n+1}\otimes\beta_1\dots\beta_r+\sum_{i=1}^r b_i\beta_1\dots\beta_i\otimes\beta_1\dots\beta_r = & \sum_{j=1}^m c^j\beta_1\dots\beta_r\otimes\alpha_k\dots\alpha_m \\ & +\sum_{j=1}^r d_j\beta_1\dots\beta_r\otimes\beta_j\dots\beta_r, \end{split}$$

therefore $b = c_0 = c^j = 0$, for all j = 1, ..., m, $d_j = b_i = 0$, for all j = 2, ..., r, i = 1, ..., r-1 and $b_r = d_1$. Using this we conclude that

$$\begin{aligned} \Delta(e_{m+n+1}) &= b\beta_1 \dots \beta_r \otimes e_{m+n+1}, \\ \Delta(e_0) &= be_0 \otimes \beta_1 \dots \beta_r + \sum_{j=1}^n c_j \alpha_{m+1} \dots \alpha_{m+j} \otimes e_0 \\ &+ \sum_{j=1}^n \sum_{k=1}^m c_{jk} \alpha_{m+1} \dots \alpha_{m+j} \otimes \alpha_k \dots \alpha_m + \sum_{j=1}^n \sum_{k=1}^r d_{jk} \alpha_{m+1} \dots \alpha_{m+j} \otimes \beta_k \dots \beta_r. \end{aligned}$$

The next step is to consider the path $\alpha_{m+1} \dots \alpha_{m+n}$, for this path the coproduct is determined by

$$\Delta(\alpha_{m+1}\ldots\alpha_{m+n}) = (\alpha_{m+1}\ldots\alpha_{m+n}\otimes 1)\Delta(e_{m+n}) = a\alpha_{m+1}\ldots\alpha_{m+n}\otimes\alpha_1\ldots\alpha_{m+n}.$$

By the other hand, this coproduct is determined by

$$\begin{split} \Delta(\alpha_{m+1}\ldots\alpha_{m+n}) &= \Delta(e_0) \left(1 \otimes \alpha_{m+1}\ldots\alpha_{m+n} \right) = \sum_{j=1}^n c_j \alpha_{m+1}\ldots\alpha_{m+j} \otimes \alpha_{m+1}\ldots\alpha_{m+n} \\ &+ \sum_{j=1}^n \sum_{k=1}^m c_{jk} \alpha_{m+1}\ldots\alpha_{m+j} \otimes \alpha_k\ldots\alpha_{m+n}. \end{split}$$

Comparing the expressions we have $c_j = 0$, for all j = 1, ..., n, $c_{jk} = 0$, for all j = 1, ..., n-1, i = 2, ..., m and $a = c_{n1}$. Consequently

$$\Delta(e_0) = \alpha \alpha_{m+1} \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_m + b e_0 \otimes \beta_1 \dots \beta_r + \sum_{j=1}^n \sum_{k=1}^r d_{jk} \alpha_{m+1} \dots \alpha_{m+j} \otimes \beta_k \dots \beta_r.$$

Finally, we consider the path $\alpha_1 \dots \alpha_m$, as before, we have two way to define the coproduct in this path

$$\begin{array}{lll} \Delta(\alpha_1\ldots\alpha_m) &=& \Delta(e_1)\big(1\otimes\alpha_1\ldots\alpha_m\big) = a\alpha_1\ldots\alpha_{m+n}\otimes\alpha_1\ldots\alpha_m\\ &=& (\alpha_1\ldots\alpha_m\otimes1)\Delta(e_0) = a\alpha_1\ldots\alpha_{m+n}\otimes\alpha_1\ldots\alpha_m\\ &+& b\alpha_1\ldots\alpha_m\otimes\beta_1\ldots\beta_r + \sum_{j=1}^n\sum_{k=1}^r d_{jk}\alpha_1\ldots\alpha_{m+j}\otimes\beta_k\ldots\beta_r, \end{array}$$

then $b = d_{jk} = 0$ for all j, k.

As b = 0 we have that $\Delta(e_{m+n+1}) = 0$, this implies that $\Delta(e_{m+n+i}) = 0$ for all i = 1, ..., rand $\Delta(e_0) = a\alpha_{m+1} \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_m$.

It is a simple calculation to prove that Δ is coassociative.

The next result is the symmetrical case to Lemma 4.6.

Lemma 4.7. The algebra associated to the quiver



with the relation $\alpha_m\beta_1=0,$ admits only one nearly Frobenius structure, this is

$$\Delta(e_{m+n+i}) = 0 \quad \text{and} \quad \Delta(\beta_i) = 0 \quad \text{for all } i = 1, \dots, r$$

Accordingly Frobdim(A) = 1.

Lemma 4.8. The algebra associated to the quiver



with relations $\beta_r \alpha_{m+1} = 0$ and $\alpha_m \beta_{r+1} = 0$, admits two independent nearly Frobenius structures and a general coproduct is determined by

$$\begin{array}{rcl} \Delta(e_1) &=& a\alpha_1 \dots \alpha_{m+n} \otimes e_1, & \Delta(\alpha_1) &=& a\alpha_1 \dots \alpha_{m+n} \otimes \alpha_1, \\ \Delta(e_i) &=& a\alpha_i \dots \alpha_n \otimes \alpha_1 \dots \alpha_{i-1}, & \Delta(\alpha_i) &=& a\alpha_i \dots \alpha_n \otimes \alpha_1 \dots \alpha_i, \\ \Delta(e_{m+n}) &=& ae_{m+n} \otimes \alpha_1 \dots \alpha_{m+n}, & \Delta(\alpha_{m+n}) &=& a\alpha_{m+n} \otimes \alpha_1 \dots \alpha_{m+n}, \\ \Delta(e_{m+n+1}) &=& b\beta_1 \dots \beta_{r+s} n \otimes e_1, & \Delta(\beta_1) &=& b\beta_1 \dots \beta_{r+s} \otimes \beta_1, \\ \Delta(e_{m+n+r+s+1}) &=& be_{m+n+r+s+1} \otimes \beta_1 \dots \beta_{r+s}, & \Delta(\beta_{r+s}) &=& b\beta_{r+s} \otimes \beta_1 \dots \beta_{r+s}, \end{array}$$

$$\Delta(e_0) = a \alpha_{m+1} \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_m + b \beta_{r+1} \dots \beta_{r+s} \otimes \beta_1 \dots \beta_r,$$

where $a, b \in k$. In this case Frobdim(A) = 2.

Proof. First, note that, we can expressed the coproduct in the vertices as

$$\begin{split} \Delta(e_1) &= a^1 e_1 \otimes e_1 + \sum_{i=1}^{m+n} a_i \alpha_1 \dots \alpha_i \otimes e_1, \\ \Delta(e_{m+n}) &= b^n e_{m+n} \otimes e_{m+n} + \sum_{i=1}^{m+n} b_i e_{m+n} \otimes \alpha_i \dots \alpha_{m+n}. \end{split}$$

Applying the coproduct in the path $\alpha_1 \dots \alpha_{m+n}$ we can prove that

$$\begin{array}{lll} \Delta(e_1) &=& a\alpha_1 \dots \alpha_{m+n} \otimes e_1, \\ \Delta(e_{m+n}) &=& ae_{m+n} \otimes \alpha_1 \dots \alpha_{m+n}. \end{array}$$

By symmetry we have that

$$\Delta(e_{m+n+1}) = b\beta_1 \dots \beta_{r+s} \otimes e_{m+n+1},$$

$$\Delta(e_{m+n+r+s+1}) = be_{m+n+r+s+1} \otimes \beta_1 \dots \beta_{r+s}.$$

Reproducing the calculus of Lemma 4.6 we can prove that

$$\begin{array}{lll} \Delta(e_{i}) &=& a\alpha_{i}\ldots\alpha_{m+n}\otimes\alpha_{1}\ldots\alpha_{i-1},\\ \Delta(e_{m+n+i}) &=& a\beta_{i}\ldots\beta_{r+s}\otimes\beta_{1}\ldots\beta_{i-1}. \end{array}$$

In the vertex e_0 the situation is more complicated.

$$\begin{split} \Delta \big(e_0 \big) &= b_0 e_0 \otimes e_0 + \sum_{i=1}^m b_i e_0 \otimes \alpha_i \dots \alpha_m + \sum_{j=1}^r c_j e_0 \otimes \beta_j \dots \beta_r \\ &+ \sum_{i=1}^n b^i \alpha_{m+1} \dots \alpha_{m+i} \otimes e_0 + \sum_{j=1}^s c^j \beta_{r+1} \dots \beta_{r+j} \otimes e_0 \\ &+ \sum_{i=1}^n \sum_{j=1}^m b_{ij} \alpha_{m+1} \dots \alpha_{m+i} \otimes \alpha_j \dots \alpha_m + \sum_{i=1}^s \sum_{j=1}^r c_{ij} \beta_{r+1} \dots \beta_{r+i} \otimes \beta_j \dots \beta_r \\ &+ \sum_{i=1}^n \sum_{j=1}^r b^{ij} \alpha_{m+1} \dots \alpha_{m+i} \otimes \beta_j \dots \beta_r + \sum_{i=1}^s \sum_{j=1}^m c^{ij} \beta_{r+1} \dots \beta_{r+i} \otimes \alpha_j \dots \alpha_m. \end{split}$$

If we determine the coproduct in the paths $\alpha_1 \dots \alpha_m$ and $\beta_1 \dots \beta_r$ we conclude that

$$\Delta(e_0) = a\alpha_{m+1} \dots \alpha_{m+n} \otimes \alpha_1 \dots \alpha_m + b\beta_{r+1} \dots \beta_{r+s} \otimes \beta_1 \dots \beta_r$$

The coproduct in the arrows is determined by the value in the vertices.

The coassociativity is an easy exercise.

Now, if we consider a gentle algebra A associated to Q, a finite connected quiver without oriented cycles, we can produce an algorithm that permit us to determine the number independent nearly Frobenius structures that the algebra A admits. Next we develop the algorithm.

As Q is finite we can suppose that $\#Q_0 = n$ and $\#Q_1 = m$. The quiver Q is triangular, because it has not cycles. In particular, there exist a partial order \preccurlyeq of $\{1, 2, ..., n\}$ and the arrows such that

$$\begin{cases} i \prec j & \text{if } i \rightsquigarrow j \ (i \neq j, i \text{ precede to } j) \\ i \preccurlyeq j & \text{if } i = j \text{ or } i \prec j \end{cases}$$

and

$$\left\{ \begin{array}{ll} \alpha \prec \beta \quad {\rm if} \quad t(\alpha) \preccurlyeq (\beta), \\ \alpha \preccurlyeq \beta \quad {\rm if} \quad \alpha = \beta \ {\rm or} \ \alpha \prec \beta \end{array} \right.$$

Let $\mathcal{F} = \{$ sources of $Q \}$. Note that the sources of the quiver Q are the minimal elements with the order \preccurlyeq . In addition, any vertex of the quiver Q is a source or one of the following



 $Q_0= \mathfrak{F}\cup V_0\cup V_1\cup V_2\cup V_3\cup V_4\cup V_5\cup V_6.$

This decomposition permit us to define the *type* of a vertex as:

$$\operatorname{type}(\mathfrak{i})\coloneqq\mathfrak{j}\quad \mathrm{if}\quad \mathfrak{i}\in V_{\mathfrak{j}},\ \mathfrak{j}=0,\ldots,6.$$

With the previous order we can define vectors associated to the vertexes and associated to the arrows, these are $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$ and $x = (x_1, \ldots, x_m) \in \mathbb{Z}^m$. In the next paragraph we describe the process of construction.

Step 0: Let $f \in \mathcal{F}$ and $gr_s(f) = 1$, that is there exists a unique $\beta \in Q_1$, with $s(\beta) = f$, we define $y_f = 1$ and $x_\beta = 1$.

If $f \in \mathcal{F}$ and $\operatorname{gr}_s(f) = 2$, that is there exist $\beta_1, \beta_2 \in Q_1$, with $s(\beta_1) = s(\beta_2) = f$, we define $y_f = 0$ and $x_{\beta_1} = x_{\beta_2} = 0$.

We introduce a counter $d \in \mathbb{N}$ starting in 0.

Step 1: Let $U = Q_0 - \mathcal{F}$ and $i \in U$ minimal.

- If type(i) = 0 we define $y_i = x_{\alpha}$, $x_{\beta} = y_i$ and the new set is $U := U \{i\}$.
- If type(i) = 1 we define $y_i = l_l(i)l_r(i) + 2 + x_\alpha 1$, $x_\beta = y_i$, where $l_r(i) = \max\{\log(w) : w \text{ path }, s(w) = i\}$ and $l_l(i) = \max\{\log(w) : w \text{ path }, t(w) = i\}$ and $U := U \{i\}$.
- $\bullet \ \mathrm{If} \ \mathrm{type}(\mathfrak{i})=2 \ \mathrm{we} \ \mathrm{define} \ y_\mathfrak{i}=x_{\alpha_1}, \ x_\beta=y_\mathfrak{i} \ \mathrm{and} \ U:=U-\{\mathfrak{i}\},$

$$\mathbf{d} := \mathbf{x}_{\alpha_2} - \mathbf{1} + \mathbf{d}$$

- If type(i) = 3 we define $y_i = x_\alpha$, $x_{\beta_2} = 0$, $x_{\beta_1} = x_\alpha$ and $U := U \{i\}$.
- $\bullet \ \mathrm{If} \ \mathrm{type}(\mathfrak{i})=4 \ \mathrm{we} \ \mathrm{define} \ y_\mathfrak{i}=y_{j_1}+y_{j_2}, \ x_{\beta_2}=x_{\alpha_1}, \ x_{\beta_1}=x_{\alpha_2} \ \mathrm{and} \ U:=U-\{\mathfrak{i}\}.$
- If type(i) = 5 we define $y_i = 0$ and $d := d + x_{\alpha_1} + x_{\alpha_2} \delta(\alpha_1) \delta(\alpha_2)$, where $\delta: Q_1 \to \mathbb{Z}$ is defined by $\delta(\alpha) = 0$ if $x_\alpha = 0$ and $\delta(\alpha) = 1$ if $x_\alpha \ge 1$.
- If type(i) = 6 we define $y_i = x_{\alpha}$ and $d = d + x_{\alpha}$.

We repeat this process recursiveness over the set U, and finally the number d is the total of nearly Frobenius structures that the algebra admits.

Corollary 4.9. Let A be a gentle algebra associated to Q, a finite connected quiver without oriented cycles. Then, A has finite Frobenius dimension

We apply the previous algorithm in the next two examples.

Example 4.1. The gentle algebra A associated to the quiver



with relations $\alpha_{9}\alpha_{6} = 0$, $\alpha_{8}\alpha_{1} = 0$, $\alpha_{2}\alpha_{7} = 0$ and $\alpha_{6}\alpha_{3} = 0$, has Frobdim(A) = 0, that is the only nearly Frobenius structure that this algebra admits is the trivial ($\Delta \equiv 0$).

Applying the algorithm we have the next situation



and the counter d is zero. Then, in this example, we have only the trivial nearly Frobenius coproduct.

Example 4.2. If we consider the algebra A associated to the quiver



with relations $\alpha_1 \alpha_2 = 0$, $\alpha_2 \alpha_3 = 0$, $\alpha_3 \alpha_4 = 0$ and $\alpha_{10} \alpha_5 = 0$. It is possible to determine all the nearly Frobenius structures, they are

$\Delta(e_1)$	=	$a_1 \alpha_1 \otimes e_1$
$\Delta(\alpha_1)$	=	$a_1 \alpha_1 \otimes \alpha_1$
$\Delta(e_2)$	=	$a_1e_2\otimes \alpha_1+a_2\alpha_2\otimes \alpha_1+a_3\alpha_2\alpha_9\otimes \alpha_1+a_4\alpha_2\alpha_9\alpha_{10}\otimes \alpha_1+a_5\alpha_2\alpha_9\alpha_{10}\alpha_4\otimes \alpha_1,$
Δ	=	0 on the other cases

In this case $\operatorname{Frobdim}(A) = 5$. We determine, applying the algorithm, that the counter d is five and we conclude that $\operatorname{Frobdim}(A) = 5$. In the next diagram we represent the vectors

given in the algorithm associated to the vertex and arrows



then d = 5.

4.3 Comparing dimensions

In this subsection we determine the Frobenius dimension of algebras associated to cyclic quivers. Using this result we exhibit a family of algebras with Frobenius dimension great that it's dimension over k.

Let $C(n_1, n_2, \ldots, n_m)$ the quiver



where $\mathfrak{m},\mathfrak{n}_1,\mathfrak{n}_2,\ldots,\mathfrak{n}_\mathfrak{m}\in\mathbb{N}^*$ and $A_C=\frac{\Bbbk C}{I_C}$ with

$$I_{C} = \big\langle \alpha_{n_{\mathfrak{m}}}^{\mathfrak{m}} \alpha_{1}^{\mathfrak{l}}, \ \alpha_{n_{\mathfrak{i}}}^{\mathfrak{i}} \alpha_{1}^{\mathfrak{i}+1}, \ \mathfrak{i} = 1, \dots, \mathfrak{m}-1 \big\rangle.$$

Theorem 4.10. Frobdim(A) = $m + \sum_{i=1}^{m} n_i n_{i+1}$, with $n_{m+1} = n_1$.

Proof. We will determine the coproduct in the vertices of the algebra A using the formula

$$\Delta(e_{\mathfrak{i}}) = \Delta(e_{\mathfrak{i}})(1 \otimes e_{\mathfrak{i}}) = (e_{\mathfrak{i}} \otimes 1)\Delta(e_{\mathfrak{i}}).$$

• If i = 1 then

$$\begin{split} \Delta(e_1) &= a_1 e_1 \otimes e_1 + \sum_{k=1}^{n_m} a_1^k e_1 \otimes \alpha_k^m \dots \alpha_{n_m}^m + \sum_{j=1}^{n_1} a_j^1 \alpha_1^j \dots \alpha_j^1 \otimes e_1 \\ &+ \sum_{j=1}^{n_1} \sum_{k=1}^{n_m} a_{jk}^1 \alpha_1^j \dots \alpha_j^1 \otimes \alpha_k^m \dots \alpha_{n_m}^m. \end{split}$$

• If the index is $n_1+\dots+n_i+1,\,i=1,\dots,m-1$ then

$$\begin{split} \Delta \big(e_{n_1 + \dots + n_i + 1} \big) &= a_i e_{n_1 + \dots + n_i + 1} \otimes e_{n_1 + \dots + n_i + 1} + \sum_{k=1}^{n_i} a_k^i e_{n_1 + \dots + n_i + 1} \otimes \alpha_k^i \dots \alpha_{n_i}^i \\ &+ \sum_{j=1}^{n_{i+1}} a_j^j \alpha_1^{i+1} \dots \alpha_j^{i+1} \otimes e_{n_1 + \dots + n_i + 1} + \sum_{j=1}^{n_i} \sum_{k=1}^{n_i} a_{jk}^i \alpha_1^{i+1} \dots \alpha_j^{i+1} \otimes \alpha_k^i \dots \alpha_{n_i}^i. \end{split}$$

• If $k=2,\ldots n_i$ then

$$\begin{split} \Delta \big(e_{n_1+\ldots n_{i-1}+k}\big) &= b_k^i e_{n_1+\ldots n_{i-1}+k} \otimes e_{n_1+\ldots n_{i-1}+k} + \sum_{l=1}^{k-1} b_l^{ik} e_{n_1+\ldots n_{i-1}+k} \otimes \alpha_l^i \ldots \alpha_{k-1}^i \\ &+ \sum_{j=k}^{n_i} b_k^{ij} \alpha_k^i \ldots \alpha_j^i \otimes e_{n_1+\ldots n_{i-1}+k} + \sum_{j=k}^{n_i} \sum_{l=1}^{k-1} b_j^{ik} \alpha_k^i \ldots \alpha_j^i \otimes \alpha_l^i \ldots \alpha_{k-1}^i. \end{split}$$

Now, we consider the maximal path $\alpha_1^i\ldots\alpha_{n_i}^i,$ for this the coproduct satisfies

$$\Delta(\alpha_1^{i}\dots\alpha_{n_i}^{i}) = \Delta(e_{n_1+\dots+n_{i-1}+1})(1\otimes\alpha_1^{i}\dots\alpha_{n_i}^{i}) = (\alpha_1^{i}\dots\alpha_{n_i}^{i}\otimes1)\Delta(e_{n_1+\dots+n_{i+1}})$$
(9)

By substitution in (9) we get:

$$a_{i-1}e_{n_1+\dots+n_{i-1}+1} \otimes \alpha_1^i \dots \alpha_{n_i}^i + \sum_{j=1}^{n_i} a_{i-1}^j \alpha_1^i \dots \alpha_j^i \otimes \alpha_1^i \dots \alpha_{n_i}^i$$

$$=$$

$$a_i \alpha_1^i \dots \alpha_{n_i}^i \otimes e_{n_1+\dots+n_i+1} + \sum_{k=1}^{n_i} a_k^i \alpha_1^i \dots \alpha_{n_i}^i \otimes \alpha_k^i \dots \alpha_{n_i}^i$$

then $a_i = a_{i-1} = 0$, $a_k^i = 0 \ \forall \ k = 2, \dots, n_i$, $a_{i-1}^j = 0 \ \forall \ j = 1, \dots, n_i - 1$ and $a_1^i = a_{i-1}^{n_i}$.

Therefore

$$\begin{array}{lll} \Delta \big(e_{n_1 + \dots + n_i + 1} \big) &=& a_i e_{n_1 + \dots + n_i + 1} \otimes \alpha_1^i \dots \alpha_{n_i}^i + a_{i+1} \alpha_1^{i+1} \dots \alpha_{n_{i+1}}^{i+1} \otimes e_{n_1 + \dots + n_i + 1} \\ &+& \sum_{j=1}^{n_{i+1}} \sum_{k=1}^{n_i} a_{jk}^i \alpha_1^{i+1} \dots \alpha_j^{i+1} \otimes \alpha_k^i \dots \alpha_{n_i}^i \end{array}$$

If we study the particular paths $\alpha_1^i \dots \alpha_{k-1}^i$ we can determine the relation between the coproduct values in the vertices $e_{n_1+\dots+n_{i-1}+1}$ and $e_{n_1+\dots+n_{i-1}+k}$.

The coproduct in $\alpha_1^i \dots \alpha_{k-1}^i$ satisfies

$$\Delta(\alpha_1^{i}\dots\alpha_{k-1}^{i}) = \Delta(e_{n_1+\dots+n_{i-1}+1})(1\otimes\alpha_1^{i}\dots\alpha_{k-1}^{i}) = (\alpha_1^{i}\dots\alpha_{k-1}^{i}\otimes1)\Delta(e_{n_1+\dots+n_{i-1}+k})$$

then

$$a_{i}\alpha_{1}^{i}\ldots\alpha_{n_{i}}^{i}\otimes\alpha_{1}^{i}\ldots\alpha_{k-1}^{i}$$

$$=$$

$$\sum_{k=1}^{k-1} i^{ik} i^{k-1}$$

$$b_k^i \alpha_1^i \dots \alpha_{k-1}^i \otimes e_{n_1 + \dots n_{i-1} + k} + \sum_{\substack{l=1 \\ n_i}}^{k-1} b_l^{ik} \alpha_1^i \dots \alpha_{k-1}^i \otimes \alpha_l^i \dots \alpha_{k-1}^i$$
$$+ \sum_{j=k}^{n_i} b_k^{ij} \alpha_1^i \dots \alpha_j^i \otimes e_{n_1 + \dots n_i + k} + \sum_{j=k}^{n_i} \sum_{l=1}^{k-1} b_j^{ik} \alpha_1^i \dots \alpha_j^i \otimes \alpha_l^i \dots \alpha_{k-1}^i$$

If we compare the expressions we conclude that $b_k^i = b_k^{ij} = b_l^{ik} = 0 \forall j, l, b_{jl}^{ik} = 0 \forall j = k, ..., n_i - 1, l = 2, ..., k - 1 and <math>b_{n_i l}^{ik} = a_i$ Consequently

$$\Delta(e_{n_1+\cdots+n_{i-1}+k}) = a_i \alpha_k^i \dots \alpha_{n_i}^i \otimes \alpha_1^i \dots \alpha_{k-1}^i.$$

The coassociativity is satisfied by a simple calculus. Then, Counting the independent coefficients we determine that A admits $m + \sum_{i=1}^{m} n_i n_{i+1}$ independent nearly-Frobenius structures.

 $\textbf{Corollary 4.11. } \textit{If } n_1 = n_2 = \dots = n_m = t, \textit{ with } t \geq 3, \textit{ then } \mathrm{Frobdim}(A) > \dim_\Bbbk(A).$

Proof. In this case the dimension of A, as vector space, is

$$\dim_{\Bbbk}(\mathsf{A}) = \frac{\mathsf{m}(\mathsf{t}^2 + 3\mathsf{t})}{2}$$

and $\operatorname{Frobdim}(A) = \mathfrak{m}(1+t^2)$. If we compare these expressions we conclude that $\operatorname{Frobdim}(A) > \dim_{\Bbbk}(A)$ if t > 2.

References

- [ASS06] I. Assem, D. Simson, and A. Skowronski, Elements of the Representation Theory of Associative Algebras Volume 1 Techniques of Representation Theory, Cambridge University Press, 2006.
- [CG04] Ralph L. Cohen and Veronique Godin, A polarized view of string topology, Topology, geometry, and quantum field theory, Cambridge: Cambridge University Press. London Mathematical Society Lecture Notes 308 (2004), 127-154.
- [Gab72] P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972), 71-103.

- [Gab73] P. Gabriel, Indecomposable representations II, Symposia Mat. Inst. Naz. Alta Mat.11 (1973), 81-104.
- [GLSU13] A. González, E. Lupercio, C. Segovia, and B. Uribe, Orbifold Topological Quantum Field Theories in Dimension 2, Book finished, 2013.
- [Lév04] Jessica Lévesque, *Produits Fibrés d'algebres et inclinaison*, Ph.D. thesis, Faculté des Sciencies, Université de Sherbrooke, 2004.