MONOTONE TWO-SCALE METHODS FOR A CLASS OF INTEGRODIFFERENTIAL OPERATORS AND APPLICATIONS

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Abstract. We develop a monotone, two-scale discretization for a class of integrodifferential operators of order $2s$, $s \in (0,1)$. We apply it to develop numerical schemes, and convergence rates, for linear and obstacle problems governed by such operators. As applications of the monotonicity, we provide error estimates for free boundaries and a convergent numerical scheme for a concave fully nonlinear, nonlocal, problem.

1. Introduction

In recent times, nonlocal models have gained a lot of popularity in the pure and applied sciences. The reason behind this boom is manifold. From the point of view of applications, it is claimed that they are able to encode a wider range of phenomena when compared to their local counterparts. In this regard, for instance, one can refer to the creation of peridynamics [65, 66], nonlocal diffusion reaction equations [70], fractional Cahn-Hilliard models [4, 3], fractional porous media equations [31], the fractional Schrödinger equation [68], fractional viscoelasticity [26], fractional Monge-Ampère [18] and many more. The interested reader can refer to the many existing overviews [12, 48, 32] for further references and insight.

In our opinion, nonlocal models started to gain the attention of the mathematical community after the seminal work [20], where the authors showed that the so-called fractional Laplacian (in the whole space) can be realized as a degenerate elliptic operator in one more dimension. With this, many of the techniques that were used to analyze local problems could now be applied to nonlocal ones. Later, purely nonlocal techniques for many problems were also introduced; see, for instance, [19, 45].

Regarding the numerical treatment of nonlocal problems, some early attempts could be found in [41, 42, 43, 34, 33]. However, references [56, 14, 13, 1] deserve

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special attention. In [56] the extension technique of [20] was exploited to develop a numerical method, and its analysis, for the so-called *spectral* fractional Laplacian. References [14, 13] for the spectral and integral fractional Laplacian, respectively, develop a nonconforming approximation on the basis of an integral representation, a quadrature formula, and spatial discretization. Finally, closer to our discussion here is [1] where the *integral fractional Laplacian* is considered. On the basis of weighted Hölder regularity results developed in [61], the authors of this work provide weighted Sobolev regularity, and construct a direct discretization over graded meshes that, in two dimensions, is optimal with respect to degrees of freedom.

The error estimates of [1, 15] are in the energy and $L^2(\Omega)$–norms. Pointwise error estimates for the integral fractional Laplacian were obtained in [39], where a monotone two-scale method for this operator was developed. The error estimates in this work were based on weighted Hölder regularity and the construction of suitable barriers.

Two-scale methods, such as those developed in [58, 46, 52, 54] and [39, 40] naturally inherit a discrete maximum principle from the continuous operator. This is in general not true for finite element approximations. For instance, the finite element approximation of the classical Laplacian possesses a discrete maximum principle only under certain geometric constraints on the mesh [29, Section III.20]. We remark that for nonlocal operators, the lack of monotonicity for the finite element approximation is not expected to be fixed by only a geometric constraint as for its local counterpart, especially when $s \approx 0$, where the stiffness matrix approaches a standard mass matrix.

This brings us then to the main goals of the present work. We will consider an integrodifferential operator of order $2s$, $s \in (0,1)$,

$$\mathcal{L}_\eta[w](x) = \text{v.p.} \int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x-y|^{d+2s}} \eta\left(\frac{x-y}{|x-y|}\right) \mathrm{d}y, \tag{1}$$

where the so-called *kernel* $\eta : \mathbb{S}^{d-1} \to \mathbb{R}$ is assumed to verify several properties that will be specified below. Operators of this form have been extensively studied in probability and finance, as they represent the generator of a $2s$–stable Lévy processes, see e.g., [30, 7]. The simplest, and most important, example is the aforementioned integral fractional Laplacian $(-\Delta)^s$, which is defined by

$$\eta(\theta) \equiv C(d,s) = \frac{4^s s \Gamma(s + d/2)}{\pi^{d/2} \Gamma(1-s)} > 0.$$

In [39], the authors propose a monotone discretization for the fractional Laplacian based on the splitting of the integral in the definition of the operator into a singular and a tail parts; the former is approximated by a scaled finite-difference Laplacian. This, in essence, amounts to truncating the kernel in a neighborhood of the origin. Inspired by that work, for a broader class of operators with kernel $\eta \in \mathcal{C}(\lambda, \Lambda)$ (cf. Definition 1), we develop a monotone, two-scale method for the discretization of the operator (1) and show its consistency under weighted Hölder regularity assumptions. Instead of truncation, we propose to *regularize* the kernel in (1) near the origin. From an analysis perspective, this means that we are using a zero-order operator for the approximation of the singular part of the integral. Our motivation is to obtain approximations of (1) that are robust in the limit $s \uparrow 1$, which may require wide stencils.

As applications of the proposed scheme, we then consider three applications of increasing difficulty. First we obtain pointwise rates of convergence for a linear problem where the operator is of the form (1). Next, we consider an obstacle problem for (1) and obtain pointwise rates of convergence. These pointwise rates of convergence, in turn, allow us to provide error estimates for free boundaries. Finally, we develop a numerical scheme for a class of concave fully nonlinear integrodifferential operators of order $2s$. Rates of convergence for problems of this type, however, are at this stage beyond our reach. We need to point out that the rates of convergence are obtained under realistic regularity assumptions. Namely, up to the boundary C^s regularity, and interior C^β regularity, with $\beta \in (2s, 4)$. In the linear case, the interior regularity is solely dictated by the regularity of the forcing function, whereas the obstacle problem exhibits a reduced regularity of $\beta \leq 1 + s$.

Our presentation is organized as follows. Notation and the functional framework we shall operate in is presented in Section 2. There we also introduce the class of integrodifferential operators we shall be interested in, and their most elementary properties. The two-scale discretization of our operators is introduced in Section 3. We describe the action of each one of the scales, namely regularization and discretization, and study the consistency of approximations. The first application of our two-scale discretization is the content of Section 4, where we study a linear problem and provide pointwise rates of convergence for its numerical scheme. These rates of convergence are obtained under realistic regularity assumptions. Next, in Section 5, we study an obstacle problem. We propose and analyze a pointwise convergent scheme. The rates are, once again, obtained under realistic regularity assumptions. We then continue, in Sections 5.2 and 5.3, with the study of the approximation of free boundaries. We provide a rate of convergence for discrete boundaries, both in the presence of regular and singular points. As a final application of our scheme, Section 6 studies a concave fully nonlinear integrodifferential equation. We propose a convergent scheme, albeit without explicit rates of convergence. Finally, Appendix A provides some intuition, justification, and consequences for our design choices behind the regularization scale.

2. Notation and preliminaries

Let us introduce some notation and terminology that will be used throughout our discussion. For $A, B \in \mathbb{R}$ the relation $A \lesssim B$ means that, for a nonessential constant c, we have $A \leq cB$. The value of this constant may change in every occurrence. $A \gtrsim B$ means $B \lesssim A$. If $A \lesssim B \lesssim A$ we abbreviate this by saying $A \approx B$. The Landau symbols, big-O and little-o, respectively, are $\mathcal{O}$ and $\mathfrak{o}$.

For $r > 0$ and $x \in \mathbb{R}^d$ we denote by $B_r(x)$ the (open) Euclidean ball of radius r centered at x. We set $B_r = B_r(0)$. The unit sphere in $\mathbb{R}^d$ is $\mathbb{S}^{d-1} = \partial B_1$.

Throughout our discussion $\Omega \subset \mathbb{R}^d$, with $d \in \mathbb{N}$, is a bounded Lipschitz domain which we assume satisfies an exterior ball condition. We denote its boundary by $\partial\Omega$ and its complement by $\Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$.

For $x \in \Omega$ we define $\delta(x) = \operatorname{dist}(x, \partial\Omega)$ and if $x, y \in \Omega$, then we set $\delta(x, y) = \min\{\delta(x), \delta(y)\}$.

2.1. Function spaces and their norms.

We will adhere to standard notation for the Lebesgue spaces, and their norms, when they are defined either over the whole space or some domain. Since we are concerned with pointwise estimates, we must work with functions that are at least continuous. The space of functions $w : \bar{\Omega} \to \mathbb{R}$

that are continuous is denoted by $C(\bar\Omega)$. We recall that this space endowed with the norm

$$\|w\|_{C(\bar\Omega)} = \sup_{x\in\bar\Omega} |w(x)|,$$

is a Banach space. We also need spaces of continuously differentiable functions. For $k \in \mathbb{N}$ we set

$$C^k(\bar\Omega) = \left\{ w : \bar\Omega \to \mathbb{R} \;\middle|\; D^\beta w \in C(\bar\Omega), |\beta| \le k \right\}.$$

We also set $C^0(\bar\Omega) = C(\bar\Omega)$. The norm on these spaces is

$$\|w\|_{C^k(\bar\Omega)} = \sup_{|\beta|\le k} \|D^\beta w\|_{C(\bar\Omega)}.$$

To provide a more refined characterization of smoothness, for $k \in \mathbb{N}_0$, and $\alpha \in (0,1]$ we define the Hölder spaces via

$$C^{k,\alpha}(\bar\Omega) = \left\{ w \in C^k(\bar\Omega) \;\middle|\; [D^\beta w]_{C^{0,\alpha}(\bar\Omega)} < \infty, |\beta| = k \right\},$$

where

$$[w]_{C^{0,\alpha}(\bar\Omega)} = \sup_{x,y\in\bar\Omega: x\neq y} \frac{|w(x)-w(y)|}{|x-y|^\alpha}.$$

The norm in the Hölder spaces is

$$\|w\|_{C^{k,\alpha}(\bar\Omega)} = \max\left\{ \|w\|_{C^k(\bar\Omega)}, \max_{|\beta|=k} [D^\beta w]_{C^{0,\alpha}(\bar\Omega)} \right\}.$$

These spaces are complete.

We set $C^{k,0}(\bar\Omega) = C^k(\bar\Omega)$. In addition, whenever $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$ with $k+\alpha \notin \mathbb{N}_0$, we may denote $C^{k+\alpha}(\bar\Omega) = C^{k,\alpha}(\bar\Omega)$. Finally, for $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$ we will say that $w \in C^{k,\alpha}(\Omega)$ if $w_{|\bar U} \in C^{k,\alpha}(\bar U)$ for all $U \Subset \Omega$. A further refinement of these spaces will be detailed when needed.

We will also deal with fractional Sobolev spaces. For $r \in (0,1)$ we set

$$H^r(\mathbb{R}^d) = \left\{ w \in L^2(\mathbb{R}^d) \;\middle|\; \|w\|_{H^r(\mathbb{R}^d)} < \infty \right\},$$

with

$$\|w\|^2_{H^r(\mathbb{R}^d)} = \|w\|^2_{L^2(\mathbb{R}^d)} + [w]^2_{H^r(\mathbb{R}^d)},$$
$$[w]^2_{H^r(\mathbb{R}^d)} = \frac{C(d,r)}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|w(x)-w(y)|^2}{|x-y|^{d+2s}} \,\mathrm{d}x\,\mathrm{d}y,$$

for which it is a Hilbert space.

We also set, again for $r \in (0,1)$,

$$\widetilde H^r(\Omega) = \left\{ w \in L^2(\Omega) \;\middle|\; \widetilde w \in H^r(\mathbb{R}^d) \right\},$$

where by $\widetilde w$ we denote the extension by zero onto Ω^c. Owing to the fractional Poincaré inequality,

$$\|w\|_{L^2(\Omega)} \lesssim [\widetilde w]_{H^r(\mathbb{R}^d)}, \quad \forall w \in \widetilde H^r(\Omega),$$

the seminorm $[\cdot]_{H^r(\mathbb{R}^d)}$ is actually an equivalent norm in $\widetilde H^r(\Omega)$ which makes it Hilbert. The dual of $\widetilde H^r(\Omega)$ is denoted by $H^{-r}(\Omega)$, and the duality pairing will be $\langle\cdot,\cdot\rangle$. In what follows, if confusion does not arise, we shall suppress the explicit mention of zero extensions.

2.2. **The integrodifferential operator.** We shall consider integrodifferential operators of the form (1). Regarding the kernel, we encode its assumptions in the following definition.

Definition 1 (class $\mathcal{C}(\lambda,\Lambda)$)**.** Let $\lambda,\Lambda \in (0,\infty)$ with $\lambda \le \Lambda$. We will say that the kernel $\eta : \mathbb{S}^{d-1} \to \mathbb{R}$ belongs to the class $\mathcal{C}(\lambda,\Lambda)$ if it is:
1. Symmetric, i.e., $\eta(\theta) = \eta(-\theta)$ for all $\theta \in \mathbb{S}^{d-1}$.
2. Elliptic, i.e., we have

$$0 < \lambda \le \eta(\theta) \le \Lambda, \quad \text{a.e.} \quad \theta \in \mathbb{S}^{d-1}.$$

Owing to these assumptions, if $\eta \in \mathcal{C}(\lambda,\Lambda)$, a fractional integration by parts shows that

$$[w]^2_{H^s(\mathbb{R}^d)} \approx \langle \mathcal{L}_\eta[w], w\rangle, \quad \forall w \in \widetilde{H}^s(\Omega),$$

with equivalence constants that depend on η only through the constants λ and Λ. In other words, for every $\eta \in \mathcal{C}(\lambda,\Lambda)$ the expression

$$\|w\|^2_{\eta,s} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|w(x)-w(y)|^2}{|x-y|^{d+2s}} \eta\left(\frac{x-y}{|x-y|}\right) \mathrm{d}y \, \mathrm{d}x,$$

is an equivalent Hilbertian norm on $\widetilde{H}^s(\Omega)$.

Owing to the positivity of the kernel, the operator (1) satisfies a comparison principle.

Proposition 2 (comparison)**.** *Let $K\colon \mathbb{R}^d \to \mathbb{R}$ be a positive function. Define the operator $\mathcal{L}_K$ via*

$$\mathcal{L}_K[w](x) = \text{v.p.} \int_{\mathbb{R}^d} (w(x)-w(y))\, K(x-y)\, \mathrm{d}y.$$

Let $w\colon \mathbb{R}^d \to \mathbb{R}$ be such that, in the weak sense, $\mathcal{L}_K[w] \ge 0$ almost everywhere in Ω and $w \ge 0$ in Ω^c. Then, $w \ge 0$ in Ω.

Proof. See [60, Proposition 4.1]. □

As a final piece of preliminary notation we must introduce a set of parameters, relations between them, and a space of functions with a prescribed behavior away from the boundary. We let $\beta \in \mathbb{R}$, and assume it satisfies

$$\beta > 2s, \qquad \beta, \beta - 2s \notin \mathbb{N}_0. \tag{2}$$

Given such β we define

$$\mathcal{S}^\beta(\Omega) = \left\{ w \in C^\beta(\Omega) \cap C^{0,s}(\bar{\Omega}) \;\middle|\; \|w\|_{C^\beta(\{x\in\Omega \,\mid\, \delta(x)\ge\rho\})} \lesssim \rho^{s-\beta} \right\}. \tag{3}$$

The usefulness of this function class shall become clear once we perform the error analysis. Indeed, as we shall show (cf. Lemma 11 below), solutions of problems involving the operator $\mathcal{L}_\eta$ typically belong to such class.

3. Two-scale discretization

We now begin the discretization of our integrodifferential operator (1). Besides consistency, a fundamental necessity, we wish to preserve its comparison property detailed in Proposition 2.

These two requirements stand at odds of each other. For instance, when dealing with (local) elliptic second order differential operators and their finite element discretization, very stringent mesh requirements must be imposed to retain a comparison principle, see [29, Section III.20] and [49, Section 3.5]. If, on the contrary, we discretize via finite differences, it is known that wide stencils must be employed, see [49, Section 3.2]. To fulfill these two conditions then, we will employ a two-scale discretization.

Two-scale discretizations have become a popular choice to develop schemes that preserve the comparison principle for (local) elliptic second order differential operators. As an example, the reader is referred to [58, 46, 52, 54]. Regarding the two-scale discretization of nonlocal problems we mention [39, 40], where the authors proposed a two-scale discretization for the fractional Laplacian.

3.1. **The regularization scale.** The first step in the approximation of (1) is regularization. The idea is that, for some $\varepsilon > 0$, we split the integral that defines the operator in two parts: $\mathbb{R}^d \setminus B_\varepsilon$ and B_ε. The integral in the small ball B_ε is then regularized by introducing a non singular kernel with suitable approximation properties.

We begin by observing that, since the kernel η is symmetric, we have

$$\mathcal{L}_\eta[w](x) = \frac{1}{2}\int_{\mathbb{R}^d} \left(2w(x) - w(x-y) - w(x+y)\right)\eta\left(\frac{y}{|y|}\right)\mathcal{K}(|y|)\,\mathrm{d}y, \tag{4}$$

where $\mathcal{K}(r) = \frac{1}{r^{d+2s}}$. Let now $\varepsilon > 0$. We define

$$\mathcal{L}_{\eta,\varepsilon}[w](x) = \frac{1}{2}\int_{\mathbb{R}^d} \left(2w(x) - w(x-y) - w(x+y)\right)\eta\left(\frac{y}{|y|}\right)\mathcal{K}_\varepsilon(|y|)\,\mathrm{d}y,$$

where the radial, nonsingular kernel $\mathcal{K}_\varepsilon$ is defined as

$$\mathcal{K}_\varepsilon(r) = \begin{cases} \dfrac{1}{r^{d+2s}}, & r \geq \varepsilon, \\ \dfrac{1}{\varepsilon^{d+2s}} + \gamma(r^2 - \varepsilon^2) + \nu(r^3 - \varepsilon^3), & r < \varepsilon. \end{cases} \tag{5}$$

We point out that (5) gives

$$\lim_{r\uparrow\varepsilon} \mathcal{K}_\varepsilon(\varepsilon) = \varepsilon^{-d-2s}, \qquad \lim_{t\downarrow 0} \frac{\mathcal{K}_\varepsilon(t) - \mathcal{K}_\varepsilon(0)}{t} = 0.$$

The constants γ and ν are chosen so that $\mathcal{K}_\varepsilon \in C^1([0,\infty))$ and

$$\mathcal{L}_\eta[q] = \mathcal{L}_{\eta,\varepsilon}[q], \quad \forall q \in \mathbb{P}_2.$$

The smoothness requirement implies that

$$2\gamma\varepsilon + 3\nu\varepsilon^2 = -(d+2s)\varepsilon^{-d-2s-1}.$$

On the other hand, by the symmetry of η we will have exactness provided that

$$\int_0^\varepsilon \frac{r^2}{r^{d+2s}} r^{d-1}\,\mathrm{d}r = \int_0^\varepsilon r^2 \mathcal{K}_\varepsilon(r) r^{d-1}\,\mathrm{d}r.$$

In short, the parameters γ and ν have the values

$$\gamma = -\frac{(4+d)(d+2s)(3+d+2s)}{4(1-s)}\varepsilon^{-d-2s-2} < 0,$$
$$\nu = \frac{(5+d)(d+2s)(2+d+2s)}{6(1-s)}\varepsilon^{-d-2s-3} > 0.$$

The following result studies the interior consistency of this regularization. By interior we mean that we consider points $x \in \Omega$, such that $B_\varepsilon(x) \subset \Omega$, i.e., all the points where the regularization of the kernel takes place are contained in Ω.

Theorem 3 (interior consistency)**.** *Let Ω be a bounded Lipschitz domain that satisfies the exterior ball condition. Assume that $\beta \in (2s, 4]$ and $\eta \in \mathcal{C}(\lambda, \Lambda)$. Let $w \in \mathcal{S}^\beta(\Omega)$, where this class is defined in (3). Let $\alpha_0 > 1$, $\varepsilon > 0$, and $x \in \Omega$ be such that $\delta(x) \geq \alpha_0 \varepsilon$. Then, it holds that*

$$|\mathcal{L}_\eta[w](x) - \mathcal{L}_{\eta,\varepsilon}[w](x)| \lesssim \varepsilon^{\beta-2s}\delta(x)^{s-\beta}.$$

The implicit constant depends on d, s, α_0, β, λ, and Λ.

Proof. We need to estimate

$$\begin{aligned}
&|\mathcal{L}_\eta[w](x) - \mathcal{L}_{\eta,\varepsilon}[w](x)| = \\
&\frac{1}{2}\left|\int_{B_\varepsilon} (2w(x) - w(x-y) - w(x+y))\,\eta\left(\frac{y}{|y|}\right)\left(\frac{1}{|y|^{d+2s}} - \mathcal{K}_\varepsilon(|y|)\right)\mathrm{d}y\right| \lesssim \\
&\qquad \int_0^\varepsilon \|w\|_{C^\beta(B_r(x))} r^\beta \left|\frac{1}{r^{d+2s}} - \mathcal{K}_\varepsilon(r)\right| r^{d-1}\,\mathrm{d}r \\
&\qquad\qquad \lesssim \int_0^\varepsilon (\delta(x) - r)^{s-\beta}\left|\frac{1}{r^{d+2s}} - \mathcal{K}_\varepsilon(r)\right| r^{\beta+d-1}\,\mathrm{d}r,
\end{aligned}$$

where, in the last step, we used the interior Hölder estimate that is assumed of w. Because $\delta(x) \geq \alpha_0\varepsilon$,

$$\begin{aligned}
&\int_0^\varepsilon (\delta(x) - r)^{s-\beta}\left|\frac{1}{r^{d+2s}} - \mathcal{K}_\varepsilon(r)\right| r^{\beta+d-1}\,\mathrm{d}r \lesssim \\
&\qquad \delta(x)^{s-\beta}\int_0^\varepsilon r^{\beta+d-1}\left(\frac{1}{r^{d+2s}} + \mathcal{K}_\varepsilon(r)\right)\mathrm{d}r \lesssim \varepsilon^{\beta-2s}\delta(x)^{s-\beta}. \quad \square
\end{aligned}$$

As it can be seen from the proof of Theorem 3, the interior consistency of our regularization depends on how close we are to the boundary. In particular, we need to have $B_\varepsilon(x) \Subset \Omega$. For this reason, given $\varepsilon > 0$ we let $\boldsymbol{\varepsilon} : \Omega \to (0, \infty)$ be sufficiently smooth and satisfy

$$\boldsymbol{\varepsilon}(x) \leq \min\left\{\frac{\delta(x)}{2}, \varepsilon\right\}, \quad \forall x \in \Omega.$$

With this function at hand, we then define

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w](x) = \mathcal{L}_{\eta,\boldsymbol{\varepsilon}(x)}[w](x).$$

In order to properly leverage the boundary regularity of solutions, the choice of $\boldsymbol{\varepsilon}$ is made more precise below; depending on the problem under consideration.

Owing to the positivity of $\mathcal{K}_\varepsilon$, the comparison principle from Proposition 2 holds for $\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}$. Thus, pointwise consistency estimates for this operator can be achieved by combining a comparison principle with a suitable barrier function. We thus construct a barrier.

Lemma 4 (barrier)**.** *Let Ω be a bounded, Lipschitz domain. Define*

$$b(x) = \chi_\Omega(x), \quad \forall x \in \Omega.$$

Then,

$$\delta(x)^{-2s} \lesssim \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](x).$$

Proof. Since, by construction, we have $\boldsymbol{\varepsilon}(x) \leq \delta(x)$ for all $x \in \Omega$ we may write

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](x) = \int_{\mathbb{R}^d} (1 - b(y))\, \eta \left(\frac{x-y}{|x-y|} \right) \mathcal{K}_{\boldsymbol{\varepsilon}(x)}(|x-y|) \,\mathrm{d}y \geq \lambda \int_{\Omega^c} \frac{1}{|x-y|^{d+2s}} \,\mathrm{d}y.$$

Since

$$\delta(x)^{-2s} \lesssim \int_{\Omega^c} \frac{1}{|x-y|^{d+2s}} \,\mathrm{d}y,$$

with a hidden constant that depends on d, s, and Ω, the result follows. $\square$

We are now in position to estimate the consistency of our regularized operator.

Theorem 5 (consistency)**.** *In the setting of Theorem 3 assume, in addition, that* $w, w_\varepsilon \in \mathcal{S}^\beta(\Omega)$ *verify*

$$\mathcal{L}_\eta[w] = \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w_\varepsilon] \quad \text{in } \Omega, \qquad w = w_\varepsilon = 0, \quad \text{in } \Omega^c.$$

Then we have

$$\|w - w_\varepsilon\|_{L^\infty(\Omega)} \lesssim \max\{\varepsilon^s, \varepsilon^{\beta-2s}\}.$$

Proof. Let $x \in \Omega$. Since $\boldsymbol{\varepsilon}(x) \leq \frac{1}{2}\delta(x)$, arguing as in the proof of Theorem 3 we get

$$\begin{aligned}
\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w - w_\varepsilon](x) &= \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w](x) - \mathcal{L}_\eta[w](x) \lesssim \|w\|_{C^\beta(B_{\boldsymbol{\varepsilon}(x)}(x))} \boldsymbol{\varepsilon}(x)^{\beta-2s} \\
&\lesssim (\delta(x) - \boldsymbol{\varepsilon}(x))^{s-\beta} \boldsymbol{\varepsilon}(x)^{\beta-2s} \lesssim 2^{s-\beta} \delta(x)^{s-\beta} \boldsymbol{\varepsilon}(x)^{\beta-2s} \\
&\lesssim \delta(x)^{3s-\beta} \boldsymbol{\varepsilon}(x)^{\beta-2s} \delta(x)^{-2s}.
\end{aligned}$$

If $3s - \beta \geq 0$ continue the estimate as

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w - w_\varepsilon](x) \lesssim \varepsilon^{\beta-2s} \delta(x)^{-2s} \lesssim \varepsilon^{\beta-2s} \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](x),$$

where we used the barrier function of Lemma 4. If, on the other hand, $3s - \beta < 0$ we use that

$$\delta(x)^{3s-\beta} \boldsymbol{\varepsilon}(x)^{\beta-2s} = \left(\frac{\boldsymbol{\varepsilon}(x)}{\delta(x)} \right)^{\beta-3s} \boldsymbol{\varepsilon}(x)^s \lesssim \varepsilon^s.$$

Gathering both cases we conclude that, for every $x \in \Omega$, we have

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w - w_\varepsilon](x) \lesssim \max\{\varepsilon^{\beta-2s}, \varepsilon^s\} \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](x).$$

An application of the comparison principle for the operator $\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}$ allows us to conclude. $\square$

3.2. **The discretization scale.** We assume that Ω is a convex polytope and let $\mathscr{T} = \{T\}$ be a conforming and shape regular simplicial triangulation of Ω. The elements $T \in \mathscr{T}$ are assumed to be closed. We set, for $T \in \mathscr{T}$, $h_T = \mathrm{diam}(T)$. We denote by $\mathscr{N}_\mathscr{T}$ the set of vertices of $\mathscr{T}$. The interior and boundary vertices are, respectively,

$$\mathscr{N}_\mathscr{T}^i = \mathscr{N}_\mathscr{T} \cap \Omega, \qquad \mathscr{N}_\mathscr{T}^\partial = \mathscr{N}_\mathscr{T} \cap \partial\Omega.$$

For each interior vertex $z \in \mathscr{N}_\mathscr{T}^i$ we define its patch to be

$$\omega_z = \bigcup \{ T \in \mathscr{T} \mid z \in T \}.$$

By h_z we denote the radius of the ball of maximal radius, centered at z, that can be inscribed in ω_z. Over such a triangulation we define the following spaces of functions

$$\begin{aligned}
\mathbb{V}_\mathscr{T} &= \left\{ w_\mathscr{T} \in C(\bar{\Omega}) \;\middle|\; w_{\mathscr{T}|T} \in \mathbb{P}_1, \ \forall T \in \mathscr{T} \right\}, \\
\mathbb{V}_\mathscr{T}^0 &= \left\{ w_\mathscr{T} \in C(\bar{\Omega}) \;\middle|\; w_{\mathscr{T}|\partial\Omega} = 0 \right\},
\end{aligned}$$

Notice that any function $w_{\mathscr{T}} \in \mathbb{V}^0_{\mathscr{T}}$ can be trivially extended to Ω^c by zero. When this causes no confusion, we shall not make a distinction between a function and its extension.

It is a general fact that solutions to problems involving integrodifferential operators, like (1), exhibit an algebraically singular behavior near the boundary, independently of the smoothness of the problem data. The problems that we shall be interested in are no exception; see the regularity results of Sections 4.1 and 5. To compensate this we will consider a mesh that is graded towards the boundary as it was studied in [1]. We consider a mesh size $h > 0$ and parameter $\mu \geq 1$. Our mesh $\mathscr{T}$ is assumed to satisfy

$$h_T \approx \begin{cases} h^{\mu}, & T \cap \partial\Omega \neq \emptyset, \\ h \operatorname{dist}(T, \partial\Omega)^{\frac{\mu-1}{\mu}} & T \cap \partial\Omega = \emptyset. \end{cases} \tag{6}$$

As shown in [17, Remark 4.14] we have that

$$\dim \mathbb{V}^0_{\mathscr{T}} \approx \begin{cases} h^{(1-d)\mu}, & \mu \geq \dfrac{d}{d-1}, \\ h^{-d}|\log h|, & \mu = \dfrac{d}{d-1}, \\ h^{-d}, & \mu < \dfrac{d}{d-1}. \end{cases} \tag{7}$$

Finally, we observe that, under the condition (6), we have

$$h_z \approx h\delta(z)^{1-1/\mu} \quad \forall z \in \mathscr{N}^i_{\mathscr{T}}. \tag{8}$$

At this point we impose that, at least, $\frac{1}{2}h_z \leq \boldsymbol{\varepsilon}(z)$. We shall later refine this choice; see Definition 12 and 20 for the linear and obstacle problems, respectively.

3.2.1. *Consistency of interpolation.* Let $I_h : C(\bar{\Omega}) \to \mathbb{V}_{\mathscr{T}}$ be the Lagrange interpolation operator. For $z \in \mathscr{N}^i_{\mathscr{T}}$ we wish to estimate the consistency error

$$\mathcal{E}[w, z] = \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h w](z) - \mathcal{L}_{\eta}[w](z),$$

provided the function w possesses suitable, but realistic, smoothness. We begin by rewriting this error as

$$\mathcal{E}[w, z] = (\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w](z) - \mathcal{L}_{\eta}[w](z)) + \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h w - w](z).$$

The first term entails the regularization error, and it was estimated in Theorem 3. Our immediate goal shall be to estimate the second term.

Lemma 6 (refined interpolation estimate)**.** *Let* $\beta > 0$ *satisfy* (2), $\bar{\beta} = \min\{\beta, 2\}$, *and* $w \in \mathcal{S}^{\beta}(\Omega)$. *Furthermore, assume the mesh* $\mathscr{T}$ *satisfies* (6). *Then, for all* $T \in \mathscr{T}$, *we have*

$$\|w - I_h w\|_{L^\infty(T)} \lesssim \begin{cases} h^{\bar{\beta}} \operatorname{dist}(T, \Omega)^{s-\bar{\beta}/\mu}, & T \cap \partial\Omega = \emptyset, \\ h^{\mu s}, & T \cap \partial\Omega \neq \emptyset. \end{cases}$$

Consequently, we have the global interpolation estimate

$$\|w - I_h w\|_{L^\infty(\Omega)} \lesssim \max\{h^{\mu s}, h^{\bar{\beta}}\}. \tag{9}$$

Proof. If $w \in C^{\alpha}(T)$ for some $\alpha \in (0,2]$, we have
$$\|w - I_h w\|_{L^\infty(T)} \lesssim \|w\|_{C^\alpha(T)} h_T^\alpha.$$
Now, if $T \cap \partial\Omega = \emptyset$ the definition of the class $\mathcal{S}^\beta(\Omega)$, given in (3), and the mesh grading imply that
$$\|w - I_h w\|_{L^\infty(T)} \lesssim \operatorname{dist}(T,\partial\Omega)^{s-\bar\beta} h^{\bar\beta} \operatorname{dist}(T,\partial\Omega)^{\bar\beta(\mu-1)/\mu} \lesssim h^{\bar\beta} \operatorname{dist}(T,\Omega)^{s-\bar\beta/\mu}.$$
If, on the contrary, $T \cap \partial\Omega \neq \emptyset$, we estimate
$$\|w - I_h w\|_{L^\infty(T)} \lesssim \|w\|_{C^{0,s}(T)} h_T^s \lesssim h^{\mu s}.$$

Estimate (9) follows from a closer inspection of the case $T \cap \partial\Omega = \emptyset$. If $s \geq \bar\beta/\mu$ then there is nothing to prove, so we assume $s < \bar\beta/\mu$. Let $z \in \mathscr{N}_{\mathscr{T}}^i$ be a vertex of T. By shape regularity and (8), we have $h_T \approx h_z \approx h\delta(z)^{1-1/\mu}$, and $\operatorname{dist}(T,\partial\Omega) \approx \delta(z)$. Therefore, we can write
$$\|w - I_h w\|_{L^\infty(T)} \lesssim h^{\mu s} h^{\bar\beta - \mu s} \delta(z)^{s-\bar\beta/\mu} \approx h^{\mu s} \left(\frac{h_z}{\delta(z)}\right)^{\bar\beta-\mu s} \lesssim h^{\mu s}. \qquad \square$$

We can now estimate the remaining consistency term.

Proposition 7 (consistency of interpolation)**.** *Let the function $\boldsymbol{\varepsilon} : \Omega \to \mathbb{R}$ be such that $\frac{1}{2}h_z \leq \boldsymbol{\varepsilon}(z) \leq \frac{1}{2}\delta(z)$ for all $z \in \mathscr{N}_{\mathscr{T}}^i$. In the setting of Lemma 6 we have, for all $z \in \mathscr{N}_{\mathscr{T}}^i$,*
$$|\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h w - w](z)| \lesssim h^{\bar\beta} \delta(z)^{s-\bar\beta/\mu} \boldsymbol{\varepsilon}(z)^{-2s} + \max\{h^{\mu s}, h^{\bar\beta}\} \delta(z)^{-2s}. \tag{10}$$

Proof. Let $z \in \mathscr{N}_{\mathscr{T}}^i$. We observe that $I_h w(z) = w(z)$ and that $I_h w \equiv w \equiv 0$ on Ω^c. Therefore, we have
$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h w - w](z) = \int_\Omega (w(y) - I_h w(y))\, \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \eta\left(\frac{z-y}{|z-y|}\right) \mathrm{d}y$$
We define
$$\Omega^\partial = \bigcup \{T \in \mathscr{T} \mid T \cap \partial\Omega \neq \emptyset\}, \tag{11}$$
and partition the integration domain into
$$\bar\Omega = \bigcup_{i=1}^3 D_i,$$
where
$$D_1 = \Omega^\partial \cap B_{\delta(z)/2}(z)^c,$$
$$D_2 = (\Omega \setminus \Omega^\partial) \cap B_{\delta(z)/2}(z)^c,$$
$$D_3 = B_{\delta(z)/2}(z),$$
and estimate each term separately.

Estimate on D_1: Since every point $y \in D_1$ belongs to a boundary-touching element, we use the first part of Lemma 6 to write
$$\left| \int_{D_1} (w(y) - I_h w(y))\, \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \eta\left(\frac{z-y}{|z-y|}\right) \mathrm{d}y \right| \lesssim h^{\mu s} \int_{D_1} \frac{1}{|z-y|^{d+2s}} \mathrm{d}y \lesssim$$
$$h^{\mu s} \int_{B_{\delta(z)/2}(z)^c} \frac{1}{|z-y|^{d+2s}} \mathrm{d}y \lesssim h^{\mu s} \delta(z)^{-2s}.$$

Estimate on D_2: Notice now that every element $y \in D_2$ belongs to a non-boundary-touching element. We can then invoke the other case in the first part of Lemma 6 to estimate

$$\left| \int_{D_2} (w(y) - I_h w(y)) \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \eta \left(\frac{z-y}{|z-y|} \right) \mathrm{d}y \right| \lesssim h^{\bar{\beta}} \int_{D_2} \frac{\delta(y)^{s-\bar{\beta}/\mu}}{|z-y|^{d+2s}} \mathrm{d}y.$$

Now, if $s - \frac{\bar{\beta}}{\mu} \geq 0$ we simply estimate

$$h^{\bar{\beta}} \int_{D_2} \frac{\delta(y)^{s-\bar{\beta}/\mu}}{|z-y|^{d+2s}} \mathrm{d}y \leq h^{\bar{\beta}} \int_{B_{\delta(z)/2}(z)^c} \frac{1}{|z-y|^{d+2s}} \mathrm{d}y \lesssim h^{\bar{\beta}} \delta(z)^{-2s}.$$

If, instead, $s - \frac{\bar{\beta}}{\mu} < 0$ then

$$h^{\bar{\beta}} \int_{D_2} \frac{\delta(y)^{s-\bar{\beta}/\mu}}{|z-y|^{d+2s}} \mathrm{d}y \lesssim h^{\bar{\beta}} h^{\mu(s-\bar{\beta}/\mu)} \int_{D_2} \frac{1}{|z-y|^{d+2s}} \mathrm{d}y \lesssim h^{\mu s} \delta(z)^{-2s}.$$

Estimate on D_3: Observe that $\delta(y) \geq \delta(z) - |z-y|$ and, if $y \in D_3$, we also have that $\frac{1}{2}\delta(z) \leq \delta(y) \leq \frac{3}{2}\delta(z)$. Now, since $z \in \mathscr{N}_{\mathscr{T}}^i$ we have that $\delta(z) \gtrsim h^\mu$. Consequently, if $y \in T \cap D_3$ for some T such that $T \cap \partial\Omega = \emptyset$,

$$h_T \lesssim h\delta(y)^{1-1/\mu} \lesssim h\delta(z)^{1-1/\mu}.$$

We, once again, invoke the estimates of Lemma 6 to obtain

$$\begin{aligned} &\left| \int_{D_3 \cap (\Omega \setminus \Omega^\partial)} (w(y) - I_h w(y)) \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \eta \left(\frac{z-y}{|z-y|} \right) \mathrm{d}y \right| \lesssim \\ &\qquad h^{\bar{\beta}} \delta(z)^{s-\bar{\beta}/\mu} \int_{D_3 \cap (\Omega \setminus \Omega^\partial)} \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \mathrm{d}y \lesssim \\ &\qquad h^{\bar{\beta}} \delta(z)^{s-\bar{\beta}/\mu} \left(\int_{\boldsymbol{\varepsilon}(z)}^{\delta(z)/2} \frac{1}{r^{1+2s}} \mathrm{d}r + \int_{B_{\boldsymbol{\varepsilon}(z)}(z)} \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \mathrm{d}y \right) \lesssim \\ &\qquad h^{\bar{\beta}} \delta(z)^{s-\bar{\beta}/\mu} \boldsymbol{\varepsilon}(z)^{-2s}. \end{aligned}$$

On the other hand, if $D_3 \cap \Omega^\partial \neq \emptyset$ it means that $\mathrm{dist}(B_{\delta(z)/2}(z), \partial\Omega) \lesssim h^\mu$, and therefore $\delta(z) = \mathrm{dist}(z, \partial\Omega) \lesssim \delta(z)/2 + h^\mu$, namely $\delta(z) \approx h^\mu \approx h_z$. Therefore, it must be $\boldsymbol{\varepsilon}(z) \approx \delta(z)$. In such a case, on $D_3 \cap \Omega^\partial$, we have

$$\begin{aligned} &\left| \int_{D_3 \cap \Omega^\partial} (w(y) - I_h w(y)) \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \eta \left(\frac{z-y}{|z-y|} \right) \mathrm{d}y \right| \lesssim \\ &\qquad h^{\mu s} \int_{D_3 \cap \Omega^\partial} \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \mathrm{d}y \lesssim \\ &\qquad h^{\mu s} \left(\int_{\boldsymbol{\varepsilon}(z)}^{\delta(z)/2} \frac{1}{r^{1+2s}} \mathrm{d}r + \int_{B_{\boldsymbol{\varepsilon}(z)}(z)} \mathcal{K}_{\boldsymbol{\varepsilon}(z)}(|z-y|) \mathrm{d}y \right) \lesssim \\ &\qquad h^{\mu s} \boldsymbol{\varepsilon}(z)^{-2s} \approx h^{\mu s} \delta(z)^{-2s}. \end{aligned}$$

Gathering all the obtained estimates leads to the claim. □

4. The linear problem

Having studied a consistent and monotone discretization of the operator $\mathcal{L}_\eta$, defined in (1), we proceed to use this discretization to propose and analyze numerical methods for problems of increasing complexity. The first one shall be a linear one.

We consider the following problem: Let $s \in (0,1)$ and $\eta \in \mathcal{C}(\lambda,\Lambda)$ for some $\lambda,\Lambda > 0$. Given $f \in H^{-s}(\Omega)$, find $u \in \widetilde{H}^s(\Omega)$ such that

$$\mathcal{L}_\eta[u] = f, \qquad \text{in } \Omega. \tag{12}$$

Notice that, by virtue of the definition of the solution space, we are implicitly providing the exterior condition $u = 0$ on Ω^c. Owing to the fact that $\|\cdot\|_{\eta,s}$ is an equivalent norm on $\widetilde{H}^s(\Omega)$, existence and uniqueness of a weak solution follows immediately. In addition, since η is positive, a nonlocal maximum principle holds; see Proposition 2.

4.1. **Regularity.** The regularity properties of u, solution of (12), are of utmost relevance for its numerical approximation. In contrast to local elliptic operators, it is well known that solutions to (12) possess limited regularity near the boundary, regardless of the smoothness of Ω and f; see [69, 38, 16]. The following is an *optimal* regularity result.

Lemma 8 (optimal Hölder regularity)**.** *Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain that satisfies the exterior ball condition. Let $f \in L^\infty(\Omega)$ and $u \in \widetilde{H}^s(\Omega)$ be the (weak) solution to* (12)*. Then $u \in C^{0,s}(\bar{\Omega})$. Moreover, we have*

$$\|u\|_{C^{0,s}(\bar{\Omega})} \le C\|f\|_{L^\infty(\Omega)},$$

where the implicit constant depends only on d, Ω and s.

Proof. See [63, Proposition 4.5]. □

The previous regularity is optimal, as exemplified by existing explicit solutions to the fractional Laplace problem over a ball (cf. [37]). The reason for this limited regularity gain lies in the fact that there is an algebraic boundary singularity present in the solution, which can be characterized via

$$u(x) \approx \operatorname{dist}(x,\partial\Omega)^s,$$

as x approaches $\partial\Omega$. Higher order regularity estimates can be obtained if one takes into account such boundary behavior. In [62] such results were obtained for the fractional Laplacian over weighted Hölder spaces, where the weights are given by powers of the distance to the boundary.

Definition 9 (weighted Hölder space)**.** Let $\sigma > -1$ and $\beta = k + \gamma > 0$ with $k \in \mathbb{N}_0, \gamma \in (0,1]$. For $w \in C^k(\Omega)$ define the seminorm

$$|w|^{(\sigma)}_{\beta,\Omega} = \sup_{x,y\in\Omega: x\neq y} \left(\delta(x,y)^{\beta+\sigma} \frac{\left|D^k w(x) - D^k w(y)\right|}{|x-y|^\gamma} \right),$$

and the norm

$$\|w\|^{(\sigma)}_{\beta,\Omega} = \sum_{\ell=1}^{k} \sup_{x\in\Omega} \left(\delta(x)^{\ell+\sigma}|D^\ell w(x)|\right) + |w|^{(\sigma)}_{\beta,\Omega} + \begin{cases} \sup_{x\in\Omega} \left(\delta(x)^\sigma |v(x)|\right), & \sigma \ge 0, \\ \|v\|_{C^{0,-\sigma}(\bar{\Omega})}, & \sigma < 0. \end{cases}$$

The methods used in [61] can be adapted to operators of the form (1) to obtain the following weighted Hölder regularity.

Lemma 10 (weighted Hölder estimates)**.** *Let Ω be a bounded domain that satisfies the exterior ball condition, and let $\beta > 0$ satisfy* (2)*. Moreover, we have*

$$\|u\|_{\beta,\Omega}^{(-s)} \lesssim \|u\|_{C^{0,s}(\mathbb{R}^d)} + \|f\|_{\beta-2s,\Omega}^{(s)},$$

where the implicit constant only dependsLet $f \in C^{\beta-2s}(\Omega)$ be such that $\|f\|_{\beta-2s,\Omega}^{(s)} < \infty$. If, for $k \in \mathbb{N}$, $\beta \geq k$, then we additionally assume that $\eta \in C^k(\mathbb{S}^{d-1})$. In this setting, the solution of (12) *satisfies $u \in C^\beta(\Omega)$. on d, Ω, λ, Λ and s.*

Proof. The result can be obtained by using the same procedure as in [61, Proposition 1.4] for $\eta \equiv 1$. For completeness, we repeat the main parts of the argument.

Let $x_0 \in \Omega$ and $R = \frac{\delta(x_0)}{K}$ for some $K \in \mathbb{N}$. We define $\tilde{u}(x) = u(x_0 + Rx) - u(x_0)$ and, following [61, Proposition 1.4], it is possible to show that

$$\begin{aligned} \|\tilde{u}\|_{C^{0,s}(B_1)} &\leq CR^s [u]_{C^{0,s}(B_R(x_0))} \\ \|(1+|\cdot|)^{-d-2s}\tilde{u}(\cdot)\|_{L^1(\mathbb{R}^n)} &\leq CR^s [u]_{C^{0,s}(\mathbb{R}^d)}. \end{aligned} \tag{13}$$

In addition, if $\beta \leq 1$,

$$\sup_{x,y\in B_1} \frac{|\mathcal{L}_\eta[\tilde{u}](x) - \mathcal{L}_\eta[\tilde{u}](y)|}{|x-y|^\beta} = R^{2s+\beta} \sup_{x,y\in B_R(x_0))} \frac{|\mathcal{L}_\eta[u](x) - \mathcal{L}_\eta[u](y)|}{|x-y|^\beta}.$$

We also observe that, since the coefficient η is independent of x, for any $k \in \mathbb{N}_0$

$$D^k \mathcal{L}_\eta[\tilde{u}](x) = R^k \mathcal{L}_\eta[D^k u](x).$$

We use this in the case $\beta > 1$ to assert that, for any $\beta > 0$ that satisfies (2),

$$\|\mathcal{L}_\eta[\tilde{u}]\|_{C^\beta(B_1)} \lesssim R^{2s+\beta} \|\mathcal{L}_\eta[u]\|_{C^\beta(B_R(x_0))} \leq R^s \|\mathcal{L}_\eta[u]\|_{\beta,\Omega}^{(s)}. \tag{14}$$

The estimates so far do not use the regularity of the coefficient η, but only its boundedness and translation invariance. Now we can repeat the proof [61, Corollary 2.4] using the additional regularity on η together with the estimates (13)—(14) to get

$$\begin{aligned} \|\tilde{u}\|_{C^\beta(B_{1/2})} &\lesssim \|(1+|\cdot|)^{-d-2s}\tilde{u}(\cdot)\|_{L^1(\mathbb{R}^d)} + \|\tilde{u}\|_{C^{\beta-2s}(B_2)} + \|\mathcal{L}_\eta[\tilde{u}]\|_{C^{\beta-2s}(B_2)} \\ &\lesssim R^s \left([u]_{C^{0,s}(\mathbb{R}^d)} + \|f\|_{\beta-2s,\Omega}^{(s)} \right) + \|\tilde{u}\|_{C^{\beta-2s}(B_2)}, \end{aligned} \tag{15}$$

where we assumed that $K \geq 2$ to bound the term $\|\mathcal{L}_\eta[\tilde{u}]\|_{C^{\beta-2s}(B_2)}$ appropriately. We repeat that the regularity of the coefficient η is only used in order to be able to stress the arguments used in the proof of [61, Corollary 2.4].

Now, for $\beta - 2s \leq s$, the claim follows from estimate (15). Indeed, using that, if $y \in B_R(x_0)$, we have

$$\|D^\ell \tilde{u}\|_{L^\infty(B_{1/2})} = R^\ell \|D^\ell u\|_{L^\infty(B_{R/2}(x_0))},$$

and $\delta(x,y)/|x-y| \leq K$ if x and y are far away from each other.

If, instead, we have that $\beta - 2s > s$, we repeat the argument in (15) for $\|\tilde{u}\|_{C^{\beta-2s}(B_2)}$ in total m times until $\beta - 2sm \leq s$. Choosing K large enough, but finite (depending on s), finishes the proof. □

As a consequence of Lemmas 8 and 10, we have the following regularity estimate away from the boundary.

Lemma 11 (interior Hölder estimate)**.** *Let Ω be a bounded Lipschitz domain that satisfies the exterior ball condition and let f, β, and η satisfy the same assumptions as in Lemma 10. For every $\rho > 0$ we have*

$$\|u\|_{C^\beta(\{x\in\Omega:\delta(x)\geq\rho\})} \lesssim \rho^{s-\beta},$$

for a constant only on $\|f\|^{(s)}_{\beta-2s,\Omega}, \|f\|_{L^\infty(\Omega)}$, s, and Ω.

Proof. Repeat the proof of [39, Corollary 2.5]. □

In short, the previous results show that u, the solution of (12), satisfies $u \in \mathcal{S}^\beta(\Omega)$ and $\|u\|^{(-s)}_{\beta,\Omega} < \infty$.

4.2. **Pointwise error estimates.** We have reached the point where we can propose a numerical scheme for (12) and provide an error analysis for it. To begin, we need to make a precise choice of the regularization scale $\boldsymbol{\varepsilon}$.

Definition 12 ($\boldsymbol{\varepsilon}$ for the linear problem)**.** Let the vertex $z \in \mathcal{N}^i_{\mathcal{T}}$. We set

$$\boldsymbol{\varepsilon}(z) = \frac{1}{2} h_z^{1/2} \delta(z)^{1/2}.$$

At this point we make some comments regarding the choice in Definition 12. First, notice that close to the boundary we have

$$\boldsymbol{\varepsilon}(z) \approx h_z \approx \delta(z),$$

which ensures that $B_{\boldsymbol{\varepsilon}(z)}(z) \subset \Omega$. On the other hand, in the interior of the domain we have $\delta(z) \approx 1$, so that

$$\boldsymbol{\varepsilon}(z) \approx h_z^{1/2}.$$

This scaling bears resemblance to the scalings used in two-scale methods for (local) linear second order elliptic problems; see; [49, 50, 64, 46, 58, 53, 54].

We now describe the scheme. For $\boldsymbol{\varepsilon}$ given according to Definition 12, our scheme seeks for $u_h \in \mathbb{V}^0_{\mathcal{T}}$ such that

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h](z) = f(z), \quad \forall z \in \mathcal{N}^i_{\mathcal{T}}. \tag{16}$$

We comment that, since this is a finite dimensional problem, existence and uniqueness of a solution are implied immediately by the following discrete comparison principle for the operator $\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}$.

Proposition 13 (discrete comparison principle)**.** *Let $\boldsymbol{\varepsilon}$ be such that, for every $z \in \mathcal{N}^i_{\mathcal{T}}$, we have $\boldsymbol{\varepsilon}(z) \geq \frac{1}{2} h_z$. Assume that $v_h, w_h \in \mathbb{V}^0_{\mathcal{T}}$ are such that*

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[v_h](z) \geq \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w_h](z), \quad \forall z \in \mathcal{N}^i_{\mathcal{T}}. \tag{17}$$

Then,

$$v_h(z) \geq w_h(z) \quad \forall z \in \mathcal{N}^i_{\mathcal{T}}.$$

Proof. The proof follows by a simple argument. Suppose the inequality (17) is strict, and that the function $v_h - w_h$ attains a (non-positive) minimum at an interior node $z \in \mathcal{N}^i_{\mathcal{T}}$. Then, we have

$$v_h(z) - w_h(z) \leq v_h(y) - w_h(y) \implies v_h(z) - v_h(y) \leq w_h(z) - w_h(y) \quad \forall y \in \mathbb{R}^d$$

and, consequently,

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[v_h](z) \leq \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w_h](z).$$

This shows that, in case of strict inequality in (17), it must be that $v_h > w_h$ in Ω.

Assume next that the inequality (17) is not strict. Consider the discrete barrier function $b_h = I_h b$, where $b = \chi_\Omega$ and I_h denotes the Lagrange interpolant. We have $\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b_h](z) > 0$ for all $z \in \mathscr{N}_{\mathscr{T}}^i$. Therefore, for $\epsilon > 0$, we have the strict inequality

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[v_h + \epsilon b_h](z) > \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w_h](z) \quad \forall z \in \mathscr{N}_{\mathscr{T}}^i,$$

from which it follows that $v_h(z) + \epsilon b_h(z) > w_h(z)$ for all $z \in \mathscr{N}_{\mathscr{T}}^i$. Letting $\epsilon \to 0$, we obtain the desired result. □

We now proceed with the error analysis of our scheme. Using the choice of $\boldsymbol{\varepsilon}$ given by Definition 12, we begin by making the results from Proposition 7 more precise.

Proposition 14 (consistency of interpolation for the linear problem)**.** *Let the regularization scale $\boldsymbol{\varepsilon}$ verify Definition 12, β satisfy* (2)*, $\bar\beta = \min\{\beta, 2\}$, and $w \in \mathcal{S}^\beta(\Omega)$. We have*

$$|\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h w - w](z)| \lesssim \max\{h^{\mu s}, h^{\bar\beta - s}\}\delta(z)^{-2s} \quad \forall z \in \mathscr{N}_{\mathscr{T}}^i.$$

Proof. Let $z \in \mathscr{N}_{\mathscr{T}}^i$. By (10), it suffices to show that

$$h^{\bar\beta}\delta(z)^{s-\bar\beta/\mu}\boldsymbol{\varepsilon}(z)^{-2s} \lesssim \max\{h^{\mu s}, h^{\bar\beta - s}\}\delta(z)^{-2s}.$$

We consider the set Ω^∂ given by (11).

If $z \in \Omega^\partial$, then $\delta(z) \approx h_z \approx h^\mu$. Combining this with $\boldsymbol{\varepsilon}(z) = \frac{1}{2}h_z^{1/2}\delta(z)^{1/2} \approx \delta(z)$, we obtain

$$h^{\bar\beta}\delta(z)^{s-\bar\beta/\mu}\boldsymbol{\varepsilon}(z)^{-2s} \approx h^{\mu s}\delta(z)^{-2s}.$$

In contrast, if $z \notin \Omega^\partial$, then $z \in T$ for some T with $T \cap \partial\Omega = \emptyset$ and therefore $h_z \approx h_T \approx h\delta(z)^{1-1/\mu}$ and $\boldsymbol{\varepsilon}(z) = \frac{1}{2}h_z^{1/2}\delta(z)^{1/2}$ If $s \geq (\bar\beta - s)/\mu$, then we have

$$h^{\bar\beta}\delta(z)^{s-\bar\beta/\mu}\boldsymbol{\varepsilon}(z)^{-2s} \approx h^{\bar\beta-s}\delta(z)^{s-(\bar\beta-s)/\mu}\delta(z)^{-2s} \lesssim h^{\bar\beta-s}\delta(z)^{-2s}.$$

Otherwise, we write $h \approx h_z\delta(z)^{-1+1/\mu}$ and

$$h^{\bar\beta}\delta(z)^{s-\bar\beta/\mu}\boldsymbol{\varepsilon}(z)^{-2s} \approx h_z^{\bar\beta-s}\delta(z)^{-\bar\beta+2s}\delta(z)^{-2s} \approx h^{\mu s}\left(\frac{h_z}{\delta(z)}\right)^{\bar\beta-s-\mu s}\delta(z)^{-2s}.$$

Because $h_z \leq \delta(z)$ and $\bar\beta - s - \mu s > 0$, the second term in the right hand side is uniformly bounded above. □

It remains then to obtain error estimates. This is the content of the following result.

Theorem 15 (error estimate)**.** *Let Ω be a convex polytope, $s \in (0,1)$, $\beta \leq 4$ is such that* (2) *holds. Define $\bar\beta = \min\{\beta, 2\}$. Let $f \in C^{\beta-2s}(\Omega) \cap L^\infty(\Omega)$ be such that $\|f\|_{\beta-2s,\Omega}^{(s)} < \infty$. Assume that $\eta \in \mathcal{C}(\lambda, \Lambda)$ and that if, for some $k \in \mathbb{N}$, $\beta > k$, then $\eta \in C^k(\mathbb{S}^{d-1})$. Let $u \in \mathcal{S}^\beta(\Omega)$ solve* (12) *and $u_h \in \mathbb{V}_{\mathscr{T}}^0$ solve* (16)*. If $\mathscr{T}$ satisfies* (6) *and $\boldsymbol{\varepsilon}$ is chosen as in Definition 12, we have*

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \max\{h^{\mu s}, h^{\bar\beta - s}, h^{\beta/2 - s}\}.$$

Proof. We consider $u - u_h = (u - I_h u) + (I_h u - u_h)$. By (9), to prove the claim, it suffices to estimate the second term. We do so by estimating the consistency

$$|\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h u - u_h](z)| \leq |\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h u - u](z)| + |\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u - u_h](z)| = \mathrm{I} + \mathrm{II},$$

for $z \in \mathscr{N}_{\mathscr{T}}^i$, and then applying the discrete comparison principle from Proposition 13.

Proposition 14 yields

$$\mathrm{I} \lesssim \max\{h^{\mu s}, h^{\bar\beta - s}\}\delta(z)^{-2s} \approx \max\{h^{\mu s}, h^{\bar\beta - s}\}\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z),$$

where we recall that b is the barrier function introduced in Lemma 4. Additionally, the choice of $\boldsymbol{\varepsilon}$, identity (8), and Theorem 3 with $\alpha_0 = 2$, yield

$$\begin{aligned}\mathrm{II} = |\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u](z) - \mathcal{L}_\eta[u](z)| &\lesssim \boldsymbol{\varepsilon}(z)^{\beta-2s}\delta(z)^{s-\beta} \\ &\approx h^{\beta/2-s}\delta(z)^{s-\frac{\beta/2-s}{\mu}}\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z).\end{aligned}$$

Thus, if $s - \frac{\beta/2-s}{\mu} \geq 0$ we are done. In case $s < \frac{\beta/2-s}{\mu}$, we write instead $h \approx h_z\delta(z)^{1/\mu-1}$ and obtain

$$\begin{aligned}\mathrm{II} &\lesssim h^{\beta/2-s}\delta(z)^{s-\frac{\beta/2-s}{\mu}}\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z) \\ &\approx h^{\mu s}h_z^{\beta/2-\mu s-s}\delta(z)^{(1/\mu-1)(\beta/2-\mu s-s)}\delta(z)^{s-\frac{\beta/2-s}{\mu}}\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z) \\ &\approx h^{\mu s}\left(\frac{h_z}{\delta(z)}\right)^{\beta/2-\mu s-s}\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z).\end{aligned}$$

This concludes the proof. □

Remark 16 (complexity estimate)**.** Let us try to interpret the estimates of Theorem 15 in terms of degrees of freedom. To shorten the notation we set, only for this discussion, $N = \dim \mathbb{V}^0_{\mathscr{T}}$. Furthermore, we will assume that the right hand side f is as smooth as possible, i.e., we let $\beta = 4$ and satisfying (2). This regularity assumption implies that $\beta/2 = \bar\beta = 2$.

First, if $d = 2$, we can choose $\mu = 2$ so that, according to (7), $N \approx h^{-2}|\log h|$. As a consequence,

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \max\left\{N^{-s}|\log N|^s, N^{-1+s/2}|\log N|^{1-s/2}\right\}.$$

Therefore, with respect to the number of degrees of freedom, and up to logarithmic factors, we obtain convergence with order s for $s \leq 2/3$ and with order $1 - s/2$ for $s > 2/3$. Additionally we observe that, for $s > 2/3$, one does not need to take $\mu = 2$ in two dimensions, and that the maximal convergence rate is attained whenever $\mu s = 2 - s$ i.e., $\mu = \frac{2-s}{s}$; any extra mesh refinement does not reflect in an improvement of convergence rates, although it affects the conditioning of the resulting system.

On the other hand, for $d = 3$ we must choose $\mu = \frac{3}{2}$ if we wish to maintain a near optimal number of degrees of freedom, i.e., $N \approx h^{-3}|\log h|$. Thus,

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \max\left\{N^{-s/2}|\log N|^{s/2}, N^{-2/3+s/3}|\log N|^{2/3-s/3}\right\}.$$

Again, we observe that, if $-s/2 \leq -2/3 + s/3$, namely if $s \geq 4/5$, the maximal convergence order $s/2$ is attained for meshes graded with $\mu = \frac{2-s}{s}$.

Remark 17 (relationship between $\boldsymbol{\varepsilon}$ and f)**.** Definition 12 is suitable for sufficiently smooth right hand sides, namely, it formally delivers optimal convergence rates in case $f \in C^{\beta-2s}(\Omega) \cap L^\infty(\Omega)$ be such that $\|f\|^{(s)}_{\beta-2s,\Omega} < \infty$ with $\beta \geq 4$. We recall that, since we are using a second difference formula, the interior consistency of the regularized operator cannot exploit any regularity beyond $\mathcal{S}^4(\Omega)$, cf. Theorem 3. Let us briefly comment on what one can obtain when f satisfies the assumptions above but with $\beta \in (2s, 4)$ and satisfying (2).

We let $\boldsymbol{\varepsilon}(z) = \frac{1}{2} h_z^{\alpha} \delta(z)^{1-\alpha}$ with $\alpha \in [0,1]$, that clearly satisfies $\frac{1}{2} h_z \le \boldsymbol{\varepsilon}(z) \le \frac{1}{2}\delta(z)$. By doing the same calculations as in Proposition 14, we obtain

$$|\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[I_h w - w](z)| \lesssim \max\{h^{\mu s}, h^{\bar{\beta}-2\alpha s}\} \delta(z)^{-2s} \quad \forall z \in \mathcal{N}_{\mathcal{T}}^{i}.$$

Arguing then as in the proof of Theorem 15, we obtain the error estimate

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \max\{h^{\mu s}, h^{\bar{\beta}-2\alpha s}, h^{\alpha(\beta-2s)}\}.$$

Now, if $\beta \in (2,4)$, we have $\bar{\beta} = 2$ and

$$2 - 2\alpha s = \alpha(\beta - 2s) \;\Rightarrow \alpha = \frac{2}{\beta}.$$

Therefore, choosing $\boldsymbol{\varepsilon}(z) = \frac{1}{2} h_z^{\frac{2}{\beta}} \delta(z)^{1-\frac{2}{\beta}}$ yields the error estimate

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \max\{h^{\mu s}, h^{2-\frac{4s}{\beta}}\}.$$

In contrast, if $\beta \in (2s, 2)$, we observe $\bar{\beta} = \beta$ and we get that

$$\beta - 2\alpha s = \alpha(\beta - 2s) \quad \Longrightarrow \quad \alpha = 1.$$

In this low regularity case, setting $\boldsymbol{\varepsilon}(z) = \frac{1}{2} h_z$ gives rise to

$$\|u - u_h\|_{L^\infty(\Omega)} \lesssim \max\{h^{\mu s}, h^{\beta-2s}\}.$$

The latter will be of interest in the approximation of the obstacle problem in the next section, and justifies Definition 20 below.

5. The obstacle problem

As the next application of our two-scale discretization, we will consider the following nonlinear problem. In the setting of Section 4 we assume that, in addition, we have $\psi : \bar{\Omega} \to \mathbb{R}$ that satisfies $\psi < 0$ on $\partial\Omega$. We seek for $u \in \widetilde{H}^s(\Omega)$ that satisfies

$$\min\{\mathcal{L}_\eta[u] - f, u - \psi\} = 0, \quad \text{a.e. } \Omega. \tag{18}$$

While existence and uniqueness of a weak solution is classical, the regularity of such solution is more delicate. Following [17] we introduce the classes

$$\mathcal{F}_s(\bar{\Omega}) = C^{3-2s+\epsilon}(\bar{\Omega}), \qquad \Psi = \left\{\psi \in C(\bar{\Omega}) : \psi_{|\partial\Omega} < 0\right\} \cap C^{2,1}(\Omega),$$

where $\epsilon > 0$ is sufficiently small, so that $1 - 2s + \epsilon \notin \mathbb{N}$.

For future use we define the contact and non-contact sets as follows:

$$\Omega^0 = \{x \in \Omega \mid u(x) = \psi(x)\}, \qquad \Omega^+ = \{x \in \Omega \mid u(x) > \psi(x)\}.$$

In order to obtain rates of convergence, we must understand the regularity of the solution. This can be achieved by combining the arguments in [17], [19], and the regularity of the linear problem presented in Section 4.1. Namely, one first proves the result for a problem in the whole space $\mathbb{R}^d$, and then use a localization argument; see [17] for details in the case $\eta \equiv 1$.

Proposition 18 (regularity). *Assume that $f \in \mathcal{F}_s$ and $\psi \in \Psi$. Then, $u \in \widetilde{H}^s(\Omega)$, the solution of* (18) *satisfies $u \in C^{1,s}(\Omega)$ and*

$$\|\mathcal{L}_\eta[u]\|_{1-s,\Omega}^{(s)} < \infty.$$

Remark 19 (pointwise evaluation)**.** Notice that, since $\psi < 0$ on $\partial\Omega$, the solution to (18) solves the linear problem $\mathcal{L}_\eta[u] = f$ in a neighborhood of the boundary. By the interior regularity of the previous result we additionally have $\mathcal{L}_\eta[u] \in C^{0,1-s}(\Omega)$. As a consequence, we have that $\mathcal{L}_\eta[u] \in C^{0,1-s}(\bar{\Omega})$, meaning that pointwise evaluation of the operator is meaningful.

5.1. **Pointwise error estimates.** Let us now provide a numerical scheme for the obstacle problem (18) and pointwise error estimates for it. We seek for $u_h \in \mathbb{V}^0_{\mathscr{T}}$ such that

$$\min\left\{\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h](z) - f(z), u_h(z) - \psi(z)\right\} = 0, \quad \forall z \in \mathscr{N}^i_{\mathscr{T}}. \tag{19}$$

For this problem, we shall make a different choice of $\boldsymbol{\varepsilon}$ than for the linear problem. The reason behind this is that even if the data is sufficiently smooth, the solution to the obstacle problem possesses a limited interior regularity, compare Lemma 10 with Proposition 18; see also the discussion of the case $\beta \in (2s, 2)$ in Remark 17.

Definition 20 (choice of $\boldsymbol{\varepsilon}$ for the obstacle problem)**.** Let $z \in \mathscr{N}^i_{\mathscr{T}}$. We set

$$\boldsymbol{\varepsilon}(z) = \frac{1}{2} h_z.$$

Existence and uniqueness of u_h follow from the fact that we are in finite dimensions and the comparison principle for $\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}$. Of interest here is the derivation of pointwise error estimates. The technique that we will use is rather classical and can be traced back to [5, 51], see also [55]. We begin by introducing the notions of sub- and supersolutions to the obstacle problem.

Definition 21 (sub- and supersolution)**.** We say that $u_h^+ \in \mathbb{V}_{\mathscr{T}}$ is a supersolution to (19) if $u_h^+ \geq 0$ in Ω^c and, for all $z \in \mathscr{N}^i_{\mathscr{T}}$, we have

$$u_h^+(z) \geq \psi(z), \qquad \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h^+](z) \geq f(z).$$

On the other hand, we say that $u_h^- \in \mathbb{V}_{\mathscr{T}}$ is a subsolution to (19) if $u_h^- \leq 0$ in Ω^c and, for every $z \in \mathscr{N}^i_{\mathscr{T}}$, if $u_h^-(z) \geq \psi(z)$, then

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h^-](z) \leq f(z).$$

The comparison principle of the operator $\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}$ gives a comparison for sub- and supersolutions.

Lemma 22 (discrete comparison)**.** *Let $u_h^+, u_h^- \in \mathbb{V}_{\mathscr{T}}$ be super- and subsolutions to* (19)*, and $u_h \in \mathbb{V}^0_{\mathscr{T}}$ be the solution to* (19)*. Then, for every $z \in \mathscr{N}_{\mathscr{T}}$ we have*

$$u_h^-(z) \leq u_h(z) \leq u_h^+(z).$$

Proof. We consider each inequality separately. Let u_h^- be a subsolution and consider the set of nodes

$$C_- = \left\{ z \in \mathscr{N}_{\mathscr{T}} \;\middle|\; u_h^-(z) \geq \psi(z) \right\}.$$

Now, if $z \in C_-$, we have

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h^-](z) \leq f(z) \leq \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h](z).$$

If, on the other hand $z \notin C_-$, then

$$u_h^-(z) < \psi(z) \leq u_h(z).$$

In summary, the function $w_h = u_h^- - u_h \in \mathbb{V}_{\mathscr{T}}$ verifies

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w_h](z) \leq 0, \quad z \in C_-, \qquad w_h(z) \leq 0, \quad z \notin C_-.$$

A discrete comparison principle then implies that $w_h \leq 0$.

Let now u_h^+ be a supersolution. Consider now the discrete contact set

$$C_+ = \{z \in \mathscr{N}_{\mathscr{T}} \mid u_h(z) = \psi(z)\},$$

and observe that, if $z \in C_+$,

$$u_h(z) = \psi(z) \leq u_h^+(z).$$

On the other hand, if $z \notin C_+$ we have

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h^+](z) \geq f(z) = \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h](z).$$

In conclusion, the function $w_h = u_h^+ - u_h \in \mathbb{V}_{\mathscr{T}}$ satisfies

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[w_h] \geq 0, \quad z \notin C_+, \qquad w_h(z) \geq 0, \quad z \in C_+.$$

Once again, a discrete comparison principle yields that $w_h \geq 0$. □

Next we need to present a suitable discrete proxy for u. Namely, we consider $R_h u \in \mathbb{V}_{\mathscr{T}}^0$ to be such that, for every $z \in \mathscr{N}_{\mathscr{T}}^i$,

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[R_h u](z) = \mathcal{L}_\eta[u](z). \tag{20}$$

Recall that, as detailed in Remark 19, the right hand side is meaningful. The approximation power of $R_h u$ is the content of the following result.

Corollary 23 (projection error). *Let u be the solution of* (18) *and $R_h u$ be defined in* (20)*. If the regularization scale $\boldsymbol{\varepsilon}$ is chosen according to Definition 20, then we have*

$$\|u - R_h u\|_{L^\infty(\Omega)} \lesssim \mathfrak{e}(h) = \max\{h^{\mu s}, h^{1-s}\}, \tag{21}$$

with an implicit constant that is independent of h.

Proof. We begin by recalling that, as indicated by Proposition 18, we must set $\beta = 1 + s$.

Observe that, owing to (20),

$$\mathrm{I} = |\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u - R_h u]| = |\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u] - \mathcal{L}_\eta[u]| \lesssim \boldsymbol{\varepsilon}(z)^{1-s}\delta(z)^{-1},$$

where, in the last step, we used Theorem 3.

Using the the current choice of $\boldsymbol{\varepsilon}$ we then continue this estimate as

$$\mathrm{I} \lesssim \left(h\delta(z)^{1-\frac{1}{\mu}}\right)^{1-s} \delta(z)^{-1} = h^{1-s}\delta(z)^{-2s}\delta(z)^{s+\frac{s-1}{\mu}} \lesssim h^{1-s}\delta(z)^{-2s},$$

provided that $s + \frac{s-1}{\mu} \geq 0$.

If, on the other hand, we have $s + \frac{s-1}{\mu} < 0$ we use that $\delta(z) \gtrsim h_z \gtrsim h^\mu$ to obtain that

$$h^{1-s}\delta(z)^{s+\frac{s-1}{\mu}} \lesssim h^{1-s}h^{\mu\left(s+\frac{s-1}{\mu}\right)} \approx h^{\mu s},$$

so that, in all cases,

$$\mathrm{I} \lesssim \max\{h^{\mu s}, h^{1-s}\}\delta(z)^{-2s}.$$

The remaining of the proof follows exactly as the one of Theorem 15. □

With this proxy of the solution at hand we are ready to obtain error estimates.

Theorem 24 (error estimates)**.** *In the setting of Proposition 18 let u solve* (18) *and $u_h \in \mathbb{V}^0_{\mathscr{T}}$ solve* (19)*. If $\mathscr{T}$ satisfies* (6) *and $\boldsymbol{\varepsilon}$ is chosen as in Definition 20, we have that*

$$\|u - u_h\|_{L^\infty(\Omega)} = \tilde{\mathfrak{e}}(h) \lesssim \mathfrak{e}(h),$$

where the quantity $\mathfrak{e}(h)$ is defined in (21)*.*

Proof. We will construct suitable super- and subsolutions to the discrete obstacle problem (19) and apply the comparison principle of Lemma 22 to conclude. Notice that, owing to (21), there is a sufficiently large $C > 0$ for which

$$u - C\mathfrak{e}(h) \le R_h u \le u + C\mathfrak{e}(h).$$

Supersolution: Let $u_h^+ = R_h u + C_1 \mathfrak{e}(h) \in \mathbb{V}_{\mathscr{T}}$, where the constant $C_1 > 0$ is to be chosen, and notice that $u_h^+ \ge 0$ in Ω^c. Moreover, for $z \in \mathscr{N}^i_{\mathscr{T}}$, with the barrier function $b = \chi_\Omega$ and Lemma 4,

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h^+](z) = \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[R_h u](z) + C_1 \mathfrak{e}(h) \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z) > \mathcal{L}_\eta[u](z) \ge f(z).$$

In addition, if $C_1 \ge C$, we have, for any $z \in \mathscr{N}^i_{\mathscr{T}}$,

$$u_h^+(z) \ge u(z) + (C_1 - C)\mathfrak{e}(h) \ge \psi(z) + (C_1 - C)\mathfrak{e}(h) \ge \psi(z).$$

We have then shown that u_h^+ is a supersolution for the obstacle problem. This implies, via Lemma 22, that for every $x \in \Omega$ we have

$$u_h(x) \le u_h^+(x) \le u(x) + (C + C_1)\mathfrak{e}(h),$$

so that

$$u_h(x) - u(x) \lesssim \mathfrak{e}(h).$$

Subsolution: We now define $u_h^- = R_h u - C_2 \mathfrak{e}(h) \in \mathbb{V}_{\mathscr{T}}$ where $C_2 > 0$ is to be chosen. Notice that $u_h^- \le 0$ in Ω^c. We will show that it is a subsolution. To see this, let $z \in \mathscr{N}^i_{\mathscr{T}}$ and assume that $u_h^-(z) \ge \psi(z)$. Then,

$$\psi(z) \le u_h^-(z) \le u(z) - (C_2 - C)\mathfrak{e}(h) < u(z),$$

provided $C_2 > C$. The fact that this inequality is strict shows that

$$\mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[u_h^-](z) = \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[R_h u](z) - C_2 \mathfrak{e}(h) \mathcal{L}_{\eta,\boldsymbol{\varepsilon}}[b](z) < \mathcal{L}_\eta[u](z) = f(z),$$

so that indeed this is a subsolution. We invoke once again Lemma 22 to obtain that

$$u(x) - (C + C_2)\mathfrak{e}(h) \le u_h^-(x) \le u_h(x), \quad \forall x \in \Omega,$$

as we needed to show. □

Remark 25 (optimality and complexity)**.** We comment that the rate of convergence of Theorem 24, as expressed by the quantity $\mathfrak{e}(h)$ is optimal for our proof technique. To see this, we recall that near the boundary the solution to the obstacle problem behaves like that of the linear problem, i.e., $u(z) \approx \delta(z)^s$, so that the rate of interpolation is at best $h^{\mu s}$. On the other hand, the interior regularity of the solution is, at best, $C^{1,s}$. Since our operator $\mathcal{L}_\eta$ is of order $2s$, and our proof technique is based on comparison principles, the rate in the interior can be at best $h^{1+s-2s} = h^{1-s}$, as we have obtained.

Finally, with the notation and conventions of Remark 16 let us present the following complexity estimate

$$\|u-u_h\|_{L^\infty(\Omega)} \lesssim \begin{cases} \max\left\{N^{-s}|\log N|^{s}, N^{(s-1)/2}|\log N|^{(1-s)/2}\right\}, & d=2,\\ \max\left\{N^{-s/2}|\log N|^{s/2}, N^{(s-1)/3}|\log N|^{(1-s)/3}\right\}, & d=3. \end{cases}$$

5.2. **Regularity of free boundaries.** Of particular importance in applications is the so-called free boundary, which is the boundary of the contact set

$$\Gamma = \partial\Omega^0 \cap \Omega.$$

In this section we are concerned with the regularity of this set.

5.2.1. *Regular points.* We begin with the regularity of the free boundary near *regular points* which, roughly speaking, are those at which the function $u-\psi$ grows at a rate $1+s$. Define $d(x) = \operatorname{dist}(x,\Omega^0)$ to be the distance from $x\in\Omega$ to the contact set. The following result is proved in [19, Theorem 1.1].

Theorem 26 (regular points I)**.** *Let $\alpha\in(0,\min\{s,1-s\})$. Let $\psi\in\Psi$ and $x_0\in\Gamma$ be a regular point. Then, there exist positive constants $a(x_0)$ and $r(x_0)$ such that*

$$u(x)-\psi(x) = a(x_0)d(x)^{1+s} + \mathfrak{o}(|x-x_0|^{1+s+\alpha})$$

for all $x\in B_{r(x_0)}(x_0)\cap\Omega^+$. Moreover, the set of points satisfying this property is an open set of Γ and is locally a $C^{1,\gamma}$ graph for all $\gamma\in(0,s)$. Finally,

$$u \in C^{1,s}\left(B_{r(x_0)}(x_0)\right)$$

Remark 27 (singular points)**.** The points of Γ that are not regular are called *singular.* According to [19, Theorem 1.1], these points satisfy

$$u(x)-\psi(x) = \mathfrak{o}(|x-x_0|^{1+s+\alpha}).$$

Singular points do in fact occur and are characterized in [6] for the case $\eta\equiv 1$.

Remark 28 ($C^{1,\gamma}$ smoothness)**.** Consider an interface Γ, a point $x_0\in\Gamma$ with normal vector $\boldsymbol{\nu}$, and $x = x_0 + r_0\boldsymbol{\nu}$ for a sufficiently small $r_0>0$. If the interface Γ was C^2, then the closest point to x in Γ is x_0. However, if Γ is of class $C^{1,\gamma}$ with $\gamma<1$ (as in the conclusion of Theorem 26), then this is not the case anymore and the distance $d(x,\Gamma)$ may be realized at a point different than x_0.

The following variant of Theorem 26, which avoids the use of the distance function d is stated in [36, Theorem 4.4.1].

Theorem 29 (regular points II)**.** *Let $\alpha\in(0,\min\{s,1-s\})$, $\theta>\max\{0,2s-1\}$, and $\gamma\in(0,s)$. Let the obstacle $\psi\in C_0^{2,\theta}(\mathbb{R}^d)$. Let $x_0\in\Gamma$ be a free boundary point. Then, there is $r_0>0$ such that, for all $x\in B_{r_0}(x_0)$, we have:*

(i) either

$$u(x)-\psi(x) = a_0\left((x-x_0)\cdot\boldsymbol{\nu}\right)_+^{1+s} + \mathcal{O}(|x-x_0|^{1+s+\gamma}), \tag{22}$$

for some $a_0>0$, $\boldsymbol{\nu}\in\mathbb{S}^{d-1}$, and $\gamma>0$,

(ii) or

$$u(x)-\psi(x) = \mathcal{O}(|x-x_0|^{1+s+\alpha}).$$

Moreover, the set of points satisfying (22) *(regular points) is an open set of Γ, and it is locally a $C^{1,\gamma}$ manifold.*

The following result is presented in [36, Corollary 4.5.3] without proof. Since this estimate will be useful in our constructions we present a proof.

Lemma 30 (Hölder continuity)**.** *Let $x_0 \in \Gamma$ be a regular point, namely, one that satisfies* (22)*. In the setting of Theorem 29, the vector and scalar*

$$\mathbf{b}(z) = \lim_{x \to z} \frac{1}{d(x)^s} \nabla(u(x) - \psi(x)), \qquad a(z) = \frac{|\mathbf{b}(z)|}{1+s}$$

are of class $C^{0,\gamma}$ in $B_{r_0}(x_0) \cap \Gamma$.

Proof. Let $x_1, x_2 \in \Gamma \cap B_{r_0}(x_0)$ be two arbitrary free boundary points which we assume to be regular points and so to satisfy Theorem 29. Since the set of regular points is open in Γ, this may require further restricting $r_0 > 0$.

Denote $v = u - \psi$. It is possible to show that [36, Proposition 4.4.15]

$$\left\| \frac{1}{d^s} \nabla v \right\|_{C^{0,\gamma}(B_{r_0}(x_0))} \le c_0,$$

whence

$$\left| \frac{1}{d(x)^s} \nabla v(x) - \frac{1}{d(y)^s} \nabla v(y) \right| \le c_0 |x-y|^\gamma, \qquad \forall x, y \in B_{r_0}(x_0).$$

Since Γ is $C^{1,\gamma}$ within $B_{r_0}(x_0)$ we let $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \mathbb{S}^{d-1}$ be the unit normals to Γ at $x_1,\ x_2$ pointing towards Ω^+. Let $x \to x_1$, $y \to x_2$ and use the fact that

$$\lim_{x \to x_1} \frac{1}{d(x)^s} \nabla v(x) = \mathbf{b}(x_1), \qquad \lim_{y \to x_2} \frac{1}{d(y)^s} \nabla v(y) = \mathbf{b}(x_2),$$

to deduce that

$$|\mathbf{b}(x_1) - \mathbf{b}(x_2)| \le c_0 |x_1 - x_2|^\gamma,$$

whence $\mathbf{b} \in C^{0,\gamma}(B_{r_0}(x_0) \cap \Gamma)$. Since $a(x_i) = \frac{1}{1+s} |\mathbf{b}(x_i)|$, we infer that

$$|a(x_1) - a(x_2)| \lesssim ||\mathbf{b}(x_1)| - |\mathbf{b}(x_2)|| \le |\mathbf{b}(x_1) - \mathbf{b}(x_2)| \le c_0 |x_1 - x_2|^\gamma.$$

This is the desired estimate for a. □

To exploit the previous result we make the following convenient, but realistic, regularity assumption on Γ.

Assumption 31 (regular points)**.** The free boundary Γ consists only of regular points, namely those that satisfy Theorem 26 or (22).

Next we discuss the fundamental *nondegeneracy properties* (NDP). Given $\varepsilon > 0$ we let $\mathfrak{S}(\Gamma, \varepsilon)$ denote a strip of thickness ε around the free boundary Γ, namely,

$$\mathfrak{S}(\Gamma, \varepsilon) = \{ x \in \Omega \mid d(x) = \operatorname{dist}(x, \Gamma) < \varepsilon \}, \qquad \mathfrak{S}^+(\Gamma, \varepsilon) = \mathfrak{S}(\Gamma, \varepsilon) \cap \Omega^+.$$

The following result is known as an NDP in distance. It prescribes a pointwise behavior of $u - \psi$ (growth with rate at least $1+s$) if one is close to a regular point of the free boundary Γ.

Corollary 32 (NDP in distance)**.** *Let $K \Subset \Omega$ and set $\tilde{\Gamma} = \Gamma \cap K$. If Assumption 31 is valid, then there exists constants $a, \varepsilon_0 > 0$ such that*

$$u(x) - \psi(x) \ge a d(x)^{1+s}, \qquad \forall x \in \mathfrak{S}^+(\tilde{\Gamma}, \varepsilon_0).$$

Proof. We proceed in several steps.

1. Since, by Assumption 31, every point $x_0 \in \tilde{\Gamma}$ is regular we have that $a(x_0) = \frac{|\mathbf{b}(x_0)|}{1+s} > 0$, and a and $\mathbf{b}$ are of class $C^{0,\gamma}$ in view of Lemma 30. We then deduce that
$$a = \frac{1}{2}\min_{x_0\in\tilde{\Gamma}} a(x_0) > 0,$$
because $\tilde{\Gamma}$ is compact.
2. Let $v = u - \psi$, $x_0 \in \tilde{\Gamma}$, and $r_0 > 0$ be such that
$$\left\| \frac{1}{d^s}\nabla v \right\|_{C^{0,\gamma}(B_{r_0}(x_0))} \le C_0 = C(x_0).$$
Given $x \in B_{r_0/2}(x_0)$ we let $x_1 \in \tilde{\Gamma} \cap B_{r_0}(x_0)$ be a point at minimal distance, i.e.,
$$d(x) = |x - x_1| = (x - x_1)\cdot \boldsymbol{\nu}_1.$$
Then, for y close to x_1, we have
$$\left| \frac{1}{d(x)^s}\nabla v(x) - \frac{1}{d(y)^s}\nabla v(y) \right| \le C_0|x-y|^\gamma,$$
whence, upon computing the limit as $y \to x_1$, we get
$$\left| \frac{1}{d(x)^s}\nabla v(x) - \mathbf{b}_1 \right| \le C_0|x-x_1|^\gamma,$$
This implies that
$$\frac{1}{d(x)^s}\partial_{\boldsymbol{\nu}_1} v(x) \ge \mathbf{b}_1\cdot\boldsymbol{\nu}_1 - C_0|x-x_1|^\gamma = |\mathbf{b}_1| - C_0|x-x_1|^\gamma \ge (1+s)a,$$
provided $r_0^\gamma C_0 \le (1+s)a$, upon restricting r_0 if necessary. Therefore, we deduce the nondegeneracy property for the normal derivative
$$\partial_{\boldsymbol{\nu}_1} v(x) \ge (1+s)ad(x)^s = (1+s)a\left((x-x_1)\cdot\boldsymbol{\nu}_1\right)_+^s.$$
3. Let $x(t) = tx + (1-t)x_1$ denote any point in the segment joining x_1 and x, $t \in [0,1]$. Then $x_1 \in \tilde{\Gamma}$ is again a point in $\tilde{\Gamma}$ at a minimal distance to $x(t)$. The previous point implies then that
$$\frac{\mathrm{d}v(x(t))}{\mathrm{d}t} = \partial_{\boldsymbol{\nu}_1} v(x(t))|x-x_1| \ge (1+s)at^s|x-x_1|^{1+s},$$
or
$$v(x(t)) = v(x(1)) - v(x(0)) = \int_0^1 \frac{\mathrm{d}v(x(t))}{\mathrm{d}t}\,\mathrm{d}t \ge a|x-x_1|^{1+s},$$
for all $x \in B_{r_0/2}(x_0)$. This is the desired local nondegeneracy property.
4. We cover $\tilde{\Gamma}$ with balls $B_{r_0/2}(x_0)$ for every $x_0 \in \tilde{\Gamma}$. Since $\tilde{\Gamma}$ is compact, there is a finite subcovering
$$\tilde{\Gamma} \subset \bigcup_{m=1}^M B_{r_m/2}(x_m).$$
Finally, let $\varepsilon_0 > 0$ be the distance from $\tilde{\Gamma}$ to the complement of $\cup_{m=1}^M B_{r_m/2}(x_m)$. Then every $x \in \mathfrak{S}^+(\tilde{\Gamma}, \varepsilon_0)$ belongs to a ball $B_{r_m/2}(x_m)$ for which the previous step applies.

This concludes the proof. □

Remark 33 (NDP)**.** Observe that Corollary 32 implies the following weaker form of nondegeneracy: For all $x_0 \in \tilde{\Gamma}$ and $r \in (0, \varepsilon_0]$ we have

$$\sup_{x \in B_r(x_0)} (u(x) - \psi(x)) \geq ar^{1+s}. \tag{23}$$

This inequality is also valid for all $\varepsilon_0 < r \leq \operatorname{diam}(\Omega)$ because

$$\varepsilon_0 = \frac{\varepsilon_0}{r} r \geq \frac{\varepsilon_0}{\operatorname{diam}(\Omega)} r,$$

which yields (23) with the constant a replaced by

$$\widetilde{a} = a \left(\frac{\varepsilon_0}{\operatorname{diam}(\Omega)} \right)^{1+s}.$$

We now make an explicit assumption about the boundary behavior of $u - \psi$, which is not only useful for the subsequent argument, but it also has been used in deriving regularity via a localization argument; see Proposition 18.

Assumption 34 (boundary behavior)**.** The obstacle is strictly negative on $\partial\Omega$, i.e., there is $c_0 > 0$ for which

$$\psi(x) \leq -c_0 < 0, \quad \forall x \in \partial\Omega.$$

The assumption above, by continuity, assumes that the free boundary Γ is uniformly away from the boundary of the domain $\partial\Omega$, and thus the problem is linear in a neighborhood of $\partial\Omega$; see [17] for details in the case $\eta \equiv 1$. In addition, this implies that the NDP in distance of Corollary 32 holds for all $x \in \mathfrak{S}^+(\Gamma, \varepsilon_0)$.

Let now $\varepsilon \in (0, \varepsilon_0]$. We define a ε–neighborhood of the free boundary Γ

$$\mathfrak{N}(\Gamma, \varepsilon) = \left\{ x \in \Omega^+ \;\middle|\; 0 < u(x) - \psi(x) < \varepsilon^{1+s} \right\}.$$

Corollary 35 (comparing $\mathfrak{N}(\Gamma, \varepsilon)$ and $\mathfrak{S}(\Gamma, \varepsilon)$)**.** *If Assumptions 31 and 34 hold, then there is $\varepsilon_1 \in (0, \varepsilon_0]$, with $\varepsilon_0 > 0$ defined in Corollary 32, such that*

$$\mathfrak{N}\left(\Gamma, a^{\frac{1}{1+s}} \varepsilon \right) \subset \mathfrak{S}(\Gamma, \varepsilon), \qquad \forall \varepsilon \in (0, \varepsilon_1].$$

Proof. Consider the compact set ω between $\mathfrak{S}^+(\Gamma, \varepsilon_0)$ and $\partial\Omega$, namely,

$$\omega = \overline{\Omega \setminus (\mathfrak{S}^+(\Gamma, \varepsilon_0) \cap \Omega^0)}.$$

Set again $v = u - \psi$. In view of Assumption 34 and the fact that $v \in C^{0,s}(\bar{\Omega})$, we deduce

$$v \geq c_0, \text{ on } \partial\Omega, \qquad v \geq a\varepsilon_0^{1+s} \text{ on } \partial\mathfrak{S}^+(\Gamma, \varepsilon_0) \cap \Omega^+,$$

and that there is a constant $c_1 > 0$ such that

$$v(x) \geq c_1, \qquad \forall x \in \omega.$$

Let $\varepsilon_1 \in (0, \varepsilon_0]$ be given by $a\varepsilon_1^{1+s} = c_1$. The definition of $\mathfrak{S}(\Gamma, \varepsilon)$ in conjunction with the NDP in distance of Corollary 32 guarantee that for all $\varepsilon \in (0, \varepsilon_1]$

$$v(x) \geq a\varepsilon^{1+s}, \qquad \forall x \in \mathfrak{S}(\Gamma, \varepsilon)^c \cap \Omega^+.$$

Therefore, for all $x \in \mathfrak{N}(\Gamma, a^{\frac{1}{1+s}} \varepsilon)$ we have

$$0 < v(x) < \left(a^{\frac{1}{1+s}} \varepsilon \right)^{1+s} = a\varepsilon^{1+s},$$

whence $x \in \mathfrak{S}(\Gamma, \varepsilon)$ as asserted. This concludes the proof. $\square$

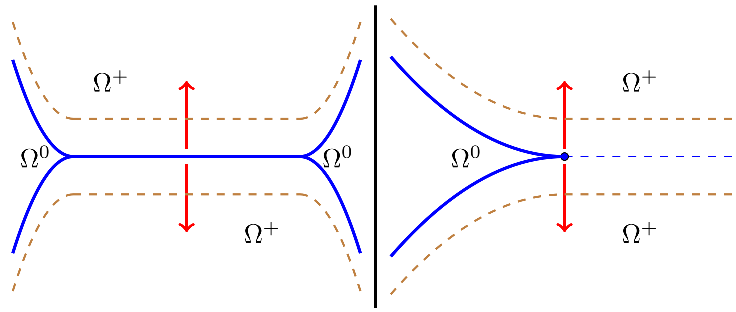

FIGURE 1. Examples of singular free boundary points where the function $u - \psi$ has strict quadratic growth. We expect quadratic growth in the direction of the red arrows. The case on the right is more degenerate than the case on the left, and worse for the approximation of the free boundary Γ (depicted in blue). In both cases we have that $\dim\ker \mathbf{A} = 1$. A discrete free boundary $\Gamma_{\mathscr{T}}$, which is at a distance $\mathcal{O}(\delta(h)^{1/2})$ is depicted in dashed brown.

We comment that the inclusion asserted in the previous Corollary is the typical assumption that is made for the error estimates in distance for the classical obstacle problem. See, for instance, [55, Section 2(d)].

5.2.2. *Singular points.* We now examine nondegeneracy for singular points. We base our discussion on [6], which requires that Theorem 26 holds. We recall that a singular point $x_0 \in \Gamma$ corresponds to

$$a(x_0) = \frac{|\mathbf{b}(x_0)|}{1+s} = 0$$

The characterization of singular points given in [6] hinges on the following structural assumption.

Assumption 36 (singular points)**.** There is $c_0 > 0$ and $\gamma > 0$ such that $\psi \in C^{3,\gamma}(\Omega)$, $f + \Delta\psi \le -c_0 < 0$ in $\{x \in \Omega \mid \psi(x) > 0\}$ and $\emptyset \ne \{x \in \Omega \mid \psi(x) > 0\} \Subset \Omega$.

For the rest of the discussion regarding singular points we restrict our attention to the *fractional Laplacian*, i.e., $\eta \equiv 1$. The following result is proved in [6, Lemma 3.2].

Lemma 37 (general growth)**.** *If Assumption 36 holds, then there are constants* $a, r_0 > 0$ *such that, for all* $x_0 \in \Gamma$

$$\sup_{x \in B_r(x_0)} (u(x) - \psi(x)) \ge ar^2, \qquad \forall r \in (0, r_0).$$

Notice that this is similar to (23) but with exponent 2 instead of $1+s$.

Consider now the following class of homogeneous polynomials of degree two:

$$\mathbb{P}_2^+ = \left\{ p_2(x) = \frac{1}{2} x^\intercal \mathbf{A} x \;\middle|\; \mathbf{A} \in \mathbb{R}^{d\times d} \setminus \{\mathbf{O}\},\ \mathbf{A}^\intercal = \mathbf{A}, \sigma(\mathbf{A}) \subset [0,\infty) \right\}.$$

The following result, which can be found in [6, Proposition 7.2], is crucial to characterize singular points but does not play an important role in our discussion.

Proposition 38 (growth at singular points)**.** *If Assumption 36 holds, then there is a modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any $x_0 \in \Gamma$ singular, we have*

$$u(x) - \psi(x) = p_2^{x_0}(x - x_0) + \omega(|x - x_0|)|x - x_0|^2,$$

for some $p_2^{x_0} \in \mathbb{P}_2^+$.

Remark 39 (growth at singular points)**.** In the setting of Proposition 38 it is important to notice that, if $\dim\ker \mathbf{A} = k$, with $k \in \{0, \dots, d-1\}$, then $u - \psi$ exhibits strict quadratic growth in the directions orthogonal to $\ker \mathbf{A}$. Since there is at least one such direction, we conclude that Proposition 38 implies Lemma 37.

The first example of this scenario is when $\dim\ker \mathbf{A} = 0$. This corresponds to an isolated contact point x_0, with the function $u - \psi$ growing quadratically in all directions emanating from x_0.

As a second example we consider the situations given in Figure 1. Although the geometry is quite distinct, in both cases we have $\dim\ker \mathbf{A} = 1$. We expect quadratic growth in the directions of the red arrows. The case on the right is more degenerate than the case on the left, and worse for the approximation of the free boundary Γ. This will be discussed further below.

5.3. **Error estimates for free boundaries.** Let us now put the pointwise error estimates of the previous sections to use and, on the basis of the discussions of Section 5.2, obtain approximation properties for the free boundary Γ. To start we mention that there is $\psi_h = I_h\psi \in \mathbb{V}_{\mathscr{T}}$ (the Lagrange interpolant) such that $\psi_h(z) = \psi(z)$ for all $z \in \mathscr{N}_{\mathscr{T}}$ and, more importantly,

$$\|\psi - \psi_h\|_{L^\infty(\Omega)} = \sigma(h), \tag{24}$$

for some function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sigma(h) \downarrow 0$ as $h \downarrow 0$. Typically, and this shall be the case if $\psi \in \Psi$,

$$\sigma(h) \lesssim |\psi|_{W^{2,\infty}(\Omega)} h^2.$$

Notice that (19), and its analysis, remain unchanged if we replace ψ by ψ_h.

The next step is to define the discrete free boundary Γ_h. To do so, instead of looking at the zero level set of $u_h - \psi_h$ we consider the level set at height

$$\delta(h) = \tilde{\mathfrak{e}}(h) + \sigma(h),$$

where $\tilde{\mathfrak{e}}$ was defined in Theorem 24, see also (21). Thus we define the discrete noncontact and contact sets, respectively, to be

$$\Omega^+_{\mathscr{T}} = \{x \in \Omega \mid u_h(x) - \psi_h(x) > \delta(h)\}, \quad \Omega^0_{\mathscr{T}} = \{x \in \Omega \mid u_h(x) - \psi_h(x) \le \delta(h)\}.$$

The free boundary is then,

$$\Gamma_{\mathscr{T}} = \partial\Omega^+_{\mathscr{T}} \cap \Omega = \partial\Omega^0_{\mathscr{T}} \cap \Omega.$$

We intend to prove that, in a sense, Γ and $\Gamma_{\mathscr{T}}$ are close. This will be quantified by means of the *Hausdorff distance.*

Definition 40 (Hausdorff distance)**.** Let $A, B \subset \mathbb{R}^d$. Their Hausdorff distance is

$$d_H(A, B) = \max\left\{ \max_{x\in A} \operatorname{dist}(x, B), \max_{y \in B} \operatorname{dist}(y, A) \right\}.$$

The error estimate for free boundaries reads as follows.

Theorem 41 (free boundary approximation)**.** *Let* Γ *satisfy Assumptions 31 and 34, and let* $a > 0$ *be given in Corollary 32 and* (23)*. Then, there is* $h_0 > 0$ *such that*

$$d_H(\Gamma, \Gamma_{\mathscr{T}}) \le \left(\frac{2}{a} \delta(h) \right)^{\frac{1}{1+s}}, \qquad \forall h \in (0, h_0].$$

Proof. We proceed in three steps.

1. We first show that $\Omega^0 \subset \Omega^0_{\mathscr{T}}$. To achieve this consider $x \in \Omega^0$, so that $u(x) = \psi(x)$. Thus, using (21) and (24) we have

$$u_h(x) - \psi_h(x) = (u_h(x) - u(x)) + (\psi(x) - \psi_h(x)) \le \tilde{\mathfrak{e}}(h) + \sigma(h) = \delta(h).$$

   In other words, $x \in \Omega^0_{\mathscr{T}}$.
2. Next we show that

$$\Gamma_{\mathscr{T}} \subset \mathfrak{S}\left(\Gamma, \left(\frac{2}{a} \delta(h) \right)^{\frac{1}{1+s}} \right).$$

   Let then $x \in \Gamma_{\mathscr{T}}$ and $x_0 \in \Gamma$ be the closest point to it, i.e.,

$$|x - x_0| = d(x) = \operatorname{dist}(x, \Gamma).$$

   From the previous step we know that $x \in \Omega^+$. Since

$$u_h(x) - \psi_h(x) = \delta(h),$$

   and, in view of Theorem 24, $u(x) - u_h(x) \le \tilde{\mathfrak{e}}(h)$ we realize that

$$\begin{aligned} u(x) - \psi(x) &\le (u(x) - u_h(x)) + (u_h(x) - \psi_h(x)) + (\psi_h(x) - \psi(x)) \\ &\le \tilde{\mathfrak{e}}(h) + \delta(h) + \sigma(h) = 2\delta(h). \end{aligned}$$

   Let now $h_0 > 0$ be sufficiently small so that $\delta(h_0) \le \frac{a}{2} \varepsilon_1^{1+s}$, where $\varepsilon_1 > 0$ is given in Corollary 35. This implies

$$x \in \mathfrak{N}(\Gamma, a^{\frac{1}{1+s}} \varepsilon_1) \subset \mathfrak{S}(\Gamma, \varepsilon_1).$$

   Therefore, Corollary 32 yields

$$u(x) - \psi(x) \ge a \operatorname{dist}(x, \Gamma)^{1+s},$$

   whence,

$$a \operatorname{dist}(x, \Gamma)^{1+s} \le 2\delta(h),$$

   i.e.,

$$\operatorname{dist}(x, \Gamma) \le \left(\frac{2}{a} \delta(h) \right)^{\frac{1}{1+s}},$$

   as desired.
3. We show that

$$\Gamma \subset \mathfrak{S}\left(\Gamma_{\mathscr{T}}, \left(\frac{2}{a} \delta(h) \right)^{\frac{1}{1+s}} \right).$$

   Indeed, let $x_0 \in \Gamma$ and assume, for the sake of contradiction, that

$$R = \operatorname{dist}(x, \Gamma_{\mathscr{T}}) > \left(\frac{2}{a} \delta(h) \right)^{\frac{1}{1+s}}.$$

Notice that the first step of the proof yields that if $B_R(x_0) \subset \Omega^0_{\mathscr{T}}$, then

$$u_h(y) - \psi_h(y) \le \delta(h), \qquad \forall y \in B_R(x_0).$$

We now recall (23) which is a consequence of Corollary 32: for all $r \le \mathrm{diam}(\Omega)$

$$\sup_{y \in B_r(x_0)} (u(y) - \psi(y)) \ge a r^{1+s}.$$

In other words, by compactness and continuity, there is $y \in B_R(x_0)$ such that

$$u(y) - \psi(y) \ge a R^{1+s}.$$

On the other hand,

$$\begin{aligned} u_h(y) - \psi_h(y) &= (u_h(y) - u(y)) + (u(y) - \psi(y)) + (\psi(y) - \psi_h(y)) \\ &\ge -\tilde{\mathfrak{e}}(h) + aR^{1+s} - \sigma(h) > -\tilde{\mathfrak{e}}(h) + 2\delta(h) - \sigma(h) > \delta(h), \end{aligned}$$

which is a contradiction. This proves the assertion and concludes the proof. □

Remark 42 (stability)**.** We observe that the nondegeneracy constant $a > 0$, defined in Corollary 32, acts as a stability parameter in the estimate of Theorem 41.

Remark 43 (localized estimate)**.** Let $K \Subset \Omega$ be a compact so that $\Gamma \cap K$ is made only of regular points. We thus allow Γ to have singular points in $\Gamma \cap K^c$. Since the set of regular points is relatively open in Γ, we realize that Corollary 32 is valid in K, i.e.,

$$u(x) - \psi(x) \ge a_K d(x)^{1+s}, \quad \forall x \in \mathfrak{S}^+(\Gamma, \varepsilon) \cap K,$$

for some constant $a_K > 0$ that, in particular, depends on the compact K. We can thus repeat the proof of Theorem 41 locally to deduce

$$d_H(\Gamma \cap K, \Gamma_{\mathscr{T}} \cap K) \le \left(\frac{2}{a_K} \delta(h) \right)^{\frac{1}{1+s}}, \qquad \forall h \in (0, h_0].$$

Theorem 41 and Remark 43 assume that, at least locally, there are no singular free boundary points. Let us conclude the discussion by presenting some results about the general case, namely when Γ contains singular points. An inspection of the proof of Theorem 41 shows that the second step cannot be carried out, but the first and third one remain valid. In fact, the first step hinges on the definition of $\Gamma_{\mathscr{T}}$ and the last step relies on the growth condition (23) which, in principle, could be replaced by the general growth condition provided in Lemma 37. This brings about the following result.

Theorem 44 (error estimates for singular points)**.** *If Assumption 36 holds and $\eta \equiv 1$, then for every $x \in \Gamma$ we have that $x \in \Omega^0_{\mathscr{T}}$ and*

$$\mathrm{dist}(x, \Gamma_{\mathscr{T}}) \le \left(\frac{2}{a} \delta(h) \right)^{\frac{1}{2}}. \tag{25}$$

Remark 45 (a posteriori error estimation)**.** Estimate (25) establishes an error of Γ relative to $\Gamma_{\mathscr{T}}$. This is the spirit of an *a posteriori* error estimate. We refer to [57] for similar estimates for the classical obstacle problem for the Laplace operator.

Remark 46 (regularity)**.** Estimate (25) requires no regularity of the free boundary Γ beyond the nondegeneracy property of Lemma 37, which relies on Assumption 36. The free boundary regularity stated in Proposition 38 is not needed to assert (25). It is then natural to wonder how the quadratic growth in certain directions established in Proposition 38 can be exploited for free boundary approximation. For instance, one may use the approximations that are depicted in Figure 1.

In the left panel of Figure 1 the discrete free boundary is uniformly close to Γ and a global estimate of the form

$$\operatorname{dist}(x,\Gamma) \lesssim \delta(h)^{1/2}, \qquad \forall x \in \Gamma_{\mathscr{T}}$$

is expected.

On the other hand, the right panel of Figure 1 shows that points of $\Gamma_{\mathscr{T}}$ may be far away from Γ and, thus, the estimate above cannot be obtained.

To conclude the discussion of approximation of free boundaries at singular points we mention that the scenarios depicted in Figure 1 also illustrate the relevance of the localized error estimates alluded to in Remark 43.

6. A concave, fully nonlinear, nonlocal, problem

As a final application of our two-scale discretization, we shall consider a nonlocal Hamilton Jacobi Bellman equation. Let $s \in (0,1)$, and $\eta_i \in \mathcal{C}(\lambda,\Lambda)$ $(i \in \{1,2\})$ for some λ, Λ. Given $f \in C(\bar{\Omega})$ we must find $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$\min\{\mathcal{L}_{\eta_1}[u], \mathcal{L}_{\eta_2}[u]\} = f, \quad \text{in } \Omega, \qquad u = 0, \quad \text{in } \Omega^c. \tag{26}$$

Equations of this form have gathered a lot of attention in recent times. We refer the reader to, for instance, [21, 23, 44] for details regarding the existence, uniqueness, and regularity of solutions. Regarding numerics, in the case $\Omega = \mathbb{R}^d$, see [8, 59, 25, 10, 9, 27, 28]. To our knowledge, however, no reference addresses the numerical approximation in the case of a bounded domain, as in (26). Our goal here will be to provide then the first convergent method for such problem. To achieve this we will get inspiration from [47, 35, 2, 11], and relate this problem to a sequence of obstacle problems.

6.1. Existence and uniqueness.

Since it will be useful for our numerical purposes, we begin by discussing the existence of solutions. Below, by $\div$ we mean the remainder of integer division.

Theorem 47 (existence and uniqueness). *Assume that $f \in C(\bar{\Omega})$ and that $\mathcal{L}_{\eta_i}$ have a common supersolution, i.e., there is $U \in C^2(\mathbb{R}^d)$ for which*

$$\mathcal{L}_{\eta_i}[U] \geq f \quad \text{in } \Omega, \qquad U \geq 0, \quad \text{in } \Omega^c.$$

Then, problem (26) has a unique solution. Moreover, this solution can be obtained as the uniform limit of the following sequence: $u_0 \in \widetilde{H}^s(\Omega)$ solves

$$\mathcal{L}_{\eta_1}[u_0] = f, \quad \text{in } \Omega.$$

For $k \in \mathbb{N}$, let $i = k \div 2 + 1$. The function $u_k \in \widetilde{H}^s(\Omega)$ solves

$$\min\{\mathcal{L}_{\eta_i}[u_k] - f, u_k - u_{k-1}\} = 0, \quad \text{in } \Omega. \tag{27}$$

Proof. We split the proof in several steps.
<u>Convergence:</u> Notice that the sequence $\{u_k\}_{k\in\mathbb{N}_0}$, as solutions of an obstacle problem with common right hand side, lie uniformly in $C^{0,s}(\bar{\Omega})$. Moreover, by construction, the sequence is nondecreasing, i.e., $u_k \geq u_{k-1}$ and bounded above. Indeed, since

$$\mathcal{L}_{\eta_1}[U] \geq f = \mathcal{L}_{\eta_1}[u_0] \quad \text{in } \Omega, \qquad U - u_0 \geq 0, \quad \text{in } \Omega^c,$$

we must have, by comparison, that $U \geq u_0$. Assume next that, for some $k \in \mathbb{N}$, we have $u_{k-1} \leq U$. Let $x \in \Omega$. We either have $u_k(x) = u_{k-1}(x) \leq U(x)$ or $u_k(x) > u_{k-1}(x)$. However, at such points we must have

$$\mathcal{L}_{\eta_i}[u_k](x) = f(x) \leq \mathcal{L}_{\eta_i}[U](x).$$

Therefore, if we denote $\Omega_k^+ = \{x \in \Omega \mid u_k(x) > u_{k-1}(x)\}$ we see that

$$\mathcal{L}_{\eta_i}[U - u_k] \geq 0, \quad \text{in } \Omega_k^+, \qquad U - u_k \geq 0, \quad \text{in } (\Omega_k^+)^c.$$

Consequently, by comparison, we must have that $u_k \leq U$ in Ω_k^+. By Dini's theorem then, there is $u \in C(\bar{\Omega})$ such that $u_k \rightrightarrows u$.
Solution: The uniform convergence also implies that, for $i \in \{1, 2\}$,

$$\mathcal{L}_{\eta_i}[u] \geq f, \quad \text{in } \Omega.$$

On the other hand, if we show that for every $k \in \mathbb{N}_0$

$$\min\{\mathcal{L}_{\eta_1}[u_k], \mathcal{L}_{\eta_2}[u_k]\} \leq f, \quad \text{in } \Omega, \tag{28}$$

we may pass to the limit and obtain that

$$\mathcal{L}_{\eta_1}[u] \geq f, \quad \mathcal{L}_{\eta_2}[u] \geq f, \quad \min\{\mathcal{L}_{\eta_1}[u], \mathcal{L}_{\eta_2}[u]\} \leq f,$$

and thus u must be a solution.
Proof of (28): We argue by induction. By the way the iterative scheme is initialized, (28) holds for $k = 0$. Assume now that the inequality holds for some $k \in \mathbb{N}$. Consider the noncoincidence set

$$x_n \in \Omega_k^+ = \{x \in \Omega \mid u_{k+1}(x) > u_k(x)\}.$$

By the complementarity conditions we must have, for some $i \in \{1, 2\}$, that

$$\mathcal{L}_{\eta_i}[u_{k+1}](x_n) = f(x_n).$$

If, on the other hand, we consider the coincidence set

$$x_c \in \Omega_k^0 = \{x \in \Omega \mid u_{k+1}(x) = u_k(x)\} \cap \Omega,$$

we see that $u_{k+1}(x_c) = u_k(x_c)$ and $u_{k+1}(y) \geq u_k(y)$ for all $y \in \mathbb{R}^d$. By the inductive hypothesis, there is $i_0 \in \{1, 2\}$ for which

$$\mathcal{L}_{\eta_{i_0}}[u_k](x_c) \leq f(x_c),$$

then

$$\begin{aligned}
\mathcal{L}_{\eta_{i_0}}[u_{k+1}](x_c) &= \text{v.p.} \int_{\mathbb{R}^d} (u_{k+1}(x_c) - u_{k+1}(y)) \frac{1}{|x-y|^{d+2s}} \eta\left(\frac{x_c - y}{|x_c - y|}\right) \mathrm{d}y \\
&= \text{v.p.} \int_{\mathbb{R}^d} (u_k(x_c) - u_{k+1}(y)) \frac{1}{|x-y|^{d+2s}} \eta\left(\frac{x_c - y}{|x_c - y|}\right) \mathrm{d}y \\
&\leq \text{v.p.} \int_{\mathbb{R}^d} (u_k(x_c) - u_k(y)) \frac{1}{|x-y|^{d+2s}} \eta\left(\frac{x_c - y}{|x_c - y|}\right) \mathrm{d}y \\
&= \mathcal{L}_{\eta_{i_0}}[u_k](x_c) \leq f(x_c),
\end{aligned}$$

as claimed.
Uniqueness: Follows by comparison. □

Remark 48 (obstacle)**.** Notice that the iterative scheme presented in Theorem 47 requires, at every step the solution of an obstacle problem like the one described in Section 5.

Remark 49 (supersolution)**.** Theorem 47 relies on the existence of a common supersolution for the operators. A possible common supersolution is

$$U(x) = \frac{A}{2}|x|^2 + B.$$

The constant B can be chosen so that $U \geq 0$ in Ω^c, whereas we can choose A, depending only on d, s, λ, Λ, and $-\|f\|_{L^\infty(\Omega)}$, to obtain a supersolution.

Remark 50 (rate of convergence)**.** It is not known to us whether a rate of convergence for the iteration of Theorem 47 can be established.

6.2. **A convergent scheme.** As a final application of our constructions we present a scheme for problem (26). Inspired by the proof of Theorem 47 we consider the following iterative scheme: $u_{h,0} \in \mathbb{V}^0_{\mathscr{T}}$ is such that

$$\mathcal{L}_{\eta_1,\boldsymbol{\varepsilon}}[u_{h,0}](z) = f(z), \quad \forall z \in \mathscr{N}^i_{\mathscr{T}}.$$

For $k \in \mathbb{N}$, let $i = k \div 2 + 1$. The function $u_{h,k} \in \mathbb{V}^0_{\mathscr{T}}$ solves

$$\min\left\{\mathcal{L}_{\eta_i,\boldsymbol{\varepsilon}}[u_{h,k}](z) - f(z), u_{h,k}(z) - u_{h,k-1}(z)\right\} = 0, \quad \forall z \in \mathscr{N}^i_{\mathscr{T}}. \tag{29}$$

We immediately observe that, for every k, (29) has a unique solution. We would now like to obtain convergence of $u_{h,k}$ to u, the solution to (26). In order to achieve this, we introduce, for $k \in \mathbb{N}$, the function $\widetilde{u}_{h,k} \in \mathbb{V}^0_{\mathscr{T}}$ that is the solution of

$$\min\left\{\mathcal{L}_{\eta_i,\boldsymbol{\varepsilon}}[\widetilde{u}_{h,k}](z) - f(z), \widetilde{u}_{h,k}(z) - I_h u_{k-1}(z)\right\} = 0, \quad \forall z \in \mathscr{N}^i_{\mathscr{T}}. \tag{30}$$

Notice that $\widetilde{u}_{h,k}$ is nothing but an approximation to the function u_k from the scheme of Theorem 47. As such we expect that $\widetilde{u}_{h,k} \to u_k$ with a given rate. We quantify this by introducing the following assumption.

Assumption 51 (approximation)**.** There is a continuous function $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\sigma(h) \downarrow 0$ as $h \downarrow 0$ for which

$$\sup_{k \in \mathbb{N}_0} \|u_k - \widetilde{u}_{h,k}\|_{L^\infty(\Omega)} \leq \sigma(h),$$

where $\{u_k\}_{k \in \mathbb{N}_0}$ are defined in Theorem 47 and $\{\widetilde{u}_{h,k}\}_{k \in \mathbb{N}_0}$ in (30).

Remark 52 (smoothness and compatibility)**.** While we would like to assert that the rate $\sigma(h)$ of Assumption 51 is that given by Theorem 24 we are unable to prove this. The reason is that the error estimates of this Theorem hinge on the regularity of Proposition 18 which need the obstacle to belong to the class Ψ. This means that, for every $k \in \mathbb{N}$, we must be able to assert that:

1. $u_k \in C^{2,1}(\Omega)$. While we are not able to verify this directly, we comment that this smoothness assumption is taken from [17], which in turn follows the arguments of [24, 67]. This may not be sharp.
2. $u_{k|\partial\Omega} < 0$. However, we have $u_{k|\partial\Omega} = 0$.

Clearly

$$\tilde{\mathfrak{e}}(h) \leq \sigma(h),$$

where $\tilde{\mathfrak{e}}(h)$ was defined in Theorem 24; see also (21). Let us now show convergence.

Theorem 53 (convergence)**.** *In the setting of Theorem 47 and under Assumption 51, let u solve* (26)*, and $\{u_{h,k}\}_{k \in \mathbb{N}_0, h>0}$ be the solutions to* (29)*. Assume that, as $k \uparrow \infty$ and $h \downarrow 0$, we have that*

$$k\,\sigma(h) \to 0.$$

Then, $u_{h,k} \rightrightarrows u$.

Proof. As mentioned above, the family $\{\widetilde{u}_{h,k}\}_{k\in\mathbb{N}_0} \subset \mathbb{V}^0_{\mathscr{T}}$, defined in (30), is an approximation of the obstacle problem defined in (27).

Consider now the difference $\widetilde{u}_{h,k} - u_{h,k}$. We claim that

$$\|\widetilde{u}_{h,k} - u_{h,k}\|_{L^\infty(\Omega)} \le \|u_{h,k-1} - I_h u_{k-1}\|_{L^\infty(\Omega)}. \tag{31}$$

Indeed, if we define $w_h = \widetilde{u}_{h,k} + \|u_{h,k-1} - I_h u_{k-1}\|_{L^\infty(\Omega)}$ we see that

$$w_h \ge \widetilde{u}_{h,k} = 0, \quad \text{in } \Omega^c.$$

Moreover, for $z \in \mathscr{N}^i_{\mathscr{T}}$,

$$w_h(z) \ge \widetilde{u}_{h,k}(z) + u_{h,k-1}(z) - I_h u_{k-1}(z) \ge u_{h,k-1}(z)$$

and $\mathcal{L}_{\eta_i,\boldsymbol{\varepsilon}}[w_h](z) \ge f(z)$. In other words, the function w_h is a supersolution to the obstacle problem (29). Lemma 22 then implies that $u_{h,k} \le w_h$, i.e.,

$$u_{h,k} - \widetilde{u}_{h,k} \le \|u_{h,k-1} - I_h u_{k-1}\|_{L^\infty(\Omega)}.$$

A similar argument shows the lower bound.

Using Assumption 51 we now iterate (31) to obtain

$$\begin{aligned}
\|u_k - u_{h,k}\|_{L^\infty(\Omega)} &\le \|u_k - \widetilde{u}_{h,k}\|_{L^\infty(\Omega)} + \|\widetilde{u}_{h,k} - u_{h,k}\|_{L^\infty(\Omega)} \\
&\le \sigma(h) + \|u_{h,k-1} - I_h u_{k-1}\|_{L^\infty(\Omega)} \\
&\le \sigma(h) + \|u_{k-1} - I_h u_{k-1}\|_{L^\infty(\Omega)} + \|u_{h,k-1} - u_{k-1}\|_{L^\infty(\Omega)} \\
&\le 2\sigma(h) + \|u_{h,k-1} - u_{k-1}\|_{L^\infty(\Omega)} \\
&\le (k+1)\sigma(h) + \|u_{h,0} - u_0\|_{L^\infty(\Omega)} \le (k+2)\sigma(h),
\end{aligned}$$

where, in the last step, we used that $u_{h,0}$ is an approximation to the solution of the linear problem.

The triangle inequality, and the assumption on k and h, yield the result. □

Appendix A. Approximation of integrodifferential operators of order $2s$ by second order differential operators

To gain some intuition about what properties of the operator we would like to preserve after regularization; and, in addition, to highlight the differences between the case of constant η (i.e., a fractional Laplacian) and a variable one, here we inspect the approximation of the integrodifferential operator

$$\mathcal{I}_\varepsilon[w](x) = \int_{B_\varepsilon} \frac{w(x+y) - 2w(x) + w(x-y)}{|y|^{d+2s}} \eta\left(\frac{y}{|y|}\right) \mathrm{d}y$$

by considering its action on a quadratic, i.e., $w \in \mathbb{P}_2$. This may help in justifying our choice, showing the stark difference between our approach and that of [39, 40], as well as providing some intuition into existing works that operate in a reverse way. For instance, [22, 58, 64] approximate (local) second order elliptic differential operators by integral operators like $\mathcal{I}_\varepsilon$. We emphasize, however, that to obtain consistent, monotone, finite-difference schemes for arbitrary (local) second order elliptic operators, one requires the use of wide stencils [49].

We begin by observing that, for $w \in \mathbb{P}_2$,

$$w(x+y) - 2w(x) + w(x-y) = D^2 w(x) : y \otimes y,$$

where $D^2 w(x)$ is the Hessian of w at x, $\otimes$ is the outer product, and $:$ is the Frobenius inner product. This means that

$$\begin{aligned}\mathcal{I}_\varepsilon[w](x) &= \int_{B_\varepsilon} D^2 w(x) : y \otimes y \frac{1}{|y|^{d+2s}} \eta\left(\frac{y}{|y|}\right) \mathrm{d}y \\ &= D^2 w(x) : \int_{B_\varepsilon} y \otimes y \frac{1}{|y|^{d+2s}} \eta\left(\frac{y}{|y|}\right) \mathrm{d}y = \mathbf{A} : D^2 w(x),\end{aligned}$$

where

$$\mathbf{A} = \int_{B_\varepsilon} y \otimes y \frac{1}{|y|^{d+2s}} \eta\left(\frac{y}{|y|}\right) \mathrm{d}y \in \mathbb{R}^{d\times d}, \qquad \mathbf{A}^\intercal = \mathbf{A}.$$

Consider now the particular case of $\eta \equiv 1$. The change of variables $y = \varepsilon z$ with $z \in B_1$ shows that

$$\mathbf{A} = \varepsilon^{2(1-s)} \int_{B_1} z \otimes z \frac{1}{|z|^{d+2s}} \,\mathrm{d}z = \frac{\varepsilon^{2(1-s)}\omega_d}{2d(1-s)} \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix, and $\omega_d = |\mathbb{S}^{d-1}|$. Therefore,

$$\mathcal{I}_\varepsilon[w](x) = \frac{\varepsilon^{2(1-s)}\omega_d}{2d(1-s)} \Delta w(x), \qquad \forall w \in \mathbb{P}_2.$$

Assume now that the coefficient η is not constant, but even, i.e., $\eta(z) = \eta(-z)$. Notice that this assumption is included in the class $\mathcal{C}(\lambda, \Lambda)$ of Definition 1. The change of variables $y = \varepsilon z$ implies that

$$\mathbf{A} = \varepsilon^{2(1-s)} \int_{B_1} z \otimes z \frac{1}{|z|^{d+2s}} \eta\left(\frac{z}{|z|}\right) \mathrm{d}z.$$

We see that $\mathbf{A}$ is symmetric positive definite. Let us use polar coordinates to obtain that

$$\begin{aligned}\mathbf{A} &= \varepsilon^{2(1-s)} \int_{\mathbb{S}^{d-1}} \theta \otimes \theta \eta(\theta) \int_0^1 \frac{r^2}{r^{d+2s}} r^{d-1} \,\mathrm{d}r \,\mathrm{d}\theta \\ &= \frac{\varepsilon^{2(1-s)}}{2(1-s)} \int_{\mathbb{S}^{d-1}} \theta \otimes \theta \eta(\theta) \,\mathrm{d}\theta = \frac{\varepsilon^{2(1-s)}}{2(1-s)} \mathbf{A}_0,\end{aligned}$$

where, since η is even, $\mathbf{A}_0$ is diagonal but anisotropic. In this case

$$\mathcal{I}_\varepsilon[w](x) = \frac{\varepsilon^{2(1-s)}}{2(1-s)} \mathbf{A}_0 : D^2 w(x) = \frac{\varepsilon^{2(1-s)}}{2(1-s)} \nabla\cdot(\mathbf{A}_0 \nabla w(x)), \qquad \forall w \in \mathbb{P}_2. \tag{32}$$

Let us, finally, consider the general case of a coefficient of the form $\eta(x, \frac{z}{|z|})$. In this case

$$\mathbf{A}_0 = \mathbf{A}_0(x) = \int_{\mathbb{S}^{d-1}} \theta \otimes \theta \eta(x, \theta) \,\mathrm{d}\theta,$$

and

$$\mathcal{I}_\varepsilon[w](x) = \frac{\varepsilon^{2(1-s)}}{2(1-s)} \mathbf{A}_0(x) : D^2 w(x), \qquad \forall w \in \mathbb{P}_2. \tag{33}$$

Remark 54 (anisotropy)**.** As we have already mentioned, it is not easy to construct approximations of second order anisotropic operators, like those in (32) and (33), that are monotone. In particular, (33) requires wide stencils.

Let us mention now what are the implications of the previous considerations for the approximation of the operator $\mathcal{L}_\eta$, introduced in (1). First, owing to the symmetry of the coefficient η, which is part of Definition 1, we have (4), with $\mathcal{K}(r) = r^{-d-2s}$. Next, if we assume that $w \in \mathbb{P}_2$, we have

$$\mathcal{L}_\eta[w](x) = \frac{1}{2}\int_{\mathbb{R}^d \setminus B_\varepsilon} (2w(x) - w(x+y) - w(x-y))\,\eta\left(\frac{y}{|y|}\right)\mathcal{K}_\varepsilon(|y|)\,\mathrm{d}y \\ - \frac{1}{2}\mathcal{I}_\varepsilon[w](x),$$

where the smooth kernel $\mathcal{K}_\varepsilon$ was introduced in (5), and

$$\mathcal{I}_\varepsilon[w](x) = \int_{B_\varepsilon} \frac{w(x+y) - 2w(x) + w(x-y)}{|y|^{d+2s}}\eta\left(\frac{y}{|y|}\right)\mathrm{d}y.$$

Using (32) we then conclude that, if $w \in \mathbb{P}_2$,

$$\mathcal{L}_\eta[w](x) = \frac{1}{2}\int_{\mathbb{R}^d \setminus B_\varepsilon} (2w(x) - w(x+y) - w(x-y))\,\eta\left(\frac{y}{|y|}\right)\mathcal{K}_\varepsilon(|y|)\,\mathrm{d}y \\ - \frac{\varepsilon^{2(1-s)}}{1-s}\nabla\cdot(\mathbf{A}_0\nabla w(x)).$$

In summary, by regularizing our operator, we have replaced it by a (local) second order differential operator in divergence form, with a small coefficient, and an operator of order zero. The monotonicity properties can then be driven by the zero order operator without imposing any mesh restrictions.

Finally, expression (4) motivates the consistency for quadratics. The symmetry of the kernel transforms the difference inside the integral into a second difference. A quadratic is the highest order polynomial for which a second difference gives exactly the value of the (directional) derivative. Notice that the approach from [39, 40], that merely truncates the kernel, does not preserve this property.

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