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*A Raquel, Felipe y Mateo
por los capítulos que escribimos
y los que restan por escribir.*

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Esta tese só não tem mais erros devido à valiosa colaboração de Massae e Fernando nas etapas finais de edição.

The most merciful thing in the world, I think,
is the inability of the human mind
to correlate all its contents.

We live on a placid island of ignorance
in the midst of black seas of infinity,
and it was not meant that we should voyage far.
The sciences, each straining in its own direction,
have hitherto harmed us little; but some day
the piecing together of dissociated knowledge
will open us such terrifying vistas of reality,
and of our frightful position therein,
that we shall either go mad from the revelation
or flee from the deadly light,
sheltering us in the safety
and peace
of a new dark age.

H. P. Lovecraft

Resumo

Estudamos a estrutura ressonante da região transnetuniana mediante métodos analíticos e numéricos. Para isso reformulamos a Teoria de Librações de Alta Excentricidade de Ferraz–Mello (1988) e incluímos no tratamento analítico as perturbações não ressonantes dos outros planetas gigantes. Com esta base teórica e também mediante integrações numéricas das equações exatas do movimento estudamos em detalhe a ressonância externa 2:3 com Netuno. O movimento de libração das partículas em ressonância com Netuno e perturbadas pelos planetas do Sistema Solar Exterior, apresenta uma complexidade maior do que a prevista pelas teorias construídas com o modelo planar de três corpos. Para poder estudar a evolução do movimento ressonante nas integrações numéricas, desenvolvemos um método de análise temporal de freqüências. Com esta técnica estudamos a estabilidade de várias ressonâncias de primeira e segunda ordem exteriores a Netuno e analisamos as possíveis consequências na evolução da população de objetos transnetunianos.

Summary

We study the resonant structure of the transneptunian region by analytical and numerical methods. In order to achieve this objective, we reformulate the High–Eccentricity Libration Theory of Ferraz–Mello (1988) and include the non–resonant perturbations of the other giant planets. With this theoretical base and with the numerical integrations of the exact equations of motion, we study in detail the exterior **2:3** resonance with Neptune. The libration motion of particles in resonance with Neptune and perturbed by the planets of the Outer Solar System, is more complex than that predicted by the theories based on the planar three–body problem. In order to study the evolution of the resonant motion resulting from the numerical integrations, we develop a time–frequency analysis method. Using this technic, we study the stability of several first and second order exterior resonances with Neptune and analyse the possible consequences in the evolution of the transneptunian population.

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Capítulo 1

Introdução.

O debate sobre a região transnetuniana começa quando Oort (1950) resolve (parcialmente) o problema da origem dos cometas, argumentando em favor da existencia da nuvem. Mas, como a nuvem se originou? Oort propôs que a nuvem poderia ser o resultado da dispersão de objetos originalmente localizados na região do cinturão de asteróides. Pouco depois Kuiper (1951) em um pormenorizado trabalho relativo à origem do Sistema Solar argumenta no sentido de que os cometas não se formaram na região dos asteróides mas na região dos planetas gigantes, e que foram ejetados para a nuvem no processo de formação do Sistema Solar Exterior. Nada impede que essa população primordial continue até regiões além Netuno. Kuiper argumenta que a população transnetuniana não poderia agrupar-se para formar outros planetas devido as baixas velocidades relativas que fazem o processo de acresção ineficiente; portanto essa população existiria até hoje. Pouco tempo antes, Edgeworth (1949) tinha sugerido que na região transnetuniana a nebulosa original deveria ter se condensado em pequenos corpos e que, dada a lentidão do processo de acresção, hoje eles deveriam existir em aglomerados que de quando em quando aparecem pelo Sistema Solar interior na forma de cometas.

Na década dos anos 60 alguns autores como Whipple (1964) e Hamid et al. (1968) intentaram estabelecer limites para a massa atual nessa região baseados nas perturbações que tal massa geraria em objetos do Sistema Solar. Na década dos 70 começa a ficar evidente que a família de cometas de curto período tinha características dinâmicas que dificilmente podiam ser reproduzidas a partir de cometas da nuvem de Oort capturados pelas perturbações planetárias. Fernández (1980) mostra que uma fonte localizada no cinturão de Kuiper seria muito mais eficiente que a nuvem de Oort na geração de cometas de curto período. Nos anos seguintes uma sucessão de trabalhos (especialmente Duncan et al., 1988 e Quinn et al., 1990) aponta para o cinturão de Kuiper como fonte dos cometas de curto período. Desde 1984 existe evidencia observacional (Aumann et al., 1984) da existencia de discos de poeira entorno de estrelas. Weissman (1984) sugeriu que esses discos estão compostos de cometas que formariam a nuvem de Oort primordial ainda sem se dispersarem a valores maiores do semieixo e a valores aleatórios da inclinação. O processo de formação da nuvem a partir de um disco de planetesimais perturbado pelos planetas gigantes, pela passagem de estrelas e pelo potencial galático, consolidou-se a partir

de uma simulaqão feita por Duncan et al. (1987).

É preciso dizer que existem autores que defendem a hipótese de que os cometas de curto período provêm da captura de cometas da nuvem e até de cometas interestelares (Valtonen et al., 1992). Mas, quando compararmos as características dinâmicas da populacão de cometas de curto período com as que podem ser obtidas mediante simulações de captura de cometas da nuvem para órbitas de curto período, vemos que elas são muito diferentes. Nossa projeto começou em 1992 exatamente neste ponto. Apresentamos no Anexo IV um trabalho onde estudamos a evolução orbital de cometas provenientes da nuvem de Oort até se transformar em cometas de curto período. Concluímos que sob determinadas hipóteses é possível reproduzir alguma característica dos cometas de curto período, mas ao analisar o conjunto dos parâmetros tais como a distribuição do parâmetro de Tisserand, da distancia periélica, da inclinação e o número da populacão de cometas em estado estacionário, resulta claro que a origem na nuvem de Oort não é viável.

Os Centauros, objetos que cruzam as órbitas de vários planetas gigantes e com propriedades físicas que sugerem uma origem não asteroidal, são considerados 'mensageiros' da região transnetuniana, e provavelmente futuros cometas de curto período. O primeiro deles a ser descoberto foi Chiron (Kowal, 1989), e depois de algumas tentativas fracassadas (Luu e Jewitt 1988, Levison e Duncan 1990), em agosto de 1992, o primeiro objeto da região transnetuniana, 1992QB₁, foi descoberto por Jewitt e Luu (1992). Na Fig. 1.1 vemos a populacão atual, composta por corpos de aproximadamente 200 kms de diâmetro. Observações recentes do telescópio Hubble indicam a existencia de objetos de diâmetro 5 – 10 kms que na região entre 30 e 40 UA constituiriam uma população da ordem de 10^8 corpos (Cochran et al. 1995).

Ainda resta explicar o mecanismo de transferencia da a região exterior para a interior do Sistema Solar. Isto fez crescer o interesse pelo estudo da dinâmica da região transnetuniana. Os primeiros resultados numéricos foram apresentados por Torbett (1989) e Torbett e Smoluchowski (1990). Mediante cálculo de expoentes de Lyapunov eles determinam que todas as partículas com distancia periélica menor ou igual a 45 UA seguem uma evolução caótica. Mas, um expoente de Lyapunov positivo somente é uma medida da instabilidade local e não necessariamente implica um comportamento caótico em grande escala. Por isto, os seguintes estudos (Gladman e Duncan 1990, Holman e Wisdom 1993) foram direcionados ao cálculo do tempo de vida até cruzar a órbita de Netuno. A intrincada estrutura de faixas estáveis e instáveis na região transnetuniana ficou bem clara na figura de Levison e Duncan (1993) que aqui reproduzimos na Fig. 1.2. Nesse trabalho foi sugerido, pela primeira vez, que através dos anos essa estrutura poderia ter sido 'esculpida' pelas ressonâncias de movimentos médios exteriores a Netuno de uma forma similar ao que acontece no cinturão de asteróides.

No Capítulo 2 veremos quais são as principais ressonâncias no cinturão de Kuiper e apresentamos as ferramentas teóricas que usaremos para estuda-las. No Capítulo 3 estudamos a ressonância **2:3** no caso plano, dentro do esquema do problema de tres corpos, e no Capítulo 4 faremos um estudo baseado em integrações numéricas

do Sistema Solar Exterior. As outras ressonâncias são analisadas no Capítulo 5, com especial ênfase na estabilidade das mesmas.

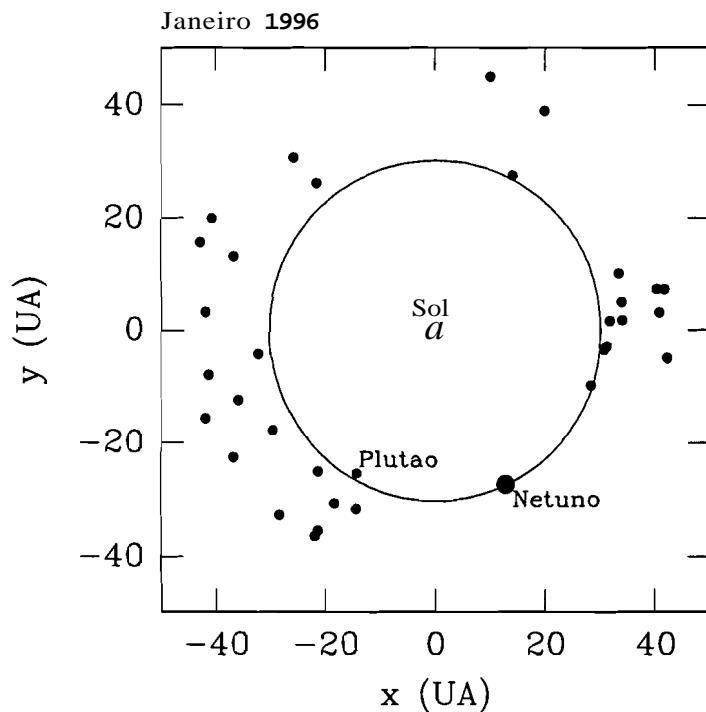


Figura 1.1: População de objetos transnetunianos descobertos até janeiro de 1996, (Marsden, 1996).

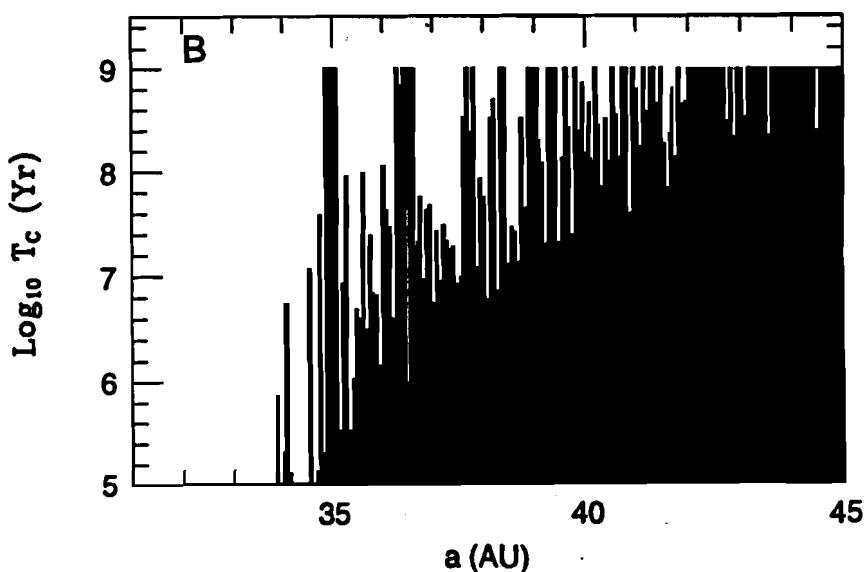


Figura 1.2: Diagrama de 'tempo de cruzamento' em função do semieixo inicial. O tempo de cruzamento é definido como o tempo necessário para que a partícula cruce à órbita de Netuno. A excentricidade inicial é 0.1 , (Levison e Duncan, 1993).

Capítulo 2

Ressonâncias no cinturão de Kuiper.

2.1 Natureza e localização das ressonâncias.

As regiões estáveis e instáveis da Fig. 1.2 podem ser associadas às ressonâncias com Netuno. Uma delas corresponde à ressonância 2:3 ($a \sim 39$ UA) que é precisamente onde se encontra o planeta duplo Plutão–Caronte. Plutão tem uma distância periélica de 29 UA, bem menor que o limite ~ 45 UA obtido por Torbett e Smoluchowski (1990) para o movimento estável. Porém, Plutão evita as aproximações com Netuno devido ao mecanismo que o liga à ressonância. Plutão e Caronte são considerados os objetos mais massivos da população atual de transnetunianos. Não é improvável que os objetos transnetunianos se encontrem evoluindo em regiões estáveis como as oferecidas por alguns movimentos ressonantes. Desde sua descoberta, o movimento de Plutão foi estudado com intensidade. Mas a estrutura completa da ressonância 2:3 somente começou a ser analisada a partir de 1995, junto com outras ressonâncias exteriores a Netuno (Morbidelli et al. 1995, Malhotra 1995a, Duncan et al. 1995).

Em forma esquemática, podemos dizer que a estabilidade do movimento das partículas em ressonância depende de três fatores: 1) estabilidade intrínseca do sistema Sol–Netuno–partícula, 2) estabilidade frente às perturbações dos outros planetas gigantes, e 3) estabilidade frente à proximidade de outras ressonâncias. Na Fig. 2.1 apresentamos as posições das ressonâncias de até ordem 20 com Netuno localizadas entre 25 e 65 UA. Se q é a ordem da ressonância, o potencial ressonante é proporcional a e^q , quer dizer que quanto maior é a ordem, o potencial ressonante é mais débil. Para de alguma forma levar em conta a ordem, cada ressonância é representada na figura com uma linha de comprimento igual a $1/q$. Quanto menor o comprimento da linha, menor é a importância da ressonância. As ressonâncias interiores ($a < 30$ UA) estão fortemente perturbadas pelos planetas Júpiter, Saturno e Urano, o que faz supor que devem ser menos estáveis que as exteriores. No esquema muito simplificado da Fig. 2.1, podemos supor que a proximidade entre as ressonâncias de baixa ordem entre 28 e 32 UA é uma indicação de que o movimento ali deve ser caótico (Wisdom, 1980), com transições de uma para outra ressonância. Uma exceção seriam os hipotéticos Troianos de Netuno (ressonância 1:1), pois se

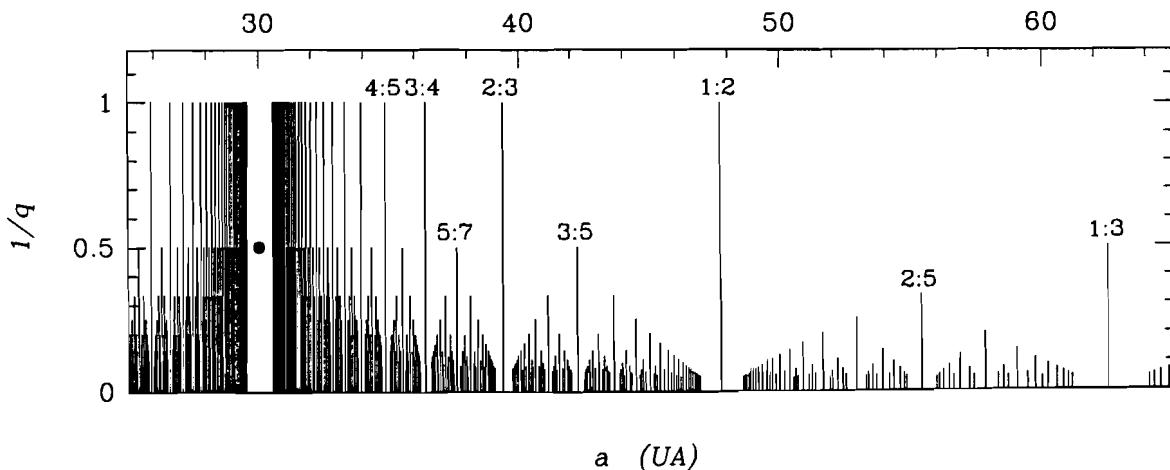


Figura 2.1: Localização das ressonâncias de movimentos médios com Netuno no cinturão de Kuiper. Observe-se que a ressonância de primeira ordem mais afastada é a 1:2 localizada em $a \sim 48$ UA.

encontram isolados de outras ressonâncias. Também fica claro que a partir de 34 UA temos várias ressonâncias de primeira, segunda e até terceira ordem que estão isoladas de outras ressonâncias de baixa ordem, e então, segundo este ponto de vista seriam estáveis. Obviamente, a dinâmica de cada ressonância dirá se elas são ou não estáveis.

As ressonâncias exteriores também despertaram o interesse a partir dos trabalhos de captura em ressonância pela ação de forças dissipativas como os de Beaugé e Ferraz-Mello (1993, 1994) e Beaugé et al. (1994) e do trabalho de captura em ressonância de Malhotra (1995b) por efeito da expansão da órbita de Netuno no processo de acresção dos planetas gigantes (Fernández e Ip, 1984, 1995). O processo de captura ideado por Malhotra é tão eficiente que, se for verdadeiro, a maioria dos objetos transnetunianos devem se encontrar em órbitas com excentricidade relativamente alta nas ressonâncias 2:3, 3:5 e 1:2.

Outro mecanismo importante na região do cinturão de Kuiper, as ressonâncias seculares foram estudadas por Knežević et al. (1991), concluindo que elas não existem além das 42 UA exceto a ressonância de periélio ν_8 associada à ressonância 1:2 de movimentos médios localizada em $a \sim 48$ UA (Fig. 2.2). A ressonância de Kozai (oscilações de grande amplitude em e e i associadas à libração ou circulação de w), foram analisadas por Thomas e Morbidelli (1996) concluindo que, no cinturão de Kuiper, elas somente afetam órbitas de alta excentricidade e inclinação.

2.2 Revisão da Teoria de librações.

O problema de três corpos ressonante plano é o ponto de partida imprescindível para entender o comportamento básico das partículas em ressonância. Porém, as teorias

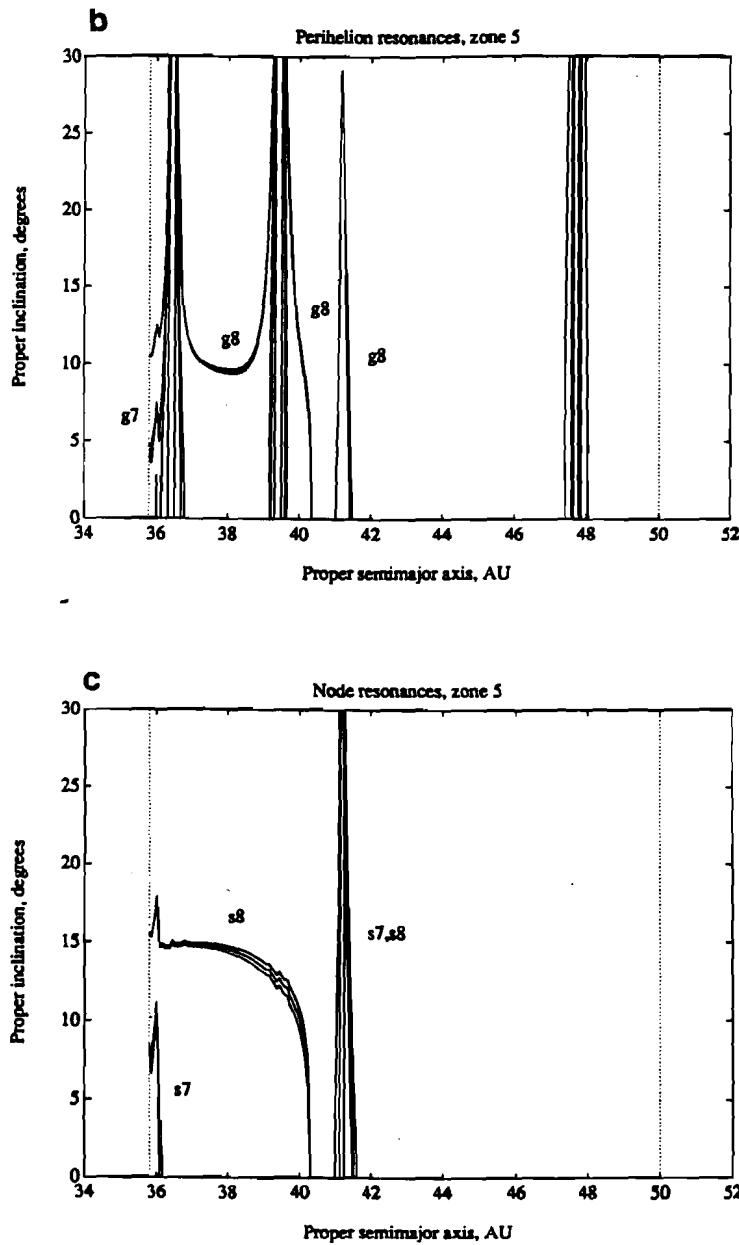


Figura 2.2: Ressonâncias seculares do periélio ν_7 e ν_8 (representadas como g_7 e g_8) e as ressonâncias do nodo ν_{17} e ν_{18} (representadas como s_7 e s_8) segundo Knežević et al. (1991).

existentes dão uma solução analítica muito simplificada do movimento ressonante. Por exemplo, no plano (l_c, h) onde $k = e \cos a$ e $h = e \sin a$, sendo a o ângulo crítico, as soluções sempre são apresentadas como um termo constante (que define o *centro de libracão*) mais duas componentes circulares. Uma é devida à libracão e a outra é o *segundo modo forçado*. Nas integrações numéricas é evidente que nem a libracão nem o segundo modo forçado são circunferências. Procurando uma solução analítica simples que descreva as características do movimento em forma mais completa, reformulamos a teoria de libracões de alta excentricidade (Ferraz-Mello 1988, Ferraz-Mello e Sato 1989, Ferraz-Mello et al. 1993). A nova teoria, apresentada no Anexo I, também é baseada no desenvolvimento assimétrico da função perturbadora, mas as equações canônicas são resolvidas de forma diferente. Basicamente conseguimos uma melhor representação do segundo modo forçado e da libracão que agora são oscilações quase-elípticas, obtemos as oscilações forçadas no semieixo da partícula ressonante e predizemos variações periódicas no período instantâneo de libração. A teoria é válida para qualquer excentricidade mas limitada às libracões de pequena amplitude. Na versão que apresentamos no Anexo I, a teoria também está limitada às libracões entorno dos eixos k ou h .

No caso das ressonâncias com Netuno resulta difícil pensar que é possível desprezar no tratamento analítico as perturbações dos outros planetas gigantes. Segundo uma idéia desenvolvida por Ferraz-Mello e Brunini (comunicação pessoal), implementamos um método para a inclusão do efeito não ressonante de outros planetas perturbadores no desenvolvimento assimétrico da função perturbadora. Os detalhes do método encontram-se no Apêndice. As leis características das ressonâncias, obtidas desta forma representam mais adequadamente o movimento ressonante, especialmente no cinturão de Kuiper pela magnitude das perturbações dos outros planetas gigantes.

A um estudo superficial de todas as ressonâncias, preferimos um estudo em profundidade de uma delas para poder chegar a conclusões melhor fundamentadas. Além disso, talvez algumas características sejam comuns a várias ressonâncias.

Capítulo 3

Ressonância 2:3. Modelos de três corpos.

Sem dúvida, esta é a ressonância que desperta mais interesse pelo fato de conter o planeta duplo Plutão–Caronte, que é objeto de várias polêmicas em torno de sua origem e natureza de movimento (Applegate et al. 1986, Olsson–Steel 1988, Sussman e Wisdom 1988, Milani et al. 1989, Wisdom e Holman 1991, Kinoshita e Nakai 1995, Malhotra 1993 e 1995b, Levison e Stern 1995). Uma primeira visão global da ressonância foi dada por Malhotra (1995a) no marco do problema circular restrito de três corpos. Estudos analíticos usando modelos mais completos foram feitos por Morbidelli et al. (1995), entretanto explorações numéricas nas proximidades da ressonância foram feitas por Levison e Stern (1995) e Duncan et al. (1995). Neste capítulo, consideraremos dois modelos dentro do esquema do problema de três corpos plano. Em primeiro lugar aplicaremos a teoria desenvolvida no Anexo I incluindo os efeitos dos outros planetas gigantes como é explicado no Apêndice. Posteriormente analisaremos a topologia da ressonância mediante superfícies de secção obtidas através da integração numérica das equações exatas do movimento do problema plano de três corpos.

Consideremos o modelo plano Sol–Netuno–partícula, sendo a órbita de Netuno elíptica ($e_N = 0.01$) e com seu periélio circulando uniformemente com velocidade g_N . Seguindo Ferraz–Mello et al. (1993), definimos as variáveis angulares

$$\begin{aligned}\sigma &= -2\lambda_N + 3\lambda - \varpi \\ \sigma_1 &= -2\lambda_N + 3\lambda - \varpi_N \\ \sigma_2 &= \lambda - \lambda_N\end{aligned}\tag{3.1}$$

e seus momentos canônicamente conjugados. Depois de fazer uma média com respeito ao ângulo sinódico σ_2 , para efeito de eliminar termos de curto período, a função Hamiltoniana resulta:

$$\mathcal{H} = -\frac{\mu}{2a} + \frac{2}{3}\sqrt{\mu a}(g_N - n_N) - R,\tag{3.2}$$

sendo R a função perturbadora média desenvolvida em torno do ponto (k_0, h_0) do plano (k, h) e onde

$$\begin{aligned} k &= e \cos \alpha \\ h &= e \sin \alpha \end{aligned} \quad (3.3)$$

O sistema resultante possui dois graus de liberdade com a energia ($E = \mathcal{H}$) como constante do movimento.

3.1 Librações de pequena amplitude.

Nesta secção, consideraremos as órbitas que apresentam libracão do ângulo α , usando a teoria de librações de alta excentricidade do Anexo I e incluindo as perturbações dos planetas exteriores como é explicado no Apêndice. Para efeito de comparar os resultados desta secção com os do Capítulo 4, tomamos Netuno com periélio circulando com velocidade $g_N = g_8 = 8.9288 \times 10^{-9}$ rads/dia e o semi-eixo, a_N , a partir de

$$a_N = \left[\frac{k^2(1 + m_{int} + m_N)}{n_N^2} \right]^{\frac{1}{3}} \quad (3.4)$$

sendo k a constante de Gauss, m_{int} a massa total dos planetas interiores, m_N a massa de Netuno e n_N o valor médio de $n_N(t)$, obtido das integrações numéricas do Capítulo 4. Da Eq.(3.4) obtemos $a_N = 30.0583$ UA.

A teoria brinda uma aproximação simples do movimento de libracão. No espaço (k, h, α) , a solução é uma trajetória cujos principais termos são uma componente fixa (k_0, h_0, α_0) (centro de libração) mais uma oscilação livre em torno de (k_0, h_0, α_0) chamada *libração* de amplitudes A_k, A_h, A_α , e uma oscilação forçada com amplitudes Z_k, Z_h, Z_α devida à excentricidade da órbita de Netuno (segundo modo *forçado*). Na ressonância exterior 2:3, os centros de libracão estão localizados em $h_0 = 0$ (Beaugé, 1994), em consequência a libra em torno de $\sigma_0 = 0^\circ$ ($k_0 > 0$), ou em torno de $\sigma_0 = 180^\circ$ ($k_0 < 0$), e as trajetórias de pequena amplitude de libracão são dadas por

$$k(t) = k_0 + A_k \cos(\nu_l t + \phi) + Z_k \cos \sigma_1 \quad (3.5)$$

$$h(t) = A_h \sin(\nu_l t + \phi) + Z_h \sin \sigma_1 \quad (3.6)$$

$$\alpha(t) = \alpha_0 + A_\alpha \cos(\nu_l t + \phi) + Z_\alpha \cos \sigma_1. \quad (3.7)$$

onde ν_l é a freqüência de libracão e ϕ uma constante de integração. Na teoria assume-se $\dot{\sigma}_1 = cte \neq 0$ sendo ν_f a freqüência de circulação de σ_1 . É importante notar que, como $A_\alpha = \alpha - \omega_N = \sigma_1 - \alpha$, quando o apresenta pequena amplitude de libração, o período de circulação de σ_1 , P_1 , é igual ao período de circulação de A_α . Os resultados da aplicação da teoria ao problema em consideração estão resumidos nas Figs. 3.1–3. Salientamos que esses resultados correspondem a librações com amplitudes próximas de zero.

A análise do Hessiano de \mathcal{H} , obtido depois de fazer a média com respeito ao ângulo sinódico, mostra que as librações em torno de centros com $0.017 < k_0 < 0.25$ não são estáveis. A análise é restrita a centros com $-0.5 < k_0 < 0.5$ devido ao fato

de que para $\epsilon > 0.5$ podem acontecer encontros próximos com Urano. No plano (k, h) as librações estáveis podem acontecer em torno de centros localizados no eixo k indicados com uma linha tracejada na Fig. 3.1a. As trajetórias que correspondem às partículas em libração não podem cruzar a curva de colisão ilustrada com uma linha continua (a curva completa pode ser encontrada em Beaugé e Ferraz-Mello, 1994). Existem famílias de centros de libração: duas dentro da região limitada pela curva de colisão e outra fóra, com diferentes trajetórias relativas à posição de Netuno (ver Malhotra 1995a, Figs. 6e-f). Da Fig. 3.1a vemos que o movimento de libração fóra da curva de colisão no problema completo deve ser muito instável dado que as trajetórias no plano (k, h) são próximas à curva de colisão.

A lei de estrutura (Fig. 3.1b) dá a relação entre k_0 e a_0 para as librações estáveis. Existem três ramos nesta curva. Um é para valores negativos de k_0 . Outro é para valores de k_0 próximos de zero mas positivos, e o outro é para os centros localizados fóra da curva de colisão. Estes três ramos correspondem a três diferentes regimes dinâmicos. Geralmente é considerado o ramo negativo ϵ , neste caso, a lei de estrutura mostra que a adoção de um valor fixo a_{res} , derivado de $2n_N - 3n = 0$, como usualmente acontece (ver Malhotra, 1995a) é justificado se as pequenas excentricidades não são consideradas. Para baixas excentricidades, o movimento do periélio torna-se rápido e o valor de a_0 para o qual $\dot{\sigma} = 0$ (centro de libração), cresce rapidamente.

Para librações de amplitude zero ilustramos na Fig. 3.1c a mínima distância a Netuno. Ali fica claro por que as librações fóra da curva de colisão são instáveis no problema completo: somente amplitudes de libração muito baixas escapam da colisão com Netuno. O período de libração é ilustrado na Fig. 3.1d sendo que os sinais indicam o sentido no qual a trajetória é percorrida (negativo é horário e positivo antihorário).

A razão A_h/A_k (Fig. 3.2a) dá a forma das librações no plano (k, h) . Por exemplo, quando $A_h/A_k = 1$, a libração é uma circunferência (isto acontece somente para pequenas excentricidades). Para $\epsilon \geq 0.05$, as trajetórias de libração no plano (k, h) tornam-se muito finas transformando-se em arcos de circunferência. Isto significa que a excentricidade tem variações muito pequenas durante o período de libração. A teoria também da a razão A_a/A_σ entre a amplitude de libração de a e σ (Fig. 3.2b). Nas baixas excentricidades, a amplitude de libração é pequena. Por outro lado, são grandes fora da curva de colisão. Integrações numéricas mostram que, para um dado centro de libração, a relação A_a/A_σ é aproximadamente independente da amplitude de libração (veja também Franklin et al. 1993). Em consequência, os valores A_a/A_σ nos permitem estimar os valores $\Delta\sigma$ necessários para romper a libração mediante uma colisão com Netuno ($\Delta\sigma = \sigma_{colisão} - \sigma_0$) ou mediante a chegada à separatrix ($\Delta\sigma \approx 180^\circ$). Estes valores $\Delta\sigma$ estabelecem os limites da ressonância e são mostrados mediante linhas tracejadas junto à lei de estrutura na Fig. 3.1b. Nas baixas excentricidades, a amplitude de libração de a é pequena, então a região no espaço de fase das condições iniciais que levam a librações estáveis também o será. A largura da região de libração mostrada na Fig. 3.1b está em bom acordo com os resultados de Morbidelli et al. (1995) e Malhotra (1995a) e foram

confirmadas pelas integrações numéricas do Capítulo 4.

O período de circulação de σ_1 (ou $\Delta\varpi$) é dado na Fig. 3.2c. O período de circulação (P_1) de σ_1 tende ao período de libração (P) de σ quando a excentricidade tende para zero (Fig. 3.2d). Quando a relação P_1/P é igual a um número inteiro, acontece uma *ressonância secular* entre σ e σ_1 . Da Fig. 3.2c vemos que para órbitas com centros de libração $k_0 < -0.40$ temos $\dot{\sigma}_1 \rightarrow 0$. O *centro de corrotação* (libração de σ e σ_1 ambos com amplitude zero) acontece para $k_0 = -0.48$, e as librações de σ_1 acontecem em torno de $\sigma_1 = 0^\circ$. Em consequência $\Delta\varpi = \sigma_1 - \sigma$ oscila em torno de 180° .

As amplitudes (Z_k, Z_h, Z_a) do segundo modo forçado (Fig. 3.3a–c) são em geral muito pequenas devido ao pequeno valor da excentricidade da órbita de Netuno. Existe um comportamento especial nas baixas excentricidades pois $Z_h \sim Z_k$, $A_h \sim A_k$ e $P_1 \sim P$. Em consequência, os dois movimentos, libração e segundo modo forçado, são quase-circulares com períodos similares e resulta difícil distingui-los. Para trajetórias com $|k_0| \leq 0.01$ o ângulo σ sempre circula pois o segundo modo forçado tem amplitude Z_k maior do que $|k_0|$. O segundo modo forçado também produz pequenas variações no período de libração *instantâneo*, as quais são estimadas aproximadamente na Fig. 3.3d assumindo $P/P_1 \rightarrow 0$ (ver Anexo I).

Dentro deste modelo, podemos fazer alguns comentários a respeito da estabilidade. Os centros de libração que estudamos são pontos de equilíbrio estável, em consequência as oscilações em torno deles são estáveis. Quando uma pequena perturbação externa é introduzida, espera-se que essas oscilações próximas ao centro de libração sejam as últimas a serem destruídas. Os pequenos valores de A_a/A_σ quando $e \rightarrow 0$ (Fig. 3.2b), por exemplo, significam que uma pequena perturbação δa leva a uma grande $\delta\sigma$, e isto pode ser suficiente para destruir a libração alcançando a separatrix. Nesta circunstância é produzida uma descontinuidade na evolução dos elementos orbitais da partícula. Na realidade, quando a excentricidade do centro de libração é próxima a zero o termo ressonante (proporcional a e) se anula e a topologia característica da ressonância é destruída. Nas excentricidades muito baixas não existe separatrix, e alguns autores dizem que não há mais ressonância. Isto é somente um problema de denominação. De fato, a família de oscilações em torno dos centros de libração não é afetada pela existência ou não da separatrix e as leis da ressonância e a teoria de librações de pequena amplitude são plenamente válidas, além de qualquer problema de denominação.

Quando consideramos um modelo dinâmico mais completo, as outras perturbações de Netuno ou as de Júpiter, Saturno e Urano podem dominar sobre os da ressonância de movimentos médios nas baixas excentricidades. A partícula será facilmente ejetada da ressonância, ou capturada em movimentos mais complexos como ressonâncias secundárias ou ressonâncias seculares. Da Fig. 3.2b podemos dizer que a ressonância é *débil* (ou *superficial*) para centros de libração que verificam $|k_0| < 0.04$. Quando $|k_0| > 0.04$ dizemos que a ressonância é *forte* (ou *profunda*), pois neste caso a partícula pode suportar maiores perturbações δa sem ser destruído o movimento de libração.

O efeito dos outros planetas gigantes incluído em R não é muito importante (Fig.

A.4). Ele produz um pequeno deslocamento na lei de estrutura (0.0024 em unidades de a_N) no sentido dos valores crescentes de a (Fig. 4.2a), e pequenas modificações no período de circulação de σ_1 , na posição do centro de corrotação e na amplitude do segundo modo forçado. Estes efeitos, mesmo sendo pequenos, foram detectados nas integrações numéricas do Capítulo 4.

3.2 Dinâmica da ressonância.

Para entender a dinâmica da ressonância 2:3 sem restringirnos as pequenas librações analisaremos o problema restrito elíptico e plano de três corpos seguindo as mesmas técnicas usadas por Michtchenko e Ferraz-Mello (1993) e Ferraz-Mello (1994). Integramos numéricamente as equações exatas de movimento de partículas localizadas na ressonância tomando agora uma órbita elíptica e fixa para Netuno, com $\varpi_N = 0$ e os mesmos valores de e_N e a_N da secção anterior. Calculamos a energia $E = \mathcal{H}$ usando a Eq. (3.2) onde agora R é a função perturbadora de Netuno calculada numéricamente. Mediante filtragem digital eliminamos da solução e de E os termos de período menor que 1500 anos, obtendo elementos e energia médios e . Tomando como condições iniciais $\sigma = 180^\circ$ e $\Delta\varpi = 0^\circ$ ou 180° e variando os valores iniciais de a e e , construímos níveis de igual E , que em forma aproximada se ilustram na Fig. 3.4. Cada nível fornece uma série de condições iniciais de trajetórias com a mesma energia média. Para trajetórias com a mesma energia média, calculamos os valores $(e \cos \Delta\varpi, e \sin \Delta\varpi)$ cada vez que $\sigma = 180^\circ$ com $\dot{\sigma} < 0$. Ao representá-los graficamente surgem as superfícies de secção da Fig. 3.5. Como em Ferraz-Mello (1994), obtivemos uma alta resolução entre trajetórias do mesmo nível de energia. As integrações compreenderam períodos da ordem de 10^7 anos.

As trajetórias com os maiores valores de E são geradas por condições iniciais em torno do ponto localizado em $e_i = 0.47$, $\Delta\varpi_i = 180^\circ$ e $a_i = 39.40$ no lado esquerdo da Fig. 3.4 (o máximo no lado direito tem uma energia menor). Em particular, uma trajetória gerada por condições iniciais correspondentes ao máximo E terá σ e $\Delta\varpi$ constante (*centro de corrotação*). O nível $E = -1.5034955$ está muito perto do máximo. Algumas das secções correspondentes a este nível são mostradas na Fig. 3.5a. Como estamos perto do centro de corrotação, a amplitude de libração de σ é próxima de zero e o ângulo $\Delta\varpi$ libra em torno de 180° . Esta é a ressonância secular ν_8 . A seguinte secção, mostrada na Fig. 3.5b corresponde ao nível de energia dado pelas duas curvas mais internas a ambos os lados da Fig. 3.4. Neste caso, algumas soluções apresentam circulação de $\Delta\varpi$ (aqueles com valores iniciais $\Delta\varpi \sim 0^\circ$). Para valores menores de E , o limite superior para a amplitude de libração de σ cresce. O movimento ocupa maiores regiões do espaço de fase e soluções com menor excentricidade são permitidas. Para valores decrescentes de E , a região entre as separatrizes diminui e as librações do ângulo $\Delta\varpi$ somente podem acontecer em uma região muito fina do plano de fase (Figs. 3.5e-f). Se a amplitude de libração fôr muito grande, podem acontecer encontros próximos com Netuno e a partícula sai da ressonância.

Para as energias mais baixas, as librações de \mathbf{Aw} em torno de 180° tendem a desaparecer, mas surge uma estrutura interessante nas baixas excentricidades. Quando a excentricidade do centro de libração decresce, a bifurcação que gera a separatrix que limita a região de libração no plano (k, h) desaparece (Ferraz-Mello, 1985). Devido à proximidade entre o centro de libração e a origem, a solução geralmente encerra ambos os pontos e, neste caso, a circulará. A circulação acontece com $\dot{\sigma} < 0$, mas o iingulo σ_1 também circula com $\dot{\sigma}_1 < 0$ e com período similar ao de a (Fig. 3.2d). O resultado é que $\mathbf{Aw} = \sigma_1 - a$ libra em torno de 0° (curvas fechadas em torno da cruz na Fig. 3.6). As trajetórias da Fig. 3.6 mostram uma variação $\Delta e \sim 0.015$ que está em acordo com os valores do segundo modo forçado da Fig. 3.3. Dado que \mathbf{Aw} libra, temos mais uma vez a ressonância secular ν_8 dentro da ressonância de movimentos médios. A origem desta ressonância secular é a ressonância secundária 1:1 entre a e σ_1 . Morbidelli et al. (1995) argumentam que para $e \rightarrow 0$ a ressonância secular ν_8 não pode existir pois $\overline{\mathbf{Aw}} \rightarrow -\infty$. Mas, isto é válido somente para librações de amplitude nula, onde $\sigma(t) \equiv 180^\circ$, e daí $\overline{\Delta\omega} = \dot{\sigma}_1 \rightarrow -\infty$. E, como vimos, um pequeno afastamento do centro de libração é suficiente para produzir uma circulação de a com $\dot{\sigma} \sim \dot{\sigma}_1$ dando origem à ressonância secular ν_8 . Este é um ponto importante pois, como será visto mais adiante, esta ressonância secular interage com a ressonância secular ν_{18} produzindo um dos mais importantes mecanismos de instabilidade na ressonância 2:3.

Temos-nos referido aos centros de libração localizados no ramo negativo da lei de estrutura, mas um mecanismo similar gera a ressonância ν_8 no ramo positivo.

Em todas as superfícies de secção, as trajetórias mostram-se muito regulares, inclusive perto das separatrizes. Não existe o menor indicio de difusão. A região caótica que geralmente aparece perto da separatrix, aqui não é detectada. O espaço de fase analisado, incluindo a ressonância secular ν_8 , é regular. Calculamos o máximo expoente de Lyapunov para algumas trajetórias das Figs. 3.5 e 3.6. Para obter convergência no computo do expoente precisamos integrar as equações de movimento por períodos maiores que 10^8 anos. Achamos valores compreendidos entre 10^{-6} e 10^{-7} anos $^{-1}$. Mas em escalas de tempo de 10^8 anos não foi detectada nenhuma difusão nas trajetórias das superfícies de secção.

3.3 Influencia de outros perturbadores.

Aqui analisamos brevemente o efeito de incluir diferentes perturbadores na integração numérica das equações de movimento do problema de quatro corpos. Analisamos separadamente os efeitos de Urano, Saturno e Júpiter nas partículas que apresentam librações do iingulo crítico no modelo restrito Sol–Netuno–partícula. Mostramos na Fig. 3.7, os efeitos de Urano, Saturno e Júpiter em uma partícula com energia inicial $E = -1.503500$ e excentricidade inicial $e \sim 0.25$ (Fig. 3.7a). Agora que incluímos os efeitos de outros planetas, seguimos o mesmo procedimento descrito anteriormente mas as superfícies de secção resultantes já não são mapas de Poincaré pois existem outros graus de liberdade envolvidos. Porém, podemos

fazer projeções das secções $a = 180^\circ$ sobre o plano ($e \cos Aw$, $e \sin Aw$). Nos quatro casos da Fig. 3.7 o período de libração e a amplitude manteve-se aproximadamente a mesma, mas aparecem oscilações forçadas na excentricidade. No caso em que se incluem os efeitos de Júpiter, a excentricidade apresenta oscilações de largo período e grande amplitude ($0.10 < e < 0.28$). Estas oscilações forçadas não modificam as librações de a , mas perturbam substancialmente a evolução de Aw e e . A dinâmica do problema restrito é fortemente afetada pelo Júpiter. A Fig. 3.8 corresponde a uma órbita mais excêntrica que inicialmente apresentava uma libração de Aw a qual acaba sendo destruída pela influencia do Júpiter.

A importancia dos efeitos dos planetas gigantes na ressonância 2:3 nos motivou a explorar um modelo mais próximo do caso real, o que fazemos no capítulo seguinte.

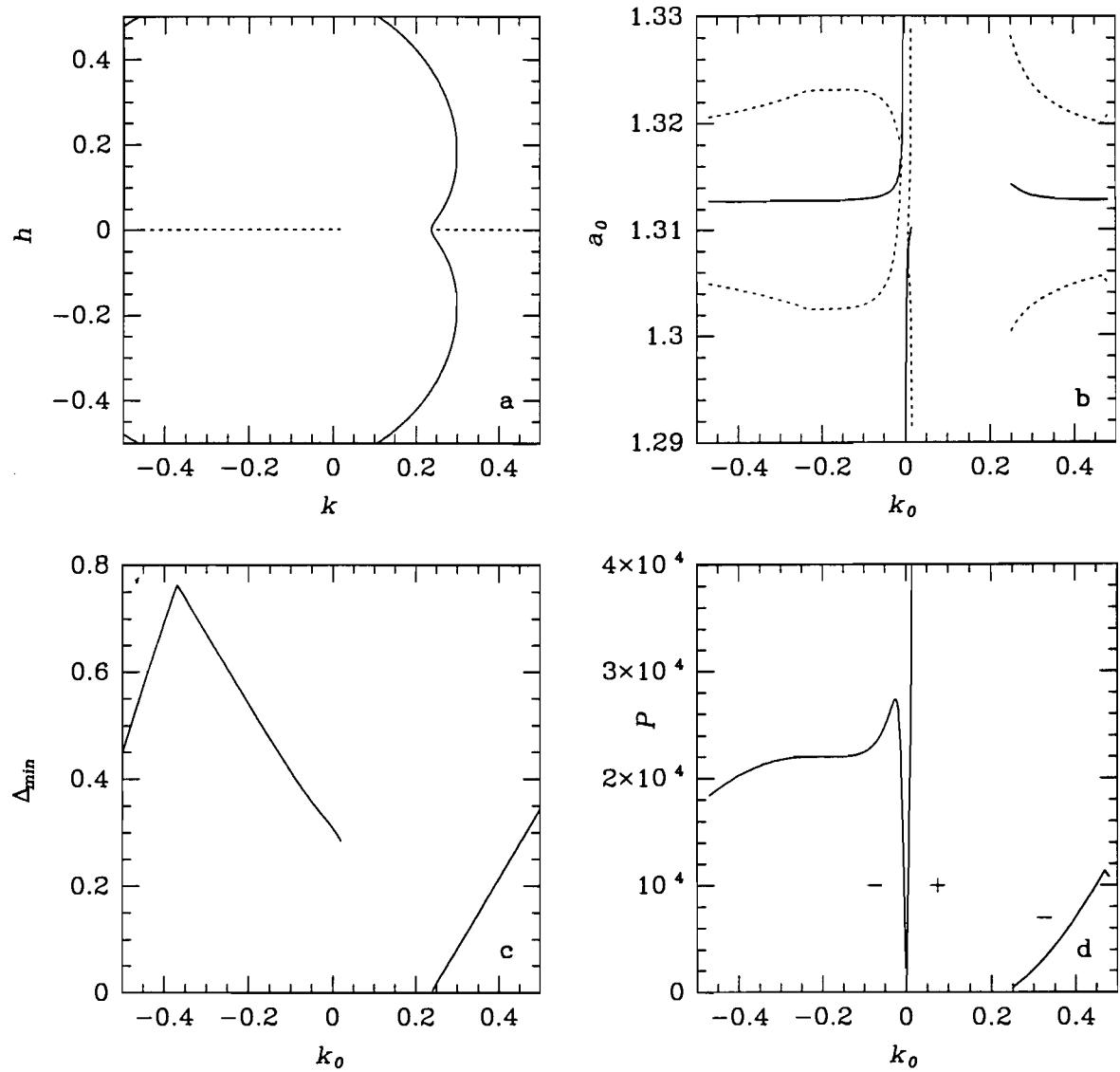


Figura 3.1: a)Curva de colisão. Em linha tracejada se indicam os centros de libração estáveis no plano (k, h) . b)Lei de estrutura em linha continua e limites para as librações em linhas tracejadas. Semi-eixo em unidades de a_N . c)Distância mínima a Netuno para partículas com amplitude de libração zero (unidade= a_N). d)Período de libração de σ em anos. Sinal – indica sentido horário e + anti-horário.

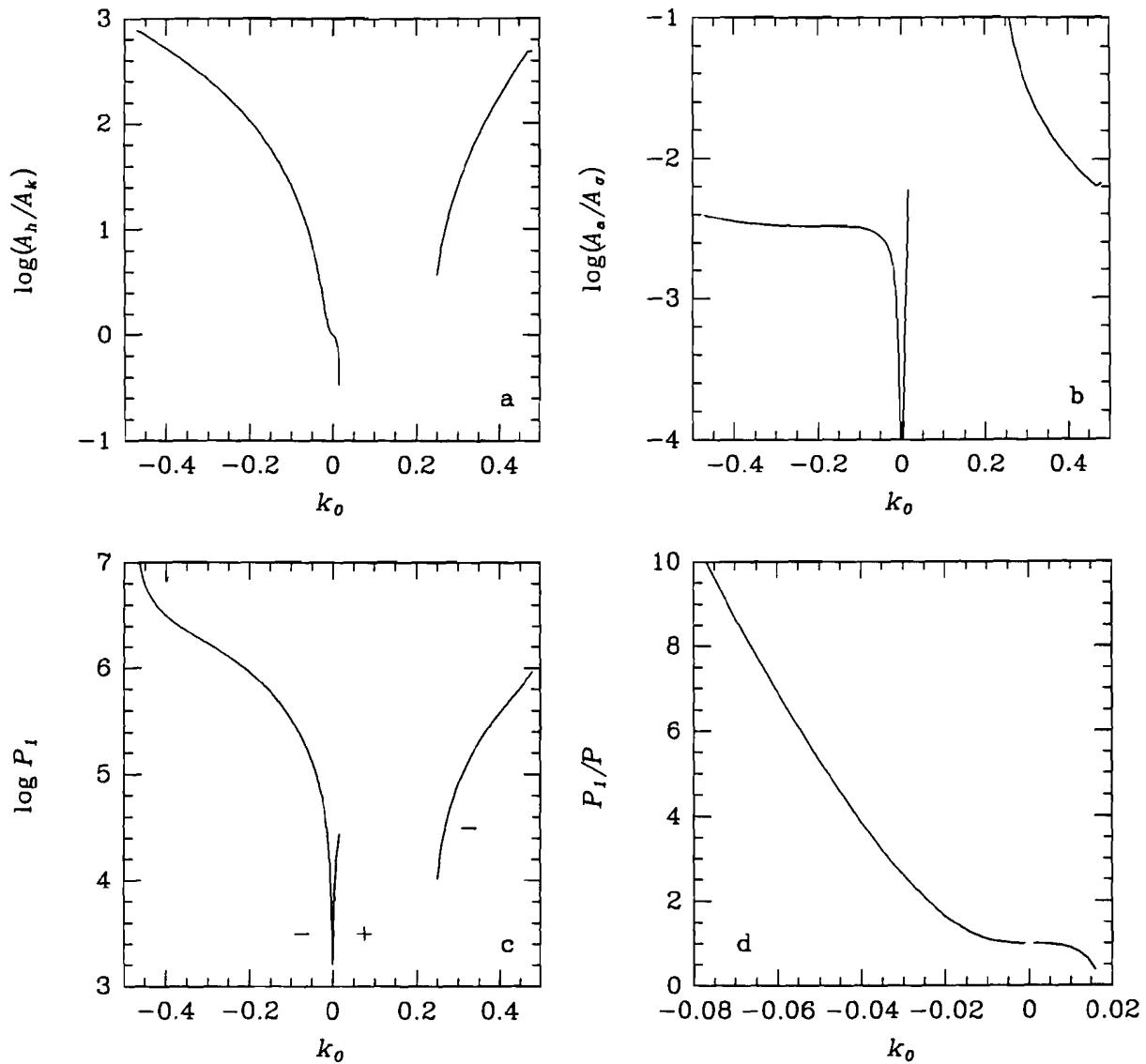


Figura 3.2: a) Razão A_h/A_k , que dá a forma da figura de libração no plano (k, h) . b) Relação entre a amplitude de libração em a (em unidades de a_N) e em σ (em radianos). c) Período de circulação de σ_1 em anos. d) Razão entre os períodos de circulação de σ_1 e de libração de σ para amplitudes de libração zero nas baixas excentricidades.

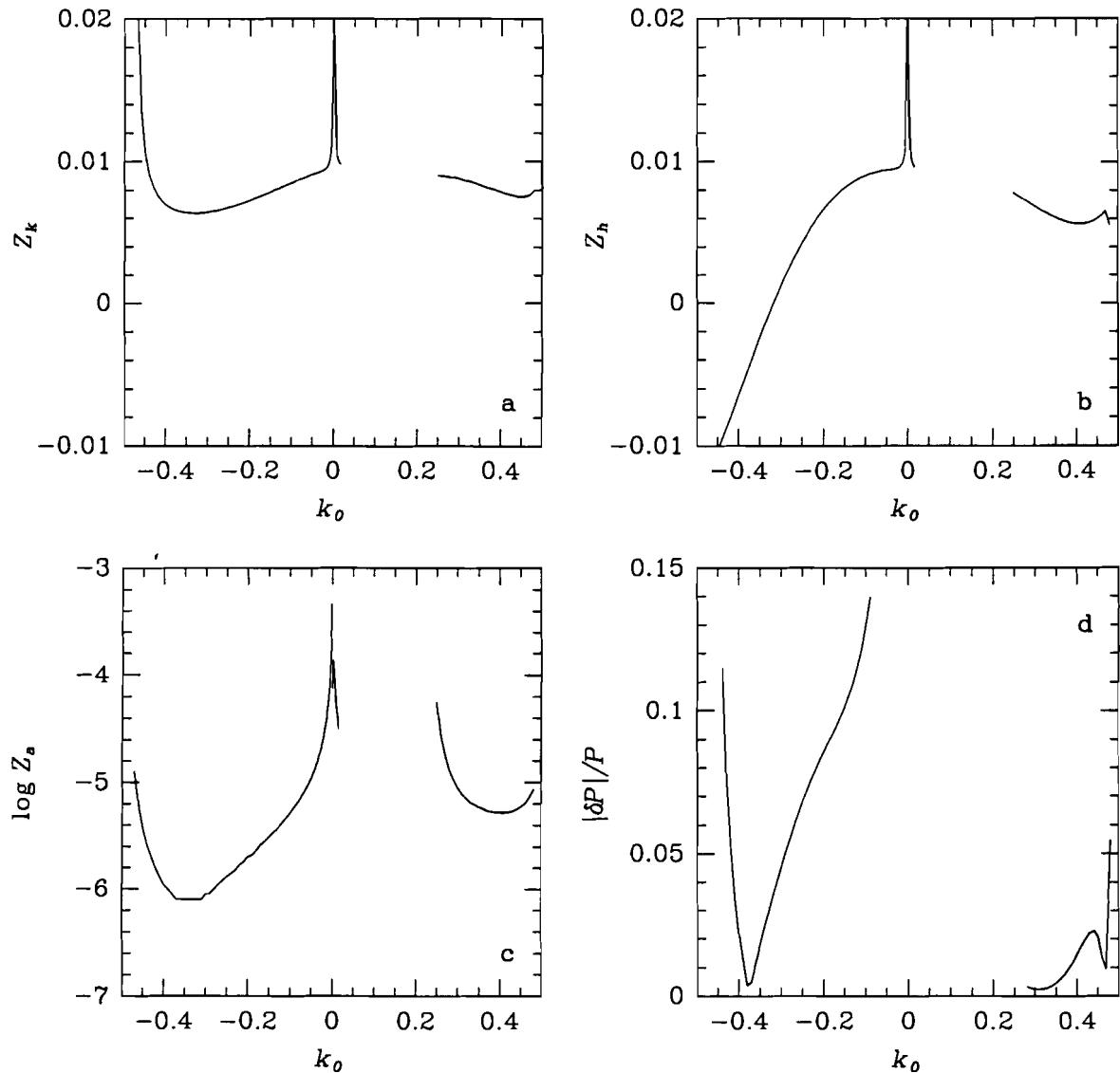


Figura 3.3: a,b,c)Amplitudes para o segundo modo forçado em k , h e a . Os picos em ~ 0 foram confirmados nas integrações numéricas. d)Estimação da variação no período instantâneo de libração devido às oscilações forçadas assumindo $P/P_1 \rightarrow 0$.

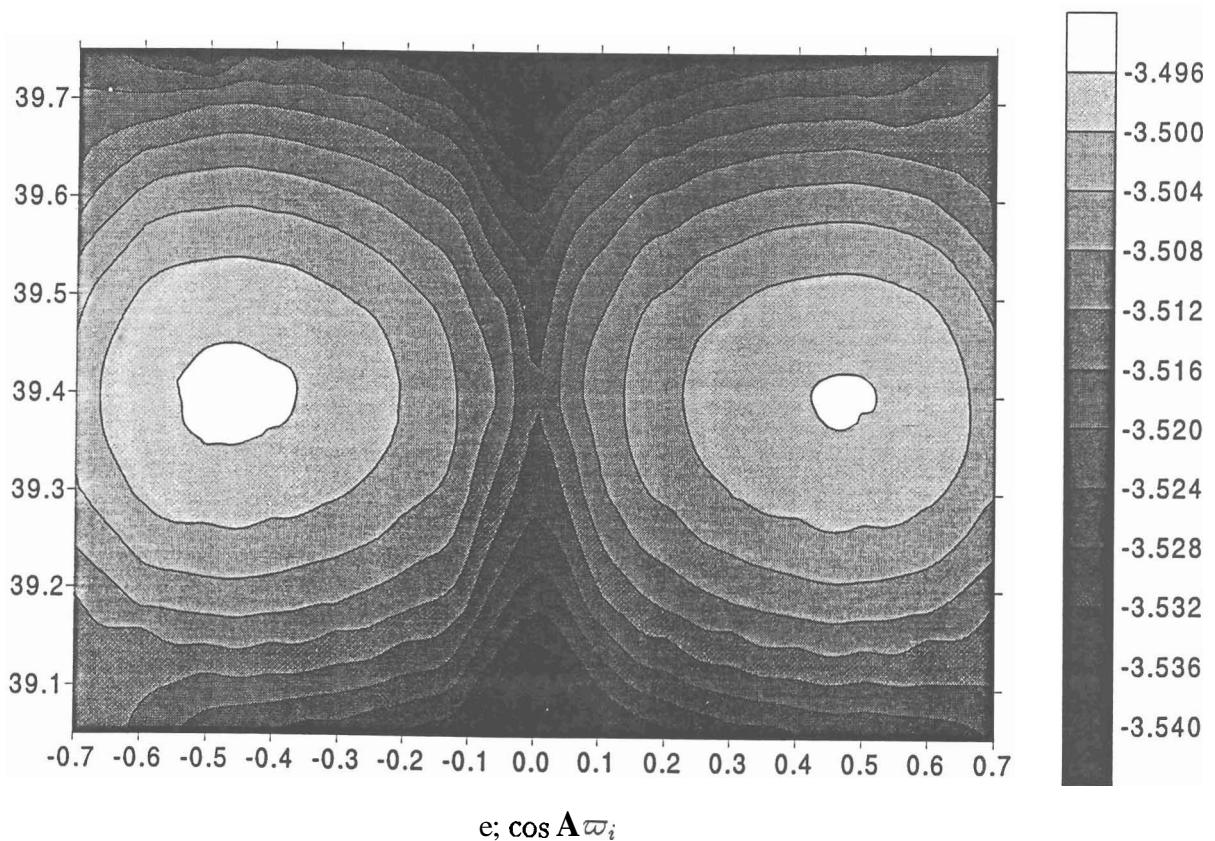


Figura 3.4: Curvas de nível para a energia, E , em função das condições iniciais. O pico mais alto esta dentro da pequena curva fechada no lado esquerdo da figura e corresponde ao centro de corrotação. A escala de cinzas indica os valores de $(E + 1.5) \times 10^3$.

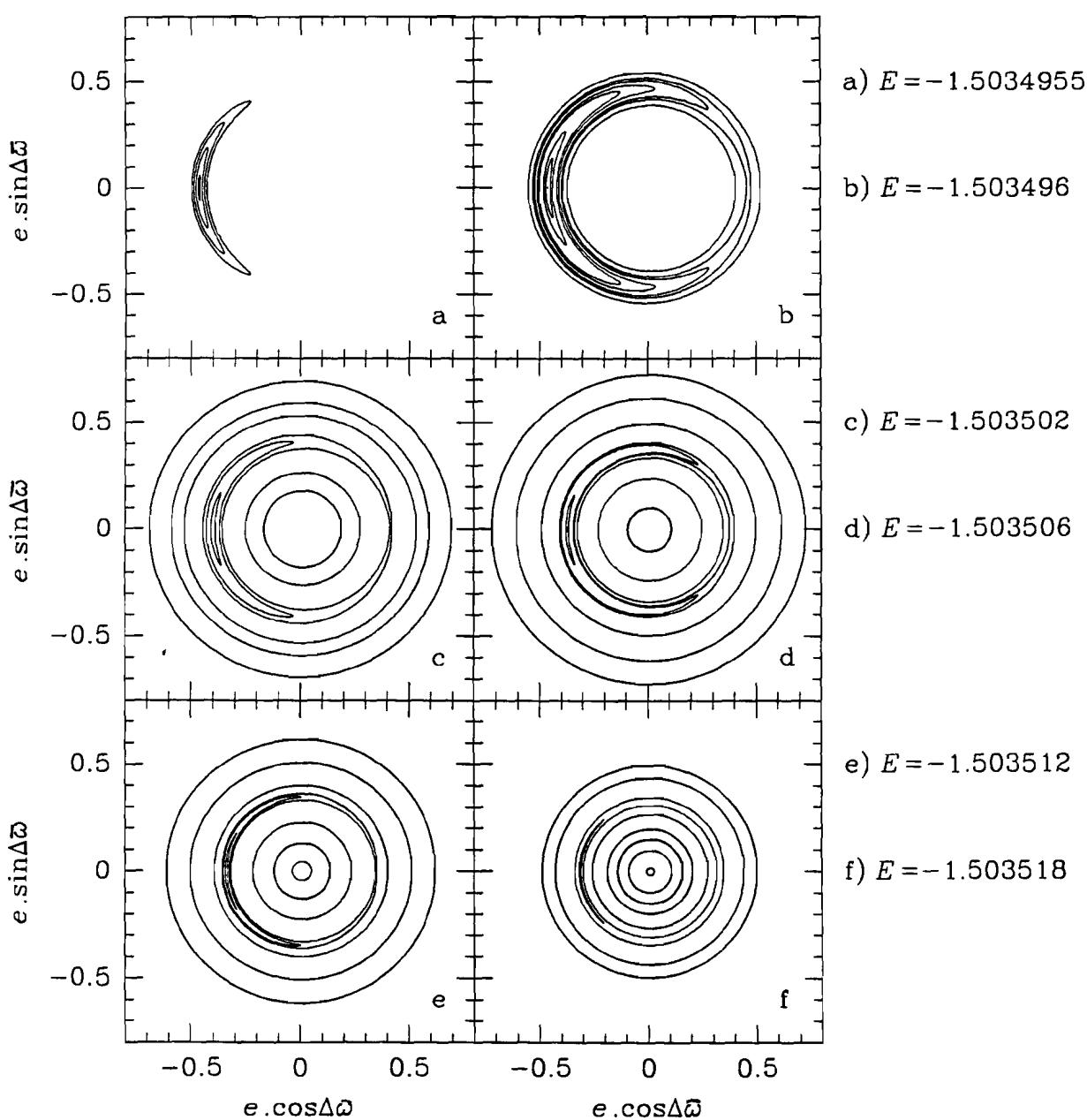


Figura 3.5: Superfícies de secção em $\alpha = 180^\circ$ e para $\dot{\sigma} < 0$ para seis diferentes níveis de energia. A amplitude de libração para o primeiro nível (próximo da corrotação) é quase zero, e para o ultimo nível pode chegar até 130° . As trajetórias foram calculadas por vários 10^7 anos. Em cada figura, o limite superior obtido para a excentricidade é devido à colisão com Netuno.

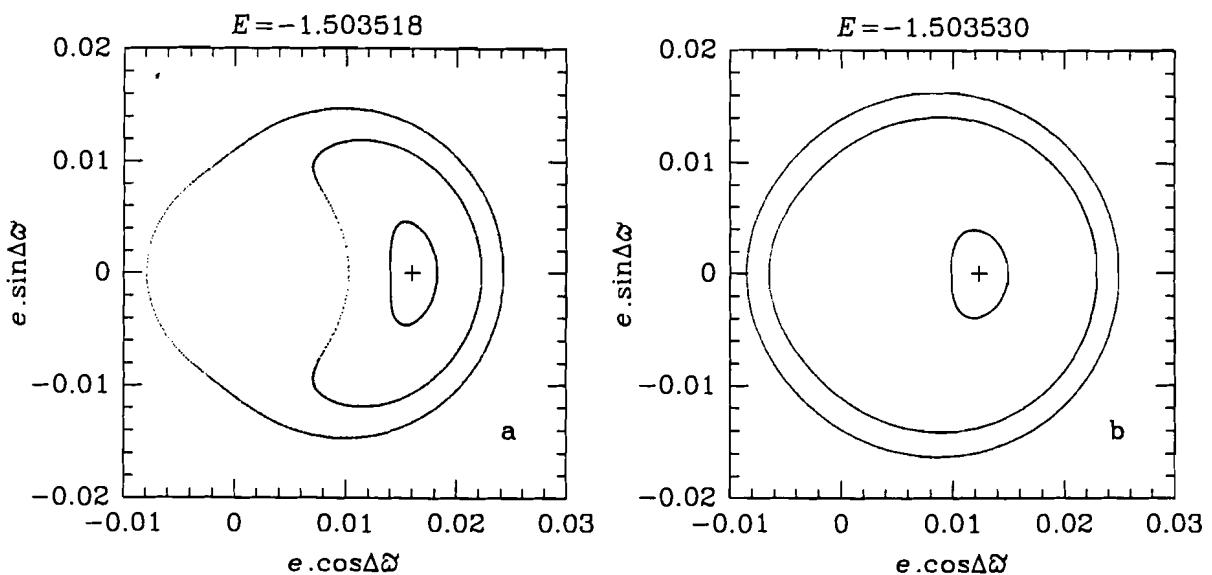


Figura 3.6: Ressonânciia secular ν_8 dentro da ressonânciia de movimenti médios, devida à ressonânciia secundaria $P_1/P \rightarrow 1$ para pequenas excentricidades. O ângulo $\Delta \tilde{\omega}$ oscila em torno de 0° . As superfícies foram construídas como na Fig. 3.5. Trajetórias por 30 milhões de anos (a) e por 77 milhões de anos (b). O centro de libração destas trajetórias se encontra em $k_0 \sim 0$ no ramo negativo da lei de estrutura, e $a_0 \sim 1.3192$.

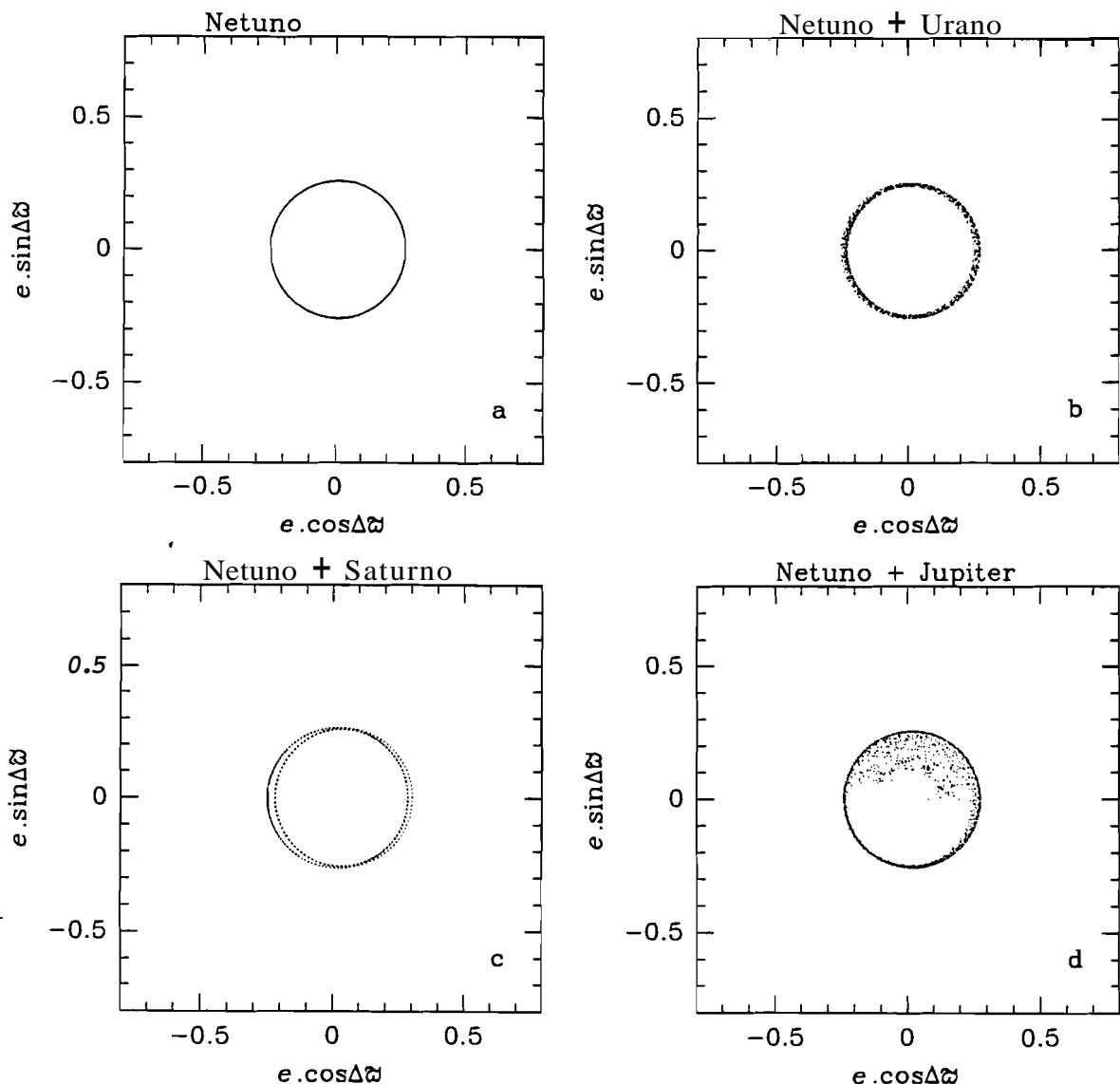


Figura 3.7: Projeções no plano ($e \cos \Delta \omega$, $e \sin \Delta \omega$) de uma secção de uma solução com periélio circulando sem outros perturbadores que Netuno (a), com Netuno e Urano (b), com Netuno e Saturno (c) e com Netuno e Júpiter (d). Trajetórias por 25 milhões de anos.

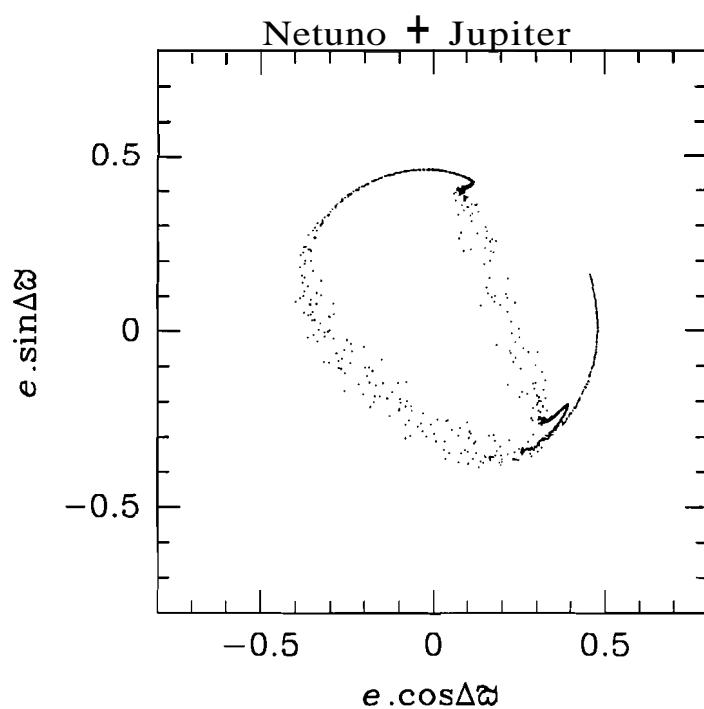


Figura 3.8: Projeções no plano ($e \cos A\omega$, $e \sin A\omega$) de uma secção de uma solução com periélio originalmente librando em torno de 180° . A libração de $A\omega$ é destruída pelas perturbações de Júpiter.

Capítulo 4

Ressonância 2:3. Modelo de N corpos,

4.1 Procedimento numérico.

Para completar o estudo da dinâmica desta ressonância, executamos integrações numéricas do Sistema Solar Exterior incluindo Júpiter, Saturno, Urano e Netuno. incluímos as massas de Mercúrio, Venus, Terra e Marte na do Sol. Tomamos as condições iniciais de Choen et al. (1973) e usamos o plano invariável do Sistema Solar como referencia. Seguimos os elementos heliocêntricos dos planetas e partículas usando o integrador RA15 (Everhart, 1985) com parâmetro de precisão, $L = 10$. Integramos centenas de partículas por períodos de vários 10^7 anos, com valores iniciais $w = 90^\circ$, $M = 0^\circ$ e $i = 0^\circ$. Levando em conta que a longitude média inicial de Netuno foi de $\lambda \sim 176^\circ$, o valor inicial do ângulo crítico resulta $a; \sim 180^\circ$. O semi-eixo maior original foi escolhido no intervalo 39.1 – 40.4 UA e a excentricidade inicial entre 0 e 0.5.

O estudo do problema em consideração mediante integrações numéricas apresenta, como a mais séria dificuldade, o grande número de variáveis envolvidas e a impossibilidade de um estudo metódico completo. Usamos o conhecimento da dinâmica do problema de três corpos para superar algumas dessas dificuldades. Segundo Milani et al. (1989), estudamos a evolução dos ângulos a , A_w , 2ω e $AR = R - \Omega_N$ e também os elementos i , e , a , k e h . O primeiro ângulo está associado com a ressonância de movimentos médios com Netuno. O segundo é o relacionado ao movimento do periélio e a ressonância secular ν_8 que produz oscilações na excentricidade. O terceiro está relacionado à ressonância de Kozai, que produz oscilações na inclinação. O último está relacionado com a ressonância secular ν_{18} , a qual também produz oscilações na inclinação.

Obtivemos os espectros dos elementos orbitais e ângulos críticos usando o método apresentado no Anexo II. Este método é especialmente preciso na região do espectro correspondente às baixas freqüências, tanto para a determinação das freqüências como das amplitudes. Na Tabela 4.1 apresentamos os intervalos de variação das amplitudes A , e A , dos componentes harmônicos mais importantes achados em a e e . Os termos de curto período, originados por combinações lineares das freqüências orbitais dos corpos envolvidos, são devidos principalmente a Júpiter (com a exceção

Termo	Período (anos)	A_e	A_a
curto período libração	$10^1 - 10^2$	0.0083 - 0.0087	0.010 - 0.017
segundo modo forçado	10^4	0.0000 - 0.0120	0.000 - 0.012
	$10^5 - 10^6$	0.0030 - 0.0090	10^{-6}

Tabela 4.1: Amplitudes máximas das linhas mais importantes nos espectros de e e a. As mesmas freqüências também existem em a, mas, devido a razões geométricas simples, elas são dependentes da e. A unidade de A_a é a_N .

das excentricidades muito grandes, onde as perturbações de Urano predominam). Suas amplitudes são proporcionais à excentricidade da partícula e são independentes da amplitude de libracão. Corroborando os resultados do capítulo anterior, vemos que os termos forçados apresentam uma amplitude muito pequena. Eles dependem de e e também observou-se uma pequena dependencia com a amplitude da libracão.

4.2 Librações.

A região na qual as librações sobrevivem em escalas de tempo de 10^7 anos está definida aproximadamente pelas condições iniciais $39.35 < a < 40.05$ UA, $0 < e < 0.37$ e também está restrita à região de $A_a < 150^\circ$. A máxima excentricidade permitida é relativamente pequena se comparada com os resultados do capítulo anterior. Os limites em A_a são consistentes com os de Duncan et al. (1995) e Malhotra (1995a). Na Fig. 4.1 mostramos a amplitude de libração em a, a e e, e o período de libracão, em função dos valores iniciais da excentricidade e semi-eixo. As amplitudes mostradas na figura foram obtidas das linhas correspondentes no espectro como foi mencionado. As áreas em cinza correspondem a ejeção em menos de 10^7 anos. Os resultados que apresentamos nas Figs. 4.1–3 correspondem a trajetórias com centros de libração no ramo negativo da lei de estrutura. Também procuramos soluções onde a libra em torno de 0° , as quais são estáveis no problema circular restrito. Encontramos uma evolução estável para $0 < k_0 < 0.017$ mas no ramo após a curva de colisão ($k_0 > 0.24$) as librações são destruídas em tempos da ordem de 10^5 anos.

Da análise espectral de a, e e k obtivemos os centros de libracão (k_0, a_0). Esses valores são colocados no gráfico na Fig. 4.2a junto com o ramo negativo da lei de estrutura da Fig. 3.1b. O acordo é muito bom e confirma o pequeno deslocamento em a devido as perturbações médias dos outros planetas gigantes. Para o caso de integrações numéricas do Sistema Solar Exterior, Dvorak e Lohinger (1996) também obtiveram um deslocamento (da ordem de 0.01 em unidades de a_N) da região de movimento estável, com respeito a região obtida com o modelo restrito de tres corpos.

Para as soluções com um mesmo centro de libracão encontramos as seguintes

relações aproximadas:

$$\Delta\sigma \propto \Delta a \quad (4.1)$$

$$\Delta e \propto \Delta a \quad (4.2)$$

$$\nu_l \propto (\Delta a)^2. \quad (4.3)$$

A relação linear (4.1) é válida até para amplitudes de libração da ordem de 120°. Este resultado está em bom acordo com o valor A_a/A_σ calculado para amplitude de libração zero na aproximação semi-analítica e justifica o cálculo dos limites da ressonância feito no Capítulo 3. A relação linear (4.2) está também em bom acordo com a relação dada pela teoria semi-analítica. Veja-se que na Fig. 4.1c a libração em e tem amplitude muito pequena. Já tínhamos deduzido isto a partir dos valores de A_k/A_h no capítulo anterior. A Fig. 4.1d mostra que o período de libração depende fortemente da amplitude da libração. A relação (4.3), válida para $e > 0.04$, explica a dispersão no gráfico da lei dos períodos na Fig. 4.2b. Para $e < 0.04$, o período de libração cresce junto a A_a . Este efeito particular nas baixas excentricidades já foi detectado nas ressonâncias interiores com Júpiter por Michtchenko e Ferraz-Mello (1993) e para esta ressonância por Malhotra (1995a, Fig. 6g). A Fig. 4.3 dá o valor médio da inclinação, relativa ao plano invariável. A inclinação máxima que alcança uma partícula é aproximadamente o dobro desse valor.

Os resultados numéricos estão em bom acordo com os resultados teóricos do Capítulo 3. O método de incluir em R as perturbações dos outros planetas gigantes fornece resultados realistas e os pequenos efeitos nas leis da ressonância são confirmados. Mas o efeito mais importante devido aos outros perturbadores não é aquele referido nas leis. O efeito mais importante é uma forte redução na máxima excentricidade permitida para as librações estáveis e a desestabilização do movimento ressonante nas regiões do espaço de fase onde a ressonância é superficial (pequenas excentricidades ou grande amplitude de libração). Estes pontos serão discutidos nas seguintes secções.

4.3 Escape nas altas excentricidades.

O limite em $e < 0.37$ para as librações estáveis é imposto pelas perturbações de Urano, que induz a libração a crescer. Este fato está claramente ilustrado na Fig. 4.4 onde mostramos $\sigma(t)$ e $a(t)$ junto com a intensidade das perturbações de Urano, calculada como $P_U = \text{cte} \cdot m_U/\Delta^2$. Todas as maiores variações em a acontecem quando P_U é grande (os picos de curto período que aparecem a intervalos regulares estão associados à conjunção com Júpiter). O instante no qual a começa a circular está associado a um salto grande em a produzido por uma série de máximos em P_U . A Fig. 4.5 mostra quatro partículas com uma evolução típica em alta excentricidade. A amplitude de libração cresce devido às perturbações de Urano e, surpreendentemente, em lugar de ser ejetada por um encontro próximo com Netuno, a partícula sai brevemente da ressonância (circulação de a e salto no semi-eixo) e é recapturada temporariamente em uma libração em torno de $a = 0^\circ$, do outro lado da curva de

colisão. Esta captura temporária é um fenômeno muito comum que unicamente pode ser obtido com integradores de alta precisão. Por exemplo, se usamos $L = 9$ na RA15, não é possível obter a captura temporária. Depois de algumas transições entre circulação e os dois modos de libração, acontece um encontro próximo com Netuno e a partícula é removida da ressonância.

4.4 Movimento do periélio e do nodo. Ressonâncias secundares.

O ângulo **AR** é o que apresenta a evolução mais lenta. Ele libra (ressonância secular ν_{18}) para pequenas excentricidades ($e < 0.05$) e também para $e > 0.28$, acoplado com oscilações em i . Na região $0.05 < e < 0.30$ alterna entre circulação e libração, ou circula com tendência a librar em casos de grande amplitude de libração de a , como foi observado por Morbidelli et al. (1995). Observamos que as oscilações em i são importantes somente para trajetórias com centros de libração com $e < 0.08$ localizados no ramo negativo da lei de estrutura (Fig. 4.3).

O ângulo 2ω tem uma evolução muito irregular. Para $e < 0.15$ circula em sentido horário e para $e > 0.30$ em sentido antihorário. Como foi previsto por Morbidelli et al. (1995), para $e \sim 0.24$, w libra em torno de 90° . Nas outras regiões, alterna entre libração e circulação. Quando w libra (ressonância de Kozai) aparecem oscilações forçadas de pequena amplitude em e e i , e a evolução destes elementos está acoplada. Em muitos casos achamos um acoplamento entre o período de libração de w e o de circulação de **AR** (Milani et al., 1989). Para as partículas librando na ressonância 2:3 com baixas inclinações, não achamos nenhum incremento significativo em e , i ou **Aa** associado com a ressonância de Kozai. Concordamos com Levison e Stern (1995) em que a libração de w não parece ser uma condição fundamental para ter evolução estável na ressonância 2:3.

O período de circulação de **Aw** é sempre menor que 3 milhões de anos. Para as pequenas amplitudes de libração ele é bem reproduzido pela Fig. 3.2c, mas para amplitudes altas obtivemos diferenças de até 20%. Existe um acoplamento entre variações de pequena amplitude em e e o período de circulação de **Aw** devido ao segundo modo forçado. Nas integrações numéricas do modelo de N corpos achamos que não existe libração deste ângulo em torno de 180° dentro da ressonância pois, como vimos no Capítulo 3, a excentricidade requerida é maior que 0.37, e as librações de a ali são instáveis.

Quando consideramos partículas librando em torno de $a = 180^\circ$ com valores cada vez menores de e , o ângulo **Aw** começa a librar em torno de 0° devido ao mesmo mecanismo explicado no capítulo anterior. Mas agora, no caso espacial, observamos que $\Delta\Omega$ libra e a inclinação cresce. Quando consideramos valores ainda menores da excentricidade (conseqüentemente, valores crescentes de a_0 nos limites do ramo negativo da lei de estrutura na Fig. 3.1b), as librações de **Aw** e **AR** se acoplam produzindo a ressonância secundária $\nu_8 + \nu_{18}$ que leva a altas inclinações e excentricidades (Fig. 4.6). Sabemos que a partícula estava inicialmente na ressonância de

movimentos médios pois o centro de libração estava localizado na lei de estrutura, pelo menos no inicio da integraçáo. Para libraqóes em torno de $a = 0^\circ$ também existem libraqóes de $\Delta\varpi$ em torno de 0° e de $\Delta\Omega$ em torno de 0° , mas não acopladas, e não se observa nenhum incremento sistemático nem na i nem na e (Fig. 4.7). Parece ser então que o crescimento na excentricidade e inclinaçáo está associado à ressonancia secundária $\nu_8 + \nu_{18}$. A superposição das ressonâncias seculares ν_8 e ν_{18} foi achada por Knežević et al. (1991) e Duncan et al. (1995). A ressonancia secundária $\nu_8 + \nu_{18}$ foi prevista por Morbidelli et al. (1995) (a Fig. 13 deles é muito parecida com nossa Fig. 4.6). Um comportamento análogo foi encontrado por Kinoshita e Nakai (1995) em Plutão, onde está presente a ressonancia $\nu_8 + 3\nu_{18}$.

4.5 Captura e escape.

Comeqando com diferentes valores iniciais (a, λ), estudamos a evolução de órbitas com excentricidade e inclinação inicial zero. Nos primeiros instantes da evolução os parametros das órbitas oscilam em torno de valores que podem ser associados a centros de libração localizados a ambos ramos da lei de estrutura. Encontramos que a evolução futura da partícula depende da posição (k_0, a_0) do centro de libração no começo da integração. Três diferentes evoluções podem acontecer, dependendo do valor inicial de a_0 (ver Fig. 3.1b):

i) $a_0 > 1.330$. Movimento dominado por $\nu_8 + \nu_{18}$ levando a excentricidade e inclinação a valores altos em escalas de tempo de alguns milhões de anos (Fig. 4.6). Em escalas de tempo de 100 milhões de anos, depois de um encontro próximo com Netuno, a partícula é ejetada da ressonancia.

ii) $1.314 < a_0 < 1.330$. Aqui está presente a ressonancia ν_{18} que leva a inclinação a crescer. Segue uma evolução caótica com transições entre circulação e libração de a (Fig. 4.8). A partícula também é ejetada em escalas de tempo de 100 milhões de anos. O segundo modo forçado é em parte responsável pela evolução caótica.

iii) $a_0 < 1.310$. Estes valores correspondem ao ramo positivo na lei de estrutura. A ressonancia ν_8 pode estar presente mas o movimento é muito regular com e e i restritos a valores pequenos. Em escalas de tempo de 100 milhões de anos não aparece nenhuma indicação de instabilidade (Fig. 4.7).

Em resumo, a evolução das partículas com excentricidade inicial zero está fortemente relacionada com a lei de estrutura. O ramo positivo é estável e o ramo negativo é instável. É interessante fazer notar que em aproximadamente 50 órbitas calculadas para o estudo da região de baixas excentricidades, não achamos nenhuma partícula com a_0 localizado no vazio da lei de estrutura determinado por $1.310 < a_0 < 1.314$. Vários autores acharam um drástico câmbio na evolução das partículas com inicial zero em $a \sim 39.8$ UA (Holman e Wisdom 1993, Morbidelli et al. 1995, Levison e Stern 1995). Aqui mostramos que essa diferença no comportamento está relacionada com a própria estrutura da ressonancia nas baixas excentricidades.

Um dos pontos mais importantes a resolver é achar mecanismos que levem à ejeção da partícula da ressonância, pois isso poderia explicar a migração de objetos primordiais para as regiões internas do Sistema Solar, como algumas teorias sobre a origem dos cometas sugerem. Exploramos aqui alguns possíveis mecanismos que levam à ruptura das librações. Para analisar a estabilidade do movimento de libração, seguimos a evolução temporal do espectro de a , usando o método do Anexo II com uma janela constante, T_w , e variando seu centro, τ . Observando a evolução temporal do pico correspondente à libração no espectro podemos saber se o movimento é estável ou não. No caso de um movimento estável, a freqüência (ν_l) e a amplitude do pico oscilam com freqüências bem definidas devido à influência das linhas tipo $\nu_l \pm N\nu_f$, onde ν_f é a freqüência do segundo modo forçado. Geralmente são estas linhas as que aparecem mas também podem estar presentes perto de ν_l outras linhas relacionadas com as outras variáveis angulares. Se o movimento é instável, observa-se um incremento sistemático na amplitude de libração, $A(t)$, ou aparecem componentes estranhas em $\nu_l(t)$. A origem da instabilidade pode ser determinada através da identificação das freqüências envolvidas na evolução temporal de A ou ν_l .

Usando uma janela com $T_w = 0.5$, e variando τ , podemos obter a evolução temporal da freqüência *instantânea* de libração, do centro *instantâneo* de libração (σ_0) e da amplitude *instantânea* de libração. A Fig. 4.9 mostra o que acontece no caso de um movimento estável de uma partícula librando em torno de $k_0 = -0.196$. O centro instantâneo de libração (Fig. 4.9b) oscila em torno de 180° devido ao segundo modo forçado. As oscilações de $\nu(t)$ (em Fig. 4.9a) tem freqüência ν_f e são originadas nas combinações das linhas $\nu_l \pm \nu_f$ do espectro. Estas são as variações no período de libração a que fizemos referência no Capítulo 3. A magnitude das variações mostradas na Fig. 4.9a são menores que as que calculamos na Fig. 3.3d devido ao fato de que estamos usando um intervalo de tempo $T_w = 0.5$ milhões de anos na análise de freqüências e as curvas da Fig. 3.3d correspondem a períodos instantâneos (ou seja $T_w \rightarrow 0$) e foram calculadas assumindo $P/P_1 \rightarrow 0$. A evolução temporal da amplitude de libração (Fig. 4.9c) também está modulada pela freqüência de circulação de AR. Na Fig. 4.10 mostramos a evolução temporal do espectro de uma partícula com pequena excentricidade cuja libração se torna instável depois de 20 milhões de anos de evolução estável. A transição começa com a aparição de duas linhas crescentes ($\nu_l \pm \nu_f$) a ambos lados de ν_l . Isto mostra que o segundo modo forçado também é responsável pela desestabilização de librações de pequena excentricidade. É interessante notar que a evolução inversa também é possível, como foi achado em alguns casos. No exemplo de Fig. 4.8 a análise espectral também mostrou que as transições entre libração e circulação de $\sigma(t)$ estão relacionadas à interação entre as linhas ν_l e $\nu_l \pm \nu_f$ do espectro. Parece ser então que o segundo modo forçado (e não necessariamente a ressonância secular ν_8) junto a ν_{18} são responsáveis pelas instabilidades da região $1.314 < a_0 < 1.330$ nas baixas excentricidades que mencionamos mais acima.

Nas altas excentricidades (mas sempre menores do que 0.37) podemos ter movimento estável para pequenas amplitudes de libração. Na Fig. 4.11 mostramos a evolução ao longo de 25 milhões de anos das freqüências ν_l e $(\nu_l \pm \nu_f)$ de uma

partícula desse tipo. A freqüencia instantanea de libração e o centro instantaneo de libracão apresentam oscilações periódicas devido ao segundo modo forçado como é de se esperar para um movimento ressonante estável. A amplitude tem pequenas oscilações de freqüencia $2\nu_f$ e uma componente de longo período da ordem de 10 milhões de anos. Para altas excentricidades e altas amplitudes de libracão, a ruptura do movimento ressonante é inevitável. Este processo é ilustrado na Fig. 4.12 para uma partícula com $e_0 = 0.25$. Duas linhas não identificadas a ambos lados de ν_f começam a crescer e interagem com a linha de libração produzindo uma interferencia construtiva que leva a valores de a suficientemente altos para produzir um encontro com Netuno e ter uma ejeção da ressonância no instante $t = 15.25$ milhões de anos. Ainda no problema restrito circular de três corpos, amplitude de libracão grande sempre significa evolução caótica (Malhara, 1995a). Achamos um comportamento similar em nossas integrações numéricas. No espectro de a, o comportamento caótico começa a aparecer como um ruído nas baixas freqüências ($\nu < 2$ em unidades de milhões de anos $^{-1}$) com amplitude da ordem de 0.0004 (em unidades de a_N), o qual é grande se comparado com a amplitude do segundo modo forçado. Na evolução temporal da amplitude de libração também é possível identificar uma componente caótica que é tanto maior quanto maior é a amplitude de libracão. A evolução temporal da amplitude de libração é um dos primeiros indicadores de que o movimento é instável.

As perturbações de curto período podem ser uma fonte de instabilidade. A linha de alta freqüência mais importante no espectro de a (ver Tabela 4.1) é $\nu_J - \nu_P$ (diferença entre as freqüências orbitais de Júpiter e da partícula) cuja amplitude é sempre 0.007. Para maiores excentricidades, as outras linhas ($\nu_J - N\nu_P$) começam a crescer e então a amplitude total também cresce. O mesmo acontece com as perturbações dos outros planetas gigantes e os termos de alta freqüência na excentricidade apresentam um comportamento análogo. As amplitudes destes termos de alta freqüência em a e e são grandes quando comparadas com as amplitudes de libracão. Mas cancelam em uma pequena escala de tempo. Em consequência, sua contribuição ao movimento de libração é desprezível se a partícula está numa região onde a ressonância é profunda, onde o movimento é dominado pelo termo ressonante do potencial perturbador de Netuno. Mas é possível que nas regiões onde a ressonância é superficial (ou débil), os termos de alta freqüência destabilizem as librações e produzam a ruptura das mesmas. A ressonância 2:3 está localizada dentro de uma região caótica do Sistema Solar exterior (Duncan et al., 1995). Em consequência a evolução posterior das partículas ao deixar a ressonância é caótica.

4.6 Conclusões.

Os principais resultados da teoria de librações de alta excentricidade incluindo as perturbações de Júpiter, Saturno e Urano são reproduzidas nas integrações numéricas do Sistema Solar Exterior. As superfícies de secção obtidas com o modelo plano elíptico de três corpos mostram uma estrutura muito regular sem sinais de difusão

em escalas de tempo de $10^7 - 10^8$ anos. Porém, no modelo de N corpos, aparece uma drástica redução no espaço de fase das condições iniciais que geram librações estáveis. A máxima excentricidade permitida pelas perturbações de Urano é $e \sim 0.37$. Nas baixas excentricidades a evolução é fortemente relacionada aos ramos de libração. O ramo positivo é estável e o negativo é instável devido ao efeito do segundo modo forçado e a superposição e interação entre as ressonâncias seculares ν_8 e ν_{18} . Isto explica os diferentes comportamentos que vários autores acharam para partículas com $a \sim 39.8$ UA. Partículas com centros de libração inicialmente localizados no ramo negativo da lei de estrutura podem ser temporariamente capturados em movimento de libração para a com valores de e e i similares aos de Plutão, mas essas órbitas não são estáveis. Parece ser necessário outro mecanismo para estabilizar as órbitas, como sugeriram Levison e Stern (1995).

O semi-eixo e a excentricidade das partículas evoluindo na ressonância apresentam oscilações de alta freqüência e grande amplitude devido fundamentalmente a Júpiter. Este efeito não é importante se a partícula está localizada numa região profunda da ressonância. Transições entre librações em torno do ramo negativo e o positivo na lei de estrutura são comuns para librações de alta excentricidade e amplitude. Antes de acontecer um encontro próximo com Netuno, a partícula alterna seu modo de libração através de um delicado mecanismo que somente pode ser seguido com integrações numéricas de alta precisão. Librações em torno de centros tais que $k_0 > 0.25$, do outro lado da curva de colisão, somente são possíveis por escalas de tempo de 10^5 anos.

A análise temporal de freqüências mostrou-se muito útil para detectar e identificar instabilidades. Esta técnica mostrou por exemplo, que o segundo modo forçado contribui às instabilidades das librações na região $1.314 < a_0 < 1.330$.

A ressonância de Kozai parece não ter muita importância na evolução das partículas localizadas dentro da ressonância 2:3. Finalmente, podemos indicar três possíveis saídas por onde as partículas podem deixar a ressonância: librações de grande amplitude, librações com $e \sim 0.37$ e librações de baixa excentricidade em torno de centros localizados no ramo negativo da lei de estrutura.

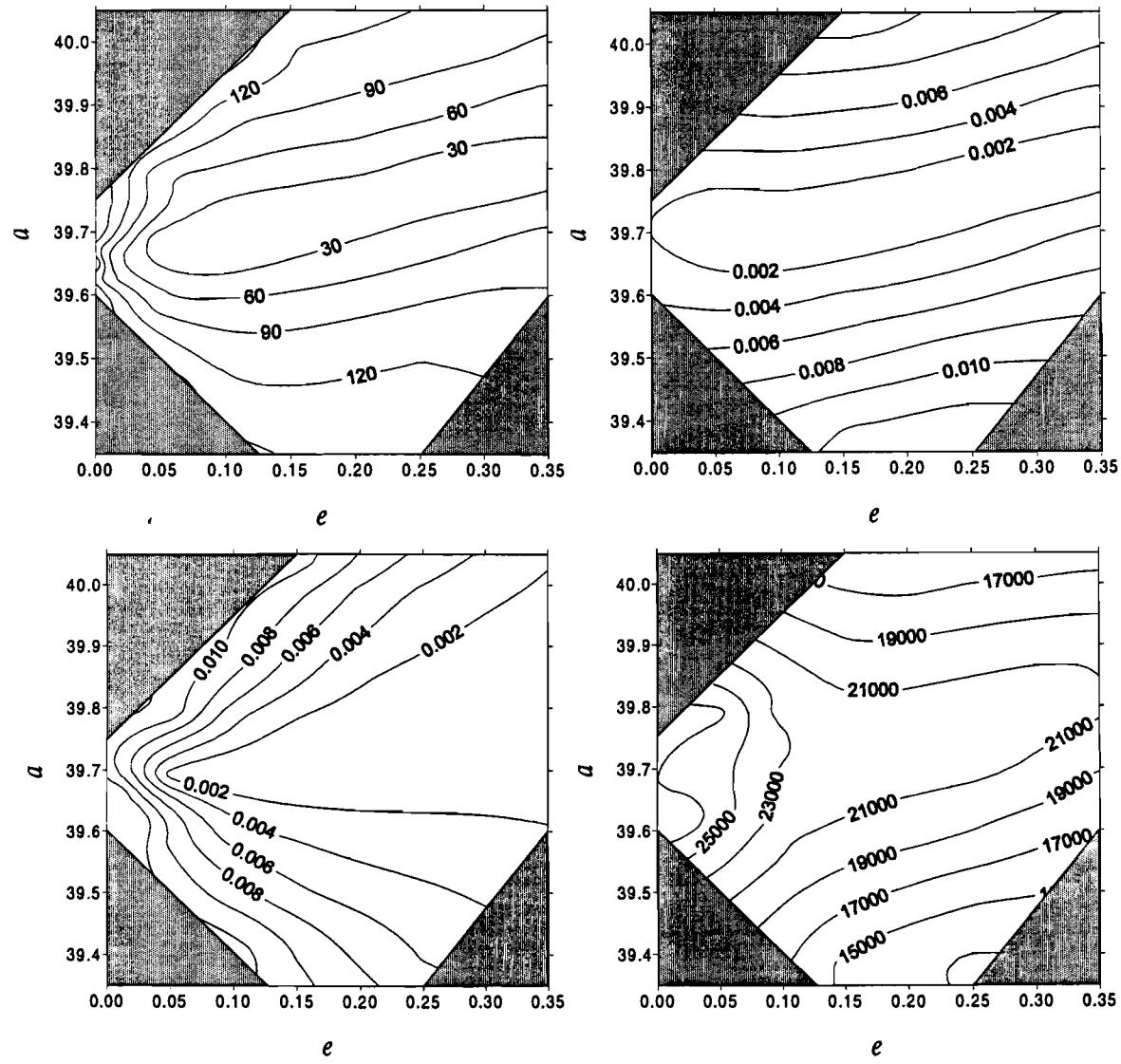


Figura 4.1: Regiões de libração em função da excentricidade e semi-eixo iniciais.
 a)Amplitude de libração de a em graus. b)Amplitude de libração de a em unidades de a_N . c)Amplitude de libração de e . d)Período de libração em anos.

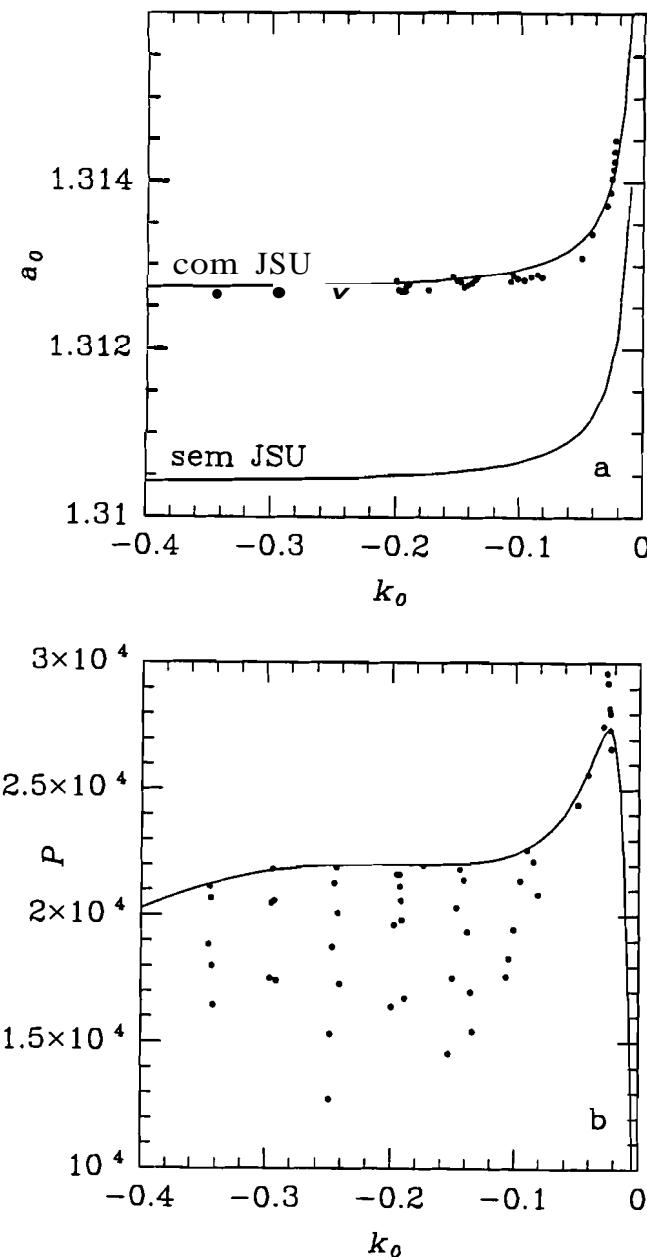


Figura 4.2: a)Lei de estrutura deduzida da analise espectral das integrações numéricas (pontos) e calculada com e sem o efeito dos outros planetas gigantes. b)Lei dos períodos dada pela teoria (para amplitude de libração zero) e pela analise espectral (pontos).

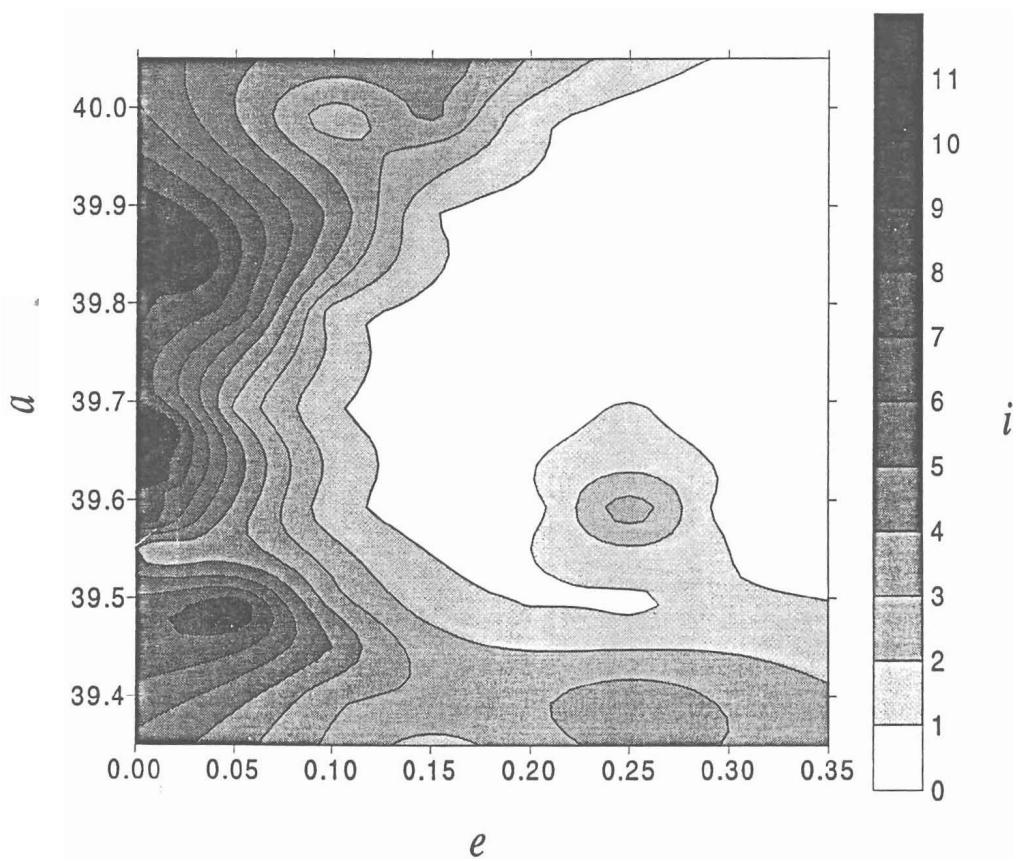


Figura 4.3: Valor médio da inclinação respeito do plano invariável, em função das condições iniciais.

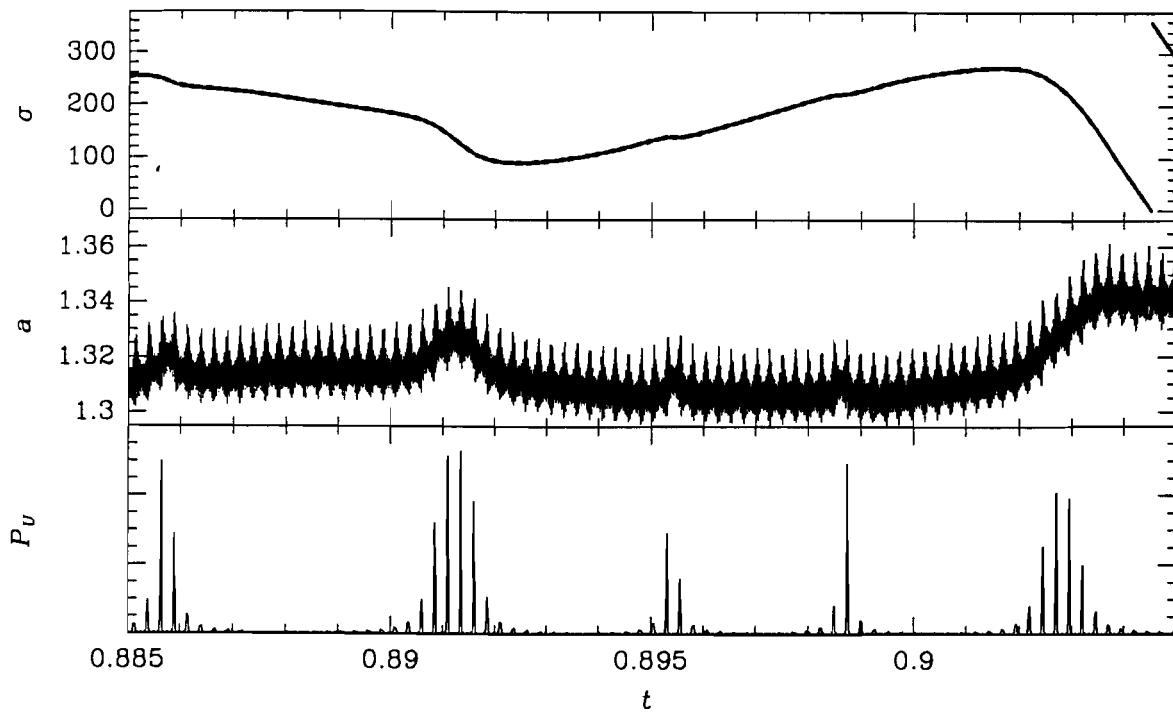


Figura 4.4: Librações de alta excentricidade destruídas pelas perturbações de Urano. Mostramos a evolução imediatamente anterior ao instante em que a começa a circular e o semi-eixo salta. Tempo em milhões de anos. As oscilações regulares rápidas de a são devidas as perturbações de Júpiter e tem amplitude um pouco menor que as de Urano.

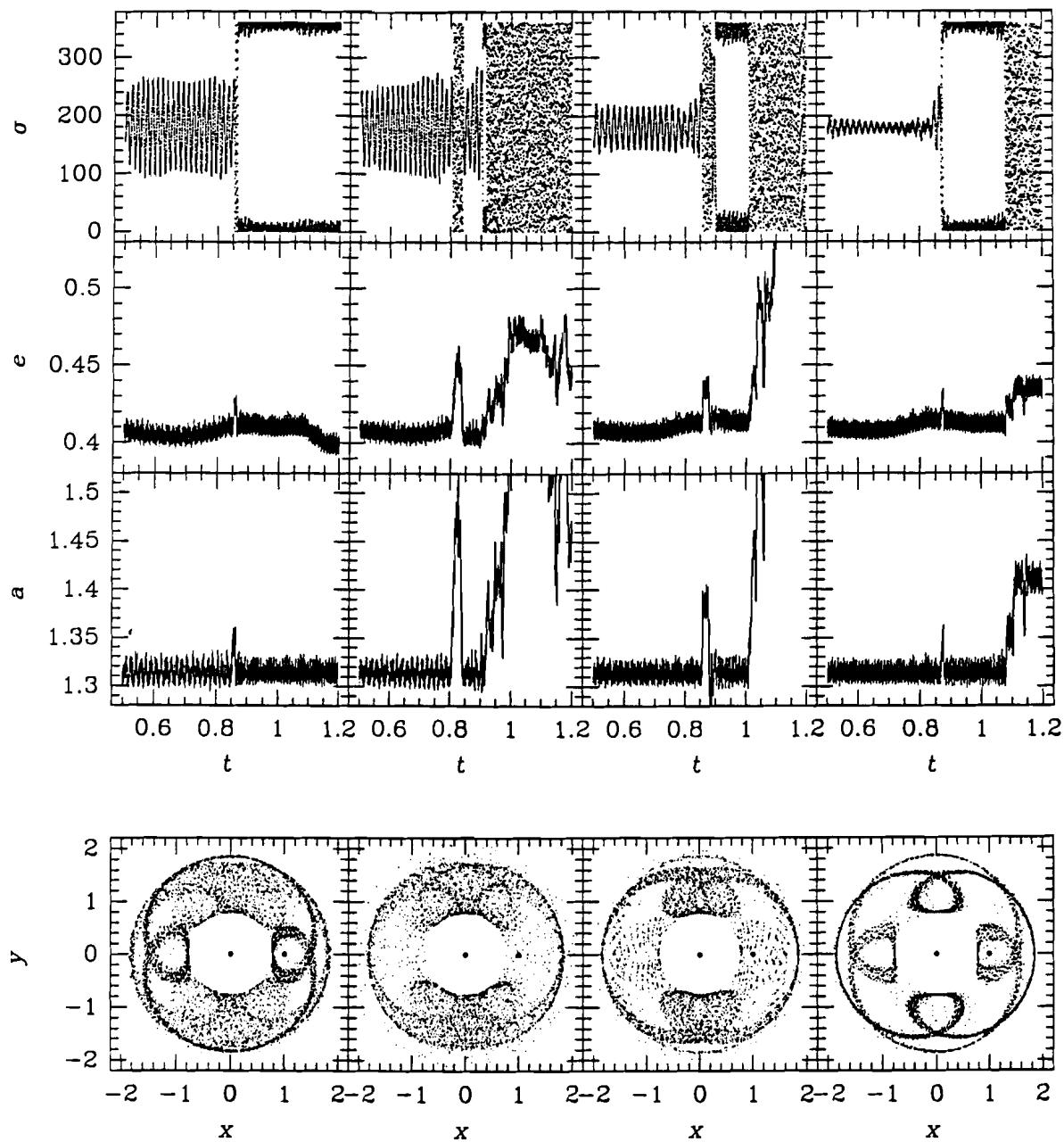


Figura 4.5: Mecanismo de escape nas altas excentricidades. Quatro partículas seguindo evoluções típicas mostrando transições entre libração em torno de $a = 180^\circ$ e $a = 0^\circ$. Na parte inferior da figura se mostra as posições das partículas tomadas a intervalos regulares respeito do sistema Sol–Netuno, sendo o Sol a origem.

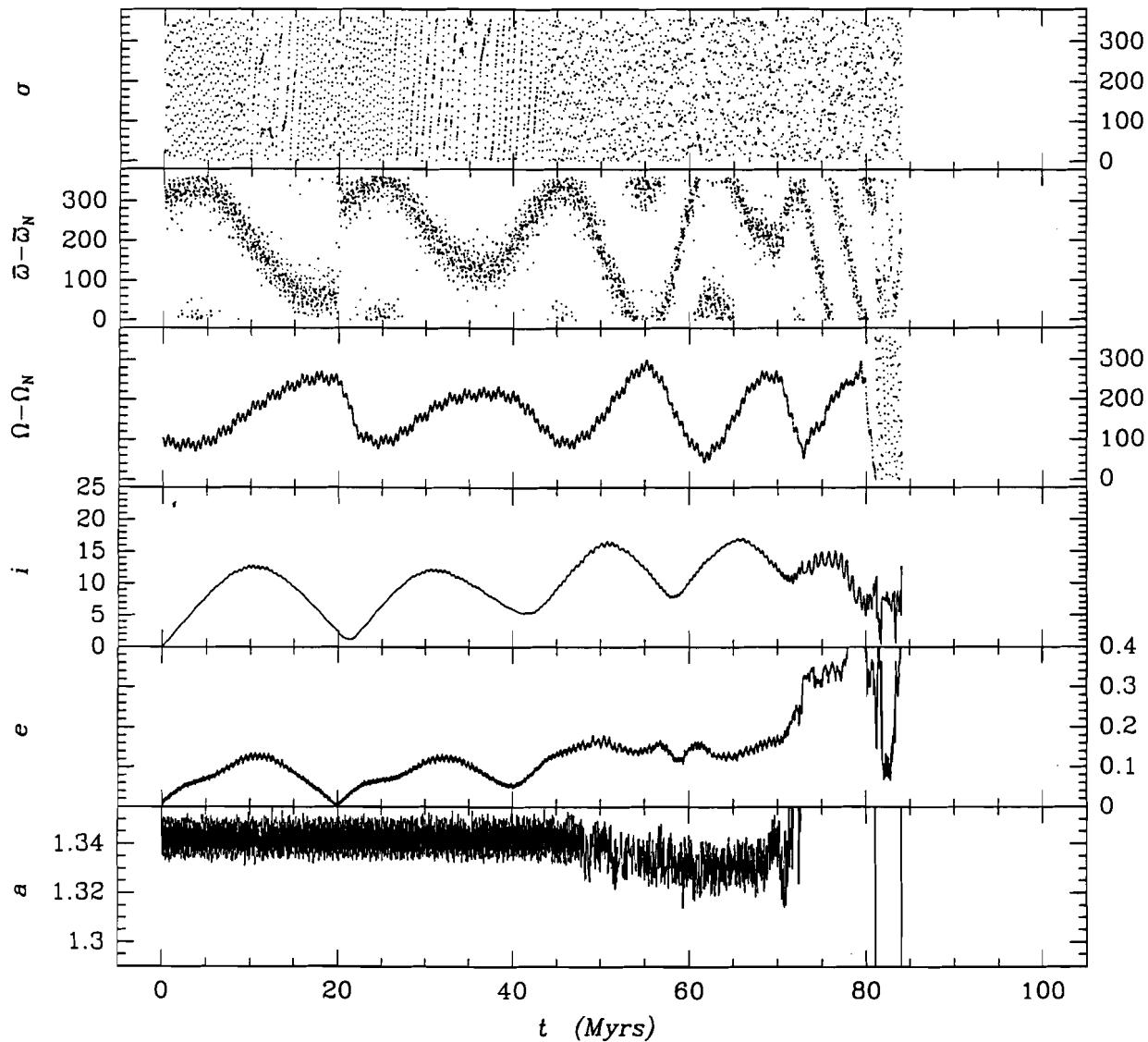


Figura 4.6: Ressonância secundária secular $\nu_8 + \nu_{18}$. Ressonância secular ν_8 acoplada com a ressonância secular ν_{18} . O centro de libração desta partícula estava originalmente localizado no ramo negativo da lei de estrutura com $a_0 = 1.334$

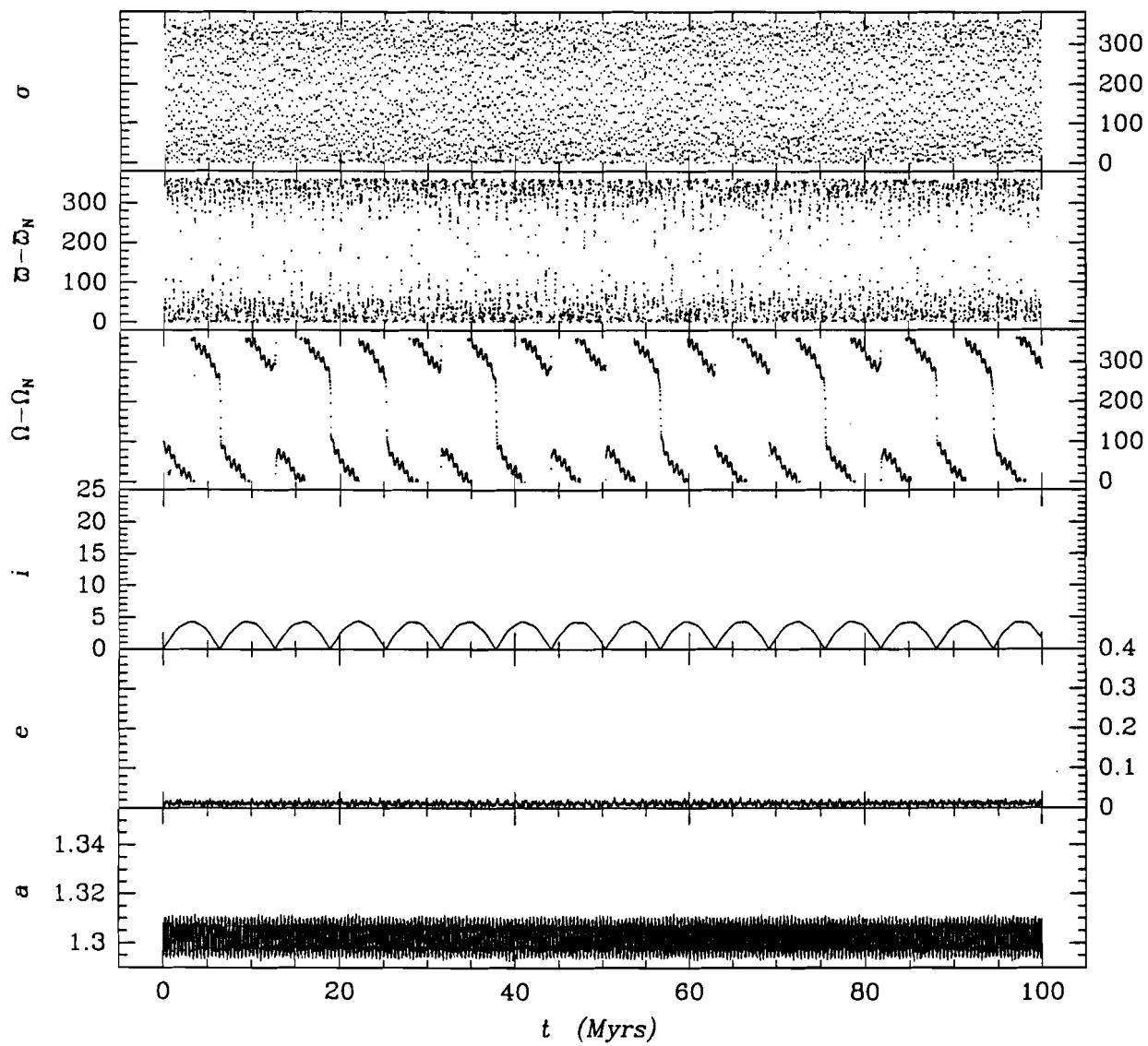


Figura 4.7: Partícula em evolução estável no ramo positivo da lei de estrutura.

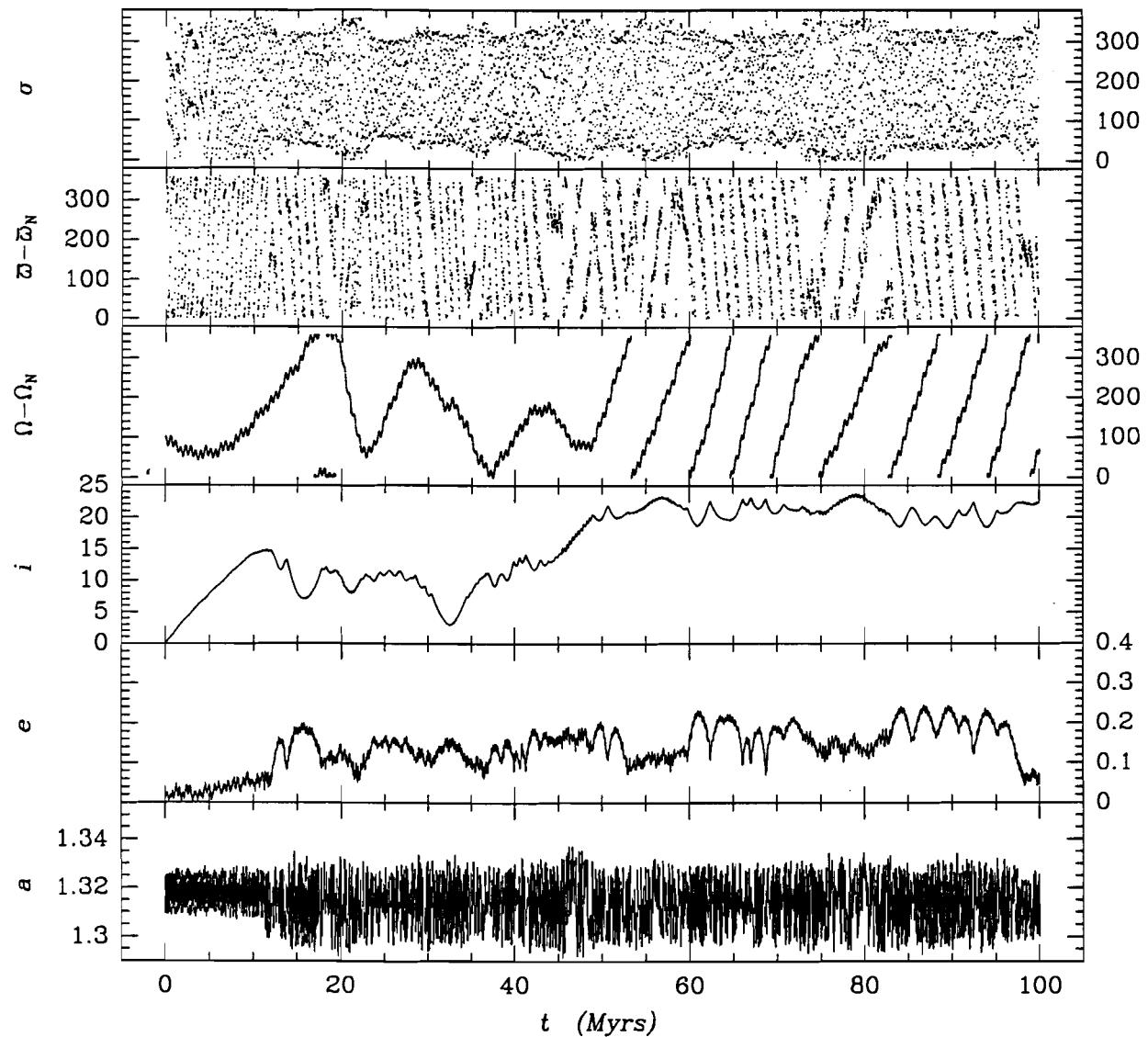


Figura 4.8: Evolução caótica no ramo negativo de librações, na região $1.314 < a_0 < 1.330$.

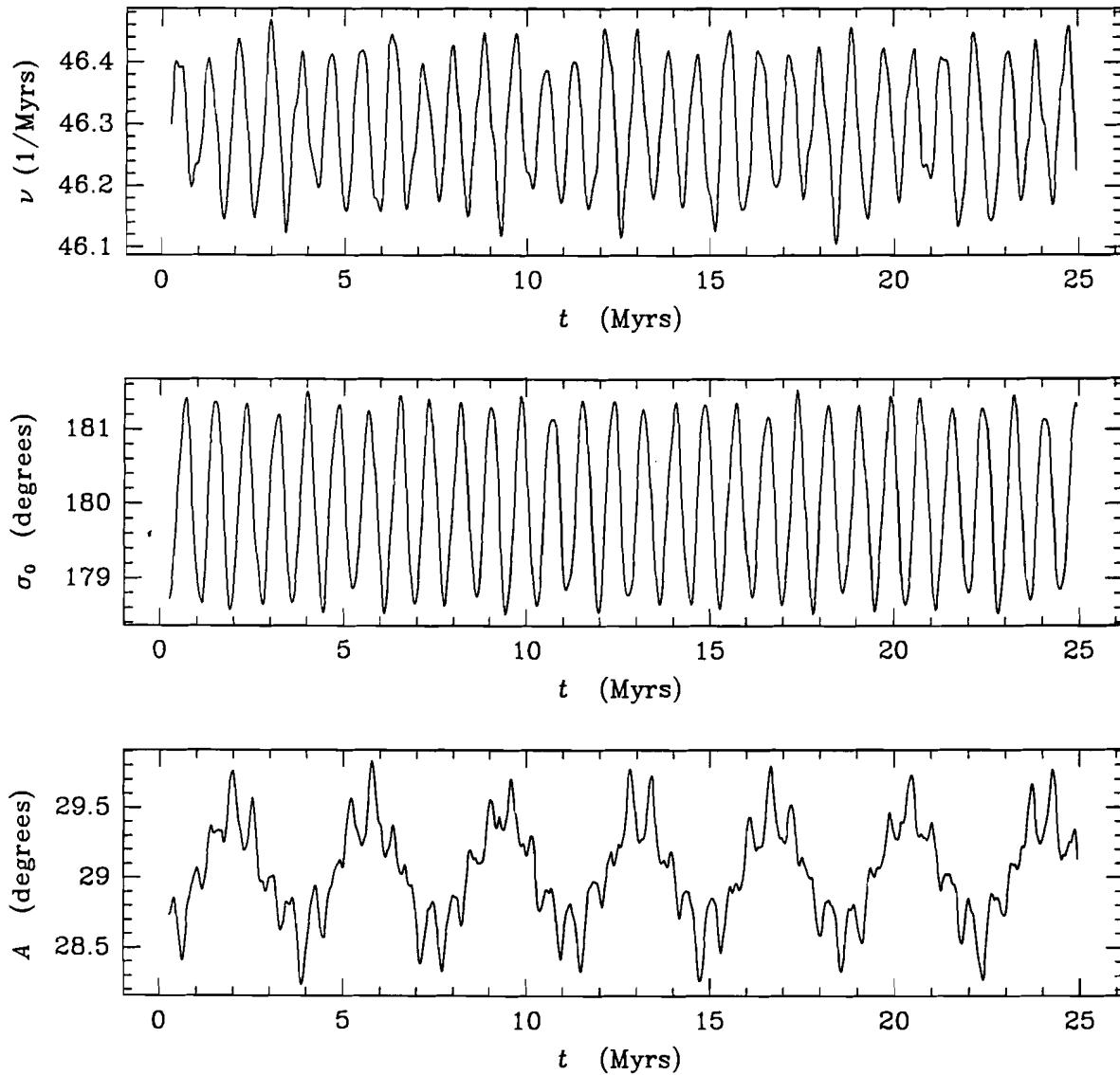


Figura 4.9: Evolução temporal da freqüéncia (a), centro (b) e amplitud (c) instantâneos de libração para o caso de un movimento estável. Cada ponto na figura é calculado usando $T_w = 0.5$ milhões de anos. Observe-se a evolução periódica em $\nu(t)$, $\sigma_0(t)$ e $A(t)$.

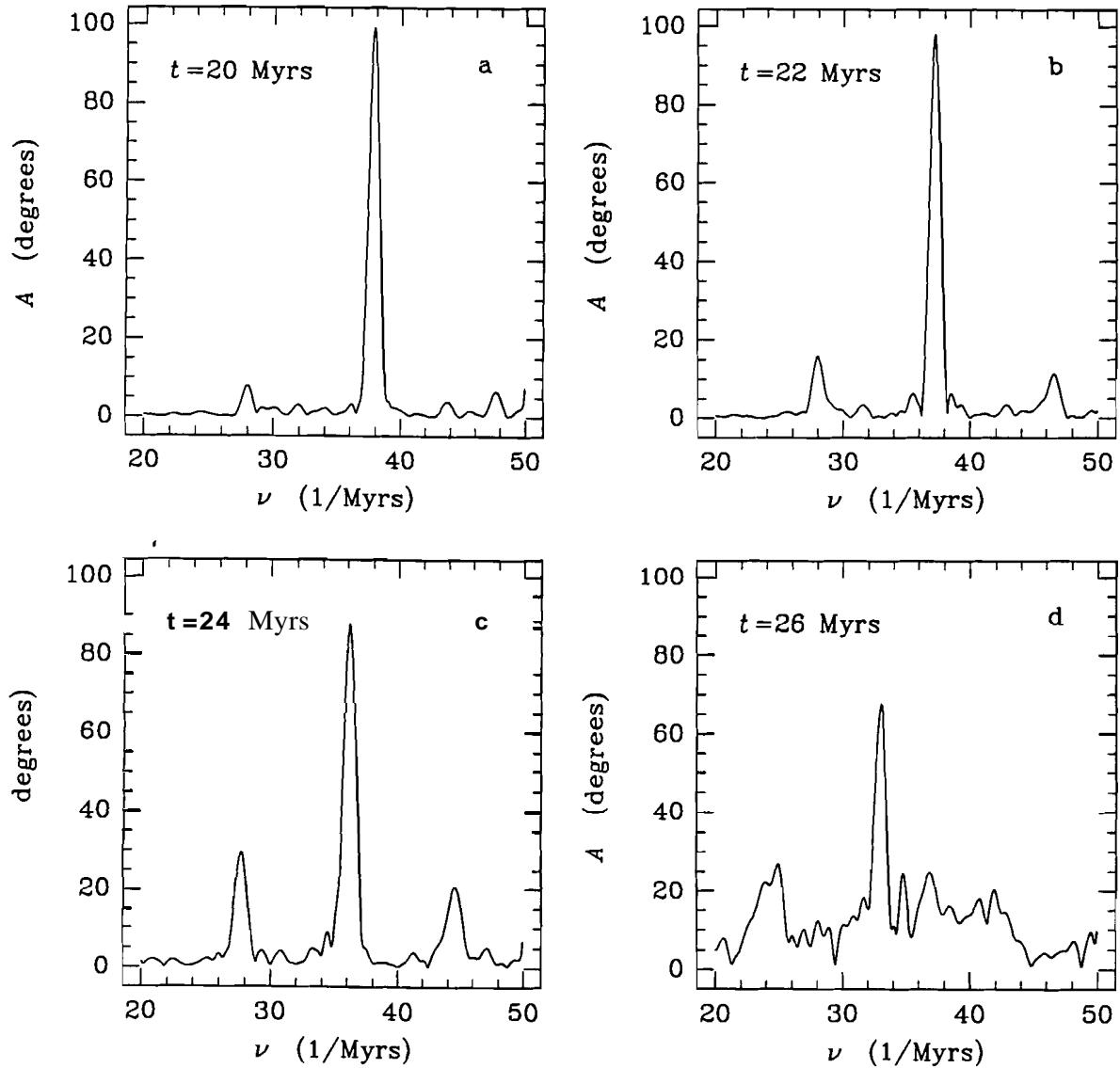


Figura 4.10: Transição para o caos nas baixas excentricidades. Evolução temporal do espectro na região da linha de libração para uma partícula com $e = 0.04$ e grande amplitude de libração. A seqüência mostra a destruição das librações. Cada espectro é obtido usando $T_w = 2$ milhões de anos. As linhas presentes são $\nu_l - \nu_f$, ν_l e $\nu_l + \nu_f$.

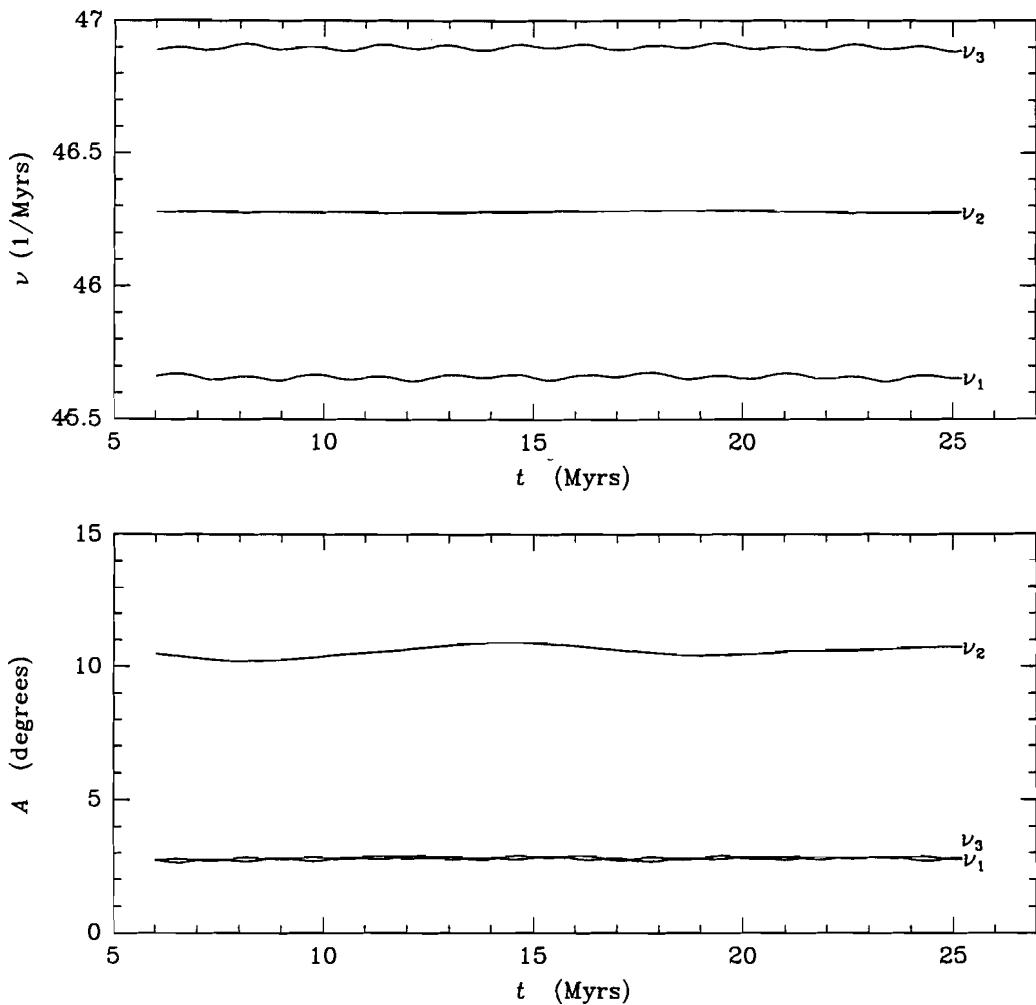


Figura 4.11: Partícula em evolução estável com librações de pequena amplitude e $\epsilon \sim 0.29$. (a) Evolução das freqüências ν_l e $\nu_l \pm \nu_f$. (b) Evolução das amplitudes. Foi usado $T_w = 6$ milhões de anos.

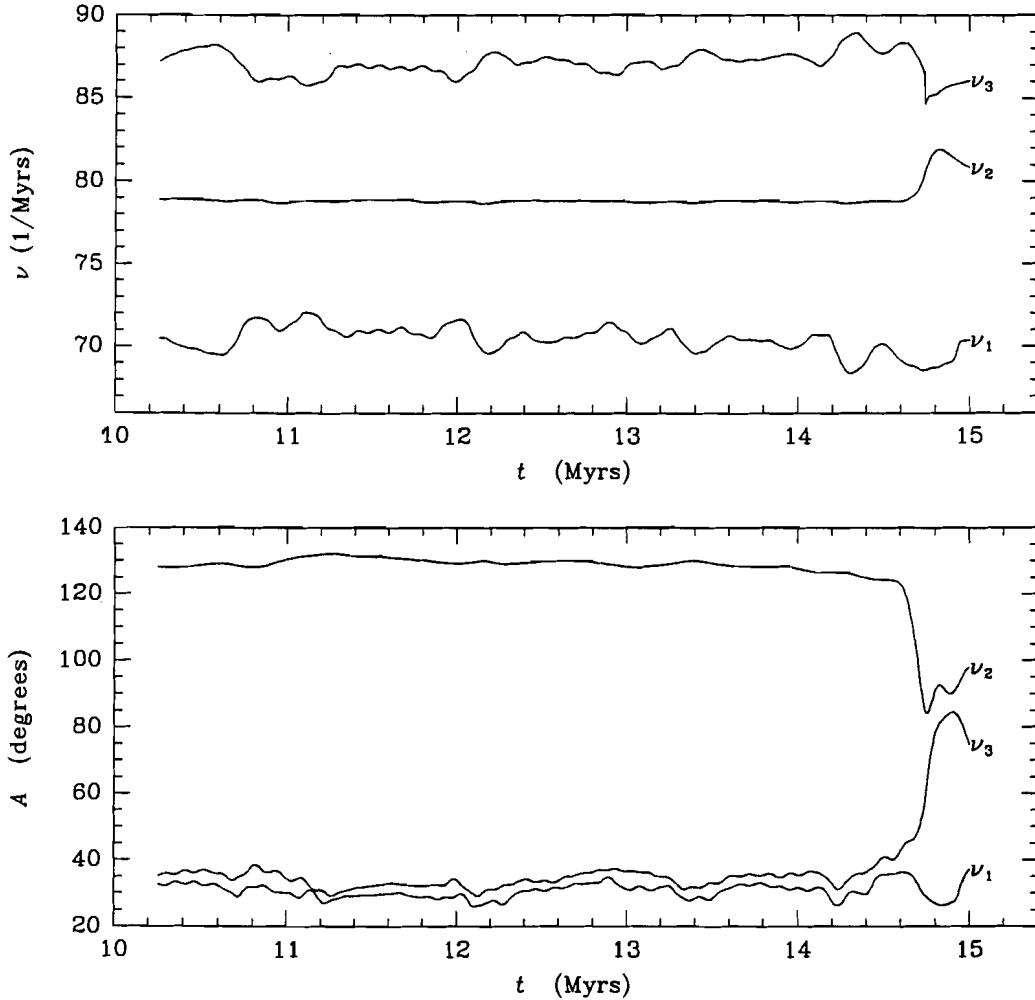


Figura 4.12: Partícula em evolução instável com librações de grande amplitude e $e \sim 0.25$. (a) Evolução das três freqüências principais do espectro de \mathbf{a} . (b) Evolução das amplitudes. A freqüência de libração é ν_2 . A ejeção da ressonância acontece em $t = 15.25$ milhões de anos. Usamos $T_w = 0.5$ milhões de anos.

Capítulo 5

Outras ressonâncias.

Vimos que a ressonância 2:3 permite a existência de partículas de excentricidade até 0.37 evoluindo em forma estável. As outras ressonâncias da região transneptuniana também podem, em princípio, ter regiões de evolução estável ainda para altas excentricidades. Nossa interesse é determinar os limites dessas regiões de librações estáveis pois, dada a caoticidade do cinturão de Kuiper, essas regiões podem ser as mais populosas. Reciprocamente, interessa determinar se a estabilidade das órbitas quase-circulares localizadas além das 45 UA é destruída por alguma ressonância. Desta forma teríamos um mecanismo de geração de órbitas excêntricas a partir de movimentos circulares, o qual é importante para estudar a evolução colisional da região.

Neste capítulo resumiremos as principais características das ressonâncias e procuraremos os limites das regiões estáveis, mediante integrações numéricas análogas as feitas no Capítulo 4 e analisadas seguindo a metodologia descrita no Anexo II. Os resultados e o significado do ponto de vista cosmogônico são apresentados e discutidos no Anexo III.

5.1 Aplicaçáo da teoria de librações.

Para as ressonâncias em destaque na Fig. 2.1 calculamos as leis respectivas em forma análoga à feita para a ressonância 2:3 no Capítulo 3, introduzindo as perturbações de Júpiter, Saturno e Urano. Como a teoria somente vale para centros de libração nos eixos k ou h , as leis correspondentes às librações assimétricas nas ressonâncias 1:2 e 1:3 não foram calculadas. Se definimos o ângulo crítico como

$$\sigma = (p/q + 1)\lambda_N - p/q\lambda - \varpi \quad (5.1)$$

é possível demonstrar que os centros de libração ('simétricos') verificam

$$\sigma_0 = 0^\circ \quad \text{módulo } \frac{180^\circ}{q}.$$

Alguns desses centros serão estáveis e outros instáveis. Nas figuras somente apresentamos os ramos de librações estáveis.

Nas Figs. 5.1–7 apresentamos a lei de estrutura, lei dos períodos, modos forçados Z_k e Z_h , e período de circulação de σ_1 para todas as ressonâncias. A lei dos períodos pode ser comparada com os resultados numéricos de Malhotra (1995a). Em todas as ressonâncias, o ramo da direita dos gráficos corresponde às librações estáveis em torno de $a = 0^\circ$ localizados do lado de fora da curva de colisão. Portanto, como vimos no caso da 2:3, essas librações são facilmente destruídas por aproximações a Netuno, mas também podem atuar como centro de libração temporário para as partículas librando nos outros ramos, como foi visto no Capítulo 4. Os limites máximos de excentricidade nos gráficos correspondem a colisão com Urano. As trajetórias das partículas com respeito ao sistema Sol–Netuno podem ser achadas em Malhotra (1995a) (nós apresentamos algumas nas Figs. 5.8–13) e as curvas de colisão para algumas ressonâncias em Beaugé e Ferraz–Mello (1994).

Observamos que somente as ressonâncias de primeira ordem apresentam dois ramos na lei de estrutura quando $e \sim 0$. Também observamos que quando $e \rightarrow 0$ se verifica $P/P_1 \rightarrow 1$, $A_h/A_k \rightarrow 1$ e $Z_h/Z_k \rightarrow 1$. Isto parece ser exclusivo das ressonâncias de primeira ordem. O movimento para centros de libração com $e \sim 0$ resulta ser a soma de dois movimentos quase-circulares de diferentes raios mas com períodos semelhantes. Nestas circunstâncias é difícil distinguir entre o que é libração e o que é modo forçado. Esta ressonância secundária 1:1 entre a e σ_1 produz, como foi visto no caso da 2:3, uma libração de $\Delta\varpi$ (ressonância secular ν_8). Para que isto aconteça é preciso que a amplitude de libração não seja exatamente zero, portanto não se trata de uma corrotação. As corrotações podem ser identificadas nos gráficos de P_1 quando este tende para infinito.

Morbidelli et al. (1995) argumenta que nas ressonâncias de primeira ordem não pode existir a ressonância secular ν_8 nas baixas excentricidades pois $\overline{\Delta\varpi} \rightarrow -\infty$. Isso é válido estritamente para amplitude de libração zero e, como vimos no caso da ressonância 2:3, basta uma pequena amplitude de libração para produzir uma libração de Aw. Aliás, o resultado de Morbidelli et al. é somente válido para o ramo negativo das librações. Para o ramo positivo temos ao contrário $\overline{Aw} \rightarrow +\infty$. Esta descontinuidade no movimento do periélio na verdade é originada da indeterminação do periélio para $e = 0$.

Outra característica das ressonâncias de primeira ordem é que tanto o período de libração de a como o período de circulação de σ_1 tendem para zero quando $e \rightarrow 0$, enquanto que em todas as outras ressonâncias de ordem superior estudadas o período de libração tende para infinito.

5.2 Exploração numérica.

Fizemos integrações numéricas do Sistema Solar Exterior análogas às do Capítulo 4 por períodos de 10 até 100 milhões de anos com diversas condições iniciais para as partículas. Todas as integrações se referem às librações em torno de centros localizados dentro da região limitada pela curva de colisão. Por ser a região dinamicamente mais ativa, estudamos o limite máximo de excentricidade para as ressonâncias lo-

calizadas até 50 UA. As ressonâncias 2:5 e 1:3 estão fóra desta região. O resultado destas integrações é discutido no Anexo III, mas alguns casos merecem ser comentados.

A dinâmica da ressonância 4:5, por exemplo, mostrou-se como a mais complexa. As librações são fortemente perturbadas pela ressonância de Kozai (libração de w) e a ressonância secular ν_{18} que excita a inclinação até aproximadamente 15 graus (Fig. 5.8). Para obter librações estáveis é preciso limitar-se a excentricidades menores do que 0.12. Nas outras ressonâncias, ao contrário, sempre foi possível estabelecer claramente um limite máximo na excentricidade (Figs. 5.9–12). Os resultados numéricos estão em bom acordo com os resultados teóricos no que se refere às leis das ressonâncias.

Ao procurar mecanismos de geração de órbitas excêntricas a partir de órbitas quase-circulares (como acontece na 2:3) centramos a atenção nas ressonâncias 1:2 e 1:3 por duas razões. Em primeiro lugar elas estão afastadas da região ativa do cinturão e portanto seriam uma possível fonte de projétils que excitariam a população além das 50 UA. Em segundo lugar, apresentam uma topologia complexa no espaço (k, h) com o surgimento de centros de libração assimétricos em excentricidades relativamente baixas (Beaugé, 1994). Para ter maior certeza nos resultados todas estas integrações foram feitas por períodos de 100 milhões de anos partindo sempre de órbitas quase-circulares com inclinação inicial zero com respeito ao plano invariável.

No caso da ressonância 1:2 achamos um comportamento análogo ao da ressonância 2:3 nas baixas excentricidades. O ramo negativo é instável, e o ramo positivo é estável. A instabilidade acontece para centros de libração que verificam $1.596 < a_0 < 1.590$ (ver Fig. 5.3a), e leva a excentricidade a crescer até e ~ 0.15 . A ressonância secular ν_8 está presente em ambos os ramos de libração, mas a ressonância ν_{18} somente aparece na região instável do ramo negativo. Então, como na 2:3, a presença das duas ressonâncias secundares ν_8 e ν_{18} parece ser a causa da instabilidade. Na Fig. 5.13 temos um exemplo. Será esta uma propriedade que se repete nas ressonâncias de primeira ordem? Pensamos que esta possibilidade merece ser explorada no futuro. Tanto na 2:3 como na 1:2 a instabilidade está associada a ν_{18} , portanto nos modelos planos a instabilidade não deveria surgir. Isto está em acordo com o fato de que nas integrações numéricas no caso plano do Capítulo 3 não foi achada nenhuma instabilidade nas baixas excentricidades.

A ressonância 1:3 não apresentou instabilidade importante nas órbitas quase-circulares. As maiores variações na excentricidade são da ordem de 0.04. As librações assimétricas aparecem para $e > 0.12$ mas esta região não foi analisada. Concluímos que a ressonância 1:3 não gera órbitas instáveis a partir de movimentos quase-circulares.

Para finalizar, fizemos algumas integrações de partículas com periódicos $q = 45$ UA e $q = 40$ UA localizadas fóra das ressonâncias e dentro da região caótica, determinada por Torbett e Smoluchowski (1990) mediante cálculo de expoentes de Lyapunov. Usando a metodologia do Anexo II, e com tempos de integração dez vezes superiores aos de Torbett e Smoluchowski, obtivemos os espectros resultando

ser típicos de movimentos quase-regulares para o caso de partículas com $q = 45$ UA. No caso das partículas com $q = 40$ UA alguns espectros são notoriamente caóticos, mas não todos. Utilizando um procedimento diferente para o cálculo de expoentes de Lyapunov (neste caso integrando as equações variacionais), Roig (1996) achou que os expoentes correspondentes a ambos grupos de partículas não apresentavam convergência para valores positivos, na mesma escala de tempo em que trabalharam Torbett e Smoluchowski (integrações feitas por 10 milhões de anos). Talvez seja prematuro dizer que nossos resultados e os de Roig estão em contradição com os de Torbett e Smoluchowski, mas evidentemente existe uma diferença, que é notória nos resultados correspondentes às partículas com periélio de 45 UA (Fig. 5.14). Uma possível explicação é que os resultados de Torbett e Smoluchowski estejam contaminados pois é sabido que os expoentes de Lyapunov são muito sensíveis ao integrador usado, ao passo de integração, ao método de cálculo do expoente, às variáveis usadas, à separação inicial das partículas, etc. Nossos resultados, ao contrário, estão em bom acordo com os de Duncan et al. (1995, Fig. 1), quem acharam que em escalas de tempo de 10^9 anos as partículas com $q = 45$ não são ejetadas do Sistema Solar, enquanto algumas com $q = 40$ são.

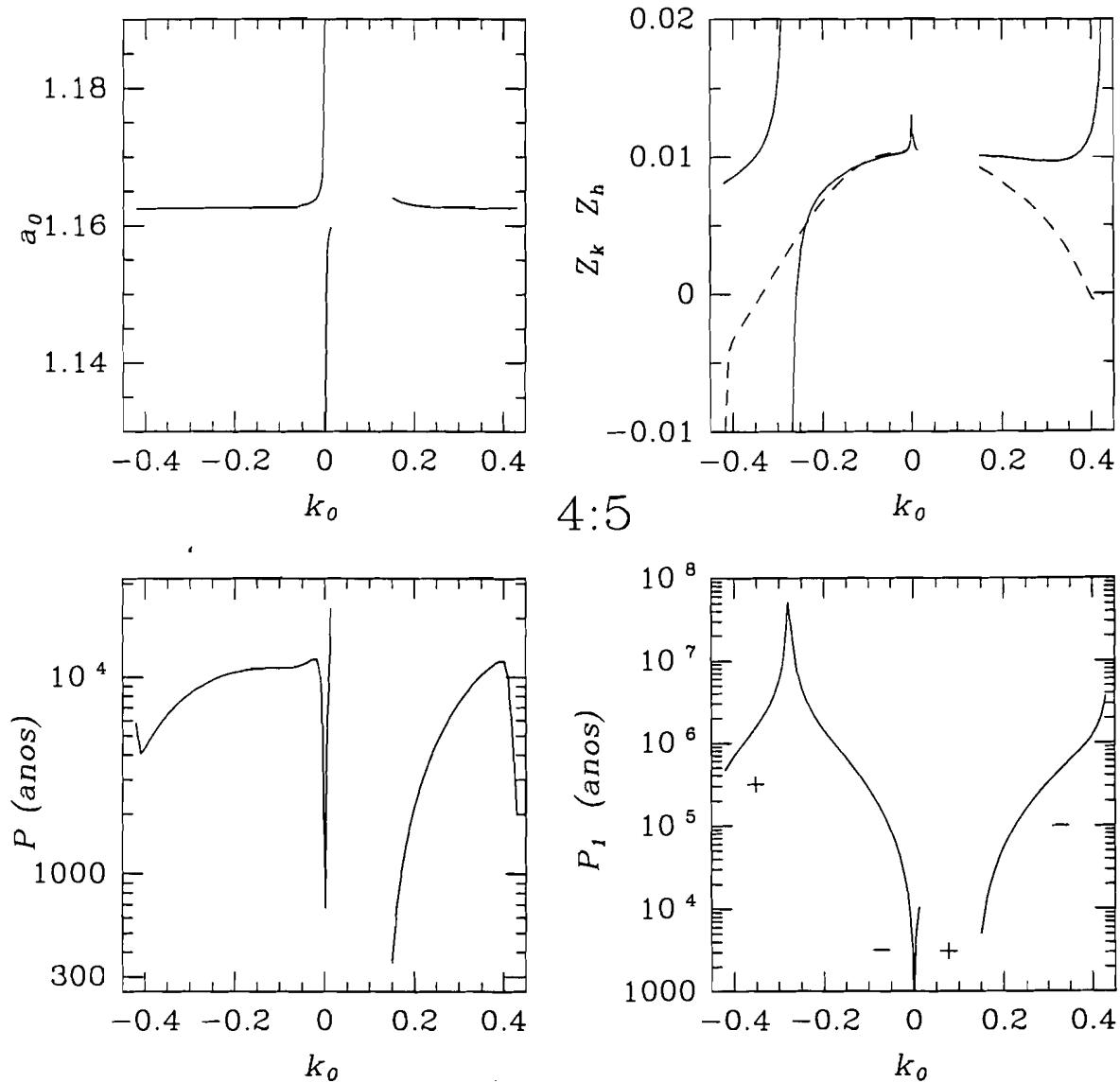


Figura 5.1: Ressonância 4:5. Lei de estrutura (a), modos forçados (b), período de libração (c) e período de circulação de σ_1 (d).

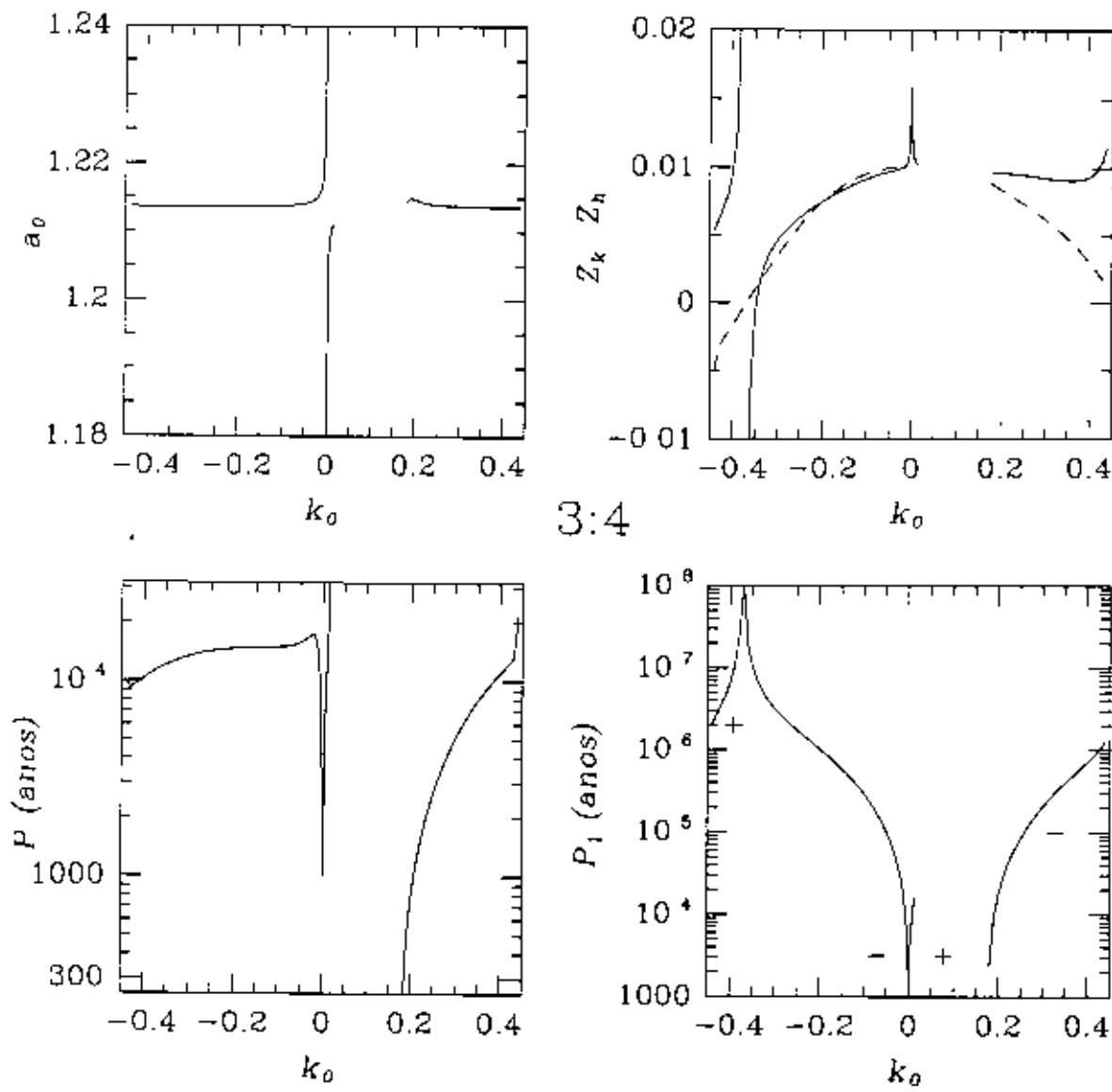


Figura 5.2: Ressonância 3:4. Lei de estrutura (a), modos forçados (b), período de libração (c) e período de circulação de σ_1 (d).

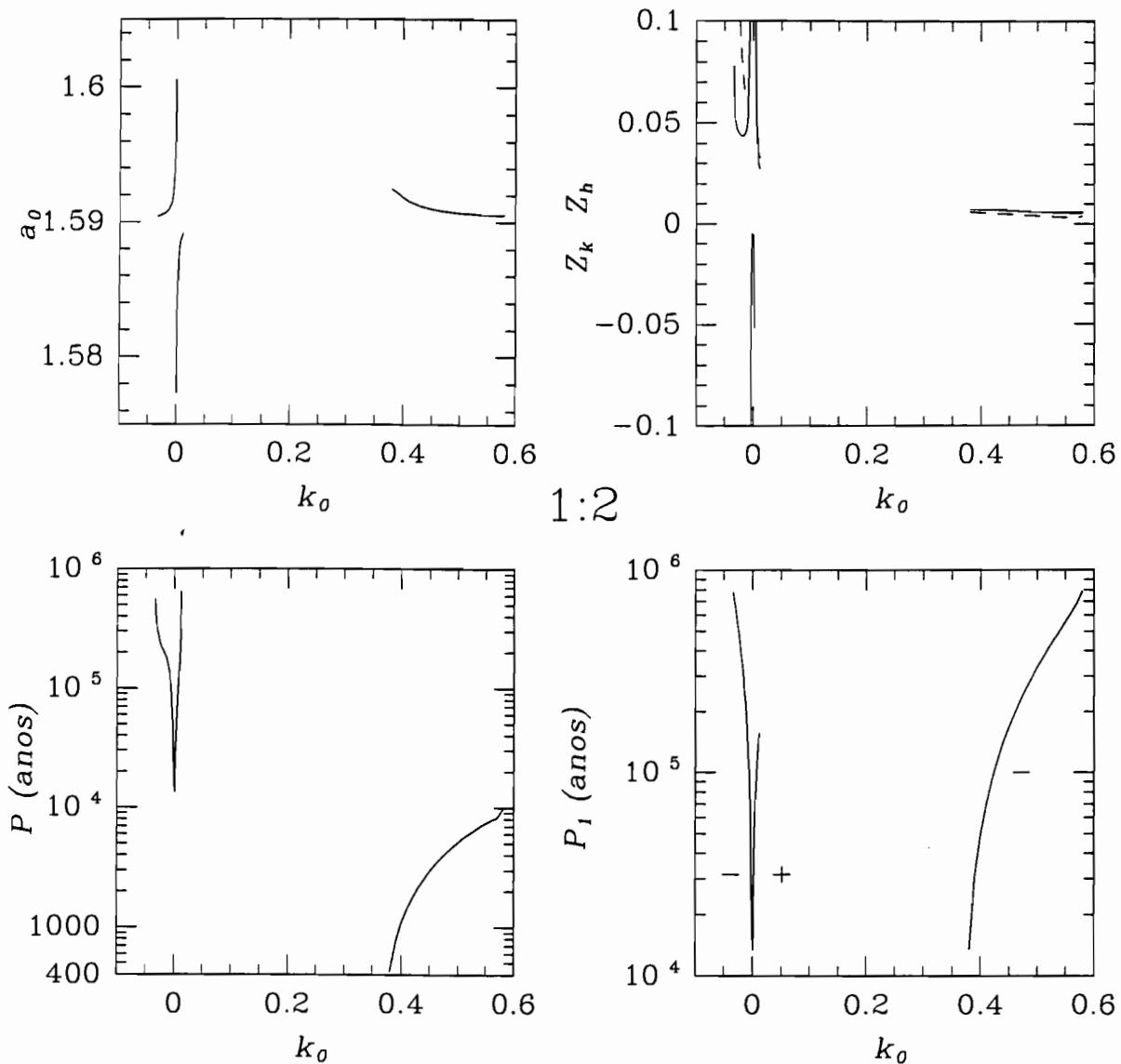


Figura 5.3: Ressonância 1:2. Lei de estrutura (a), modos forçados (b), período de libração (c) e período de circulação de σ_1 (d).

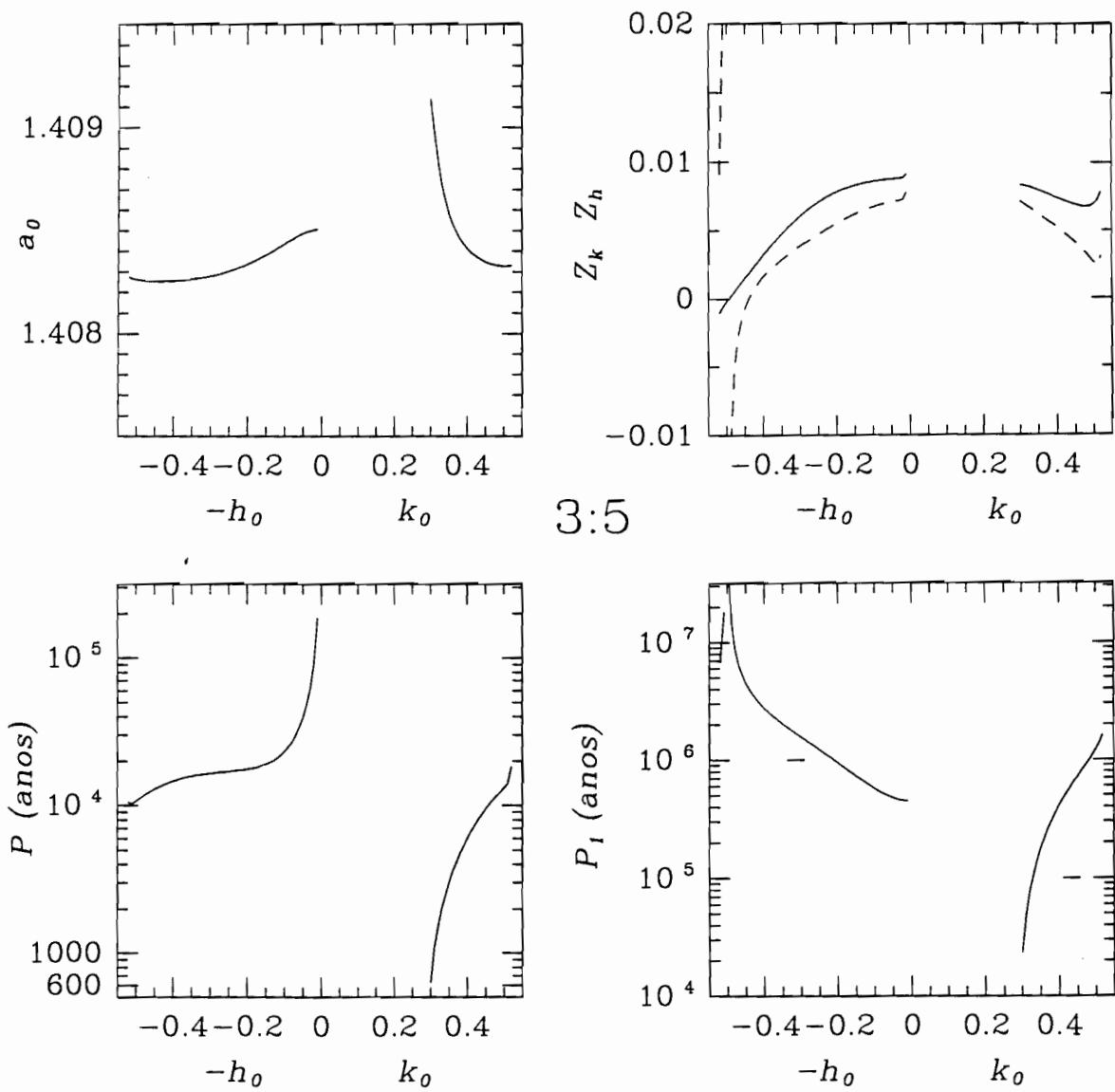


Figura 5.4: Ressonância 3:5. Lei de estrutura (a), modos forçados (b), período de libração (c) e período de circulação de σ_1 (d).

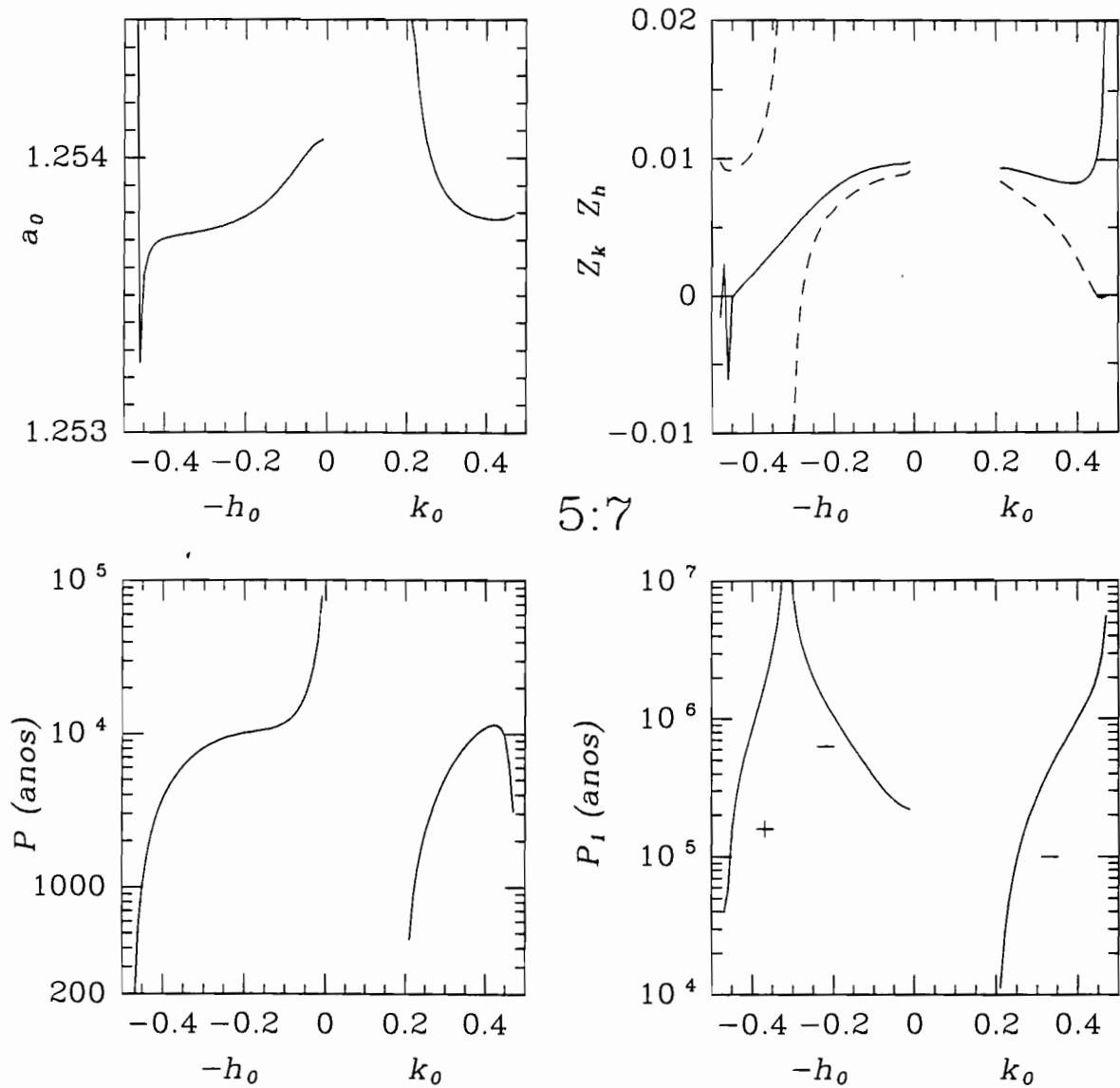


Figura 5.5: Ressonância 5:7. Lei de estrutura (a), modos forçados (b), período de libração (c) e período de circulação de σ_1 (d).

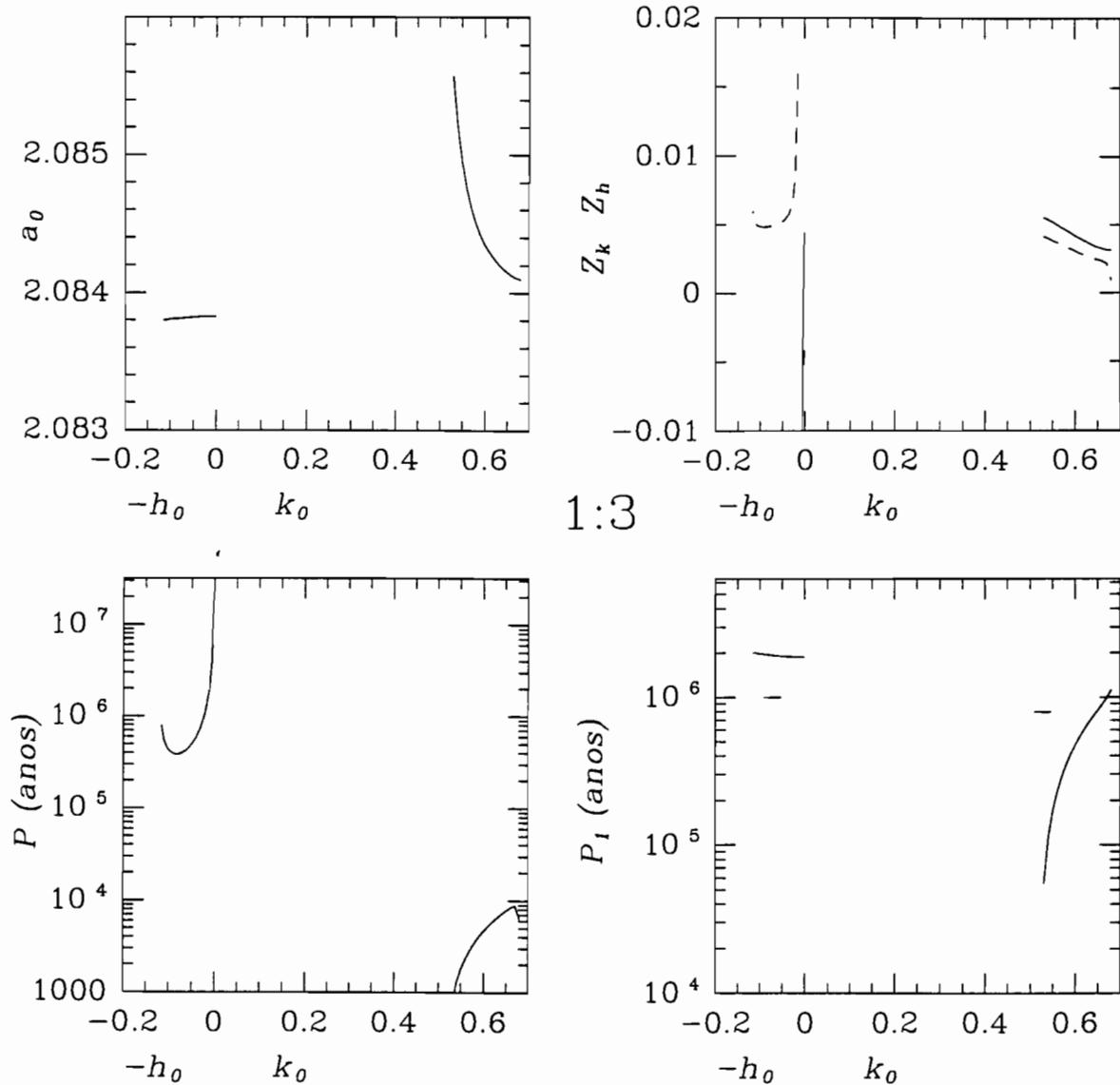


Figura 5.6: Ressonância 1:3. Lei de estrutura (a), modos forçados (b), período de libração (c) e período de circulação de σ_1 (d).

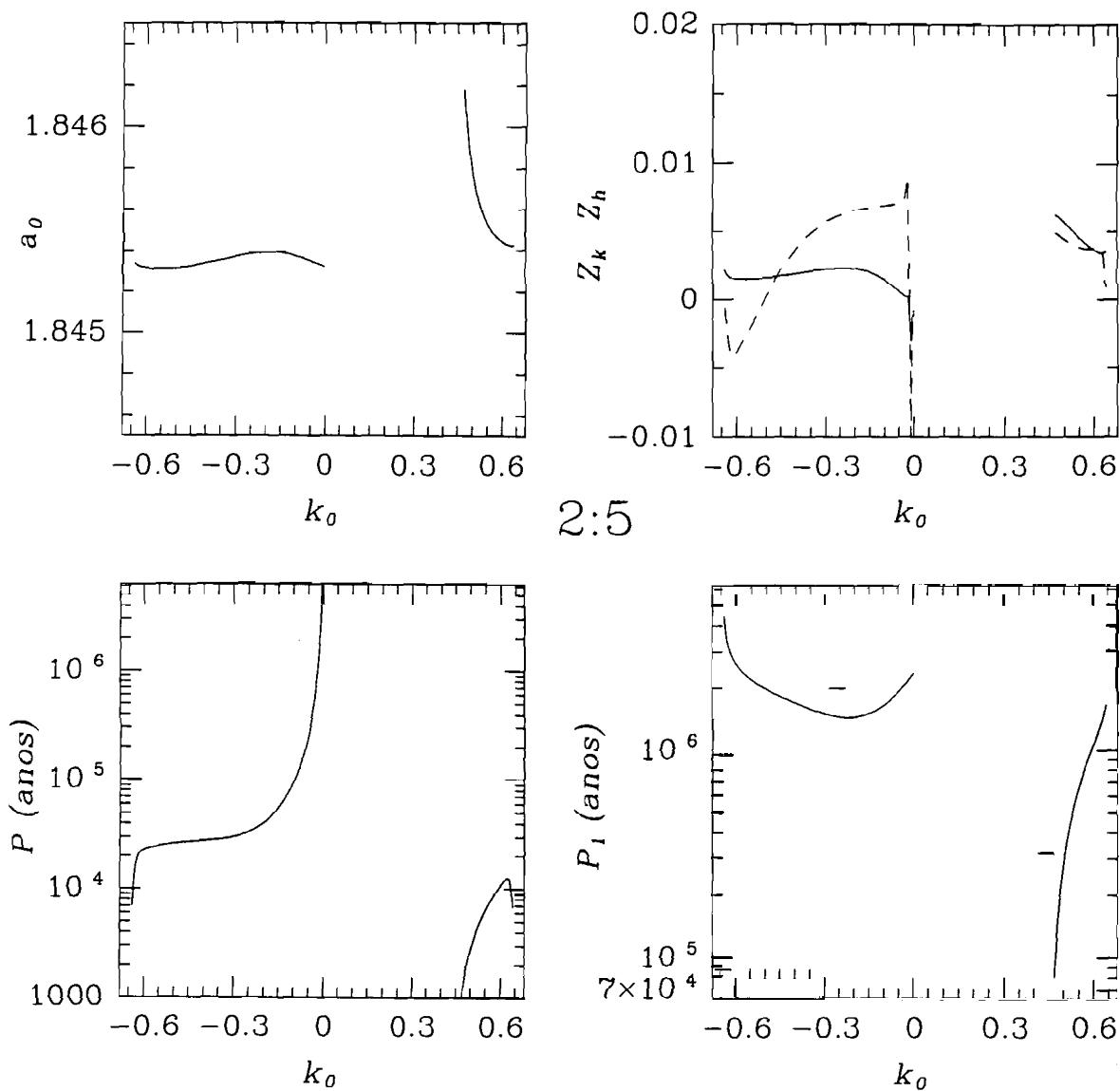


Figura 5.7: Ressonância 2:5. Lei de estrutura (a), modos forçados (b), período de libraqão (c) e período de circulaqão de σ_1 (d).

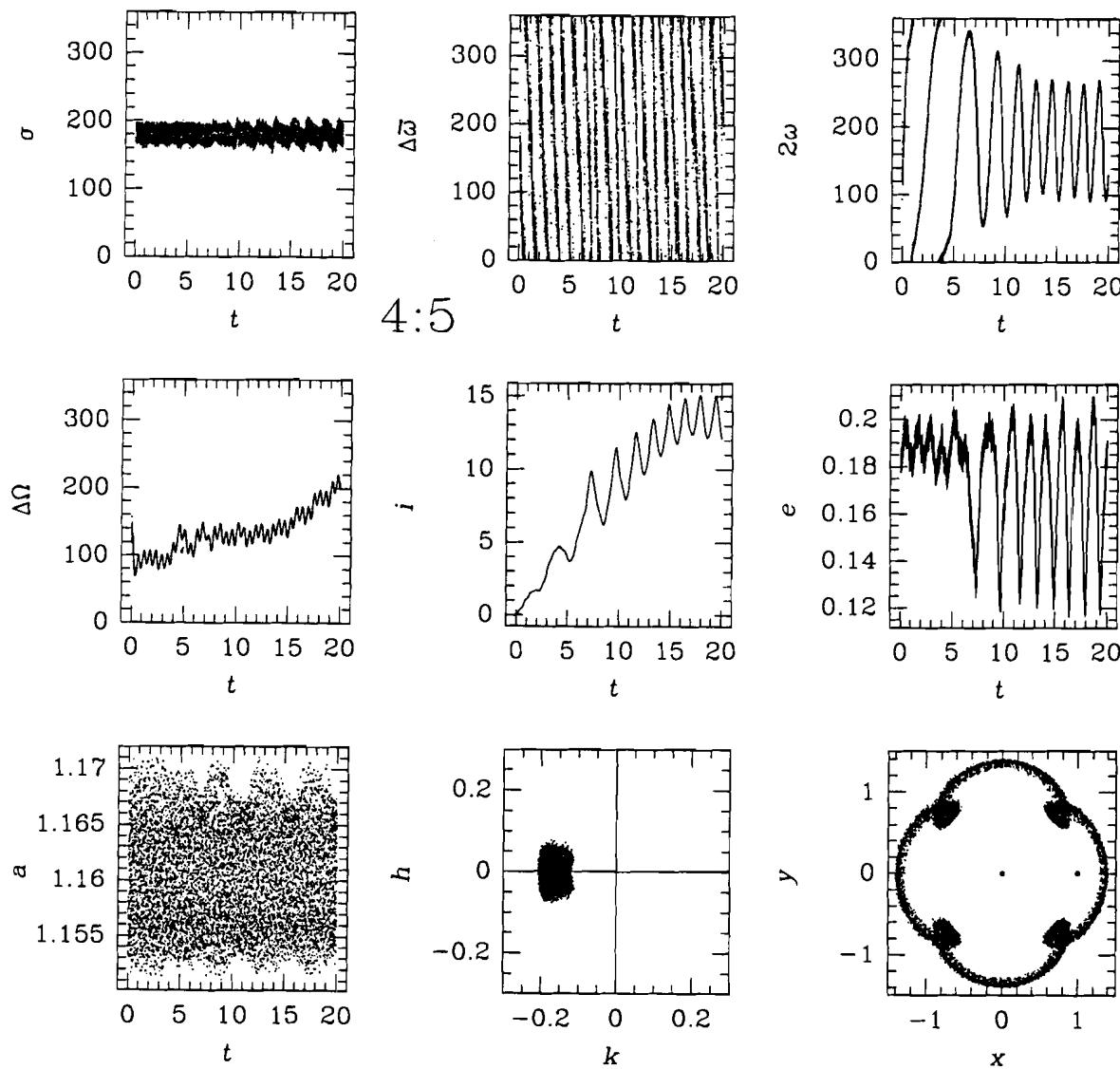


Figura 5.8: Ressonância 4:5. Partícula capturada na ressonância de Kozai. Os dois últimos gráficos mostram a evolução no plano (k, h) e as posições relativas ao sistema Sol–Netuno.

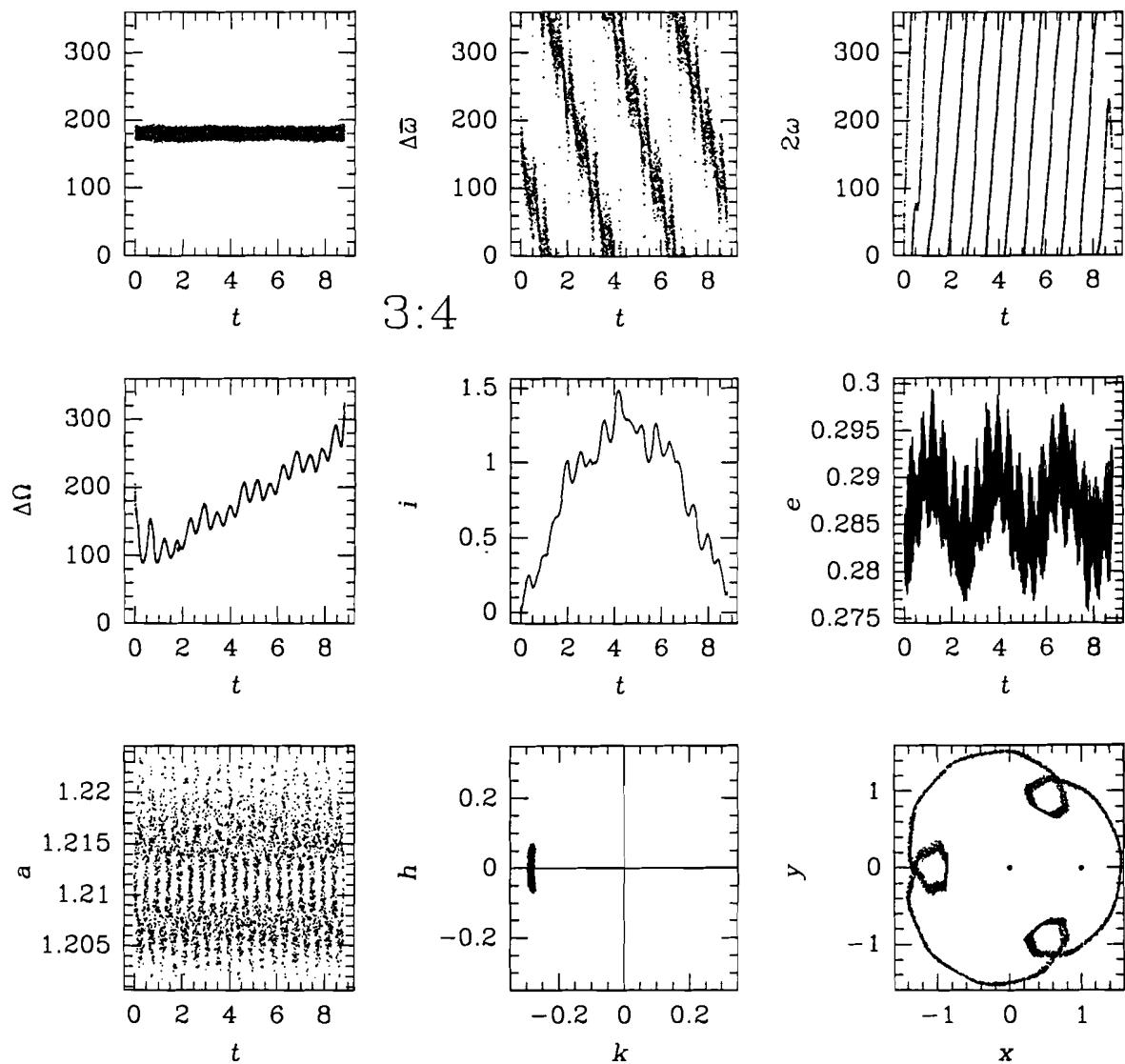


Figura 5.9: Ressonância 3:4. Partícula evoluindo no limite da região estável. Os dois últimos gráficos mostram a evolução no plano (k, h) e as posições relativas ao sistema Sol–Netuno.

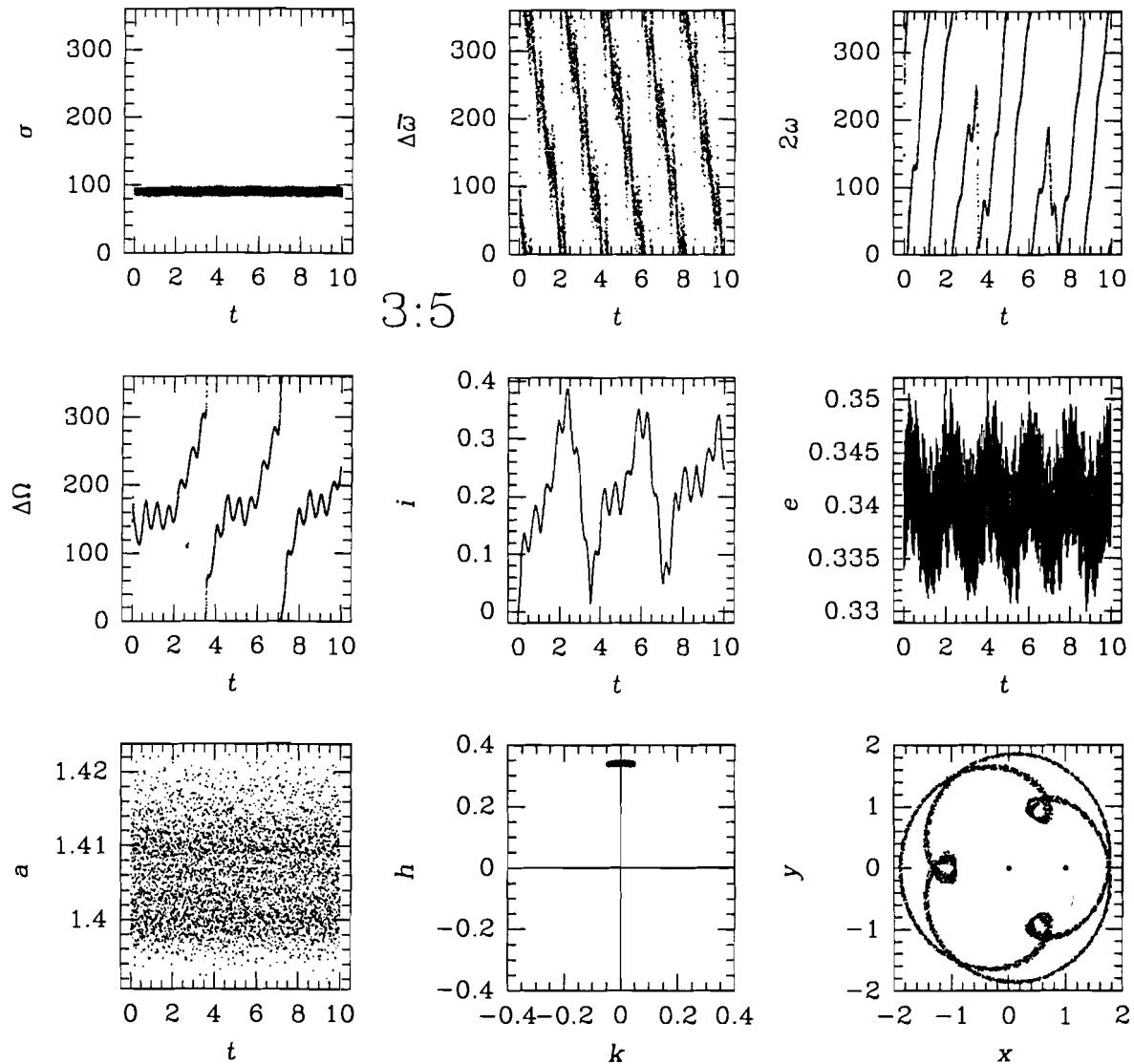


Figura 5.10: Ressonância 3:5. Partícula evoluindo no limite da região estável. Os dois últimos gráficos mostram a evolução no plano (k, h) e as posições relativas ao sistema Sol–Netuno. O estudo da evolução desta partícula por 100 milhões de anos encontra-se no Anexo II.

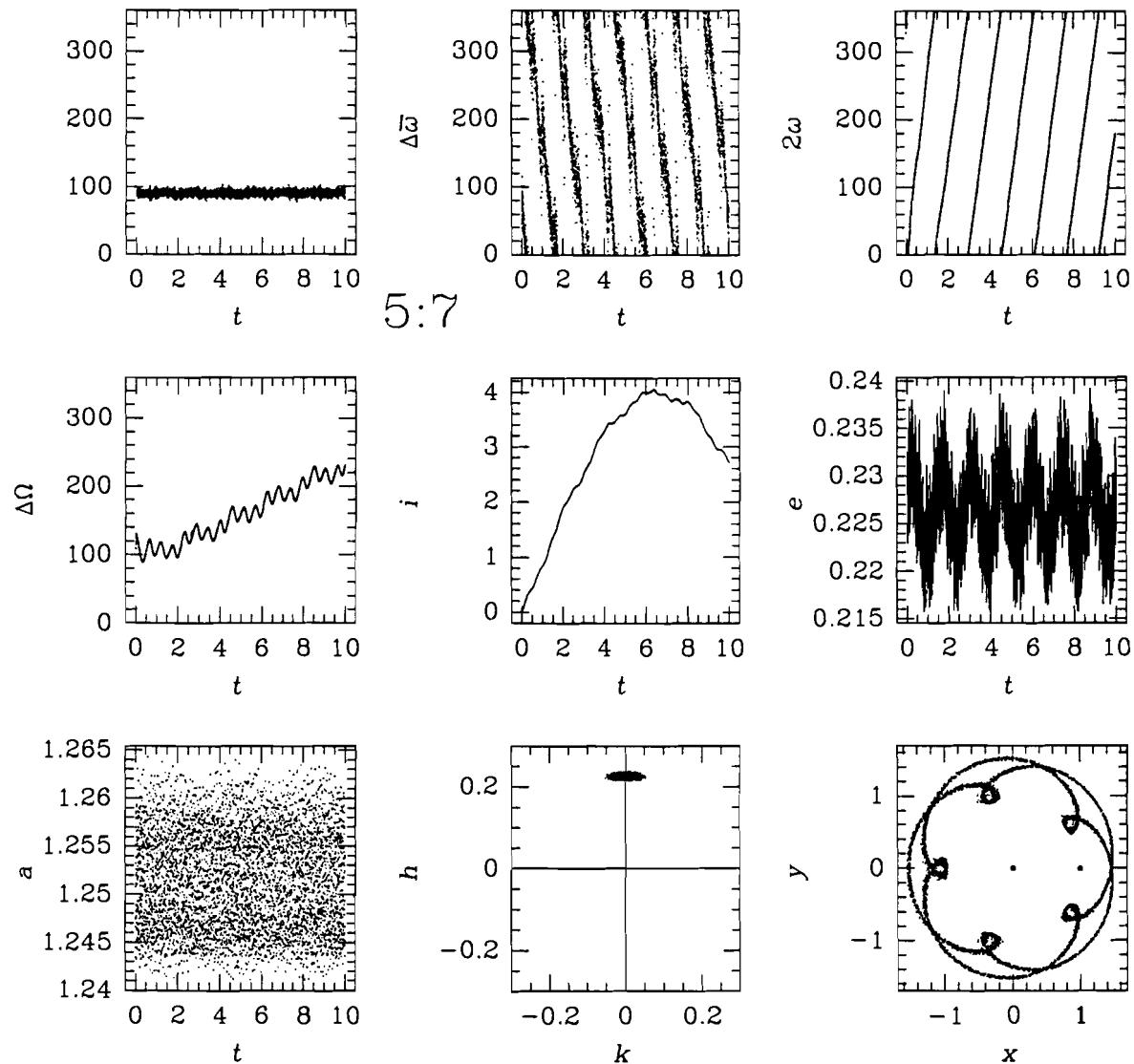


Figura 5.11: Ressonância 5:7. Partícula evoluindo no limite da região estável. Os dois últimos gráficos mostram a evolução no plano (k, h) e as posições relativas ao sistema Sol–Netuno.

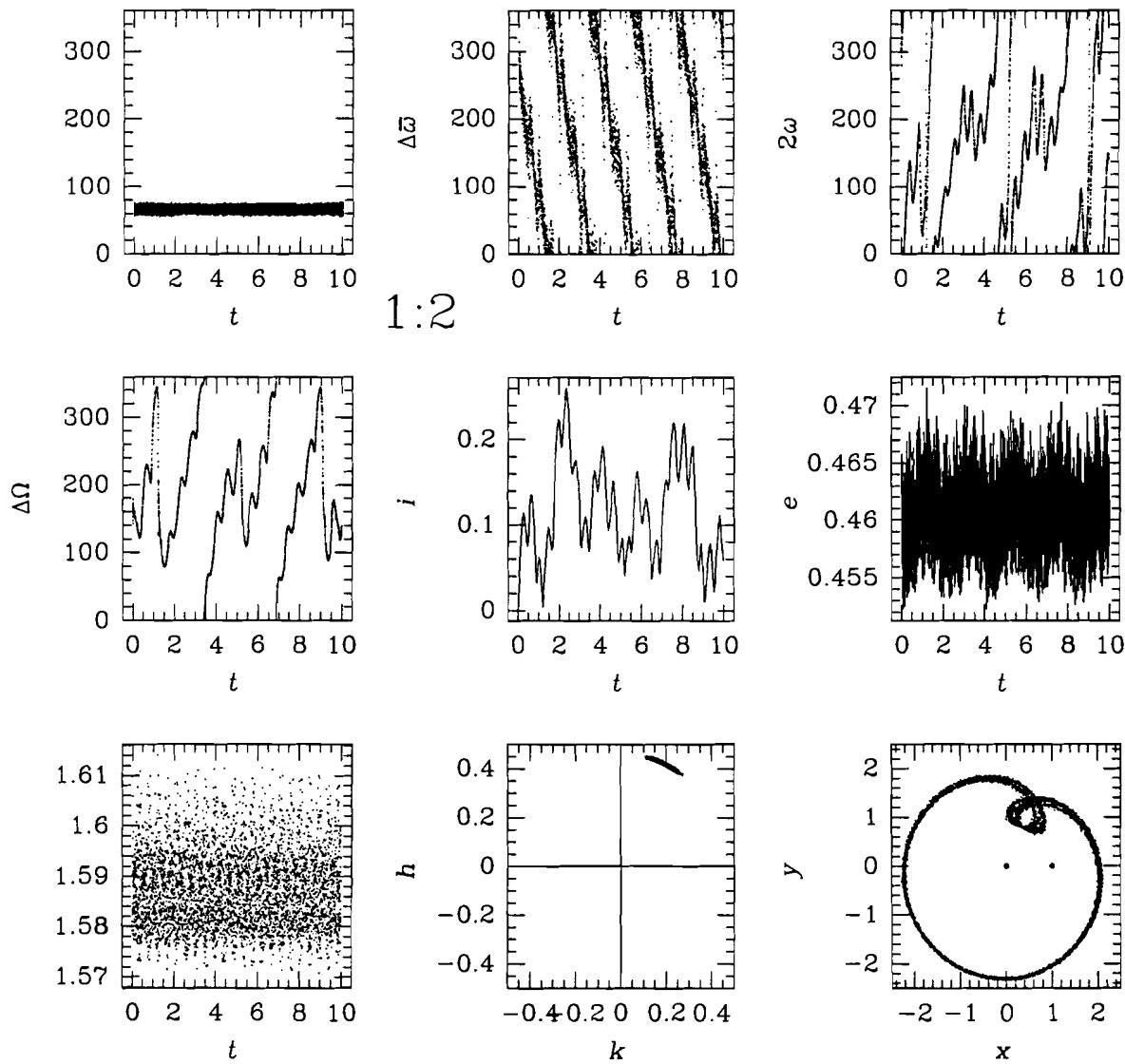


Figura 5.12: Ressonância 1:2. Partícula evoluindo no limite da região estável. Os dois últimos gráficos mostram a evolução no plano (k, h) e as posições relativas ao sistema Sol–Netuno.

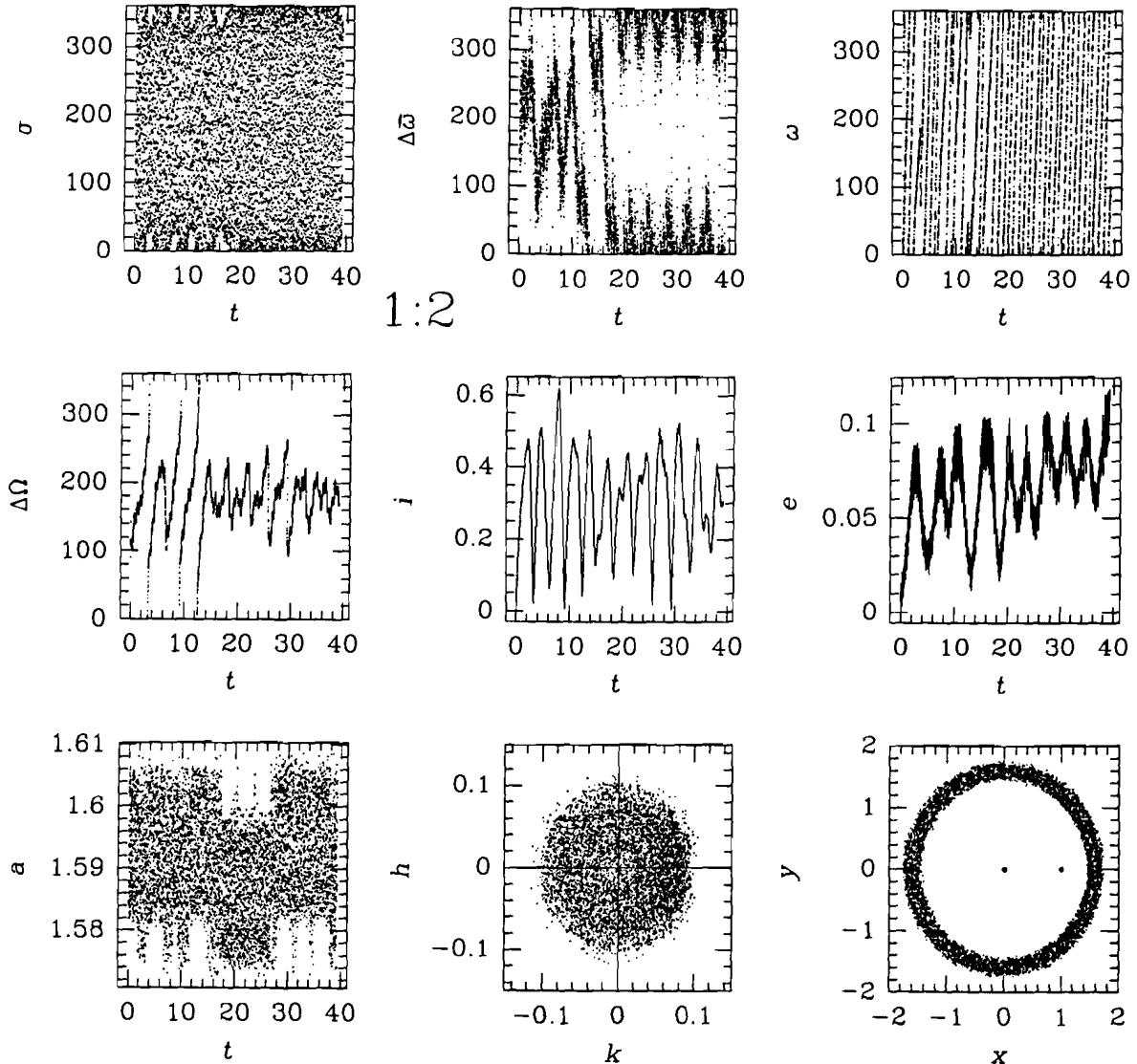


Figura 5.13: Ressonância 1:2. Evolução instável a partir do ramo negativo da lei de estrutura nas baixas excentricidades. Observem-se as ressonâncias seculares ν_8 e ν_{18} . Os dois últimos gráficos mostram a evolução no plano (k, h) e as posições relativas ao sistema Sol–Netuno.

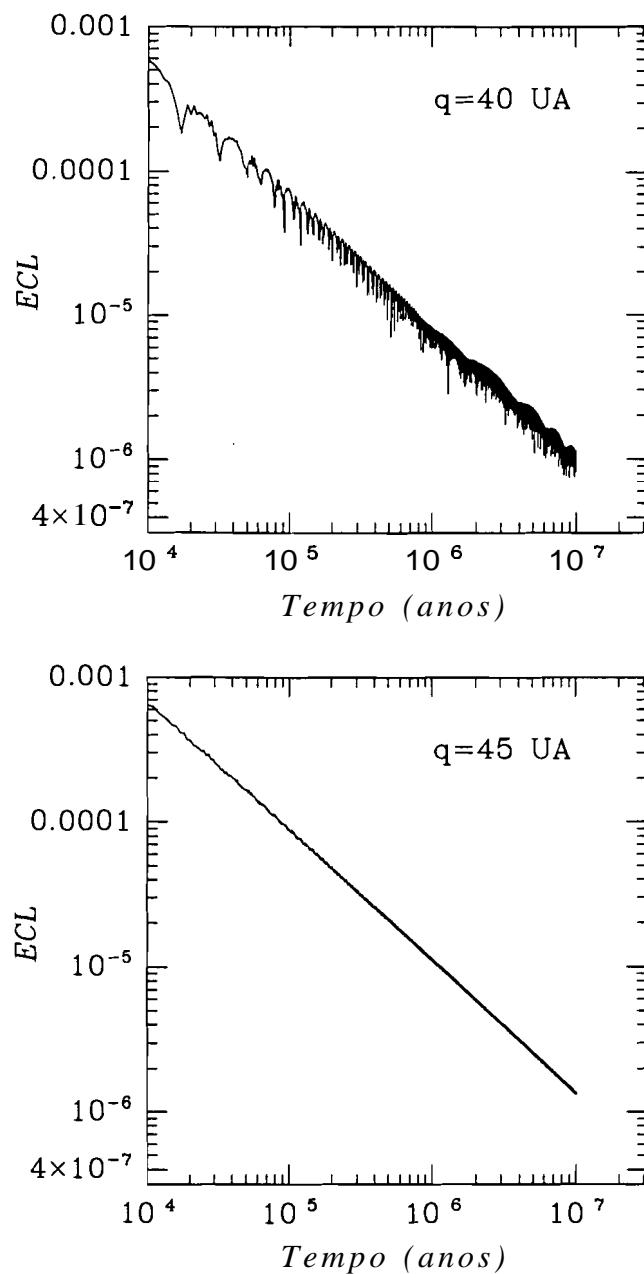


Figura 5.14: Evolução do expoente característico de Lyapunov (ECL), calculado integrando equações variacionais (Roig, 1996). As duas partículas tem $a = 55$ UA, uma com $q = 40$ UA e a outra com $q = 45$ UA. O espectro da primeira é caótico e o da segunda é regular. As duas partículas estão dentro de região definida como caótica por Torbett e Smoluchowski.

Conclusões.

Quando consideradas em conjunto, as características da população de cometas de curto período eventualmente capturados a partir da nuvem de Oort não se correspondem com as características da população conhecida, mesmo adotando os mais variados modelos de captura. Vários autores apontam para o cinturão de Kuiper como fonte mais provável.

Revisamos e ampliamos a teoria de librações de alta excentricidade, obtendo uma melhor representação do movimento ressonante plano. No desenvolvimento assimétrico da função perturbadora foi introduzido o efeito não ressonante de outros planetas perturbadores. Nesta aproximação os efeitos de Júpiter, Saturno e Urano nas ressonâncias exteriores a Netuno são pequenos, mas verificados numericamente. Os parâmetros que definem um movimento ressonante, dados pelo centro de libração, período de libração, segundo modo forçado, etc., ficam praticamente inalterados pelo efeito de outros perturbadores. Quando as perturbações dos outros planetas são suficientemente fortes, a libração, antes de ser significativamente alterada é destruída.

As integrações numéricas das equações exatas de movimento de partículas em movimento ressonante mostram o que já fora previsto pela teoria de librações do Anexo I: o movimento de libração não pode ser resumido a uma linha no espectro, mas a um pacote de linhas. Para uma melhor compreensão do movimento de libração desenvolvemos um método de análise temporal de freqüências, que se mostrou eficiente no seguimento de librações tanto nos casos caóticos como nos quase-regulares.

A ressonância 2:3 no modelo de três corpos plano mostrou-se regular. No caso do modelo de N corpos espacial, o espaço de fase correspondente a evoluções quase-regulares reduziu-se notoriamente, mas mesmo assim conservou uma região de movimento altamente estável. O limite máximo permitido na excentricidade para as librações estáveis é determinado pelas perturbações de Urano. As partículas possuindo excentricidades menores também podem ser removidas da ressonância desde que a amplitude de libração seja suficientemente grande. Nestes casos observa-se um crescimento da amplitude de libração até acontecer um encontro com Netuno. A captura temporária entorno dos centros de libração localizados em $a = 0^\circ$ após a curva de colisão, é um fenômeno comum mas difícil de obter numericamente se o integrador não for suficientemente preciso. Tanto nesta ressonância como na 1:2 existe no ramo negativo das librações de baixa excentricidade um mecanismo de instabilidade associado às ressonâncias seculares ν_8 e ν_{18} . O ramo positivo, no entanto, é estável. Para uma adequada compreensão do movimento nas baixas excentricidades

é necessário considerar o modelo espacial.

As ressonâncias 3:4, 5:7, 3:5 e 1:2 também apresentaram regiões estáveis até valores altos da excentricidade. Do ponto de vista dinâmico estas ressonâncias podem conter objetos evoluindo de forma estável. De fato, no cinturão de Kuiper, as órbitas excêntricas mais estáveis são as que estão em ressonância. Podemos dizer que nas ressonâncias de primeira ordem estudadas podem existir partículas até com $q \sim 25.5$ UA evoluindo de forma estável. Nas de segunda ordem o limite é $q \sim 28.5$ UA. Os objetos transnetunianos presos nas ressonâncias exteriores a Netuno são os que estariam em melhores condições de suportar o bombardeio de planetesimais ejetados da região de Urano e Netuno nas etapas iniciais de formação do Sistema Solar.

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Apêndice A

Efeito de um segundo perturbador nas librações.

A.1 As leis das librações.

As leis que caracterizam uma ressonância podem ser obtidas através de diferentes tratamentos analíticos. Elas foram e são usadas como ponto de partida para o estudo das ressonâncias interiores com Júpiter e, recentemente, no estudo das ressonâncias exteriores a Netuno (Morbidelli et al., 1995). As teorias geralmente são construídas no marco do problema restrito de três corpos e podemos perguntar-nos o que acontece com os efeitos não ressonantes dos outros planetas nestas leis. Acaso as perturbações de Marte são suficientemente importantes para produzir alterações no movimento de libração dos asteróides em ressonância com Júpiter? Pode Urano produzir mudanças no movimento de libração de partículas nas ressonâncias exteriores a Netuno?

Voltamos ao desenvolvimento da teoria de librações de pequena amplitude como é apresentada em Ferraz–Mello (1988) e no Anexo I. Consideremos o problema restrito plano elíptico de três corpos, encontrando-se a partícula na ressonância de movimentos médios ($p + q : p$ com o planeta P_1 , onde q é a ordem da ressonância e p é o grau, sendo positivo para as interiores e negativo para as exteriores. O sistema é definido pelo conjunto de variáveis canônicas $(a, \sigma_1, \sigma_2, J, J_1, J_2)$ onde as variáveis angulares são

$$\begin{aligned}\sigma &= (p/q + 1)\lambda_1 - p/q\lambda - \varpi \\ \sigma_1 &= (p/q + 1)\lambda_1 - p/q\lambda - \varpi_1 \\ \sigma_2 &= (\lambda - \lambda_1)/q = Q\end{aligned}\tag{A.1}$$

sendo λ e λ_1 as longitudes médias da partícula e do planeta e ϖ e ϖ_1 as longitudes respectivas dos periélios. Como em Ferraz–Mello et al. (1993), podemos tomar ϖ_1 como função linear do tempo, com $d\varpi_1/dt = g_1$. Para eliminar um grau de liberdade e os termos de curto período, fazemos uma média com respeito a Q (Ferraz–Mello,

1988). A função Hamiltoniana média fica

$$\mathcal{H} = -\frac{\mu}{2a} + \frac{(p+q)}{p} \sqrt{\mu a} (g_1 - n_1) - R, \quad (\text{A.2})$$

sendo μ o quadrado da constante de Gauss, a o semi-eixo da partícula e n_1 o movimento médio do planeta. (Note-se que definimos \mathcal{H} com o sinal trocado com respeito à definição do Anexo I). O primeiro termo de 31 é a parte Kepleriana, o segundo vem da extensão do espaço de fase e o ultimo, R , é a função perturbadora média

$$R = \frac{\mu m_1}{2\pi} \int_0^{2\pi} F dQ \quad (\text{A.3})$$

onde

$$F = \frac{1}{\Delta_1} - \frac{r \cos S_1}{r_1^2} \quad (\text{A.4})$$

sendo r e r_1 as distâncias heliocêntricas da partícula e do planeta, Δ_1 é a distância partícula-planeta, S_1 o ângulo partícula-Sol-planeta e m_1 a massa do planeta. Em Ferraz-Mello e Sato (1989), F é escrita como $F(k, h, a, \sigma_1, Q)$ e prova-se que é uma função periódica de período 2π em Q devido à comensurabilidade de movimentos médios. Esta periodicidade justifica a média feita na Eq.(A.3). São definidas as variáveis não singulares k e h como

$$\begin{aligned} k &= e \cos a \\ h &= e \sin a \end{aligned} \quad (\text{A.5})$$

e é usado o desenvolvimento assimétrico para R . Trata-se de um desenvolvimento em série de Taylor em torno de um ponto arbitrário (k_0, h_0, a_0) e em torno de $e_1 = 0$:

$$\begin{aligned} R = & A_0 + A_1(k - k_0) + A_2(h - h_0) \\ & + \frac{1}{2}[A_3(k - k_0)^2 + A_4(h - h_0)^2] + A_5(k - k_0)(h - h_0) \\ & + e_1 \cos \sigma_1 [A_6 + A_8(k - k_0) + A_{10}(h - h_0)] \\ & + e_1 \sin \sigma_1 [A_7 + A_9(k - k_0) + A_{11}(h - h_0)] \\ & + \frac{1}{2}e_1^2[A_{12} + A_{13} \cos 2\sigma_1 + A_{14} \sin 2\sigma_1] + \text{etc}, \end{aligned} \quad (\text{A.6})$$

(ver Ferraz-Mello et al., 1993). Para o caso $e_1 = 0$, é possível obter famílias de soluções estacionárias (os chamados centros de libração) no espaço (k, h, a) , e, para e_1 genérico, soluções analíticas aproximadas para as oscilações de pequena amplitude em torno dos centros (Ferraz-Mello, 1988 e Anexo I). Estas soluções seguem uma série de leis que dependem dos coeficientes A ; e de algumas de suas derivadas parciais $\frac{\partial A_i}{\partial a}$ e $\frac{\partial^2 A_i}{\partial a^2}$. As expressões analíticas para A ; podem ser achadas em Ferraz-Mello e Sato (1989) mas também podem ser calculadas numéricamente.

A.2 Inclusão de um segundo planeta perturbador.

Consideremos um segundo planeta perturbador, P_2 , em órbita circular de semi-eixo a_2 com movimento médio, n_2 , não comensurável com a partícula nem com o planeta principal. Somente consideraremos as perturbações de P_2 na partícula. Em nosso modelo aproximado não consideraremos as perturbações mútuas entre os planetas. As equações canônicas são as mesmas mas o Hamiltoniano tem um termo adicional igual a $-R'$ onde

$$R' = \mu m_2 \langle F' \rangle_Q \quad (\text{A.7})$$

sendo

$$F' = \frac{1}{\Delta_2} - \frac{r \cos S_2}{a_2^2}. \quad (\text{A.8})$$

Podemos fazer um desenvolvimento de R' do mesmo tipo que R , mas agora os coeficientes A_i com $i > 5$ se anulam pois as perturbações de P_2 são independentes de e_1 (e o novo planeta perturbador é suposto em órbita circular). Então temos a expressão

$$\begin{aligned} R' = & A_0 + A_1(k - k_0) + A_2(h - h_0) \\ & + \frac{1}{2}[A_3(k - k_0)^2 + A_4(h - h_0)^2] + A_5(k - k_0)(h - h_0). \end{aligned} \quad (\text{A.9})$$

A função F' pode ser escrita como $F'(k, h, a, Q)$, mas não será mais periódica em Q . Para calcular $\langle F' \rangle_Q$, teremos que usar a definição do valor médio para as funções quase-periódicas (Bohr, 1933):

$$\langle F' \rangle_Q = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F'(Q) dQ. \quad (\text{A.10})$$

Permita-se-nos agora introduzir a variável $Q' = \lambda_1 - \lambda_2$ que está relacionada com S_2 através de

$$S_2 = 4 - \lambda_2 = (4 - \lambda) + (qQ + Q') \quad (\text{A.11})$$

ou

$$S_2 = (v - \sigma) - (pQ - Q') \quad (\text{A.12})$$

onde 4 e v são a longitude e anomalia verdadeiras da partícula. Q' está relacionada com Q através de:

$$Q' = \lambda_1 - \lambda_2 = (1 - \frac{n_2}{n_1})(pQ + \sigma_1) - n_2 T_1 + \varpi_1 - \lambda_{20}. \quad (\text{A.13})$$

onde T_1 é o instante da passagem pelo periélio de P_1 e λ_{20} é a longitude inicial (em $t = 0$) de P_2 . Daqui em diante podemos seguir os mesmos passos que Ferraz-Mello e Sato (1989) para o cálculo dos coeficientes de R' substituindo S por S_2 e trabalhando com $F'(k, h, a, Q, Q')$. Pode-se provar que as fórmulas são periódicas

em Q e Q' e as mesmas regras de paridade continuam sendo válidas. Então, se $A_i = 0$ no desenvolvimento assimétrico para o planeta principal, o mesmo vale para o desenvolvimento assimétrico de P_2 .

Se não existe interseção entre a órbita da partícula e do planeta P_2 , então $F'(k, h, a, Q, Q')$ não diverge e podemos provar que

$$\langle F' \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T F'(Q, Q'(Q)) dQ = \frac{1}{(2\pi)^2} \int_0^{2\pi} dQ' \int_0^{2\pi} F'(Q', Q) dQ. \quad (\text{A.14})$$

Para provar isto primeiro escrevemos F' como uma série dupla de Fourier:

$$F' = \sum_{N,M}^{\infty} C_{NM} e^{i(NQ+MQ')} \quad (\text{A.15})$$

com N e M inteiros. Se introduzirmos esta expressão no lado direito da Eq.(A.14), se reduz a C_{00} . Por outro lado, da Eq.(A.13) temos $Q' = Qp(1 - n_2/n_1) + \alpha$ então

$$F' = \sum_{N,M}^{\infty} C_{NM} e^{iQ\beta} e^{iM\alpha} \quad (\text{A.16})$$

onde $\beta = N + Mp(1 - n_2/n_1)$. Introduzindo esta expressão no lado esquerdo da Eq.(A.14) e tendo em conta que β nunca se anula exceto para $N = M = 0$, obtemos novamente C_{00} . Provada a igualdade, em lugar de calcular a integral infinita da Eq.(A.10) podemos calcular a função perturbadora média através da integral dupla

$$R' = \frac{\mu m_2}{(2\pi)^2} \int_0^{2\pi} dQ' \int_0^{2\pi} F'(Q', Q) dQ \quad (\text{A.17})$$

Se a órbita da partícula cruza a órbita de P_2 , existirá um ponto onde $F' = \infty$, o teorema do valor médio para funções quase-periódicas não poderá ser aplicado e também não poderá ser feita a expansão em série dupla de Fourier. Em consequência, consideraremos perturbadores que não cruzam a órbita da partícula. Os coeficientes que aparecem nas leis são: $A_1, A_2, A_3, A_4, A_5, \frac{\partial A_0}{\partial a}, \frac{\partial A_1}{\partial a}, \frac{\partial A_2}{\partial a}$ e $\frac{\partial^2 A_0}{\partial a^2}$. Trata-se de integrais duplas de derivadas parciais. Calculamos a integral dupla com a fórmula de Simpson e as derivadas parciais também as determinamos numericamente calculando R' em 23 pontos do espaço (k, h, a) separados por $\Delta k = \Delta h = \Delta a = 10^{-4}$ (Δa em UA) e usando as fórmulas dadas em Abramowitz e Stegun (1972). No problema de três corpos, a lei de estrutura é obtida por aproximações sucessivas aos valores (k_0, h_0, a_0) . Em nosso problema, seguimos este processo sendo que em cada passo da iteração recalculamos os A ; para o planeta principal e para P_2 . Uma vez que o centro de libração (k_0, h_0, a_0) é obtido, todos os A ; e suas derivadas são calculadas no ponto (k_0, h_0, a_0) para obter as outras leis. É fácil ver que é possível levar em conta o efeito de vários planetas perturbadores simultaneamente. A seguir apresentamos algumas aplicações.

A.3 Ressonância 3:1 corn Júpiter.

Nas ressonâncias interiores de segunda, os centros de libração estáveis antes da curva de colisão estão no eixo h , e ali verifica-se $A_1 = A_5 = 0$. Calculamos os coeficientes A_i e suas derivadas para a Terra, Marte e Saturno e comparamos com os coeficientes devidos a Júpiter (Fig. A.1). Somente aos efeitos da comparação, os coeficientes são calculados em $(0, h_0, a_0)$ onde a_0 é o semi-eixo correspondente a ressonância exata (deduzido de $3n_1 - n = 0$). Os valores dos coeficientes correspondentes a Terra, Marte e Saturno são desprezíveis quando comparados com os coeficientes devidos a Júpiter (pontos pretos). O efeito de Marte e a Terra é importante somente quando existe um encontro próximo com a partícula, mas neste caso a nossa aproximação não representa adequadamente o fenômeno, pois acontece uma perturbação instantânea grande demais para poder ser modelada por qualquer procedimento de média. Os perturbadores contribuem notoriamente a A_0 , mas este coeficiente não aparece nas leis. Para calcular as leis, supomos a Júpiter em uma órbita elíptica com periélio circulando uniformemente com velocidade $g_1 = 5.6308 \times 10^{-8}$ rads/dia. O efeito nas leis de libração (Fig. A.2) é desprezível. As linhas contínuas na figura correspondem ao problema de três corpos e as tracejadas ao caso 'perturbado'. Não podemos esperar variações notórias no movimento de libração, exceto se acontecer um encontro próximo com os planetas.

A.4 Ressonância 2:3 com Netuno.

Os centros de libração estáveis estão localizados no eixo k , e ali verifica-se $A_2 = A_5 = 0$. Calculamos os coeficientes A_i e suas derivadas para Urano, Saturno e Júpiter, e comparamos com os respectivos valores de Netuno (Fig. A.3). Existem importantes alterações em $\frac{\partial A_0}{\partial a}$ e $\frac{\partial^2 A_0}{\partial a^2}$ com respeito aos valores correspondentes a Netuno, mas o efeito nas leis (Fig. A.4) somente é apreciável na lei de estrutura (Fig. A.4a), em Z_k (Fig. A.4e) e no período de circulação de σ_1 (Fig. A.4h). Para o cálculo das leis adotamos para Netuno uma órbita elíptica com periélio circulando uniformemente com velocidade $g_1 = 8.9288 \times 10^{-9}$ rads/dia. Estes pequenos efeitos nas leis foram confirmados pelas integrações numéricas do Capítulo 4.

A.5 Conclusões.

Um deslocamento na lei de estrutura é o efeito mais evidente nos casos estudados (Figs A.2a e A.4a). Os movimentos não são afetados no período de libração (Figs. A.2b e A.4b) nem na forma da figura de libração (Figs. A.2c-d e A.4c-d). O período de circulação de σ_1 é levemente modificado (Figs. A.2h e A.4h) e o mesmo podemos dizer para as amplitudes das componentes do segundo modo forçado (Figs. A.2e-g e A.4e-g). Essas modificações são fortes somente quando acontece um encontro próximo com algum dos planetas perturbadores, mas nesta circunstância o método de média não é mais válido. Por exemplo, nas librações em torno dos eixos k ou

h , o coeficiente A_5 é zero no problema médio. Este coeficiente aparece na fórmula do período de libração que por sua vez está relacionado com a estabilidade da libração (Ferraz–Mello et al., 1993). No caso de acontecer um encontro próximo, se calcularmos A_5 usando a Eq. (A.10) em lugar da Eq.(A.17), usando um intervalo longo, mas finito, obteríamos $A_5 \neq 0$, modificando o período de libração e eventualmente fazendo o movimento instável. Em consequência, o método de média fracassa em predizer librações estáveis até excentricidades muito grandes, onde as perturbações de curto período, eliminadas pelo procedimento de média, são suficientemente grandes para destruir as librações. Mas se o movimento for estável, as modificações previstas pelo método podem ser confirmadas, como de fato foi feito no caso da ressonância exterior 2:3 com Netuno nesta tese.

É preciso notar a pequena magnitude dos efeitos que obtivemos nos casos analisados. Podemos interpretar isto da seguinte forma: se a partícula está na ressonância, os efeitos dos outros perturbadores no movimento de libração são desprezíveis exceto quando as perturbações são suficientemente grandes para destruir as librações. E, neste caso, não haverá librações.

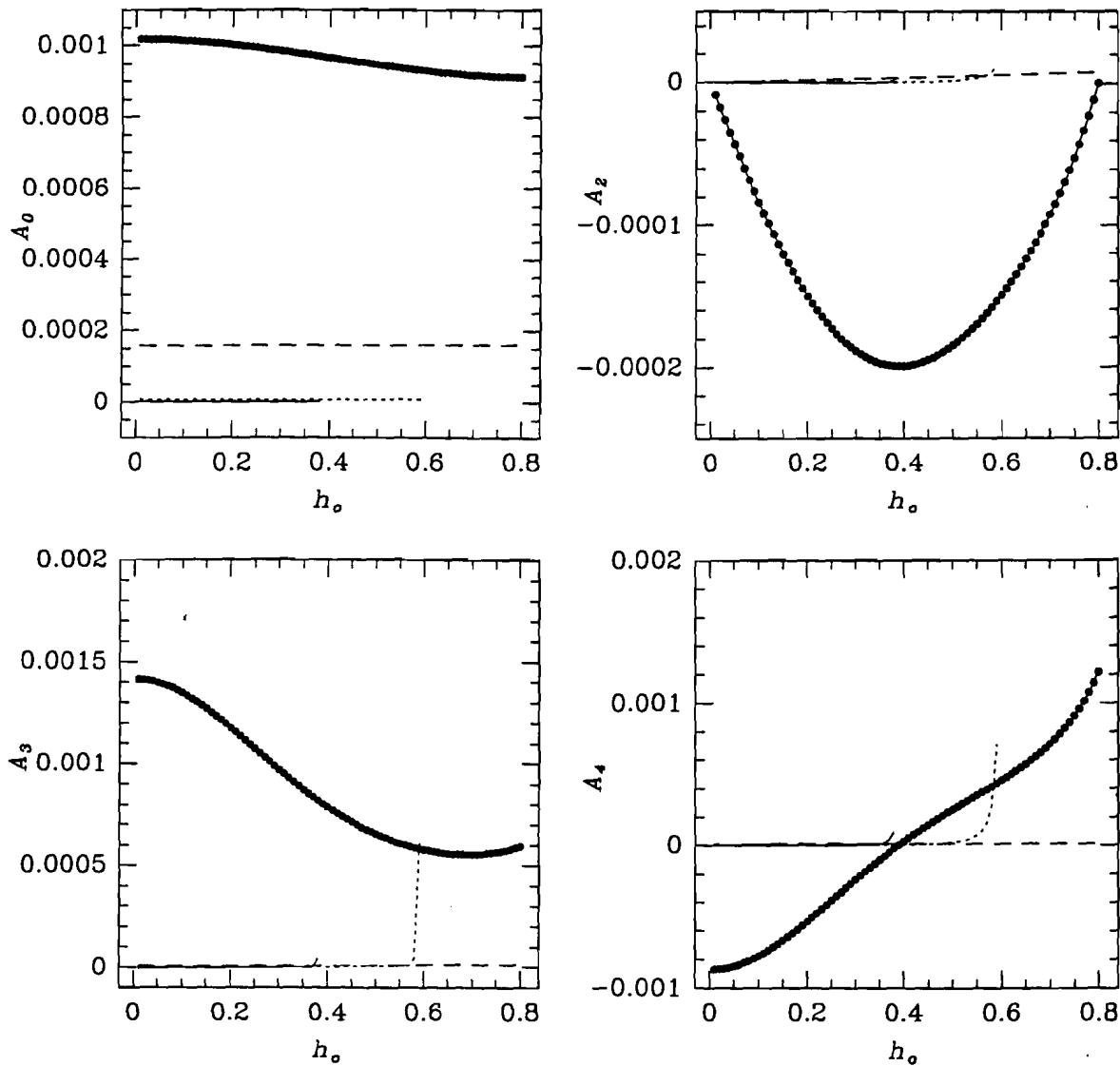


Figura A.1: Ressonância 3:1 com Júpiter. Coeficientes do desenvolvimento assimétrico da função perturbadora devidos a Júpiter (pontos), Marte (linha contínua), Terra (linha a traços pequenos) e Saturno (linha a traços longos) calculados em $(0, h_0, a_0)$, sendo a_0 o valor para a ressonância exata, e variando h_0 desde zero até a intersecção das órbitas da partícula e perturbador. Os coeficientes A_1 e A_5 são zero. Sub-índice a significa $\partial/\partial a$.

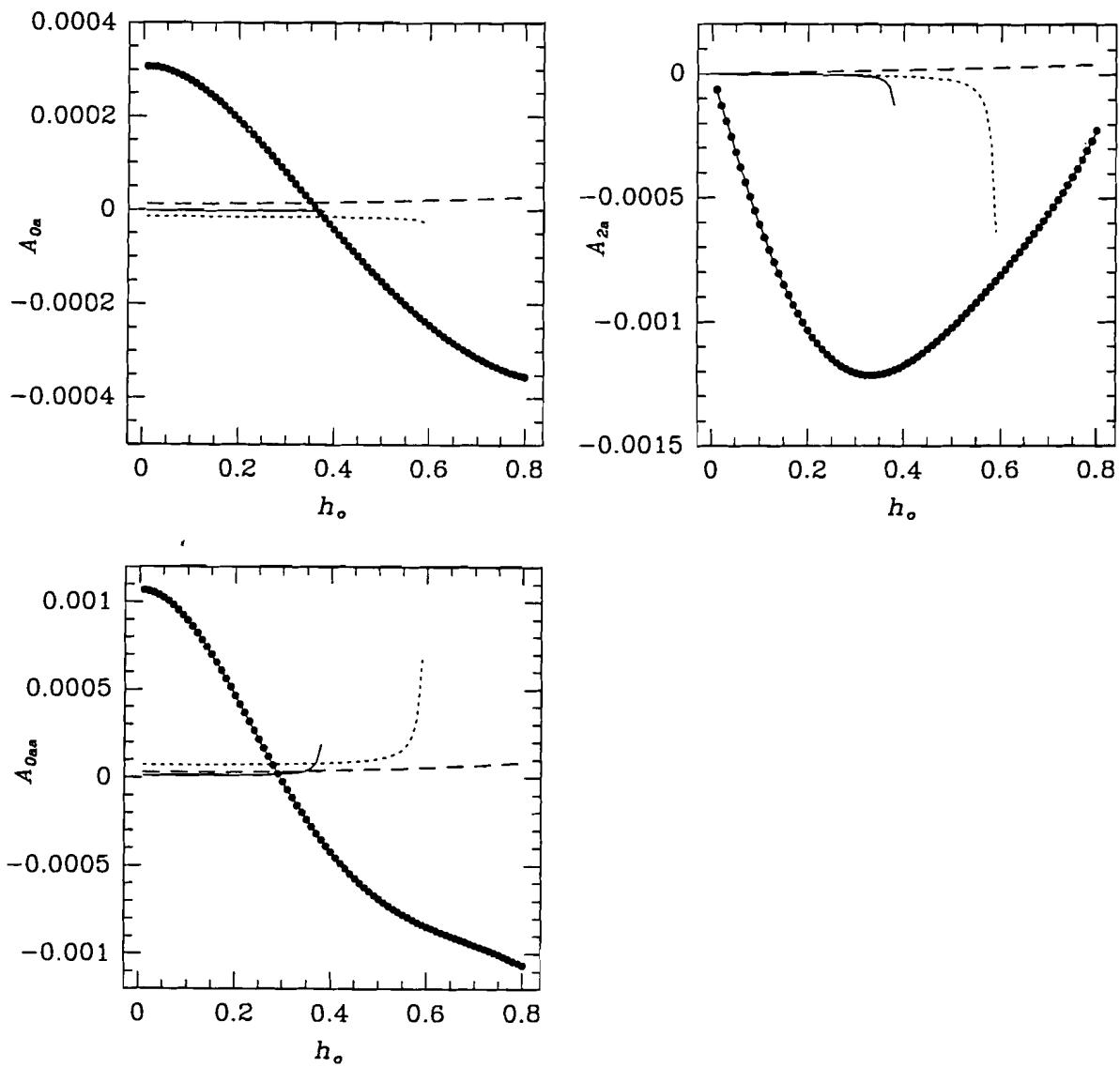


Figura A.1: Continuação.

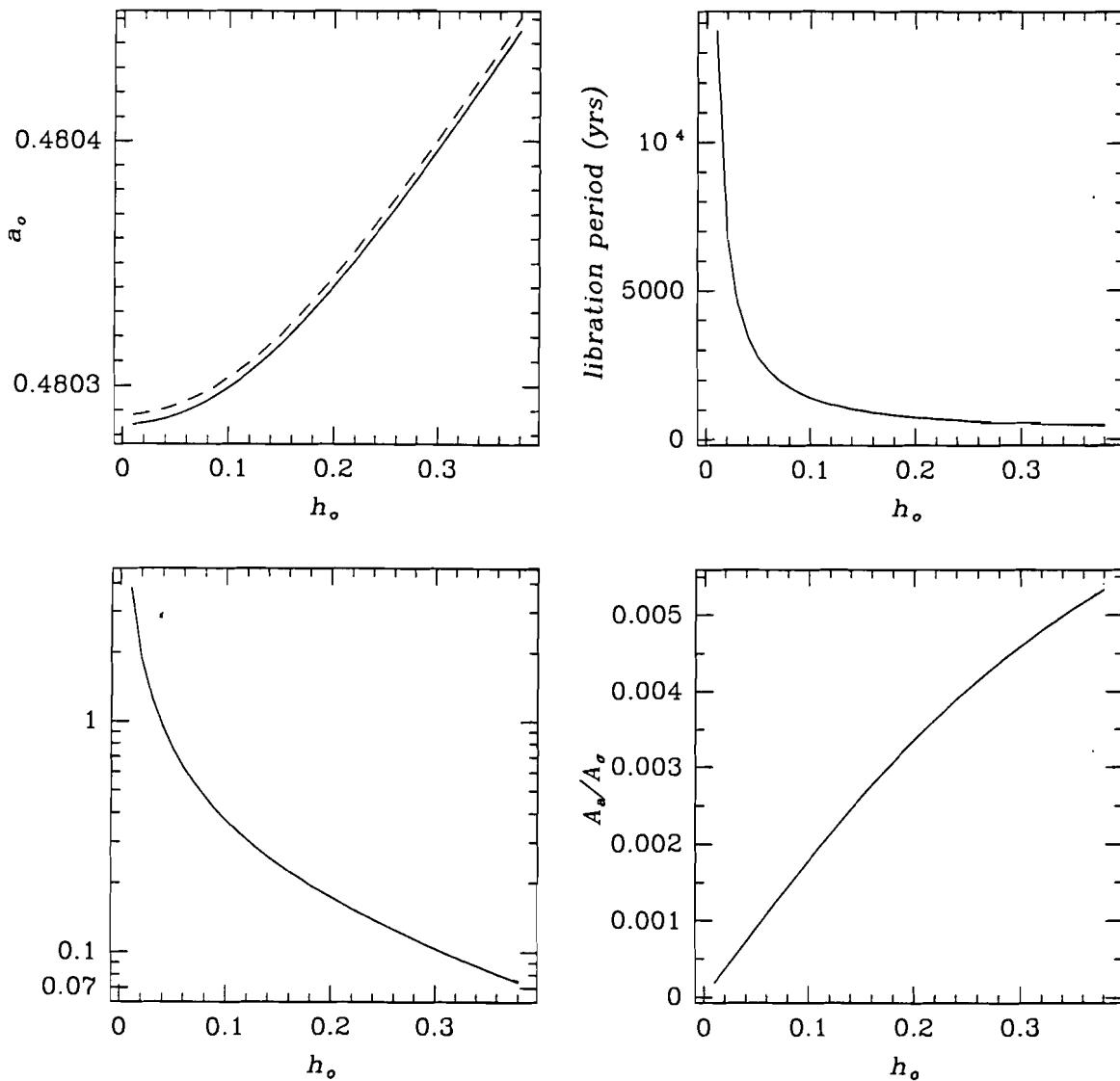


Figura A.2: Ressonância 3:1 com Júpiter. As leis são calculadas para valores de h_0 até a colisão com Marte. Linha tracejada: incluindo Júpiter, Marte, Terra e Saturno. Linha continua: somente Júpiter. Não há modificações no período de libração (b) nem na geometria das trajetórias de libração (c-d). Existe um pequeno deslocamento na lei de estrutura (a), no segundo modo forçado (e-g) e no período de circulação de σ_1 (h).

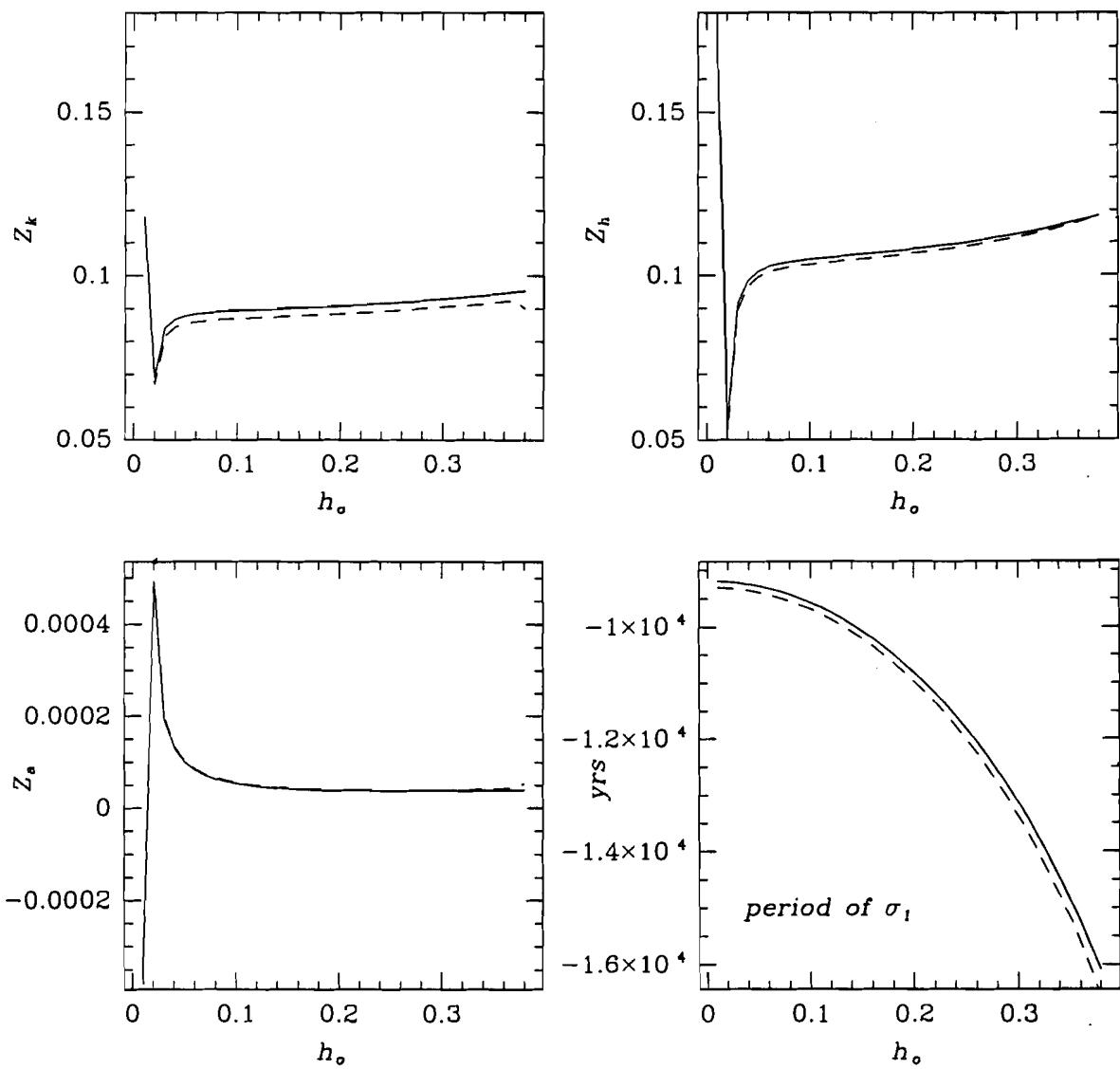


Figura A.2: Continuação.

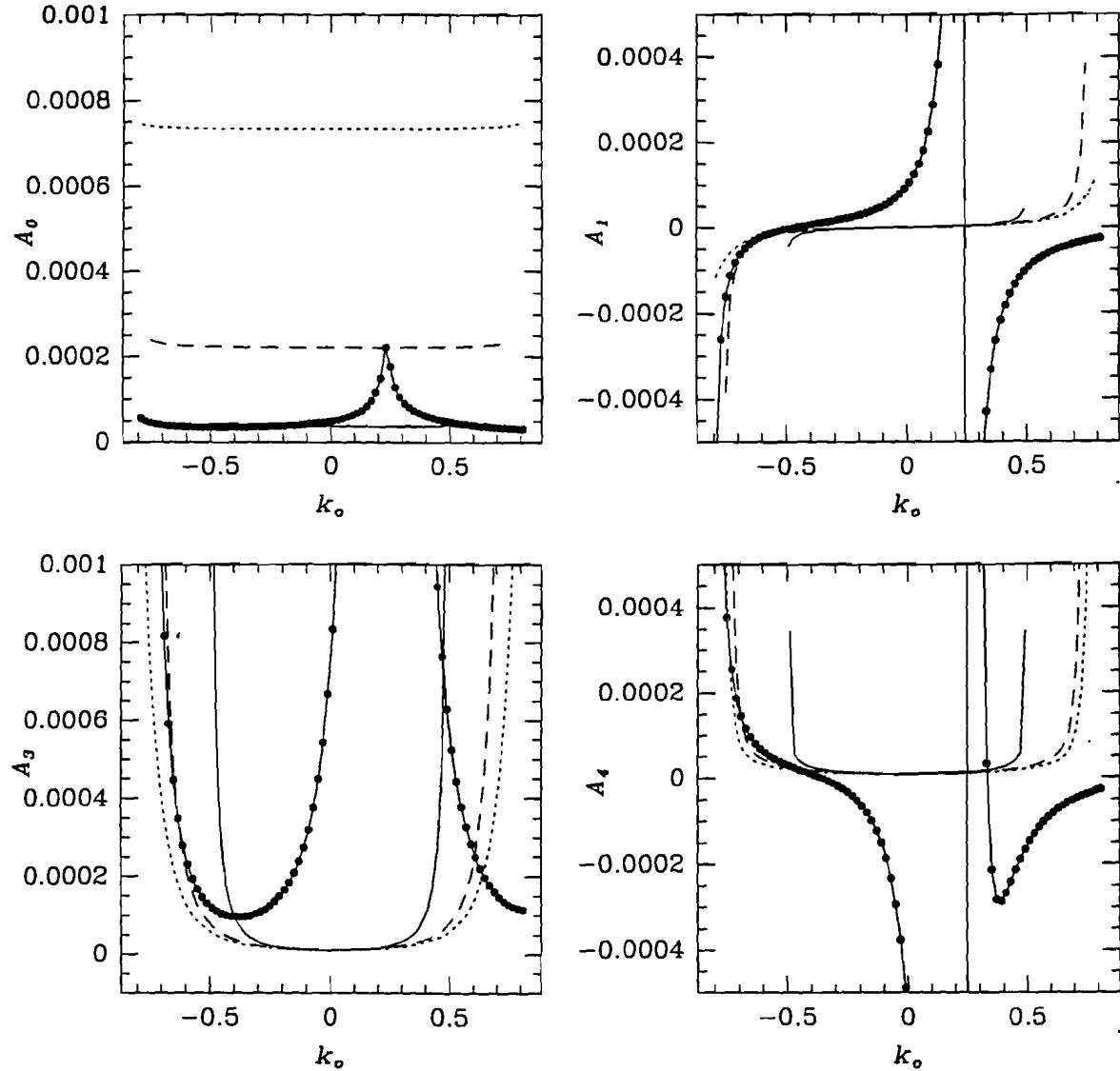


Figura A.3: Ressonância 2:3 com Netuno. Coeficientes do desenvolvimento assimétrico da função perturbadora devidos a Netuno (pontos), Urano (linha continua), Júpiter (linha a traços pequenos) e Saturno (linha a traços longos) calculados em $(k_0, 0, a_0)$, sendo a_0 o valor para a ressonância exata, e variando k_0 desde zero até a intersecção das órbitas da partícula e perturbador. Os coeficientes A_2 e A_5 são zero. Sub-índice a significa $\partial/\partial a$.

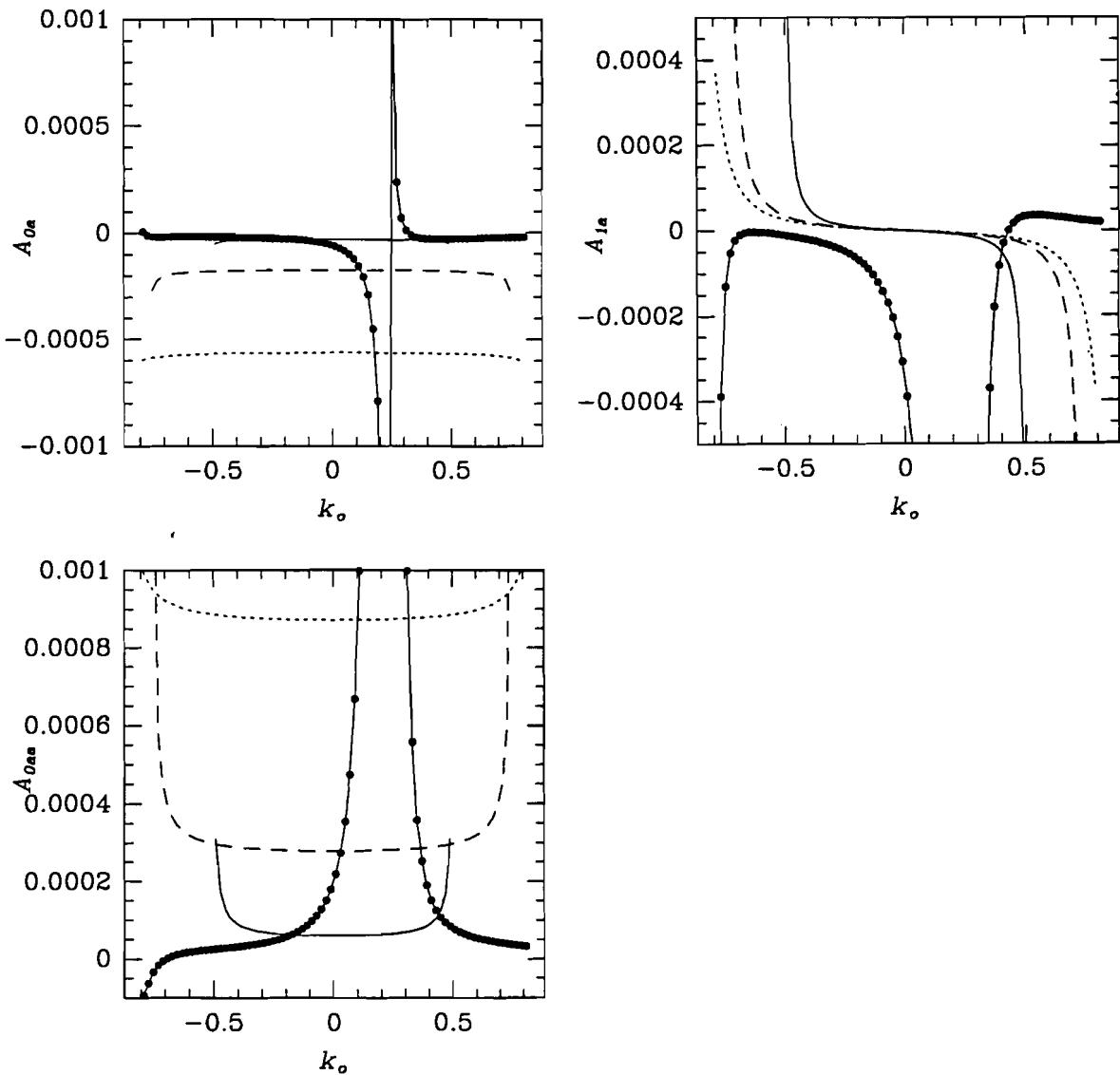


Figura A.3: Continuação.

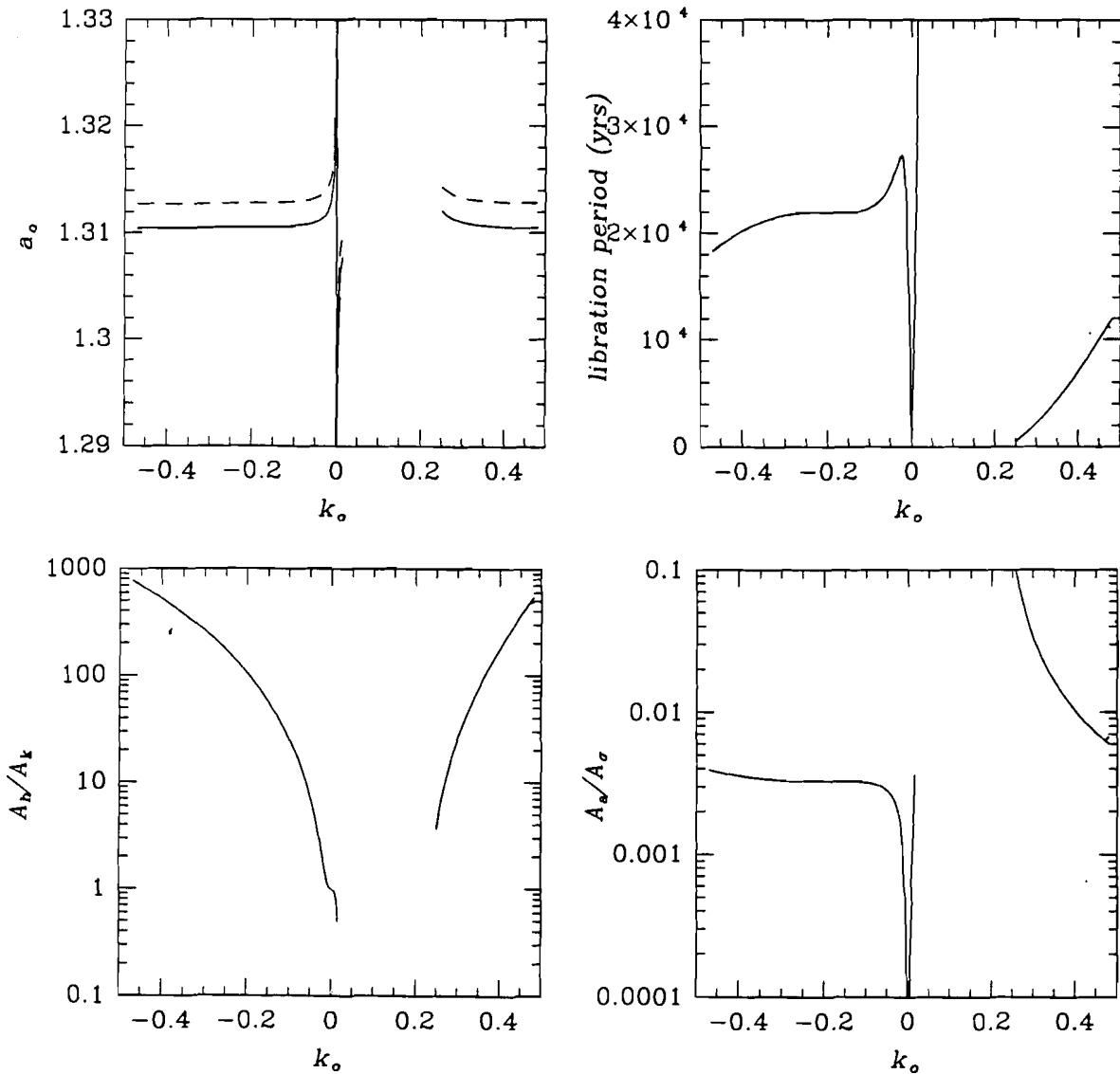


Figura A.4: Ressonância 2:3 com Netuno. As leis são calculadas para valores de k_0 até a colisão com Urano. Linha tracejada: incluindo Júpiter, Saturno, Urano e Netuno. Linha continua: somente Netuno. Não há modificações no período de libração (b) nem na geometria das trajetórias de libração (c-d). Existe um notório deslocamento na lei de estrutura (a), uma alteração no segundo modo forçado (e-g) e no período de circulação de σ_1 (h).

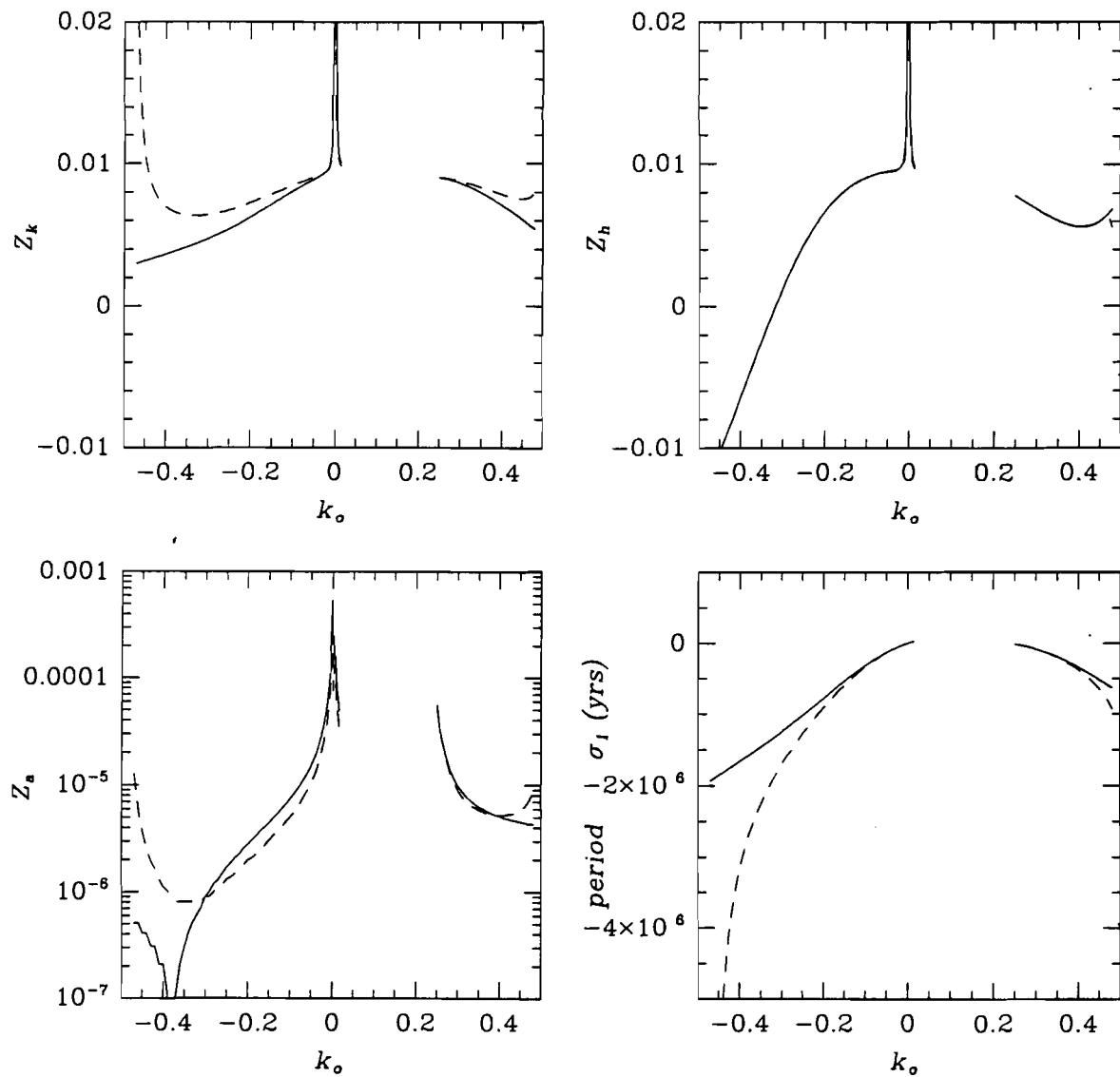


Figura A.4: Continuação.

Anexo I

The High-Eccentricity Libration Theory revisited

THE HIGH-ECCENTRICITY LIBRATION THEORY REVISTED

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Abstract. Using the local asymmetric expansion of the disturbing function for the planar elliptic restricted three-body problem up to degree 1 in e_1 we develop a small amplitude libration theory. We review the laws that characterize the mean-motion resonances of asteroids with Jupiter and we obtain other new laws. Special attention is paid to the second forced mode whose equations are reformulated and new consequences of this component are discussed. An analytical expression for the trajectories in the phase space is obtained. The predictions are compared with numerical experiments which confirm the results.

Key words: Asteroids, librations, resonances.

1. Introduction

Analytical studies done in the past have given a fair description of the principal features of the motion of an asteroid while in a mean-motion resonance with Jupiter. These approximated studies show that the solutions expressed in the complex variable

$$k + ih = e \exp(ia) \quad (1)$$

where e is the eccentricity and a the critical angle (see Equation (2)), for example, have one fixed and two circular components with frequencies equal to the libration frequency (ν) and to the circulation frequency of the perihelion (ν_1). The fixed component gives the libration center which is a stationary solution for the circular case ($e_1 = 0$). The first circular component is a free oscillation around the libration center and the second is a forced mode due to the disturbing function terms depending on e_1 . The classical methods were successfully applied to asteroids with small eccentricities but failed at higher eccentricities since they were based on expansions of the disturbing function around $e = 0$ which have known convergence problems (Ferraz-Mello, 1994). An improvement was possible by using local expansions around arbitrary values of the asteroid eccentricity (Ferraz-Mello, 1987a, 1988; Ferraz-Mello and Sato, 1989; Morbidelli and Giorgilli 1990a). Here, we obtain the principal components of the time variation of the elements (k, h, a) for the planar elliptic restricted three-body problem, by perturbing the solutions for the circular case and using the formalism of Ferraz-Mello, Tsuchida and Klaafke (1993) (hereafter FTK). The method does not use any expansions in the powers of the asteroid eccentricity but uses an expansion around the libration center (k_0, h_0).

After writing the system of canonical equations we neglect second order terms in $\delta k = k - k_0$, $\delta h = h - h_0$ and e_1 . The results were applied to the resonances 3:1, 2:1 and 3:2. Nevertheless, the formulae are valid for any librations around centers in the k - or h -axis.

2. Mean Motion Resonances

Let us consider the planar ($i = 0$) elliptic ($e_1 \neq 0$ restricted ($m = 0$)) three-body problem, being the asteroid in the mean-motion resonance $(p+q) : p$ with Jupiter. We can work out the problem with the canonical variables

$$\begin{aligned}\sigma &= (r + 1)\lambda_1 - r\lambda - \varpi & J &= L - G \\ \sigma_1 &= (r + 1)\lambda_1 - r\lambda - \varpi_1 & J_1 &= G + \frac{\Lambda}{n_1} \\ \sigma_2 &= (\lambda - \lambda_1)/q & J_2 &= (p + q)L + p\frac{\Lambda}{n_1}\end{aligned}\tag{2}$$

where $r = p/q$, λ and λ_1 are the mean longitudes of the asteroid and Jupiter, ϖ and ϖ_1 are the longitudes of the perihelion of the asteroid and Jupiter, L , G and Λ are the Delaunay elements of the asteroidal orbit, A is the variable canonically conjugated to t and n_1 is the mean motion of Jupiter. We average the disturbing function over σ_2 in order to eliminate the high-frequency oscillations and the Hamiltonian function becomes:

$$\mathcal{H} = \frac{\mu^2}{2L^2} - \Lambda + R\tag{3}$$

where R is the averaged disturbing function expanded about an arbitrary point (k_0, h_0) :

$$\begin{aligned}R &= A_0 + A_1(k - k_0) + A_2(h - h_0) \\ &\quad + \frac{1}{2}[A_3(k - k_0)^2 + A_4(h - h_0)^2] + A_5(k - k_0)(h - h_0) \\ &\quad + e_1 \cos \sigma_1[A_6 + A_8(k - k_0) + A_{10}(h - h_0)] \\ &\quad + e_1 \sin \sigma_1[A_7 + A_9(k - k_0) + A_{11}(h - h_0)] \\ &\quad + \frac{1}{2}e_1^2[A_{12} + A_{13} \cos 2\sigma_1 + A_{14} \sin 2\sigma_1] + \text{etc},\end{aligned}\tag{4}$$

(see Ferraz-Mello, 1988 and FTK). We may note that not all seconddegree terms were conserved in the actual solution of the equations. For instance, the terms in

e_1^2 are not used. The Hamiltonian function is $\mathcal{H}(J, J_1, J_2, u, \sigma_1)$ with $J_2 = \text{cte.}$, so we have a system with two degrees of freedom. After some approximations, it is possible to find another integral (Sessin 1981; Ferraz-Mello, 1987b), which makes this system integrable, and a straightforward calculation solves the problem. It is clear that this approach cannot give rise to chaotic motion but it can help to predict and explain the origin of chaotic behavior (Henrard, 1988; Henrard and Lemaître, 1986; Lemaître and Henrard, 1988, 1990).

3. Libration Laws

For $e_1 = 0$ there exist stationary solutions (libration centers) of the resonant averaged restricted three-body problem. The main parameters of the libration centers were given by Ferraz-Mello (1988) by means of two laws, the law of structure and the law of periods, valid even for very high eccentricities. The effect of e_1 was fixed by a third law, the law of the second forced mode. This third law has its origin in a canonical transformation, first introduced in the study of this problem by Sessin (1981), which is a rotation in the space phase. It eliminates the variable σ_1 from the first-order terms of the disturbing function, making the problem integrable in this approximation. The resulting motion has three components in the plane (k, h). They are: one fixed component (the libration center or first forced mode), one free oscillation (libration) and one forced circular oscillation (the so-called second forced mode). The libration centers are defined by the law of structure which is a relation between the eccentricity and the semi-major axis of the asteroid. The period of the free oscillation is given by the law of periods while the third law gives the amplitude of the second force mode. However, the canonical transformation involves an approximation (Ferraz-Mello, 1988 Eq. (15)) that we would like to avoid.

We may deduce the law of periods and the law of structure from the circular case ($e_1 = 0$) because they are independent of the eccentricity of Jupiter and this result is the same as this obtained in the high-eccentricity theory of Ferraz-Mello (1988). In this paper, we obtain a new and more general expression for the second forced mode by means of a perturbative method developed from the equations of FTK. We use the same set of variables as FTK in order to make use of some calculations done there. First, we consider the canonical transformation $(J, J_1, \sigma, \sigma_1) \rightarrow (K, J_1, H, \sigma_1)$ defined by:

$$K = \sqrt{2J} \cos \sigma \quad H = \sqrt{2J} \sin u. \quad (5)$$

The system of canonical equations becomes:

$$\frac{dK}{dt} = \frac{\partial \mathcal{H}}{\partial H} \quad \frac{dH}{dt} = -\frac{\partial \mathcal{H}}{\partial K}$$

$$\frac{dJ_1}{dt} = \frac{\partial \mathcal{H}}{\partial \sigma_1} \quad \frac{d\sigma_1}{dt} = -\frac{\partial \mathcal{H}}{\partial J_1} \quad (6)$$

or, in an explicit way, neglecting second order terms in δk , δh and e_1 :

$$\frac{dK}{dt} = -\mathcal{A}H + BF_h \quad (7)$$

$$\frac{dH}{dt} = \mathcal{A}K - BF_k \quad (8)$$

$$\frac{dJ_1}{dt} = -e_1 A_6 \sin \sigma_1 + e_1 A_7 \cos \sigma_1 \quad (9)$$

$$\frac{d\sigma_1}{dt} = \mathcal{A} - \frac{KF_k + HF_h}{4BL^2} \quad (10)$$

where

$$\mathcal{A} = (r+1)n_1 - rn + \frac{2rL}{\mu} \frac{\partial R}{\partial a} - (KF_k + HF_h) \frac{\partial B}{\partial J} \quad (11)$$

$$B = \sqrt{\frac{2L-J}{2L^2}} = \frac{e}{\sqrt{2J}} \quad (12)$$

$$n = \frac{\mu^2}{L^2} \quad (13)$$

$$L = \frac{J_2}{q} - r(J + J_1) = \sqrt{\mu a} \quad (14)$$

$$k = KB \quad (15)$$

$$h = HB \quad (16)$$

and F_k and F_h are the derivatives of R with respect to k and h , respectively (see FTK). We will also use the notation $\nu_1 = d\sigma_1/dt$ (noted as S_1 in FTK).

3.1. LAW OF STRUCTURE

For $e_1 = 0$, the forced stationary solutions $K = \text{cte}$, $H = \text{cte}$ and $J_1 = \text{cte}$, called *libration centers*, are obtained solving iteratively Eqns.(7) and (8). As a result, each resonance, has a family of points (k_0, h_0, a_0) called *law of structure*. For the most important resonance they are plotted in Ferraz-Mello (1990) and Morbidelli and Giorgilli (1990b), for example. Message (1966), looking for stationary solutions

TABLE I
Libration centers with secondary resonances. At first order resonances, when $\nu/\nu_1 \rightarrow 1$, $k_0 \rightarrow 0$ but, in $k_0 = 0$ there is no libration center because a_0 is not determined.

ν/ν_1	$h_0(3:1)$	$k_0(2:1)$	$k_0(3:2)$
1/2	0.0073	- 0.0570	- 0.0465
1	0.0145		
2	0.0290	0.0836	0.0650
3	0.0433	0.1146	0.0888
4	0.0576	0.1393	0.1073
5	0.0717	0.1605	0.1230

of the averaged **restricted** three-body problem near a 2:1 resonance, found a similar relationship between the eccentricity and the **asteroid** semi-major **axis** valid for small eccentricities. Searching for periodic orbits of the restricted three-body problem near a resonance and by numerical integration of the exact equations of motion, Colombo et al. (1968) found **some** curves relating osculating values of e and a which **have** trends similar to those given by this law; their curves vary according to the epoch where the elements are calculated. In the averaged problem the relation between a and e is unique.

3.2. LAW OF PERIODS AND STABILITY

The **square** of the proper frequency, ν , of the small oscillations about the libration center **is** given by the Hessian of Hamiltonian function for the circular case:

$$\nu^2 = \text{hess}(\mathcal{H}_0) = S_{11}S_{33} - S_{13}^2 \quad (17)$$

where S_{11} , S_{33} and S_{13} are $\frac{\partial^2 \mathcal{H}_0}{\partial K^2}$, $\frac{\partial^2 \mathcal{H}_0}{\partial H^2}$ and $\frac{\partial^2 \mathcal{H}_0}{\partial K \partial H}$ calculated at the libration center (see Ferraz-Mello, 1988, **FTK**). If $\nu^2 < 0$ the equilibrium solution is unstable **and** if $\nu^2 > 0$ it is stable. For stable solutions, the libration period, in Julian **years**, is obtained from $P = \frac{2\pi}{365.25} \frac{1}{\nu}$. This is the *law of periods* plotted for the resonances 3:1, 2:1 and 3:2 in Figures 1a-c.

The law of structure and the law of periods were extensively tested by Michtchenko (1993) for the resonances 2:1 and 3:2. For small amplitude librations, the **numerical** results of Michtchenko (1993) are in very good agreement with the **high-eccentricity libration theory**.

3.3. THE SECOND FORCED MODE

A second forced component of motion, due to Jupiter's eccentricity, appears in all numerical studies and its amplitude was analytically deduced for small eccentricity asteroids by Message (1966) and Greenberg and Franklin (1975) and, for high eccentricities, by Ferraz-Mello (1988). However, in the high-eccentricity libration theory of Ferraz-Mello, in order to set an integrable approximation, some simplifications were done which, in fact, limited the validity range of his third law. We follow here a different approach, valid for all eccentricities, perturbing the system of equations for the circular case ($e_1 = 0$). Let \mathcal{H}_0 be the Hamiltonian for the circular case. The terms δR linear with respect to e_1 are responsible for a perturbation $(\delta K, \delta H, \delta J_1)$ about the libration center (K_0, H_0, J_{10}) . The variational equations for \mathbf{I}' and \mathbf{H} are:

$$\frac{d\delta K}{dt} = \delta \frac{\partial \mathcal{H}}{\partial H} = \frac{\partial^2 \mathcal{H}_0}{\partial H \partial K} \delta K + \frac{\partial^2 \mathcal{H}_0}{\partial H^2} \delta H + \frac{\partial^2 \mathcal{H}_0}{\partial H \partial J_1} \delta J_1 + \frac{\partial \delta R}{\partial H} \quad (18)$$

$$\frac{d\delta H}{dt} = -\delta \frac{\partial \mathcal{H}}{\partial K} = -\frac{\partial^2 \mathcal{H}_0}{\partial K^2} \delta K - \frac{\partial^2 \mathcal{H}_0}{\partial K \partial H} \delta H - \frac{\partial^2 \mathcal{H}_0}{\partial K \partial J_1} \delta J_1 - \frac{\partial \delta R}{\partial K}. \quad (19)$$

The calculation of the second-order partial derivatives of \mathcal{H}_0 appearing in these equations is cumbersome. However they are given in FTK. We may use the equations given in Table I of that paper calculated at the libration center and $e_1 = 0$. The second derivatives appear under the notation S_{ij} where the subscripts 1 to 4 mean $K, J_1, \mathbf{H}, \sigma_1$, respectively. We can take δR directly from Eqn. (4) but it is necessary to transform it to the variables K, \mathbf{H} . We prefer to give it indirectly from

$$\delta R = \int \frac{\partial \mathcal{H}}{\partial \sigma_1} d\sigma_1 \quad (20)$$

then

$$\frac{\partial \delta R}{\partial H} = \int \frac{\partial^2 \mathcal{H}}{\partial H \partial \sigma_1} d\sigma_1 = \int S_{34}^1 d\sigma_1 \quad (21)$$

$$\frac{\partial \delta R}{\partial K} = \int \frac{\partial^2 \mathcal{H}}{\partial K \partial \sigma_1} d\sigma_1 = \int S_{14}^1 d\sigma_1 \quad (22)$$

where the superscript' means up to degree 1 in e_1 . The Equation for J_1 is taken from Eqn. (9):

$$\frac{d\delta J_1}{dt} = -e_1 A_6 \sin \sigma_1 + e_1 A_7 \cos \sigma_1. \quad (23)$$

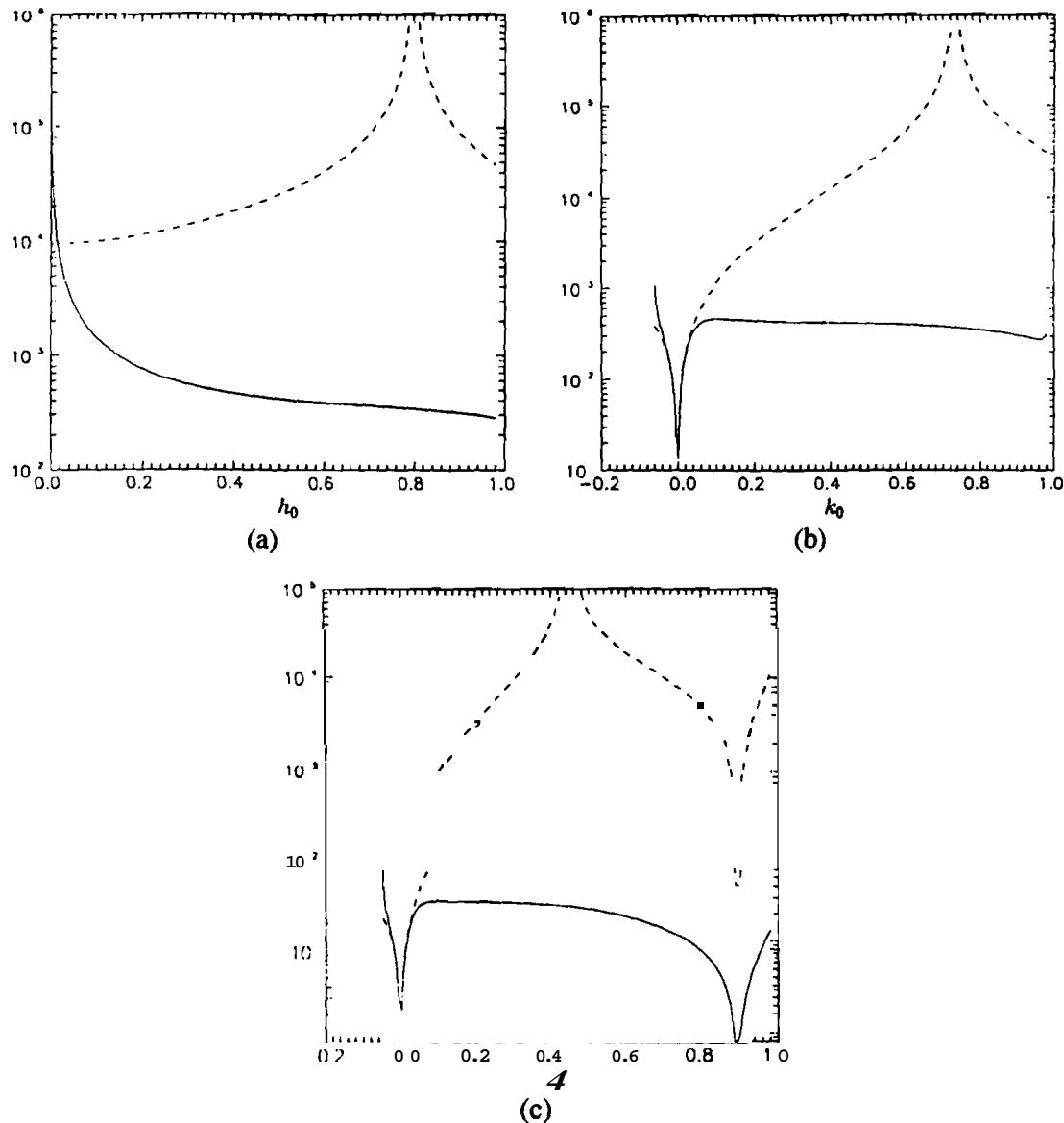


Fig. 1. (a,b,c). Resonances 3:1 (a).2:1 (b) and 3:2 (c). Solid line: libration period (P) of u . Dashed line: circulation period (P_1) of σ_1 . For the 3:1 resonance, at very small eccentricities, the libration period is greater than the circulation period of the perihelion.

At last, for σ_1 , we have:

$$\frac{d\sigma_1}{dt} = \nu_{10} + \delta\nu_1 \quad (24)$$

where ν_{10} is $d\sigma_1/dt$ calculated at the libration center. If we assume $\nu_{10} \neq 0$ and $|\delta\nu_1| \ll |\nu_{10}|$ we can neglect $\delta\nu_1$ (this point is discussed later) and use this equation to change the independent variable; so the system of equation becomes:

$$\frac{d\delta K}{d\sigma_1}\nu_{10} = S_{13}^0\delta K + S_{33}^0\delta H + S_{23}^0\delta J_1 + \int S_{34}^1 d\sigma_1 \quad (25)$$

$$\frac{d\delta H}{d\sigma_1}\nu_{10} = -S_{11}^0\delta K - S_{13}^0\delta H - S_{12}^0\delta J_1 - \int S_{14}^1 d\sigma_1 \quad (26)$$

$$\frac{d\delta J_1}{d\sigma_1}\nu_{10} = -e_1 A_6 \sin \sigma_1 + e_1 A_7 \cos \sigma_1 \quad (27)$$

where all S_{ij} are calculated at the libration center and the superscripts 0 and 1 indicate, respectively, that we consider the terms up to degree 0 and 1 in e_1 . The last equation is easily integrated:

$$\delta J_1 = \frac{e_1 A_6}{\nu_{10}} \cos \sigma_1 + \frac{e_1 A_7}{\nu_{10}} \sin \sigma_1 \quad (28)$$

and this result is replaced in the equations for K and H giving:

$$\frac{d\delta K}{d\sigma_1}\nu_{10} = S_{13}^0\delta K + S_{33}^0\delta H + c \cos \sigma_1 + d \sin \sigma_1 \quad (29)$$

$$\frac{d\delta H}{d\sigma_1}\nu_{10} = -S_{11}^0\delta K - S_{13}^0\delta H + g \cos \sigma_1 + h \sin \sigma_1 \quad (30)$$

where c , d , g and h are known functions calculated at the libration center and independent of al . From now on, we only consider the case of libration centers located at the axes k or h (that means $k_0 \cdot h_0 = 0$ and $\sigma_0 = N\frac{\pi}{2}$) since the libration centers in first and second-order resonances are always on these axes (except, at least, for the 1:2 and 1:3 exterior resonances; Beaugé, 1994). On the other hand, in these situations the coefficients c , h and S_{13}^0 vanish and the algebra is simplified. For these cases the system of Equations is reduced to

$$\frac{d\delta K}{d\sigma_1}\nu_{10} = S_{33}^0\delta H + d \sin \sigma_1 \quad (31)$$

$$\frac{d\delta H}{d\sigma_1}\nu_{10} = -S_{11}^0\delta K + g \cos \sigma_1 \quad (32)$$

with

$$d = \left\{ A_{11} \left[B + H^2 \frac{\partial B}{\partial J} \right] - \frac{2rLH}{\mu} \frac{\partial A_7}{\partial a} + \frac{S_{23}^0 A_7}{\nu_{10}} \right\} e_1 \quad (33)$$

$$g = - \left\{ A_8 \left[B + K^2 \frac{\partial B}{\partial J} \right] - \frac{2rLK}{\mu} \frac{\partial A_6}{\partial a} + \frac{S_{12}^0 A_6}{\nu_{10}} \right\} e_1 \quad (34)$$

where K and A_6 or H and A_7 vanish, for h-axis or k-axis libration centers, respectively. The system has the particular solution:

$$\delta K = \alpha \cos \sigma_1 \quad (35)$$

$$\delta H = \beta \sin \sigma_1 \quad (36)$$

with

$$\alpha = \frac{d\nu_{10} + gS_{33}^0}{(\nu^2 - \nu_{10}^2)} \quad (37)$$

$$\beta = \frac{dS_{11}^0 + g\nu_{10}}{(\nu_{10}^2 - \nu^2)}, \quad (38)$$

all expressions being calculated at the libration center. When $\nu^2 \simeq \nu_{10}^2$ this solution is no longer valid. For the resonances studied, this situation only happens in the 3:1 resonance when $h_0 \simeq 0.0145$ and gives rise to the discontinuity in the second forced mode shown in Fig. 2a at very small eccentricities for this resonance. At resonances 2:1 and 3:2, the frequencies of libration and of the forced mode may be very near one of another but they are different enough so as to have always $(\nu^2 - \nu_{10}^2) \neq 0$.

The expressions for δk and δh are obtained after transforming the variables and variations in Eqns.(35) and (36) by means of:

$$\delta k = B\delta K + K\delta B \quad (39)$$

$$\delta h = B\delta H + H\delta B \quad (40)$$

$$\delta B = \frac{\partial B}{\partial J}\delta J + \frac{\partial B}{\partial J_1}\delta J_1 \quad (41)$$

$$\delta J = K\delta K + H\delta H. \quad (42)$$

There results

$$\delta k = Z_k \cos \sigma_1 \quad (43)$$

$$\delta h = Z_h \sin \sigma_1 \quad (44)$$

where

$$Z_k = \alpha \left[B + K^2 \frac{\partial B}{\partial J} \right] + K \frac{e_1 A_6}{\nu_{10}} \frac{\partial B}{\partial J_1} \quad (45)$$

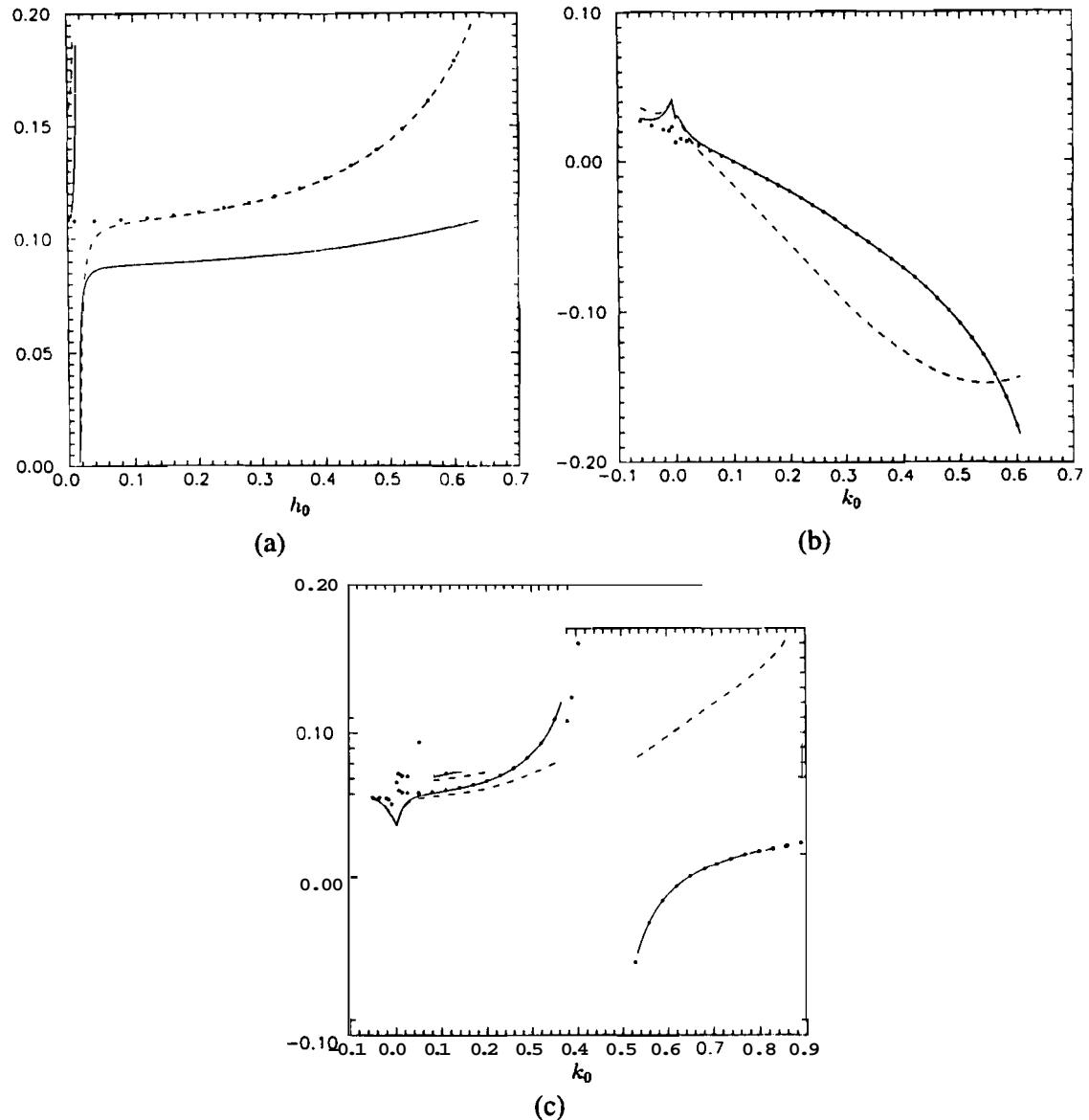


Fig. 2. (a,b,c). The second forced mode in the resonances 3:1(a), 2:1(b) and 3:2(c). Solid line: Z_k . Dashed line: Z_h . Points: old approximation(Ferraz-Mello, 1988).

$$Z_h = \beta \left[B + H^2 \frac{\partial B}{\partial J} \right] + H \frac{e_1 A_7}{\nu_{10}} \frac{\partial B}{\partial J_1}. \quad (46)$$

Again, all expressions are calculated at the libration center. The forced component of k has not the same amplitude as the forced component of h . This oscillation mode describes a quasi-elliptical trajectory in the plane (k , h) which may degenerate into a line when one component is zero. This is a better modelling of those trajectories

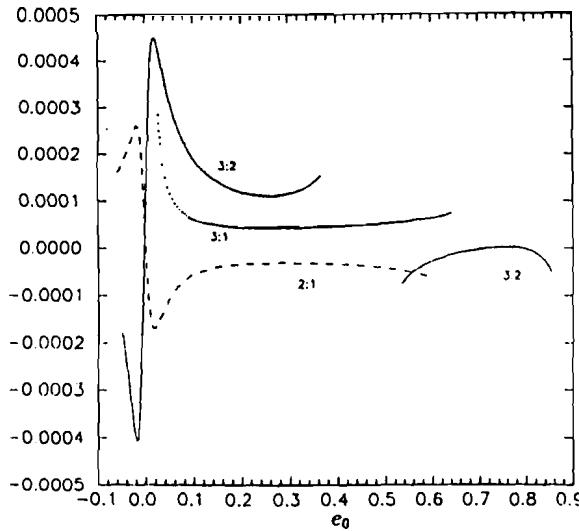


Fig. 3. Forced oscillation (Z_a) in α (in units of a_1), for the resonances 3:1 (points), 2:1 (dashed line) and 3:2 (solid line).

than the previous circular approximations. Figures 2a-c show the amplitudes of these forced modes, as well as for the sake of comparison, those given by the old approximation (Ferraz-Mello, 1988). The new approximation gives good results even at small eccentricities. For example, Greenberg and Franklin (1975), for the 2:1 resonance, calculated a forced component of amplitude 0.038 at $k_0 = -0.026$ while we obtain, for this center, $Z_k = 0.030$ and $Z_h = 0.032$. For zero-amplitude librations, the second forced component of the eccentricity was defined as a vector of constant modulus rotating around the libration center (or first forced component) in the space (k, h) , or as a fixed constant component in the space $(e \cos \varpi, e \sin \varpi)$. Due to the fact that $Z_k \neq Z_h$, the modulus of this second component is no more constant and the actual forced oscillation in the eccentricity cannot be well represented by a circular model. It is worth mentioning that the amplitude of the forced oscillations in the eccentricity are better represented by $|Z_k|$ for the first order resonances and by $|Z_h|$ for second order resonances. The old model, for $e_0 > 0.1$, coincides with Z_k for first-order resonances and with Z_h for second-order resonances. This means that the old model succeeded in calculating the forced amplitude variations in the eccentricity for $e_0 > 0.1$ but failed in assuming $Z_h = Z_k$. It is important to stress that all theories using Sessin's rotation of the phase space in order to eliminate σ_1 , are based on the circular model for the forced mode.

The period, P_1 , of this forced oscillation, is directly deduced from ν_{10} and is shown in Figures 1a-c, together with the curves giving P .

The method proposed in this paper is no longer valid when $\nu_{10} \approx 0$ since the change of variable t to σ_1 is no longer allowed in this case. It is worthwhile

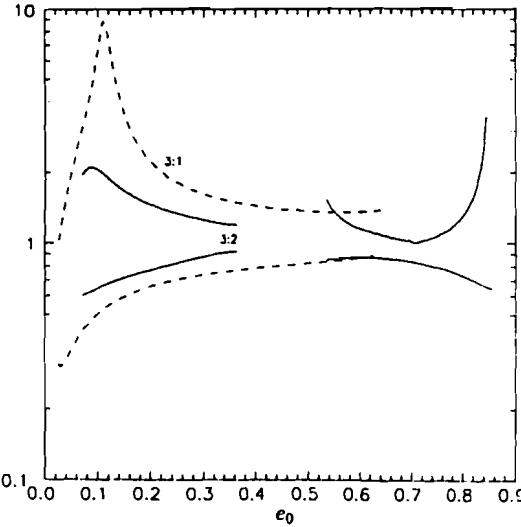


Fig. 4. Relative extremal variations $\frac{\delta P}{P}$ of the instantaneous libration period for resonances 3:1 and 3:2 corresponding to centers with $P_l > 2P$. In the resonance 2:1 there are no significative changes in P .

emphasizing that $\nu_{10} \simeq 0$ is the characteristic of a different regime of movement called *corotation*, with a different topology, and that was exhaustively studied in FTK. Corotation is a motion with two librating critical angles a and $(\sigma_1 - a)$. If we assume $a = \text{cte}$ (libration center) then σ_1 librates and ν_1 oscillates around a mean value $\nu_{10} = 0$. In order to have an indicator of the validity of our results, we have calculated $E = \nu_{10}^{-1} \max(\delta\nu_1)$ at each libration center, being $\delta\nu_1 = \nu_1 - \nu_{10}$ where ν_1 is calculated in the forced trajectory. Even for $E \sim 1$, we have found satisfactory results but, for greater values, we are in the corotation zone and the theory is no longer valid. In Figures 2-4 we only plot the values corresponding to $\epsilon < 1$.

3.4. FORCED OSCILLATION OF THE SEMI-MAJOR AXIS

The perturbation method of the previous section allows us to obtain also the forced oscillation in the semi-major axis of the asteroid:

$$\delta a = \frac{2L}{\mu} \delta L = -\frac{2rL}{\mu} (\delta J + \delta J_1), \quad (47)$$

that is

$$\delta a = Z_a^c \cos \sigma_1 + Z_a^s \sin \sigma_1 \quad (48)$$

with

$$Z_a^c = -\frac{2rL}{\mu} (K\alpha + \frac{elA_6}{\nu_{10}}) \quad (49)$$

$$Z_a^s = -\frac{2rL}{\mu}(H\beta + \frac{e_1 A_7}{\nu_{10}}). \quad (50)$$

For *k*-axis libration centers, $Z_a^s = 0$, and for h-axis libration centers, $Z_a^c = 0$. This new law is, in general, a small oscillation as compared with the free librations, except for small eccentricities in which case it becomes strongly dependent on e (see Figure 3). This component appears in the periodograms of Michtchenko (1993) and Michtchenko and Ferraz-Mello (1993). The present theory is in agreement with the forced oscillations found by Schubart (1968, Table I) for the 3:2 resonance.

3.5. VARIATIONS IN THE LIBRATION PERIOD

Because of the second forced node, librations occur around instantaneous moving points that describe a quasi-elliptical trajectory around the libration center. As this forced node is going on, the libration in fact happens to be centered at different points of the space (*k*, *h*, *a*). If we assume $P_1 \gg P$, the libration frequency is given by Eqn.(17) calculated in each point of the forced trajectory. As a result, we can calculate the libration period as a function of σ_1 . The external values $P^{-1} \min(P)$ and $P^{-1} \max(P)$ thus obtained are plotted in Figure 4 for resonances 3:2 and 3:1. The last one shows very high variations at small eccentricities, (the motion in this region is known to be chaotic). There are important variations in the libration period for $e_0 < 0.1$ in the 3:2 resonance. It is interesting to remind that no asteroids are librating with $e_0 < 0.1$ and that Franklin et al. (1993) and Michtchenko (1993) have found chaotic regime in this region. There are no significant changes in P in the 2:1 resonance.

3.6. SMALL AMPLITUDE LIBRATIONS FOR THE CIRCULAR CASE

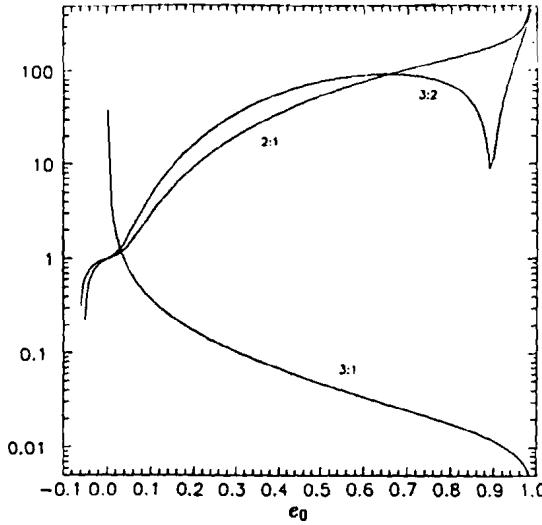
The same perturbation method used to obtain the forced mode may be applied to obtain the trajectories in the (*k*, *h*, *a*) space, for small-amplitude librations, when assuming $e_1 = 0$. In this case, if the trajectory of the libration is described by the points $(K + \delta K, H + \delta H)$, being (K, H) the center, the same perturbative equations with $6R = 0$ are valid and the system becomes:

$$\frac{d\delta K}{dt} = S_{33}^0 \delta H \quad (51)$$

$$\frac{d\delta H}{dt} = -S_{11}^0 \delta K \quad (52)$$

where terms involving e_1 and σ_1 are not present. This system is valid for libration centers in the *k* or *h* axis. The solution in terms of *k* and *h* is:

$$\delta k = A_k \cos(\nu t + \phi) \quad (53)$$


 Fig. 5. Values of $-A_h/A_k$.

$$\delta h = A_h \sin(\nu t + \phi) \quad (54)$$

where $\nu = \sqrt{S_{11}^0 S_{33}^0}$ is the frequency of stable librations, ϕ is a constant of integration and

$$\frac{A_h}{A_k} = -\frac{(B + \frac{\partial B}{\partial J} H^2)}{(B + \frac{\partial B}{\partial J} K^2)} \sqrt{\frac{S_{11}^0}{S_{33}^0}} \quad (55)$$

which corresponds to a quasi-elliptical shape. This relation is plotted in Figure 5 and it makes evident that the libration trajectories are not always elongated in the same direction. For example, for the apocentric librations ($\sigma_0 = a$) of first-order resonances, the libration trajectories are more elongated along the k -axis. Nevertheless, this trajectory is difficult to distinguish from the forced one because both have nearly equal frequencies, but they can be separated by a Fourier analysis. Numerical results of Michtchenko (1994) for apocentric librators at the 2:1 resonance show that $|A_h/A_k| < 1$ in accordance with the present theory. There is an analogous behavior for small eccentricities at the 3:1 resonance.

The libration in the semi-major axis of the asteroid is calculated in a similar way as the variations in the forced mode and we obtain

$$\delta a = A_a^c \cos(\nu t + \phi) + A_a^s \sin(\nu t + \phi) \quad (56)$$

with

$$\frac{A_a^c}{A_k} = -\frac{2rLK}{\mu(B + \frac{\partial B}{\partial J} K^2)} \quad (57)$$

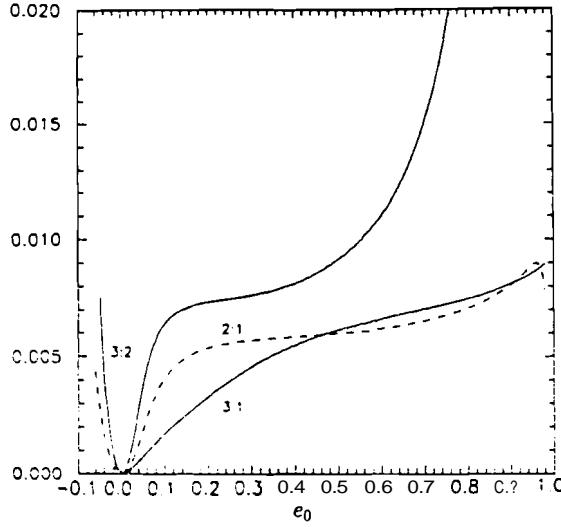


Fig. 6. Values of A_a/A_σ . A_a is in units of a_1 and A_σ in radians.

$$\frac{A_a^s}{A_h} = -\frac{2rLH}{\mu(B + \frac{\partial B}{\partial J}H^2)}. \quad (58)$$

For k-axis libration centers, $A_a^s = 0$ and for h-axis libration centers, $A_a^c = 0$. The amplitudes A_a^c and A_a^s can be expressed as functions of A_k , A_h or A_σ (libration amplitude of σ). Franklin et al. (1993) found a linear relation between the libration amplitude of the semimajor axis (A_a) and the libration amplitude in σ (Ao) for the Hildas. Also a linear relation was found by Gallardo (1995) for exterior resonances to Neptune. In both cases the linearity is valid even for very-high amplitude librations and the coefficients $\Delta a/\Delta \sigma$ are in very good agreement with the values A_a/A_σ deduced from this theory (Figure 6).

The time can be eliminated from the perturbations (k , δh , δa) defining a relation (e, a) , which is satisfied during the libration and which is the same as found by Ferraz-Mello (1988, Figure 8). This motion in the (e, a) plane is an oscillation around the libration center (e_0, a_0) determined by the law of structure and the value $\Delta a/\Delta e$ is approximately given by Eqn.(57) for k-axis libration centers and by Eqn.(58) for h-axis libration centers, except for very small eccentricities.

3.7. GENERAL SOLUTION FOR THE ELLIPTIC CASE

The principal features of the trajectories in the (k, h, a) phase space are done by the addition of the three components:

$$k = k_0 + A_k \cos(\nu t + \phi) + Z_k \cos \sigma_1 \quad (59)$$

$$h = h_0 + A_h \sin(\nu t + \phi) + Z_h \sin \sigma_1 \quad (60)$$

$$a = a_0 + A_a^c \cos(\nu t + \phi) + A_a^s \sin(\nu t + \phi) + Z_a^c \cos \sigma_1 + Z_a^s \sin \sigma_1. \quad (61)$$

The expressions for k and h are more general than others deduced before because we have $A_k \neq A_h$ and $Z_k \neq Z_h$, and the formula (61) is original in the sense of giving an approximate analytical expression for a . These expressions are valid as a first approximation for small amplitude librations. The interest of these expressions is not the exact representation of the motion (which is obtained better by numerical integration) but the general description of the trajectories in the space (k, h, a) .

3.8. SECONDARY RESONANCES AT THE LIBRATION CENTERS

When $\nu/\nu_1 = N/M$, with N and M integers, **secondary resonances** occur (see Henrard and Lemaître, 1986). They are more concentrated at small eccentricities in the 3:1 resonance than in the other resonances. Table 1 shows the eccentricity of the libration centers at some secondary resonances as found with the present model. For resonances 2:1 and 3:2 we can compare these results with the ones obtained by numerical methods by Michtchenko (1993). Michtchenko gives the values f_σ/f_ω which are the same as ν/ν_1 because $w - \omega_1 = \sigma_1 - \phi$ and, in our case, we consider ω_1 and a as constants (libration centers). The agreement with our calculations is very good. It is important to stress that our model can predict the positions of secondary resonances but it cannot predict any special behavior associated with them.

4. Numerical Experiments

We have done some numerical integrations for the sake of comparison with the analytical results. We used the numerical integration program with digital filtering of Michtchenko (1993) to remove high-frequency terms and, in order to determine the relevant components in the evolution of each variable, we made periodograms using the subroutines of Press et al. (1986).

4.1. EXAMPLE 1: THE ELLIPTIC FORCED MODE

In this first example we choose a particle in the resonance 2:1 with initial conditions generating an evident non-circular force mode. In Table II we compare the set of results obtained by the periodograms with the set of theoretical values which best fit them. The amplitude $|Z_h|$ is greater than twice $|Z_a|$.

4.2. EXAMPLE 2: VARIATIONS IN THE LIBRATION PERIOD

Variations in the libration period are more evident in the 3:1 libration. In this case, the Fourier analysis of the output shows a great number of broadened spectral lines and it is impossible to identify fundamental frequencies from this. Nevertheless,

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Table II Results of the numerical integrations compared with the set of theoretical values which best fit them. Z, and \mathbf{A}_n , in units of a_1 .

k_0	h_0	a_0	P	P_1	$ Z_k $	$ Z_h $	$ Z_a $	$ A_h/A_k $	$ A_a/A_h $
Example 1 (2:1)									
.280	0	.629394	438	6140	.0351	.0895	.000020	18.5	.0213
.286	0	.629397	433	6100	.0401	.0894	.000031	18.2	.0198
Example 3 (3:2)									
.621	0	.763287	173.7	14900'	.0025	.1300	.000019	6.3'	.0187
.620	0	.763304	173.7	16130	.0019	.1321	.000020	90.9	.0187
Example 4 (3:2)									
.031	0	.753063	176.2	210	.0394	.0385	.000272	1.31	.0718
.031	0	.753158	180.5	211	.0562	.0549	.000411	1.30	.0721

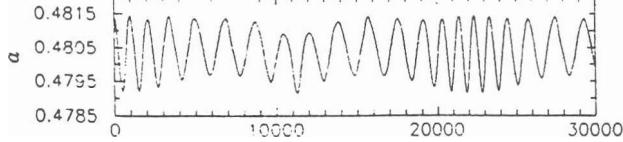


Fig. 7. Example 2: Time evolution of a showing variations in the libration period. Time in years. (Resonance 3:1).

from the trajectory in the (k, h) plane it is possible to estimate the center to be in $h_0 \simeq 0.14$. For this center our theory indicates a libration period of 1024 yrs and predicts that the period of the instantaneous libration varies from 600 to 4200 yrs. Figure 7 shows the evolution of a in the first 30,000 years and variations in the libration period are evident. From the periodogram of a obtained with an integration of 501,600 years, it is clear that it has important components with periods from ~ 900 to ~ 2300 years. The time evolution of the elements of this fictitious asteroid over 500,000 years of the numerical integration shows variable frequencies. This asteroid is near the chaotic region of the resonance and we can interpret this variation as instabilities in the librations introduced by the second forced mode. In fact, e_1 is the responsible for the chaos, as was suggested by Wisdom (1983). Numerical integrations of Yoshikawa (1990) shows a similar time evolution of a for this resonance. Our results are valid only when $P_1 \gg P$, and in the numerical experiments we detect smaller variations than the predicted ones because, in general, P_1 cannot be neglected as compared with P . As in the case of secondary resonances, our model cannot generate chaotic motion. In this case we only observe variations in the libration period.

4.3. EXAMPLE 3: A SINGULAR SECOND FORCED MODE

We look for a second forced mode with one component equal to zero. The results are summarized in Table 2. In Figure 8 we show the resulting trajectory in the (k, h) plane with approximately zero forced mode along the k -axis and a superposed libration with $A_h \sim 0.06$. Our model predicts nearly zero variations in k for libration and force mode, but due to the fact that the whole trajectory is better fitted by an arc of circumference than a straight line, there exists a k component in the periodogram and this is the reason of the larger values of Z_k and the discrepancies between predicted and numerical values of A_h/A_k . The error in the frequency of the forced mode is due to the resolution of the periodogram. The predicted period is 16,130 yrs but this value is not constant since $\epsilon = 0.15$, allowing us to expect some difference.

4.4. EXAMPLE 4: SMALL ECCENTRICITIES

We have done a numerical integration in the 3:2 resonance with low eccentricity in the region where the old model failed in predicting a correct value for the second

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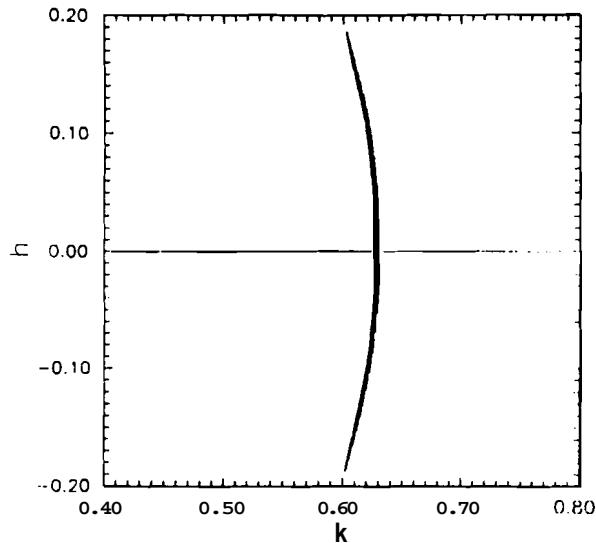


Fig. 8. Example 3: Singular force mode a superimposed to a small amplitude libration during 16,600 yrs. (Resonance 3:2).

force' mode ($k_0 < 0.1$). At small eccentricities, both movements, libration and forced mode, are confounded because they are nearly circular and have similar periods. The values of Z_k and Z_h are slightly smaller than the predicted ones and this is due to the following facts: 1) both movements are similar in frequency and amplitude thus producing a mixture where both movements are confounded, 2) we neglected in the analytical model the components at combined frequencies as ($\nu - \nu_1$) and 3) proper imprecision in the amplitude determinations of the periodogram. Considering those facts, the agreement is very good. Michtchenko and Ferraz-Mello (1993) have found values smaller than our predictions for $k_0 < 0.06$ because they calculated the second forced mode as the amplitude corresponding to the frequency ν_1 in the periodogram of e . According to this definition of forced eccentricity we have:

$$e_f = \frac{e_{\max} - e_{\min}}{2} = \frac{k_0 + Z_k - |k_0 - Z_k|}{2}. \quad (62)$$

If $k_0 > Z_k$ (that means $k_0 > 0.06$ for our theory) we have $e_f = Z_k$ and both definitions of second forced mode are the same. But, if $k_0 < Z_k$, we have $e_f = k_0$, and this is evident in Figure 5 of Michtchenko and Ferraz-Mello (1993). Then, these discrepancies are only consequences of the definition of second forced mode. Similar discrepancies with Michtchenko (1993) appear at small eccentricities in the 2:1 resonance.

5. Conclusions

The use of Sessin's canonical rotation in the phase space leads to a solution giving a good prediction of the forced variations amplitude of the eccentricity for libration centers with $e_0 > 0.1$, at the resonances studied, but **does** not allow to predict the **trajectories** in the (k, h) phase space. Using the asymmetric expansion of the disturbing function and a first-order theory on δk , δh and e_1 , we obtain a better description of the forced mode and, moreover, we obtain the effect of this forced oscillation on the **semimajor axis** of the **asteroid** and on the libration period. Those effects, especially the variations δP , are indicators of instabilities introduced by Jupiter's eccentricity. We **have** found **large** perturbations in the librations at resonances 3:1 and 3:2 at regions where chaotic motion was already detected. In the resonance 2:1 we did not find significative variations in the libration period. For **pericentric** and apocentric librators with small eccentricity, the forced oscillation in a , k and h are greater in the 3:2 resonance than in the 2:1 resonance (see Figures 2b, 2c and 3). For apocentric librators in the 3:2 resonance we **have** always $|Z_k| > |k_0|$; this could be related to the non-existence of apocentric librators in this resonance, while they are abundant in the 2:1 resonance. The study is completed with approximate analytical expressions for the small amplitude (**free**) librations. Libration centers at **some** secondary resonances are determined and they are **in** very good agreement with **numerical** determinations. This theory is **valid** for amplitude librations with a circulating perihelion, so the case of corotations and those phenomena associated with high-amplitude librations as the peculiar **orbits** of Scholl and **Froeschlé** (1974, 1975) cannot be considered. Chaotic **trajectories** cannot be obtained, but suspected by means of variations in the libration period or locations of secondary resonances. The formulae presented **here were** deduced for libration centers in the **axes** k or h , but it is not difficult to obtain more general formulae **valid** for libration centers located at any point in the (k, h) plane.

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Anexo II

Understanding libration via time-frequency analysis

UNDERSTANDING LIBRATION VIA TIME-FREQUENCY ANALYSIS

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March 22, 1996

Abstract: Numerical integrations of resonant particles in the Solar System show that the libration is in general a more complex motion than that given by simplified analytical models. To obtain a more comprehensive view of the libration we propose a time-frequency analysis, valid for regular and chaotic motion, based on the time evolution of the local behaviour of the motion. The local behaviour is characterized by the frequency, amplitude and center around which the analyzed variable oscillates. These parameters are deduced from the power spectrum obtained with a relative small time-span of the total numerical integration. We applied this method to the planet Pluto obtaining a chaotic component in the time evolution of its libration amplitude. We also analize other exterior resonances with Neptune in the Kuiper disc.

Key words: frequency analysis, resonances, Kuiper disc, Pluto.

1 INTRODUCTION

The study of the dynamical behaviour of a system by means of the analysis of the power spectrum of the output of numerical integrations is a powerful tool that has been successfully applied to the case of the Solar System since the work of Applegate et al. (1986). One of the most important applications was the construction of synthetic theories for the planetary motion (Carpino et al. 1987, Laskar 1988). A systematic study of mean-motion resonances in the asteroid belt by means of the spectral analysis was first done by Michtchenko (1993). The time-frequency analysis was introduced in the study of the dynamics of the Solar System by Laskar (1990) who proposed to follow the time evolution of some well determined frequencies of the system to obtain a measure of the chaotic regions. After that (Laskar et al. 1992,

Laskar 1993), a very precise method for obtaining the topology of the phase space and its regular and chaotic region was devised . An adaptation of this method to the 2:1 asteroidal resonance can be found in Nesvorný and Ferraz-Mello (1995). The wavelet analysis (Bendjoya and Slezak 1993) has also been shown to be an alternative time-frequency analysis, specially for chaotic motion (Michtchenko and Nesvorný 1995).

1.1 The power spectrum.

There are several ways of obtaining the power spectrum starting with an n -dimensional data vector, \vec{x} , composed by n equally spaced sampled values of the unknown function $x(t)$, obtained in a total time span T . We can mention:

1) Signal representation by orthonormal series expansion. The idea is to decompose the vector \vec{x} onto an n -dimensional orthonormal base. Fourier series are the most common expansions where the orthogonal functions used are: $\cos(2\pi\nu_k t)$ and $\sin(2\pi\nu_k t)$, where $\nu_k = k/T$ is the k th Fourier harmonic frequency and $0 \leq k \leq n/2$. For each ν_k , the projection of $x(t)$ onto the normalized functions cosine and sine is computed giving coefficients c_k and s_k respectively. The spectral power at that frequency is defined as $P(\nu_k) = c_k^2 + s_k^2$ and gives a discrete spectrum. If $x(t)$ is a pure sinusoidal signal but with frequency u different from the ν_k , its spectral power will appear distributed in the Fourier frequencies located near u . This effect is known as leakage and is more important in the region $(\nu_1 - 2/T, \nu_1 + 2/T)$. We can estimate the original u by interpolation and its power adding the power of the different terms of the decomposition. This is possible because the base is orthonormal. Windows are used in order to soothe the leakage. We obtain the decomposition of the data onto an orthonormal base but we cannot always find precisely the frequencies present in $x(t)$.

2) Projection onto $\cos(2\pi\nu t)$ and $\sin(2\pi\nu t)$ in the continuous space of u . Each u defines the simple base given by the two functions $\cos(2\pi\nu t)$ and $\sin(2\pi\nu t)$. In general, these two functions are not orthonormal nor orthogonal in the interval $[0, T]$ for arbitrary u . It is necessary to introduce a shift in the time axis (see Laskar et al. 1992) and normalization coefficients to make them orthonormal. After that, the power can be defined as $P(\nu) = c^2 + s^2$ where c and s are the projections of $x(t)$ onto $\cos(2\pi\nu t)$ and $\sin(2\pi\nu t)$ respectively. If the orthonormality between the functions sine and cosine is

guaranteed, it will be possible to use the complex notation $\exp(i2\pi\nu)$. If $x(t)$ has a unique periodic component with frequency ν_1 , $P(\nu_1)$ is a maximum, but the power at other frequencies ν will be different from zero because, in general, the bases generated by the ν 's will be not orthogonal to the base generated by ν_1 (leakage). For a multiperiodic $x(t)$ the resulting power spectrum is the summation of the different lines with their leakages. Projection onto other bases is also possible and wavelet transforms are an example.

3) DCDFT or Date–Compensated Discrete Fourier Transform (Ferraz-Mello, 1981). In this method, an orthonormal base with the functions 1 , $\cos(2\pi\nu t)$ and $\sin(2\pi\nu t)$ is constructed with the Gram–Schmidt algorithm. The projections c and s are done over the basis functions and the usual definition of power is adopted. The projection onto 1 gives the central value (or constant term) of $x(t)$. By this method, the errors introduced by the use of the Pythagorean rule to combine the projection on the basis functions is removed. Also, the spectrum has a better precision because it is not contaminated by the projection of $x(t)$ onto 1 as happens in method 2, where the orthogonality to 1 (that means to the base corresponding to $\nu = 0$) was not guaranteed. So, the leakage that appears in the region of low frequencies when using method 2, now is removed and this region of the spectra is better resolved.

4) CLEANEST (Foster, 1995). In its most simple form defines a model function $y(t)$ that is the projection of $x(t)$ onto the non orthogonal basis 1 , $\cos(2\pi\nu t)$ and $\sin(2\pi\nu t)$ and defines the power as the (squared) projection of $y(t)$ onto the 2-dimensional subspace orthogonal to 1 . The Gram–Schmidt diagonalization is substituted by the inversion of a matrix. When only one frequency is involved it is mathematically equivalent to DCDFT. As the algebra involved is simpler than in DCDFT it may be easily generalized to treat several frequencies at a time.

These four methods analyse $x(t)$ with sinusoids, but other bases can be adopted. Thus, if $x(t)$ is a non-sinusoidal periodic process of frequency ν the harmonics $N\nu$ will also appear in the spectrum. There are some methods to extract from the spectrum the set of frequencies, amplitudes and phases that best fit the data. See for example: Carpino et al. (1987), Laskar (1988) and Foster (1995). Nevertheless, for our purposes it is enough to consider the information contained in the spectrum.

When we want to detect a transient phenomena or a very low frequency, and do not have many cycles to analyze, the effects of the non-orthogonality

of 1, $\cos(2\pi\nu t)$ and $\sin(2\pi\nu t)$ grow and methods like DCDFT and CLEANEST are necessary if a precise spectrum is aimed. They also have a good performance when working with unequally spaced and noisy data or when the number of available data is small. We perform our time-frequency analysis following the ideas of the method CLEANEST with some modifications as explained in Section 2.

1.2 Frequency analysis for Solar System bodies.

Let us suppose that we want to analize the libration of a resonant motion. If we take a small time interval, we generally can fit satisfactorily the output to a unique frequency. But, due to leakage, we will obtain a very poorly resolved line. If we take a longer interval, the discrepancies between the output and model will grow and it will be necessary to consider, in the model, other components than the previous determined frequency. We could say that in the second case the model is more complete. This reasoning is valid only for regular motion where the fundamental frequencies are well determined and fixed. As the Solar System is chaotic as a whole (Laskar 1990, Sussman and Wisdom 1992), it is hard to think of a particle in the Solar System that is locked in a regular motion forever. Actually, we have degrees of chaos and instead of assuming a regular motion it is better to say that the motion is quasi-regular in a given time-scale. Instead of trying to obtain a very precise spectrum of the total time-span of the numerical integration's output, it is preferable to study the local behaviour of the output and to look at its evolution. By local behaviour we mean the frequency, amplitude and other parameters that characterize the variable analyzed in a relative small time-span. This is the same idea of the time-frequency analysis of Laskar (1990) but it is applied in very small time-scales and without any filtering process.

The time evolution of the local behaviour will be affected by the other components of the motion that we are not considering. The local frequency will be affected by those frequencies that cannot be distinguished from the main frequency in the small time-scale spectrum. The method is not devised to measure the chaotic behaviour but to obtain a more comprehensive view of the time-evolution of a feature in the spectrum as the libration line. We show that the analysis of the time evolution of the local behaviour gives us more information than a unique spectrum obtained from the total time interval of the numerical integration and this is true in stable or chaotic

motion as well. In Section 3 we illustrate our time–frequency analysis with some applications to the exterior resonances with Neptune.

2 TIME–FREQUENCY ANALYSIS

As Foster (1995) we use here the vectorial representation of signals. The output of the numerical integrations of the equations of motion of a particle gives us the data vector

$$\vec{x} = (x(t_1), \dots, x(t_n)) \quad (1)$$

and we try to describe the unknown function $x(t)$ by the model function

$$y(t) = \sum_{\alpha=1} c_{\alpha} \phi_{\alpha}(t) \quad (2)$$

where the ϕ_{α} ($\alpha = 1, \dots, r$) are given linearly independent trial functions and the coefficients c_{α} are undetermined. Each ϕ_{α} defines a vector

$$\vec{\phi}_{\alpha} = (\phi_{\alpha}(t_1), \dots, \phi_{\alpha}(t_n)) \quad (3)$$

and we impose that the model vector

$$\vec{y} = \sum_{\alpha=1} c_{\alpha} \vec{\phi}_{\alpha} \quad (4)$$

is the projection of \vec{x} onto the r –dimensional subspace defined by the vectors $\vec{\phi}_{\alpha}$. So we can write:

$$\vec{x} = \vec{y} + \vec{\theta} \quad (5)$$

where $\vec{\theta}$ is a residual vector orthogonal to the subspace of the vectors $\vec{\phi}_{\alpha}$. If we calculate the inner product of \vec{x} with $\vec{\phi}_{\alpha}$ we obtain:

$$(\vec{\phi}_{\alpha}, \vec{x}) = (\vec{\phi}_{\alpha}, \vec{y}) + 0 = \sum_{\beta=1} c_{\beta} (\vec{\phi}_{\alpha}, \vec{\phi}_{\beta}) \quad (6)$$

and solving this system of r linear equations we obtain the coefficients c_{α} . As in Ferraz–Mello (1981) we define the inner product as

$$(\vec{f}, \vec{g}) = \sum_{i=1}^n f_i g_i w_i \quad (7)$$

where $w_i = w(t_i)$ are the components of a weight vector (or window vector) such that

$$\sum_{i=1}^n w_i = 1. \quad (8)$$

The trial functions are parameter dependent and are chosen to be such that the norm $|\vec{\theta}|$ is as small as possible, i.e. $|\vec{y}|$ maximum. The norm is defined

$$|\vec{y}| = \sqrt{(\vec{y}, \vec{y})}. \quad (9)$$

In this paper, we use the most simple basis: we try to describe $x(t)$ using the trial functions

$$\begin{aligned} \phi_1 &= 1 \\ \phi_2 &= \cos(2\pi\nu t) \\ \phi_3 &= \sin(2\pi\nu t) \end{aligned} \quad (10)$$

that are adequate for oscillating variables, but others trial functions can be used. The model vector can be written as the addition of two parts: a projection onto the fixed $\vec{\phi}$ and a projection onto the 2-dimensional subspace orthogonal to $\vec{\phi}_1$:

$$\vec{y} = (\vec{\phi}_1, \vec{y}) \vec{\phi}_1 + \vec{y}_{\perp}. \quad (11)$$

(Note that $|\phi_1| = 1$). From Eq.(6) we have

$$(\vec{\phi}_1, \vec{y}) = (\vec{\phi}_1, \vec{x}) = \langle x \rangle \quad (12)$$

where $\langle x \rangle$ is the mean (weighted) value of $x(t_i)$ and it is independent of ν . The only part of y that depends on ν is \vec{y}_{\perp} , so, varying ν , we look for the maximization of:

$$A(\nu) = \sqrt{2} |\vec{y}_{\perp}| \quad (13)$$

where, from Eqs.(9), (11) and (12)

$$|\vec{y}_{\perp}| = \sqrt{(\vec{y}, \vec{y}) - \langle x \rangle^2}. \quad (14)$$

We use $A(\nu)$ instead of the usual definition of power because, when working with the basis given by (10), $A(\nu)$ is approximately equal to the amplitude A_M of the model function y , which is actually given by

$$A_M = \sqrt{c_2^2 + c_3^2}. \quad (15)$$

Then, the determination of the maxima of $A(\nu)$ gives us not only the location of the frequencies of the periodic components of the motion but also an estimation of the amplitude of these sinusoid components. It is important to stress that $A(\nu)$ and A_M can be very different for low frequencies or when working with a small data set and it is a conceptual error to look for the maximization of A_M . The statistical significance and the minimum residual fit are related to $A(\nu)$ and not to A_M (see Ferraz-Mello 1981, Foster 1995, 1996).

Following this procedure, we solve system (6) for each ν obtaining the coefficients (c_1, c_2, c_3) and defining a component of motion with amplitude given by (15) and centered at c_1 . Note that $c_1 \neq (x)$. The best fit is obtained looking for the maximization of Eq.(13) but the amplitude of the model function is strictly given by Eq.(15). The amplitude given by the method is the one of the best fit and we may expect that some difference with the amplitude observed by plotting the data $x(t_i)$ exist. In order to reduce the side lobes of the peaks of $A(\nu)$, produced by the leakage, we use a Hanning window vector centered at $t = r$:

$$w(t) = \begin{cases} b(1 + \cos(\frac{2\pi(t-\tau)}{T_w})) & \text{if } |t - \tau| \leq T_w/2 \\ 0 & \text{otherwise} \end{cases} \quad (16)$$

where b is taken so that condition (8) is verified and T_w is the time width of the window. The local maxima of $A(\nu)$ are determined using subroutines from Press et al. (1992). The values of $A(\nu)$ depends on the chosen T_w which is fixed so that the principal peak of $A(\nu)$ is as great as possible compatible with the leakage produced. Using the same T_w and varying r we obtain the time evolution of these peaks, that is, the time evolution of the frequency (ν), of the amplitude (A_M) and of the center (c_1) of this periodic component of the motion.

3 TIME EVOLUTION OF THE LOCAL BEHAVIOR

The most representative and well-known feature in the time evolution of the semi-major axis and eccentricity of a resonant particle are the presence of an oscillation with period equal to the libration period of the critical angle, α . More precisely, the small amplitude libration theories for the planar restricted three body problem with an elliptic orbit for the planet (P) and

averaged over the synodic angle show that the first terms of the solutions for the resonant motion are represented by a constant (the libration center or *first* forced mode) plus two periodic terms. These terms are sine or cosine functions, one with frequency ν_l (the free libration) and the other with a small frequency ν_f (the second forced mode) equal to the circulation frequency of the angle ($w - \varpi_P$). So, these solutions must have only two peaks in the spectra $A(\nu)$, at ν_f and ν_l respectively. But, numerical integrations of the exact equations of the same problem show the existence of the lines $(\nu_l \pm N\nu_f)$ with N small, near the line ν_l in the spectra of the variables like a , e and the critical angle α (Michtchenko and Ferraz-Mello 1993, Gallardo and Ferraz-Mello 1995). This is an expected result because an integrable system with two degrees of freedom has two fundamental frequencies (ν_f and ν_l) but their linear combinations can also be present in any set of chosen variables in which the two degrees of freedom are not separated. In this last (idealized) case, one angle-variable circulates with frequency ν_f and the other circulates with frequency ν_l (Laskar 1990).

To see the effect of the lines $(\nu_l \pm N\nu_f)$ in the libration motion we first consider a function given by the sum of two components with two close frequencies ν and $\nu + \delta\nu$:

$$A e^{i2\pi\nu t} + B e^{i2\pi(\nu+\delta\nu)t} = (A + B e^{i2\pi\delta\nu t}) e^{i2\pi\nu t} \quad (17)$$

where A and B are complex constants. We can write:

$$A + B e^{i2\pi\delta\nu t} = C(t) e^{i\Phi(t)} \quad (18)$$

where $C(t)$ and $\Phi(t)$ are real periodic functions with frequency $\delta\nu$. The instantaneous amplitude $C(t)$ is a periodic function with frequency $\delta\nu$, and the instantaneous frequency is:

$$\nu_i = \nu + \frac{1}{2\pi} \frac{d\Phi}{dt} \quad (19)$$

which is also a periodic function with frequency $\delta\nu$. Then, the presence of the lines $(\nu_l \pm \nu_f)$ means that the instantaneous amplitude and frequency of the libration motion of variables like a , e and α have oscillations with frequency equal to ν_f . These oscillations in the libration period were already pointed out by Gallardo and Ferraz-Mello (1995) where a rough method to estimate these variations is also given for the case $\nu_f/\nu_l \rightarrow 0$. This behaviour is due to the fact that the instantaneous libration does not happen around a fixed

point (the *libration center*) but around a moving point of the phase space that follows the *second forced mode*. It is important to stress that the lines ($\nu_l \pm N\nu_f$) in the spectra cannot be obtained by adding two periodic terms with fixed frequencies ν_f and y , but adding one term with a fixed frequency ν_f and other with a frequency $\nu_l(t)$ which oscillates with frequency ν_f .

When considering the spatial case and the action of other perturbing planets, the complexity of the spectra near the libration line increases. The strategy adopted here is to include all these lines close to ν_l into the libration motion. The result is an oscillating $\nu_l(t)$. In a stable resonant motion, these oscillations must be periodic and related to the other frequencies of the motion. If the motion is chaotic, there are not fundamental fixed frequencies and the behaviour of the time evolution of the libration will be more complex. Here we analyse the time evolution of the libration in order to obtain these oscillations. From this analysis we can conclude whether the motion does have or not evidence of instabilities.

We will illustrate with three examples of resonant motion outside the orbit of Neptune. We integrated numerically the three dimensional outer Solar System taking the initial conditions from Cohen et al. (1973) and the invariable plane of the Solar System as reference. The mass of the inner planets was added to the Sun. We used the RA15 integrator with $L = 10$ (Everhart, 1985), and the results are expressed in a heliocentric frame.

3.1 Resonance 2:3. The case of Pluto.

We integrated for 100 Myrs the four outer planets and a massless Pluto with initial conditions from Cohen et al. (1973). It is known that Pluto is locked in the resonance 2:3 and the critical angle

$$\sigma = 3\lambda - 2\lambda_N - \varpi \quad (20)$$

librates around $\sigma_0 = 180^\circ$ with an amplitude of approximately 80° and a period of approximately 20.000 yrs. Taking the total time span of the numerical integration, using $T_w = 100$ Myrs and $\tau = 50$ Myrs, we obtained the spectrum of a near the libration line (Fig.1a). This figure can be compared with Fig.9 of Milani et al. (1989). They are not exactly equal due to small differences in the constants and initial conditions. We can see in Fig.1a several clusters of lines and it is hard to find a unique line for the libration frequency.

Table 1: Frequencies which are present in the libration motion of Pluto obtained by different ways. Unit in 1/Myrs.

freq.	Fig.1a	Fig.3	phenomenon
ν_1	50.3560	50.3559	libration of σ
ν_2	hidden	0.7903	circulation of $\Delta\varpi$
ν_3	0.0294	0.0294	super-resonance 1:1
ν_4	0.2635	0.2631	libr. w , circ. $\Delta\Omega$

Now, if we take $T_w = 1$ Myrs centered at $\tau = 50$ Myrs we obtain the poorly resolved spectra of Fig.1b. Nevertheless, we can find the frequency of maximum amplitude (ν_{lib}), the amplitude A_M given by the model for ν_{lib} and the constant c_1 which is the center around which the motion happens in that time span. These values are the ones that best reproduce the *local* behaviour of the libration motion at $\tau = 50$ Myrs and in a time-scale of 1 Myrs. The amplitude is well reproduced but there is a poor resolution in frequency due to the small time interval employed. The most important frequencies present in Fig.1a are in fact present but unresolved in Fig.1b. Now, using the same T_w and varying τ from 0.5 to 99.5 Myrs we obtain the time evolution of ν_{lib} , c_1 and A_M (Fig.2). In Fig.2a we have the evolution of the libration frequency as it can be determined in time scales given by T_w . It is not constant but oscillates with a regular trend. The local (or instantaneous) center of the libration, c_1 (Fig.2b), is not fixed and shows regular oscillations. We must distinguish the position of the center of the libration in the time interval T_w , which is a function $c_1(\tau)$ and the 'libration center' which is a constant and, in fact, equal to the center of the oscillations of $c_1(\tau)$. For the sake of avoiding confusion the word *local* is used throughout this paper to indicate the parameters corresponding to the interval T_w . The amplitude given by the model (Fig.2c) oscillates around a mean value of 79.3 degrees. All the frequencies that appear in Fig.1a will appear in this time evolution because the values ν_{lib} , c_1 and A_M obtained at each instant τ are affected by the frequencies located inside the curve of Fig.1b.

Fig.3 gives the spectra of the time evolution of ν_{lib} , c_1 and A_M on the whole time span. One may see that there is no leakage in frequency ~ 0 because in the definition of $A(\nu)$ there is no contamination with the projection of $\$$ into the vector 1. The value ν_1 is obtained as the constant

term of the model in the analysis of $\nu_{lib}(\tau)$. The libration center is the constant term in the analysis of $c_1(\tau)$ and is $\sigma_0 = 179.9992$. In the spectra of the local libration frequency and center, the lines of the libration period of the argument of the perihelion, w , and circulation of $Aw = w - \varpi_N$ are well defined. The amplitude of the model shows a more noisy spectrum, but without high amplitude terms. Using the notation of Milani et al. (1989) we show in Table 1 the frequencies ν_1 (libration of u), ν_2 (circulation of Aw), ν_3 (super-resonance 1:1) and ν_4 (libration of $w =$ circulation of $AR = R - \Omega_N$) as deduced from the complete spectra of Fig.1a and from the spectra of Fig.3a (which give us the same lines of Fig.3b). Milani's values are nearly equal to those of the Table and the discrepancies are mainly due to our differences in the integration. From Fig.1a we deduced the frequency ν_1 as the one of maximum amplitude. Frequency ν_4 is the separation between the different clusters, and ν_3 is the difference between the lines inside the clusters (Milani found that the lines in each cluster are separated not by ν_3 , but $\nu_3/2$). As can be seen in the Table, all the features in Fig.1a are well reproduced in Fig.3 and the frequency ν_2 , which is hidden in Fig.1a, also appears. In Fig.3a, the line $2\nu_4$ is also important.

Kinoshita and Nakai (1995), in a 5.5 Gyrs integration of a massless Pluto obtained a libration for the angle $\theta = Aw + 3AR$ with a period of 570 Myrs. From our results, we obtain $\nu_2 - 3\nu_4 \approx 0.0010$, which is consistent with a libration.

The main lines in the spectra of $\nu_{lib}(\tau)$ and $c_1(\tau)$ are ν_4 and ν_2 , so the most important variations in the libration period are due to the Kozai resonance (Morbidelli et al. 1995) and to the second forced mode. We can say that the libration motion of Pluto is the result of the addition of several lines or we can say that Pluto has an instantaneous libration frequency that oscillates around a mean value. In fact these oscillations are implicit in Eq.(5) of Milani et al. (1989).

As ν_{lib} and c_1 have a very clean spectra with well defined periodic components and considering that there is no secular increase in the libration amplitude, A_M , we conclude that the motion of Pluto has no indication of instabilities in time-scales of 100 Myrs. This does not mean regular motion. On the contrary, looking at Fig.2c and Fig.3c it is clear there are other components with amplitude of the order of some 10^{-2} degrees in the libration amplitude, not due to errors in the spectral analysis, which in this case are of the order of 10^{-5} degrees, which can be related to chaos. The evolution of the constant term $c_1(\tau)$ for the semi-major axis and eccentricity do not show any secular increase, in agreement with a long term stability.

From this analysis we conclude that the spectra of Fig.3 contains more information than Fig.1 (the frequency ν_2 , for example) and that the time evolution of ν_{lib} , c_1 and A_M gives us details of how the libration motion evolves. If we take smaller values for T_w , small scale features appear in Fig.2 and more high frequency lines in the spectra of Fig.3. It is worth noting that correlation between the *local* libration center and *local* libration frequency (Fig.4) exists.

3.2 Stable motion in the resonance 3:5.

Stable resonant orbits at very high eccentricities can exist in the region of the Kuiper disc. Fernández and Gallardo (1996) obtained the maximum limits in the eccentricity for small inclination orbits at several exterior resonances with Neptune. We followed up to 100 Myrs the evolution of a particle librating with small amplitude in the 3:5 resonance with Neptune. Its eccentricity remains almost constant in 0.34. In Fig.5a we show the first 80 Myrs of the evolution of the libration frequency as deduced from the analysis of the critical angle

$$\sigma = \frac{5}{2}\lambda - \frac{3}{2}\lambda_N - \varpi \quad (21)$$

and using a window with $T_w = 1$ Myr. The evolution of the local libration center c_1 and the amplitude A_M of the libration line are showed in Fig.5b-c. The spectra of $\nu_{lib}(\tau)$, $c_1(\tau)$ and $A_M(\tau)$ are shown in Fig.6a-c. Except for $A_M(\tau)$ which has very long period components, the others have a very regular evolution that is confirmed by the spectra of Fig.6a-b. The most prominent line in Fig.6a-b corresponds to the frequency of the second forced mode, ν_f , that is the circulation frequency of $\mathbf{A}w$. There exists a correlation between local libration frequency and center that is showed in Fig.7. From this analysis we can conclude that this particle has also a very stable motion in time-scales of 100 Myrs.

3.3 Chaotic motion in the resonance 1:2.

Fernández and Gallardo (1996) also studied this resonance and found stable orbits at high eccentricities. They argued that it is possible that a cluster of ten discovered bodies located approximately at 70° ahead of Neptune are trapped in high eccentric orbits with small amplitude librations in this resonance. The phase space for this resonance in the variables $k = e \cos a$ and $h = e \sin a$ where

$$\sigma = 2\lambda - \lambda_N - \varpi, \quad (22)$$

can be found in Beaugé (1994, Fig.1). Here, we reproduce that figure in Fig.8. It has asymmetric librations which means that the libration center is not at 180° but at a value σ_0 that, unlike other resonances, depends on the eccentricity. For high amplitude librations, we expect a chaotic behaviour because the trajectories will occur near the separatrix Γ_1 on Fig.8 and a small perturbation can produce intermittences involving libration around the center located in the subspace $h > 0$ and the one located in the subspace $h < 0$. Also, trajectories can occur temporarily between both separatrices Γ_1 and Γ_2 . This phenomena is well illustrated in the numerical integration of a particle with small inclination (always less than 0.7 degrees) and eccentricity ($e < 0.1$) but with high libration amplitude. We show in Fig.9 the time evolution of the local libration frequency, center and amplitude obtained using a window with $T_w = 0.4$ Myrs. In Fig.9d we plot the evolution of the critical angle, $\sigma(t)$. For the sake of allowing a comparison with the results of the time-frequency analysis, we superpose with this plot the graphs of the local center $c_1(\tau)$ and the libration limits $c_1 \pm A_M$ obtained from Fig.9b-c. All the features in $\sigma(t)$ are well represented. The particle starts with a trajectory with high amplitude and low frequency located between the separatrices, so the libration center is near 180° . At $t = 0.2$ Myrs it is captured in a high frequency libration around $\sim 120^\circ$ and starts a transition to a libration center at $\sim 240^\circ$. At $t = 1.6$ Myrs a trajectory between both separatrices as at the beginning of the integration librating around 180° with high amplitude and so on. This example illustrates an application of the time-frequency analysis to an highly chaotic trajectory. In situations like this one must chose a small T_w to obtain the small scale features in $\sigma(t)$. In this circumstance, when we have a few cycles (or even a fraction of cycles) of a, the methods based in DCDFT and CLEANEST are those given the best results.

4 DISCUSSION

The libration motion can be more complex in the numerical integrations than in the analytical approximations made obtained for small amplitude librations and using a simplified perturbing function. In order to study the evolution of this component of the motion, we propose not to analyse the well resolved libration line in the spectrum (which could not exist), but the time evolution of the poorly resolved feature appearing, in the place of the libration line, when using a small time interval. This is done fitting the data

to a model given by a constant plus sinusoid (Ferraz-Mello, 1981; Foster, 1995) and following it in time using a small-width and time-dependent window. After obtaining the time evolution of the ν_{lib} , c_1 and A_M we analyze them calculating new spectra and looking for sharp frequencies, secular trends or chaotic components. The method is also valid for highly chaotic motion.

This method is an alternative way of analyzing the output of a numerical integration. The resolution in frequency in Fig.3 is the same as in Fig.1a for example, so there is no increase in the frequency resolution. But Fig.2 and 3 give us a more complete and ordered picture of the librating motion. Features hidden in a unique and well resolved spectrum of $\sigma(t)$ can appear in the spectra of the time evolution of the local behaviour.

The use of a low-pass filter to eliminate short period terms with frequency greater than the Nyquist frequency is recommendable but, is not always necessary. If high frequency lines are aliased near the libration line their aliases can be identified in the spectra of $\nu_{lib}(\tau)$, $c_1(\tau)$ and $A_M(\tau)$.

In the examples given, we have analyzed the libration line of the critical angle but the same analysis can be done for other lines in the spectrum and for other variables like semi-major axis, eccentricity and also other angular variables like A_w for example, as done by Michtchenko and Nesvorný (1995) with the wavelet transforms.

In the case of Pluto, we found an stable evolution but with a small chaotic component in the libration amplitude. The libration is affected by several frequencies and the angle $\theta = A_w + 3AR$ is consistent with a libration. In the example of the resonance 3:5 we followed a very stable evolution. The libration amplitude shows a chaotic behaviour but without compromising the stability. We found a highly chaotic behaviour at small excentricities in the resonance 1:2. This chaos is related to the apparition of the asymmetric libration centers.

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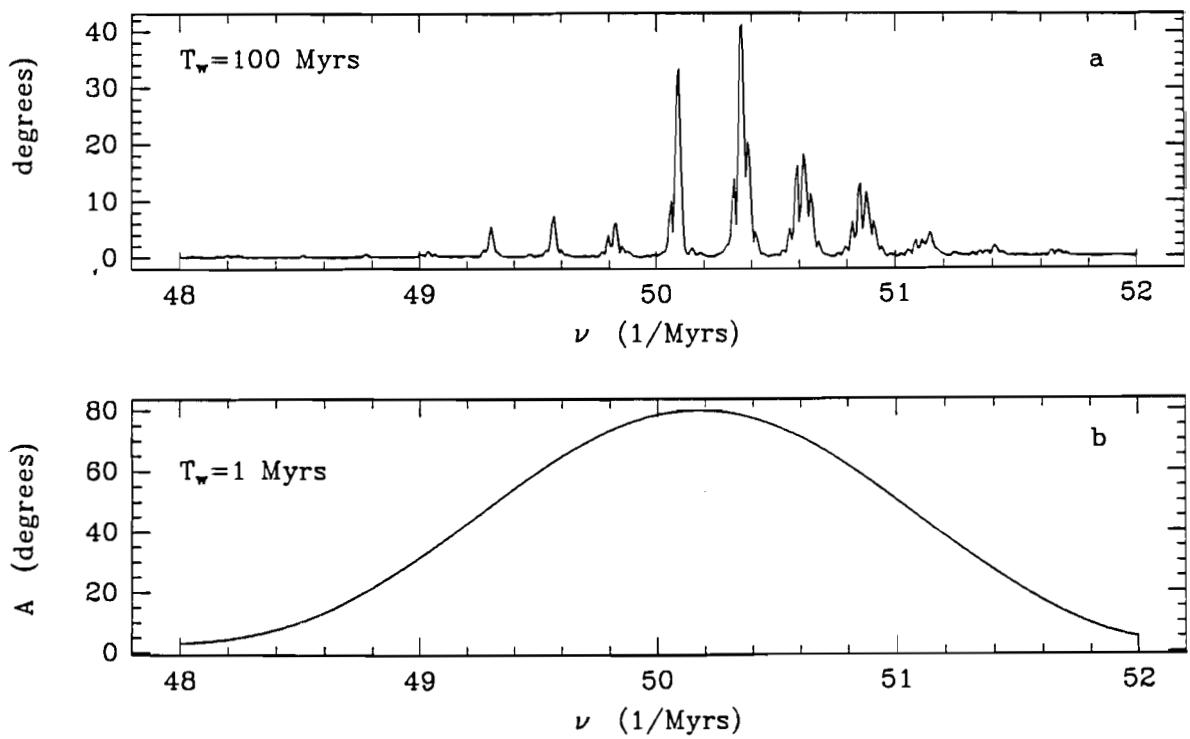


Figura 1: Pluto.(a):spectrum of $\sigma(t)$ obtained with 100 Myrs of a numerical integration. (b):The same region of the spectrum using $T_w = 1$ Myrs and $\tau = 50$ Myrs. Leakage covers a region approximately equal to $4/T_w$ in the frequency axis.

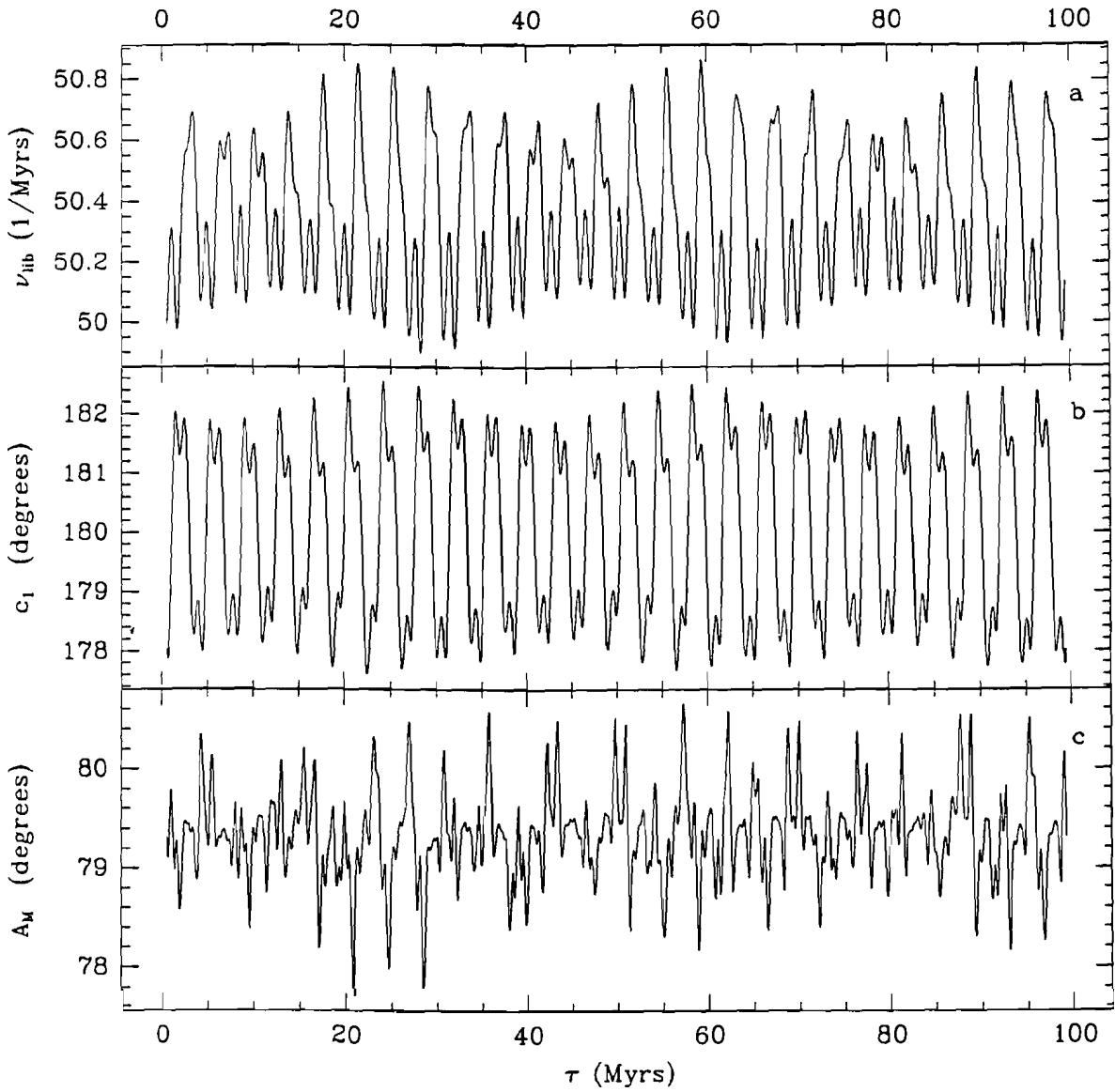


Figura 2: Libration frequency (a), local center of the libration (b) and libration amplitude (c) deduced from spectra like Fig.1b but with r varying from 0.5 to 99.5 Myrs. See the differences in the evolution of A_M compared with ν_{lib} and c_1 .

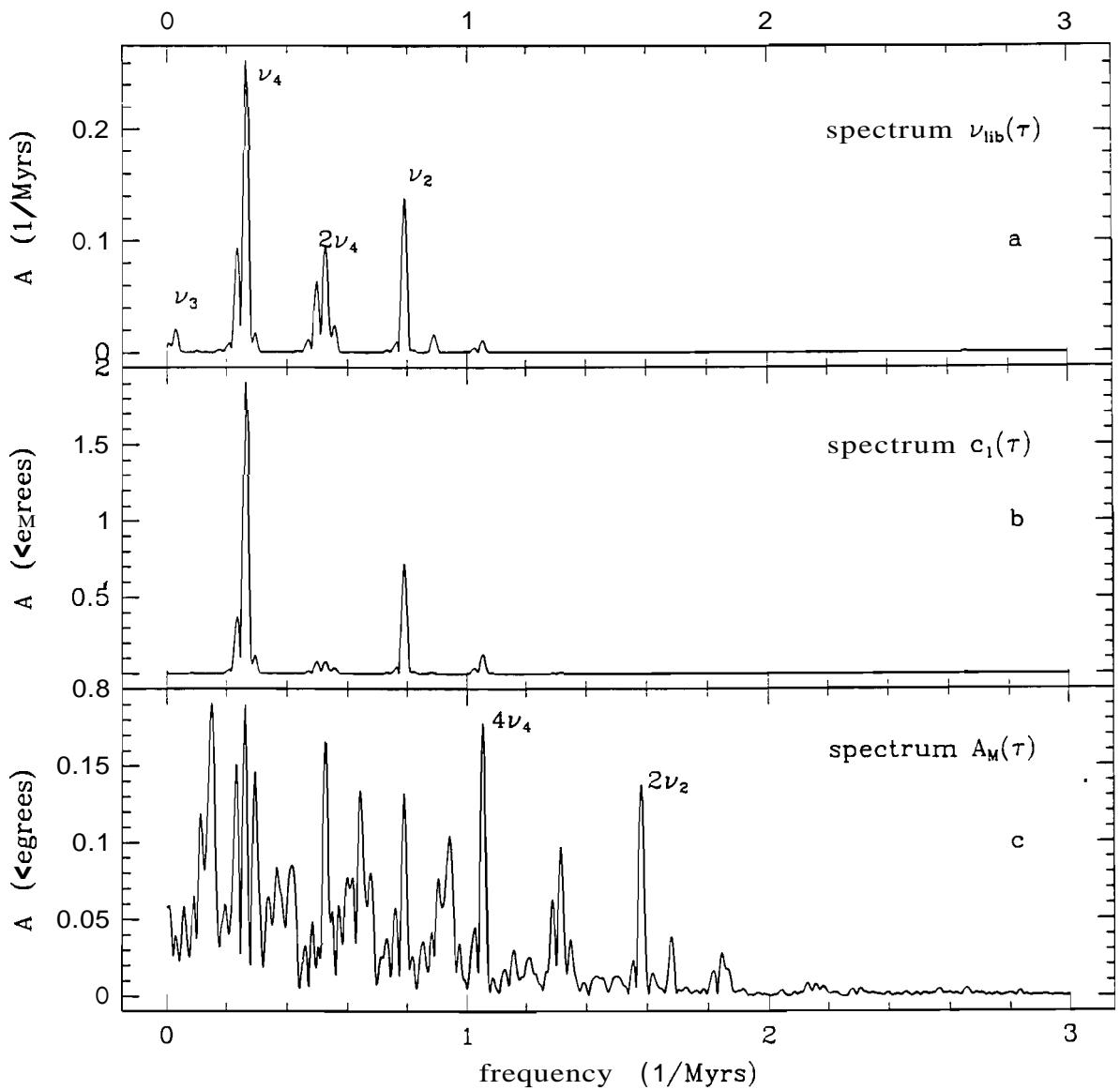


Figura 3: Spectra of Fig.2a–c. The most important lines in (a) and (b) are ν_4 (libration of ω) and ν_2 (circulation of A). It is possible to see in (c) the same frequencies and its harmonics.

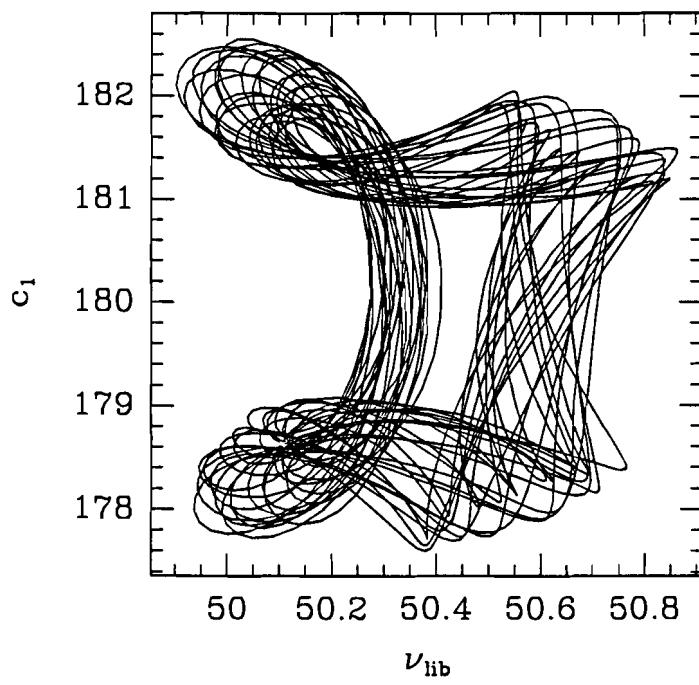


Figura 4: Correlation between the instantaneous (or local) center of the libration and the frequency of the libration for planet Pluto.

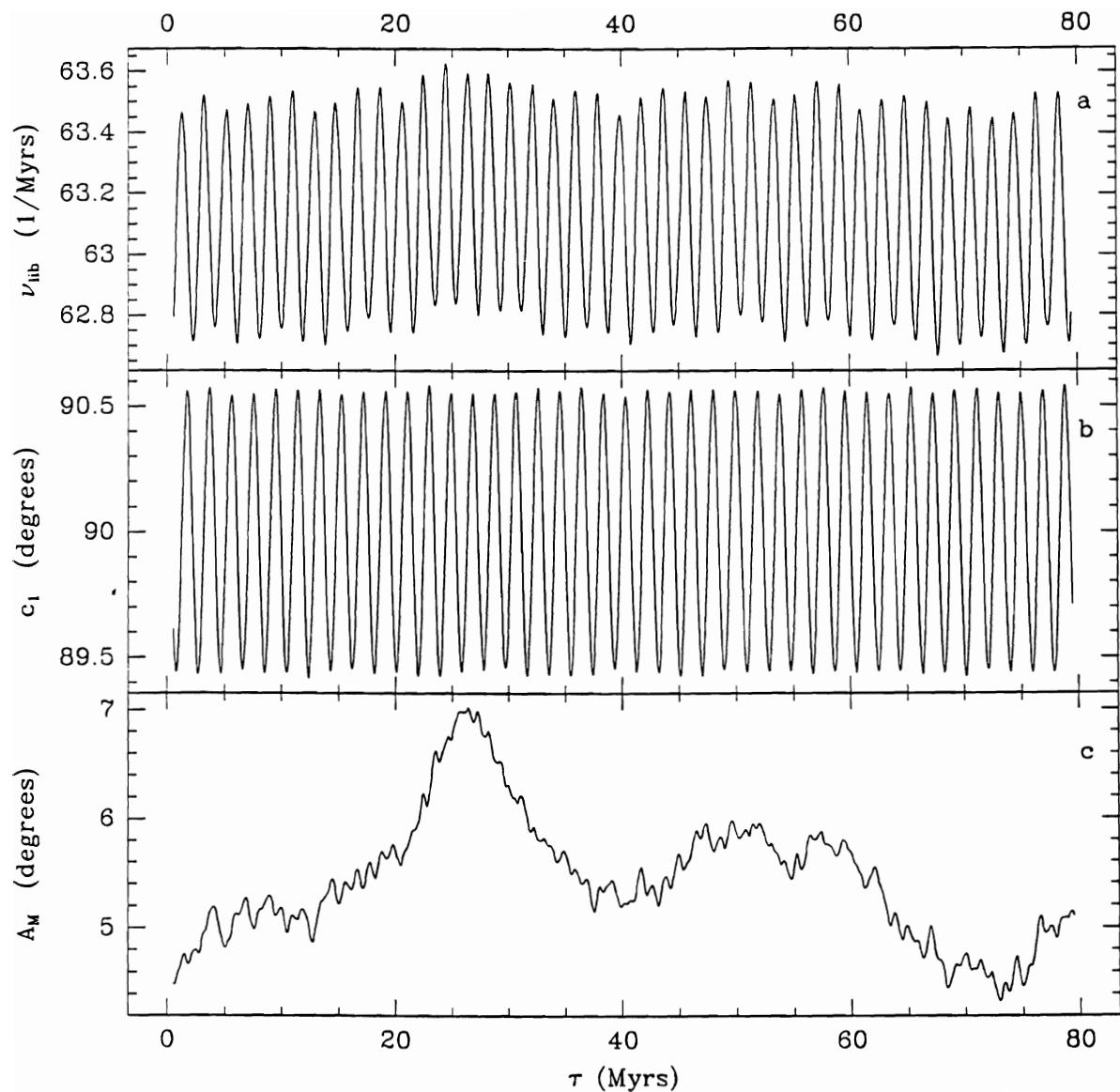


Figura 5: Stable evolution at high eccentricities in the resonance 3:5. Same as Fig.2 for a small amplitude librations of a particle in the resonance 3:5 exterior to Neptune. In this case was used a window with $T_w = 1$ Myr.

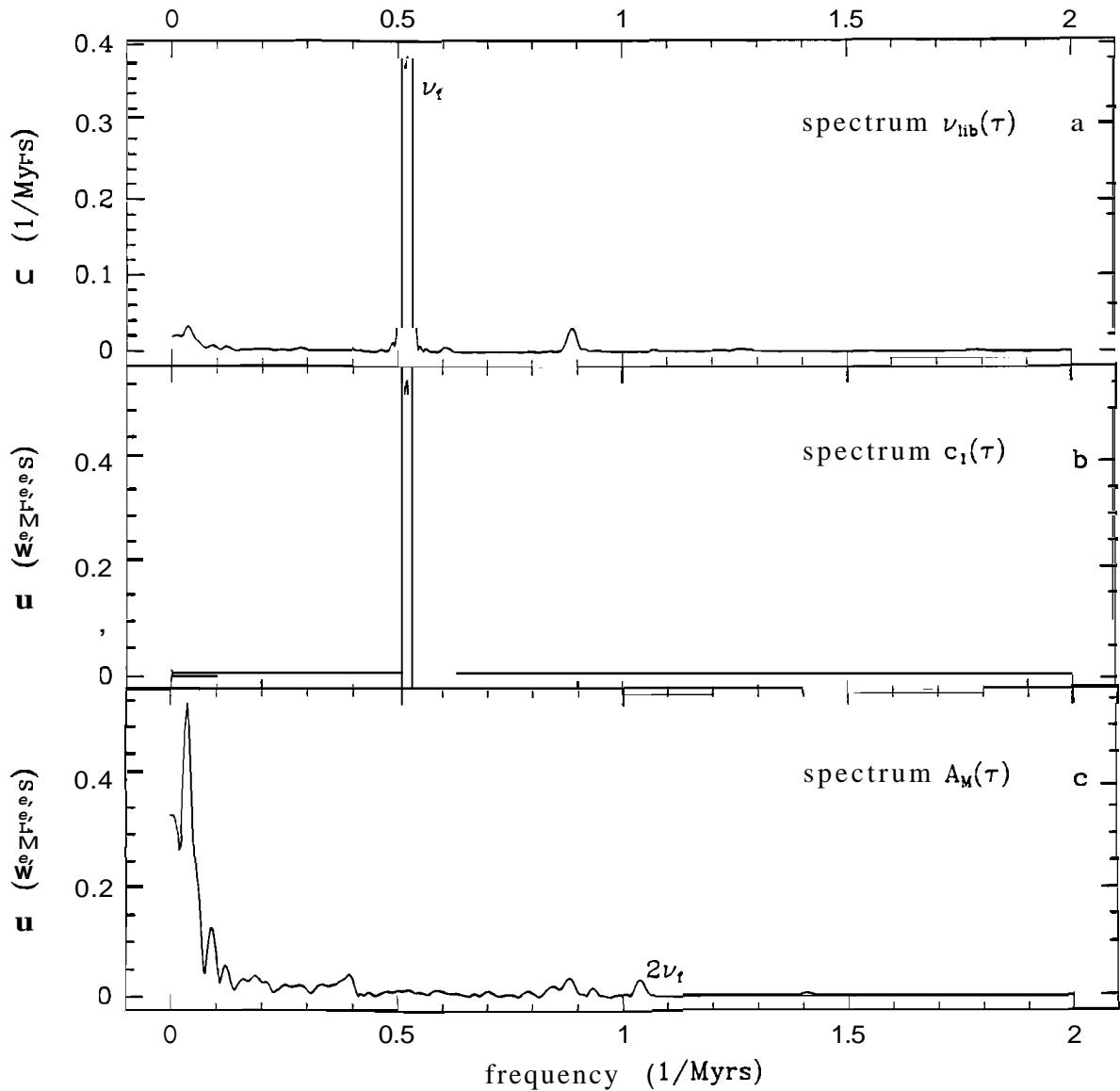


Figura 6: Spectra of Fig.5. The prominent line in the spectra of $\nu_{lib}(\tau)$ and $c_1(\tau)$ is due to ν_f , the circulation frequency of $\Delta\varpi$. In (c) this frequency is not present but it appears its first harmonic near 1.04 Myrs^{-1} with small amplitude. The most important line in the spectrum of A_M corresponds with the period of transition from circulation to libration around 180° of the angle AR.

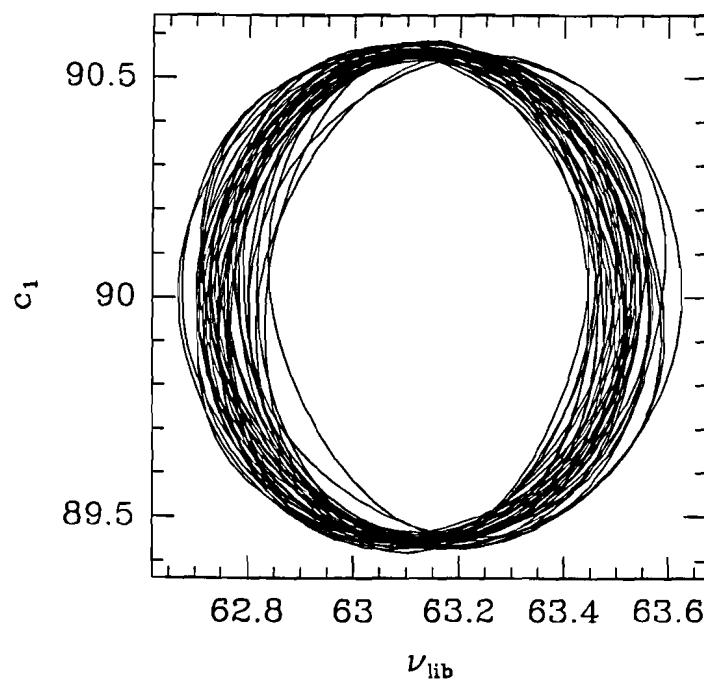


Figura 7: Correlation between the instantaneous (or local) center of the libration and the frequency of the libration for the particle in the **resonance 3:5**.

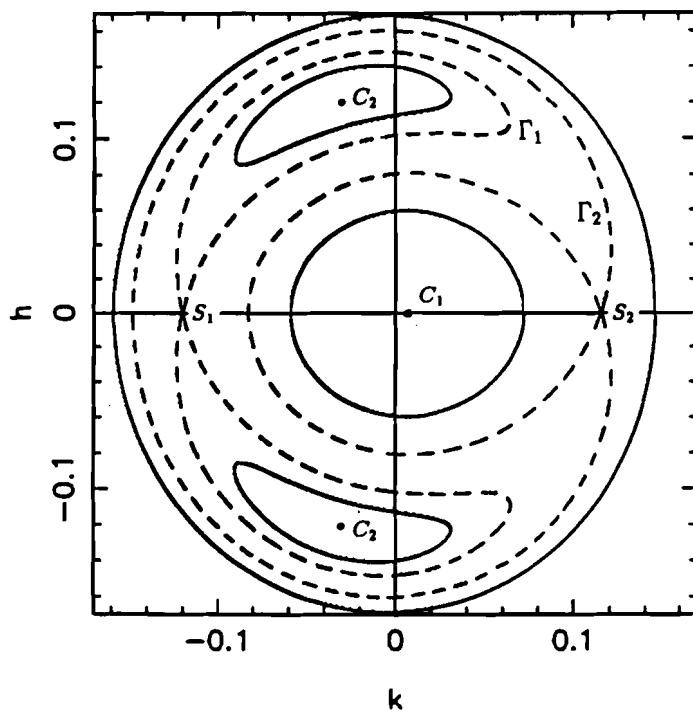


Figura 8: The phase space of the **resonance 1:2** for a certain energy level. Simplified plot indicating the separatrices (dashed lines), stable libration centers (C_i) and unstable equilibrium points (S_i). (From Beaugé, 1994).

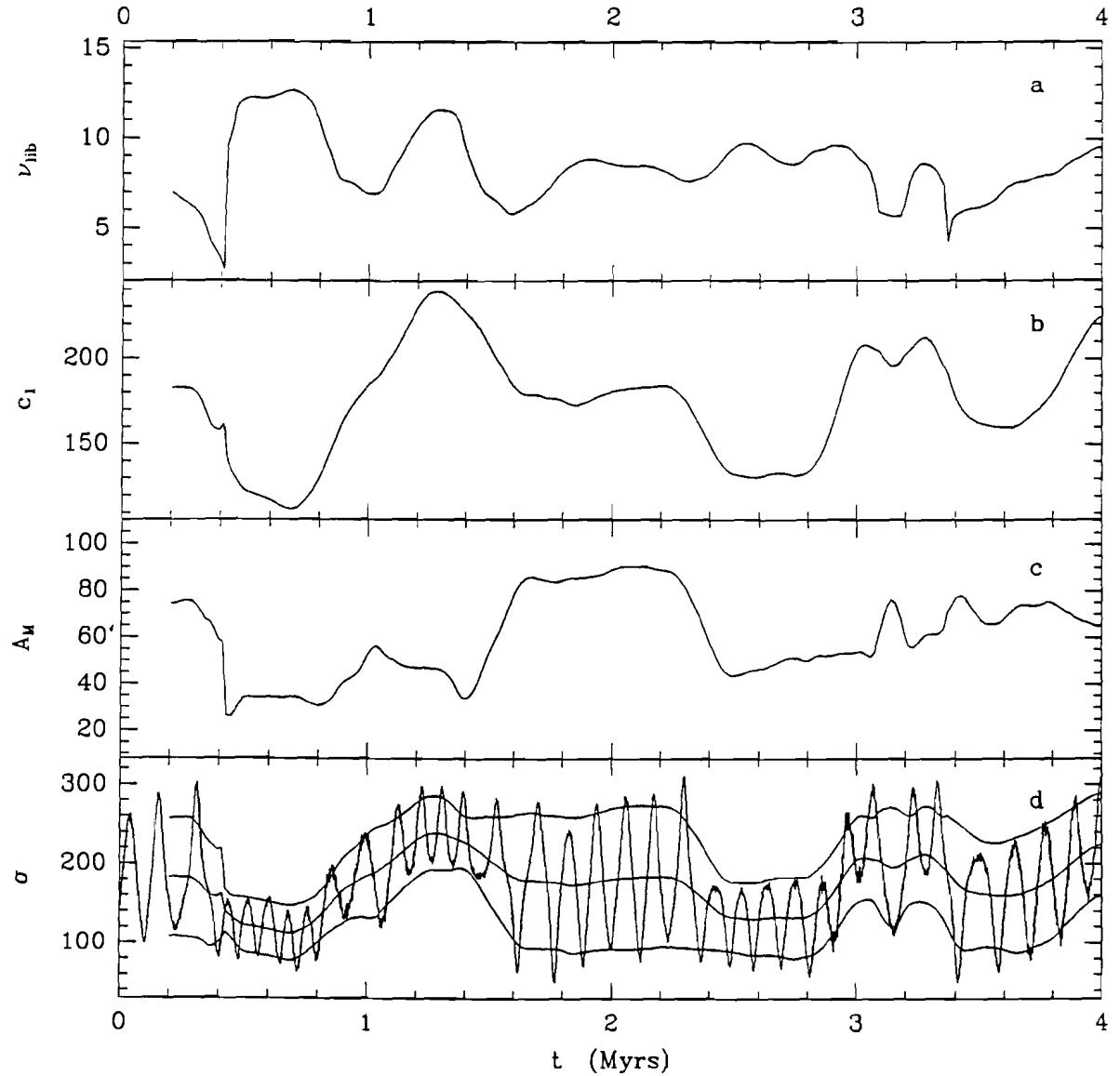


Figura 9: Chaotic trajectory at small eccentricities in the resonance 1:2. (a): $\nu_{lib}(\tau)$, (b): $c_1(\tau)$, (c): $A_M(\tau)$, (d): the critical angle $\sigma(t)$ (points) and also $c_1(\tau)$ and $c_1 \pm A_M$ for comparison. There are intermittence between libration centers located at $\sigma_0 > 180^\circ$ and $\sigma_0 < 180^\circ$. Was used a window with $T_w = 0.4$ Myrs.

Anexo III

On the origin of the discovered trans-neptunian objects

ON THE ORIGIN OF THE DISCOVERED TRANS-NEPTUNIAN OBJECTS

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Abstract: We study the dynamical properties **and** possible origins of the trans-Neptunian population. We argue that its current dynamical structure may be related to the final stages of the accretion of Uranus and Neptune, that were probably accompanied by the radial expansion of their orbits and ejection of a substantial amount of **solid** matter (Fernández and Ip 1984). The inner regions of the Kuiper disk (**e.g.**, from 30 AU to about 100 AU) may **have** suffered an intense bombardment of bodies scattered outwards by the accreting Jovian planets, thus disrupting the regular structure of near-circular, near-coplanar orbits of Kuiper-disk objects (KDOs). In the process, the KDO population **was** severely depleted, so that its current mass may be about 3 orders of magnitude smaller than the primordial one. The residual trans-Neptunian population may **have** mingled with bodies driven away through resonances by the outwardly displacing proto-Uranus and proto-Neptune (Malhotra 1995a). Most of the detected trans-Neptunian objects probably move in low-order exterior resonant orbits, such as 2:3 (like Pluto's). We perform numerical integrations of bodies moving in the 2:3 and other low-order exterior resonances that suggest a long dynamical stability even for high eccentricities. Most of the observed KDOs **have positions relative** to Neptune in agreement with those provided by test particles in 1:2, 2:3 **and/or** 3:4 resonances, which supports the prevalence of resonant motion among the discovered KDOs.

1 The trans-Neptunian population: Theoretical background and observations

The existence of a trans-Neptunian population **was** discussed by Kuiper (1951) on the grounds that it could correspond to the outer portions of the protoplanetary disk, too distant to accumulate into a large planet or be swept by the accreting proto-Neptune. Hamid *et al.* (1968) estimated

an upper limit of about one M_{\oplus} (Earth's mass) for the total mass of the disk between 35 and 50 AU, based on the lack of observed perturbations on Halley's orbit. Fernández (1980) argued that such a disk may be the source of short-period comets. Duncan *et al.* (1988) showed that an origin of the Jupiter-family comets in the Kuiper disk would lead to a flat distribution of inclinations in agreement with the observed one, whereas an origin from an isotropic source (the Oort cloud) would leave a fraction of short-period comets in retrograde orbits. The observation of nebular disks around other young stars lend support to the idea that our own Solar System might have had an extended disk of particulate matter. Recent Hubble Space Telescope observations of young stars in the Orion nebula show that planetary disks are a quite common byproduct of star formation, having been detected in 50% of the surveyed sample of 110 stars (O'Dell and Wen 1994).

A major step forward was achieved in 1992 when the first trans-Neptunian object – 1992QB1 – was discovered by Jewitt and Luu (1992, 1993) after years of unsuccessful searches by several teams (e.g. Luu and Jewitt 1988, Kowal 1989, Levison and Duncan 1990). Thirty KDOs have been discovered through January'95, suggesting that we are before a fast-growing population. The KDOs have heliocentric distances between 29 and 47 AU and a flat distribution of inclinations (Fig.1). The Kuiper belt (or disk, since it might well extend to large heliocentric distances) is no longer a hypothesis, but already part of our Solar System.

Jewitt and Luu (1995) have recently presented a comprehensive review of the discovered KDOs. They estimate that about 35,000 KDOs with diameters ≥ 100 km exist between 30 and 50 AU; their total mass may be about 1.8×10^{22} kg or $\sim 0.003M_{\oplus}$. Some more mass may be hidden in smaller bodies, though it is difficult that it can alter the above result by more than an order of magnitude (Weissman 1995). So a total mass of a few $\sim 0.01M_{\oplus}$ may be a reasonable estimate for the considered region. Such a population may be about two orders of magnitude larger than the corresponding population of main-belt asteroids. Even though Jewitt's and Luu's results may partly depend on the assumed origin and dynamical structure of the disk (see below), the total solid mass enclosed there seems to be a tiny fraction of the mass that could have been present in the early protoplanetary disk. For example Bailey (1994) assumes a density falloff of solid matter in the nebula $\propto r^{-3/2} - r^{-2}$ with the formation of 100-km-size planetesimals in the 30–50 AU zone, in agreement with the size of the KDOs. Bailey's estimate leads to a total primordial mass of solid matter in the 30–50 AU zone of a few

tens M_{\oplus} , i.e. about 10^3 times larger than the current mass.

2 Resonances in the Kuiper disk

The dynamical stability of bodies in the trans–Neptunian region up to ~ 100 AU has been the subject of several recent studies (e.g. Torbett 1989, Levison and Duncan 1993, Holman and Wisdom 1993, Morbidelli et al. 1995, Malhotra 1995b, Duncan et al. 1995). According to these studies, KDOs up to ~ 45 AU can be transferred to Neptune–crossing orbits through chaotic motion in time scales of $10^7 - 10^9$ yr. Torbett (1989) showed that the stability strongly depends on the initial eccentricity and inclination. Thus, objects in near-circular, low-inclination orbits would remain stable over time scales comparable to the Solar System age for semimajor axes $a \gtrsim 45$ AU (Torbett 1989, Levison and Duncan 1993, Duncan et al. 1995).

By adopting the initial conditions from Cohen et al. (1973), we have performed numerical integrations of the Outer Solar System over periods of 10 to 100 Myrs. In order to find regions of stability, we have followed the dynamical evolution of test particles located at several first and second order mean-motion exterior resonances with Neptune with initial zero inclination relative to the invariable plane. For studies of resonant motion it is convenient to introduce the critical angle α . For the $(p+q) : p$ resonance it is given by

$$\sigma = (p/q + 1)\lambda_N - p/q\lambda - \varpi \quad (1)$$

where p and q are small integers, being p negative for exterior resonances and q is the order of the resonance.

In the analytical solutions of the restricted circular three–body problem there is a family of stationary solutions (a_*, e_*, α_*) for the resonant motion called libration centers. When the orbit of the planet is assumed elliptic the resonant motion has two well-known components: a free oscillation (or libration) around the libration center and a forced component (or second forced mode) induced by Neptune’s eccentricity with a period equal to the circulation period of $(\varpi - \varpi_N)$ (Ferraz–Mello 1988, Gallardo and Ferraz–Mello 1995). In the solutions of our numerical integrations there are also other components due to the other perturber planets and to the introduction of the inclination. When we calculate the power spectra of a , a and e , besides the frequencies of the libration and the second forced mode, all resonant

particles exhibit short period components which are due to the perturbations by Jupiter, Saturn and Uranus. We find the sum of the amplitudes of short period terms in a are proportional to the semimajor axis of the test particles and the maximum values of these sums are found to go from 0.009 for the 4:5 resonance to 0.019 for the 1:2 resonance (in units of a_N). The amplitudes of these short period perturbations are also proportional to the eccentricity. These amplitudes are in general greater than the libration amplitude of a , but due to the very small period of the perturbation they do not disturb the resonant motion if the particle is in a deep region of the resonance.

The resonant term of the perturber potential is proportional to e^q so at very small eccentricities the resonance is shallow, and the libration can be easily disrupted by other perturbations. If the libration is broken and the resonant term is small the subsequent evolution is known to be chaotic in the region $a < 45$ AU, depending strongly on the existence of other mechanisms like secular resonances in the region of the phase space where the particle is located (Knežević *et al.* 1991, Morbidelli *et al.* 1995). As a general rule, e increases to the point that close encounters with Neptune become possible; which leads to the particle's ejection. For orbits of small eccentricity ($e \lesssim 0.05$), the perturbation in semimajor axis A_a required for a body to pass from a zero-amplitude libration to an amplitude such that the resonance breaks down or the body collides with Neptune, is more or less the same for several first-order resonances, but it is a few times larger than the corresponding perturbation for some second-order resonances (Fig.2). This effect would suggest that the first-order resonances are somewhat more stable to the perturbations of the other giant planets than the second-order ones.

At high eccentricities, particles with small amplitude librations are in deeper regions of the resonance, but we found in our numerical integrations the perturbations by Jupiter, Saturn and Uranus are approximately twice as large as those for small eccentricities, producing a progressive increase in libration amplitude until close encounters with Neptune become possible. It is clear that the instability at high eccentricities is not due to Neptune because in the case of the restricted three body problem it is possible to find stable librations up to much greater values of e (Malhotra 1995b, Gallardo and Ferraz-Mello 1996a). As was found by Gallardo and Ferraz-Mello (1996a) for the 2:3 resonance, in our numerical integrations, the main responsible for the disruption of the librations is Uranus. Consequently, for high eccentricities the region of the phase space with initial conditions compatible with

stable librations shrinks.

At moderate eccentricities, where the resonance is fairly deep and the short period perturbations are small, the libration in a can reach the greatest values without compromising the stability.

We determined the upper limits in e for stable librations using the time-frequency analysis method of Gallardo and Ferraz-Mello (1996b). We analyzed the time evolution of the libration frequency, libration amplitude and librational central values of a , a and e . All particles in resonant motion have oscillations in the time evolution of the frequency, amplitude and central values of the libration components. When these oscillations are periodic and related to the evolution of the other orbital elements, we considered the motion to be stable because these oscillations are always present in regular resonant motion and this behavior is predicted by the libration theory (Gallardo and Ferraz-Mello 1995). On the other hand, when we detected a systematic increase of the libration amplitude or unexpected components in the time evolution of the frequency we considered the motion to be unstable. As seen in Table I, bodies trapped in resonant motion with Neptune are allowed to increase their eccentricities to rather high values before the resonance becomes unstable. For stable librations, the minimum perihelion distances for first order resonances seems to be $q \sim 25.5$ AU and for second order resonances $q \sim 28.5$ AU. This is coherent with the analytical results of Fig.2 and with the already mentioned fact that first order resonances are more stable to external perturbations (in this case due mainly to Uranus) than second order ones.

For bodies trapped in the resonances we obtained very small oscillations in e (approximately 0.01). This is due to the narrowness of the librations trajectories in the space ($\cos a, e \sin \sigma$) which is a characteristic of the external resonances with Neptune (Gallardo, 1996).

In integrations for 20 Myr, we obtained several particles captured in the 4:5 resonance with eccentricities up to 0.22. Yet, for this particular resonance we cannot set a reliable limit to the eccentricity for stable librations because our test particles were frequently captured in the secular resonance ν_{18} and/or the Kozai resonance. The first one is a long period libration of $(O - \Omega_N)$ and produces high amplitude oscillations in inclination; the second one is a libration of ω which we found it produces high amplitude oscillations in the eccentricity. The complexity of the 4:5 resonance deserves a more rigorous study and we decided to remove it from Table I.

Table 1: Exterior resonances with Neptune. Upper limit of the eccentricity $e_{\text{,}}$, and minimum q for stable librations.

<i>res</i>	a	e_l	σ_c	e_{up}	q
3:4	1.21	0.17	180°	0.29	0.86
5:7	1.25	0.20	90°	0.23	0.96
2:3	1.31	0.23	180°	0.36	0.84
3:5	1.41	0.29	90°	0.34	0.93
1:2	1.59	0.37	±66°*	0.46	0.86

Notes: a and q are in units of the semimajor axis of Neptune; e_l is the limiting eccentricity of the body for crossing Neptune's orbit; u , is the central value of the librating critical angle.

* The value of u , for res. 1:2 depends on the eccentricity; the quoted values are for e_{up}

The locations of most of the discovered KDOs fall within the librating regions of the first-order resonances: (a) 2:3, (b) 3:4, and/or (c) 1:2 (Fig.3), which make them possible candidates to be in resonant orbits with Neptune. The cluster of 10 bodies located approximately 70° ahead of Neptune are well inside the only region where detection is favourable for high-eccentricity, 1:2 resonant bodies (Fig.3c). It is known this resonance has asymmetric librations (Beaugé 1994). For eccentricities between 0.15 and 0.75 the critical angle u librates around a value σ_c where $60^\circ < \sigma_c < 90^\circ$.

In summary, even though several numerical studies show that the inner region of the Kuiper disk quickly evolved under the perturbing action of the Jovian planets, bodies in mean-motion, low-order resonances with Neptune have a very stable evolution and could have survived through the present time, even at rather high eccentricities.

As counterpart, we followed up to 100 Myrs several particles in the 1:2 resonance ($a \sim 47.9$ AU) with initial $e \sim 0$. We found a region of width 0.3 AU in the initial semimajor axis of the particles which leads to $e \sim 0.15$ in 10^7 years. This region of instability was also found by Levison and Duncan (1993) and we found is realted to the negative branch of the law of structure of this resonance, which can be found in Beaugé (1994, fig 3b). A similar behaviour is also present at small eccentricities at the 2:3 resonance (Gallardo and Ferraz-Mello, 1996a). This effect at the 1:2 resonance is

important because it generates particles which contributed to the depletion of the region up to distances of $r \sim 55$ AU. We also analyzed the 1:3 resonance at $e \sim 0$ looking for another possible source of high eccentric particles but the motion there is very stable in time-scales of 100 Myrs. Then, the farthest mechanism generator of high eccentric orbits seems to be the 1:2 resonance.

3 Collisional depletion of the Kuiper disk

According to Torbett and Smoluchowski (1990) and Duncan et al. (1995) it seems to be a minimum q allowed for stable evolution, with value between 40 and 45 AU. We followed up to 100 Myrs 8 particles with $45 \leq a \leq 75$ AU, four with $q = 45$ AU and four with $q = 40$ AU. We analyzed the spectra of the semimajor axis and obtained a typical spectra of quasi-regular motion for the first group and for the second group we obtained two typical spectra of chaotic regime. Then, KDOs in near-circular, near-coplanar orbits at distances $r \gtrsim 45$ AU seem to be protected from planetary perturbations (with the exception of the 1:2 resonance). External perturbers, such as passing stars or giant molecular clouds, could only induce weak changes in relative energies and eccentricities of ~ 0.01 (Stern 1990, Brunini and Fernández 1995). But such an orderly-structured Kuiper disk could form only in a quiescent formation environment. In particular, we have to analyze what would be the outcome of a primordial heavy bombardment of the Kuiper disk in the 30–50 AU zone as regards to the orbital properties of the residuals KDOs. We will argue below that a strong depletion of the Kuiper disk should have occurred during the late stages of the formation of the outer planets, in the course of which the primordial Kuiper-disk population was partially replaced by bodies driven from inner regions through resonances.

Safronov (1969) argued that the formation of Uranus and Neptune was very inefficient, because once the accreting protoplanets acquired powerful gravitational fields, scattering of bodies became a much more likely event than mutual collisions. Fernández and Ip (1981) concluded that a mass comparable to that of Uranus or Neptune was ejected during the formation of these planets. The scattering of planetesimals towards the influence zones of Jupiter and Saturn was accompanied by an exchange of angular momentum, by which Jupiter – the main ejector – lost angular momentum and drove inward, while Saturn, Uranus and Neptune gained angular momentum and moved outwards (Fernández and Ip 1984, 1995). The radial displacement of

Uranus and Neptune – the less massive among the Jovian planets – could have attained several AU.

A byproduct of the massive scattering of bodies by the Jovian planets was the late heavy bombardment of the terrestrial planets (Wetherill 1975), with important effects on the cratering rate of their surfaces and the contribution of volatiles to their atmospheres (Chyba 1987, Ip and Fernández 1988).

As the orbit of proto–Neptune expanded, it swept across mean–motion resonances with exterior planetesimals, which got trapped and accompanied proto–Neptune in its outward displacement, increasing at the same time their eccentricities. This may have been the origin of Pluto and other bodies in the 2:3 resonance and other low–order resonances, mainly 3:5 and 1:2 (Malhotra 1993, 1995a).

Levison and Stern (1995) have presented an alternative model in which the Pluto–Charon binary system started in a near-circular orbit at ~ 40 AU, close to the 2:3 resonance, from which it evolved to its current high-i, high-e orbit by secular effects of the Jovian planets. Yet the authors require some damping mechanism (collisions or close interactions with KDOs) to decrease the libration amplitude of the critical angle, otherwise the computed orbits are always unstable over the age of the Solar System. Despite some uncertainties with the probabilities of the different events involved, Levison and Stern seem indeed to provide a plausible path to Pluto–type orbits. On the other hand, the appeal of the model developed here is just that it arises naturally from a consistent cosmogonic scenario. Furthermore, a Neptune's birthplace closer to the Sun, from where it swept and accreted matter as it moved outwards, can explain satisfactorily why the last planet is so large as compared to the sizes of the trans–Neptunian bodies.

There was a trans–Neptunian zone subject to an incessant bombardment by scattered bodies. There could have been a few Mars– to Earth–size bodies among them that gravitationally stirred the Kuiper disk (Ip 1989, Ip and Fernández 1991). Yet, once bodies reached encounter velocities with Neptune of ~ 0.3 (in units of Neptune's orbital velocity, assuming for it a circular orbit), they were also able to reach Saturn's influence zone, where their dynamical evolution proceeded much faster. We can conclude that the trans–Neptunian zone was mainly stirred by bodies having encounter velocities with Neptune $\lesssim 0.3$. Since the maximum semimajor axis that can be reached by a body of encounter velocity 0.3 is ~ 100 AU, we find that the trans–Neptunian region most intensely bombarded extended to ~ 150

AU.

If the mass scattered by the accreting proto-Uranus and proto-Neptune was of the order of their own current masses, the scattered mass could well largely exceed the mass contained in the Kuiper disk (say to heliocentric distances ~ 100 AU), thus producing a massive bombardment of KDOs. To put some numbers into this reasoning, let us assume that the initial number of scattered bodies was $N_{\text{sc}} = 10^{11}$, of mass $\sim 10^{18}$ g each (i.e. the total scattered mass is assumed to be of the order of Neptune's mass). Let N_{K} be the initial population of KDOs in the considered range 30 – 100 AU. The interaction time between both populations is taken to be $T = 10^8$ yr, which roughly corresponds to the dynamical time scale for dispersal of planetesimals from the accretion zones of Uranus and Neptune (Fernández and Ip 1981).

By using a "particle-in-a-box" approach, the number of collisions, dN , between scattered bodies and the Kuiper disk population in the range 30–100 AU during the time interval $(t, t + dt)$ can be computed as

$$dN = S u N_s \frac{N}{V} dt \quad (2)$$

where $S (\simeq 10^{13} \text{ cm}^2)$ is the collision cross-section of the bodies; $u \simeq 1.7 \text{ km/s}$ is the encounter velocity; N is the KD population at time t ; and $V (\simeq 10^{45} \text{ cm}^3)$ is the volume of the Kuiper disk between 30–100 AU and thickness 10 AU assumed to contain the population N .

Let us assume that collisions only occur between bodies of the population N_{sc} and the population N_{K} , and that all collisions are catastrophic with the consequent disruption and removal of all the fragments of the colliding bodies from the Kuiper zone. Therefore, after a time dt both populations N_{sc} and N_{K} will decrease by the same number dN . After a time t , the initial number of scattered bodies will consequently have decreased by

$$N_{\text{sc}} = N_{\text{sc},0} - (N_{\text{sc},0} - N_{\text{sc}}) \quad (3)$$

Substituting this expression into eq.(2) and integrating between $0 < t < T$ (with $T = 10^8$ yr), we finally obtain the ratio of the number of KDOs that underwent collisions, N_{col} , to the initial population $N_{\text{sc},0}$ as

$$\frac{N_{\text{col}}}{N_{\text{sc},0}} = \frac{y \{ 1 - \exp[\Gamma(1 - 1/y)] \}}{1 - y \exp[\Gamma(1 - 1/y)]} \quad (4)$$

where $y = N_{\text{scattered}} / N_{\text{initial}}$ is the ratio between initial populations and $\Gamma = SuTN_{\text{so}}/V$.

The fraction of the KD population that underwent collisions is shown in Fig.4 as a function of the parameter Γ and for $y = 1, 2$ and ∞ . According with the argumentation given above it is correct to take $y \geq 1$. In this case, Fig.4 shows that y is not a critical parameter. For the numerical values given above we obtain $\Gamma \sim 0.54$, and from Fig.4 we conclude that $\sim 40\%$ of the KD population would have collided with scattered bodies.

Mutual collisions among KDOs also contributed to deplete the Kuiper disk (Stern 1995). It is hard at this point to compare the depletion effect of bombardment from bodies scattered by the outer planets with the depletion from mutual collisions, since they depend on the population size, mass distribution and impact velocity. But one should note that typical encounter velocities among KDOs are about $\sqrt{e^2 + i^2} v_{\text{cir}} \approx 0.1 v_{\text{cir}}$ (for KDOs with $e \approx \sin i \approx 0.1$), whereas typical encounter velocities with bodies scattered by the outer planets may be $\sim 2 - 3$ times larger, i.e. impact energies $\sim 4 - 10$ times larger for projectiles of the same mass. Therefore, we should expect more catastrophic effects in the latter case (if gravitational focusing effects can be neglected, which in effect can be done for small planetoids of radii \lesssim a few hundred km).

Stern (1995) points out that mutual collisions cannot explain the existence of QB1s-class objects (i.e. $\sim 150 - 200$ km diameter), unless we assume that a much larger population was present there and/or that encounter velocities were lower than at present. Our model does explain the existence of QB1-class bodies in a natural way under the framework of the accretion and scattering of bodies by Uranus and Neptune. Thus, the ensuing heavy bombardment by the scattered planetesimals had a twofold effect: first, a severe depletion of KDOs and, second, a stirring effect with the consequent increase in $\langle e \rangle$ and $\langle i \rangle$ and therefore in the encounter velocities. Therefore, the Kuiper disk probably was much crowded and "colder" at the beginnings, favoring the formation of larger bodies. The same heavy bombardment event may have been responsible for the excitation of the asteroid population, inducing relative velocities of the order of 5 km s^{-1} . Furthermore, a large fraction of the bodies in resonant orbits may have been driven from the denser planetary region, where conditions existed for the growth of a substantial population of QB1-class bodies.

The heavy bombardment of the Kuiper disk may have also depleted the primordial population of Pluto-like objects in resonant motion with

Neptune. A simple calculation allows us to show that if Pluto is assumed to be the largest of a population of bodies with a mass distribution $n(m) \propto m^{-\alpha}$ with $\alpha \simeq 1.8$, then the primordial number of Pluto-type bodies with sizes ranging from 1 km to 1000 km might have been around $10^7 - 10^8$ objects. Most of them may have been lost via impacts with scattered planetesimals. The largest objects of this population, like Pluto, have remained because they are too massive to be unlocked from the 2:3 resonance by collisions with small planetesimals. Malhotra (1993) estimates that a mass $\sim 10^{-2}$ of Pluto's is required to unlock it, that means masses of the order $\sim 10^{22}$ gm are required to unlock bodies in the range 100–1000 km size.

Higher eccentricities in the orbits of KDOs are induced from collisions with scattered planetesimals. Numerical integrations by Torbett (1989) and Levison and Duncan (1993) show that KDOs in more eccentric orbits will evolve faster. Consequently, after the initial heavy depletion, an enhanced erosion rate of the remaining KD population in the 30–50 AU zone has persisted up to the present owing to the induced fairly high eccentricities. If higher eccentricities were induced in more distant KDOs ($a \gtrsim 50$ AU) by the bombarding planetesimals, then they might well evolve toward Neptune-crossing orbits in time scales comparable or shorter than the Solar System age, virtually extending the dynamically active portion of the Kuiper disk beyond the standard 50-AU limit.

4 Concluding remarks

It is then possible that the discovered KDOs represent the remains of a substantially larger population, mingled with bodies drove there from the planetary region via resonant transfer of angular momentum from the outward-displacing proto-Neptune. Since then they have been locked in resonant motion with Neptune. The accretion process of Uranus and Neptune also led to a massive bombardment of the Kuiper disk, inducing higher eccentricities up to distances ~ 100 AU. This stirring effect may have increased the erosion rate of the Kuiper disk, since planetary perturbations may have driven many of the high-e KDOs to Neptune-crossing orbits. On the other hand, other KDOs may have been injected in low-order resonances. Exterior, low-order resonances are islands where bodies settled and remained up to the present, as our numerical studies suggest (unless collisions removed them). As mentioned, some of the discovered KDOs seem to librate around low-order, mean-motion commensurabilities with Neptune's orbit.

In short, we may ask the question: is the discovered trans-Neptunian population mainly formed by planetesimals of the accretion zone of Neptune trapped in resonant orbits and drove away along the orbital expansion of Neptune?. Yet, some preliminary orbit determinations suggest that some KDOs move in low-eccentricity, non-resonant orbits. They might be *bona fide* Kuiper disk members, perhaps in slow evolution toward Neptune-crossing orbits. From the scant observational data, a scenario can be built in which the Kuiper-disk population may be viewed as a mixture of bodies captured by Neptune in resonant orbits and drove away during its late accretion stages, and leftovers of a substantially larger population in the primordial Kuiper disk subject to a heavy bombardment by bodies scattered by the Jovian planets. A better observational sample will be required to assess the relative importance of both sources.

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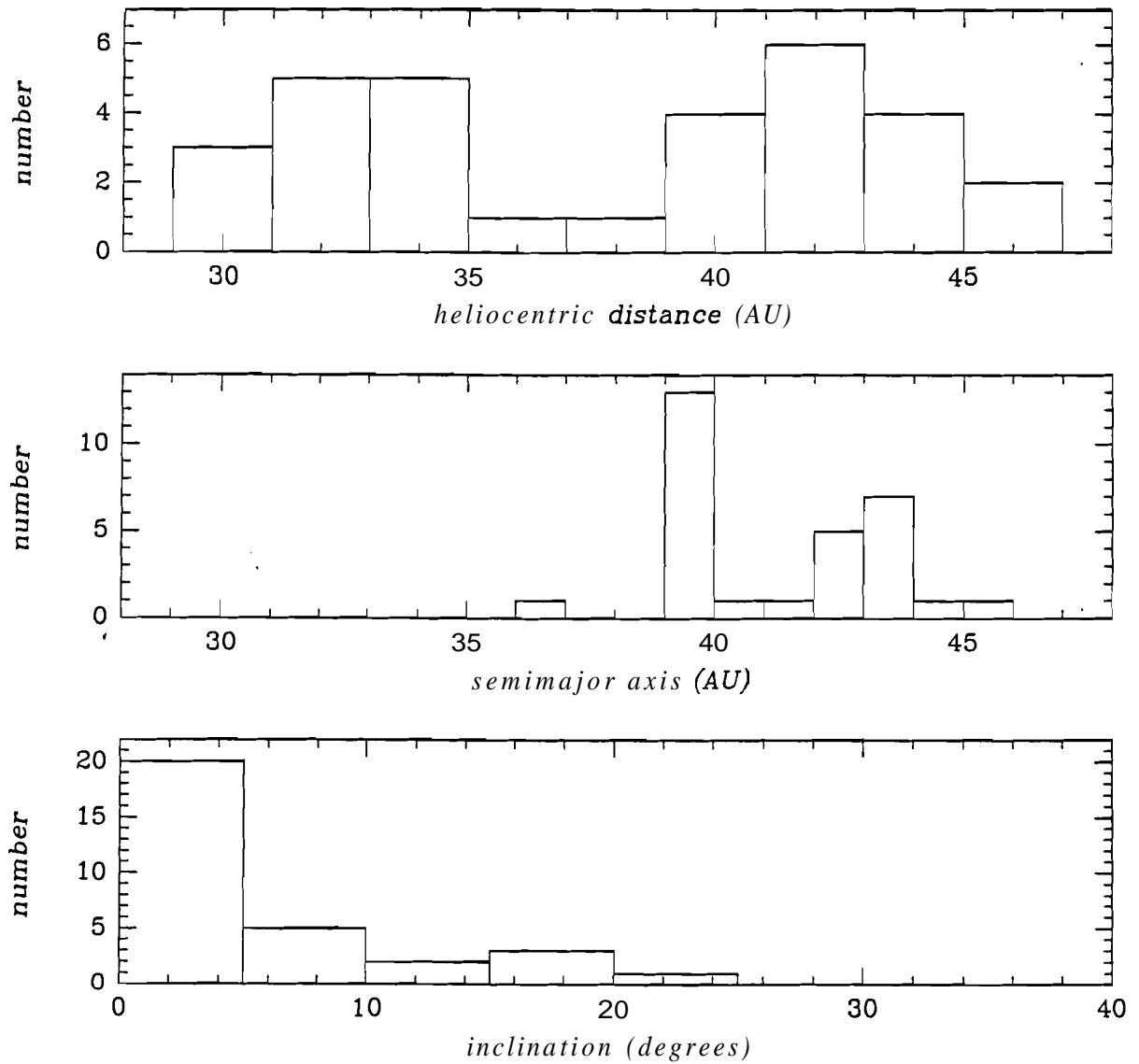


Figura 1: Some properties of the discovered trans-Neptunian objects: (a) distribution of heliocentric distances; (b) distribution of estimated semimajor axes; (c) distribution of inclinations. Source: Marsden (1996) and MPECs.

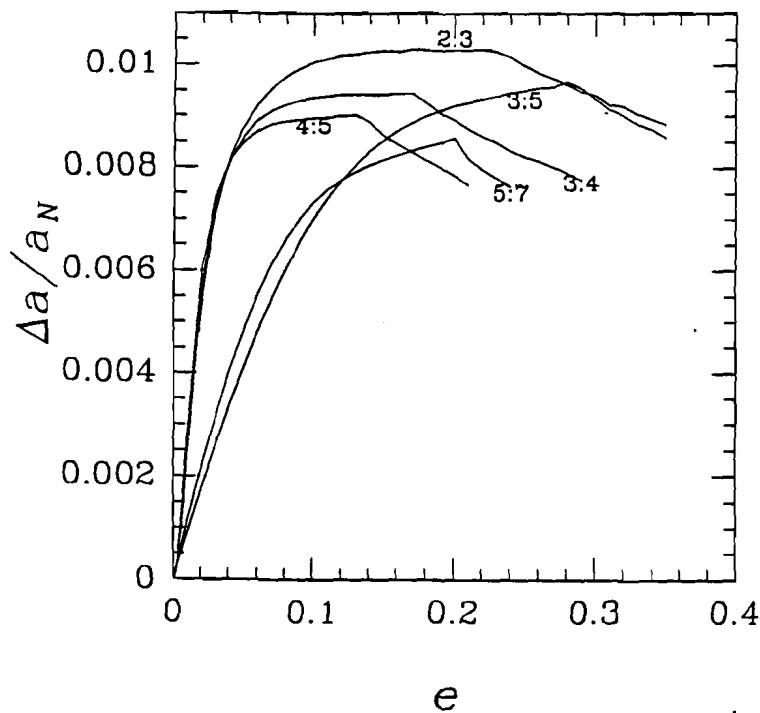


Figura 2: Perturbation in semimajor axis required to excite a particle from a zero-amplitude libration to a circulation on σ or to a libration with such an amplitude that the particle collides with Neptune, as a function of the orbital eccentricity and for different **resonances**. These calculations were performed for the planar case using the theory of Gallardo and Ferraz-Mello (1995) that can be applied to librations around $u = N\pi/2$. The **resonance 1:2** is not included.

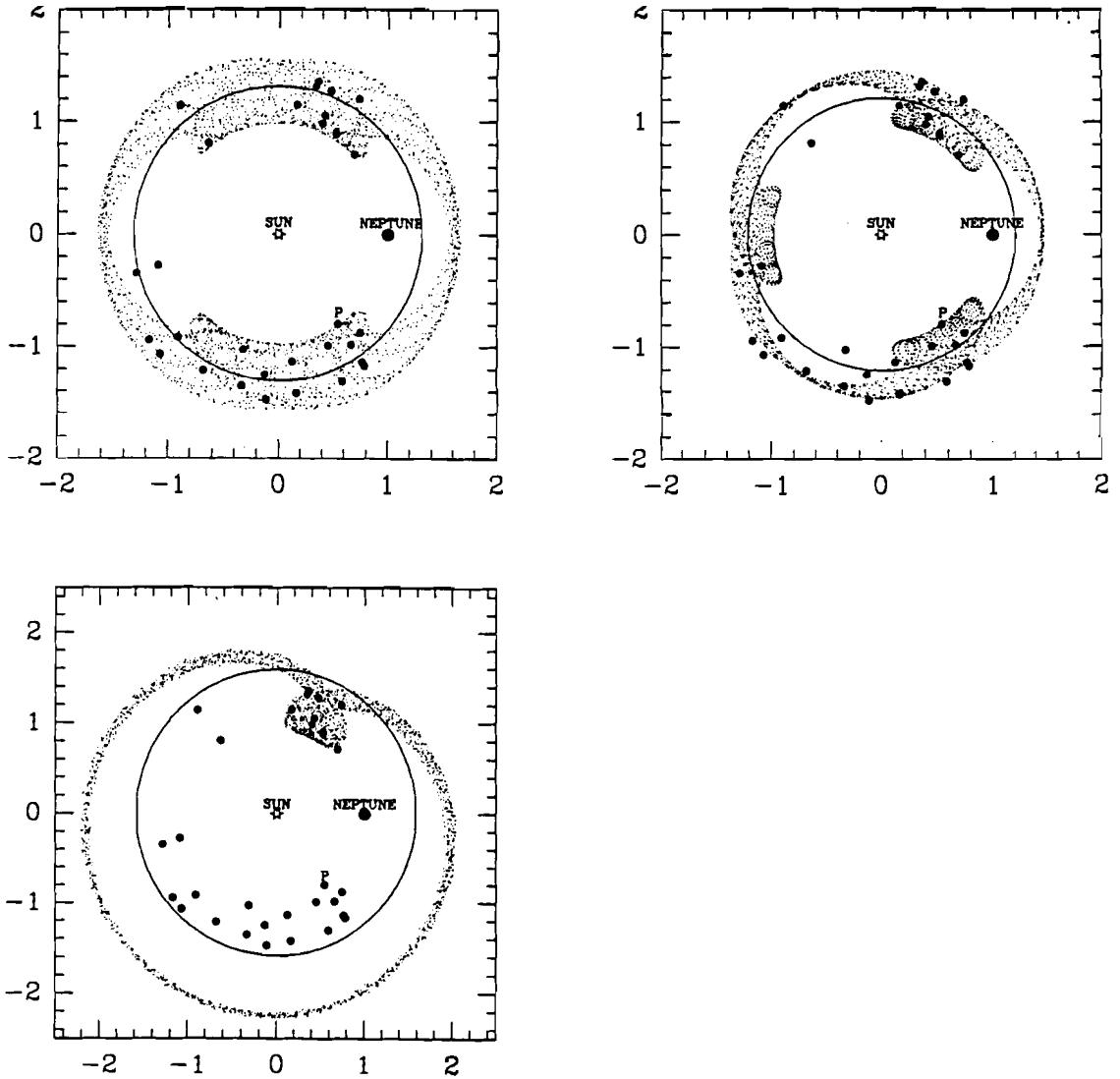


Figura 3: (a) Relative positions with respect to the system Sun–Neptune of a particle with $e = 0.25$ librating in the 2:3 resonance with amplitude $\Delta\sigma = 90^\circ$. In this rotating frame we plot the current relative positions of the discovered KDO including Pluto. The circular orbit is for a particle with zero eccentricity in the same resonance. (b) Same as (a) but for a particle with $e = 0.20$ librating in the 3:4 resonance with amplitude $\Delta\sigma = 62^\circ$. (c) Same as (a) but for a particle with $e = 0.42$ librating in the 1:2 resonance around $a_r = 66^\circ$ with amplitude $\Delta\sigma = 7^\circ$. See that the only region where the detection of this particle is favourable is the small lobe located ahead Neptune and where in fact ten objects have been discovered. a_N was taken as unit of length.

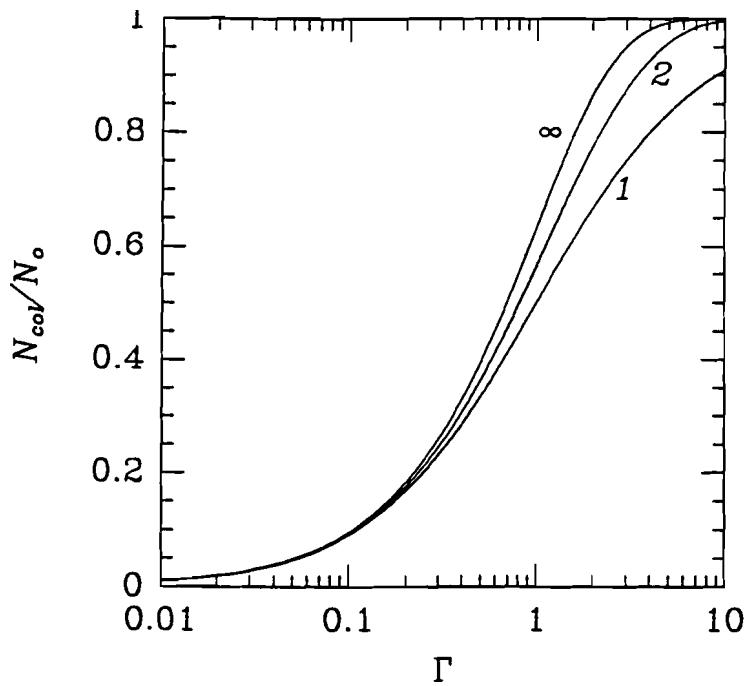


Figura 4: Fraction of the KDO population colliding with bodies scattered by the Jovian planets as a function of the parameter Γ for three values (1, 2 and ∞) of the original size ratio between the two colliding populations, $y = N_{so}/N_o$. For the case we studied, $\Gamma \sim 0.54$.

Anexo IV

The transfer of comets from parabolic orbits to short-period orbits

The transfer of comets from parabolic orbits to short-period orbits: numerical studies

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Abstract. The transfer process of long-period (LP) comets from parabolic orbits to short-period (SP) orbits is studied numerically for small perihelion **distances** ($q < 2$ AU) and for perihelia in the Jupiter's zone ($4 < q < 5.2$ AU). Two different numerical approaches are used: (1) For the evolution throughout the LP stage in which comets move in very eccentric orbits the perturbations are assumed to be stochastic, so each comet passage by the planetary region is **simulated** by adding random sets of increments Ax , Ai , Aq to the orbital energy x , inclination i and perihelion **distance** q , respectively. (2) For the late evolution of comets in periodic orbits ($P < 200$ yr), Öpik's two-body approximation is used.

For $q < 2$ AU, it is shown that the inclination **distribution** of a population of LP comets with initial random orientations of their orbital planes (i.e. a sinusoidal **distribution** of i) evolves toward a flatter distribution by the combination of **two** dynamical effects: (1) the dependence of the transfer probability to a certain energy **level** on the typical energy change, which itself depends on the **inclination**, and (2) the tendency of comets in near-perpendicular orbits to shift toward retrograde orbits. When the number of passages is limited by physical causes, less comets in retrograde orbits are found to reach evolved states owing to their slower evolution. This can explain the depletion of retrograde comets among the observed old LP comets ($200 < P < 1000$ yr) and intermediate-period (IP) comets ($20 < P < 200$ yr) with $q < 2$ AU, which suggests average physical lifetimes of about 300–500 revolutions for comets moving through the LP path. For $q < 2$ AU the probability of capture of a LP comet into an IP orbit is found to be about 0.01 when an upper **limit** of 400 revolutions is considered. This gives a steady-state population of intermediate-period comets of the Halley type with $q < 2$ AU of ≈ 300 members.

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Long-period comets evolving in the Jupiter's zone ($4 < q < 5.2$ AU) from an initial population of randomly oriented, parabolic comets reach the intermediate-period stage with an inclination distribution already biased toward direct orbits without considering physical **losses**. This is due to the stronger dependence of the typical energy change on the inclination, which results in a much lower transfer probability for retrograde orbits.

Randomly-oriented LP comets **captured** in SP orbits, either from the small- q zone or from the Jupiter's zone, show dynamical properties that do not match the **observed** ones. For instance, a fraction of the **captured** SP comets are found to move in retrograde orbits in contradiction with their observed i -distribution, which is strongly concentrated toward **small** inclinations. There are also serious discrepancies with the values of the Tisserand constant, the derived distribution of the perihelion **distances** and the steady-state population of SP comets. Low-inclination comets moving in the trans-jovian region, like P/Schwassmann-Wachmann 1 and Chiron, may be the appropriate precursors of the Jupiter-family comets.

Key words: celestial mechanics – comets: general

I. Observational background

The observed comets show quite different orbital periods that reflect different dynamical ages. Some comets move in near-parabolic orbits, presumably coming from the Oort cloud by the first time whereby Oort (1950) called them "new". Comets are usually divided into three dynamical classes: long-period (LP) comets (for orbital periods $P > 200$ yr); intermediate-period (IP) comets or Halley-type comets (for $20 < P < 200$ yr); and short-period (SP) comets or Jupiter-family comets (for $P < 20$ yr). Even though this may be a rather arbitrary classification of

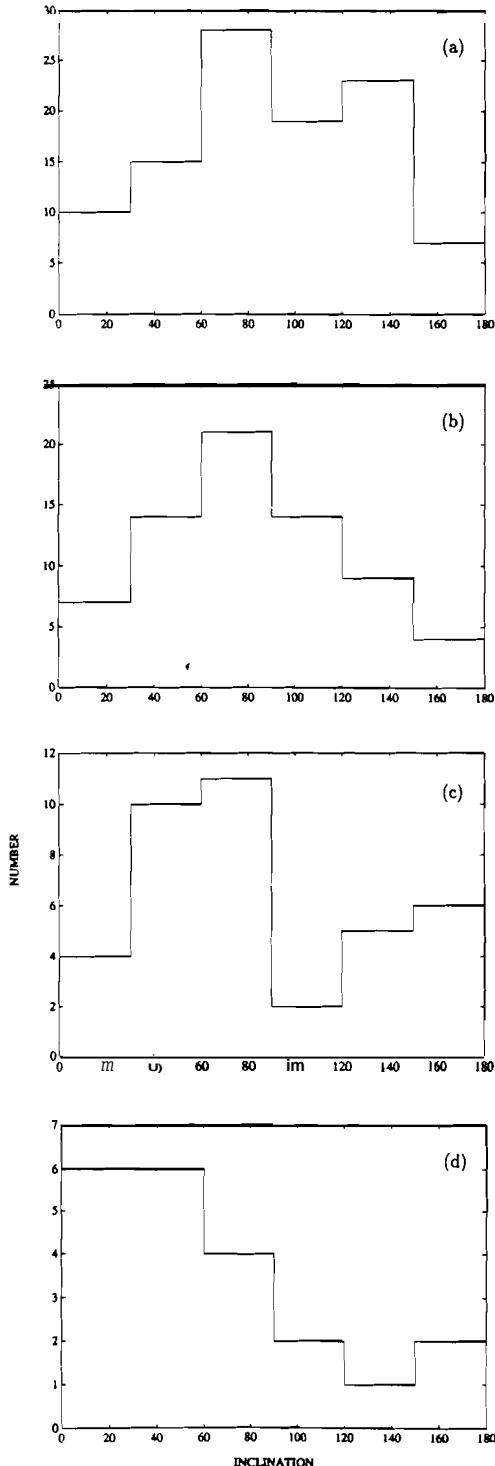


Fig. 1a-d. Inclination distributions of the observed comets with $q < 2 \text{ AU}$ appearing in Marsden's (1992) catalogue: a comets with $(1/a)_{\text{orig}} < 10^{-3} \text{ AU}^{-1}$ ($P > 31600 \text{ yr}$), b $10^{-3} < (1/a)_{\text{orig}} < 10^{-2} \text{ AU}^{-1}$ ($10^3 < P < 31600 \text{ yr}$), c $10^{-2} < (1/a)_{\text{orig}} < 10.03 \text{ AU}^{-1}$ ($200 < P < 10^3 \text{ yr}$), and d intermediate-period

periods, it is nevertheless a very useful one which has been used extensively by comet researchers (sometimes with slight variations in the limiting periods), so we will stick to these standard dynamical classes in this paper. We will mainly refer to the orbital properties of comets with perihelion distances $q < 2 \text{ AU}$ for which the record of comet discoveries is more complete. Furthermore, for LP comets we will require to know their original semimajor axes, a_{orig} , i.e., their semimajor axes before entering the planetary region and being perturbed by the planets.

For comets coming from the Oort cloud we should expect to find a random distribution of their orbital planes (a sinusoidal distribution of their inclinations i). This is what is indeed observed for dynamically young comets (say with $a_{\text{orig}} > 10^3 \text{ AU}$ or periods $P > 31600 \text{ yr}$) (Fig. 1a). Yet, LP comets more evolved dynamically show an ever increasing excess of direct orbits as well as a depletion of near-perpendicular orbits as observed, for instance, in the samples with $100 < a_{\text{orig}} < 10^3 \text{ AU}$ ($1000 < P < 31600 \text{ yr}$) (Fig. 1b), and $34 < a_{\text{orig}} < 100 \text{ AU}$ ($200 < P < 1000 \text{ yr}$) (Fig. 1c). In other words, the i -distribution tends to be flatter than that expected for a sinusoidal distribution. Finally, the i -distribution of IP comets is flat and with a clear excess of direct orbits (Fig. 1d). The orbits of all SP comets so far discovered are direct and strongly concentrated toward the ecliptic plane (Fig. 2).

The Tisserand criterion can be very useful for distinguishing among different dynamical sources of comets. Strictly speaking, it is applicable to the circular, restricted three-body problem, but since Jupiter is generally the main perturber of a comet crossing or approaching Jupiter's orbit, we can apply the Tisserand criterion to the observed comets with accuracy enough. For a comet with orbital elements q , a , and i it can be expressed as

$$T = a_J/a + 2 \cos i [q/a_J(2 - q/a)]^{1/2}, \quad (1)$$

where a_J is the radius of Jupiter's orbit assumed to be circular and T is a constant under the above conditions (the Tisserand constant).

The encounter velocity U of a comet with Jupiter can be expressed in terms of the Tisserand constant as

$$U = (3 - T)^{1/2}, \quad (2)$$

where U is expressed in terms of Jupiter's orbital velocity. Equation (2) shows that encounters with Jupiter are possible only if $T \leq 3$. Indeed, this is what is actually observed in the great majority of comets. There are only a few SP comets with values of T slightly above three.

Comets in more eccentric orbits and higher inclinations will have larger encounter velocities with Jupiter,

comets ($20 < P < 200 \text{ yr}$). For LP comets ($P > 200 \text{ yr}$) we considered only those discovered after 1758. The 22 members of the Kreutz family of sungrazing comets were taken as a single comet, as well as 1987XXX and 1988I that share practically the same orbit

which means smaller values of T . As shown in Fig. 3, all the T values of LP and IP comets with $q < 2$ AU are in the range $-2 < T < +2$. By contrast, the great majority of SP comets have $T > 2$ and most of them are in the range $2.5 < T < 3$.

There are 149 SP comets discovered up to the end of 1991 in Marsden & Williams (1992) Catalogue of Cometary Orbits, of which very few cross Earth's orbit. Analytical results based on the ability of Jupiter to scatter comets to the Sun's vicinity (Fernández 1984), as well as orbital computations of the observed SP comets showing that they remain in small- q orbits for a very small fraction of time (Nakamura & Yoshikawa 1992), suggest that the number of SP comets should increase very fast with q . This conclusion is strengthened by the ever increasing discovery rate of SP comets with $q \gtrsim 1.5$ AU, while it seems to be reaching a ceiling for $q \lesssim 1.5$ AU (Fernández et al. 1992).

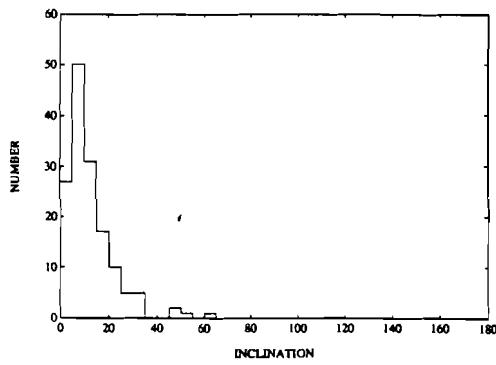


Fig. 2. Inclination distribution of the observed SP comets ($P < 20$ yr) appearing in Marsden & Williams (1992)

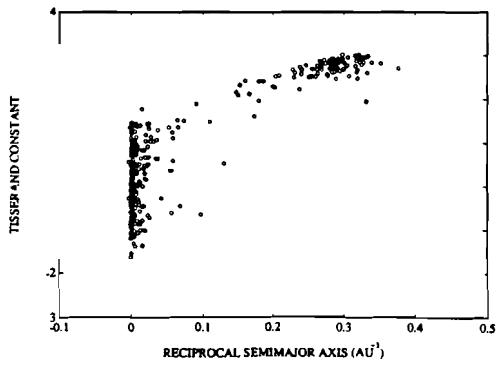


Fig. 3. Plot of the Tisserand constant versus the reciprocal semimajor axis for the observed LP comets [$(1/a) < 0.03$ AU $^{-1}$], IP comets [$0.03 < (1/a) < 0.136$ AU $^{-1}$] and SP comets [$(1/a) > 0.136$ AU $^{-1}$]. For all the dynamical classes we considered only those with $q < 2$ AU and LP comets with well determined osculant values of $(1/a)$

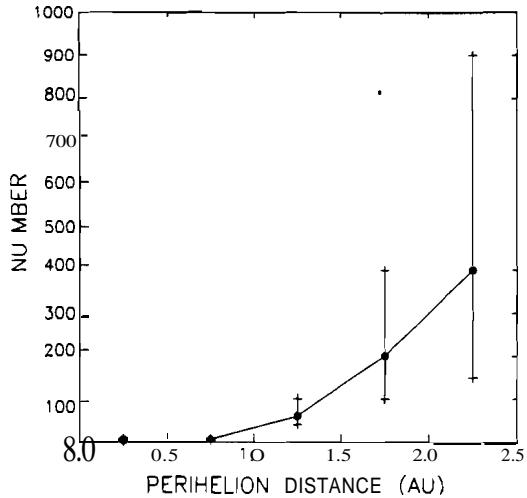


Fig. 4. Computed number of SP comets with perihelion distances within a half AU bins from reasonable extrapolations of the number of SP comets that have so far been detected, according to the results by Fernández et al. (1992). The respective error bars are indicated

The estimated number of SP comets having perihelion distances within 0.5 AU bins is shown in Fig. 4.

2. Orbital diffusion to a certain energy level

Let $x = 1/a$ be the binding energy of an object on a Keplerian orbit, so $x > 0$ for elliptic orbits and $x < 0$ for hyperbolic ones. Comets coming from the Oort cloud (i.e. energies $x \approx 0$) can reach a certain energy level x under the action of planetary perturbations. For LP comets the orbital energy x is the parameter that suffers the greatest variation during a passage by the planetary region. The changes in the other orbital parameters are much smaller, though their long-term effects might be non-negligible, as we will see below for the inclination variation. The evolution of the comet's energy is conceived as a diffusion process under the assumption that the energy change per orbital revolution is stochastic and small in magnitude, i.e. $\Delta x \ll x_f - x_i \approx x_f$, where x_i, x_f are the initial and final energy states, respectively. The stochastic nature of the energy changes can be justified on the basis of the very long orbital period of an evolving LP comet, so it will meet a planetary configuration completely different from the previous one.

Therefore, a comet starting in a parabolic orbit will random-walk in the energy space (where each step will correspond to a passage by the planetary region), reaching an energy x_f after an average number of passages,

$$n_{x_f} \approx (x_f/\sigma)^2, \quad (3)$$

where a is the typical energy change per passage, which can be expressed by the standard deviation of the distribution function of energy changes $\psi(\Delta x)$, assumed to be symmetrical with respect to $\Delta x = 0$.

During the random-walk many comets will be ejected along hyperbolic orbits (i.e. they get binding energies $x < 0$). The probability of a LP comet to survive n revolutions bound to the solar system, p_n , is approximately given by

$$p_n \approx 0.5 n^{-1/2} \quad (4)$$

(Everhart 1976).

The statistical results given by Eqs. (3) and (4) should be taken with caution bearing in mind the limitations of the diffusion approach. For instance, if the typical energy step a is extremely small as compared with the energy range between the two absorbing barriers, the rare strong perturbations in close encounters [i.e. changes Δx falling in the tails of the distribution function $\psi(\Delta x)$] will control the dynamical evolution, since the longer dynamical time scales involved allow these events to occur (Stagg & Bailey 1989). This constraint should be of greater concern for Uranus and Neptune than for Jupiter and Saturn. Actually, as we will see below, our numerical results for Jupiter are in reasonable good agreement with the diffusion approach.

By combining Eqs. (3) and (4) we obtain the probability of capture, p_c , of a LP comet starting in a parabolic orbit to an orbit with energy x_f (assumed to be elliptic, i.e. $x_f > 0$). We find

$$p_c \approx 0.5 \sigma / x_f. \quad (5)$$

Therefore, according to the diffusion theory the capture probability to a certain energy level x_f turns out to be proportional to a . As mentioned, for Eq. (5) to be applicable the energy step a must be much smaller than the energy interval covered by the evolution $x_f - x_i \sim x_f$, so we require $a \ll x_f$ (though, as mentioned, a should not be too small as to allow the occurrence of very strong planetary perturbations). For instance, for random LP comets in Jupiter-crossing orbits, $a = 6 \cdot 10^{-4} \text{ AU}^{-1}$, so Eq. (5) can be applicable for $x_f \gtrsim 10^{-2} \text{ AU}^{-1}$.

Stagg & Bailey (1989) point to a seemingly paradox in the result given by Eq. (5). They argue that an object injected with energy x_0 between two absorbing barriers at energies x_i and x_f , where $x_i < x_0 < x_f$, will diffuse to an energy $x > x_f$ with a probability

$$p_c = \frac{x_n - x_i}{x_f - x_i}, \quad (6)$$

which turns out to be independent of a . Now, for comets injected in near-parabolic orbits we have $x_0 \approx 0$. Furthermore $x_i \approx 0$. Equation (6) will have any meaning after one orbital revolution, where roughly half the population will be lost to the interstellar space and the other half will

remain bound with an average energy o . Applying Eq. (6) to this half population, where $x_0 \approx \sigma$, we obtain $p_c \sim 1/2$ σ/x_f , which is in agreement with Eq. (5).

3. Transition from parabolic orbits to intermediate-period orbits

3.1. The numerical method

It is not possible to design a single numerical procedure to simulate the dynamical evolution of comets from the Oort cloud to SP orbits. For LP comets, it is reasonable to assume that the reciprocal semimajor axis or energy x varies at random at each perihelion passage due to planetary perturbations as discussed before. As a first approximation we can conceive the orbital evolution of a LP comet as an one-dimensional random-walk in the energy space, neglecting second-order changes in the other orbital elements. Long-term changes in the inclination will be discussed later. The assumption of randomness in the variation of x breaks down when the comet's orbital period decreases to values of the order of the orbital periods of the Jovian planets. Furthermore, the changes in the other orbital parameters: q , i , w and Ω become significant. We thus divided the numerical analysis in two parts: (1) the comet's evolution in a LP orbit, and (2) its evolution in a periodic orbit ($P < 200$ yr).

We considered two ranges of perihelion distances: (A) $0.1 < q < 2$ AU and (B) $4 < q < 5.2$ AU, and six ranges of inclinations: (1) $0 < i < 30^\circ, \dots, (6) 150^\circ < i < 180^\circ$, so we produced 12 combinations of (q, i) ranges: A1, ..., A6, B1, ..., B6. For each combination we took a sample of 2000 fictitious parabolic comets with initial values of q and $\cos i$ taken at random within the respective ranges. The values of w and Ω were taken at random between 0° and 360° . We integrated numerically the orbits of each set of 2000 comets through a passage by the planetary region, which allowed us to build up data bases of perturbations $(\Delta x, \Delta i, \Delta q)$ per perihelion passage for each combination A1, ..., B6 to use later in the simulations. We adopted a model of solar system planar and circular with all the planets but Pluto. We used the integration algorithm RA15 developed by Everhart (1985).

We then considered a fictitious comet starting out on a parabolic orbit with initial q, i values within the ranges A1. The random-walk of x, q , and i was simulated by adding sets $(\Delta x, \Delta q, \Delta i)$ drawn at random from the corresponding data base A1. Each addition of a random set $(\Delta x, \Delta q, \Delta i)$ represented a new passage by the planetary region. The evolution was followed until the comet was ejected along a hyperbolic orbit (i.e. it reached $x < 0$) or was transferred to a periodic orbit (i.e. when $x \geq x_p \approx 0.03 \text{ AU}^{-1}$, or periods $P < 200$ yr). In the latter case, the initial and final values q, i were stored. Then we started again with another fictitious comet belonging to A1 until completing a sample of 500 comets reaching

energies $x \geq x_p$. This procedure was repeated for the remaining combinations A2, ..., B6, so we produced an overall sample of 6000 captured periodic comets for further studies.

Let N_{Htot} be the total number of comets ejected along hyperbolic orbits during the capture of 500 comets into periodic orbits. The probability p_c that a comet starting in a parabolic orbit be finally captured into a periodic orbit will then be expressed as

$$p_c(x_p) = 1/(\bar{N}_H + 1), \quad (7)$$

where $\bar{N}_H = N_{\text{Htot}}/500$.

We also computed the probabilities: $p_c(x_1)$ and $p_c(x_2)$ that a LP comet reached two other intermediate energy levels $x_1 = 0.001 \text{ AU}^{-1}$ and $x_2 = 0.01 \text{ AU}^{-1}$ in a similar way as that defined by Eq. (7).

We also considered the influence of physical losses in the samples of small- q comets A1, ..., A6 by adopting an upper limit $n_{\text{MAX}} = 400$ for the number of passages. In this case, when a comet reached $n = n_{\text{MAX}}$ the simulation was terminated and the comet assumed to be lost.

The population of comets $N_j(x)$ that reach a certain energy level x within the inclination range j will be given by

$$N_j(x) = N_0 \{ f_j p_{cj}(x) [1 - R_j(x) - A_j(x)] \\ + f_{j-1} p_{cj-1}(x) A_{j-1}(x) + f_{j+1} p_{cj+1}(x) R_{j+1}(x) \}, \quad (8)$$

where N_0 is the initial population of parabolic comets, p_{cj} is the capture probability within the range j , A_j is the probability that a LP comet within the inclination range j advances to the range $j+1$ during its random-walk to x , whereas R_j is the probability that it recedes to the range $j-1$. f_j is the fraction of the initial population of comets in the range j over the total population ($0 < i < 180^\circ$) assuming that the initial distribution of inclinations is sinusoidal.

3.2. Results

The capture probability to a certain energy level x , $p_c(x)$, is more or less independent of the inclination for very small values of x (e.g. $x = 0.001 \text{ AU}^{-1}$ in Fig. 5). The reason is that comets need only a few steps to reach that small energy level, so the condition $a \ll x_f - x_i$ for a diffusion regime is not fulfilled yet. For larger x , say -0.01 AU^{-1} , the differences among the different values of p_c are found to be approximately inversely proportional to x (or $1/a$) and proportional to their corresponding typical energy changes a for the different i -ranges, in agreement with the previous analysis [cf. Eq. (5)]. When we limit the number of revolutions in samples A1, ..., A6 the capture probability for retrograde orbits will decrease in a much steeper way (Fig. 5), as they require a larger number of revolutions to reach a certain energy level. The capture probabilities for the samples B1, ..., B6 are of the same order as those

for samples A1, ..., A6 (no upper limit to the number of passages), though the differences among the p_c values for direct and retrograde orbits are larger, owing to the larger differences among the respective a values.

Values of a were computed from the distribution of energy changes (including close encounters) of the 2000 parabolic comets of each one of the samples A1, ..., A6, B1, ..., B6. The results are shown in Table 1. The good correlation between p_c and a can be seen in Fig. 6.

Let us now consider the evolution of the i -distribution with x . If p_c were independent of i the i -distribution would keep unchanged throughout the dynamical evolution, namely a sine-law distribution (leaving apart second-order changes in the inclination). This is what happens with dynamically young LP comets reaching the level

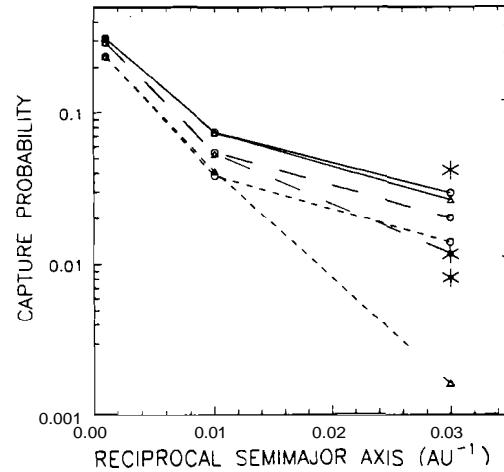


Fig. 5. Capture probability of near-parabolic comets with initial perihelion distances in the range $0.1 < q < 2 \text{ AU}$ into orbits with energies (or reciprocal semimajor axes) greater than the values indicated in the abscissae. Three inclination ranges are considered: $(0-30^\circ)$ — ; $(90-120^\circ)$ - - - ; $(150-180^\circ)$ - - - . The symbols represent: no upper limit to n_{MAX} (open circles); evolution with $n_{\text{MAX}} = 400$ (open triangles). The three star-symbols are for comets with initial perihelion distances in the range $4 < q < 5.2 \text{ AU}$ and the inclination ranges (from top to bottom) indicated before in the same order

Table 1. Typical energy change per orbital revolution (10^{-4} AU^{-1})

i-range	$0.1 < q < 2$	$4 < q < 5.2$
0-30	13.4 (A1)	25.1 (B1)
30-60	9.7 (A2)	12.7 (B2)
60-90	8.5 (A3)	9.8 (B3)
90-120	5.7 (A4)	5.1 (B4)
120-150	6.9 (A5)	4.0 (B5)
150-180	8.7 (A6)	3.7 (B6)

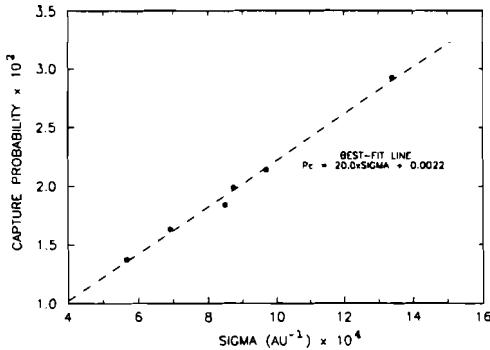


Fig. 6. Capture probability of near-parabolic comets with perihelion distances in the range $0.1 < q < 2$ AU into orbits with orbital energies $> 0.03 \text{ AU}^{-1}$ as a function of their typical energy changes (which are computed for the inclination ranges $0\text{--}30^\circ, \dots, 150\text{--}180^\circ$)

$x=0.001 \text{ AU}^{-1}$ (Fig. 7). The i -distribution of more evolved LP comets (larger x) tends to get flatter and slightly biased toward direct orbits as a result of the combination of two effects: (1) the dependence of p_c on a (and thus on i), and (2) the migration of comets from an i -bin to the next to it (see cases for $x=0.01$ and 0.03 AU^{-1} in Fig. 7). Effect (1) will tend to deplete retrograde orbits more than direct ones, while effect (2) will tend to flatten the i -distribution by shifting near-perpendicular comets mainly to the retrograde branch, thus tending to compensate for the greater losses of retrograde comets. This effect can be understood from the Tisserand criterion. If we substitute in Eq. (1) $1/a$ by x , differentiate it and assume $x \approx 0$ (valid for highly eccentric orbits), we get

$$(a_i^{3/2} - q^{3/2}/2^{1/2} \cos i) \Delta x - (2^{3/2} q^{1/2} \sin i) \Delta i + (2^{1/2}/q^{1/2} \cos i) \Delta q = 0. \quad (9)$$

For near-perpendicular orbits we have $\cos i \approx 0$, so Eq. (9) becomes

$$a_i^{3/2} \Delta x - 2^{3/2} q^{1/2} \Delta i \sim 0, \quad (10)$$

which means $\Delta x/\Delta i > 0$, so an increase in the binding energy x (larger $1/a$) will generally be accompanied by an increase in i . Therefore, near-perpendicular LP comets evolving toward larger $1/a$ will tend to increase their inclinations.

The average change in the comet's perihelion distance from the beginning in a parabolic orbit to the end when it acquires an energy $x=0.03 \text{ AU}^{-1}$ is only a few tenths AU. It is too low to have in general any significant influence in the dynamical evolution of LP comets so it will be no longer considered in our study. Nevertheless, in some cases secular perturbations can cause the perihelion distance to drift to very small values that could explain, for instance, the family of sungrazers (Bailey et al. 1992).

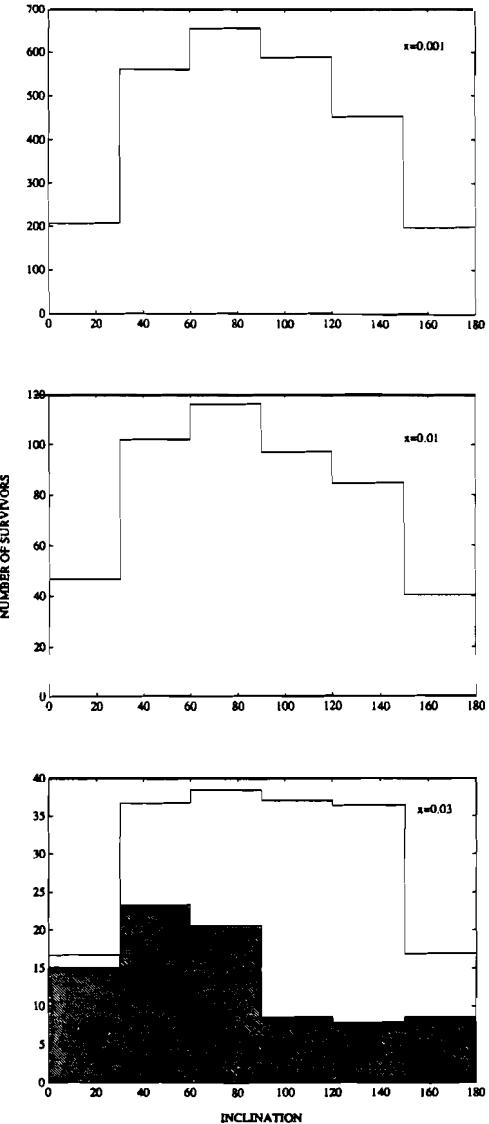


Fig. 7. Computed number of survivors reaching binding energies greater than the ones indicated at the upper right corner of each histogram (in AU^{-1}) per 10^4 initial parabolic cornets with $0.1 < q < 2$ AU within inclination ranges of 30° width (samples A1, ..., A6). The shaded histogram is for a simulation that includes physical losses by setting a limit $n_{MAX}=400$ to the number of revolutions

If we add physical losses to samples A1, ..., A6, as represented by an upper limit to the number of revolutions (400 in our case), we find their effect to be quite significant for evolved comets (see the shaded histogram for $x=0.03 \text{ AU}^{-1}$ in Fig. 7). The number of survivors here is smaller for retrograde orbits, which means an i -distribution clearly biased toward direct orbits, in a closer match with the observed i -distribution for old LP comets

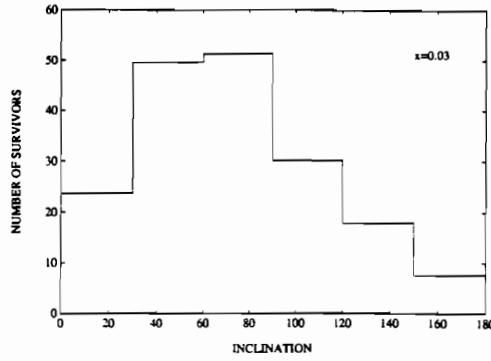


Fig. 8. Idem as Fig. 7 but for initial parabolic comets in the range $4 < q < 5.2$ AU (samples B1, ..., B6) reaching energies $> 0.03 \text{ AU}^{-1}$

($200 < P < 10^3$ yr) and IP comets (cf. Figs. 1c, d). Comets of samples B1, ..., B6 show a clear depletion of retrograde orbits for $x=0.03 \text{ AU}^{-1}$ (Fig. 8), without considering physical losses (i.e. no limit to the number of revolutions), which can be explained by the stronger dependence of σ on i (cf. Table 1).

4. Transition from intermediate-period orbits to short-period orbits

4.1. The numerical method

For the simulation of the latest stages of comet evolution we used Öpik's two-body approximation that can handle the strong planetary perturbations in close encounters (see Öpik 1976 for a general review of the method). We considered all the perturbations produced in planetary approaches within a cross section of radius equal to 100 times the gravitational radius for collision. We took into consideration the four Jovian planets. The method has some important shortcomings as, for instance, it neglects the smooth orbital changes in more distant encounters, and it is not possible either to deal with resonances or non-crossing orbits. Furthermore, for Jupiter the time scale between close encounters can be significantly shorter than the precession period of the argument of the comet's perihelion, one of the principles of Öpik's method (Wetherill 1991). Nevertheless, despite the shortcomings the Öpik method can be a useful tool to give a quick diagnostic of the dynamical state of ensembles of stray objects (like comets) evolving in the planetary region. In this regard, it is reassuring to find that it reproduces the asymmetric tails in the distribution of energy changes in agreement with numerical results (Carusi et al. 1990).

For the computed population of comets with $0.1 < q < 2$ AU reaching IP orbits, as shown in Fig. 7 for $x=0.03 \text{ AU}^{-1}$ and $n_{\text{MAX}}=400$ (shaded histogram), we considered sets of six sub-samples of $N=10^4$ fictitious comets each with the following initial conditions:

(1) Perihelion distances in the range $0 < q < 2$ AU, aphelion distance = 50 AU and inclinations randomly taken in $\cos i$ within one of the six ranges: $0-30^\circ, \dots, 150-180^\circ$. Bearing in mind that comets with small q may have rather limited physical lifetimes, we adopted upper limits of 200, 10^3 , and 10^4 for the number of passages.

For the computed population of comets with $4 < q < 5.2$ AU reaching IP orbits, as shown in Fig. 8, we considered similar sets of six sub-samples with the initial conditions:

(2) Perihelion distances in the same range, aphelion distance and inclinations like case 1. We considered two cases: no upper limit to the number of revolutions and an upper limit of 10^3 .

For all the studied cases the simulation was terminated if the comet was ejected or transferred to a LP orbit ($P > 200$ yr). We computed the time span $\tau_{jlk}(q)$ that a fictitious comet k ($k=1, \dots, N$) remained in a SP orbit with a perihelion distance within the interval $(q, q+\delta q)$ and within the inclination range j ($j=1, \dots, 6$). The suffix l means that the comet k comes from the sub-sample with initial inclination in the range l ($l=1, \dots, 6$). If T_{lk} is the total time that the comet k remained bound in a periodic orbit, we then computed the fraction $\gamma_{jlk}=\tau_{jlk}/T_{lk}$. The average fraction of time for the sub-sample l will be $\bar{\gamma}_{jl}=\sum_{k=1}^N \gamma_{jlk}/N$. Finally, we summed the contributions $\bar{\gamma}_{jl}$ for the six sub-samples l appropriately weighted so as to take into account the distribution of inclinations of the source population of IP comets. In other words, we obtain $\Gamma_j=\sum_{l=1}^6 \bar{\gamma}_{jl} \times w_l$, where w_l are the weights derived from either the distribution of Fig. 7 for $x=0.03 \text{ AU}^{-1}$ and physical losses (shaded histogram) or Fig. 8. For both cases we compute $w_l=N_l/N_{\text{ALL}}$, where N_l is the number of comets in the inclination range l and N_{ALL} the number for all the inclinations.

For the chosen interval $(q, q+\delta q)$, the number of SP comets within the inclination range j will be proportional to the fraction Γ_j .

4.2. Results

The values of the Tisserand constant T of the captured SP comets with $q < 2$ AU from the population with $0.1 < q < 2$ AU are mostly concentrated in the interval $0 < T < 2$ (Fig. 9a) in fairly good agreement with the T values for the observed LP comets and IP comets, though quite different from those for the observed SP comets (cf. Fig. 3). The T -distribution of the captured SP comets with $q < 2$ AU from the population with $4 < q < 5.2$ AU spreads through a rather large range of values (mostly in the range $0 \leq T \leq 2.75$) when no upper limit to the number of passages is considered. When we limit the maximum number of revolutions to 10^3 , the greatest concentration in the T -distribution shifts to the

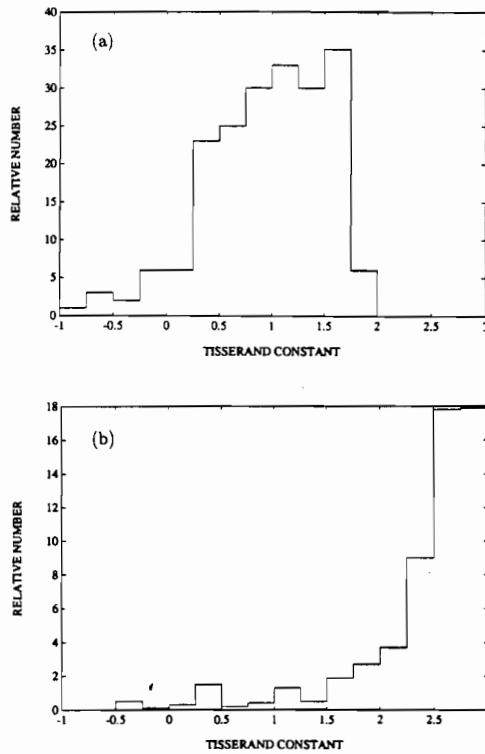


Fig. 9a and b. Distribution of Tisserand constant values for the computed SP comets with $q < 2$ AU obtained from the population of IP comets derived from: **a** parabolic comets with $0.1 < q < 2$ AU and a limiting number of 400 revolutions, and **b** parabolic comets with $4 < q < 5.2$ AU. The transfer IP comets \rightarrow SP comets is followed up to a limiting number of: **a** 200 revolutions, and **b** 10^3 revolutions

range $2.5 < T < 3$ (Fig. 9b), in closer agreement with that observed for SP comets, though the fraction of captured SP comets with $T < 2.5$ is still significant, while among the observed SP comets very few have $T < 2.5$.

It is interesting now to analyze the fraction of time $\Phi(q < 2)$ that a periodic comet will remain captured in a SP orbit with $q < 2$ AU. This fraction will be obtained by summing all the Γ_j ($j=1, \dots, 6$) derived for $q < 2$ AU. Some values of Φ are shown in Table 2 for different values of n_{MAX} . There is an increase of Φ with n_{MAX} as more IP comets are allowed to evolve to SP orbits. For comets starting with $q < 2$ AU, the fraction Φ will roughly represent the ratio SP comets/IP comets for $q < 2$ AU as very few comets will evolve toward larger q during their dynamical evolution.

For comets evolving from the Jupiter's zone ($4 < q < 5.2$ AU), the fraction Φ strongly decreases when we limit the number of passages as less comets have the chance to evolve toward small- q orbits. More than 90% of the comets reaching $q < 2$ AU are already in SP orbits.

Table 2. Fraction of time that periodic comets remain in SP orbits with $q < 2$ AU

n_{MAX}	$\Phi(q < 2)$
<i>Comets with initial $q < 2$ AU</i>	
200	$3.2 \cdot 10^{-3}$
10^3	$4.4 \cdot 10^{-3}$
10^4	$1.7 \cdot 10^{-2}$
<i>Comets with initial $4 < q < 5.2$ AU</i>	
10^3	$1.3 \cdot 10^{-3}$
No limit	$2.1 \cdot 10^{-2}$

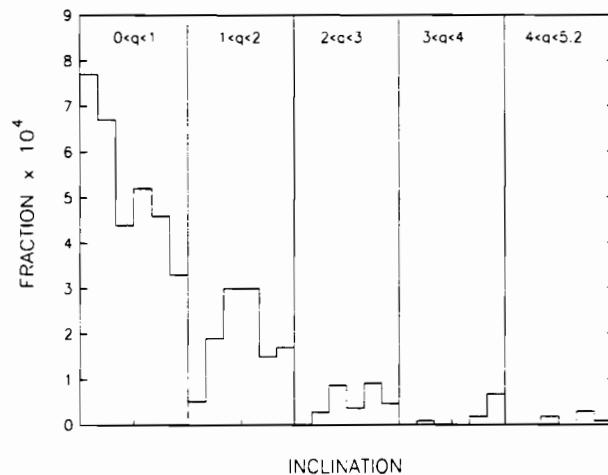


Fig. 10. Inclination distributions of the computed SP comets obtained during the dynamical evolution of the sample of fictitious IP comets with initial perihelion distances $q < 2$ AU followed up to a maximum of 10^3 revolutions and the initial i -distribution shown in Fig. 7 for $x = 0.03 \text{ AU}^{-1}$ and $n_{\text{MAX}} = 400$ revolutions (shaded histogram). The range of perihelion distances corresponding to each histogram is shown at the top

There are SP comets captured in retrograde orbits for all the considered cases. The fraction of retrograde orbits is larger for SP comets captured from an original population with $q < 2$ AU for which is nearly 50% (Fig. 10). It decreases to about 30% for SP comets coming from the Jupiter's zone (Figs. 11 and 12).

SP comets captured from the range $0 < q < 2$ AU will tend to remain in the same range. SP comets captured from the Jupiter's zone will spread more or less uniformly over all the range of q interior to Jupiter's orbit over a large number of revolutions. When we limit the maximum number of revolutions ($n_{\text{MAX}} = 10^3$), the number of captured SP comets will drop toward smaller q (Fig. 12), in better agreement with the observations (cf. Fig. 4).

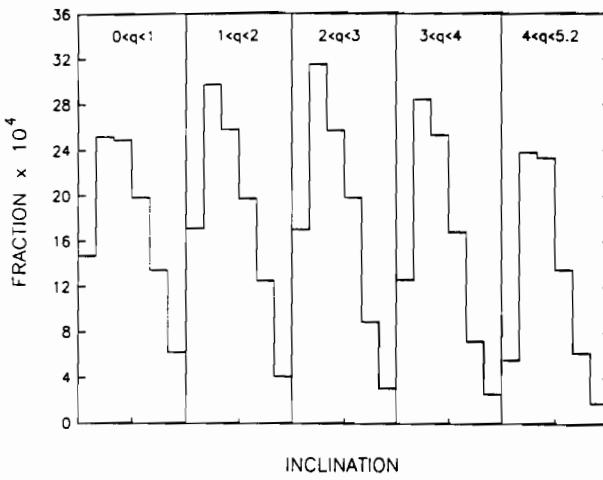


Fig. 11. Idem as Fig. 10 but for the sample of fictitious IP comets with initial q in the range $4 < q < 5.2$ AU. There is no limit to the number of passages for the simulation in the stage IP orbit \rightarrow SP orbit

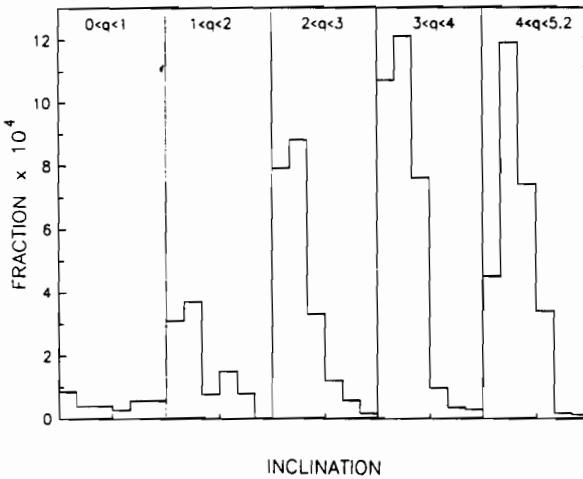


Fig. 12. Idem as Fig. 11 but for a limiting number of 10^3 revolutions for the stage IP orbit \rightarrow SP orbit.

5. The steady-state populations of IP comets and SP comets

Let us try to use the above results to evaluate the resultant steady-state populations of IP comets and SP comets from a source of nearly parabolic ("new") comets coming from the Oort cloud. From the observed frequency of apparitions of new comets and making allowance for missed comets, the influx rate of new comets interior to Jupiter's orbit brighter than absolute magnitude 11 is $\dot{N}_{\text{NEW}} \approx 0.5 \text{ yr}^{-1} \text{ AU}^{-1}$ (Fernández & Ip 1991). From our simulations with samples A1, ..., A6 we obtain the num-

ber of survivors reaching energies $x > x_p \approx 0.03 \text{ AU}^{-1}$ and within inclination ranges of 30° width (see the corresponding histograms of Fig. 7). From these results we can easily derive an average capture probability into an IP orbit: $p_c(x_p) \sim 0.018$ for randomly oriented comets (no limit to the number of passages), or $p'_c(x_p) \sim 0.0085$ when we set the limit $n_{\text{MAX}} = 400$.

From the previous results we can compute the supply of IP comets with $q < 2$ AU averaged over all the inclinations as

$$\dot{N}_{\text{IP}}(q < 2) = \dot{N}_{\text{NEW}}(q < 2) \times p'_c(x_p)$$

$$\sim 1 \text{ yr}^{-1} \times 0.0085 \sim 0.01 \text{ yr}^{-1}, \quad (11)$$

where the choice of p'_c arises from considering the upper limit of 400 revolutions as the best fit to the observed i -distribution (see the discussion in the previous section). We note that this upper limit actually represents an oversimplification of a complex process in which LP comets may have quite different physical lifetimes, some can last much less while others much more than 400 revolutions. Thus, n_{MAX} can be taken as a sort of average for a wide range of physical lifetimes.

If the average period of an IP comet is ~ 80 yr and its typical lifetime is of the order of n_{MAX} , say of a few hundreds revolutions, then the comet will last a few 10^4 yr. The steady-state population of IP comets with $q < 2$ AU will then be

$$N_{\text{IP}}(q < 2) = \dot{N}_{\text{IP}}(q < 2) \times \tau_{\text{IP}} \sim 300 \text{ comets}, \quad (12)$$

where τ_{IP} is the typical lifetime in years of an IP comet for which we have adopted a value of 3×10^4 yr.

We have so far discovered 21 IP comets with $q < 2$ AU, but they surely represent a small fraction of the whole population of IP comets with $q < 2$ AU. If we define the "detectable region" as the sphere centered on the Sun of radius 2.5 AU, we find that a typical IP comet of $P = 80$ yr and $q = 1$ AU will spend only $\sim 1\%$ of the time within the detectable region. In the last 150 yr – where the record of comet apparitions is more complete – it had the chance to pass only a couple of times by its perihelion, thus making difficult its detection. Thus the degree of completeness of IP comet discoveries of 21/300 $\sim 7\%$ seems reasonable. For instance, Kresák & Pittich (1978) give degrees of completeness of LP comets ranging from $\sim 60\%$ for $q < 1$ AU down to $\sim 10\%$ for $q = 2$ AU. It is reasonable to think that it should be lower for more evolved – and thus presumably fainter – comets. For SP comets the degree of completeness is $\sim 30\%$ for $q < 2$ AU (Fernández et al. 1992), but we should bear in mind that SP comets pass more often by their perihelia which increases their probability of discovery. For instance, a typical SP comet of $P = 7$ yr and $q = 1$ AU will spend $\sim 15\%$ of the time within the detectable region. In short, the lower degree of completeness of the sample of IP comets can be the result of

two unfavorable circumstances: first they are on the average less active than LP comets owing to their longer dynamical ages, and second they have only 1–3 perihelion passages in the last 150 yr as compared to as many as 15–25 for SP comets.

Let us now examine how many IP comets evolving from the initial range $q < 2$ AU are expected to end up in SP orbits. For instance, if we assume an upper limit of several 10^2 revolutions (e.g. Fernández 1985), then we have $\Phi(q < 2) \sim 4 \times 10^{-3}$ (see Table 2), which leads to

$$N_{\text{SP}}(q < 2) = N_{\text{IP}}(q < 2) \times \Phi(q < 2) \sim 300 \times 4 \times 10^{-3} = 1.2. \quad (13)$$

The steady-state number of SP comets coming from the flux of LP comets turns out to be two orders of magnitude too short, so the observed Jupiter family must have another dynamical origin.

Let us now analyze the results for comets evolving in the Jupiter's zone ($4 < q < 5.2$ AU) as derived from the samples B1, ..., B6. The average probability of capture Π_c into an IP orbit for a comet with random orientation can be easily computed from the number of survivors within inclination ranges of 30° width shown in Fig. 8, we get approximately $\Pi_c(x_p) \simeq 0.02$. The flux of new comets in the range $4 < q < 6$ AU (like that defined by Everhart 1972) is again about 1 yr^{-1} . If we assume that the validity of Π_c can be extended to the range $4 < q < 6$ AU, the capture rate of IP comets in that interval will then be given by

$$\dot{N}_{\text{IP}}(4 < q < 6) \sim 1 \text{ yr}^{-1} \times 0.02 = 0.02 \text{ yr}^{-1}. \quad (14)$$

If we only consider a dynamical lifetime $\tau_{\text{dyn}} \sim 10^6$ yr (Olsson-Steel 1988) we obtain

$$N_{\text{IP}}(4 < q < 6) = \dot{N}_{\text{IP}}(4 < q < 6) \times \tau_{\text{dyn}} \sim 2 \times 10^4 \text{ comets}. \quad (15)$$

Some IP comets can reach the region $q < 2$ AU, thus adding to the number derived in Eq. (12). Yet, if we take into account their physical lifetimes $\tau_{\text{phys}} \ll \tau_{\text{dyn}}$ and the low probability that a comet reaches an IP orbit with $q < 2$ AU we can estimate that they will not alter significantly the previous result of 300 comets.

The fraction of comets captured in SP orbits with $q < 2$ AU from a population in the Jupiter's zone is $\sim 2.1 \times 10^{-2}$ when physical losses are neglected (cf. Table 2). In this case we obtain

$$N_{\text{SP}}(q < 2) \sim 2 \times 10^4 \times 2.1 \times 10^{-2} = 420 \text{ comets}, \quad (16a)$$

which turns out to be of the same order as the estimated number of SP comets from observations (cf. Fig. 4).

However, if we consider that the maximum number of revolutions is limited by physical decay, for instance to 10^3 revolutions (Fernández 1985), the average lifetime will decrease by at least one order of magnitude, while the fraction of captured SP comets will decrease to $\sim 1.3 \times 10^{-3}$ so that we obtain

$$N_{\text{SP}}(q < 2) \sim 2 \times 10^3 \times 1.3 \times 10^{-3} \simeq 3 \text{ comets} \quad (16b)$$

the steady-state population of SP comets will be in this case two orders of magnitude too small.

6. Discussion

The i -distribution of LP comets with $q < 2$ AU evolving toward periodic orbits suffers changes that mainly imply a depletion of high-inclination orbits. As mentioned, this results from the combination of two dynamical effects: (1) according to the diffusion theory, the capture probability to a certain energy level x is proportional to the typical energy change σ which is larger for direct orbits than for retrograde ones, (2) as x evolves toward larger values during its random-walk (smaller orbital periods), an appreciable migration of near-perpendicular comets toward retrograde orbits takes place. By combining effects (1) and (2), the greater losses of retrograde comets by effect (1) will be roughly offset by the gains by effect (2), so the balance between direct and retrograde orbits is more or less kept throughout the dynamical evolution. When we allow for physical losses by taking a limiting number of revolutions, the resulting i -distribution shows a clear depletion of retrograde orbits in better agreement with the observed i -distribution of the old LP comets with $200 < P < 1000$ yr and that for IP comets (cf. Fig. 1). These results suggest that the observed i -distribution of old LP comets and IP comets with $q < 2$ AU is the result of dynamical causes and physical losses in which the average number of revolutions before disintegration or deactivation is of a few hundreds (Gallardo & Fernández 1992).

Random nearly parabolic comets starting in the interval $4 < q < 5.2$ AU reach the IP stage with an i -distribution already strongly biased toward direct orbits. This is due to the stronger dependence of σ on i (cf. Table 1). Given the large values of q , no physical losses were considered in this case.

The computed steady-state population of IP comets with $q < 2$ AU turns out to be one order of magnitude larger than the observed one. We can interpret this seemingly conflicting result in terms of a very low degree of completeness of the observed sample of IP comets owing to the difficulties to detect them. Since IP comets have on the average passed many times by the planetary region, they may have already lost most of its volatile material and/or built insulating dust mantles, so they should be intrinsically faint. Even an active comet like P/Halley has only $\sim 10\%$ of its surface area still active (Keller et al. 1987). Others may be defunct objects looking asteroidal like 1991 DA (Bailey & Hahn 1992; Steel & Asher 1992). Furthermore, IP comets spend only a tiny fraction of time in the detectable region. For instance, Kresák (1978) estimates a rate of encounters with the Earth to within 0.1 AU of 1 and 0.2 per century for SP comets and IP comets, respectively. When we take into account the much lower encounter probability of IP comets, their number will have to be corrected by a factor ~ 10 to compensate for this

effect. This gives a population of IP comets of the same size as that of SP comets, namely of a few hundreds for $q < 2$ AU.

A fraction of the captured SP comets are found to be in retrograde orbits in agreement with some previous numerical results (Duncan et al. 1988), but in contradiction with the observations (cf. Fig. 2). Our results are independent of the initial range of perihelion distances and of whether we consider physical losses by limiting the number of passages to a value n_{MAX} . One possibility may be to consider very low values of n_{MAX} in the hope of getting only captures of direct orbits (say $n_{\text{MAX}} \lesssim 100$). But this may be too short as compared to the estimated physical lifetimes of a few 10^2 revolutions for LP comets. Furthermore, for $n_{\text{MAX}} < 100$ the capture probability in SP orbits will be too low to account for the observed number of SP comets.

The computed steady-state number of SP comets with $q < 2$ AU from a flux of near-parabolic orbits in the same range of q turns out to be of only a few. This points to another dynamical origin for the Jupiter family comets and helps to explain why there are no recorded SP comets in retrograde orbits. In the sample of observed SP comets with $q < 2$ AU, there are only three that might directly come from the orbital evolution of small- q LP comets as inferred from their low values of T and high inclinations; they are: P/Tuttle ($T = 1.61$, $i = 54^\circ .2$), P/IRAS ($T = 1.96$, $i = 46^\circ .2$), and P/Machholz ($T = 1.94$, $i = 60^\circ .1$).

The capture of SP comets from the Jupiter's region (say $4 < q < 6$ AU) can supply the right steady-state number of SP comets when no upper limit to the number of passages is imposed. However, there are several other results in conflict with the observations as, for instance, a fraction of $\sim 1/3$ of retrograde orbits, values of the Tisserand constant with predominance of $T < 2.5$ and a q -distribution of SP comets more or less uniform (Fig. 11) in contradiction with the observed steep increase in the number of SP comets with q (at least to $q \sim 2.5$ AU). When we limit the number of passages to 10^3 , we obtain distributions of q and T in better agreement with the observed ones (Figs. 9b and 12), though the steady-state number of SP comets decreases by about two orders of magnitude and thus falls too short to explain the size of the observed population. Furthermore, there is the persisting problem of $\sim 1/3$ SP comets in retrograde orbits.

The capture of Oort cloud comets by Jupiter does not seem to be a suitable source of the Jupiter family comets, given the difficulties in getting computed populations in close match with the observed one. This confirms some previous results based on the failure to produce the right number of SP comets (Joss 1973; Fernández & Ip 1981) or from conflicting results between the observed i -distribution and the computed ones with a significant fraction of SP comets in retrograde orbits (Duncan et al. 1988; Wetherill 1991). We have shown in addition that the distributions of T and q also play against this source.

The above objections lead us to discuss possible alternative sources for the Jupiter family. Comets like

P/Schwassmann-Wachmann 1 in the Jupiter-Saturn region may be suitable precursors. And P/S-W 1 may be only the brightest member of a transient population of 30 000 – 100 000 comets (Delsemme 1973), large enough to keep the population of SP comets in steady-state. But comets in the Jupiter–Saturn region have rather short dynamical lifetimes (Gladman & Duncan 1990), so a suitable more distant source has to be found for them. A possibility is that Saturn, Uranus and Neptune capture a sufficient number of comets coming from the Oort cloud to maintain this transient population. The capture efficiencies of these giant planets should be studied in more detail as well as their dependence on the orbital inclination; in other words, its capability to screen out high- i comets in order to produce a flat population. Although preliminary results (Duncan et al. 1988; Wetherill 1991) seem to indicate that a fraction of comets with initial random inclinations in Neptune-crossing orbits will be captured in retrograde SP orbits with $q < 1.5$ AU, some objections as, e.g. artificially high masses for the giant planets (Stagg & Bailey 1989), strongly favors to re-analyze the whole subject.

Another interesting possibility is that the Jupiter-family comets do not come from the capture of Oort cloud comets by the giant planets but from another reservoir located beyond Neptune, the Kuiper belt (Fernández 1980; Duncan et al. 1988). The Kuiper belt would have a strongly flattened, ring-shaped structure, which would explain the i -distribution of the observed SP comets.

7. Concluding remarks

When we take together into consideration several properties of comets of different dynamical classes as, for instance: (a) their i -distributions, (b) their T -distributions, (c) the ratio between the number of SP comets to that of IP comets, (d) the capture probability into a SP orbit with $q < 2$ AU, and (e) the q -distribution of SP comets, we conclude that the capture of Oort cloud comets by Jupiter is able to explain the observations down to IP comets, but is unable to explain the observed population and orbital properties of SP comets. Another more distant source is required as, for instance, capture of Oort cloud comets by Saturn, Uranus and Neptune or comets coming from a transneptunian belt.

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The capture of SP comets from the Jupiter's region (say $4 < q < 6 \text{ AU}$) can supply the right steady-state number of SP comets when no upper limit to the number of passages is imposed. However, there are several other results in conflict with the observations as, for instance, a fraction of $\sim 1/3$ of retrograde orbits, values of the Tisserand constant with predominance of $T < 2.5$ and a q -distribution of SP comets more or less uniform (Fig. 11) in contradiction with the observed steep increase in the number of SP comets with q (at least to $q \sim 2.5 \text{ AU}$). When we limit the number of passages to 10^3 , we obtain distributions of q and T in better agreement with the observed ones (Figs. 9b and 12), though the steady-state number of SP comets decreases by about two orders of magnitude and thus falls too short to explain the size of the observed population. Furthermore, there is the persisting problem of $\sim 1/3$ SP comets in retrograde orbits.

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The above objections lead us to discuss possible alternative sources for the Jupiter family Comets like

P/Schwassmann-Wachmann 1 in the Jupiter-Saturn region may be suitable precursors. And P/S-W 1 may be only the brightest member of a transient population of 30 000 – 100 000 comets (Delsemme 1973), large enough to keep the population of SP comets in steady-state. But comets in the Jupiter-Saturn region have rather short dynamical lifetimes (Gladman & Duncan 1990), so a suitable more distant source has to be found for them. A possibility is that Saturn, Uranus and Neptune capture a sufficient number of comets coming from the Oort cloud to maintain this transient population. The capture efficiencies of these giant planets should be studied in more detail as well as their dependence on the orbital inclination; in other words, its capability to screen out high- i comets in order to produce a flat population. Although preliminary results (Duncan et al. 1988; Wetherill 1991) seem to indicate that a fraction of comets with initial random inclinations in Neptune-crossing orbits will be captured in retrograde SP orbits with $q < 1.5 \text{ AU}$, some objections as, e.g. artificially high masses for the giant planets (Stagg & Bailey 1989), strongly favors to re-analyze the whole subject.

Another interesting possibility is that the Jupiter-family comets do not come from the capture of Oort cloud comets by the giant planets but from another reservoir located beyond Neptune, the Kuiper belt (Fernández 1980; Duncan et al. 1988). The Kuiper belt would have a strongly flattened, ring-shaped structure, which would explain the i -distribution of the observed SP comets.

7. Concluding remarks

When we take together into consideration several properties of comets of different dynamical classes as, for instance: (a) their i -distributions, (b) their T -distributions, (c) the ratio between the number of SP comets to that of IP comets, (d) the capture probability into a SP orbit with $q < 2 \text{ AU}$, and (e) the q -distribution of SP comets, we conclude that the capture of Oort cloud comets by Jupiter is able to explain the observations down to IP comets, but is unable to explain the observed population and orbital properties of SP comets. Another more distant source is required as, for instance, capture of Oort cloud comets by Saturn, Uranus and Neptune or comets coming from a transneptunian belt.

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