STABILITY OF THE DOUBLE-CUSP SPACETIMES AND LONG-TIME GEOMETRIZATIONS

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Abstract. Since the early years of General Relativity, understanding the long-time behavior of the cosmological solutions of Einstein's vacuum equations has been a fundamental yet challenging task. Solutions with global symmetries, or perturbations thereof, have been extensively studied and are reasonably understood. On the other hand, thanks to the work of Fischer-Moncrief and M. Anderson, it is known that there is a tight relation between the future evolution of solutions and the Thurston decomposition of the spatial 3-manifold. Consequently, cosmological spacetimes developing a future asymptotic symmetry should represent only a negligible part of a much larger yet unexplored solution landscape. In this work, we revisit a program initiated by the second named author, aimed at constructing a new type of cosmological solution first posed by M. Anderson, where (at the right scale) two hyperbolic manifolds with a cusp separate from each other through a thin torus neck. Specifically, we prove that the so-called double-cusp solution, which models the torus neck, is stable under $\mathbb{S}^1 \times \mathbb{S}^1$ - symmetry-preserving perturbations. The proof, which has interest on its own, reduces to proving the stability of a geodesic segment as a wave map into the hyperbolic plane and partially relates to the work of Sideris on wave maps and the work of Ringström on the future asymptotics of Gowdy spacetimes.

1. Introduction

Since the early years of General Relativity, understanding the long-time behavior of the cosmological solutions of Einstein's equations has been a fundamental yet quite challenging task. Solutions with spatial symmetries, like the spatially homogeneous Bianchi models or the Gowdy $\mathbb{T}^2$-symmetric spacetimes, have been extensively studied over the decades and are reasonably well understood [8], [5], [9]. All these models are very valuable and provide explicit examples of future dynamics but fall short when the goal is to describe the full set of possible future behaviors. In this work, we revisit a program initiated by the second named author, aimed at constructing a new type of cosmological solution first posed by M. Anderson with a qualitative behavior that is pretty different from any other model known. As we will explain below, such a solution would provide strong support to some ideas developed by Fischer-Moncrief and Anderson relating fundamentally the topology of the Cauchy 3-hypersurfaces to the dynamics of the cosmological solutions [3], [4], [1], [2].

Motivated by certain considerations on the Thurston geometrization conjecture, Anderson posed in [2] (see paper's bottom) the problem of finding a cosmological solution where coarsely speaking, two hyperbolic 3-manifolds with a cusp[1] separate from each other through a thin torus neck. In Anderson's picture, hypersurfaces Σ_k of mean curvature $k \in (-\infty, 0)$ evolve in the expanding direction $k \uparrow 0$, but the geometry at each time k is scaled so that the mean curvature of Σ_k is -3. Under this scaling, the two hyperbolic pieces with their corresponding cusp should emerge over time, separating from each other along an increasingly thin torus neck that develops asymptotically a $\mathbb{T}^2$-symmetry. Figure 1 schematizes that behavior. This spacetime would comprise a new and non-trivial example of a cosmological solution of the vacuum Einstein equations whose spatial geometry (at the mentioned scale) evolves towards the Thurston decomposition of its Cauchy hypersurface. Furthermore, for this solution, the Fischer-Moncrief's reduced volume would decay towards its topological lower bound given by $(-\sigma/6)^{3/2}$, where σ is the Yamabe invariant of the Σ's.

The double-cusps solutions were introduced in [6] and are explicit $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$-symmetric solutions on $\mathbb{R} \times \mathbb{R} \times \mathbb{T}^2$ tailored to model the evolution of the torus neck. As we will see, they enjoy all the required global and asymptotic properties and are therefore a crucial piece. But before attempting to study the combined evolution of the torus neck and the two hyperbolic manifolds with a cusp, it becomes necessary to prove that the double cusps are future-stable and provide sharp decaying estimates. In this article, we do that but for perturbations preserving the $\mathbb{T}^2$-symmetry. Quite interestingly, this problem reduces to proving the stability of a parametrized geodesic segment (that models the double-cusp) as a wave map from a (flat) 3-dimensional spacetime into the hyperbolic plane. The stability problem of geodesics as wave maps was studied by Sideris in [10], but the problem considered in that work is different from ours. Also, the future evolution of $\mathbb{T}^3$-Gowdy spacetimes was studied by Ringström in [7], through a wave map that, of course, has the same origin as ours. In that case, any solution defines a loop in hyperbolic space, while here, it defines a curve with the same ends as the geodesic segment.

In section 2 we introduce the double cusp spacetimes and describe their global properties. This analysis is not relevant for the technical part of the paper, sections 4 to 6, but helps to understand the geometric motivation of the article and to point out certain subtleties that appear when we discuss, in section 3, the statements of the main results and the stability of the double cusp as spacetimes.

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[1]A cusp is a 3-manifold $(-\infty, 0] \times \mathbb{T}^2$ with a hyperbolic metric of the form $g_H = dx^2 + e^{2x} g_T$, with g_T flat $x-$independent on $\mathbb{T}^2$. Cusps are discussed later in the article.

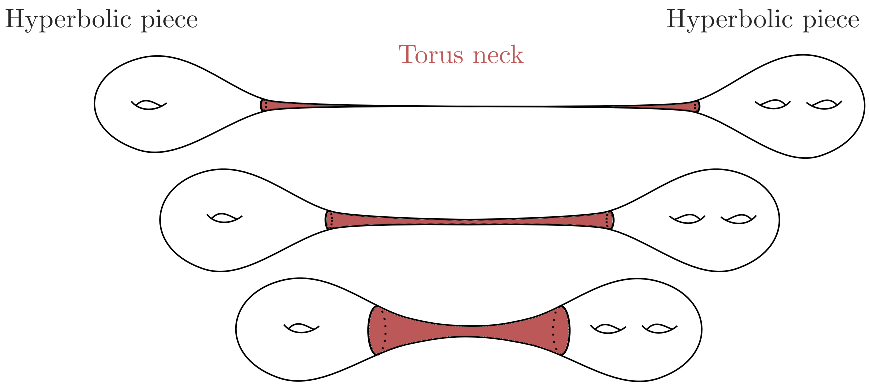

FIGURE 1. Expected behavior of the type of solution posed in [2].

2. Double-cusps and their global properties

The double cusp spacetimes have metrics of the form [6],

$$g = e^{2a}(-dt^2 + dx^2) + R(e^{2W} + q^2 e^{-2W})d\theta_1^2 - 2Rqe^{-2W}d\theta_1 d\theta_2 + Re^{-2W}d\theta_2^2, \quad (1)$$

over the manifold $\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{S}^1_{\theta_1} \times \mathbb{S}^1_{\theta_2}$, where a, R, and q depend only on t and x. Cauchy hypersurfaces, for instance, those with t constant, are diffeomorphic to $\mathbb{R} \times \mathbb{T}^2$ and are thus "torus necks". Metrics of this form are similar to the $\mathbb{T}^3$-Gowdy's metric, [5], but differ from them in that R is not taken as a coordinate and that x is not periodic. The double-cusps are non-stationary spacetimes. We will explicitly present the forms of a, R, and q later below. We will analyze their global geometry, explaining how their geometry behaves along the CMC foliation (hypersurfaces with t constant are not CMC). However, before doing that and to motivate how these solutions arise, we first present the equations for R, W, q, and a. These equations, derived from the Einstein equations, are,

$$R_{xx} - R_{tt} = 0, \quad (2)$$

$$W_{tt} - W_{xx} + \frac{R_t}{R}W_t - \frac{R_x}{R}W_x + \frac{(q_t^2 - q_x^2)}{2}e^{-4W} = 0, \quad (3)$$

$$q_{tt} - q_{xx} + \frac{R_t}{R}q_t - \frac{R_x}{R}q_x - 4q_tW_t + 4q_xW_x = 0, \quad (4)$$

$$a_{tt} - a_{xx} + \tfrac{R_x^2 - R_t^2}{4R^2} + W_t^2 - W_x^2 + \tfrac{1}{4}(q_t^2 - q_x^2)e^{-4W} = 0, \quad (5)$$

and,

$$a_t\frac{R_t}{R} + a_x\frac{R_x}{R} + \frac{1}{4}\left(\frac{R_x^2}{R^2} + \frac{R_t^2}{R^2}\right) - \frac{R_{xx}}{R} - (W_x^2 + W_t^2) - \frac{1}{4}e^{-4W}(q_x^2 + q_t^2) = 0, \quad (6)$$

$$a_x\frac{R_t}{R} + a_t\frac{R_x}{R} - \frac{R_{tx}}{R} + \frac{R_xR_t}{2R^2} + 2W_tW_x - \frac{1}{2}e^{-4W}q_xq_t = 0. \quad (7)$$

The equations (2), (3), (4) and (5) are the dynamical equations for R, W, q and a, and (6) and (7) are the constraint equations. The dynamical equation for R decouples from all the others, and the dynamical equations for W and q decouple from that of a. In certain cases, one can solve globally for a_x and a_t from (6)-(7)

and then simply perform line integrals to find a. In this case, a is determined from R, W, and q up to an integration constant.

It is crucial but also well known that the equations (3) and (4) are wave map equations into the hyperbolic plane. This is seen as follows. Think of the hyperbolic plane $\mathbb{H}$ as $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$ endowed with the metric $h = 4dy^2 + e^{4y}dx^2$. On the other hand consider the manifold $\mathbb{R}_t \times \mathbb{R}_x \times \mathbb{S}^1_\phi$ endowed with the metric $k = 4e^{4t}(-dt^2 + dx^2) + R^2(t,x)d\phi^2$ and denote this Riemannian manifold as $\mathbb{K}$. Then W and q satisfy equations (3) and (4) if and only if the map $\chi : \mathbb{K} \longrightarrow \mathbb{H}$ given by,

$$\chi(t,x,\phi) = (q(t,x), -W(t,x)), \tag{8}$$

is a wave map between the two manifolds. Another way of expressing this is that equations (3) and (4) are the Euler-Lagrange equations of the action,

$$S = \int \partial_l \chi^i \partial_m \chi^j h_{ij} k^{lm}\, dV_k = 2\pi \int R(4(W_x^2 + W_t^2) + (q_x^2 + q_t^2)e^{-4W})dtdx. \tag{9}$$

Let us now see the explicit form of the double cusps. First, for all the double-cusp solutions, one takes $R(t,x) = R_0 e^{2t}\cosh 2x$ with R_0 a constant, which of course solves (2). Second, one requires W and q to be t-independent, i.e. $W = W(x)$ and $q = q(x)$. The Euler-Lagrange equations for such particular data are equivalent to the Euler-Lagrange equations of the action,

$$F = \int_{\mathbb{R}} |\gamma'|^2 \cosh(2x)dx \tag{10}$$

where $\gamma(x) = \chi(x)$, and whose solutions are well known to be parametrized geodesic segments of the hyperbolic plane. When the geodesic segment is vertical and thus has q constant, we say that the double cusp is *polarized*. Their explicit form is,

$$R = R_0 e^{2t}\cosh(2x), \tag{11}$$

$$W = W_1 + W_0 \arctan(e^{2x}), \tag{12}$$

$$q = q_0, \tag{13}$$

$$a = a_0 - \left(\frac{1}{2} + \frac{W_0^2}{2}\right)\frac{1}{2}\ln(\cosh(2x)) + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t, \tag{14}$$

with $R_0 > 0, W_0 \neq 0, W_1$, q_0 and a_0 constants. The non-polarized double-cusps are created by transforming polarized ones by an isometry of the hyperbolic plane (see Figure 2), and the explicit expression won't be particularly relevant. From now on, double cusp solutions will be denoted by R_b, W_b, q_b and a_b, where 'b' stands for 'background'.

Formally speaking, the stability problem that we face amounts to the stability of R_b, W_b, q_b and a_b as particular solutions of a system of partial differential equations, and in fact this is pretty much the viewpoint that we take. We will study first the wave equation for R, then the wave map equation for (W,q), and finally, we will study a, which will be determined entirely from them. As it turns out, to control the perturbations of R and of (W,q), we will use some natural norms that may seem very suitable and standard from a PDE point of view but that may not guarantee the stability of a, and thus of the spacetime, even for perturbations of R, W and q arbitrarily small with respect to them. In the next section, we will see that this apparently conflicting point can be solved using finer norms to measure the smallness of the initial data for R, W, and q over the Cauchy surface $t = 0$. The double cusp spacetimes are stable for small perturbations in the sense of these finer norms, even though the evolution of R, W, and q is controlled with more coarse ones. All these subtleties have their origin in the very nature of the coordinates (t, x) and the nature of the Cauchy hypersurface $\{t = 0\}$, (where we are perturbing the initial data). In the rest of this section, we review the global properties of the double cusps. As said, this information will not play a role when studying the stability of R and of (W, q), which will be treated as a standard PDE problem, but it will help to understand the discussion about the stability of a and therefore of the double cusp as a spacetime.

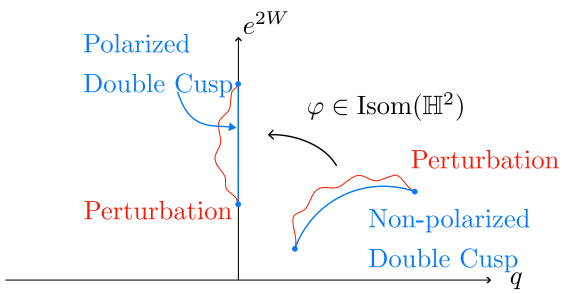

FIGURE 2. In blue, two double cusps are represented. A perturbation is represented in red, as the curve $x \mapsto \chi(t, x)$ for a fixed t. As time evolves, this curve will move. The figure also illustrates how an arbitrary double cusp can be seen as a polarized double cusp.

A main property of the double cusps is that at each of their two ends, one can define spacetime coordinates (t', x') and (t'', x'') where one can observe the spatial scaled metrics converge towards hyperbolic cusps. This is one of the main properties making double cusps adequate to model the necks of the solutions posed in [2]. To explain all that, we begin recalling certain notions on hyperbolic manifolds and flat cone spacetimes. If $(\mathbb{M}, g_H)$ is a hyperbolic manifold, then $\mathbb{R}_\tau \times \mathbb{M}$ endowed with the metric $g = -d\tau^2 + \tau^2 g_H$ is a flat spacetime (hence a solution of the Einstein equations) called a flat cone. The mean curvature of the hypersurface $\tau = \tau_0$ is $k_0 = -3/\tau_0$. Therefore, when the spacetime metric g is scaled as $(k_0/3)^2 g = \tau_0^{-2} g = d(\tau/\tau_0)^2 + (\tau/\tau_0)^2 g_H$, then the mean curvature of the hypersurface $\tau = \tau_0$ becomes -3 and the induced 3-metric g_H. This is called CMC scaling and can be made at any CMC hypersurface Σ_k of mean curvature k inside a spacetime. Hyperbolic manifolds of finite volume can be non-compact. When this is so, the manifold has a finite number of truncated "cusps" of the form $\mathbb{C} = (-\infty, x_0]_x \times \mathbb{T}^2$ with $g_H = dx^2 + e^{2x} g_T$, where g_T is an $x-$independent flat metric on $\mathbb{T}^2$. A cusp spacetime is a flat cone with $\mathbb{M} = \mathbb{R}_x \times \mathbb{T}^2$ and $g_H = dx^2 + e^{2x} g_T$, with g_T $x-$independent and flat. As mentioned a few lines above, the two ends of double cusps are asymptotic to cusp spacetimes as $t \to \infty$. This behavior is not observed in the coordinates t, x but rather in new coordinates linearly related to them. On a double cusp solution, consider the new coordinates,

$$t' = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)x + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)t, \tag{15}$$

$$x' = -\left(\frac{1}{2} + \frac{W_0^2}{2}\right)t + \left(\frac{3}{2} + \frac{W_0^2}{2}\right)x. \tag{16}$$

These coordinates are plotted in Figure 3. When we fix x' and increase t', or when we fix t' and increase x', both x and t increase. In this sense, these new coordinates are adapted to the 'right' end. It is on these coordinates that the double-cusp metric approaches a cusp spacetime metric ($\tau = e^{t'}$). This is easy to show and has been done in [6] in detail. On the 'left' end, one can also define coordinates x'', t'' where the evolution displays a similar behavior. The whole picture is represented in Figure 3. This phenomenon is best observed globally along the CMC foliation. Indeed, double cusps admit a global CMC foliation of Cauchy hypersurfaces Σ_k covering the whole spacetime, where the mean curvature k ranges on $(-\infty, 0)$, [6]. More specifically, there is a Cauchy hypersurface Σ_{-3} of mean curvature -3 defined by a graph $t = s(x)$ and any other leaf of the CMC foliation is obtained translating in t the graph of $s(x)$. Furthermore, the graph of $s(x)$ approaches a $t' = \text{const}$ and $t'' = \text{const}$ line as x goes to ∞ and $-\infty$ respectively. Hence, one can simultaneously observe the convergence to the flat cones on the right and left ends by following the CMC foliation Σ_k, $k \uparrow 0$. If one performs CMC scalings so that the mean curvature of each leaf Σ_k becomes -3, then a convergence-collapse picture emerges. Roughly speaking, the scaled metric over the Σ_k converges to a hyperbolic cusp metric on each of the two ends (one must follow the $x' = \text{const}$ and $x'' = \text{const}$ directions), while the central part collapses its volume while keeping its curvature bounded so that the narrow necks appear to look like thin and long lines. Figure 3 depicts this phenomenon.

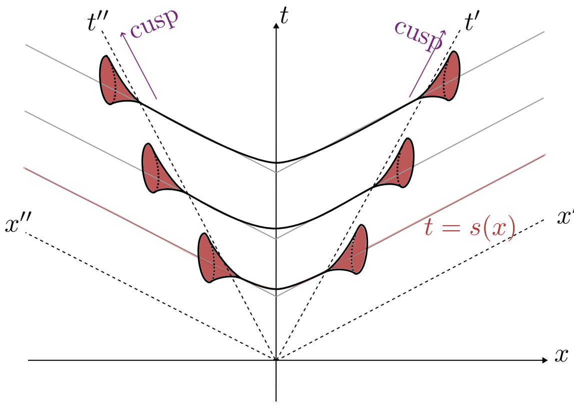

FIGURE 3. Double cusp's behavior over the CMC foliation after CMC scalings.

There is a relevant but standard change of variables $(t, x) \to (R, V)$ given by,

$$R = R_0 e^{2t} \cosh(2x), \quad V = R_0 e^{2t} \sinh(2x), \tag{17}$$

where one can better observe certain important global facts. The spatial coordinate V is the conjugate of R and satisfies the wave equation too. With this change, we have,

$$-dt^2 + dx^2 = 4R^{-2}(-dR^2 + dV^2) \tag{18}$$

so light rays in the plane (t, x) are mapped into light rays in the (R, V) plane. The spacetime region is the region $R > |V|$ and the slice $t = 0$ maps into the hyperbola $R^2 - V^2 = R_0^2$, $R > 0$. This is displayed in Figure 4. The whole picture proves that $t = 0$ is a Cauchy surface for the double cusp spacetime. It can be seen that the spacetime cannot be smoothly extended to the past boundary $R = |V|$, which is singular. The CMC curve $t = s(x)$ is mapped into a curve that is also asymptotic to the lines $R = V, V > 0$ and $R = -V, V < 0$ but is not a hyperbola, of course. Though these two Cauchy hypersurfaces look similar, there is a clear distinction between them: the former approaches the past boundary faster than the latter and in such a way that the Lorentzian distance to the past boundary tends to zero for the former but remains bounded away from zero for the latter. The significant consequence of this is that when considering the Cauchy problem for the Einstein equations (2)-(7) over the hypersurface $t = 0$, perturbations of the initial data that do not fall to zero sufficiently fast as $x \to \infty$ may cause large distortions of the spacetime. This point will reappear when discussing the double cusp's stability statement as a spacetime.

Based on the discussion above, it may seem that making the analysis on the coordinates (t, x) is inconvenient. However, the great advantage of it is that it displays (W_b, q_b) as a time-independent parametrized geodesic segment in the hyperbolic plane. This proves to be very useful. Given the motivation of the stability problem we have discussed, the natural gauge to work would be the CMC gauge. This choice, however, entangles a number of difficulties, for instance, finding and dealing with the right shift, which makes it more complex. We plan to do that analysis elsewhere.

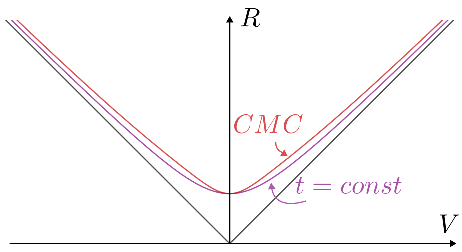

FIGURE 4. Different Cauchy surfaces seen in the $R - V$ coordinates.

3. Statements of the main results

To state the main results, we must first introduce a few norms and spaces. Let,

$$m_0(t) := \|R - R_b\|_{C^0} + \|R_t - R_{bt}\|_{C^0}, \tag{19}$$

$$m_k(t) := \|R - R_b\|_{C^k} + \|R_t - R_{bt}\|_{C^{k-1}}, \text{ for } k \geq 1, \tag{20}$$

where R, R_b, R_t and R_{bt} are considered at time t. The quantity $m_0(t)$ measures the C^0 norm between R and R_b and between R_t and R_{bt}. The quantity $m_k(t)$ instead measures, at time t, the C^k norm between R and R_b and the C^{k-1} norm between R_t and R_{bt}. We also define for $k \geq 1$,

$$\widetilde{\mathcal{M}}_k(t) := \|W - W_b\|_{\widetilde{H}_k} + \|\partial_t(W - W_b)\|_{\widetilde{H}_{k-1}} + \|q\|_{\widetilde{H}_k} + \|\partial_t q\|_{\widetilde{H}_{k-1}}, \tag{21}$$

where again, the functions inside the norms are considered at time t. Here $\widetilde{H}_k$ is the completion of the space of smooth and compactly supported functions, $C_c^\infty(\mathbb{R})$, with respect to the norm,

$$\|f\|^2_{\widetilde{H}_k} := \sum_{i=0}^{k} \int_{\mathbb{R}} (f^{(k)}(x))^2 \cosh(2x)\ dx. \tag{22}$$

This is a weighted Sobolev space with weight $\cosh(2x)$. Lastly, let $C_0^k(\mathbb{R})$ be the space of C^k functions f such that for every $i \leq k$, $f^{(i)}(x) \to 0$ when $|x| \to +\infty$.

We will first discuss some basic statements about the perturbations of R. Regarding (W, q), we will state only the stability results for the polarized double cusp ($q_b = 0$) as an isometry of the hyperbolic plane relates polarized and non-polarized double cusps. We will present two separate stability statements, one for polarized perturbations (i.e. with $q = 0$) and one for general non-polarized perturbations. The former is stronger than the latter. After stating these results, we will discuss what they imply for the stability of a and double cusp spacetimes.

Lemma 3.1. *For every solution R to equation (2), if $m_0(0) < 2R_0/3$, then $R(t,x) > 0$ when $t \geq 0$. Moreover,*

$$\|R - R_b\|_\infty(t) \leq (t+1)m_0(0) \quad \textit{for } t \geq 0,$$

and for all multi-index $\alpha \neq (0,0)$, there is a constant C such that, for $t \geq 0$,

$$\|\partial^\alpha(R - R_b)\|_\infty(t) \leq m_\alpha(0),$$

and

$$\left\|\partial^\alpha\left(\frac{R_t}{R} - \frac{R_{bt}}{R_b}\right)\right\|_\infty, \left\|\partial^\alpha\left(\frac{R_x}{R} - \frac{R_{bx}}{R_b}\right)\right\|_\infty(t) \leq C\frac{t+1}{e^{2t}\cosh(2x)} m_{|\alpha|+1}(0).$$

The first statement below is for polarized perturbations, i.e., $q = 0$. It shows the exponential decay of $\tilde{\mathcal{M}}_k$.

Theorem 6.3. *Let $k \geq 3$. Let (R, W) be a C^2 solution of the system eqs. (2) and (3) with $q = 0$. Suppose also that $(R - R_b(0,\cdot), \partial_t(R - R_b)(0,\cdot)) \in C_0^k \times C_0^{k-1}$, $m_0(0) < 2R_0/3$ and $((W - W_b)(0,\cdot), \partial_t(W - W_b)(0,\cdot)) \in \widetilde{H}^k \times \widetilde{H}^{k-1}$. Then, the solution is defined for every $t \geq 0$ and*

$$\widetilde{\mathcal{M}}_k(t) \leq Ce^{-t}(t+1)\left(\widetilde{\mathcal{M}}_k(0) + m_k(0)\right). \tag{23}$$

Moreover, the constant C depends only on an upper bound on $m_k(0)$ and k.

Observe that except for the condition $m_0(0) \leq 2R_0/3$, which is somehow unavoidable, no particular smallness condition is required. In this sense, this proves that the double cusp is somehow a global attractor among polarized data.

For non-polarized perturbations ($q \neq 0$) a similar estimate is obtained but for $\widetilde{\mathcal{M}}_3(t)$. A smallness hypothesis must also be provided.

Theorem 6.4. *There is a number $\delta > 0$ such that, any C^2 solution of the system eqs.* (2) *to* (4)*, (R, W, q), with initial data satisfying $(R - R_b(0,\cdot), \partial_t(R - R_b)(0,\cdot)) \in C_0^k \times C_0^{k-1}$, $m_0(0) < 2R_0/3$, $((W - W_b)(0,\cdot), \partial_t(W - W_b)(0,\cdot)) \in \widetilde{H}^k \times \widetilde{H}^{k-1}$, $(q(0,\cdot), \partial_t q(0,\cdot)) \in \widetilde{H}^k \times \widetilde{H}^{k-1}$, and $m_3(0) < \delta$, $\widetilde{\mathcal{M}}_3(0) < \delta$, is defined for all $t \geq 0$ and*

$$\widetilde{\mathcal{M}}_3(t) \leq C e^{-t}(t+1)(\widetilde{\mathcal{M}}_3(0) + m_3(0)). \tag{24}$$

Moreover, the constant C depends only on an upper bound on $m_3(0)$ and on $\widetilde{\mathcal{M}}_3(0)$.

This last theorem implies that the red curve in Figure 2 approaches the blue one exponentially fast as $t \to +\infty$.

We now discuss the consequences of the last two results on a and the stability of the double cusp spacetimes.

A basic consequence of the previous results is that if the initial data for (R, W, q) satisfies the hypotheses of either Theorem 6.3 or Theorem 6.4, and (R, W, q, a) satisfies eqs. (2) to (5), then a is defined for all $t \geq 0$. This happens because the equation (5) is a wave equation with a source defined for all $t \geq 0$. Furthermore, if the constraint equations are satisfied at $t = 0$, then, by standard arguments, they are satisfied for every $t \geq 0$. This, in turn, proves that the system, eqs. (2) to (7), gives a Cauchy development of the perturbed initial data on $[0, +\infty) \times \mathbb{R} \times \mathbb{T}^2$. The argument is standard as the system (2)-(7) is equivalent to the Einstein equations, and hence we omit it.

Corollary 3.2. *Consider initial data to the Einstein's equations on $\mathbb{R} \times \mathbb{T}^2$, induced by eq.* (1) *on $t = 0$. Suppose that R, R_t, W, W_t, q_t and q_t, at $t = 0$, satisfy the hypotheses of either Theorem* 6.3 *or Theorem* 6.4*. Then, the solutions of the system given by eqs.* (2) *to* (5) *are defined for all $t \geq 0$, and the metric, eq.* (1)*, given by these functions on $[0, +\infty) \times \mathbb{R} \times \mathbb{T}^2$ gives a Cauchy development of the data.*

Regarding the stability of a, note that the equation (5) implies

$$(a - a_b)_{tt} - (a - a_b)_{xx} = F(R, W, q) - F(R_b, W_b, q_b). \tag{25}$$

where the source $F(R, W, q) - F(R_b, W_b, q_b)$ is controlled by eq. (24). Using this and D'Alembert, we deduce that the contribution of the source to $\|a(t,\cdot) - a_b(t,\cdot)\|_{C^0}$ is controlled by $\tilde{\mathcal{M}}_2(0) + m_2(0)$. The contribution of the homogeneous solution is controlled by the initial data norm $\|a(0,\cdot) - a_b(0,\cdot)\|_{C^0} + \|a_t(0,\cdot) - a_b(0,\cdot)\|_{L^1}$. This immediately leads to the following result.

Theorem 3.3. *For any $\epsilon > 0$ there exist $\delta > 0$ such that if the initial data for (R, W, q, a) satisfies $\widetilde{\mathcal{M}}_3(0) + m_3(0) \leq \delta$ and $\|a(0,\cdot) - a_b(0,\cdot)\|_{C^0} + \|a_t(0,\cdot) - a_{bt}(0,\cdot)\|_{L^1} < \delta$, then $\|a(t,\cdot) - a_b(t,\cdot)\|_{C^0} \leq \epsilon$ $\forall t \geq 0$.*

The spacetime perturbations of the previous theorem are future geodesically complete. Once again, the functional space for the perturbations of a is chosen from the point of view of the PDEs. The natural question is if there is a non-trivial perturbation on these spaces subjected to the constraint equations. Indeed, this is the case, for instance, if we require stricter norms for (R, W, q) at $t = 0$. By doing this, $a - a_b$ naturally belongs to the above spaces for each $t \geq 0$. An example of these norms could be $\|f\|^2_{\widetilde{H}_{p,k}} := \sum_{i=0}^{k} \int_{\mathbb{R}} (f^{(k)}(x))^2 \cosh^p(2x)\, dx$ for W an q, and $m_{l,k}(f) := m_k(\cosh^l(2x) f(x))$ for R. With these norms, using $p = 2$ instead of $p = 1$, and $l = 1$ instead of $l = 0$, one can see that small perturbations in this new sense imply that a_t and a_x can be solved out in eqs. (6) and (7). Furthermore, $\|a(t,\cdot) - a_b(t,\cdot)\|_{C^0} + \|a_t(t,\cdot) - a_{bt}(t,\cdot)\|_{L^1}$ is finite for each $t \geq 0$, and arbitrarily small by reducing the values of $m_{l,1}(R - R_b)(0)$ and $\widetilde{\mathcal{M}}_{2,3}(0)$, where

$$\widetilde{\mathcal{M}}_{p,k}(t) := \|W - W_b\|_{\widetilde{H}_{p,k}} + \|\partial_t(W - W_b)\|_{\widetilde{H}_{p,k-1}} + \|q\|_{\widetilde{H}_{p,k}} + \|\partial_t q\|_{\widetilde{H}_{p,k-1}}.$$

The proof of the Theorems 6.3, 6.4, are done first for *compactly supported perturbations*, i.e., solutions with initial data differing only on a compact set from that of the background. This is done in section 4 and 5. The latter section is the central part of the paper. Finally, in section 6, we give a rather general argument to extend this simplified versions to larger functional spaces, proving theorem 6.3 and 6.4.

4. Compactly supported polarized perturbations

In this section, we shall address compactly supported perturbations with $q = 0$, namely, a solution (R, W) of the system given by eqs. (2) and (3) whose initial data differs from that of the background only on a compact set. Although we will not use the results found in this section explicitly in the non-polarized case, the computations used in the non-polarized case rely on the ones developed here. Furthermore, here the computations are more clear and yield stronger results.

At first, the solutions are not defined for all $t \geq 0$. All the estimates concerning will be for $t \geq 0$ in the interval of existence. Furthermore, we also assume $m_0(0) < 2R_0/3$ to ensure $R > 0$ for $t \geq 0$. Throughout the work, depending on the kind of computations, we will use $\partial_x f$ or f_x. Also $\partial^\alpha f$ with α a multi-index. We say that a function $f(t, x)$ is *of locally x-compact support* if for any interval $[T_1, T_2]$ there is a compact subset $K \subset \mathbb{R}$ such that $f(t, x) = 0$ if $t \in [T_1, T_2]$ and $x \notin K$. Recall the notation m_k. We summarize some elementary properties in the following lemma.

Lemma 4.1. *If (R, W) is a solution such that $R - R_b$ and $W - W_b$ at $t = 0$ are compactly supported and $m_0(0) < \frac{2}{3}R_0$ then, for $t \geq 0$, and where defined, $R > 0$ and $(R - R_b, W - W_b)$ is of locally x-compact support.*

Proof. The use of D'Alembert's formula gives $R \geq R_b - m_0(0)(t + 1), t \geq 0$. With this, it can be seen that both claims about R are true. For W, use a finite speed propagation argument. □

4.1. **The basic energy inequality.** In order to obtain the asymptotic stability of our solution, we need some useful energy. Consider the change of variable $z = R^{1/2}(W - W_b)$. In this new variable, the equation for W becomes

$$z_{tt} - z_{xx} + zG = g \quad \text{with } G = \frac{R_t^2 - R_x^2}{4R^2} \text{ and } g = R^{1/2}\left(\frac{R_x}{R} - \frac{R_{bx}}{R_b}\right)W_{bx}. \tag{26}$$

Now z is of locally x-compact support, then it makes sense to define the energy

$$E := \frac{1}{2}\int_{\mathbb{R}} z_t^2 + z_x^2 + z^2 G_b \, dx,$$

where $G_b = \frac{1}{\cosh^2(2x)}$. We need some estimates involving $R - R_b$ so that we can control E. These properties are given in the following lemma.

Lemma 4.2 (Coefficients estimates)**.** *For all $t \geq 0$, we have the following estimates:*

$$\|R - R_b\|_\infty(t) \leq (t+1)m_0(0), \tag{27}$$

$$\|R_x - R_{bx}\|_\infty(t), \|R_t - R_{bt}\|_\infty(t) \leq m_1(0). \tag{28}$$

Furthermore, as we are assuming $m_0(0) \leq \frac{2}{3}R_0$ we also have that $R \sim R_b$, i.e, there is a constant $d > 0$ such that

$$\frac{1}{d} \leq \left\|\frac{R_b}{R}\right\|_\infty(t) \leq d, \tag{29}$$

and this, in turn, implies the existence of a constant $C > 0$ such that

$$\left\|\frac{R_t}{R} - \frac{R_{bt}}{R}\right\|_\infty(t), \left\|\frac{R_x}{R} - \frac{R_{bx}}{R_b}\right\|_\infty(t) \leq \frac{C(t+1)}{e^{2t}\cosh(2x)}m_1(0), \tag{30}$$

$$\|G_b - G\|_\infty(t) \leq \frac{C(t+1)}{e^{2t}\cosh(2x)}m_1(0), \tag{31}$$

$$\|g\|_\infty(t) \leq \frac{C(t+1)}{e^{t}\cosh^{3/2}(2x)}m_1(0). \tag{32}$$

Proof. Use D'Alembert to derive the first estimates. The other follows from direct computation using these. □

Proposition 4.3 (The basic energy inequality)**.** *Let (R, z) be a solution such that $R - R_b$, z and z_t are compactly supported at $t = 0$, with $m_0(0) < 2R_0/3$. Then there is a constant C such that, where defined,*

$$E^{1/2}(t) \leq C(E^{1/2}(0) + m_1(0)), \ \textit{with} \ \ t \geq 0. \tag{33}$$

Furthermore, the constant C depends only on a bound on $m_1(0)$.

Proof. The function z is of locally x-compact support as $W - W_b$ is. For this reason, we can derive under the integral and use integration by parts. Then, we use the equation to obtain

$$
\begin{aligned}
\dot{E} &= \int_{\mathbb{R}} z_t z_{tt} + z_x z_{xt} + z z_t G_b \; dx = \int_{\mathbb{R}} z_t (z_{tt} - z_{xx} + z G_b) \; dx \\
&= \int_{\mathbb{R}} z_t z (G_b - G) \; dx + \int_{\mathbb{R}} z_t g \; dx \\
&\leq \underbrace{\sqrt{\int_{\mathbb{R}} z_t^2 \; dx} \sqrt{\int_{\mathbb{R}} z^2 (G_b - G)^2 \; dx}}_{\text{First term}} + \underbrace{\sqrt{\int_{\mathbb{R}} z_t^2 \; dx} \sqrt{\int_{\mathbb{R}} g^2 \; dx}}_{\text{Second term}} .
\end{aligned}
$$

In the first term, the first integral is bounded by $E^{1/2}$. For the second integral, use the estimate (31) to find out that this term is less or equal to

$$
C(t+1) m_1(0) e^{-2t} E^{1/2} \sqrt{\int_{\mathbb{R}} \frac{z^2}{\cosh^2(2x)} \; dx},
$$

but the integrand in the last integral is just $4z^2 G_b$, so in the end, our first term is less or equal to

$$
C(t+1) e^{-2t} m_1(0) E,
$$

where we have adjusted the constant C. Let us control the second term. This term is a product of two integrals. The first one is less or equal to $E^{1/2}$. For the second integral, the use of the estimate (32) yields

$$
\int_{\mathbb{R}} g^2 \; dx \leq C(t+1) e^{-t} m_1(0).
$$

Using all these observations, we get

$$
\begin{aligned}
\dot{E} &\leq C(t+1) e^{-2t} m_1(0) E + C(t+1) e^{-t} m_1(0) E^{1/2} \\
&\leq C(t+1) e^{-2t} m_1(0) E + C(t+1) e^{-t} E + C(t+1) e^{-t} m_1^2(0) \\
&\leq C(t+1) e^{-t} E + C(t+1) e^{-t} m_1^2(0),
\end{aligned} \tag{34}
$$

where in the last inequality, the constant depends on $m_1(0)$. This inequality implies the thesis. □

Note that since $R \sim R_b = R_0 e^{2t} \cosh(2x)$, this implies exponential decay in our original variables plus decay as x goes to infinity. However, we do not have future existence for all $t \geq 0$. This matter is the objective of the following subsection.

4.2. **Higher order energies.** Let α be a multi-index $\alpha = (m, n)$. The first letter will refer to time derivatives, and the second will refer to spatial derivatives. Let us denote

$$
E^{\alpha}(t) := \frac{1}{2} \int_{\mathbb{R}} (\partial^{\alpha} z)_t^2 + (\partial^{\alpha} z)_x^2 + (\partial^{\alpha} z)^2 G_b \; dx.
$$

The most important of these energies are the ones with $\alpha = (0, m)$ because they are related to H^k norms of z and $\partial_t z$. This fact is important since, at $t = 0$, they involve only the initial data. One could be tempted to do an argument similar to the one made in the derivation of eq. (33), but using $E^{(0,m)}$. If we do this, we will discover that the growth of $E^{(0,m)}$ is bounded by a polynomial of degree m. This result is not bad, but we found another, longer way to obtain better estimates. First, we derive estimates for $E^{(n,0)}$, and then we pass our estimates to $E^{(0,m)}$ using the equations satisfied by z and its derivatives. Now, more coefficient estimates are needed.

Lemma 4.4. *Suppose $m_0(0) < 2R_0/3$, then:*

a) For all multi-index $\alpha \neq (0,0)$ we have

$$\|\partial^\alpha (R - R_b)\|_\infty(t) \leq m_\alpha(0) \quad t \geq 0. \tag{35}$$

b) For every multi-index α, $\partial^\alpha R/R$ and $\partial^\alpha R^{1/2}/R^{1/2}$ are bounded to the future. Moreover, the bound depends only on a bound on $m_{|\alpha|}(0)$.

c) For every multi-index α, $\partial^\alpha R_t/R$, $\partial^\alpha R_x/R$, $\partial^\alpha(R_t/R)$ and $\partial^\alpha(R_x/R)$ are bounded to the future. Moreover, the bound depends only on a bound on $m_{|\alpha|+1}(0)$.

d) For all multi-index α there is a constant C such that

$$\left\|\partial^\alpha\left(\frac{R_t}{R} - \frac{R_{bt}}{R_b}\right)\right\|_\infty, \left\|\partial^\alpha\left(\frac{R_x}{R} - \frac{R_{bx}}{R_b}\right)\right\|_\infty(t) \leq C\frac{t+1}{e^{2t}\cosh(2x)}m_{|\alpha|+1}(0). \tag{36}$$

e) Estimates for $G_b - G$: for every multi-index α there is a constant C such that

$$\|\partial^\alpha(G - G_b)\|_\infty(t) \leq C\frac{(t+1)}{e^{2t}\cosh(2x)}m_{|\alpha|+1}(0). \tag{37}$$

Moreover, the constant depends only on a bound on $m_{|\alpha|+1}(0)$.
f) For every multi-index α there is a constant C such that

$$|\partial^\alpha G| \leq \frac{C}{\cosh 2x} = 2C\sqrt{G_b}. \tag{38}$$

Furthermore, the constant C depends on α and on a bound on $m_{|\alpha|+1}(0)$.
g) For every multi-index α there is a constant C such that $|\partial^\alpha W_{bx}| \leq C/\cosh(2x)$. In addition, if α is not purely spatial then $\partial^\alpha W_{bx} = 0$.
h) Estimates for g: For every multi-index α there is a constant C such that

$$\|\partial^\alpha g\|_\infty \leq C\frac{t+1}{e^{2t}\cosh^{3/2}(2x)}m_{|\alpha|+1}(0). \tag{39}$$

Additionally, the constant C just depends on a bound on $m_{|\alpha|+1}(0)$.

Proof. Item a) is a direct consequence of D'Alembert's Formula. Item b) and c) are just computations using a) and the fact that $R \sim R_b$. For these computations, it is often helpful to use recursion formulas, such as

$$\partial^\alpha\left(\frac{R_t}{R}\right) = \frac{\partial^\alpha R_t}{R} - \sum_{0\leq\beta<\alpha}\binom{\alpha}{\beta}\frac{\partial^{\alpha-\beta}R\partial^\beta(\frac{R_t}{R})}{R},$$

or

$$\left|\frac{\partial^\alpha(R^{1/2})}{R^{1/2}}\right| \leq \left|\frac{\partial^\alpha R}{2R}\right| + \sum_{0<\alpha<\beta}\binom{\alpha}{\beta}\left|\frac{\partial^{\alpha-\beta}R^{1/2}\partial^\beta(R^{1/2})}{2R}\right|.$$

Item d) is proved similarly, and item e) is a consequence of a)-d). Item f) follows from item e). Item g) follows from direct computation, and finally, g) is a consequence of d), g), and b). $\square$

Proposition 4.5 (Energy estimates for time derivatives)**.** *Let (R, z) be a solution such that $R - R_b$, z and z_t are compactly supported at $t = 0$, with $m_0(0) < 2R_0/3$, and let $\alpha = (m, 0)$. Then, there is a constant C such that, where defined,*

$$\sqrt{E^{(m,0)}}(t) \leq C(\sqrt{E^{(m,0)}}(0) + \ldots + \sqrt{E^{(1,0)}}(0) + \sqrt{E}(0) + m_{|\alpha|+1}(0)), \; \textit{for } t \geq 0.$$

Furthermore, the constant C just depends on m and on a bound on $m_{|\alpha|+1}(0)$.

Proof. We have already proved the case $\alpha = 0$ in the Proposition 4.3. Let us proceed by induction. Deriving the equation with respect to $m > 0$ yields:

$$(\partial_t^m z)_{tt} - (\partial_t^m z)_{xx} + (\partial_t^m z)G + \sum_{i=0}^{m-1} \binom{\alpha}{\beta} \underbrace{\partial_t^i z}_{\substack{\text{in } E^{(i,0)} \\ \text{already controlled}}} \underbrace{\partial_t^{m-i}(G - G_b)}_{eq.\ (37)} = \underbrace{\partial_t^m g}_{eq.\ (39)} .$$

Here, we have used that $\partial_t^i G_b = 0$ in the last term before the equal sign. Now, as we did in the proof of Proposition 4.3), we can derive $E^{(m,0)}$ with respect to time, integrate by parts, use the equation and control each of the terms appearing. The new terms are the ones that have a curly bracket in the above equation. Below these brackets, it is specified how to control these terms. $\square$

The following lemma goes in the direction of proving the desired estimates for $E^{(0,n)}$. Here we use the notation $|\alpha| = m + n$ for $\alpha = (m, n)$.

Lemma 4.6. *In the above assumptions, suppose $n \geq 1$ and $m \geq 0$, then there is a constant C such that, where defined,*

$$\sqrt{E^{(m,n)}}(t) \leq C\sqrt{E^{(m+1,n-1)}}(t) + C \sum_{0 \leq \beta \leq (m,n-1)} \sqrt{E^{\beta}}(t) + Cm_{|\alpha|}(0) \; \textit{for } t \geq 0,$$

where $\alpha = (m, n)$. Now if $m \geq 1$ and $n \geq 0$ then there is a constant C such that, where defined,

$$\sqrt{E^{(m,n)}}(t) \leq C\sqrt{E^{(m-1,n+1)}}(t) + C \sum_{0 \leq \beta \leq (m-1,n)} \sqrt{E^{\beta}}(t) + Cm_{|\alpha|}(0) \; \textit{for } t \geq 0.$$

The constants just depend on a bound on $m_{|\alpha|+1}(0)$, and on m and n.

Proof.

$$\begin{aligned} E^{(m,n)}(t) &= \frac{1}{2} \int_{\mathbb{R}} \underbrace{[(\partial_t^m \partial_x^n z)_t]^2}_{\text{It is in } E^{(m+1,n-1)}} + [(\partial_t^m \partial_x^n z)_x]^2 + \underbrace{(\partial_t^m \partial_x^n z)^2}_{\text{It is in } E^{(m,n-1)}} \underbrace{G_b}_{\leq 1} \, dx \\ &\leq E^{(m+1,n-1)}(t) + E^{(m,n-1)}(t) + \frac{1}{2} \int_{\mathbb{R}} [(\partial_t^m \partial_n^x z)_x]^2 \, dx. \end{aligned}$$

To bound the last term, derive the eq. (26) $m-$times with respect to time and $n-1-$times with respect to x. The derived equation gives us the following estimate,

$$\int_{\mathbb{R}} (\partial_t^m \partial_x^{n+1} z)^2 \leq C \underbrace{\int_{\mathbb{R}} (\partial_t^{m+2} \partial_x^{n-1} z)^2}_{\text{It is in } E^{(m+1,n-1)}} + C \sum_{\beta=(0,0)}^{(m,n-1)} \underbrace{\int_{\mathbb{R}} (\partial^\beta z \partial^{\alpha-\beta} G)^2}_{\leq CE^\beta} + \underbrace{\int_{\mathbb{R}} (\partial_t^m \partial_x^{n-1} g)^2}_{\leq Cm_{|\alpha|}^2(0)} .$$

Using this and putting a square root, we arrive at the first claim stated in the lemma. Keeping track of the constant, we see that the assertion about its dependence is true. For the second inequality stated in the lemma, the same reasoning works. $\square$

Corollary 4.7. *In the above assumptions, given n, there is a constant C such that, where defined,*

$$\sqrt{E^{(0,n)}}(t) \le C(\sqrt{E^{(n,0)}}(t) + \ldots + \sqrt{E^{(1,0)}}(t) + \sqrt{E}(t)) + Cm_n(0), \tag{40}$$

and

$$\sqrt{E^{(n,0)}}(t) \le C(\sqrt{E^{(0,n)}}(t) + \ldots + \sqrt{E^{(0,1)}}(t) + \sqrt{E}(t)) + Cm_n(0). \tag{41}$$

The constant depends only on a bound on $m_{n+1}(0)$ and on n, m.

Proof. Use the previous lemma and induction. □

Proposition 4.8 (Energy estimates for spatial derivatives)**.** *Let (R, z) be a solution such that $R - R_b$, z and z_t are compactly supported at $t = 0$, with $m_0(0) < 2R_0/3$. Then there is a constant C such that, where defined,*

$$\sqrt{E^{(0,n)}}(t) \le C(\sqrt{E^{(0,n)}}(0) + \ldots + \sqrt{E^{(0,1)}}(0) + \sqrt{E}(0)) + Cm_{n+1}(0). \tag{42}$$

The constant C just depends on n and on a bound on $m_{n+1}(0)$.

Proof. First, note that we have already proved the corresponding theorem for $E^{(n,0)}$ in the Proposition 4.5. We first use the eq. (40), then the Proposition 4.5, and finally the eq. (41), this time evaluated at $t = 0$. □

4.3. **Relation with H^k-norms and returning to our original variables.** Let us define

$$\mathcal{M}_k[z](t) := \|z\|_{H^k}(t) + \|\partial_t z\|_{H^{k-1}}(t), \quad \mathcal{A}(t) := \frac{1}{2}\int_{\mathbb{R}} z^2 \, dx.$$

Note that $\dot{\mathcal{A}} \le E^{1/2}\mathcal{A}^{1/2} \le C(E^{1/2}(0) + m_1(0))\mathcal{A}^{1/2}$. Therefore

$$\mathcal{A}^{1/2}(t) \le \mathcal{A}^{1/2}(0) + C(E^{1/2}(0) + m_1(0))t.$$

Theorem 4.9 (H^k-evolution)**.** *Let (R, z) be a solution such that $R - R_b$, z and z_t are compactly supported at $t = 0$, with $m_0(0) < 2R_0/3$, and let $k > 0$. Then there is a constant C such that*

$$\mathcal{M}_k(t) \le C\left(\mathcal{M}_k(0) + m_k(0)\right)(t + 1), \text{ for } t \ge 0. \tag{43}$$

The constant C here depends on a bound on $m_k(0)$ and on k. Furthermore, the solution (R, z) is defined for all $t \ge 0$.

Proof. Just note that $\mathcal{M}_k$ involves $(\partial_x^i z)^2$ with $i = 0, ..., k$ and $(\partial_x^j \partial_t z)^2$ with $j = 0, ..., k-1$, and that all of these terms appear in the t-derivative or the x-derivative term of one of the followings quantities: $\mathcal{A}, E, E^{(0,1)}, \ldots, E^{(0,k-1)}$. Using this, with the above computation for $\mathcal{A}$ and eq. (42) yields eq. (43). Lastly, if we call T_+ the supreme of times T such that the solution is defined on $[0, T]$, then $\mathcal{M}_k(t)$ is bounded on $[0, T_+]$. By a continuity lemma, it follows that $T_+ = +\infty$. □

This result allows us to prove the existence of solutions with certain decay in our original variables.

Lemma 4.10 (Passage lemma)**.** *Given R a solution of eq.* (2) *with $m_0(0) < 2R_0/3$, and let $k > 0$. Then for every solution z to eq.* (26)*, of locally x-compact support, we have that $W - W_b$ is of locally x-compact support and*

$$\frac{1}{C}e^{-t}\|z\|_{H^k}(t) \leq \|W - W_b\|_{\widetilde{H}_k}(t) \leq Ce^{-t}\|z\|_{H^k}(t)$$

where $\widetilde{H}^k$ is the weighted Sobolev space, with $\cosh(2x)\ dx$ *as the weight. Here C is a constant that depends on a bound on $m_k(0)$ and k.*

Proof. The proof is just computations and the use of the estimates derived in Lemma 4.4. □

In the following corollary, we summarize what we have obtained so far. Recall the notation

$$\widetilde{\mathcal{M}}_k(t) := \|W - W_b\|_{\widetilde{H}_k}(t) + \|\partial_t(W - W_b)\|_{\widetilde{H}_{k-1}}(t). \tag{44}$$

Corollary 4.11. *Given (R, W) a smooth solution of the system eqs.* (2) *and* (3) *with $q = 0$. Suppose also that the initial data R, R_t, W, W_t differs from that of the background in a compact set and that $m_0(0) < 2R_0/3$. Then, the solution is defined for every $t \geq 0$ and*

$$\widetilde{\mathcal{M}}_k(t) \leq Ce^{-t}(t+1)\left(\widetilde{\mathcal{M}}_k(0) + m_k(0)\right).$$

Moreover, the constant C depends only on a bound on $m_k(0)$ and k.

5. Compactly supported non-polarized perturbations

The objective of this section is to study non-polarized perturbations, i.e., we allow our perturbations to have $q \neq 0$. As before, we first study compactly supported perturbations.

5.1. **Bounded distance from the background.**

Again, let us consider a solution to the system eqs. (2) to (4). We ask this solution to be smooth and to differ initially from the background only on a compact set. Due to a finite speed propagation argument, $R - R_b$, $W - W_b$, and $q - q_b$, where defined, are of locally x-compact support. These properties allow us to integrate by parts and derive under the integral.

In what follows, we will use a special connection. First of all, recall $\chi : \mathbb{K} \to \mathbb{H}$. Note that its differential, $D\chi$, can be regarded as a section of the fiber-bundle $\mathbb{K} \to T^*\mathbb{K} \otimes \chi^{-1}T\mathbb{H}$, where $\chi^{-1}T\mathbb{H}$ is the pullback bundle over $\mathbb{K}$. In $\chi^{-1}T\mathbb{H}$, we have the pullback connection and the pullback metric, and in the bundle $T^*\mathbb{K}$ we have the metric k and its connection. The product connection on $T^*\mathbb{K} \otimes \chi^{-1}T\mathbb{H}$ preserves the product metric. Using this connection the equations eqs. (3) and (4) are just $\nabla^a\partial_a\chi^i = 0$, i.e, $tr(\nabla D\chi) = 0$. Consider the energy-momentum tensor given by $T_{ab} := \partial_a\chi^i\partial_b\chi^j h_{ij} - \frac{1}{2}g_{ab}g^{\alpha\beta}h_{ij}\partial_\alpha\chi^i\partial_\beta\chi^j$. A direct computation shows that $\nabla^a T_{ab} = \partial_b\chi^i h_{ij}\nabla^a\partial_a\chi^j$. The last term, with $j = 1, 2$, are exactly the equations eq. (3) and eq. (4). Accordingly, the solutions to these equations have divergence null energy-momentum tensor. Recall the metric $k = 4e^{4t}(-dt^2 + dx^2) + R^2(t,x)d\psi^2$, and consider the the vector field $N = \partial_t/2e^{2t}$, the slice $S_t = \{t\} \times \mathbb{R} \times S^1$ and the vector field given by $Y^a = T^a_b X^b$. This vector field is the dual vector to the one form $T(\cdot, X)$. Using Stokes[2] we have:

$$\int_{[0,t_0]\times\mathbb{R}\times S^1} \nabla^a(T_{ab}X^b)\; dVol = -\int_{S_0} X^a T_{ab} X^b \; dS_0 + \int_{S_t} X^a T_{ab} X^b \; dS_t$$

Now $\nabla^a(T_{ab}X^b) = T_{ab}\nabla^a X^b$. A computation shows that

$$\begin{aligned}
\nabla^a X^b &= \frac{1}{4e^{6t}}(\partial_x)^a(\partial_x)^b + R^{-2}\frac{R_t}{R}\frac{1}{2e^{2t}}(\partial_\psi)^a(\partial_\psi)^b,\\
T_{00} = T_{11} &= \frac{1}{2}(\|\partial_t\chi\|_h^2 + \|\partial_x\chi\|_h^2),\\
T_{22} &= \frac{R^2}{2}\frac{\|\partial_t\chi\|_h^2 - \|\partial_x\chi\|_h^2}{4e^{4t}}.
\end{aligned}$$

Using this

$$\begin{aligned}
\partial_t \int_{\mathbb{R}} \frac{1}{2}(\|\partial_t\chi\|_h^2 + \|\partial_x\chi\|_h^2)\frac{R}{e^{2t}}dx = &-2\int_{\mathbb{R}} \frac{\|\partial_t\chi\|_h^2}{2}\left(1+\frac{R_t}{2R}\right)\frac{R}{e^{2t}}\; dx\\
&-2\int_{\mathbb{R}} \frac{\|\partial_x\chi\|_h^2}{2}\left(1-\frac{R_t}{2R}\right)\frac{R}{e^{2t}}\; dx.
\end{aligned}$$

Let us momentarily call the left integral $\mathcal{E}$ (without ∂_t). Since $m_0(0) < \frac{2}{3}R_0$, eq. (36) holds. Using this estimate yields $\partial_t\mathcal{E} \leq C(t+1)(m_1(0)+2)e^{-2t}\mathcal{E}$. Integrating we obtain $\mathcal{E}(t) \leq \exp(C(m_1(0)+2))\mathcal{E}(0)$ for the future. Now

$$\begin{aligned}
|W - W_b| \leq \int_{-\infty}^{x} |(W-W_b)_x| \; dx &\leq C\sqrt{\int_{\mathbb{R}}(W-W_b)_x^2\cosh(2x)\; dx}\\
&\leq C\mathcal{E}^{1/2}(t) \leq C\; exp(C(m_1(0)+2))\mathcal{E}^{1/2}(0),
\end{aligned}$$

and analogously for $q - q_b = q$. This result allows us to conclude that the distance between the background solution and a solution (W, q), whose initial data is a compactly supported perturbation of that of the background, is bounded. Geometrically, this means that if we start with a solution that is a compactly supported perturbation of a geodesic, then this perturbation evolves at a bounded distance. Moreover, this distance depends on a bound on $m_1(0)$ and on $\widetilde{\mathcal{M}}_1(0)$, where now

$$\widetilde{\mathcal{M}}_1[W-W_b, q](t) = (\|W-W_b\|_{\widetilde{H}^1} + \|(W-W_b)_t\|_{\widetilde{H}^0} + \|q\|_{\widetilde{H}^1} + \|q_t\|_{\widetilde{H}^0})(t).$$

We define $\widetilde{\mathcal{M}}_k[\cdot,\cdot]$ in the same manner but using k and $k-1$ norms. Again, whenever we say $\widetilde{\mathcal{M}}_k$ without explicit mention of the functions, the reader should interpret $\widetilde{\mathcal{M}}_k[W-W_b, q]$.

5.2. **The basic energy inequality.** In this section, we derive the basic energy inequality from which exponential decay will follow. We will consider a smooth solution to the system formed by eqs. (2) to (4), whose initial data differs only on a compact set from the data of the background, eqs. (11) to (14), and such that $m_0(0) < 2R_0/3$. These requirements are imposed throughout all the section 5, and, for the sake of concreteness, will be referred to as *the assumptions.* Consider the

[2]As the solutions we are treating now differ from the background only on a compact set, we only have these two terms contributing to the flux.

change of variable $z = R^{1/2}(W - W_b)$ and $v = \tilde{R}^{1/2}q$, where $\tilde{R} = Re^{-4W}$. This change yields the following system of PDEs for z and v:

$$z_{tt} - z_{xx} + zG + B = g, \tag{45}$$

$$v_{tt} - v_{xx} + v(G + 4W_{bx}^2) + D = 0, \tag{46}$$

where

$$\begin{cases} D = 4v(\tilde{W}_x^2 - \tilde{W}_t^2) + 8v\tilde{W}_x W_{bx}, \\ \quad + 2v(q_x^2 - q_t^2)e^{-4W} & B = R^{1/2}\left(\frac{q_t^2 - q_x^2}{2}\right)e^{-4W}, \\ q_t = \tilde{R}^{-1/2}\left(v_t - \dfrac{R_t}{2R}v + 2R^{-1/2}(z_t - \dfrac{R_t}{2R}v)\right), & G = \frac{R_t^2 - R_x^2}{4R}, \\ q_x = \tilde{R}^{-1/2}\left(v_x - \dfrac{R_t}{2R}v + 2R^{-1/2}(z_x - \dfrac{R_x}{2R}z) + 2W_{bx}v\right), & W_{bx} = \frac{W_0}{\cosh(2x)}, \\ \tilde{W} = W - W_b, & \tilde{W}_t = R^{-1/2}\left(z_t - \dfrac{R_t}{2R}z\right), \\ \tilde{W}_x = R^{-1/2}\left(z_x - \dfrac{R_x}{2R}z\right), & g = R^{1/2}\left(\frac{R_x}{R} - \frac{R_{bx}}{R_b}\right)W_{bx}. \end{cases}$$

The system displayed is exactly the system of the polarized case plus powers of first and zeroth derivatives. Our objective of finding exponential decay is translated, under this change of variable, to prove that z and v grow at most polynomially. In order to achieve this, we define the following energies:

$$\mathcal{A} := \frac{1}{2}\int_{\mathbb{R}} z^2 + v^2 \, dx, \tag{47}$$

$$\mathcal{E}_0 := E[z, v] := \frac{1}{2}\int_{\mathbb{R}} z_x^2 + z_t^2 + z^2 G_b \, dx + \frac{1}{2}\int_{\mathbb{R}} v_x^2 + v_t^2 + v^2(G_b + 4W_{bx}^2) \, dx, \tag{48}$$

$$E_1 := E[z_t, v_t] + E[z, v], \quad \mathcal{E}_1 := E[z_x, v_x] + E[z, v]. \tag{49}$$

Recall that $G_b = \frac{1}{\cosh^2(2x)}$ and $W_{bx}^2 = \frac{W_0^2}{\cosh^2(2x)}$, so all the energies defined above are positive definite. Our first objective is to control $\mathcal{E}_1$. This control will be achieved by a series of lemmas (Lemma 5.1, Lemma 5.2 and Lemma 5.3), which culminate in the Theorem 5.4. The outline here is similar to that of the polarized case. As before, we need to use the estimates for time derivatives to get control of $\mathcal{E}_1$. The reader should think of Lemma 5.1 as Proposition 4.5, the difference being that, due to the factor $\mathcal{E}_1$ in the right-hand of Equation (51), Lemma 5.1 almost controls E_1. In the same way, the reader should think of Lemmas 5.2 and 5.3 as Lemma 4.6, serving as a passage between time and spatial energies, this time being more subtle than before.

Lemma 5.1. *There are numbers n_i, m_i such that, for every solution to the system given by eqs.* (2), (45) *and* (46)*, satisfying the assumptions, there is a constant $C > 0$, depending only on a bound on $m_2(0)$ and on $\mathcal{E}_0(0) + \mathcal{A}(0)$, such that*

$$\partial_t \mathcal{A} \leq \mathcal{A}^{1/2}\mathcal{E}_0^{1/2}, \tag{50}$$

$$\begin{aligned} \dot{E}_1 \leq \; & C(t+1)e^{-2t}E_1 + C(t+1)e^{-t}m_2(0)E_1^{1/2} \\ & + \sum_i Ce^{-t}E_1\mathcal{E}_1^{n_i/2}\mathcal{A}^{m_i/2}. \end{aligned} \tag{51}$$

Proof. Since we are working with compactly supported solutions, we can derive inside the integral. Doing this with $\mathcal{A}$ and using Cauchy-Schwarz yields

$$\partial_t \mathcal{A} = \int_{\mathbb{R}} z z_t \; dx \leq \mathcal{A}^{1/2} \mathcal{E}_0^{1/2}.$$

Now let's compute $\partial_t \mathcal{E}_0$. Deriving under the integral and using parts

$$\begin{aligned}\dot{\mathcal{E}}_0 &= \int_{\mathbb{R}} z_t(z_{tt} - z_{xx} + zG_b) + v_t(v_{tt} - v_{xx} + v(G_b + 4W_{bx}^2)) \; dx \\ &= \int_{\mathbb{R}} \underbrace{z_t[z(G_b - G) + g]}_{A} - z_t B + \underbrace{v_t[v(G_b - G)]}_{C} - v_t D \; dx.\end{aligned}$$

The terms A and C are controlled as in the polarized case:

$$\int_{\mathbb{R}} A + C \; dx \leq C(t+1)e^{-2t}\mathcal{E}_0 + C(t+1)e^{-t} m_1(0) \mathcal{E}_0^{1/2}.$$

The new thing here is to control the terms involving B and D. A careful inspection of B and D, using the equations below the eqs. (45) and (46), shows that they can be written as a linear combination of the terms displayed below

$$B \quad \begin{cases} R^{-1/2} e^{-4\tilde{W}} \alpha_1, \text{where } \alpha_1 = v_t^2, \dfrac{R_t}{R} v v_t, \dfrac{R_x}{R} v v_x, v^2 G, W_{bx}^2 v^2, \\ \qquad\qquad\qquad\qquad\qquad \dfrac{R_x}{R} W_{bx} v^2, W_{bx} v_x v, v_x^2 \\ R^{-1} e^{-4\tilde{W}} \alpha_2, \text{where } \alpha_2 = v_t v z_t, \dfrac{R_t}{R} z v_t v, \dfrac{R_t}{R} v^2 z_t, G v^2 z, v_x v z_x, \\ \qquad\qquad\qquad\qquad \dfrac{R_x}{R} z v_x v, \dfrac{R_x}{R} v^2 z_x, W_{bx} v^2 z_x, W_{bx} v^2 z \dfrac{R_x}{2R} \\ R^{-3/2} e^{-4\tilde{W}} \alpha_3, \text{where } \alpha_3 = v^2 z_t^2, v^2 z^2 G, \dfrac{R_t}{R} z_t z v^2, v^2 z_x^2, \dfrac{R_x}{R} z_x z v^2 \end{cases} \tag{52}$$

and

$$D \quad \begin{cases} R^{-1/2-i/2} e^{-4\widetilde{W}} \alpha_4, \text{where } \alpha_4 = v\alpha_i, \quad \text{for } i = 1, 2, 3. \\ R^{-1} \alpha_5, \text{where } \alpha_5 = v z_x^2, G z^2 v, \dfrac{R_x}{2R} z_x z v, v z_t^2, \dfrac{R}{2R_t} z_t z v, \dfrac{R_x}{R} W_{bx} v z \\ R^{-1/2} \alpha_6, \text{where } \alpha_6 = W_{bx} v z_x \end{cases} \tag{53}$$

Because of this, to bound $\int_{\mathbb{R}} z_t B + v_t D \; dx$, it suffices to deal with a sum of terms of the form

$$\int_{\mathbb{R}} R^{-m/2} e^{-4\tilde{W} j} z_t \alpha \; dx \quad \text{or} \quad \int_{\mathbb{R}} R^{-m/2} e^{-4\tilde{W} j} v_t \alpha \; dx \quad \begin{array}{l} \text{where } m \geq 1, j = 0, 1 \\ \alpha = \alpha_i \text{ for } i = 1, .., 6 \end{array}.$$

Let us call the term z_t (or v_t) next to α the main derivative. The strategy is as follows:

1. If α has at least one derivative, apply Cauchy-Schwarz to this derivative and the main derivative and use $\|\cdot\|_\infty$ for the remaining terms inside α. The use of Cauchy-Schwarz gives a bound $\mathcal{E}_0$. For $R^{-m} e^{-4\tilde{W}}$ just use that $\tilde{W}$ is bounded and that $R^{-m} \leq C R_b^{-m} \leq C e^{-t}$. Regarding the use of $\|\cdot\|_\infty$, note that by Sobolev embedding

$$\|v\|_\infty, \|z\|_\infty \leq C(\mathcal{A}^{1/2} + \mathcal{E}_0^{1/2}), \qquad \|v_t\|_\infty, \|v_x\|_\infty, \|z_t\|_\infty, \|z_x\|_\infty \leq C \mathcal{E}_1^{1/2}.$$

Proceeding in this way, we produce bounds of the form:

$$Ce^{-t}\mathcal{E}_0\mathcal{E}_1^{n/2}\mathcal{A}^{m/2} \qquad \text{for some } n, m \geq 0. \tag{54}$$

where we have used that $\mathcal{E}_0 \leq \mathcal{E}_1$.

2. If α comprises only zero derivative terms, then looking at the terms that constitute D and B, and recalling that $G \leq \frac{C}{\cosh(2x)}$, we see that all these terms are multiplied by $\frac{1}{\cosh(2x)}$. Apply Cauchy-Schwarz between the main derivative and $\frac{v}{\cosh(2x)}$ (or $\frac{w}{\cosh(2x)}$), and $\|\cdot\|_\infty$ and Sobolev embedding for the other terms. The use of Cauchy-Schwarz gives the bound $\mathcal{E}_0$. Finally, again we bound $R^{-m}e^{-4\tilde{W}} \leq Ce^{-t}$. This procedure yields bounds of the form
$$Ce^{-t}\mathcal{E}_0\mathcal{E}_0^{n/2}\mathcal{A}^{m/2} \qquad \text{for some } n, m \geq 0. \tag{55}$$

Using the bounds (54),(55) with the fact that $\mathcal{E}_0 \leq E_1$, and the control for A and C we obtain

$$\begin{aligned} \dot{\mathcal{E}_0} \leq\ & C(t+1)e^{-2t}m_1(0)E_1 + C(t+1)e^{-t}m_1(0)E_1^{1/2} \\ & + \sum_i Ce^{-t}E_1\mathcal{E}_1^{n_i/2}\mathcal{A}^{m_i/2}, \text{ where } n_i, m_i \geq 0. \end{aligned} \tag{56}$$

Now in order to obtain an estimate for $\dot{E}_1$ we will bound $\dot{E}[z_t, q_t]$. Deriving under the integral and using parts

$$\dot{E}[z_t, q_t] = \int_{\mathbb{R}} z_{tt}((z_t)_{tt} - (z_t)_{xx} + (z_t)G_b) + v_{tt}((v_t)_{tt} - (v_t)_{xx} + v_t(G_b + 4W_{b_x}^2))\ dx$$

Now deriving the system, eqs. (45) and (46), respect to t we find that

$$\begin{aligned} &(z_t)_{tt} - (z_t)_{xx} + z_tG + G_tz + B' = g_t, \\ &(v_t)_{tt} - (v_t)_{xx} + v_t(G + 4W_{b_x}^2) + vG_t + D' = 0. \end{aligned}$$

Therefore

$$\dot{E}[z_t, q_t] = \int_{\mathbb{R}} \underbrace{z_{tt}[z_t(G_b - G) + G_tz + g_t]}_{F} - z_{tt}B' + \underbrace{v_{tt}[v_t(G_b - G) + G_tv]}_{G} - v_{tt}D'\ dx.$$

Again, the terms F and G are controlled as in the polarized case:

$$\int_{\mathbb{R}} F + G\ dx \leq C(t+1)e^{-2t}E_1 + C(t+1)e^{-t}m_2(0)E_1^{1/2}.$$

Now when we derive B and D, we find (see eqs. (52) and (53))

$$\begin{aligned} &(R^{-m/2})_te^{-4\tilde{W}}\alpha + R^{-m/2}(-4\tilde{W}_t)e^{-4\tilde{W}}\alpha + R^{-m/2}e^{-4\tilde{W}}\alpha_t \\ &= (R^{-m/2})_te^{-4\tilde{W}}\beta + R^{-m/2}\beta, \end{aligned}$$

where β could be α_t, α or $-4\tilde{W}_t\alpha$, with $\alpha = \alpha_i,\ i = 1, .., 6$. Here $\left|(R^{-m/2})_t\right| = \left|-m\frac{R_t}{R}R^{-m/2}\right| \leq CR^{-m/2}$, so it suffices to bound terms of the form

$$\int_{\mathbb{R}} R^{-m/2}e^{-4\tilde{W}}z_{tt}\beta\ dx \quad \text{or} \quad \int_{\mathbb{R}} R^{-m/2}e^{-4\tilde{W}}v_{tt}\beta\ dx \quad \beta = \alpha, \alpha_t, -4\widetilde{W}_t\alpha.$$

The fundamental fact about the products inside β is that they have, at most, a second derivative since α is a product of one or zero derivatives of v and q. In other words, if a product inside β has a second derivative, this derivative is raised to the power of 1. In addition, this second derivative could be z_{tt}, v_{tt}, z_{xt} or v_{xt}. Let us call the term z_{tt} (or v_{tt}) next to β the main derivative. The strategy is similar to the previous one:

1. If β has a second derivative, apply Cauchy-Schwarz to this derivative and the main derivative and use $\|\cdot\|_\infty$ for the remaining terms of β. If it does not have a second derivative but has at least one derivative, apply Cauchy-Schwarz to this derivative and the main derivative. The use of Cauchy-Schwarz gives a term E_1. None of the remaining terms of β will be a second derivative so, using Sobolev embedding, these $\|\cdot\|_\infty$ will be bounded precisely as before:

$$\|v\|_\infty, \|z\|_\infty \le C(\mathcal{A}^{1/2} + \mathcal{E}_0^{1/2}), \qquad \|v_t\|_\infty, \|v_x\|_\infty, \|z_t\|_\infty, \|z_x\|_\infty \le C\mathcal{E}_1^{1/2}.$$

   Once more, $R^{-m/2}e^{-4\tilde{W}} \le Ce^{-t}$. With this procedure, we obtain bounds of the form

$$\le Ce^{-t}E_1\mathcal{E}_1^{n/2}\mathcal{A}^{m/2} \qquad \text{for some } n, m \ge 0. \tag{57}$$

2. If β comprises only zero-derivative terms then, looking at the terms that constitute D' and B', and recalling that $G \le \frac{C}{\cosh(2x)}$ and $\partial_t G \le \frac{C}{\cosh(2x)}$, we see that all these terms are multiplied by $\frac{1}{\cosh(2x)}$. Then apply Cauchy-Schwarz between the main derivative and $\frac{v}{\cosh(2x)}$ (or $\frac{w}{\cosh(2x)}$). For the remaining terms inside β, use $\|\cdot\|_\infty$ and Sobolev embedding as above. The use of Cauchy Schwarz will give a term bounded by CE_1. Finally, again, we bound $R^{-m}e^{-4\tilde{W}} \le Ce^{-et}$. This procedure yields bounds of the form

$$\le Ce^{-t}E_1\mathcal{E}_0^{n/2}\mathcal{A}^{m/2} \qquad \text{for some } n, m \ge 0. \tag{58}$$

Now $\dot{E}_1 = \dot{\mathcal{E}}_0 + \dot{E}[z_t, q_t]$. Using eqs. (57) and (58), the control of $F+G$ and eq. (56), we obtain:

$$\begin{aligned}\dot{E}_1 \le\ & C(t+1)e^{-2t}m_2(0)E_1 + C(t+1)e^{-t}m_2(0)E_1^{1/2} \\ &+ \sum_i Ce^{-t}E_1\mathcal{E}_1^{n_i/2}\mathcal{A}^{m_i/2}, \text{ where } n_i, m_i \ge 0.\end{aligned}$$

□

Lemma 5.2. *There are some numbers n_i, m_i such that, if during an interval of time $[0,T]$ we have a solution satisfying the assumptions with $|z_x|, |v_x| < 1$, then, there is a constant $C > 0$, depending only on a bound on $m_1(0)$ and on $\mathcal{E}_0(0)$, such that*

$$\mathcal{E}_1 \le CE_1 + C(t+1)^2e^{-2t}m_1(0)^2 + \sum_i Ce^{-t}E_1^{1+n_i/2}\mathcal{A}^{m_i/2} \quad \textit{for } t \in [0,T]. \tag{59}$$

Proof. We know that $\mathcal{E}_1 = E[z,v] + E[z_x, v_x] \le E_1 + E[z_x, v_x]$ so in order to bound $\mathcal{E}_1$ with E_1 we need to control $E[z_x, v_x]$ by E_1. Note that

$$\begin{aligned}E[z_x, v_x] &= \int z_{xx}^2 + z_{xt}^2 + z_x^2 G_b + v_{xx}^2 + v_{xt}^2 + v_x^2(G_b + 4W_{bx}^2)\, dx \\ &\le \int_{\mathbb{R}} z_{xx}^2 + v_{xx}^2 + CE_1.\end{aligned}$$

Now, using the equation,

$$z_{xx}^2 \leq C(z_{tt}^2 + z^2G^2 + g^2 + B^2),$$
$$w_{xx}^2 \leq C(w_{tt}^2 + w^2G^2 + D^2),$$

so

$$\begin{aligned}\int_{\mathbb{R}} z_{xx}^2 + w_{xx}^2 &\leq CE_1 + \int_{\mathbb{R}} g^2 \ dx + \int_{\mathbb{R}} B^2 + D^2 \ dx \\ &\leq CE_1 + C(t+1)^2 e^{-2t} m_1(0)^2 + \int_{\mathbb{R}} B^2 + D^2 \ dx.\end{aligned}$$

So now we need to control the last integral by E_1. In order to do this, we will use the hypothesis that $|z_x|, |v_x| < 1$. Now, remember the general form of B and D (eqs. (52) and (53)). It follows that we need to bound

$$\int R^{-k}\alpha^2 e^{-8\tilde{W}j} \ dx \text{ where } k \geq 1, j = 0,1 \text{ and } \alpha = \alpha_i \text{ for } i = 1, ..., 6.$$

Here, we have used that $2ab \leq a^2 + b^2$ many times. The strategy is as follows:

1. If α is composed only by zeroth derivative terms, then α^2 always has $\frac{1}{\cosh^2(2x)}$ as one of its factors and always has at least four terms (powers of z and v). Choose two of these, for example, v and w, and use $\|\cdot\|_\infty$ with the remaining terms and Cauchy-Schwarz with these two terms together with $\frac{1}{\cosh^2(2x)}$. Using Cauchy-Schwarz gives us a term $\mathcal{E}_0$. Again, use Sobolev embedding to bound the infinite norms of the remaining terms and $R^{-k}e^{-8\tilde{W}} \leq Ce^{-2t}$. This procedure yields bounds of the form
$$Ce^{-t}\mathcal{E}_0^{1+n/2}\mathcal{A}^{m/2} \quad (\Rightarrow \leq Ce^{-t}E_1^{1+n/2}\mathcal{A}^{m/2}) \qquad \text{for some } n, m \geq 0.$$
2. If α^2 has at least one derivative then α has at least one derivative then α^2 has at least two derivatives. Bound the remaining terms by the infinite norms and apply Cauchy-Schwarz to these two derivatives. The use of Cauchy-Schwarz will give a term bounded by $C\mathcal{E}_0 < E_1$. For the infinite norms, if we have $\|z\|_\infty$, $\|z_t\|_\infty$, $\|v\|_\infty$ and $\|v_t\|_\infty$ then use Sobolev embedding to bound these terms by
$$C(\mathcal{A}^{1/2} + \mathcal{E}_0^{1/2}) \ \text{ or } \ E_1^{1/2}.$$
Regarding terms like $\|z_x\|_\infty$ and $\|v_x\|_\infty$ just use the hypothesis that they are less than 1. Lastly, $R^{-k}e^{-8\tilde{W}} \leq Ce^{-t}$. This procedure gives a bound of the form
$$Ce^{-t}E_1^{1+n/2}\mathcal{A}^{m/2} \qquad \text{for some } n, m \geq 0.$$

Putting it all together, we find

$$E[z_x, v_x] \leq CE_1 + C(t+1)^2 e^{-2t} m_1(0)^2 + \sum_i Ce^{-t}E_1^{1+n_i/2}\mathcal{A}^{m_i/2},$$

which is the desired conclusion. □

Lemma 5.3. *There are some numbers n_i, m_i such that, for every solution satisfying the assumptions, there is a constant $C > 0$, depending only on a bound on $m_1(0)$ and $\mathcal{E}_0(0)$, such that*

$$E_1 \leq C\mathcal{E}_1 + C(t+1)^2 e^{-2t} m_1(0)^2 + \sum_i C e^{-t} \mathcal{E}_1^{1+n_i/2} \mathcal{A}^{m_i/2}, \tag{60}$$

in particular, evaluating at $t = 0$,

$$E_1(0) \leq C(\mathcal{E}_1(0) + m_1^2(0)), \tag{61}$$

and now the constant also depends on a bound on $\mathcal{A}(0) + \mathcal{E}_1(0)$.

Proof. Notice that this is the inequality of the previous lemma but with E_1 and $\mathcal{E}_1$ reversed. Following the same argument leads to the need to control $\int B^2 + D^2\, dx$ by $\mathcal{E}_1$. This control is more straightforward than before. The strategy used in the previous lemma works with minor modifications. The first step already led to bounds of the form $Ce^{-t}\mathcal{E}_0^{1+n/2}\mathcal{A}^{m/2}$, which is less than $Ce^{-t}\mathcal{E}_1^{1+n/2}\mathcal{A}^{m/2}$. In the second step, the use of Cauchy-Schwarz gave $\mathcal{E}_0$, which is fine. Regarding the infinite norms:

$$\|z\|_\infty, \|v\|_\infty, \|z_t\|_\infty, \|z_x\|_\infty, \|v_t\|_\infty, \|v_x\|_\infty \leq \mathcal{A}^{1/2} + \mathcal{E}_1^{1/2},$$

which is also fine. □

Theorem 5.4. *There is some $\delta > 0$ such that, for every solution to the system satisfying the assumptions and $\mathcal{E}_1^{1/2}(0), m_2(0) < \delta$, there is a constant C, such that*

$$E_1^{1/2} < C(E_1^{1/2}(0) + m_2(0)) \quad \forall t \in [0, T), \tag{62}$$

and

$$\mathcal{E}_1^{1/2} < C(\mathcal{E}_1^{1/2}(0) + m_2(0)) \quad \forall t \in [0, T), \tag{63}$$

where T is the supremum of times T' such that the solution is defined on $[0, T']$. Furthermore, the constant just depends on a bound on $m_2(0), \mathcal{E}_1(0)$ and on $\mathcal{A}^{1/2}(0)$.

Proof. Let $\delta' > 0$ be such that $\delta' < 1$ and such that if $\mathcal{E}_1^{1/2} < \delta'$ then $|z_x|, |v_x| < 1$. The existence of δ' is justified by Sobolev embedding. Now suppose that $\mathcal{E}_1^{1/2}(0), m_2(0) < \delta << \delta'$. The value of δ will be specified in a moment. The only property that we will use now is that since $\delta < \delta'$ and $\mathcal{E}_1^{1/2}(0) < \delta$, then $\mathcal{E}_1^{1/2}(t) < \delta'$ for at least an interval of time. Consider

$$\widetilde{T} = \sup\{s : s \leq T \text{ and } \mathcal{E}_1^{1/2}(t) < \delta' \text{ for } t \in [0, \widetilde{T})\}.$$

For $t \in [0, \tilde{T})$, we have $|z_x|, |v_x| < 1$, and therefore we are allowed to apply Lemma 5.2. This lemma asserts that

$$\mathcal{E}_1 \leq CE_1 + C(t+1)^2 e^{-2t} m_1(0)^2 + \sum_i Ce^{-t} E_1^{1+n_i/2} \mathcal{A}^{m_i/2} \ \forall t \in [0, \widetilde{T}).$$

Now by the evolution equation for $\mathcal{A}$, eq. (50), we know that

$$\mathcal{A}^{1/2} \leq \mathcal{A}^{1/2}(0) + \int_0^t \mathcal{E}_1^{1/2}(s)\, ds \leq \mathcal{A}^{1/2}(0) + t \qquad \forall t \in [0, \widetilde{T}),$$

so $e^{-t}\mathcal{A}^{m_i}$ is bounded, in $[0, \widetilde{T})$, by a constant that just depends on a bound on $\mathcal{A}^{1/2}(0)$. Hence $|\mathcal{E}_1, m_1| < 1$ and $e^{-t}\mathcal{A}^{m_i}$ is bounded for $t \in [0, \widetilde{T})$. Using these bounds and the previous lemma, we see that E_1 is bounded in $[0,\widetilde{T})$, and the bound depends on a bound on $m_1(0)$, $\mathcal{E}_1^{1/2}(0)$ and on $\mathcal{A}^{1/2}(0)$. As a consequence, $E_1^{1+n_i/2} < CE_1$ $\forall t \in [0,\widetilde{T})$, where again, C depends on a bound on $m_1(0)$, $\mathcal{E}_1^{1/2}(0)$ and on $\mathcal{A}^{1/2}(0)$. Using this fact, we specialize the conclusion of the Lemma 5.2, obtaining

$$\mathcal{E}_1 \leq CE_1 + C(t+1)^2 e^{-2t} m_1^2(0) \ \forall t \in [0,\widetilde{T}). \tag{64}$$

Here the constant C depends on a bound on $m_1(0)$, $\mathcal{E}_1^{1/2}(0)$ and on $\mathcal{A}^{1/2}(0)$. Now, we proceed to control our energies. First, by the Equation (51) and the fact that $\mathcal{E}_1 < 1$ in $[0,\widetilde{T})$ we know that

$$\begin{aligned}\dot{E}_1 \leq\ & C(t+1)e^{-2t}E_1 + C(t+1)e^{-t}m_2(0)E_1^{1/2} \\ & + CE_1 \sum_i e^{-t}\mathcal{A}^{m_i/2} \quad \forall t \in [0,\widetilde{T}).\end{aligned}$$

Now, use that $(t+1)e^{-t}, (t+1)e^{-2t} < Ce^{-t/2}$ and $e^{-t}\mathcal{A}^{m_i} < Ce^{-t/2}$ $\forall i, \forall\, t \in [0,\widetilde{T})$. This yields

$$\dot{E}_1 \leq Ce^{-t/2}(E_1 + m_2(0)E_1^{1/2}).$$

Using Gronwall

$$E_1^{1/2} \leq C(E_1^{1/2}(0) + m_2(0)) \ \forall t \in [0,\widetilde{T}). \tag{65}$$

The constant here depends on a bound on $m_2(0), \mathcal{E}_1^{1/2}(0)$ and on $\mathcal{A}^{1/2}(0)$. Now, using eq. (64) and eq. (65) we get

$$\begin{aligned}\mathcal{E}_1^{1/2} &\leq CE_1^{1/2} + Cm_1(0) \leq C(E_1^{1/2}(0) + m_2(0)) \\ &\leq C(\mathcal{E}_1^{1/2}(0) + m_2(0)) \ \forall t \in [0,\widetilde{T}),\end{aligned}$$

where in the last inequality, we have used eq. (61). Summarizing

$$\mathcal{E}_1^{1/2} \leq C(\mathcal{E}_1^{1/2}(0) + m_2(0)) \ \forall t \in [0,\widetilde{T}). \tag{66}$$

As we are asking $\mathcal{E}_1^{1/2}(0), m_2(0)$ to be less than δ, we have $\mathcal{E}_1^{1/2} \leq C2\delta$ $\forall t \in [0,\widetilde{T})$. Now, if we require $\delta < \frac{\delta'}{4C}$ then $\mathcal{E}_1^{1/2}(t) < \delta'/2 < \delta'$ $\forall t \in [0,\widetilde{T})$ and hence $\widetilde{T} = T$. To sum up, if $m_2(0), \mathcal{E}_1^{1/2}(0) < \delta$ then

$$E_1^{1/2} \leq C(E_1^{1/2}(0) + m_2(0)) \ \ \forall t \in [0,T),$$

and since we have (66), then we also have

$$\mathcal{E}_1^{1/2} \leq C(\mathcal{E}_1^{1/2}(0) + m_2(0)) \ \ \forall t \in [0,T).$$

□

5.2.1. *Gaining one more derivative.* In order to obtain existence for all $t \geq 0$, what we have to do is to extend Theorem 5.4 to $E_2 := E_1 + E[z_{tt}, v_{tt}]$ and $\mathcal{E}_2 := \mathcal{E}_1 + E[z_{xx}, v_{xx}]$. In this way, we control one more derivative, and then by a continuity lemma, existence for all time to the future is guaranteed . The arguments here are essentially the same as above. For this reason, we only state the results.

Lemma 5.5. *There are numbers n_i, m_i, r_i such that, for every solution to the system given by eqs. (2), (45) and (46), satisfying the assumptions, there is a constant $C > 0$, depending only on a bound on $m_3(0)$ and on $\mathcal{E}_0(0) + \mathcal{A}(0)$, such that*

$$\begin{aligned}\dot{E}_2 \leq\ & C(t+1)e^{-2t}E_2 + C(t+1)e^{-t}m_2(0)E_2^{1/2} \\ & + \sum_i Ce^{-t}E_2^{1+r_i/2}\mathcal{E}_2^{n_i/2}\mathcal{A}^{m_i/2}.\end{aligned} \tag{67}$$

Lemma 5.6. *There are some numbers n_i, m_i such that, if during an interval of time $[0,T]$ we have a solution satisfying the assumptions with $|z_x|, |v_x|, |z_{xx}|, |v_{xx}| < 1$, then there is a constant $C > 0$, depending only on a bound on $m_2(0)$ and on $\mathcal{E}_0(0) + \mathcal{A}(0)$, such that*

$$\mathcal{E}_2 \leq CE_2 + C(t+1)^2e^{-2t}m_2(0)^2 + \sum_i Ce^{-t}E_2^{1+n_i/2}\mathcal{A}^{m_i/2} \quad \textit{for } t \in [0,T]. \tag{68}$$

Lemma 5.7. *There are some numbers n_i, m_i, r_i such that, for every solution satisfying the assumptions, there is a constant $C > 0$, depending only on a bound on $m_2(0)$ and $\mathcal{E}_0(0)$, such that*

$$E_2 \leq C\mathcal{E}_2 + C(t+1)^2e^{-2t}m_2(0)^2 + \sum_i Ce^{-t}\mathcal{E}_2^{1+n_i/2}\mathcal{A}^{m_i/2} + \sum_i Ce^{-t}\mathcal{A}^{l_i/2}\mathcal{E}_1^{r_i/2}m_2^2(0), \tag{69}$$

in particular, evaluating at $t = 0$

$$E_2(0) \leq C(\mathcal{E}_2(0) + m_2^2(0)), \tag{70}$$

and now the constant also depends on a bound on $\mathcal{A}(0) + \mathcal{E}_1(0)$.

Theorem 5.8. *There is some $\delta > 0$ such that, for every solution to the system satisfying the assumptions and $\mathcal{E}_2^{1/2}(0), m_3(0) < \delta$, the solution is defined for every $t \geq 0$. Furthermore,*

$$E_2^{1/2} < C(E_2^{1/2}(0) + m_3(0)) \quad \forall t \geq 0, \tag{71}$$

and

$$\mathcal{E}_2^{1/2} < C(\mathcal{E}_2^{1/2}(0) + m_3(0)) \quad \forall t \geq 0, \tag{72}$$

for some constant C that just depends on a bound on $m_3(0), \mathcal{E}_2(0)$ and on $\mathcal{A}^{1/2}(0)$.

Proof. Having the three previous lemmas, we can repeat the proof of the Theorem 5.4, which will give the desired conclusions for $t \in [0,T)$, where T is the supremum of times T' such that the solution is defined on $[0,T']$. As a consequence, $\mathcal{A}^{1/2}(t) + \mathcal{E}_2^{1/2}(t) + E_2^{1/2}(t)$ is bounded in $[0,T)$. By Sobolev embedding, the C^2−norm of the solution is bounded in $[0,T)$. By the continuity lemma $T = +\infty$. $\square$

Corollary 5.9. *There is a number $\delta > 0$, such that for any smooth solution of the system eqs. (2) to (4), (R, W, q), with initial data that differs from that of the background only on a compact set, the following holds. If $m_3(0)$ and $\widetilde{\mathcal{M}}_3(0)$ are both less than δ, then the solution is defined for all $t \geq 0$ and*

$$\widetilde{\mathcal{M}}_3(t) \leq Ce^{-t}(t+1)(\widetilde{\mathcal{M}}_3(0) + m_3(0)).$$

Moreover, the constant C just depends on a bound on $m_3(0)$ and on $\widetilde{\mathcal{M}}_3(0)$.

Proof. The previous theorem, together with the estimate for $\mathcal{A}$, gives a bound for $\mathcal{M}_2$. Furthermore, $\mathcal{A}(0)+\mathcal{E}_2(0)$ is bounded by $\mathcal{M}_2(0)$. Now applying the passage lemma, lemma 4.10 (which also works in this non-polarized case), we have the result for $W-W_b, q$. □

6. General perturbations

So far, we have proved results concerning compactly supported perturbations. In this section, we provide an argument to generalize these results for solutions in a larger functional space. To do this, we approximate the initial data given by a sequence of initial data which, as before, differ from that of the background only on a compact set. We will see that the sequence of solutions converges to the solution with the given initial data in a suitable sense. Consider (R_1, W_1, q_1) and (R_2, W_2, q_2), two solutions with initial data that differs from that of the background only on a compact set. Define $z_1 = R_1^{1/2}(W_1 - W_b)$, $v_1 = R^{1/2}e^{-2W}q$ and similarly z_2 and v_2. Then, for each i, we have

$$(z_i)_{tt} - (z_i)_{xx} + f(G_i, R_i^{-1/2}, \tfrac{R_{it}}{R_i}, \tfrac{R_{ix}}{R_i}, v_i, v_{ix}, v_{it}, z_i, z_{it}, z_{ix}, W_{bx}) = g_i,$$
$$(v_i)_{tt} - (v_i)_{xx} + h(G_i, R_i^{-1/2}, \tfrac{R_{it}}{R_i}, \tfrac{R_{ix}}{R_i}, v_i, v_{ix}, v_{it}, z_i, z_{it}, z_{ix}, W_{bx}) = 0,$$

where f and h are polynomials on these variables. For instance, in the polarized case, $h\equiv 0$ and $f = zG$. For short, let us simply put $f(A_i, B_i, C)$ where $A_i = (G_i, R_i^{-1/2}, \frac{R_{it}}{R_i}, \frac{R_{ix}}{R_i})$, $B_i = (v_i, v_{ix}, v_{it}, z_i, z_{it}, z_{ix})$ and $C = W_{bx}$. In addition, put $\Delta z = z_2 - z_1$ and analogously, also define Δv, ΔA, ΔB, Δf, Δg and ΔR. Taking the difference of the equations for $i=1$ and $i=2$ yields

$$(\Delta z)_{tt} - (\Delta z)_{xx} + \Delta f = \Delta g, \tag{73}$$
$$(\Delta v)_{tt} - (\Delta v)_{xx} + \Delta h = 0. \tag{74}$$

In order to control the sequence, we need to introduce the following energy

$$\mathcal{H}_n = \frac{1}{2}\sum_{i=0}^{n}\int_{\mathbb{R}} [(\partial_x^i \Delta z)_x]^2 + [(\partial_x^i \Delta z)_t]^2 + (\partial_x^i \Delta z)^2 + [(\partial_x^i \Delta v)_x]^2 + [(\partial_x^i \Delta v)_t]^2 + (\partial_x^i \Delta v)^2 dx.$$

Again, some estimates are required to bound $\mathcal{H}_n$. For convenience, we introduce the notation

$$m_k[f](t) := \|f\|_{C^k}(t) + \|\partial_t f\|_{C^k}(t), \quad m_k(t) := \sup_i m_k[R_i](t).$$

Also note that we now have two definitions for $\mathcal{E}_n$, namely, one for (z_1, v_1) and one for (z_2, v_2). Now, we will call $\mathcal{E}_n$ to the maximum of these two.

Lemma 6.1. *For every α multi-index, there is a constant C such that*

$$|\partial^\alpha \Delta g| \le C\frac{m_{|\alpha|+1}[\Delta R](0)(t+1)}{\cosh(2x)^{3/2}e^t}, \tag{75}$$

and also

$$\|\partial^\alpha \Delta A\| \le C\frac{m_{|\alpha|+1}[\Delta R](0)}{e^t\cosh(2x)}(t+1). \tag{76}$$

Moreover, the constant C just depends on α, and on a bound on $m_{|\alpha|+1}(0)$.

Proof. The same kind of computations shown in the proof of the Lemma 4.4 works. □

Proposition 6.2 (Control of $\mathcal{H}_n$)**.** *Let $n \geq 1$ and suppose that for all $t \geq 0$*

$$\mathcal{E}_n(t) \leq K(\mathcal{E}_n(0) + m_{n+1}(0)),$$

for some constant K. Then, there is a polynomial $P(t)$ and a constant C such that

$$\dot{\mathcal{H}}_n \leq CP(t)(\mathcal{H}_n + m_{n+1}^2[\Delta R](0)).$$

Moreover, the constant C, and $P(t)$, just depends on a bound on K, $\mathcal{A}(0)$, $\mathcal{E}_n(0)$ and on $m_{n+1}(0)$.

Proof. Deriving inside the integral, applying parts, using the equation, Cauchy-Schwarz and the eq. (75), we obtain

$$\dot{\mathcal{H}}_n \leq C\mathcal{H}_n + C\mathcal{H}_n^{1/2} m_{n+1}[\Delta R](0) + C\sum_{i=0}^{n}\int_{\mathbb{R}} (\partial_x^i \Delta z)_t \partial_x^i(\Delta f) + (\partial_x^i \Delta v)_t \partial_x^i(\Delta h)\ dx.$$

so we need to bound the last two terms. Since these terms are completely similar, we only show how to control the first term. To do that, let us call $\zeta(\lambda) = (\Delta A, \Delta B, 0)\lambda + (A_1, B_1, C)$, and observe that

$$|\Delta f| = |f(A_1, B_1, C) - f(A_2, B_2, C)| \leq \sup_{\lambda\in[0,1]} \|\nabla f(\zeta(\lambda))\|(\|\Delta A\| + \|\Delta B\|).$$

Now, ∇f is a polynomial which is evaluated in a point between (A_1, B_1, C) and (A_2, B_2, C). Each of the components in this vectors are bounded by $C(t+1)$ where C is a constant that depends on a bound on $\mathcal{E}_1(0)$, $\mathcal{A}(0)$, $m_2(0)$ and K. As a consequence, the term involving ∇f grows polynomially. Using this and the eq. (76) we get

$$|\Delta f| \leq P(t)(C\frac{m_1[\Delta R](0)(t+1)e^{-t}}{\cosh(2x)} + \|\Delta B\|),$$

and hence

$$\int_{\mathbb{R}} (\Delta z)_t\, |\Delta f|\ \ dx \leq CP(t)(\mathcal{H}_n^{1/2} m_{n+1}[\Delta R](0) + \mathcal{H}_n).$$

For the other terms, a slightly different argument is needed. Given $i \geq 1$ we are going to control the term involving $(\partial_x^i \Delta z)_t \partial_x^i(\Delta f)$, only using that m_{i+1} and $\mathcal{E}_i$ are bounded. In this way, we can apply this up to $i = n$ without needing more hypotheses. Consider

$$\begin{aligned}\partial_x^i(\Delta f) &= \partial_x^i(f(A_1, B_1, C) - f(A_2, B_2, C)) = \partial_x\partial_x^{i-1}(f(A_1, B_1, C) - f(A_2, B_2, C))\\ &= \partial_x(\widehat{f}(\widehat{A}_1, \widehat{B}_1, \widehat{C}) - \widehat{f}(\widehat{A}_2, \widehat{B}_2, \widehat{C})),\end{aligned}$$

here $\hat{A}_1$ is a vector formed by the elements of A_1 and their $i-1$ derivatives with respect to x, and similarly for $\widehat{A}_2$, $\widehat{B}_1, \widehat{B}_2$ and $\widehat{C}$. Now $\widehat{f}$ is a polynomial on these variables. Hence

$$\begin{aligned}\partial_x^i(\Delta f) &= \nabla\widehat{f}(\widehat{A}_2, \widehat{B}_2, \widehat{C})\cdot(\partial_x\widehat{A}_2, \partial_x\widehat{B}_2, \partial_x\widehat{C}) - \nabla\widehat{f}(\widehat{A}_1, \widehat{B}_1, \widehat{C})\cdot(\partial_x\widehat{A}_1, \partial_x\widehat{B}_1, \partial_x\widehat{C})\\ &= (I) + (II), \quad \text{where}\\ &(I) = \nabla\widehat{f}(\widehat{A}_2, \widehat{B}_2, \widehat{C})\cdot(\partial_x\Delta\widehat{A}, \partial_x\Delta\widehat{B}, 0)\\ &(II) = [\nabla\widehat{f}(\widehat{A}_2, \widehat{B}_2, \widehat{C}) - \nabla\widehat{f}(\widehat{A}_1, \widehat{B}_1, \widehat{C})]\cdot(\partial_x\widehat{A}_1, \partial_x\widehat{B}_1, \partial_x\widehat{C}).\end{aligned}$$

For (II), observe that

$$(II) \leq C \sup_{i,j,t\in[0,1]} \left|\partial^i\partial^j \widehat{f}(\widehat{\zeta}(t))\right| \|(\Delta\widehat{A},\Delta\widehat{B},0)\|\|(\partial_x\widehat{A}_1,\partial_x\widehat{B}_1,\partial_x\widehat{C})\|,$$

where C is a constant that does not depend on anything. The first term in this product is a polynomial which is evaluated in a point between $(\widehat{A}_1,\widehat{B}_1,\widehat{C})$ and $(\widehat{A}_2,\widehat{B}_2,\widehat{C})$. The terms in $\widehat{A}_j$, $j=1,2$, are G_j,$R_j^{-1/2}$, R_{j_t}/R_j, R_{j_x}/R_j, and their $x-$derivatives up to order $i-1$. Since m_i is bounded, all these terms are bounded. The terms in $\widehat{B}_j$ are composed of the $x-$derivatives of $z_j, z_{j_t}, z_{j_x}, v_j, v_{j_t}, v_{j_x}$ up to order $i-1$. Since $\mathcal{E}_i$ is bounded, each term grows, at most, linearly. Lastly, $\hat{C}$ is bounded. As a result, the first term in this product is bounded by $CP(t)$, where $P(t)$ is a polynomial and C is a constant that depends on a bound on $\mathcal{E}_i(0)$, $\mathcal{A}(0)$, $m_{i+1}(0)$ and K. For the middle term, we use the eq. (76) for $\Delta\widehat{A}$, and $\|\Delta\widehat{B}\| \leq C\mathcal{H}_n^{1/2}$, which is true by Sobolev embedding. This yields

$$|(II)| \leq CP(t)(\frac{m_i[\Delta R](0)(t+1)}{e^t\cosh(2x)} + \mathcal{H}_n^{1/2})\|(\partial_x\widehat{A}_1,\partial_x\widehat{B}_1,\partial_x\widehat{C})\|.$$

Using the same ideas, we see that

$$|(I)| \leq CP(t)(C\frac{m_{i+1}[\Delta R](0)(t+1)e^{-t}}{\cosh(2x)} + \|\partial_x\Delta\widehat{B}\|).$$

Now, using Cauchy-Schwarz, and the fact that $\int_{\mathbb{R}}\|(\partial_x\widehat{A}_1,\partial_x\widehat{B}_1,\partial_x\widehat{C})\|^2\,dx$ is bounded, we get

$$\int_{\mathbb{R}}(\partial_x^i\Delta z)_t\partial_x^i(\Delta f)\,dx \leq CP(t)\left(\mathcal{H}_n^{1/2}m_{n+1}[\Delta R](0)+\mathcal{H}_n\right)$$

Putting all together, and using $ab \leq a^2+b^2$, we arrive at the conclusion. □

Now we can enhance the results given by Corollary 4.11 and Corollary 5.9.

Theorem 6.3. *Let $k \geq 3$. Let (R,W) be a C^2 solution of the system eqs.* (2) *and* (3) *with $q=0$. Suppose also that $(R-R_b(0,\cdot),\partial_t(R-R_b)(0,\cdot)) \in C_0^k \times C_0^{k-1}$, $m_0(0) < 2R_0/3$ and $((W-W_b)(0,\cdot),\partial_t(W-W_b)(0,\cdot)) \in \widetilde{H}^k \times \widetilde{H}^{k-1}$. Then, the solution is defined for every $t \geq 0$ and*

$$\widetilde{\mathcal{M}}_k(t) \leq Ce^{-t}(t+1)\left(\widetilde{\mathcal{M}}_k(0)+m_k(0)\right).$$

Moreover, the constant C depends only on an upper bound on $m_k(0)$ and k.

Proof. Consider a sequence of initial data $(R_i(0,\cdot),R_{it}(0,\cdot))$ such that for each i, $((R_i-R_b)(0,\cdot),R_{it}-R_{bt}(0,\cdot))$ are compactly supported and converges to $((R-R_b)(0,\cdot),(R-R_b)_t(0,\cdot))$ in $C_0^k \times C_0^{k-1}$. Similarly, we ask the same for the initial data of W, but this time, we require that the sequence $(z_i(0,\cdot),z_{it}(0,\cdot))$ converges to $(z(0,\cdot),z_t(0,\cdot))$ in $H^k \times H^{k-1}$.

A quick computation using D'Alembert shows that for each fixed $T>0$, (R_i,R_{it}) is a Cauchy sequence in $C([0,T],C_0^k) \times C([0,T],C_0^{k-1})$, converging to (R,R_t). Regarding z_i, first note that eq. (42) holds for every i with the same constant. Accordingly, we are in the hypothesis of Proposition 6.2, with the same K for each i. By applying this with $n=k-1$, we see that $\mathcal{H}_{k-1} \leq C^*(\mathcal{H}_{k-1}(0)+m_k(0)), 0 \leq t \leq T$, where C^* now depends also on T. Again, we can use the same constant, C^*, each time we apply this proposition. It follows that (z_i,z_{it}) is a Cauchy sequence in

$C([0,T],H^k)\times C([0,T],H^{k-1})$. Since $k\geq 3$, we have that (z_i,z_{it}) is a Cauchy sequence in $C([0,T],C^2)\times C([0,T],C^1)$. Finally, using the equation (73), we see that we also have control of $\partial_t^2 z_i$ and then, we have that z_i is a Cauchy sequence in $C^2([0,T]\times\mathbb{R},\mathbb{R})$. As result, z_i converges in $C^2([0,T]\times\mathbb{R})$ to a function $h:[0,T]\times\mathbb{R}\to\mathbb{R}$ which satisfies the equation (26) with initial data $(z(0,\cdot),z_t(0,\cdot))$. By uniqueness, h is the solution with this initial data, defined on $[0,T]\times\mathbb{R}$. From now on, we shall call it z instead of h. Now, by eq. (43), we have

$$\widetilde{\mathcal{M}}_k[z_i](t)\leq Ce^{-t}(t+1)\left(\widetilde{\mathcal{M}}_k[z_i](0)+m_k[R_i](0)\right)\ \forall t\geq 0,$$

where we have used the same constant for each i. Since (z_i,z_{it}) converges in $C([0,T],H^k)\times C([0,T],H^{k-1})$ to (z,z_t) and (R_i,R_{it}) converges in $C([0,T],C_0^k)\times C([0,T],C_0^{k-1})$ to (R,R_t), taking limit in the above estimate yields

$$\mathcal{M}_k[z](t)\leq Ce^{-t}(t+1)\left(\mathcal{M}_k[z](0)+m_k[R](0)\right)\quad 0\leq t\leq T.$$

Since this is valid for all T, by uniqueness, we have constructed a smooth solution z to the eq. (26) which satisfies this estimate for all $t\geq 0$ and which has the desired initial data, namely, that of W but seen in the $z-$variable. By the passage Lemma 4.10, this z corresponds to the smooth solution W given in the statement. The estimate in the statement is a consequence of this lemma as well. □

Theorem 6.4. *There is a number $\delta>0$ such that, any C^2 solution of the system eqs.* (2) *to* (4)*, (R,W,q), with initial data satisfying $(R-R_b(0,\cdot),\partial_t(R-R_b)(0,\cdot))\in C_0^k\times C_0^{k-1}$, $m_0(0)<2R_0/3$, $((W-W_b)(0,\cdot),\partial_t(W-W_b)(0,\cdot))\in\widetilde{H}^k\times\widetilde{H}^{k-1}$, $(q(0,\cdot),\partial_t q(0,\cdot))\in\widetilde{H}^k\times\widetilde{H}^{k-1}$, and $m_3(0)<\delta$, $\widetilde{\mathcal{M}}_3(0)<\delta$, is defined for all $t\geq 0$ and*

$$\widetilde{\mathcal{M}}_3(t)\leq Ce^{-t}(t+1)(\widetilde{\mathcal{M}}_3(0)+m_3(0)).$$

Moreover, the constant C depends only on an upper bound on $m_3(0)$ and on $\widetilde{\mathcal{M}}_3(0)$.

Proof. The argument is essentially the same, the difference being that now we work with z_i and v_i, that it is Theorem 5.8 which allows us to use Proposition 6.2 with $n=2$, and also that we use the equation given by Corollary 5.9, version $\mathcal{M}_3$, instead of the Equation (43). □

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