EXOTIC EQUILIBRIA OF HARARY GRAPHS AND A NEW MINIMUM DEGREE LOWER BOUND FOR SYNCHRONIZATION

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ABSTRACT. This work is concerned with stability of equilibria in the homogeneous (equal frequencies) Kuramoto model of weakly coupled oscillators. In Taylor, (2012 *R. J. of Physics A: Math. and Th.* **45**, pp 1–15) a sufficient condition for almost global synchronization, was found in terms of the minimum degree–order ratio of the graph. In this work an new lower bound for this ratio is given. The improvement is achieved by a concrete infinite sequence of regular graphs.

Besides, non standard unstable equilibria of the graphs studied in Wiley *et al* (2006 *Chaos* **16** 015103) are shown to exist as conjectured in that work.

1. INTRODUCTION

In [17], Wiley, Strogatz and Girvan suggested a new line of research that they hoped were "appeal to the nonlinear dynamics community". They consider a network of identical oscillators and asked for "how likely is the system to synchronize, starting from a random initial condition?" and "how does the probability of synchronization depend on the way the network is connected?" Interestedly, at least one year before in [11], P. Monzón and F. Paganini considered the same questions. Both teams studied oscillators coupled according to the model introduced by Kuramoto in [10]. While in Monzon et al. consider the almost global stability property (AGS property for short) applied to Kuramoto model, searching for densities under the conditions given by Rantzer (see [13]), Wiley et al. made numerical experiments in order to measure the size of the synchronized states' attraction basin. The last authors also presented analytics results, but over some limit equations derived from the finite ones. In this work we prove the correctness of some of these limits.

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Starting with these seminal papers, some researchers have been working on the subject, trying to classify the graphs that lead to the AGS property, i.e., to answer the second question posed in [17]. However it seems that they were not aware of each other and some repetition on the results arised. For instance, as far as 2006, in [11], it is proved that the complete graphs synchronize, but the question is conjectured two years later in [15] and proved (again) in [14]. Later on, we made some improvements: we proved that the AGS property depends only in the block of the graphs [3], we also proved that every connected graph is the induced graph of a synchronized one, and that any graph with at least one cycle is homeomorphic to a non synchronized one [4]. Besides, we proved some other less general results, for instance that the wheels synchronize [5] as well as the complete k-partite graphs [2]. Lastly, in [14], Taylor made a big progress proving that there is a non trivial upper bound for the ratio of the minimum degree over the number of nodes to assure the synchronizability of a graph. It is worth to say that *there is not an analogous bound for the average degree*, since the graph made by a large complete graph and a 6-cycle touching each other in exactly one vertex, gives an example of a non synchronizing graph with large average degree.

In this work we consider those graphs treated by Wiley et al. in [17] which are called Harary graphs. We prove that some limits consider by them are correct, we give examples of exotic equilibria of Harary graphs and we prove them to be unstable, as it was conjectured by Wiley et al. Finally we build new graphs with non trivial stable equilibria, but with minimum degree–order ratio greater than the lower bound derived in [14]. In particular, we prove that the minimum degree–order ratio that assure synchronizability should be greater than 0.618.

This works is organized in the following way. In Section 2 we present the basic definitions and results in graph theory and about homogeneous Kuramoto model. In Section 3 we give examples of exotic equilibria for Harary graphs and prove their instability, besides we show that they form an strange set. In Section 4 we study the stability of a particular important equilibrium of Harary graphs called 1-twisted equilibria. In Section 5 we prove that the asymptotical size a Harary graph must have in order for its 1-twisted equilibria to be unstable is indeed the one proved in [17]. In

Section 6 we defined an operator τ over the set of regular graph and we study its relationship with Kuramoto models properties, in particular, with the stability of their equilibria. Taking in count this operator, we found a new lower bound for the minimum degree–order ratio that asure synchronizability.

2. Basic Definitions and Results

2.1. **Graph Theory.** A *graph* G consists in a set VG of *vertices*, some of them *joined* by *edges* in a set EG. If two vertices v and w are joined by an edge e, we say they are *adjacent* and we write $e = vw$, $v \sim_G w$ or simply $v \sim w$ if no doubt about G could arise. In this work, all graphs are *simple*, i.e. there are no edge of the form vv and no two different edges join the same vertices.

The *order* $|G|$ of G is the cardinality $|VG|$ of its vertex set. We will denote by G_v the set of vertices adjacent with v in G. Thus, $w \in G_v$ iff $v \sim_G w$. The cardinal of G_v is the *degree* of v and is denoted by $d_G(v)$. The minimum degree amount the vertices of G is denoted δG and called *minimum degree* of G, so

$$\delta G = \min_{v \in VG} d_G(v).$$

Two vertices are *twins* if they have the same set of adjacent vertices. We will consider *adjacent twins*, i.e., adjacent vertices which have the same set of adjacent vertices except for themselves. More formally, two vertices v and w are adjacent twins iff $v \sim w$ and $G_v \setminus \{w\} = G_w \setminus \{v\}$.

The *circulant* graph $\mathrm{Ci}_n(S)$ is the graph with vertex set $\mathbb{Z}_n$ of integer module n and adjacencies defined by a subset of $S \subset \{1, \dots, [n/2]\}$ in the following way: two vertices $x, y \in Z_n$ are adjacent iff $x - y \in S \cup -S$.

In this work we will focus specially in *Harary Graphs* $H_{2k,n}$, which are the graphs treated in [17] and called WSG in [14]. They are the circulant graphs with order n and generator $\{1, 2, \dots, k\}$, i.e. $H_{2k,n} = \mathrm{Ci}_n(\{1, \dots, k\})$. They can be seen as graphs where the vertices are located uniformly in a circumference and connected to the nearest k (measuring distances as arc length). Figure 1 shows two different circulant graphs, one of them is a Harary graph as well.

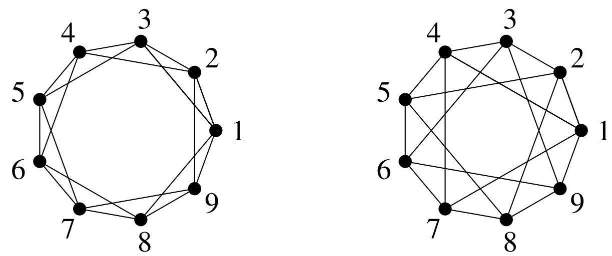

FIGURE 1. Circulant graphs $H_{4,9} = \mathrm{Ci}_9(\{1,2\})$ and $\mathrm{Ci}_9(\{1,3\})$.

Further notions of graph theory can be found in [16].

2.2. **Kuramoto Model.** The *Kuramoto model of coupled oscillators with natural frequencies ω_i coupled through a graph G with strength K*, is the system of differential equations given by:

$$\dot{\theta}_i = \omega_i + K \sum_{j \in G_i} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots, n.$$

If the ω_i are all equal, say, to ω, then the system is called homogeneous and we can suppose $\omega = 0$ and $K = 1$. Indeed, by "rotating with the oscillators" through the change of variable $\phi_i = \theta_i - \omega t$, the ω will "disappear" from the equations. On the other hand, by a change in time scale of the form $\theta_i(t) = \phi_i(Kt)$, we "cancel" the K. So, let us suppose that we have the following system of differential equations:

$$\dot{\theta}_i = \sum_{j \in G_i} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots, n. \tag{1}$$

This model is called *homogeneous Kuramoto Model* and it is the one studied in [11, 17, 14]. In this way, as observed before ([17, 8, 14, 11, 4]), the model (and all its properties) depends only upon the graph G. This is important because one can focus exactly on those properties of the topology that concern only with synchronizability. You can find more good reasons and motivations for study the homogeneous model in [17, 14].

We say that θ is a *solution* of G iff it is a solution of (1). In particular, $\theta^* \in \mathbb{R}^n$ is an *equilibrium* of G if it is an equilibrium of the system, i.e, if the constant function $\theta_i(t) = \theta_i^*$ verifies:

$$0 = \sum_{j \in G_i} \sin(\theta_j^* - \theta_i^*) \qquad i = 1, \dots, n. \tag{2}$$

For instance, *whenever* $\theta_j^* - \theta_i^* \in \{0, \pi\}$, *the point is an equilibrium*. We call these equilibria *trivial*. Among them, the *consensus* are those with null differences, i.e., $\theta^* = \theta_0 \vec{\mathbf{1}}$ where $\vec{\mathbf{1}} = (1, 1, \dots, 1)$, i.e., with all oscillators in the same phase or *synchronized*.

Let us present the non trivial equilibria studied by Wiley et al. [17]. Although they consider a kind of infinite Harary graphs, the equilibria they defined are easily extended to any circulant graph, as we will show now. Let G be a circulant graph $\mathrm{Ci}_n(S)$ and q any integer, then the "q-twisted equilibria" of G is the constant function $\theta^*(t)$ defined by

$$\theta_i^*(t) = qi\frac{2\pi}{n}.$$

Let us check that $\theta^*(t)$ *is in fact and equilibrium*. Indeed, the sum in the right hand side of (2) becomes

$$\sum_{s \in S \cup (-S)} \sin(qs2\pi/n) = \sum_{s \in S: 0 < s < n/2} [\sin(2\pi qs/n) + \sin(-2\pi qs/n)] + \sum_{s \in S \cap \{n/2\}} \sin(-2\pi qs/n),$$

which is null.

The equilibria are better seen by drawing the phasors $e^{I\theta_i^*}$, where I is the imaginary unity. For instance in Figure 2 you can see two q–twisted equilibria of $H_{4,9}$.

In [17], the authors also asked for non twisted equilibria and conjectured that if exist they should be unstable. In next section we will both show the existence of such equilibria as well as their instability.

We say that a graph *synchronizes* iff almost every solution tends to a consensus, i.e., if the set of orbits that do not tends to a consensus has Lebesgue's measure zero. This concept is exactly the one called for study in [11, 17, 14], but with another name.

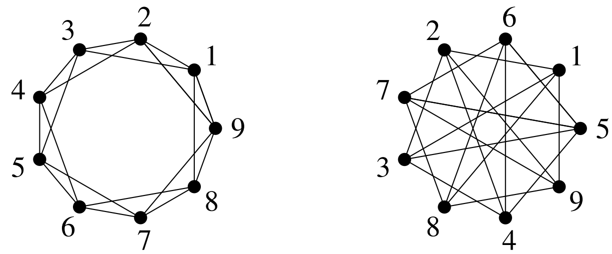

FIGURE 2. Two q–twisted equilibria of $H_{4,9}$, for $q = 1, 2$.

As noticed in [11, 17], by a theorem of La Salle, *every orbit of* (1) *must go to an equilibrium*, because the system can be seen as a gradient system running on an n-dimensional torus, which is a compact manifold. Indeed, the *energy function* defined as

$$U(\theta_1, \ldots, \theta_n) = |EG| - \sum_{ij \in EG} \cos(\theta_i - \theta_j),$$

verifies

$$\dot{\theta}_i = -\frac{\partial U}{\partial \theta_i}, \tag{3}$$

i.e., if $\theta(t) = (\theta_1(t), \ldots, \theta_n(t))$ is the vector of θ_i's, then $\dot{\theta} = -\nabla U(\theta)$. Besides, the function U is well defined in the torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. This remark answers the question about "limit cycles, limit tori or other exotic structures" posed in the conclusions of [14].

In [14] the author proved the existence of a non trivial upper bound for the number

$$\mu = \inf\{\kappa : \forall G, (\delta G \geq \kappa |G|) \Rightarrow G \text{ synchronizes}\}, \tag{4}$$

that guarantees the synchronizability of a graph with vertex degrees greater than this percent of the graph's order.

Also in [14], a non trivial lower bound for μ, based in the 1–twisted equilibrium of $H_{2k,n}$ is found. In Section 6 we will improve this lower bound.

Another important property of (1) is the orthogonality of the solutions to $\vec{\mathbf{1}}$. Indeed, the sum of the $\dot{\theta}_i$'s in (1) is zero since each $\sin(\theta_j - \theta_i)$ in $\dot{\theta}_i$ cancels with $\sin(\theta_i - \theta_j)$ of $\dot{\theta}_j$. Thus, the solution are always running in a hyperplane orthogonal to $\vec{\mathbf{1}}$, which can be see, is in fact a $(n-1)$–dimensional torus. So, we only care about the behavior in these hyperplanes, though, for simplicity, the calculation will be done in $\mathbb{R}^n$.

One way to see that a graph synchronizes is to prove that for all its non consensus equilibria θ, the Hessian matrix U''_θ of U at θ has at least one negative eigenvalue (Proposition 11 in [12])[1]. On the other hand, if U''_θ has a kernel of dimension 1 (corresponding to vector $\vec{\mathbf{1}}$) and the eigenvectors orthogonal to $\vec{\mathbf{1}}$ have positive eigenvalue, then the equilibrium is (linearly) stable. *If the last happen in a non consensus equilibrium, then the graph is not synchronizing*, because the basin of attraction for this equilibrium will have positive Lebesgue measure (in the hyperplane orthogonal to $\vec{\mathbf{1}}$).

In the particular case of circulant graphs, an explicit formula for the eigenvalues of U''_θ, when θ is a twisted equilibrium, can be given because U''_θ is itself a circulant matrix. Indeed, if $G = \mathrm{Ci}_n(S)$ and θ is its q-twisted equilibrium, then U''_θ is

$$(U''_\theta)_{xy} = \begin{cases} -c_n + 2\displaystyle\sum_{s\in S}\cos(qs2\pi/n) & x = y, \\ -\cos(q(y-x)2\pi/n) & y-x \in S \cup -S, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_n = (-1)^q$ if $n/2 \in S$ and 0 otherwise (notice that $n/2 \in S$ implies that n is even).

Thus, following [1, pp. 16], the eigenvalues of U''_θ, if $n/2 \notin S$, are

$$\lambda_{q,j} = 2\sum_{s\in S}\cos(qs2\pi/n) - 2\sum_{s\in S}\cos(qs2\pi/n)\cos(js2\pi/n) = 2\sum_{s\in S}\cos(qs2\pi/n)[1-\cos(js2\pi/n)].$$

If $n/2 \in S$ then $\lambda_{q,j} = -(-1)^q[1-(-1)^j] + 2\sum_{s\in S}\cos(qs2\pi/n)[1-\cos(js2\pi/n)]$.

[1] We notice that since U verifies $\dot{\theta} = -\nabla U$, thus U'' is opposite to the Jacobian of the vector field, so do their eigenvalues.

If the graph is $H_{2k,n}$, with $k < n/2$, the eigenvalues become

$$\lambda_{q,j} = 2\sum_{i=1}^{k} \cos(qi2\pi/n)[1-\cos(ji2\pi/n)], \qquad j = 1,\dots,n. \tag{5}$$

If $k = n/2$, we have even n and

$$\lambda_{q,j} = \sum_{i=1}^{n-1} \cos(qi2\pi/n)[1-\cos(ji2\pi/n)] = \sum_{i=0}^{n-1} -\cos(qi2\pi/n)\cos(ji2\pi/n), \tag{6}$$

since $\cos 0 = 1$ and $\sum_{i=0}^{n-1} \cos(qi2\pi/n) = 0$ for $q = 1,2,\dots$.

Finally we want to observe that instead of computing the eigenvalues of U''_θ, sometimes it is easier to consider its associated quadratic form $Q_\theta(z)$ defined by.

$$Q_\theta(z) = z^* U''_\theta z = \sum_{ij\in EG} \cos(\theta_i - \theta_j)|z_i - z_j|^2.$$

Typically, if there exists a z such that $Q_\theta(z) < 0$, then the equilibrium is unstable, and we avoid finding an eigenvalue.

3. Exotics Harary graphs' equilibria

In this section we show the existence of non-twisted equilibria for some Harary graphs and show they are all unstable. We will begin with the complete graphs since they are the base for understanding the other cases. Then we consider the cycles and finally the Harary graphs of the form $H_{(n+1)k,nh}$ with $nh < (n+1)k$ and $H_{(n-1)k,nh}$ with $nh \le (n+1)k$.

In order to derive the instability of these equilibria we will apply a Lemma proved in [2, Lemma 3.2 ii–iii], so let us recall it here.

Lemma 1. *If θ is an equilibrium of G and the i-th element $(U''_\theta)_{ii}$ in the diagonal of U''_θ is not positive, then θ is unstable.* □

It is worth to say that the strict version of this lemma was rediscovered many times and it follows directly by evaluating the quadratic form Q_θ in the indicatrix function of vertex i. However, when

$(U''_\theta)_{ii} = 0$ the argument requires a totally different approach. Also we notice that since

$$(U''_\theta)_{ii} = \sum_{j \in G_i} \cos(\theta_j - \theta_i),$$

then, if the differences $\theta_j - \theta_i$ are in the interval $(\pi/2, \pi]$, the results holds trivially.

3.1. **Complete Graphs.** In this section we give a description of the whole set of equilibria of the complete graphs K_n, i.e., those with every vertex adjacent to each other. Notice that K_n is exactly the Harary graph $H_{n-1,n}$ even when n is even, since when n even, Harary graph $H_{2k+1,n}$ is defined as the circulant graph $\mathrm{Ci}_n(\{1,2,\dots,k,n/2\})$.

Let θ be an equilibrium of the complete graph, then, from (2), we have, for each i

$$0 = \sum_{j=1}^{n} \sin(\theta_j - \theta_i).$$

Following Kuramoto, we define $Re^{I\psi} = \sum_{j=1}^{n} e^{I\theta_j}$, and see the right hand side of previous equation is the imaginary part of $\sum_{j=1}^{n} e^{I(\theta_j - \theta_i)}$, thus

$$0 = \mathrm{Im} \sum_j e^{I(\theta_j - \theta_i)} = \mathrm{Im}(e^{-I\theta_i} \sum_j e^{I\theta_j}) = \mathrm{Im}(e^{-I\theta_i} Re^{I\psi}) = \mathrm{Im}(Re^{I(\psi - \theta_i)}) = R \sin(\psi - \theta_i).$$

Thus, either $R = 0$ or $\theta_i \in \{\psi, \psi + \pi\}$. In the last case, either θ is a consensus or the oscillators are in "counter-phase", which is an unstable equilibrium as shown in [11] and easily checked by seeing that the quadratic form $Q_\theta(z)$ is negative for $z_i = 1$ if $\theta_i = \psi$ and 0 otherwise ($\theta_i = \psi + \pi$).

The more interesting case is when $R = 0$. In this situation, we have a set $M = \{e^{I\theta_j}\}$ of unit vectors which summing up 0, so they form what is known as *planar equilateral polygons* [9]. The classification of planar equilateral polygons is far from being done. In fact, this topological structures are well known only for $n \leq 6$. For $n > 6$, we only have partial information, for instance, their Betti numbers.

If n is odd, then M is a $(n-2)$–dimensional manifold (see [7]), thus U'' has at least $n-2$ null eigenvalues. Fortunately, *U'' always has a negative one:* indeed, if $R = 0$ then $\sum_{j=1}^{n} \cos(\theta_j - \theta_i) = 0$

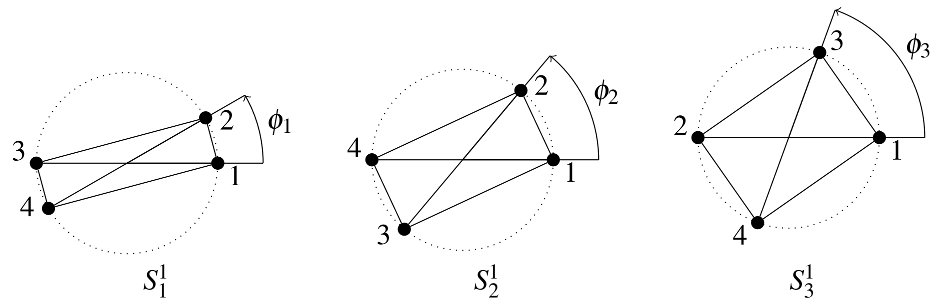

FIGURE 3. The three circumferences corresponding to the three torus of equilibria in K_4.

so $(U''_\theta)_{ii} = \sum_{j\neq i} \cos(\theta_j - \theta_i) = -1$, thus, by Lemma 1 the equilibria are unstable. It worth to say that the last argument is valid for any complete graph, including those with an even number of vertices.

When n is even, then M is not anymore a manifold, but a "manifold with singularities" (see [7]). For instance, if $n = 4$, theses singularities coincide with equilibria with oscillators in counter-phase as we will illustrate next.

If $n = 4$, i.e. K_4 we have three torus, any two of them sharing two circumferences. If we fix the phase of one vertex, for instance θ_1, then we have three circumferences any two of them sharing two points. This is illustrated in Figure 3: for $i = 1,2,3$ we parametrized the circumference S_i^1 with parameter ϕ_i. The intersection of S_1^1 and S_2^1 is given by $\phi_1 = \phi_2 = 0$, the intersection of S_1^1 and S_3^1 by $\phi_1 = \pi$ and $\phi_3 = \pi$ and the intersection of S_2^1 and S_3^1 by $\phi_2 = \pi$ and $\phi_3 = 0$.

It is interesting to observe that although the attractors of the homogeneous Kuramoto model are always single points, together they form a whole manifold of large dimension. This suggest the question of which is the relation between this large set of equilibria with the attractors that appears as soon as one relax the homogenous hypothesis.

3.2. **Cycles.** The cycles C_n with n vertices are the Harary graphs $H_{2,n}$.

If $n = 4$, the equilibria set of C_4 is made by two tori with two circumferences in common. As observed in [2], if we fix the position of one oscillator, for instance θ_1, then we obtain two circumferences crossing each other orthogonally in two points. We illustrate this configuration in

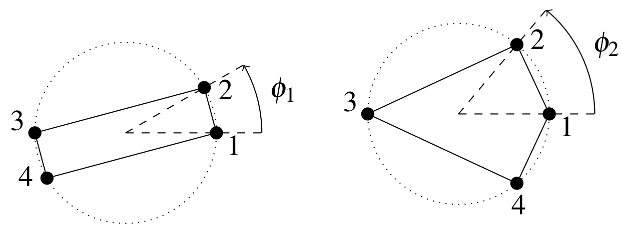

FIGURE 4. The two circumferences corresponding to the two torus of equilibria in C_4.

Figure 4. There, you can see that the circumferences intersect when $\phi_1 = \phi_2 = \pi/2$ and when $\phi_1 = \phi_2 = -\pi/2$. These points of intersection are, in fact, two 1-twisted equilibria, thus the corresponding cosines of U'' are null and we are in presence of a very rare equilibrium, one with all "its" eigenvalues zero. Again, by Lemma 1 these equilibria are unstable, since $(U'')_{ii} = 0$ for any i.

Similarly, if we consider C_{4k}, the set of equilibria is made by two (2–dimensional) tori with two circumferences in common. The tori are:

$$\mathbb{T}_1 = \{(\theta_1,\dots,\theta_n) : (\theta_{4i+1},\theta_{4i+2},\theta_{4i+3},\theta_{4i+4}) = (0,\theta_2,\pi,\pi+\theta_2)+\theta_1\vec{\mathbf{1}}_4 \qquad \theta_1,\theta_2\in[0,2\pi],\, i=0,\dots,k-1\},$$

$$\mathbb{T}_2 = \{(\theta_1,\dots,\theta_n) : (\theta_{4i+1},\theta_{4i+2},\theta_{4i+3},\theta_{4i+4}) = (0,\theta_2,\pi,-\theta_2)+\theta_1\vec{\mathbf{1}}_4 \qquad \theta_1,\theta_2\in[0,2\pi],\, i=0,\dots,k-1\},$$

where $\vec{\mathbf{1}}_4 = (1,1,1,1)$. Once again, these are essentially two circumferences with two common points where U'' is null and by Lemma 1, all these equilibria are unstable.

3.3. **$\mathbf{H_{2k,nh}}$ with $n < 2k$.** In this case and for some particular values of k and n, we can make strange equilibria from those found in K_n. The candidates are the following: for each equilibrium θ^* of K_n, we consider the point $\theta_i = \theta^*_{1+(i \mod n)}$. It remains to which of them are in equilibrium. The values of k and n we found to make θ an equilibrium of $H_{2k,nh}$ are the next ones: if n is even, then any $k \in \{n-1, 2n-1, 3n-1, \dots\}$ will work. If n is odd, then $k \in \{in + (n-1)/2 : i = 0,1,2,\dots\} \cup \{in-1 : i = 1,2,\dots\}$. It is straightforward to check these values make θ an equilibrium. The list is not exhaustive at all. Its instability follows again from Lemma 1.

4. SMALLEST EIGENVALUE OF TWISTED EQUILIBRIA

In this section we study the stability of the 1–twisted equilibrium of Harary graphs $H_{2k,n}$. The reason is that q-twisted equilibria with $q > 1$ are not "as stable" as the 1–twisted, and we want to find large k's with at least one stable equilibria. Although the last sentence is just a claim, we will not use this fact in any sense, so we leave the proof for further works.

Proposition 1. *Let U be the energy function corresponding to $H_{2k,n}$ and $\theta = (\alpha, 2\alpha, \dots, n\alpha)$ with $\alpha = 2\pi/n$ its 1–twisted equilibrium. Then the smallest eigenvalue of U''_θ among the eigenvalues with eigenvector orthogonal to $\vec{\mathbf{1}}$ is*

$$\lambda_1 = 2\sum_{i=1}^{k} \cos(i\alpha)[1-\cos(i\alpha)] \qquad \text{if } k < n/2,$$

and $\lambda_1 = \lambda_{n-1} = -n/2$ if $k = n/2$.

Proof. If $k = n/2$, then n is even and by (6) we have

$$\lambda_j = -\sum_{i=0}^{n-1} \cos(i\alpha)\cos(ij\alpha) \qquad \forall j = 1, \dots, n.$$

By the trigonometric equation

$$2\cos a \cos b = \cos(a+b) + \cos(a-b), \tag{7}$$

we have:

$$\lambda_j = -(1/2)\sum_{i=0}^{n-1} \cos[i(j+1)\alpha] + \cos[i(j-1)\alpha],$$

but $\sum_{i=0}^{n-1} \cos[ij\alpha]$ is n if $j \in \{0, n\}$ and 0 if $0 < j < n$. Thus, all λ_j are null except for $j = 1$ and $j = n-1$ for which $\lambda_j = -n/2$ as claimed. Notice that this is coherent with what we said in Section 3.1, because the trace of U''_θ is $-n$ and the dimension of the manifold is $n-2$.

If $k < n/2$ let us first consider the case $n \le 4$, i.e. $n = 3$ and $n = 4$. These cases can be verified by exhaustion, nevertheless the verification is trivial since the only possible $H_{2k,n}$ in that cases are

C_3 and C_4 respectively. But, the eigenvalues of C_3 different from 0 are equal, and for C_4 they are all zero.

Let us then consider the case $n \geq 5$ ($k < n/2$), from (5) we know that the eigenvalues of matrix U''_θ are given by

$$\lambda_j = 2\sum_{i=0}^{k}\cos(i\alpha) - 2\sum_{i=0}^{k}\cos(i\alpha)\cos(ij\alpha) \qquad \forall j = 1,\ldots,n.$$

The eigenvalue 0 corresponds to eigenvector $\vec{\mathbf{1}}$ and is λ_n, then we need to proof that the smallest value among $\lambda_1,\ldots,\lambda_{n-1}$ is λ_1. Besides, since $\lambda_j = \lambda_{n-j}$ we can suppose that $j \leq n/2$. On the other hand, since the term $2\sum_{i=0}^{k}\cos(i\alpha)$ does not depend on j it is enough to prove that the maximum of

$$S_j = 2\sum_{i=0}^{k}\cos(i\alpha)\cos(ij\alpha),$$

is attained at $j = 1$, i.e. $S_j \leq S_1$ for every $j \geq 2$.

Again, by (7), we have:

$$S_j = \sum_{i=0}^{k}\cos[i(j+1)\alpha] + \cos[i(j-1)\alpha],$$

which allows us to use the following well known formula:

$$\sum_{i=0}^{k}\cos(ix) = \begin{cases} \frac{1}{2} + \dfrac{\sin((k+\frac{1}{2})x)}{2\sin(x/2)} & x \neq 0, \\ k+1 & x = 0. \end{cases}$$

In fact, the fraction in the right hand side is one half the Dirichlet kernel $D_k(x)$, i.e.

$$D_k(x) = \frac{\sin\left((k+\frac{1}{2})x\right)}{\sin(x/2)},$$

so, we need to prove that

$$\frac{1+D_k\big((j+1)\alpha\big)}{2} + \frac{1+D_k\big((j-1)\alpha\big)}{2} \leq k+1+\frac{1+D_k(2\alpha)}{2} \qquad j \leq 2,$$

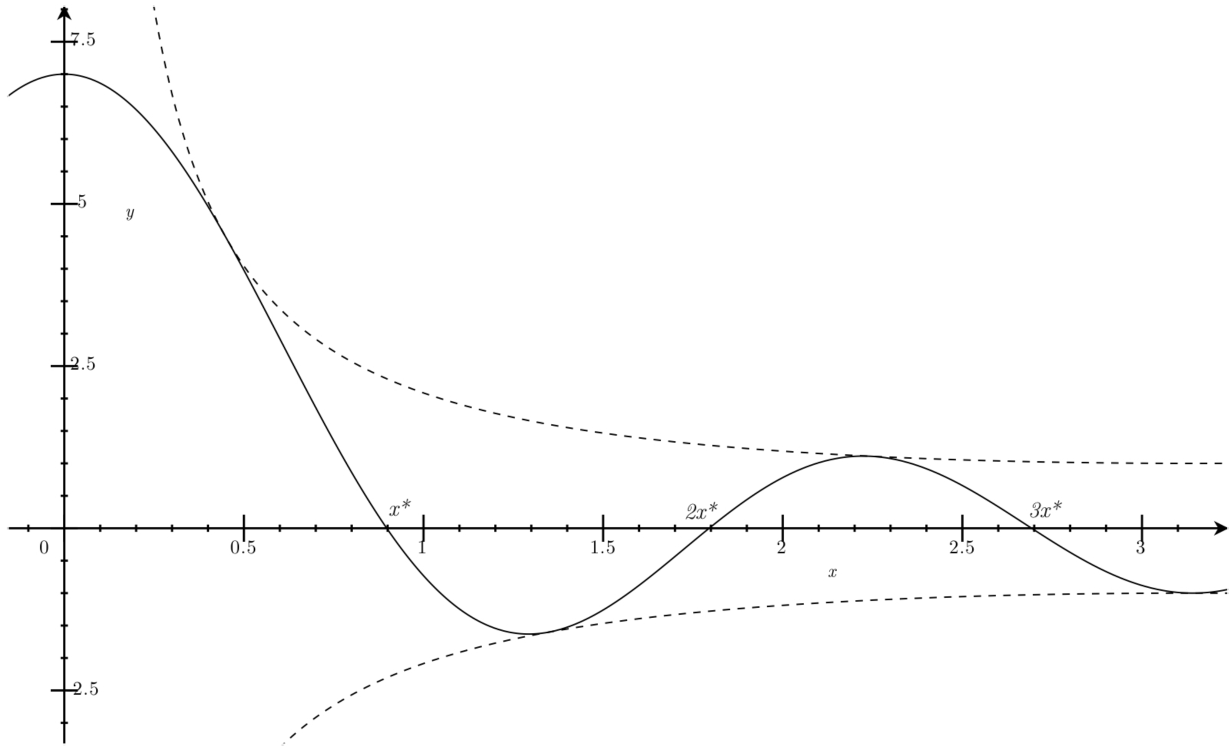

FIGURE 5. Graph of Dirichlet function $D_3(x)$ between $-f(x)$ and $f(x)$.

i.e.

$$D_k\big((j+1)\alpha\big)+D_k\big((j-1)\alpha\big) \leq 2k+1+D_k(2\alpha) \qquad j \leq 2. \tag{8}$$

Before proceeding with our argument, it is worth to figure out how the graph of $D_k(x)$ is. It suffices to see the interval in $[0,\pi]$ since $0 < j \leq n/2$ and thus $0 < j\alpha \leq \pi$. In Figure 5 we drawn $D_3(x)$. In general, the function $D_k(x)$ attains the value $2k+1$ at 0, i.e. $D_k(0) = 2k+1$ and then decreases (it can be elementary checked by computing the derivative) until reaches its first positive zero at

$x^* = 2\pi/(2k+1)$. Then it keeps decreasing until it reaches a local minimum somewhere near before $(3/2)x^*$. After that, it begins to increase until finding its second positive zero at $2x^*$. So it is negative between x^* and $2x^*$. At values greater than $2x^*$ there could be many others zeros, but we do not need to take them in count.

The idea of the argument is that near the origin the function is decreasing so the inequality is trivial and far from the origin the terms in the inequalities are much more smaller than the value of the function in the origin. Unfortunately, some extra work need to be done between these two cases.

We will employ the following bounds:

$$|D_k(x)| \leq f(x) = \frac{1}{\sin(x/2)} \leq \frac{\pi}{x} \qquad \forall x \in [0,\pi].$$

In particular we have

$$f(bx^*) \leq \frac{\pi}{bx^*} = \frac{1}{b}(k+1/2) = \frac{1}{b}\frac{D_k(0)}{2}.$$

In order to proceed with the proof we consider three cases, depending upon in which part of the partition $0 < (j-1)\alpha < (j+1)\alpha < \pi$ of the interval $[0,\pi]$ the root x^* is.

Case I: $x^* \geq (j+1)\alpha$. In this case all the arguments of $D_k(x)$ appearing in (8) belong to $[0,x^*]$ where $D_k(x)$ is decreasing, so $D_k((j+1)\alpha) < D_k(2\alpha)$ and $D_k((j-1)\alpha) < D_k(0\alpha)$ and we obtain (8).

Case II: $x^* \leq (j-1)\alpha$. In this case, both $D_k((j-1)\alpha)$ and $D_k((j+1)\alpha)$ are not greater than $f(x^*) \leq k+1/2$, thus $|D_k((j-1)\alpha)| + |D_k((j+1)\alpha)| \leq 2k+1 = D_k(0)$, so if $D_k(2\alpha)$ is positive we have (8), for instance, if $2\alpha \leq x^*$. Otherwise, let us suppose $2\alpha > x^*$. Then for each $\beta \in \{(j-1)\alpha,(j+1)\alpha\}$ either $\beta \leq 2x^*$ (so $D_k(\beta)$ is negative) or $\beta > 2x^*$. In both cases it holds $D_k(\beta) < f(2x^*)$, thus

$$\begin{aligned} D_k((j-1)\alpha) \quad &+ D_k((j+1)\alpha) - D_k(2\alpha) < f(2x^*) + f(2x^*) + f(x^*) \\ &\leq \left(\tfrac{1}{2} + \tfrac{1}{2} + 1\right)\left(k + \tfrac{1}{2}\right) = 2k+1 = D_k(0). \end{aligned}$$

As we wanted.

Case III: $(j-1)\alpha \leq x^* \leq (j+1)\alpha$. First notice that if $3 \leq j$ then $2\alpha \leq (j-1)\alpha \leq x^*$, so $D_k((j-1)\alpha) \leq D_k(2\alpha)$ and we have (8), since $D_k((j+1)\alpha) < 2k+1$. So the only remaining case is $j=2$, i.e.

$$D_k(\alpha)+D_k(3\alpha) \leq 2k+1+D_k(2\alpha), \qquad \text{with } \alpha \leq x^* \leq 3\alpha.$$

We prefer to fix x^* and consider $\alpha \in [x^*/3, x^*]$.

If $\alpha \in [x^*/3, x^*/2]$, then $3\alpha \leq 2x^*$ and $2\alpha \leq x^*$, so $D_k(3\alpha) \leq 0$ and $D_k(2\alpha) \geq 0$ implying (8).

Finally, if $\alpha \in [x^*/2, x^*]$, then $D_k(2\alpha) \leq 0$ and, if we call A to $(k+1/2)\alpha$ we have

$$D_k(\alpha)+D_k(3\alpha)-D_k(2\alpha) \leq \frac{\sin(A)}{\sin(x^*/2)}+\frac{\sin(3A)}{\sin(3x^*/2)}-\frac{\sin(2A)}{\sin(3x^*/2)}.$$

But,

$$\frac{\sin(3A)}{\sin(3x^*/2)}-\frac{\sin(2A)}{\sin(3x^*/2)}=\frac{2\sin(A/2)\cos(5A/2)}{\sin(3x^*/2)}<\frac{2}{\sin(3x^*/2)}.$$

Then

$$\begin{aligned} D_k(\alpha)+D_k(3\alpha)-D_k(2\alpha) &\leq \frac{1}{\sin(x^*/2)}+\frac{2}{\sin(3x^*/2)} \leq f(x^*)+2f(3x^*) \leq \\ &\leq (1+2/3)(k+1/2) < 2k+1. \end{aligned}$$

As we wanted to prove. □

5. Asymptotic estimation of the most stable twisted equilibrium of $H_{2k,n}$

In this section we will estimate the larger k such that the eigenvalue λ_1 of Proposition 1 is still positive. Since $k=n/2$ implies the eigenvalue to be negative, be can suppose $k<n/2$. In order to emphasize the dependance of λ_1 in k, let us write it $\lambda_{1,k}$ instead of λ_1. Then

$$\lambda_{1,k}=2\sum_{i=1}^{k}\cos(i\alpha)[1-\cos(i\alpha)].$$

with $\alpha = 2\pi/n$. If $k/n = \kappa$ then $\lambda_{1,k}$ is an approximation for the definite integral of $2\cos(x)(1-\cos(x))$ in the interval $[0, 2\pi\kappa]$, i.e.

$$I(\kappa) = \int_0^{2\pi\kappa} 2\cos(x)[1-\cos(x)]dx = 2\sin(2\pi\kappa) - \frac{1}{2}\sin(4\pi\kappa) - 2\pi\kappa.$$

The function $I(\kappa)$ is positive from 0 to $\kappa^* \approx 0.3404614171 = 0.6809228342/2$ and negative after that. However, this reasoning is similar to that presented in [17] and does not prove that the finite solution goes to κ^*. Instead, we need to compute the finite solution and then, make k and n go to infinite. So, let us consider $\lambda_{1,k}$ in terms of Dirichlet kernels:

$$\lambda_{1,k} = 1 + D_k(\alpha) - S_1 = 1 + D_k(\alpha) - \left(k + 1 + \frac{1 + D_k(2\alpha)}{2}\right) = D_k(\alpha) - k - \frac{1}{2} - \frac{1}{2}D_k(2\alpha) =$$

$$\frac{\sin((k+1/2)\alpha)}{\sin(\alpha/2)} - k - \frac{1}{2} - \frac{1}{2}\frac{\sin((k+1/2)2\alpha)}{\sin(\alpha)} = \frac{\sin((k+1/2)\frac{2\pi}{n})}{\sin(\frac{\pi}{n})} - k - \frac{1}{2} - \frac{1}{2}\frac{\sin((2k+1)\frac{2\pi}{n})}{\sin(\frac{2\pi}{n})}.$$

Now, consider κ_n^* the greatest k/n such that $\lambda_{1,k} > 0$, i.e.

$$\kappa_n^* = \frac{1}{n}\max\{k \in \mathbb{Z} : \lambda_{1,k} > 0\}.$$

Then $\lambda_{1,\kappa_n^*} > 0$ iff

$$\frac{\sin(2\pi\kappa_n^* + \pi/n)}{\sin(\pi/n)} - \kappa_n^* n - \frac{1}{2} - \frac{1}{2}\frac{\sin(4\pi\kappa_n^* + 2\pi/n)}{\sin(2\pi/n)} > 0 \iff$$

$$\frac{\sin(2\pi\kappa_n^* + \pi/n)}{n\sin(\pi/n)} - \kappa_n^* - \frac{1}{2n} - \frac{1}{2}\frac{\sin(4\pi\kappa_n^* + 2\pi/n)}{n\sin(2\pi/n)} > 0.$$

Taking $\liminf$ and $\limsup$, we deduce that both $\underline{\kappa}^* = \liminf \kappa_n^*$ and $\bar{\kappa}^* = \limsup \kappa_n^*$ verify

$$\frac{\sin(2\pi\kappa)}{\pi} - \kappa - \frac{\sin(4\pi\kappa)}{4\pi} \geq 0,$$

which is in fact $I(\kappa) \geq 0$, thus $\underline{\kappa}^*$ and $\bar{\kappa}^*$ are the same, so the limit of κ_n^* exists and it is equal to κ^*, formally

$$\lim \kappa_n^* = \kappa^*.$$

It can be seen, with a little more effort that κ_n^* is smaller than κ^* for all n.

6. The New lower bound

In order to improve the lower bound to the minimum degree–order ratio a graph should have to synchronize, it is enough to find graphs G with minimum degree greater than $0.6809|G|$ such that they have at least one stable non consensus equilibrium. We will achieve that goal by making large enough amount of adjacent twins of $H_{2k,n}$. We will proceed as general as we can.

Given an integer τ greater than 1, let us call G^τ to the graph made from G by adding τ adjacent twins to each vertex. In order to clarify the concept, let us call $v_1, v_2, \ldots, v_\tau$ the adjacent twins of v made including v itself. Remember that v could already have adjacent twins so, we are only enumerating the new twins. Then, if $T_v = \{v_1, v_2, \ldots, v_\tau\}$, we have

$$VG^\tau = \bigcup_{v \in VG} T_v, \quad \text{and} \quad G^\tau_{v_i} = T_v - v_i + \bigcup_{w \in G_v} T_w. \tag{9}$$

The next lemma says that G^τ is as stable as G in the following sense.

Lemma 2. *A graph G has a non trivial linearly stable equilibrium iff so does G^τ.*

Proof. We will prove that there is a bijection between the linearly stable equilibrium of G and G^τ. Let θ be a linearly stable equilibrium of G. From Lemma 5.1 of [2] we known that if θ^τ is a linearly stable equilibrium of G^τ, then the adjacent twins should be *synchronized*, in the sense that if v and w are adjacent twins, then $\theta^\tau_v = \theta^\tau_w$. This result suggest the candidate θ^τ for stable equilibrium of G^τ:

$$\theta^\tau_{v_i} = \theta_v \qquad \forall v_i \in G^\tau.$$

Let us first check that θ^τ is an equilibrium of G^τ and then its linear stability. In order to do the former, we need to show that θ^τ verifies (2): given $v_i \in G^\tau$, then

$$\sum_{w_j \in G^\tau_{v_i}} \sin(\theta^\tau_{w_j} - \theta^\tau_{v_i}) = \sum_{v_j \in T_{v_i} - v_i} \sin(\theta_v - \theta_v) + \sum_{w \in G_v} \sum_{w_j \in T_w} \sin(\theta_w - \theta_v) = \sum_{w \in G_v} \tau \sin(\theta_w - \theta_v) = 0.$$

In order to prove the linear stability of θ^τ we will consider the Hessian matrix U''_θ of the energy function of G at θ and the Hessian matrix U''_{θ^τ} of the energy function of G^τ at θ^τ. Notice that we make an abuse of notation by using the same letter "U" for both functions, hoping this will not give rise to confusion.

Let us compute the elements of matrix U''_{θ^τ}:

$$(U''_{\theta^\tau})_{v_j w_{j'}} = \begin{cases} -\cos(\theta^\tau_{v_j} - \theta^\tau_{w_{j'}}) = -\cos(\theta_v - \theta_w), & vw \in EG, \\ -1 & v = w, j \neq j', \\ (\tau - 1) + \tau \sum_{w \in G_v} \cos(\theta_v - \theta_w) = \tau - 1 + \tau (U''_\theta)_{vv} & v = w, j = j', \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that if x^τ is an eigenvector of U''_{θ^τ} with eigenvalue λ^τ, then either λ^τ is $\tau[1 + (U''_\theta)_{vv}]$ or λ^τ is $\tau\lambda$ for each eigenvalue λ of U''_θ.

We proceed by computing the component of $U''_{\theta^\tau} x^\tau$ in v_j:

$$(U''_{\theta^\tau} x^\tau)_{v_j} = \left[\tau - 1 + \tau (U''_\theta)_{vv}\right] x^\tau_{v_j} - \sum_{j' \in \mathbb{N}^+_\tau \setminus \{j\}} x^\tau_{v_{j'}} - \sum_{w \in G_v} \sum_{h \in \mathbb{N}^+_\tau} \cos(\theta_w - \theta_v) x^\tau_{w_h}, \tag{10}$$

where $\mathbb{N}^+_\tau = \{1, 2, \ldots, \tau\}$. Since $(U''_{\theta^\tau} x^\tau)_{v_j} = \lambda^\tau x^\tau_{v_j}$, we have

$$-\sum_{j' \in \mathbb{N}^+_\tau} x^\tau_{v_{j'}} - \sum_{w \in G_v} \cos(\theta_w - \theta_v) \sum_{h \in \mathbb{N}^+_\tau} x^\tau_{w_h} = \left[\lambda^\tau - \tau - \tau (U''_\theta)_{vv}\right] x^\tau_{v_j}.$$

Now, we observe that the left hand side of the equality does not depend on j, so does the right hand side. Then, either $\lambda^\tau - \tau - \tau (U''_\theta)_{vv}] \neq 0$ and the $x^\tau_{v_j}$'s are the same for each j or $\lambda^\tau = \tau[1 + (U''_\theta)_{vv}]$. In last case, $\lambda^\tau > 0$, because by Lemma 1 we have $(U''_\theta)_{vv} > 0$. In the former, let us say that $x^\tau_{v_j} = x_v$ for every j, then we have

$$-\tau x_v - \sum_{w \in G_v} \cos(\theta_w - \theta_v) \tau x_w = \left[\lambda^\tau - \tau - \tau (U''_\theta)_{vv}\right] x_v.$$

Thus,

$$(U''_\theta)_{vv}x_v - \sum_{w\in G_v}\cos(\theta_w-\theta_v)x_w = \frac{\lambda^\tau}{\tau}x_v.$$

This last equation says that the vector $(x_v)_{v\in VG}$ is an eigenvectors of U''_θ with eigenvalue $\lambda = \lambda^\tau/\tau$. In summary, each eigenvector x of U''_θ with eigenvalue λ gives rise to an eigenvector x^τ of U''_{θ^τ} with eigenvalue $\lambda^\tau = \tau\lambda$. Therefore, U''_{θ^τ} has one eigenvalue zero and the others positive.

Conversely, let θ^τ be a linearly stable equilibrium of G^τ, then by Lemma 5.1 of [2] $\theta^\tau_v = \theta^\tau_w$ for every pair v and w of adjacent twin vertices. By hypothesis $Q_{\theta^\tau}(x^\tau) > 0$ for every $x^\tau \in \vec{\mathbf{1}}^\perp\setminus\{0\}$. Now, given $x\in\mathbb{R}^{|VG|}\cap\vec{\mathbf{1}}^\perp\setminus\{0\}$, let $x^\tau\in\mathbb{R}^{|VG^\tau|}$ be defined by $x^\tau_{v_i} = x_v$. Then $x^\tau\in\vec{\mathbf{1}}^\perp\setminus\{0\}$ and

$$0 < Q_{\theta^\tau}(x^\tau) = \sum_{v_iw_j\in EG^\tau}\cos(\theta^\tau_{v_i}-\theta^\tau_{w_j})|x^\tau_{v_i}-x^\tau_{w_j}|^2 = \sum_{v_iw_j\in EG^\tau}\cos(\theta_v-\theta_w)|x_v-x_w|^2 =$$

$$\sum_{v\in G}\sum_{v_i,v_{i'}\in T_v}|x_v-x_v|^2 + \sum_{vw\in EG}\sum_{i,j=1}^{\tau}\cos(\theta_v-\theta_w)|x_v-x_w|^2 = \tau^2 Q_\theta(x).$$

Thus $Q_\theta(x) > 0$, as desired. □

We want to notice that the same result holds changing “stable” by “unstable”. The arguments are exactly the same, but we will not use that result anywhere.

The lemma is also true dropping the hypothesis of linearity, as we will prove next. Before proceeding with the proof, let us define an injection from the orbits of G to the orbits of G^τ in the following way: if $\theta(t)$ is a solution of G then define $\theta^\tau(t)$ as

$$\theta^\tau_{v_i}(t) = \theta_v(\tau t) \qquad \forall i = 1,\ldots,\tau.$$

We want to prove that θ^τ is a solution of G^τ as well.

Lemma 3. *If θ is a solution of G then θ^τ is a solution of G^τ as well.*

Proof. First notice that $\dot\theta^\tau_{v_i}(t) = \tau\dot\theta_v(\tau t)$. Let us check equation (1):

$$\sum_{w_j\in G^\tau_{v_i}}\sin(\theta^\tau_{w_j}(t)-\theta^\tau_{v_i}(t)) = \sum_{w_j\in T_{v_i}}\sin(\theta_v(\tau t)-\theta_v(\tau t)) + \sum_{w\in G_v}\sum_{j\in\mathbb{N}^+_\tau}\sin(\theta_w(\tau t)-\theta_v(\tau t))$$

$$= \tau \sum_{w \in G_v} \sin(\theta_w(\tau t) - \theta_v(\tau t)) = \tau \dot{\theta}_v(\tau t) = \dot{\theta}^{\tau}_{v_i}(t).$$

□

Let us observe that the orbits of G^τ are in a τn-dimensional torus, while the orbits θ^τ of the lemma form a n-dimensional invariant torus included in that torus, and have exactly the same behavior than the orbits of G except for the factor τ in time.

Lemma 4. *A graph G has a non trivial stable equilibrium iff so does G^τ.*

Proof. Let θ^* be an stable equilibrium of G and $\theta^{*\tau}$ be the corresponding equilibrium of G^τ, as defined in Lemma 4. Following the proof of this lemma, we see that $U''_{\theta^{*\tau}}$ has $\tau n - n$ positive eigenvalues $\tau[1 + (U''_\theta)_{vv}]$ and n eigenvalues of the form $\tau\lambda$ for each eigenvalue λ of U''_{θ^*}. Since θ^* is stable, then the eigenvalues of U''_{θ^*} need to be positive or zero, so do the eigenvalues of $U''_{\theta^{*\tau}}$. Thus, in order to prove the stability of $\theta^{*\tau}$, we need to study the system in any center manifold of $\theta^{*\tau}$, as it is inferred from in [6, Theorem 2, Section 1.3]. But one of such manifold is included in the set of orbits of the form θ^τ, given by Lemma 3, and its behavior is exactly the behavior of the orbits of G "accelerated" by a factor of τ. Since θ^* is stable, so does $\theta^{*\tau}$. □

6.1. **The minimum degree under the $G \mapsto G^\tau$ operation.** Recalling equation (9), we have that

$$d_{G^\tau}(v_i) = \tau - 1 + \tau d_G(v) \qquad \forall v_i \in G^\tau.$$

Thus, the minimum degree of G^τ is $\delta G^\tau = \tau - 1 + \tau\delta G$, and its minimum degree–order ratio

$$\frac{\delta G^\tau}{|G^\tau|} = \frac{\tau - 1 + \tau\delta G}{\tau|G|} = \frac{\delta G}{|G|} + \frac{\tau - 1}{\tau|G|} > \frac{\delta G}{|G|}.$$

This inequality, together with Lemma 4 proves that *if G does not synchronize due to a non trivial stable equilibrium, then there exists a graph with greater minimum degree–order ratio that does not synchronize neither.* Besides,

$$\frac{\delta G^\tau}{|G^\tau|} = \frac{1 + \delta G}{|G|} - \frac{1}{\tau|G|} \nearrow \frac{1 + \delta G}{|G|} \quad \text{if } \tau \to +\infty.$$

In particular, the 1-twisted equilibrium $\theta_i = i2\pi/22$ of $H_{14,22}$ is linearly stable and

$$\lim_{\tau\to+\infty} \frac{\delta H^{\tau}_{14,22}}{|H^{\tau}_{14,22}|} = \frac{15}{22} = 0.681818\ldots > \kappa^* \approx 0.68092$$

thus,

Proposition 2. $\mu \geq 15/22$. □

Just for curiosity, the first τ such that $\delta G^{\tau}/|G^{\tau}| > \kappa^*$ is $\tau = 51$, and for $\tau = 250\,000$ the ratio is exactly 0.681818.

There are other values of the pair (k,n) for which the limit of $\delta H^{\tau}_{k,n}/|H^{\tau}_{k,n}|$ is greater than κ^*, though smaller than 0.6818. For instance, (87, 257), (501,1473) and (9189, 3128), but we do not even know if the list is infinite.

7. Conclusion

In this work we made a summary of results about the homogeneous Kuramoto model, making some precisions about overcovered issues. Besides, we proved the correctness of some limits taken in [17] answering some questions prompted there about Harary graphs. More specifically, we showed the existence of exotics equilibria of these graphs as well as their instability. The question of classify all possible equilibria of Harary graphs remains open, but we show that even in the case of the complete graphs, which are also Harary graphs, the answer is equivalent to a difficult open topological problem: the classification of the planar equilateral polygons.

Finally we introduced an operator on graphs that open the possibility to improve the lower bound on the minimum degree–order rate μ for a graph to synchronize, a rate proved to be non trivial by Taylor in [14]. Indeed, we applied the technique successfully to improve the so far best lower bound for μ from $\kappa^* \approx 0.6809$ to $15/22$.

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REFERENCES

[1] BIGGS, N. *Algebraic Graph Theory*. Cambridge University Press, 1993.

[2] CANALE, E., AND MONZÓN, P. On the characterization of families of synchronizing graphs for Kuramoto coupled oscillators. In *1st IFAC Workshop on Estimation and Control of Networked Systems Venecia* (2009), pp. 42–47.

[3] CANALE, E. A., AND MONZÓN, P. Gluing Kuramoto coupled oscillators networks. In *46th IEEE Conference on Decision and Control , New Orleans* (2007), pp. 4596–4601.

[4] CANALE, E. A., AND MONZÓN, P. Almost global synchronization of symmetric Kuramoto coupled oscillators. In *Systems, Structure and Control.* InTech, 2008, ch. 4.

[5] CANALE, E. A., MONZÓN, P., AND ROBLEDO, F. The wheels: an infinite family of bi-conneted planar synchronizing graphs. In *The 5th IEEE Conference on Industrial Electronics and Applications (ICIEAÕ10) , Taichung, Taiwan* (2010).

[6] CARR, J. *Applications of Center Manifold Theory*, vol. 35 of *Applied Mathematical Sciences*. Springer-Verlag: New York, Heidelberg, Berlin, 1981.

[7] FARBER, M. *Invitation to Topological Robotics*. Zurich Lectures in Advanced Mathematics, 2008.

[8] JADBABAIE, A., MOTEE, N., AND BARAHONA, M. On the stability of the Kuramoto model of coupled nonlinear oscillators. In *Proc. of the American Control Conf.* (2004), vol. 5, pp. 4296–4303.

[9] KAMAYIMA, Y. The Riemannian volume of the moduli space of planar quasi-equilateral polygons with vertex number 4, 5 or 6. *Ryukyu mathematical journal 25* (2012), 1–19.

[10] KURAMOTO, Y. Self-entrainment of a population of coupled nonlinear oscillators. *Int. Symp. on Mathematical Problems in Theoretical Physics (Lecture Notes in Physics) 39* (1975), 420–422.

[11] MONZÓN, P., AND PAGANINI, F. Global considerations on the Kuramoto model of sinusoidally coupled oscillators. In *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005 Seville, Spain, December 12-15* (2005), pp. 3923–3928.

[12] MONZON, P., AND POTRIE, R. Local implications of almost global stability. *Dynamical systems an international journal 24*.

[13] RANTZER, A. A dual to Lyapunov's stability theorem. *Systems and Control Letters 42*, 3 (2001), 161–168.

[14] TAYLOR, R. There is no non-zero stable fixed point for dense networks in the homogeneous kuramoto model. *J. of Physics A: Mathematical and Theoretical 45* (2012), 1–15.

[15] VERWOERD, M., AND MASON, O. Conditions for the existence of fixed points in a finite system of Kuramoto oscillators. In *Proc. of the American Control Conf.* (2007), pp. 4613–4618.

[16] WEST, D. B. *Introduction to Graph Theory*. Prentice Hall, 1996.

[17] WILEY, D. A., STROGATZ, S. H., AND GIRVAN, M. The size of the synch basin. *Chaos 16* (2006).

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