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Extremal versus Additive Matérn Point Processes

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Abstract In the simplest Matérn point processes, one retains certain points of a Poisson point process in such a way that no pairs of points are at distance less than a threshold. This condition can be reinterpreted as a threshold condition on an extremal shot–noise field associated with the Poisson point process. This paper is focused on the case where one retains points that satisfy a threshold condition based on an *additive* shot–noise field of the Poisson point process. We provide an analytical characterization of the intensity of this class of point processes and we compare the packing obtained by the extremal and additive schemes and certain combinations thereof.

1 Introduction

Matérn type point processes were among the first point processes exhibiting repulsion and for which closed form expressions were obtained for e.g. first and second moment measures. These point processes are based on pairwise interactions between points. They are defined as a non-independent thinning of a Poisson point process, where the thinning can be seen as based on a threshold condition on the extremal shot–noise field associated with the Poisson point process.

The aim of the present paper is to define a new class of point processes where these pairwise interactions are replaced by more global interactions. More precisely, the threshold condition on the extremal shot–noise is replaced by a threshold condition on an additive shot–noise field.

Matérn type point processes are used in a variety of contexts including forestry, information theory and communication sciences to name a few. In the last domain,

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they were used to analyze exclusion mechanisms employed for medium access control [1] in wireless networks. These exclusion mechanisms aim at preventing nearby nodes to transmit simultaneously and jam each other. The practical motivations of the present paper stem from the last application domain. Within this context, these additive Matérn type point processes allow one to analyze exclusion rules based on the *interference level* rather than pairwise interactions.

The paper is structured as follows. We first define the models: the extremal shot–noise one, the additive shot–noise one and mixtures thereof. We then evaluate their first moment measures by considering first the case in general dimension and then a special case in the Euclidean plane. We conclude by an analytical comparison of the packings obtained by these point processes.

2 The models

Let $\widetilde{\Phi} = \{(X_i, m_i, F_i)\}$ be an independently marked (i.m.) Poisson point process (p.p.) with intensity λ in $\mathbb{R}^d$ where:

- $\Phi = \{X_i\}$ denotes the locations of the points;
- $\{m_i\}$ are $\mathbb{R}^+$-valued, independently and identically distributed (i.i.d.) marks with distribution H and density h;
- $\{F_i\}_i$ is an i.i.d. sequence of random vector with $F_i = (F^i_j)_j$, where the components F^i_j, $j \in \mathbb{N}$ are i.i.d. with distribution G.

We also define a deterministic function $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ where $L(X_i, X_j) = l(|X_i - X_j|)$, with $r \to l(r)$ a decreasing function. Moreover, given $X_i, X_j \in \widetilde{\Phi}$, we define $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^+$ as $f(X_i, X_j) = L(X_i, X_j) F^j_i \mathbf{1}_{\{m_j < m_i\}}$.

In this paper, we focus on three different thinnings of $\widetilde{\Phi}$ defined as follows:

$$\Phi_{ext} = \left\{ X_i \in \widetilde{\Phi} : \max_{X_j \in \widetilde{\Phi}, j \neq i} f(X_i, X_j) < M_0 \right\}, \tag{1}$$

$$\Phi_{add} = \left\{ X_i \in \widetilde{\Phi} : \sum_{X_j \in \widetilde{\Phi}, j \neq i} f(X_i, X_j) < S_0 \right\}, \tag{2}$$

$$\Phi^{add}_{ext} = \left\{ X_i \in \widetilde{\Phi} : \max_{X_j \in \widetilde{\Phi}, j \neq i} f(X_i, X_j) < M_0, \sum_{X_j \in \widetilde{\Phi}, j \neq i} f(X_i, X_j) < S_0 \right\}. \tag{3}$$

The default option is that the thresholds M_0 and S_0 are positive constants.

These processes are compatible though non-independent thinnings of the original Poisson p.p. where compatibility follows from the fact that the decision of retaining a point depends on the universal mark at (or equivalently the p.p. seen from) this point. This means that these processes are not Poisson in general.

Remark 1 Note that if the mark F_i is deterministic, then Φ_{ext} corresponds to a Matérn hard core process [5]. In this case, a point is retained by the thinning iff it has the smallest mark in a ball of fixed radius. This implies that there exists a minimum distance between any two points of the process. In the non deterministic case, when F_j^i is a random variable with distribution G, Φ_{ext} is a Matérn-like p.p. (see [1]). In this case, the retained points have the smallest mark in a random set defined from F_i rather than in a ball of fixed (or random) radius.

3 Intensities

In this section we calculate the intensity of the three thinnings previously defined. To achieve this, we relate the condition to be retained with a shot noise associated to the original process. This shot noise can be extremal or additive.

First of all, observe that the intensity of each one of the defined processes will be λp_0, where p_0 is the probability that a "typical point" (in the sense of the Palm distribution of $\widetilde{\Phi}$) is retained.

3.1 Extremal Matérn Point Processes (EMPP)

Let us concentrate first on Φ_{ext}, which, as mentioned above, is a Matérn like p.p.. If the mark of the point X_i is $m_i = t$, it is retained by Φ_{ext} iff

$$M^t(X_i) \equiv \max_{X_j \in \widetilde{\Phi}^t} F_i^j l(|X_i - X_j|) < M_0,$$

where $\widetilde{\Phi}^t = \left\{ X_j \in \widetilde{\Phi} : m_j < t \right\}$. Since Φ is a Poisson p.p. with intensity λ, $\widetilde{\Phi}^t$ is also a Poisson p.p. with intensity $\lambda H(t)$.

Assume there exists an i.i.d. sequence of random fields $y \in \mathbb{R}^2 \to F_i(y) \in \mathbb{R}^+$ such that $F_i(X_j) = F_j^i$ for all $i \neq j$. Then

$$M^t(y) = \max_{X_j \in \widetilde{\Phi}^t} F_j(y) l(|y - X_j|)$$

is an *extremal shot noise* (ESN) associated with the Poisson p.p. $\widetilde{\Phi}^t$. The distribution of $M^t(y)$ is known (see for instance [1]) and this will be used to calculate the intensity of Φ_{ext}. The key point here is that, thanks to the Slivnyak-Mecke theorem, the random variable $M^t(X_i)$ has the same distribution as $M^t(0)$.

Analogously, a point X_i with mark $m_i = t$ is retained by Φ_{add} iff

$$S^t(X_i) \equiv \sum_{X_j \in \widetilde{\Phi}^t} F_i^j l(|X_i - X_j|) < S_0.$$

As above, one can associate with the i.i.d. random fields $\{F_i(.)\}$ and the Poisson p.p. $\widetilde{\Phi}^t$ an *additive shot noise* (ASN) field

$$S^t(y) = \sum_{X_j \in \widetilde{\Phi}^t} F_i(y) l(|y - X_j|). \tag{4}$$

This additive shot noise has been extensively studied and we will use some of its main properties below. We will also use the the Slivnyak-Mecke theorem to show that the random variable $S^t(X_i)$ has the same distribution as $S^t(0)$.

Similar constructions will be used for the third model Φ_{ext}^{add}, which is a combination of the previous two cases.

Proposition 1 *The probability p_0^{ext} that a typical point is retained by Φ_{ext} is:*

$$p_0^{ext} = \frac{1 - e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)} \quad \textit{where} \quad \overline{\mathcal{N}}(M_0) = \lambda \int_{\mathbb{R}^d} \left(1 - G\left(\frac{M_0}{l(|x|)}\right)\right) dx. \tag{5}$$

Proof See [5,1].

Remark 2 Below we will assume that the integral defining $\overline{\mathcal{N}}(M_0)$ is finite so that p_0^{ext} is positive. This is not guaranteed. For instance, if we take $d = 2$, $M_0 = 1$ and $l(r) = r^{-\beta}$ with $\beta > 2$, which is natural in the applications that motivate this work, the finiteness of this integral is equivalent to

$$\int_{\mathbb{R}^+} (1 - G(v)) v^{\frac{2}{\beta}-1} dv < \infty$$

and we get for Lemma 1 p. 150 in [2] that $\overline{\mathcal{N}}(M_0)$ is finite if and only if the distribution G has a finite moment of order $2/\beta$ i.e. if

$$\int_{\mathbb{R}^+} x^{\frac{2}{\beta}} G(dx) < \infty.$$

Remark 3 Note that the probability does not depend on the distribution H of the marks m.

Remark 4 As we mentioned above, Φ_{ext} is a Matérn like p.p. and under the Palm probability of the underlying Poisson p.p., the point at the origin is retained if it has the smallest mark in the random set:

$$\begin{aligned} \mathcal{N}(M_0) &= \left\{ X_j \in \widetilde{\Phi},\ X_j \neq 0 : h(0, X_j) \geq M_0 \right\} \\ &= \left\{ X_j \in \widetilde{\Phi},\ X_j \neq 0 : F_0^j l(|X_j|) \mathbf{1}_{\{m_j < m_0\}} \geq M_0 \right\}. \end{aligned}$$

The number of point of $\widetilde{\Phi}$ in $\mathcal{N}(M_0)$ has a Poisson distribution and $\overline{\mathcal{N}}(M_0)$ can be interpreted as the mean number of points in this set.

Remark 5 The probability p_0^{ext} is asymptotically equivalent to $1/\overline{\mathcal{N}}(M_0)$ when λ tends to $+\infty$.

Remark 6 Assuming that M_0 is exponentially distributed with parameter γ, the probability $\widehat{p}_0^{ext}$ that a typical point is retained by Φ_{ext} is given by:

$$\widehat{p}_0^{ext} = \int_0^\infty \frac{1 - e^{-\overline{\mathcal{N}}(s)}}{\overline{\mathcal{N}}(s)} \gamma e^{-\gamma s} ds. \tag{6}$$

This follows directly by conditioning on the value of M_0:

$$\begin{aligned}\widehat{p}_0^{ext} &= \int_{\mathbb{R}} P(M^t(0) < M_0)h(t)dt = \int_{\mathbb{R}}\int_0^{\infty} P(M^t(0) < s)\gamma e^{-\gamma s} ds h(t)dt \\ &= \int_{\mathbb{R}}\int_0^{\infty} \frac{1-e^{-\overline{\mathcal{N}}(s)}}{\overline{\mathcal{N}}(s)}\gamma e^{-\gamma s} ds h(t)dt = \int_0^{\infty} \frac{1-e^{-\overline{\mathcal{N}}(s)}}{\overline{\mathcal{N}}(s)}\gamma e^{-\gamma s} ds.\end{aligned}$$

3.2 Additive Matérn Point Processes (AMPP)

Consider now Φ_{add} and let e_0 be the indicator function that the point at the origin is retained by this process under the Palm distribution of the underlying p.p. i.e.

$$e_0 = \mathbf{1}_{\left\{\sum_{X_j\in\widetilde{\Phi}} f(X_0,X_j)<S_0\right\}},$$

with $X_0 = 0$. Then, the Palm probability p_0^{add} that this point is retained by Φ_{add} is:

$$p_0^{add} = E^0(e_0) = \int_0^{+\infty} P^0(e_0 = 1|m_0 = t)h(t)dt = \int_0^{+\infty} P(S^t(0) < S_0)h(t)dt,$$

where $S^t(.)$ is the ASN defined in (4) and where we used the Slivnyak-Mecke theorem to get the last expression. The distribution of $S^t(0)$ is then needed. However, except for some particular cases that will be discussed later, there is no known explicit formula for this distribution. However, the Laplace transform can be calculated:

Proposition 2 *The Laplace transform of the additive shot noise $S^t(0)$ is:*

$$\mathcal{L}_{S^t(0)}(s) = \exp\{-H(t)\mathcal{K}(s)\} \quad \textit{where} \quad \mathcal{K}(s) = \lambda\int_{\mathbb{R}^d}(1-\mathcal{L}_G(sl(|x|)))\,dx, \tag{7}$$

with $\mathcal{L}_G(.)$ the Laplace transform of G.

Proof The result follows from the Laplace transform of a Poisson p.p.:

$$\begin{aligned}\mathcal{L}_{S^t(0)}(s) &= E\left(\exp\left\{-s\sum_{X_j\in\widetilde{\Phi}^t} F_j(0)l(|X_j|)\right\}\right) \\ &= \exp\left\{-\lambda H(t)\int_{\mathbb{R}^d}\left(1-\int_{\mathbb{R}^+} e^{-sul(|x|)}G(du)\right)dx\right\} \\ &= \exp\left\{-\lambda H(t)\int_{\mathbb{R}^d}(1-\mathcal{L}_G(sl(|x|)))\,dx\right\} = \exp\{-H(t)\mathcal{K}(s)\}.\end{aligned}$$

Prop. 3 below is proved in [1]. It allows one to compute p_0^{add} from $\mathcal{L}_{S^t(0)}$ in the case where the threshold S_0 is deterministic.

Proposition 3 *Under some simple conditions (see [1]), for all t, the random variable $S^t(0)$ admits a density and its distribution can be obtained from its Laplace transform by the following integral formula:*

$$P(a \leq S^t(0) \leq b) = \int_{\mathbb{R}} \mathcal{L}_{S^t(0)}(2i\pi s) \frac{e^{-2i\pi as} - e^{-2i\pi bs}}{2i\pi s} ds.$$

The last integral expression leads to closed forms in certain particular cases [3].

If the threshold S_0 is an exponential random variable, we can calculate the intensity of Φ_{add} directly from the Laplace transform:

Proposition 4 *Assume that S_0 is an exponentially distributed random variable with parameter γ, independent of everything else. Then the probability $\widehat{p}_0^{add}$ that a typical point is retained by Φ_{add} is:*

$$\widehat{p}_0^{add} = \frac{1 - e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)} \quad \textit{where} \quad \mathcal{K}(\gamma) = \lambda \int_{\mathbb{R}^d} (1 - \mathcal{L}_G(\gamma l(|x|)))\, dx. \tag{8}$$

Proof

$$\begin{aligned}
\widehat{p}_0^{add} &= \int_0^{+\infty} P^0(S^t(0) < S_0) h(t) dt \\
&= \int_0^{+\infty} \int_0^{+\infty} P(s < S_0) P_{S^t(0)}(ds) h(t) dt \\
&= \int_0^{+\infty} \int_0^{+\infty} e^{-\gamma s} P_{S^t(0)}(ds) h(t) dt \\
&= \int_0^{+\infty} \mathcal{L}_{S^t(0)}(\gamma) h(t) dt \\
&= \int_0^{+\infty} \exp\{-H(t)\mathcal{K}(\gamma)\}\, h(t) dt \\
&= \int_0^{1} \exp\{-u\mathcal{K}(\gamma)\}\, du = \frac{1 - e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)}.
\end{aligned}$$

Remark 7 Note that, as for the extremal case, the probability does not depend on the distribution H of the marks m.

Remark 8 As in the extremal case, it is not always the case that $\widehat{p}_0^{add} > 0$ (or equivalently that $\mathcal{K}(\gamma) < \infty$). Consider again the case $d = 2$ and $l(r) = r^{-\beta}$. The property $\mathcal{K}(\gamma) < \infty$ reads

$$\int_{\mathbb{R}^+} \left(1 - \mathcal{L}_G(\gamma r^{-\beta})\right) r dr = \frac{\gamma^{\frac{2}{\beta}}}{\beta} \int_{\mathbb{R}^+} (1 - \mathcal{L}_G(v))\, v^{-1-\frac{2}{\beta}} dv < \infty.$$

If G has exponential moments, i.e. if $\mathcal{L}_G(s) < \infty$ for some $s < 0$, then $\mathcal{L}_G(s)$ is analytic at $s = 0$, so that when $v \to 0$,

$$(1 - \mathcal{L}_G(v)) = E(F)v + o(v),$$

with $E(F) < \infty$. It is then easy to show that this implies that the last integral is finite.

If G is such that

$$1 - G(x) \sim_{x\to\infty} \frac{1}{\Gamma(1-a)} x^{-a} L(x), \tag{9}$$

with L slowly varying at infinity [1] and $a < 1$, then it follows from Tauberian theorems (see e.g. [2] p. 447) that

$$1 - \mathcal{L}_G(v) \sim_{v\to 0} v^a L\left(\frac{1}{v}\right). \tag{10}$$

Hence, under (9), a sufficient condition for $\mathcal{K}(\gamma) < \infty$ is that $a > \frac{2}{\beta}$.

Corollary 1 *In this random case, the probability $\widehat{p}_0^{add}$ is asymptotically equivalent to $1/\mathcal{K}(\gamma)$ when λ tends to $+\infty$.*

Remark 9 Note also that p_0^{ext} and $\widehat{p}_0^{add}$ have similar formulae. The term $\overline{\mathcal{N}}(M_0)$ is replaced by $\mathcal{K}(\gamma)$, where the difference is that the Laplace transform of the distribution G appears instead of the distribution itself.

3.3 Extremal–Additive Matérn Point Process (EAMPP)

Proposition 5 *Assume that the threshold S_0 is exponentially distributed with parameter γ, independent of everything else and that M_0 is a positive constant. Then the probability $p_0^{ext,add}$ that a typical point is retained by Φ_{ext}^{add} is:*

$$p_0^{ext,add} = \frac{1 - e^{-C(\gamma, M_0)}}{C(\gamma, M_0)}, \tag{11}$$

where

$$C(\gamma, M_0) = \lambda \int_{\mathbb{R}^d} \left(1 - E\left(e^{-\gamma F l(|x|)} \mathbf{1}_{\{Fl(|x|) \le M_0\}}\right)\right) dx, \tag{12}$$

with F a random variable with distribution G.

The proof of the proposition (given below) is based on the following result proved in [4].

Proposition 6 *For an extremal shot noise $M = M(0)$ and an additive shot noise $S = S(0)$ associated to the same realization of a Poisson process of intensity λ, let*

$$\mathcal{L}_S^{(u)}(t) = \mathrm{E}(e^{-tS} \mathbf{1}_{\{M \le u\}}).$$

Then:

$$\mathcal{L}_S^{(u)}(t) = \exp\left\{-\lambda \int_{\mathbb{R}^d} 1 - E\left(e^{-tFl(|x|)} \mathbf{1}_{\{Fl(|x|) \le u\}}\right) dx\right\}. \tag{13}$$

[1] L is said to be slowly varying at infinity if for all fixed x, $\frac{L(tx)}{L(t)} \to 1$ when $t \to \infty$.

Proof (of Proposition 5) Given that the mark of a typical point is $m_0 = t$, the indicator that it is retained by Φ_{ext}^{add} is given by:

$$e_0 = \mathbf{1}_{\{S^t(0) \leq S_0, M^t(0) \leq M_0\}}.$$

Then:

$$\begin{aligned} p_0^{ext,add} &= \int_0^{+\infty} P(e_0 = 1 | m_0 = t) h(t) dt \\ &= \int_0^{+\infty} \mathcal{L}_{S^t(0)}^{(M_0)}(\gamma) h(t) dt \\ &= \int_0^{+\infty} \exp\left\{ -\lambda H(t) \int_{\mathbb{R}^d} 1 - E\left(e^{-\gamma F l(|x|)} \mathbf{1}_{\{Fl(|x|) \leq M_0\}} \right) dx \right\} h(t) dt \\ &= \int_0^1 \exp\left\{ -u C(\gamma, M_0) \right\} du = \frac{1 - e^{-C(\gamma, M_0)}}{C(\gamma, M_0)}. \end{aligned}$$

Remark 10 As before, the probability does not depend on the distribution H of the marks m. Moreover, it is asymptotically equivalent to $1/C(\gamma, M_0)$ when λ tends to ∞.

Remark 11 The constant $C(\gamma, M_0)$ is closely related to $\overline{\mathcal{N}}(M_0)$ and $\mathcal{K}(\gamma)$. If M_0 tends to ∞, which means that the condition over the maximum is not considered, we obtain the result already proved for Φ_{add} since:

$$C(\gamma, M_0) \underset{M_0 \to \infty}{\to} \lambda \int_{\mathbb{R}^d} 1 - \mathrm{E}(e^{-\gamma F l(|x|)}) dx = \lambda \int_{\mathbb{R}^d} 1 - \mathcal{L}_G(\gamma F l(|x|)) dx = \mathcal{K}(\gamma).$$

Analogously, if γ tends to zero, which means that the condition over the sum is not considered, we obtain again the result for Φ_{ext}:

$$C(\gamma, M_0) \underset{\gamma \to 0}{\to} \lambda \int_{\mathbb{R}^d} 1 - \mathrm{E}(\mathbf{1}_{\{Fl(|x|) \leq M_0\}}) dx = \lambda \int_{\mathbb{R}^d} 1 - G(M_0 / l(|x|)) dx = \overline{\mathcal{N}}(M_0).$$

Remark 12 Here are some sufficient conditions for $C(\gamma, M_0)$ to be finite. It follows from association that for all x,

$$E\left(e^{-\gamma F l(|x|)} \mathbf{1}_{\{Fl(|x|) \leq M_0\}} \right) \geq E\left(e^{-\gamma F l(|x|)} \right) P\left(F l(|x|) \leq M_0 \right).$$

Hence, a sufficient condition for $C(\gamma, M_0)$ to be finite is that

$$\int_{\mathbb{R}^d} \left(1 - E\left(e^{-\gamma F l(|x|)} \right) P\left(F l(|x|) \leq M_0 \right) \right) dx < \infty.$$

In the particular case considered above, with $d = 2$ and $l(r) = r^{-\beta}$ with $\beta > 2$, the last integral is finite when

$$\int_{\mathbb{R}^+} \left(1 - \mathcal{L}_G(\gamma r^{-\beta}) G(M_0 r^{\beta}) \right) r dr = \int_{\mathbb{R}^+} \left(1 - \mathcal{L}_G(\gamma v) G(M_0 / v) \right) v^{-1-\frac{2}{\beta}} dv < \infty.$$

If G has exponential moments, then when $v \to 0$,

$$1 - \mathcal{L}_G(v) \sim E(F)v \quad \text{and} \quad 1 - G(1/v) \leq e^{-\kappa/v},$$

for some $\kappa > 0$. This implies that the last integral is finite.

If we assume that G is of the form (9), then, we deduce from (10) that a sufficient condition for the last integral to be finite is again that $a > \beta/2$.

4 A Particular Case

In this section we concentrate on a particular case for the dimension, the distribution of the marks and the function L. This particular choice has two purposes. On the one hand, it allows us to find closed form formulae for the intensity of Φ_{add} when the threshold S_0 is not a random variable. On the other hand, we can quantify the differences between the intensities of the introduced point processes. We will show in particular that, asymptotically, the difference only depends on the function $l(r)$.

Assume now that $d = 2$, the distribution G of the marks F_j^i is exponential with parameter μ and $l(r) = (Ar)^{-\beta}$ with $A > 0$ and $\beta > 2$. We have already seen that the results do not depend on the distribution H of the marks m.

4.1 Extremal Matérn Point Processes

We proved above that when the threshold M_0 is deterministic or exponentially distributed ((5) and (6)):

$$p_0^{ext} = \frac{1 - e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)},$$

$$\widehat{p}_0^{ext} = \int_0^\infty \frac{1 - e^{-\overline{\mathcal{N}}(s)}}{\overline{\mathcal{N}}(s)} \gamma e^{-\gamma s} ds.$$

In both cases, the result only depends on the function $\overline{\mathcal{N}}(.)$, that we calculate now for the particular case of this section:

Proposition 7 *For this particular case, we obtain that:*

$$\overline{\mathcal{N}}(s) = \frac{2\pi\lambda\Gamma(2/\beta)}{\beta A^2 (\mu s)^{2/\beta}}, \tag{14}$$

where $\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt$ *is the Gamma function.*

Proof The result follows from a direct calculation:

$$\overline{\mathcal{N}}(s) = 2\pi\lambda \int_0^\infty e^{-\mu s (Ar)^\beta} r dr = \frac{2\pi\lambda}{\beta} \int_0^\infty e^{-\mu s A^\beta t} t^{2/\beta - 1} dt = \frac{2\pi\lambda}{\beta} \frac{\Gamma(2/\beta)}{(\mu s A^\beta)^{2/\beta}}.$$

Corollary 2 *In this case, we obtain the following asymptotic results for the intensities of Φ_{ext} with deterministic and random threshold respectively:*

$$\lim_{\lambda\to\infty} \lambda p_0^{ext} = (\mu M_0)^{2/\beta} \frac{A^2}{\pi} \frac{1}{\Gamma(1+2/\beta)} \tag{15}$$

$$\begin{aligned}\lim_{\lambda\to\infty} \lambda \widehat{p}_0^{ext} &= \frac{\beta A^2}{2\pi\Gamma(2/\beta)} \mu^{2/\beta} \int_0^\infty \gamma e^{-\gamma s} s^{2/\beta} ds \\ &= \frac{\beta A^2}{2\pi\Gamma(2/\beta)} \mu^{2/\beta} \left(\frac{1}{\gamma}\right)^{2/\beta} \Gamma(1+2/\beta) \\ &= \left(\frac{\mu}{\gamma}\right)^{2/\beta} \frac{A^2}{\pi}. \end{aligned} \tag{16}$$

The last equality follows from the property of the Gamma function: $\Gamma(1+z) = z\Gamma(z)$.

Remark 13 We compare now the intensity of Φ_{ext} for the random and deterministic threshold when λ goes to infinity. The ratio depends only on the decay exponent β of the function $l(r)$:

$$\lim_{\lambda\to\infty} \frac{\lambda \widehat{p}_0^{ext}}{\lambda p_0^{ext}} = \Gamma(1+2/\beta). \tag{17}$$

See Sec.4.3 for more details in this relation.

4.2 Additive Matérn Point Processes

When the threshold S_0 is an exponentially distributed random variable, the intensity of Φ_{add} is given by (8):

$$\widehat{p}_0^{add} = \frac{1-e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)}.$$

Proposition 8 *For the particular case of this section, we obtain that:*

$$\mathcal{K}(\gamma) = \frac{2\pi\lambda\Gamma(2/\beta)\Gamma(1-2/\beta)}{\beta A^2} \left(\frac{\gamma}{\mu}\right)^{2/\beta}. \tag{18}$$

Proof The result follows from a direct calculation:

$$\begin{aligned}\mathcal{K}(\gamma) &= 2\pi\lambda \int_0^\infty 1 - \mathcal{L}(\gamma l(r)) r dr = 2\pi\lambda \int_0^\infty \frac{1}{1+\frac{\mu}{\gamma l(r)}} r dr \\ &= \frac{2\pi\lambda}{\beta} \int_0^\infty \frac{u^{2/\beta-1}}{1+\frac{\mu A^\beta}{\gamma} u} du = \frac{2\pi\lambda}{\beta A^2} \left(\frac{\gamma}{\mu}\right)^{2/\beta} \int_0^\infty \frac{v^{2/\beta-1}}{1+v} dv \\ &= \frac{2\pi\lambda}{\beta A^2} \left(\frac{\gamma}{\mu}\right)^{2/\beta} \frac{\pi}{\sin\frac{2}{\beta}\pi} = \frac{2\pi\lambda}{\beta A^2} \left(\frac{\gamma}{\mu}\right)^{2/\beta} \Gamma(2/\beta)\Gamma(1-2/\beta).\end{aligned}$$

Corollary 3 *In this case, we obtain the following asymptotic for the intensity of Φ_{add}:*

$$\lim_{\lambda\to\infty} \lambda \widehat{p}_0^{add} = \left(\frac{\mu}{\gamma}\right)^{2/\beta} \frac{A^2}{\pi} \frac{1}{\Gamma(1+2/\beta)\Gamma(1-2/\beta)}. \tag{19}$$

We have already analyzed Φ_{add} with deterministic threshold, but we did not find an explicit formula for the distribution of $S^t(0)$. However for the particular case of this section and considering $\beta = 4$, there is a closed form formula for the distribution of $S^t(0)$:

Proposition 9 *If $d = 2$, the distribution G of the marks F_j^i is exponential with parameter μ and $l(r) = (Ar)^{-\beta}$ with $\beta = 4$, then the distribution of $S^t(0)$ is:*

$$F_{S^t(0)}(s) = 1 - \mathrm{erf}\left(\frac{\lambda H(t)\pi^2}{4A^2}\sqrt{\frac{1}{s\mu}}\right), \tag{20}$$

where erf *is the error function:* $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2} ds$.

Proof From (7) and (18) we have, for $\beta = 4$:

$$\mathcal{L}_{S^t(0)}(s) = \exp\{-H(t)\mathcal{K}(s)\} = \exp\left\{-\sqrt{2c}s^{1/2}\right\}$$
$$\text{where} \quad \sqrt{2c} = \frac{\lambda H(t)\pi^2}{2A^2}\left(\frac{1}{\mu}\right)^{1/2}.$$

Then, $S^t(0)$ follows a Lévy distribution with parameter c, and the cdf is $F_{S^t(0)}(s) = 1 - \mathrm{erf}\left(\sqrt{\frac{c}{2s}}\right)$. This gives (20).

From the previous result we can calculate the intensity of Φ_{add} with a deterministic threshold in this particular case:

Proposition 10 *Under the conditions of the previous proposition, we have:*

$$p_0^{add} = \mathrm{erfc}(a) + \frac{1 - e^{-a^2}}{\sqrt{\pi}a}, \tag{21}$$

where $a = \frac{\lambda\pi^2}{4A^2}\sqrt{\frac{1}{S_0\mu}}$ *and* $\mathrm{erfc}(a) = 1 - \mathrm{erf}(a)$.

Proof By conditioning on the mark m of the typical point and by (20), it results that:

$$\begin{aligned} p_0^{add} &= \int_0^\infty P(S^t(0) \le S_0) h(t) dt \\ &= 1 - \int_0^\infty \mathrm{erf}(aF(t)) h(t) dt \\ &= 1 - \int_0^1 \mathrm{erf}(as) ds \\ &= 1 - \int_0^1 \frac{2}{\sqrt{\pi}} \int_0^a e^{-u^2 s^2} s du ds \\ &= 1 - \frac{2}{\sqrt{\pi}} \int_0^a \frac{1 - e^{-u^2}}{2u^2} du \\ &= 1 - \frac{1}{\sqrt{\pi}} \left(\frac{-1 + e^{-a^2}}{a} + \sqrt{\pi}\mathrm{erf} a \right) = \mathrm{erfc}(a) + \frac{1 - e^{-a^2}}{\sqrt{\pi} a}. \end{aligned}$$

Corollary 4 *The intensity of Φ_{add} in this case is asymptotically equivalent to* $\frac{4A^2}{\pi^2}\sqrt{\frac{S_0\mu}{\pi}}$.

Interestingly, we find the same relation between the intensity of Φ_{add} under the random or deterministic threshold assumption as for Φ_{ext} (17):

Remark 14 Using (21) with $S_0 = 1/\gamma$ and (18) for $\beta = 4$ we obtain that:

$$\lim_{\lambda \to \infty} \frac{\lambda \widehat{p}_0^{add}}{\lambda p_0^{add}} = \frac{\sqrt{\pi}}{2} = \Gamma(1 + 1/2). \quad (22)$$

Note that $\Gamma(1 + 1/2) = \Gamma(1 + 2/\beta)$. We may then conjecture that for all $\beta > 2$:

$$\lim_{\lambda \to \infty} \frac{\lambda \widehat{p}_0^{add}}{\lambda p_0^{add}} = \Gamma(1 + 2/\beta). \quad (23)$$

If this conjecture is valid, the intensity of Φ_{add} when λ goes to infinity, assuming that $S_0 = 1/\gamma$, can be deduced from the random case:

$$\begin{aligned} \lim_{\lambda \to \infty} \lambda p_0^{add} &= \frac{1}{\Gamma(1 + 2/\beta)} \lim_{\lambda \to \infty} \lambda \widehat{p}_0^{add} \\ &= \frac{1}{\Gamma(1 + 2/\beta)} \left(\frac{\mu S_0}{P} \right)^{2/\beta} \frac{A^2}{\pi} \frac{1}{\Gamma(1 + 2/\beta)\Gamma(1 - 2/\beta)}. \end{aligned} \quad (24)$$

Note that for $\beta = 4$ we find again the result obtained in (21). This conjecture, suggested by the result proved for the EMPP, is validated by simulations (see Fig.1), where the results for $\beta \neq 4$ are calculated following (24).

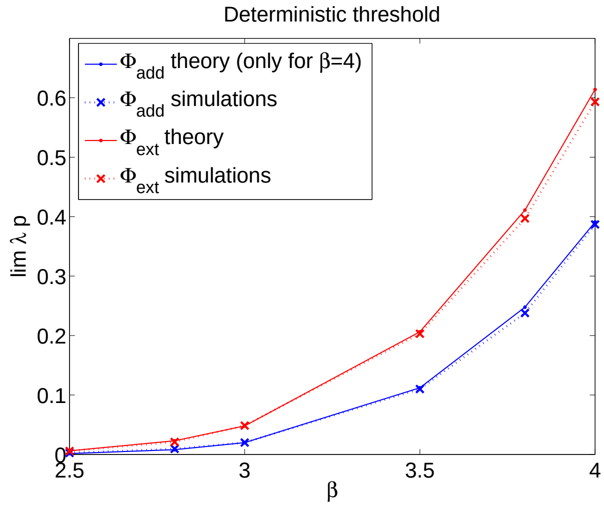

Fig. 1 Validation by simulations of the asymptotic intensity of Φ_{ext} and Φ_{add} with deterministic threshold. In particular, validation of the conjecture made in remark 14.

4.3 Comparison of EMPP and AMPP

This section is devoted to the comparison of the asymptotic intensities of Φ_{ext} and Φ_{add} with random and deterministic threshold. In Fig.**??** and Fig.**??** we plot the intensity of Φ_{ext} and Φ_{add} with random and deterministic threshold respectively, for different values of λ and β.

Proposition 11 *If S_0 is an exponentially distributed random variable with parameter γ and $M_0 = 1/\gamma$, there exists a simple relation, that depends only on the decay exponent of the function $l(r)$, between the intensities of the processes Φ_{ext} and Φ_{add} for large values of λ:*

$$\lim_{\lambda\to\infty} \frac{\lambda \widehat{p}_0^{add}}{\lambda p_0^{ext}} = \frac{1}{\Gamma(1-2/\beta)} < 1.$$

Proof *The result follows directly from the equality: $\mathcal{K}(\gamma) = \overline{\mathcal{N}}(1/\gamma)\Gamma(1-2/\beta)$.*

Some comments are in order. First of all, we should note that the ratio is always smaller than one ($\beta > 2$). This is not surprising, since for similar thresholds ($M_0 = 1/\gamma$), the condition over the sum is more restrictive than that over the maximum. The previous results give a quantification of how much smaller it is when the intensity of the original process is large enough. Second, since $\Gamma(1-2/\beta)$ is a decreasing function of β, we find that the intensities are similar if β is large enough. Once again, this is reasonable since for large values of β, the impact of distant points is almost negligible and the conditions over the sum and over the maximum are almost equivalent.

However, this compares the result obtained for Φ_{ext} with a deterministic threshold M_0, with that obtained for Φ_{add} with an exponentially distributed threshold. Let us now consider the other cases.

Proposition 12 *Assume that M_0 and S_0 are exponentially distributed random variables with parameters γ' and γ respectively. The relation between the intensities of Φ_{ext} and Φ_{add} when λ tends to infinity is:*

$$\lim_{\lambda\to\infty} \frac{\lambda \widehat{p}_0^{add}}{\lambda \widehat{p}_0^{ext}} = \frac{1}{\Gamma(1+2/\beta)\Gamma(1-2/\beta)} \left(\frac{\gamma'}{\gamma}\right)^{2/\beta}. \tag{25}$$

Proof It follows directly from (19) and (16).

Corollary 5

1. *From the previous result is possible to find the relation between the parameters γ' and γ that provides the same asymptotic intensities for Φ_{ext} and Φ_{add}:*

   $$\gamma' = \gamma \left(\Gamma(1+2/\beta)\Gamma(1-2/\beta)\right)^{\beta/2}.$$

   Clearly, the threshold for the extremal case should be smaller than for the additive case.
2. *If both parameter coincides, i.e. $\gamma' = \gamma$, the relation between both intensities is simply a constant that depends only on the decay exponent β:*

   $$\lim_{\lambda\to\infty} \frac{\lambda \widehat{p}_0^{add}}{\lambda \widehat{p}_0^{ext}} = \frac{1}{\Gamma(1+2/\beta)\Gamma(1-2/\beta)}.$$

   Note that when β tends to ∞, the right hand side function tends to 1, which means that the difference between the asymptotic intensities of both processes disappears.

Proposition 13 *When the thresholds are deterministic result only for and $\beta = 4$, we have*

$$\lim_{\lambda\to\infty} \frac{\lambda p_0^{add}}{\lambda p_0^{ext}} = \frac{2}{\pi}\left(\frac{S_0}{M_0}\right)^{1/2} = \frac{1}{\Gamma(1+2/\beta)\Gamma(1-2/\beta)} \left(\frac{S_0}{M_0}\right)^{2/\beta}.$$

Proof The result follows directly from (15) and (24) with $\beta = 4$.

Remark 15 As before we may conjecture that the relation is valid for all values of β which is validated by the simulations shown in Fig.1 and reinforced by the result proved for Φ_{ext}.

Corollary 6 *The same relation is also valid for the thresholds S_0 and M_0 in order to obtain the same asymptotic intensity of Φ_{add} and Φ_{ext}:*

$$S_0 = M_0 \left(\Gamma(1+2/\beta)\Gamma(1-2/\beta)\right)^{\beta/2}.$$

and for the same threshold, i.e. $M_0 = S_0$ we find again the same relation as for Φ_{ext} depending only on β:

$$\lim_{\lambda\to\infty} \frac{\lambda p_0^{add}}{\lambda p_0^{ext}} = \frac{1}{\Gamma(1+2/\beta)\Gamma(1-2/\beta)}.$$

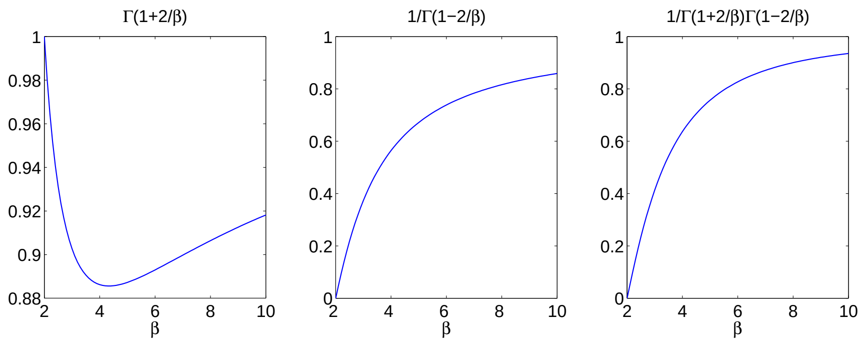

Fig. 2 Functions defining the relations given in equations (26)-(28)

4.4 Summary of the results

We have proved the following relations for the asymptotic intensities of Φ_{ext} and Φ_{add} in the particular case of the previous section:

$$\lim_{\lambda\to\infty}\frac{\lambda\widehat{p}_0^{\,add}}{\lambda p_0^{\,add}}=\lim_{\lambda\to\infty}\frac{\lambda\widehat{p}_0^{\,ext}}{\lambda p_0^{\,ext}}=\Gamma(1+2/\beta), \tag{26}$$

$$\lim_{\lambda\to\infty}\frac{\lambda\widehat{p}_0^{\,add}}{\lambda p_0^{\,ext}}=\frac{1}{\Gamma(1-2/\beta)}, \tag{27}$$

$$\lim_{\lambda\to\infty}\frac{\lambda\widehat{p}_0^{\,add}}{\lambda\widehat{p}_0^{\,ext}}=\lim_{\lambda\to\infty}\frac{\lambda p_0^{\,add}}{\lambda p_0^{\,ext}}=\frac{1}{\Gamma(1-2/\beta)\Gamma(1+2/\beta)}. \tag{28}$$

In Fig. 2, we plot the functions defined on the right hand side of these equations. First, as one can see on the left most figure, which gives the relation between the random and the deterministic case for both processes, the minimum is attained at $\beta = 4$, where the difference is of 12%. This means that the difference between a random and a deterministic threshold is only marginal in most cases.

As we mentioned earlier it can be observed that:

$$\lim_{\beta\to\infty}\frac{1}{\Gamma(1-2/\beta)}=\lim_{\beta\to\infty}\frac{1}{\Gamma(1-2/\beta)\Gamma(1+2/\beta)}=1. \tag{29}$$

On the one hand, this means that for both random and deterministic cases, the intensities of Φ_{ext} and Φ_{add} tend to be equal as β increases (see (28)). On the other hand, the same conclusion is valid when the intensity of Φ_{add} for the random case is compared with the one of Φ_{ext} for the deterministic case (27). However, as we can observe in the right part of Fig. 2, the limit is approached for very large values of β. Moreover the values obtained for β close to 2 are very small which implies that the corresponding differences on the intensities are very large. For instance, if we concentrate on the right most figure we have that for $\beta = 2.1$ and $\beta = 4$ the obtained values are 0.05 and 0.64 respectively. This means that the asymptotic intensity obtained for the AMPP will be 5% and 64% of that of the EMPP respectively. In the context of our practical motivations, β is meaningful for values less than 4. Similar conclusions can be obtained for the relation defined in (27).

5 Some Extensions

A possible way to extend the previous results is to consider the conditional probability p_r, under E^0, that the point at the origin is retained by Φ_{ext} or Φ_{add} given that there is another point of Φ at distance r. The result for Φ_{ext} that we present below is established in [1]. We sketch the proof for the making the extension to the additive case that follows more transparent.

Proposition 14 *The conditional probability p_r^{ext} for Φ_{ext} is:*

$$p_r^{ext} = p_0^{ext} + (G(M_0/l(r)) - 1)\left(\frac{1-e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)^2} - \frac{e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)}\right).$$

Proof Let us define $E_r = \{\exists X_i \in \Phi \text{ s.t. } ||X_i|| = r\}$, then:

$$p_r^{ext} = P^0(0 \in \Phi_{ext}|E_r) = \int_0^\infty P_t^0(0 \in \Phi_{ext}|E_r)h(t)dt \quad \text{where} \quad P_t(\cdot) = P(\cdot|m_0 = t)$$

$$\begin{aligned} P_t^0(0 \in \Phi_{ext}|E_r) &= P_t^0(0 \in \Phi_{ext}|E_r, m_i < t)F(t) + P_t^0(0 \in \Phi_{ext})(1-F(t)) \\ &= P_t^0(0 \in \Phi_{ext}|E_r, m_i < t)F(t) + e^{-F(t)\overline{\mathcal{N}}(M_0)}(1-F(t)) \end{aligned}$$

If $f(X_i, 0) > M_0$, the probability of being retained is zero, then:

$$\begin{aligned} P_t^0(0 \in \Phi_{ext}|E_r, m_i < t) &= P\left(\max_{X_j \in \widetilde{\Phi}^t} f(X_j, 0) < M_0 | f(X_i, 0) < M_0\right) P(f(X_i, 0) < M_0) \\ &= P\left(\max_{X_j \in \widetilde{\Phi}^t \setminus \{X_i\}} f(X_j, 0) < M_0\right) P(Fl(r) < M_0) \\ &= \exp\left\{-F(t)\overline{\mathcal{N}}(M_0)\right\} G(M_0/l(r)) \end{aligned}$$

In the last equation we used the second order statistics of the Poisson process. Using the previous results we obtain that p_r^{ext} is:

$$\begin{aligned} p_r^{ext} &= \int_0^{+\infty} [e^{-F(t)\overline{\mathcal{N}}(M_0)} G(M_0/l(r))F(t) + e^{-F(t)\overline{\mathcal{N}}(M_0)}(1-F(t))]h(t)dt \\ &= \int_0^1 e^{-s\overline{\mathcal{N}}(M_0)} G(M_0/l(r))s + e^{-s\overline{\mathcal{N}}(M_0)}(1-s)ds \\ &= \int_0^1 e^{-s\overline{\mathcal{N}}(M_0)} ds + (G(M_0/l(r)) - 1)\int_0^1 e^{-s\overline{\mathcal{N}}(M_0)} sds \\ &= \frac{1-e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)} + (G(M_0/l(r)) - 1)\left(\frac{1-e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)^2} - \frac{e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)}\right) \\ &= p_0^{ext} + (G(M_0/l(r)) - 1)\left(\frac{1-e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)^2} - \frac{e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)}\right), \end{aligned}$$

which concludes the proof.

For Φ_{add}, we have:

Proposition 15 *Assume that S_0 is exponentially distributed with parameter γ. Then, the conditional probability p_r^{add} calculated for Φ_{add} is:*

$$\widehat{p}_r^{add} = \widehat{p}_0^{add} + (\mathcal{L}_G(\gamma l(r)) - 1)\left(\frac{1-e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)^2} - \frac{e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)}\right).$$

Note that, in this case also, the distribution of the marks G is replaced by its Laplace transform.

Proof The proof is equivalent to the previous one using the following particular results for Φ_{add}:

$$P(f(X_i,0) < S_0) = \mathcal{L}_G(\gamma l(r)) \quad \text{and}$$

$$P\left(\sum_{X_j \in \widetilde{\Phi}^t \setminus \{X_i\}} f(X_j,0) < S_0\right) = P^t(0 \in \Phi_{ext}) = e^{-H(t)\mathcal{K}(\gamma)}.$$

Corollary 7 *If G is exponentially distributed with parameter μ and $l(r) = (Ar)^{-\beta}$ with $\beta > 2$, then*

$$p_r^{ext} = \frac{1-e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)} - e^{-\mu M_0/l(r)}\left(\frac{1-e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)^2} - \frac{e^{-\overline{\mathcal{N}}(M_0)}}{\overline{\mathcal{N}}(M_0)}\right),$$

$$\widehat{p}_r^{add} = \frac{1-e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)} - \frac{\gamma l(r)}{\mu + \gamma l(r)}\left(\frac{1-e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)^2} - \frac{e^{-\mathcal{K}(\gamma)}}{\mathcal{K}(\gamma)}\right),$$

where $\overline{\mathcal{N}}(M_0)$ y $\mathcal{K}(\gamma)$ are given in (14) *and* (18)*.*

In a similar way we can calculate p_r^{add} when S_0 is deterministic for the specific case analyzed above.

Proposition 16 *In the particular case of Sec.4, when S_0 is deterministic, the conditional probability p_r^{add} for Φ_{add} is*

$$\begin{aligned} p_r^{add} &= p_0^{add} + (G(S_0/l(r))) \int_0^1 (1-\mathrm{erf}(as))s ds \\ &= p_0^{add} + e^{-\mu S_0/(Ar)^{-\beta}} \int_0^1 (1-\mathrm{erf}(as))s ds \end{aligned}$$

where $a = \frac{\lambda\pi^2}{4A^2}\sqrt{\frac{1}{S_0\mu}}$.

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