Fermionic asymptotic symmetries in massless QED

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We consider soft electrons in massless QED at tree-level. The emission amplitude at leading order in the soft electron energy factorizes in a way similar to the soft photon case. We recast the soft electron factorization formula as a Ward identity of an asymptotic charge. This leads to the first example of an asymptotic fermionic symmetry in a theory with no conventional supersymmetry, suggesting that tree-level massless QED may posses an asymptotic supersymmetry algebra. Although our approach does not yet allow us to completely characterize the algebra, it suggests that subleading soft photons should feature in the anticommutator of two fermionic symmetry generators.

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I. INTRODUCTION

Asymptotic states of massless particles have their natural home at null infinity [1]. At this boundary of spacetime it is possible to unveil symmetries that are otherwise obscure from a bulk perspective [2, 3]. As first shown by Strominger and collaborators [4–6], these asymptotic symmetries manifest themselves in scattering amplitudes via so-called soft theorems. Conversely, soft theorems in scattering amplitudes can often be interpreted as arising from asymptotic symmetries, see e.g. [7–11]. In this article we present what may be the simplest example of an asymptotic *fermionic* symmetry. We consider tree-level massless Quantum Electrodynamics (QED), in which there is a notion of soft electrons, i.e. electrons with vanishingly small energy.[1] The emission amplitude for such soft electrons is given to leading order by a soft electron theorem. We will recast such theorem as a Ward identity of fermionic asymptotic charges, thus suggesting the existence of asymptotic fermionic symmetries. At this stage, however, we are unable to fully characterize the underlying symmetry algebra. In particular, we lack a trustworthy evaluation of the commutator between fermionic symmetries. We will describe what the obstacles are and discuss possible strategies to overcome them.

Fermionic asymptotic symmetries have already been discussed in other contexts. They naturally arise in supergravity theories [12–16] where they are associated to soft gravitinos [9, 10]. In [8], Dumitrescu, He, Mitra and Strominger identified asymptotic fermionic charges in supersymmetric abelian gauge theories. Our work follows closely their analysis, with the electron field here playing the role of photino field there. A comprehensive discussion of (conformally) soft fermions can be found in [17].

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[1] Alternatively, we are studying massive electrons in a regime where the soft energy E_{soft} is much larger than the electron mass yet much smaller than all other energies involved in the process: $m_e \ll E_{\mathrm{soft}} \ll E_{\mathrm{hard}}$.

The organization of the paper is as follows. In the next section we introduce notation and review basic concepts on radiative phase spaces and asymptotic symmetries, in the context of massless QED. In section III we present a soft electron theorem and interpret it as a Ward identify of an asymptotic fermionic charge. In section IV we discuss various aspects of the associated fermionic symmetries, including the commutation relations with bosonic symmetries. We conclude in section V, where we highlight the open questions left for future work. Additional material is given in three appendices: In appendix A we review the relationship between the momentum space and null infinity descriptions of fields. In appendix B we review the soft photon theorem and derive the analogous soft electron theorem. In appendix C we present a preliminary exploration on the non-linear structure of the fermionic symmetry, and observe how a naive evaluation of the commutator of two fermionic symmetries displays similarities with the asymptotic symmetry associated to subleading soft photons.

II.   PRELIMINARIES

A.   Conventions

The elementary field variables for QED are the $U(1)$ gauge field $\mathcal{A}_\mu$ and the anticommuting Dirac spinor Ψ. In the massless case the lagrangian density is[2]

$$\mathcal{L} = \sqrt{-\eta}\left(-\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + i\overline{\Psi}\gamma^\mu\mathcal{D}_\mu\Psi\right), \tag{2.1}$$

where $\sqrt{-\eta}$ is the Minkowski volume element, $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$ is the field strength, γ^μ Dirac matrices, $\overline{\Psi} = \Psi^\dagger\gamma^0$ and

$$\mathcal{D}_\nu = \partial_\mu + ie\mathcal{A}_\mu \tag{2.2}$$

the gauge covariant derivative. The electric current is defined as

$$\mathcal{J}^\mu = e\overline{\Psi}\gamma^\mu\Psi. \tag{2.3}$$

Taking variations of the lagrangian density, one finds

$$\delta\mathcal{L} = eom + \partial_\mu\theta^\mu(\delta) \tag{2.4}$$

where

$$eom = (\partial_\mu\mathcal{F}^{\mu\nu} - \mathcal{J}^\nu)\delta\mathcal{A}_\nu + (i\delta\overline{\Psi}\gamma^\mu\mathcal{D}_\mu\Psi + c.c.) \tag{2.5}$$

yield the field equations and

$$\theta^\mu(\delta) = \sqrt{-\eta}(-\mathcal{F}^{\mu\nu}\delta\mathcal{A}_\nu + i\overline{\Psi}\gamma^\mu\delta\Psi) \tag{2.6}$$

is the symplectic potential current.

Besides Poincare (and in fact Conformal) symmetries, the theory is invariant under local gauge transformations

$$\delta_\Lambda\mathcal{A}_\mu = \partial_\mu\Lambda, \quad \delta_\Lambda\Psi = -ie\Lambda\Psi, \tag{2.7}$$

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[2] We follow conventions from [18] modulo a sign in the definition of the coupling constant e.

as well as global axial rotations

$$\delta_A \mathcal{A}_\mu = 0, \quad \delta_A \Psi = i\gamma^5 \Psi. \tag{2.8}$$

The latter famously displays a 1-loop anomaly [19], but for the purposes of our tree-level discussion, we will regard (2.8) as an exact symmetry.

B. Asymptotic fields at null infinity

1. Outgoing null coordinates

In order to describe the fields near (future) null infinity, it is convenient to work in outgoing null coordinates. These can be defined as follows. First, assign a future null direction q^μ to every point $x \equiv (z, \bar{z})$ on the celestial sphere,

$$q^\mu(x) := \frac{1}{\sqrt{2}} \left(1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2 \right), \tag{2.9}$$

The specific choice (2.9) leads to a flat conformal frame on the celestial sphere, see e.g. [20] for a discussion on other possible frames. Next, choose a reference null vector k^μ transverse to q^μ that specifies the "flow of time",

$$k^\mu = \frac{1}{\sqrt{2}} \left(1, 0, 0, -1 \right), \quad k^\mu q_\mu(x) = -1. \tag{2.10}$$

Finally, parametrize cartesian coordinates X^μ by (r, u, x) according to

$$X^\mu(r, u, x) = rq^\mu(x) + uk^\mu, \tag{2.11}$$

in terms of which the spacetime metric takes the form

$$dX^\mu dX_\mu = -2dudr + 2r^2 dz d\bar{z}. \tag{2.12}$$

Future null infinity $\mathcal{I}$ is reached by taking $r \to \infty$ with constant (u, x). One can similarly define retarded null coordinates that are adapted to past null infinity. We will however focus our discussion on fields at future null infinity, with the understanding that a parallel construction is available at past null infinity.

2. Fall-offs in r

Near null infinity the electromagnetic field is described by the leading transversal components [2, 3],

$$\mathcal{A}_z(r, u, x) \stackrel{r \to \infty}{=} A_z(u, x) + \cdots, \tag{2.13}$$

where A_z is regarded as a gauge field on $\mathcal{I}$. The massless Dirac field fall-offs have been discussed in [8]. As for other massless fields, it decays as the inverse power of the radial

coordinate.[3] The Dirac equation imposes restrictions on the leading spinor components, leaving only two independent asymptotic fields:[4]

$$\Psi \stackrel{r \to \infty}{=} \frac{1}{2^{1/4} r} \begin{pmatrix} -\bar{z}\psi_- \\ \psi_- \\ \psi_+ \\ z\psi_+ \end{pmatrix} + \cdots , \tag{2.14}$$

where the overall normalization is chosen for later convenience and $\psi_\pm$ are regarded as complex fermionic fields on $\mathcal{I}$. Upon quantization, they describe electrons of positive/negative helicity. Similarly, $A_z/A_{\bar{z}}$ describe photons of positive/negative helicity. We shall use the notation

$$A_+ := A_z, \quad A_- := A_{\bar{z}}. \tag{2.15}$$

Notice that, unlike $\psi_\pm$, the gauge field satisfies the reality condition $A_+^* = A_-$.

C.   Radiative phase space

Given the fall-offs from the previous section, we can evaluate the symplectic potential current (2.6) at null infinity. For the relevant component $\mu = r$ one finds

$$\lim_{r \to \infty} \theta^r(\delta) = \dot{A}_z \delta A_{\bar{z}} + \dot{A}_{\bar{z}} \delta A_z + i\bar{\psi}_+ \delta\psi_+ + i\bar{\psi}_- \delta\psi_- \tag{2.16}$$

where $\dot{A}_z \equiv \partial_u A_z$ and $\bar{\psi}_\pm$ is the complex conjugate of $\psi_\pm$. Taking a second variation in (2.16) and integrating over $(u, z, \bar{z})$ we obtain the symplectic structure at $\mathcal{I}$ [22],

$$\Omega := \sum_{s=\pm} \int_{\mathcal{I}} du d^2x \left(\delta\dot{A}_{-s} \wedge \delta A_s + i\delta\bar{\psi}_s \wedge \delta\psi_s \right), \tag{2.17}$$

where we used the notation (2.15) for the asymptotic gauge field. Evaluating (2.17) on two variations δ_1 and δ_2 one has

$$\Omega(\delta_1, \delta_2) = \sum_s \int_{\mathcal{I}} \left(\delta_1 \dot{A}_{-s} \delta_2 A_s + i\delta_1\bar{\psi}_s \delta_2\psi_s \right) - (\delta_1 \leftrightarrow \delta_2). \tag{2.18}$$

In the above expressions it is important that $\psi_\pm$ are regarded as anticommuting fields. In particular, the reality of the symplectic form follows from the property $\overline{\psi_1 \psi_2} = \bar{\psi}_2 \bar{\psi}_1$.

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[3] We are assuming free-field fall-offs, which suffice for the tree-level considerations of this work. Loop corrections may imply slower fall-offs, see e.g. [21].

[4] Further details are given in the discussion following Eq. (C9).

1. Fall-offs in u

In order for (2.17) to be well-defined, we must impose fall-offs conditions on the fields as $|u| \to \infty$. We shall henceforth assume them to be

$$A_s(u, x) \overset{|u| \to \infty}{=} O(1) + O(1/|u|^\epsilon) \tag{2.19}$$

$$\psi_s(u, x) \overset{|u| \to \infty}{=} O(1/|u|^{1+\epsilon}) \tag{2.20}$$

for some $\epsilon > 0$. Condition (2.19) is slightly more general than the typical scattering fall-off, which corresponds to $\epsilon = 1$ [23]. Condition (2.20) is stronger than a minimal requirement for convergence of (2.17) (for which it would be enough a fall-off faster than $1/|u|^{1/2}$). We require (2.20) to ensure finiteness of the asymptotic fermionic charge defined in section III.

2. Radiative PBs

From the symplectic structure (2.17) one can obtain the elementary non-trivial Poisson brackets (PBs)[5]

$$\{A_s(u, x), \dot{A}_{-s}(u', x')\} = \frac{1}{2}\delta(u - u')\delta^{(2)}(x, x'),$$
$$\{\psi_s(u, x), \bar{\psi}_s(u', x')\} = -i\delta(u - u')\delta^{(2)}(x, x'). \tag{2.21}$$

We recall however a well known subtlety with the gauge field PBs [6], which is the occurrence of a $1/2$ discontinuity that can be expressed as

$$\int_{-\infty}^{\infty} du\{\cdot, \dot{A}_s(u, x)\} = \frac{1}{2}\{\cdot, \int_{-\infty}^{\infty} du\dot{A}_s(u, x)\}, \tag{2.22}$$

where $\{\cdot, F\} \equiv X_F$ is the Hamiltonian vector field of a functional F (see footnote 5). A fix to this problem was proposed in [6] via the isolation of the zero mode component of the gauge field and the introduction of boundary terms in the symplectic structure. For simplicity we will continue to work with the standard radiative phase space symplectic structure while keeping care when needed of the aforementioned subtlety.

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[5] Our conventions are as follows. The PBs between two functions F and G is given by $\{F, G\} := X_G(F) = -X_F(G) = \Omega(X_G, X_F)$ where X_F is defined by the condition $\Omega(\delta, X_F) = \delta F$ (and similarly for X_G). If F and G are fermionic, there are additional signs that can be determined by requiring the grasmannian Leibinitz rule on PBs, namely $\{F_1 F_2, G\} = F_1\{F_2, G\} \pm \{F_1, G\}F_2$ where the minus occurs if both F_2 and G are fermionic. In particular the PBs between two fermionic functions is symmetric rather than antisymmetric. We refer to chapter 6 of [24] for further details on fermionic PBs.

D.   Asymptotic Fock space

To obtain the asymptotic Fock space of photons and massless electrons, we start by considering the Fourier transform of the fields with respect to the u-variable,

$$A_s(u,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{A}_s(\omega,x) e^{-i\omega u},$$

$$\psi_s(u,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\psi}_s(\omega,x) e^{-i\omega u}, \tag{2.23}$$

$$\bar{\psi}_s(u,x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\bar{\psi}}_s(\omega,x) e^{i\omega u}.$$

The brackets (2.21) imply

$$\{\tilde{A}_s(\omega,x), \tilde{A}_{-s}(\omega',x')\} = \frac{i\pi}{\omega'} \delta(\omega+\omega') \delta^{(2)}(x,x'),$$

$$\{\tilde{\psi}_s(\omega,x), \tilde{\bar{\psi}}_s(\omega',x')\} = -i2\pi \delta(\omega-\omega') \delta^{(2)}(x,x'). \tag{2.24}$$

Notice that the information of $A_s(u,x)$ is contained in $\tilde{A}_s(\omega,x), \omega > 0$ since $\tilde{A}_s(-\omega,x) = (\tilde{A}_{-s}(\omega,x))^*$ due to the reality of the gauge field. On the contrary, there is no relation between the positive and negative frequency components of $\tilde{\psi}_s(\omega,x)$. An independent set of mode functions is then given by:

$$a_s(\omega,x) := \tilde{A}_s(\omega,x), \quad \omega > 0$$

$$b_s(\omega,x) := \tilde{\psi}_s(\omega,x), \quad \omega > 0 \tag{2.25}$$

$$c_s(\omega,x) := \tilde{\bar{\psi}}_{-s}(-\omega,x), \quad \omega > 0.$$

Upon quantization, these become the annihilation Fock operators of photons, electrons and positrons respectively.[6] The (anti) commutation relations of these operators with their hermitian adjoints are dictated by i times the PBs (2.24), from which one finds

$$[a_s(\omega,x), a_s^\dagger(\omega',x')] = \frac{\pi}{\omega} \delta(\omega-\omega') \delta^{(2)}(x,x'),$$

$$[b_s(\omega,x), b_s^\dagger(\omega',x')] = [c_s(\omega,x), c_s^\dagger(\omega',x')] = 2\pi \delta(\omega-\omega') \delta^{(2)}(x,x'). \tag{2.26}$$

It will be useful for later purposes to define angular-density operators for each kind of

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[6] The normalization however is different from the standard momentum-space Fock operators, see appendix A.

particle:

$$\rho_s^A(x) := \int_0^\infty \frac{d\omega}{\pi} \omega a_s^\dagger(\omega, x) a_s(\omega, x) \tag{2.27}$$

$$\rho_s^\psi(x) := \int_0^\infty \frac{d\omega}{2\pi} b_s^\dagger(\omega, x) b_s(\omega, x) \tag{2.28}$$

$$\rho_s^{\bar\psi}(x) := \int_0^\infty \frac{d\omega}{2\pi} c_s^\dagger(\omega, x) c_s(\omega, x) \tag{2.29}$$

as well as particle-number operators

$$N_s^X := \int d^2x \rho_s^X(x), \quad X = \{A, \psi, \bar\psi\}. \tag{2.30}$$

We finally note the identity

$$\int_{-\infty}^\infty du \bar\psi_s(u, x) \psi_s(u, x) = \rho_s^\psi(x) - \rho_{-s}^{\bar\psi}(x). \tag{2.31}$$

E.   Bosonic (asymptotic) symmetries

In this section we review bosonic symmetries of tree-level massless QED at null infinity: Poincare, large $U(1)$ gauge and axial rotations. We leave out of the discussion other bosonic symmetries that are more challenging to describe at null infinity: Those arising from sub^n-leading soft photons[7] [25, 26] as well as 4-dimensional conformal symmetry.

1.   Lorentz

Infinitesimal Lorentz transformations (or more generally superrotations) near null infinity are parametrized by holomorphic vector fields $Y(z)\partial_z$ (and their complex conjugate) by

$$\xi_Y = Y(z)\partial_z + \frac{1}{2}Y'(z)(u\partial_u - r\partial_r) + \cdots , \tag{2.32}$$

They act on fields via standard Lie derivatives, which for spinors include an internal rotation (see e.g. [8])

$$\delta_\xi \Psi = \left(\xi^\mu \partial_\mu - \frac{1}{8}\nabla_\mu \xi_\nu [\gamma^\mu, \gamma^\nu] \right) \Psi. \tag{2.33}$$

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[7] See however appendix C for a preliminary incorporation of subleading soft photon symmetries.

Evaluating (2.33) for $\xi = \xi_Y$ on (2.14) one finds

$$\delta_{\xi_Y}\Psi = \frac{1}{2^{1/4}r}\begin{pmatrix} -\bar{z}\delta_Y\psi_- \\ \delta_Y\psi_- \\ \delta_Y\psi_+ \\ z\delta_Y\psi_+ \end{pmatrix} + \cdots \tag{2.34}$$

where

$$\delta_Y\psi_- = \left(Y(z)\partial_z + \frac{1}{2}Y'(z) + \frac{1}{2}Y'(z)u\partial_u\right)\psi_- \tag{2.35}$$

$$\delta_Y\psi_+ = \left(Y(z)\partial_z + Y'(z) + \frac{1}{2}Y'(z)u\partial_u\right)\psi_+ \tag{2.36}$$

Similarly, for infinitesimal Lorentz transformations $\xi_{\bar{Y}}$ parametrized by $\bar{Y}(\bar{z})\partial_{\bar{z}}$ one finds

$$\delta_{\bar{Y}}\psi_- = \left(\bar{Y}(\bar{z})\partial_{\bar{z}} + \bar{Y}'(\bar{z}) + \frac{1}{2}\bar{Y}'(\bar{z})u\partial_u\right)\psi_- \tag{2.37}$$

$$\delta_{\bar{Y}}\psi_+ = \left(\bar{Y}(\bar{z})\partial_{\bar{z}} + \frac{1}{2}\bar{Y}'(\bar{z}) + \frac{1}{2}\bar{Y}'(\bar{z})u\partial_u\right)\psi_+. \tag{2.38}$$

From a 2d perspective, the above expressions determine the holomorphic/antiholomorphic conformal dimensions of the (u-independent part of) $\psi_\pm$. In particular, one concludes the 2d spin (or equivalently the 4d helicity) of $\psi_\pm$ is equal to $\pm 1/2$.

The analogous calculation on the gauge fields leads to

$$\delta_Y A_z = \left(Y(z)\partial_z + Y'(z) + \frac{1}{2}Y'(z)u\partial_u\right)A_z \tag{2.39}$$

$$\delta_Y A_{\bar{z}} = \left(Y(z)\partial_z + \frac{1}{2}Y'(z)u\partial_u\right)A_{\bar{z}} \tag{2.40}$$

$$\delta_{\bar{Y}} A_z = \left(\bar{Y}(\bar{z})\partial_{\bar{z}} + \frac{1}{2}\bar{Y}'(\bar{z})u\partial_u\right)A_z \tag{2.41}$$

$$\delta_{\bar{Y}} A_{\bar{z}} = \left(\bar{Y}(\bar{z})\partial_{\bar{z}} + \bar{Y}'(\bar{z}) + \frac{1}{2}\bar{Y}'(\bar{z})u\partial_u\right)A_{\bar{z}} \tag{2.42}$$

$$\tag{2.43}$$

from which one concludes the 2d spin/4d helicity of A_z is $+1$ and that of $A_{\bar{z}}$ is -1.

For holomorphic vector fields, the generator is given by

$$J_Y = \sum_{s=\pm}\int_{\mathcal{I}} dud^2x\left(\dot{A}_{-s}\delta_Y A_s + i\bar{\psi}_s\delta_Y\psi_s\right), \tag{2.44}$$

with similar expression holding in the antiholomorphic case. These represent the total angular momentum of the system evaluated at null infinity.

2. Translations

Spacetime translations (or more generally supertranslations) take the asymptotic form near null infinity

$$\xi_f = f(z, \bar{z})\partial_u + \cdots \tag{2.45}$$

where the sphere function f for a translation a^μ is

$$f(z, \bar{z}) = a_\mu q^\mu(z, \bar{z}) \tag{2.46}$$

with q^μ given in (2.9). The action on the asymptotic fields is obtained from the Lie derivative, leading to

$$\delta_f \psi_s = f\dot{\psi}_s, \quad \delta_f A_s = f\dot{A}_s. \tag{2.47}$$

The phase space generator of the above transformation is then given by

$$P_f = \sum_{s=\pm} \int_{\mathcal{I}} du d^2x \left(\dot{A}_{-s}\delta_f A_s + i\bar{\psi}_s \delta_f \psi_s \right), \tag{2.48}$$

and represents the total linear momentum of the system evaluated at null infinity.

3. Large $U(1)$ gauge

Gauge transformations (2.7) with asymptotic behavior

$$\Lambda(r, u, x) \overset{r \to \infty}{=} \lambda(x) + \cdots \tag{2.49}$$

induce the following action on the fields at null infinity:

$$\delta_\lambda A_a = \partial_a \lambda, \quad \delta_\lambda \psi_s = -ie\lambda\psi_s. \tag{2.50}$$

The canonical generator for (2.50) is given by

$$Q_\lambda = \int_{\mathcal{I}} du d^2x \lambda \left(-\partial^a \dot{A}_a + e(\bar{\psi}_+\psi_+ + \bar{\psi}_-\psi_-) \right). \tag{2.51}$$

By expressing the fermionic field in terms of Fock operators, the "hard" part of the charge can be written as

$$Q_\lambda^{\text{hard}} = e \int d^2x \lambda(\rho_+^\psi + \rho_-^\psi - \rho_+^{\bar{\psi}} - \rho_-^{\bar{\psi}}). \tag{2.52}$$

For $\lambda = 1$, Q_λ reduces to the total electric charge as measured at null infinity:

$$Q_{\lambda=1} = e(N_+^\psi + N_-^\psi) - e(N_+^{\bar{\psi}} + N_-^{\bar{\psi}}), \tag{2.53}$$

where we recall that $N_\pm^X$ is the number operator defined in (2.30).

4. Axial rotations

The axial symmetry (2.8) induces at null infinity the transformation

$$\delta_{\mathrm{A}}\psi_s = -is\psi_s. \tag{2.54}$$

The canonical generator is given by

$$Q_{\mathrm{A}} = (N_+^{\psi} + N_+^{\bar{\psi}}) - (N_-^{\psi} + N_-^{\bar{\psi}}) \tag{2.55}$$

and counts the excess of positive helicity fermions over negative helicity fermions.

III. SOFT ELECTRONS AND FERMIONIC ASYMPTOTIC CHARGES

In this section we present a tree-level formula for soft electrons (and soft positrons) and compute the associated asymptotic charges. For simplicity we treat all particles as outgoing, and correspondingly derive the asymptotic charge at future null infinity. A detailed derivation of the soft electron theorem is given in appendix B.

Consider a tree-level amplitude involving n hard particles and a soft electron of momentum $p^{\mu} = \omega q^{\mu}(x)$, with $\omega \to 0$ and $q(x)$ as in (2.9). As in other instances of soft theorems, the dominant diagrams are those where the soft electron is attached to an external hard leg. There are two possibilities. Either the soft electron emerges from an external hard photon, leaving behind an internal electron line, or it emerges from an external hard positron, leaving behind an internal photon line:
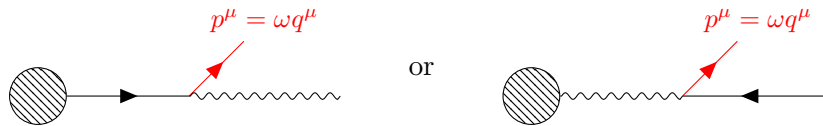
FIG. 1: The two types of diagrams contributing to a soft electron amplitude.

In both cases one obtains a result that is proportional to the n-point amplitude left behind the soft emission vertex. However, unlike the situation for soft photons, there is a change in the type of hard particle involved in the process.[8] The proportionality factor in the two types of process coincide, and is trivial unless there is certain helicity matching at the vertex. Taking for concreteness a soft electron of positive helicity and calling p_i the momentum of the hard particle, the proportionality factor is given by (see appendix B)

$$-e\frac{\bar{u}_+(p)\slashed{\epsilon}_i^+ u_+(p_i)}{2p \cdot p_i} = \frac{e}{\sqrt{\omega\omega_i}(z - z_i)}. \tag{3.1}$$

---

[8] A change in the type of hard particles also occurs for soft photinos [8] and for subleading soft photons in presence of non-minimally coupled matter [27, 28].

One thus arrives at the (positive helicty) soft electron theorem

$$
\begin{aligned}
\mathcal{A}_{n+1}(\{p_i\}, \omega q_+^{\psi})) \overset{\omega \to 0}{=} \frac{e}{\sqrt{\omega}} \sum_{i \in A_+} \frac{1}{\sqrt{\omega_i}(z - z_i)} \mathcal{A}_n(\ldots, p_{i+}^{A} \to p_{i+}^{\psi}, \ldots) \\
+ \frac{e}{\sqrt{\omega}} \sum_{i \in \bar{\psi}_-} \frac{1}{\sqrt{\omega_i}(z - z_i)} \mathcal{A}_n(\ldots, p_{i-}^{\bar{\psi}} \to p_{i-}^{A}, \ldots),
\end{aligned}
\tag{3.2}
$$

where we use the labels $\psi, \bar{\psi}$ and A for electrons, positrons and photons respectively and particle helicities are displayed by $\pm$ subscripts. The argument in $\mathcal{A}_n$ indicates the n-point amplitude involves a change in the i-th hard particle type.

Following what has been done for other soft theorems, we can interpret the above formula as a Ward identity of an asymptotic charge. Since the particle states in (3.2) are normalized according to the standard momentum-space Fock operators (see appendix A), it is simpler to first write the charge in terms of them. From (3.2) one can read-off the direction-dependent fermionic charge

$$
\begin{aligned}
\frac{4\pi i}{\sqrt{2}} Q_{\psi_+}(x) := \lim_{\omega \to 0} \sqrt{\omega} b_+^{standard}(\omega q(x)) \\
- e \int \widetilde{dp'} \frac{1}{\sqrt{\omega'}(z - z')} \left(a_+^{standard\,\dagger}(p') b_+^{standard}(p') + c_-^{standard\,\dagger}(p') a_-^{standard}(p') \right).
\end{aligned}
\tag{3.3}
$$

The overall normalization is chosen for later convenience and the label *standard* is to distinguish the momentum-space Fock operators from the ones defined in section II D.

Next, we look to express the charge in terms of the asymptotic fields $\psi_s(u, x)$ and $A_s(u, x)$. To achieve this, we first rewrite (3.3) in terms of the asymptotic Fock operators of section II D. These are related to the momentum-space operators by (see appendix A)

$$
a_s^{standard} = 4\pi i a_s, \quad b_s^{standard} = \frac{4\pi i}{\sqrt{2\omega}} b_s, \quad c_s^{standard} = \frac{4\pi i}{\sqrt{2\omega}} c_s.
\tag{3.4}
$$

Substituting (3.4) in (3.3) and using $\widetilde{dp'} = \frac{\omega'}{2(2\pi)^3} d^2 x' d\omega'$ we get

$$
\begin{aligned}
Q_{\psi_+}(x) = \lim_{\omega \to 0} b_+(\omega, x) \\
+ \frac{ie}{2\pi} \int d^2 x' \frac{1}{(z - z')} \int_0^{\infty} \frac{d\omega'}{2\pi} \left(a_+^{\dagger}(\omega', x') b_+(\omega', x') + c_-^{\dagger}(\omega', x') a_-(\omega', x') \right).
\end{aligned}
\tag{3.5}
$$

We now Fourier transform from ω to u-space using the expressions from section II D. Let us discuss the two terms in (3.5) separately. Following the standard terminology, we refer to them as "soft" and "hard" charge respectively. The "soft" part of the charge is found to be given by

$$
Q_{\psi_+}^{soft}(x) \equiv \lim_{\omega \to 0} b_+(\omega, x) = \int_{-\infty}^{\infty} du \psi_+(u, x).
\tag{3.6}
$$

The hard term is a bit more involved. We start by rewriting it as

$$
Q_{\psi_+}^{hard}(x) = ie \partial_{\bar{z}}^{-1} \sigma_{\bar{z}+}(x),
\tag{3.7}
$$

where

$$\partial_{\bar{z}}^{-1} = \frac{1}{2\pi} \int d^2x' \frac{1}{(z-z')}, \tag{3.8}$$

and

$$\sigma_{\bar{z}+}(x) := \int_0^\infty \frac{d\omega'}{2\pi} \left(a_+^\dagger(\omega', x') b_+(\omega', x') + c_-^\dagger(\omega', x') a_-(\omega', x') \right). \tag{3.9}$$

From Eqs. (2.23) and (2.25) one finds (3.9) can be written as

$$\sigma_{\bar{z}+}(x) = \int_0^\infty \frac{d\omega'}{2\pi} \left(\tilde{A}_{\bar{z}}(-\omega', x') \tilde{\psi}_+(\omega', x') + \tilde{\psi}_+(-\omega', x') \tilde{A}_{\bar{z}}(\omega', x') \right) \tag{3.10}$$

$$= \int_{-\infty}^\infty du A_{\bar{z}}(u, x') \psi_+(u, x'). \tag{3.11}$$

We finally combine the "soft" and "hard" charges by factoring out an inverse derivative in the former

$$Q_{\psi_+}^{soft}(x) = \partial_{\bar{z}}^{-1} \int_{-\infty}^\infty du \partial_{\bar{z}} \psi_+(u, x). \tag{3.12}$$

Comparing (3.12) with (3.7) and (3.11), we conclude the total charge can be written as

$$Q_{\psi_+}(x) = \partial_{\bar{z}}^{-1} \int_{-\infty}^\infty du D_{\bar{z}} \psi_+(u, x), \tag{3.13}$$

where

$$D_a \psi_s \equiv (\partial_a + ie A_a) \psi_s \tag{3.14}$$

is the gauge covariant derivative at null infinity.

Repeating the previous analysis for a negative helicity soft electron yields a charge of the form

$$Q_{\psi_-}(x) = \partial_z^{-1} \int_{-\infty}^\infty du D_z \psi_-(u, x). \tag{3.15}$$

Finally, soft positrons lead to charges that are the complex conjugates of (3.13) and (3.15).

It is natural to combine all these direction-dependent charges into a single smeared asymptotic charge which we define as

$$F_\chi := i \int d^2x du \left(\bar{\chi}_+^{\bar{z}} D_{\bar{z}} \psi_+ + \bar{\chi}_-^z D_z \psi_- + \chi_+^z D_z \bar{\psi}_+ + \chi_-^{\bar{z}} D_{\bar{z}} \bar{\psi}_- \right). \tag{3.16}$$

where $\chi_+^z(x)$ and $\chi_-^{\bar{z}}(x)$ are the components of a spinor-vector smearing parameter

$$\chi = (\chi_+^z, \chi_-^{\bar{z}}), \tag{3.17}$$

with complex conjugate $\bar{\chi} = (\bar{\chi}_+^{\bar{z}}, \bar{\chi}_-^z)$. We take χ to be grassmanian so that F_χ is real/hermitian.

IV.   ON FERMIONIC ASYMPTOTIC SYMMETRIES

Even though conserved asymptotic charges imply the existence of asymptotic symmetries, the nature of the latter may be challenging to decipher (see for example [7, 8]). One reason for this difficulty is that asymptotic symmetries are spontaneously broken [3], while the charges obtained from soft theorems are evaluated on a single vacuum sector. To characterize the symmetry one needs to make manifest the vacuum manifold, thus going beyond standard Fock-space amplitudes [29–33]. At the classical level, this usually requires an extension of the radiative phase space, see e.g. [34–42].

In this section we take the first steps towards characterizing the asymptotic fermionic symmetry implied by the soft electron theorem. We will start by evaluating the charge action according to the radiative phase space brackets (2.21). We shall see this action does not respect the $|u| \to \infty$ behavior of the asymptotic fields, thus indicating the need of a phase space extension. We leave for future work the identification of such extension as well as the related problem of fully characterizing the symmetry algebra. We will nevertheless be able to verify the commutator between fermionic and bosonic symmetries. In appendix C we shall further discuss a naive evaluation of the commutator between two fermionic symmetries.

A.   Symmetry action from radiative PBs

Infinite dimensional phase spaces may present subtleties that are absent in finite dimensions. In particular, not all phase-space functionals are guaranteed to yield well-defined PB actions. As we shall see, this is the case for the fermionic charge F_χ.

Let us for a moment ignore the aforementioned subtlety and consider the action obtained from the standard formula

$$\delta_\chi = \{\cdot, F_\chi\}. \tag{4.1}$$

From the elementary PBs (2.21) and the expression for F_χ (3.16) one finds[9]

$$
\begin{aligned}
\delta_\chi \psi_+ &= D_z \chi_+^z \\
\delta_\chi \psi_- &= D_{\bar{z}} \chi_-^{\bar{z}} \\
\delta_\chi \dot{A}_z &= \frac{e}{2}(\bar{\chi}_{z+}\psi_+ - \chi_{z-}\bar{\psi}_-) \\
\delta_\chi \dot{A}_{\bar{z}} &= \frac{e}{2}(\bar{\chi}_{\bar{z}-}\psi_- - \chi_{\bar{z}+}\bar{\psi}_+).
\end{aligned}
\tag{4.2}
$$

We first notice that $\lim_{u \to \pm} \delta_\chi \psi_s \neq 0$, and thus the transformation does not preserve the condition $\psi_s \overset{|u| \to \infty}{\to} 0$ (2.20). This suggests a phase space extension that allows for non-trivial asymptotic values of ψ_s when $u \to \pm\infty$.

The transformation rule for $\dot{A}_s$ (4.2) is compatible with the asymptotic behaviour of A_s given in (2.19) provided ψ_s decays to zero as in (2.20). The discussion from the previous paragraph however indicates that in an extended space where $\lim_{u \to \pm\infty} \psi_s \neq 0$ we would

---

[9] We recall that $D_a = \partial_a + ieA_a$ is the gauge covariant derivative at null infinity and $\chi_{z-} = \chi_-^{\bar{z}}$, $\chi_{\bar{z}+} = \chi_+^z$.
The action on $\bar{\psi}_\pm$ can be obtained from that of $\psi_\pm$ by complex conjugation. To simplify expressions we have chosen to display $\delta_\chi \dot{A}_s$ rather than $\delta_\chi A_s$ (which is non-local in u).

need to allow for non-trivial $|u| \to \infty$ values of $\dot{A}_s$. This resembles the situation for the asymptotic charges obtained from subleading photons [7], whose action creates an $O(u)$ term in A_s.

If one continues the previous considerations back and forth between $\delta_\chi \psi_s$ and $\delta_\chi A_s$, one is led to conclude that all powers of u should be allowed in the $|u| \to \infty$ behaviour of the fields. We recall however that (4.2) is at best valid only around the trivial vacuum and cannot be trusted beyond linear order in χ. Higher order iterations of δ_χ may require the inclusion of terms that are absent in (4.2) and that could modify the previous conclusion. We leave for future work the elucidation of this non-linear structure.[10] There are however checks to be made at first order in χ, namely the commutation of δ_χ with the bosonic symmetries reviewed in section II E. We discuss them next.

B.   Algebra relations with bosonic symmetries

Let δ_B be a bosonic symmetry action corresponding to a charge B, i.e.

$$\delta_B = \{\cdot, B\}. \tag{4.3}$$

We restrict attention to the bosonic symmetries discussed in section II E so that $\delta_B = \delta_Y, \delta_f, \delta_\lambda, \delta_A$ for $B = J_Y, P_f, Q_\lambda, Q_A$ respectively. As we shall see, these symmetries have a natural action on the fermionic parameter,

$$\chi \mapsto \chi + \delta_B \chi \tag{4.4}$$

such that the commutator of variations is given by

$$[\delta_\chi, \delta_B] = \delta_{\delta_B \chi}. \tag{4.5}$$

At the level of charges, this should imply the PB relations[11]

$$\{F_\chi, B\} = -F_{\delta_B \chi}. \tag{4.6}$$

There are however various subtleties in the evaluation of PBs that make (4.6) challenging to verify directly. It is for this reason that we will instead focus on the relations

$$\delta_B F_\chi = -F_{\delta_B \chi}, \tag{4.7}$$
$$\delta_\chi B = F_{\delta_B \chi}. \tag{4.8}$$

It turns out that (4.7) follows straightforwardly from the expression of F_χ and (4.5). This should lead, via (4.3), to (4.6). Finally from (4.1) one would arrive at (4.8). There are however two obstructions to this logic chain.

First, Eq. (4.3) does not hold for Q_λ if one uses the radiative PBs (2.21) (see the discussion

---

[10] A preliminary attempt go beyond linear order is presented in appendix C, where we evaluate the commutator between two fermionic variations (4.2). Although the result cannot be trusted due to the aforementioned limitations, it may still be of use in more complete treatments.

[11] In the absence of a central extension, as it turns out to be the case.

following this equation). Since these were the brackets used in the definition of δ_χ (4.2), there will be a mismatch in (4.8) when $B = Q_\lambda$. This problem should go away if δ_χ is constructed from improved PBs as the ones proposed in [6].

A second obstruction appears from the fact that the relation

$$\delta F_\chi = \Omega(\delta, \delta_\chi) \tag{4.9}$$

only holds for variations δ that decay to zero faster than the assumed ones in order to compensate for the singular behavior of δ_χ at $u = \pm\infty$. This leads to additional difficulties in verifying (4.8), for instance when $B = J_Y$. A complete fix to this second obstruction would presumably require the inclusion of soft fermionic degrees of freedom in the symplectic structure.

We now discuss in more detail the situation for each bosonic symmetry separately.

1. Lorentz

The transformation properties of χ under the Lorentz group follow directly from those of the elementary fields A_s and $\psi_\pm$. By direct evaluation one readily obtains (4.5) and (4.7) with

$$\delta_{(Y,\bar{Y})}\chi_+^z = \left(Y\partial_z + \bar{Y}\partial_{\bar{z}} + \tfrac{1}{2}\bar{Y}'\right)\chi_+^z \tag{4.10}$$

$$\delta_{(Y,\bar{Y})}\chi_-^{\bar{z}} = \left(Y\partial_z + \bar{Y}\partial_{\bar{z}} + \tfrac{1}{2}Y'\right)\chi_-^{\bar{z}} \ . \tag{4.11}$$

Thus, from a 2d perspective, χ_+^z and $\chi_-^{\bar{z}}$ have (anti-)holomorphic dimensions $(h, \bar{h}) = (0, 1/2)$ and $(1/2, 0)$ respectively.

Relation (4.8) is verified provided one discards boundary terms of the type

$$\left[A_{\bar{z}}\delta_\chi\delta_{(Y,\bar{Y})}A_z\right]_{u=-\infty}^{u=\infty}, \tag{4.12}$$

which are however non-trivial under the assumed fall-offs (2.19), (2.20).

2. Translations

It is easy to verify that δ_χ commutes with translations,

$$[\delta_\chi, \delta_f] = 0 \tag{4.13}$$

and that $\delta_f F_\chi = 0$ (assuming the fall-offs (2.19), (2.20) otherwise there could be non-zero boundary terms). The relation $\delta_\chi P_f = 0$ follows with no caveats.

3. Large $U(1)$ gauge

As in the Lorentz case, the transformation properties of χ follow from those of the elementary fields. One can easily verify (4.5) and (4.7) with

$$\delta_\lambda \chi^z_+ = -ie\chi^z_+ \tag{4.14}$$
$$\delta_\lambda \chi^{\bar{z}}_- = -ie\chi^{\bar{z}}_-. \tag{4.15}$$

As anticipated earlier, Eq. (4.8) is not verified unless one addresses the $1/2$ discontinuity in the radiative PBs (2.22). Let us describe how the issue arises in the present context.

The RHS of (4.8) takes the form

$$F_{\delta_\lambda \chi} = -e \int_{\mathcal{I}} \lambda \bar{\chi}^{\bar{z}}_+ D_{\bar{z}} \psi_+ + \cdots \tag{4.16}$$

$$= -e \int_{\mathcal{I}} \partial_{\bar{z}} \lambda \bar{\chi}^{\bar{z}}_+ \psi_+ + e \int_{\mathcal{I}} \lambda D_{\bar{z}} \bar{\chi}^{\bar{z}}_+ \psi_+ + \cdots , \tag{4.17}$$

where in the second line we integrated by parts and for simplicity we are only displaying what corresponds to the first term in F_χ (3.16) (the others follow a similar pattern).

The LHS of (4.8) can be written as a

$$\delta_\chi Q_\lambda = \delta_\chi Q^{\text{hard}}_\lambda + \delta_\chi Q^{\text{soft}}_\lambda \tag{4.18}$$

with

$$\delta_\chi Q^{\text{hard}}_\lambda = \delta_\chi \int_{\mathcal{I}} e\lambda(\bar{\psi}_+ \psi_+ + \bar{\psi}_- \psi_-) \tag{4.19}$$

$$= e \int_{\mathcal{I}} \lambda \delta_\chi \bar{\psi}_+ \psi_+ + \cdots \tag{4.20}$$

$$= e \int_{\mathcal{I}} \lambda D_{\bar{z}} \bar{\chi}^{\bar{z}}_+ \psi_+ + \cdots \tag{4.21}$$

and

$$\delta_\chi Q^{\text{soft}}_\lambda = \int_{\mathcal{I}} (\partial_{\bar{z}} \lambda \dot{A}_z + \partial_z \lambda \dot{A}_{\bar{z}}) \tag{4.22}$$

$$= \int_{\mathcal{I}} \partial_{\bar{z}} \lambda \delta_\chi \dot{A}_z + \cdots \tag{4.23}$$

$$= \frac{e}{2} \int_{\mathcal{I}} \partial_{\bar{z}} \lambda \bar{\chi}^{\bar{z}}_+ \psi_+ + \cdots , \tag{4.24}$$

where we only displayed the terms that correspond to those shown in (4.17). Comparing the expressions one sees that whereas $\delta_\chi Q^{\text{hard}}_\lambda$ reproduces the first term in (4.17), $\delta_\chi Q^{\text{soft}}_\lambda$ fails to reproduce the second term by a factor of $1/2$. The origin of this mismatch is the same as the one leading to the discontinuity discussed in Eq. (2.22) and so it would be fixed by the use of improved PBs [6].

4. Axial rotations

In this case, equations (4.7) and (4.8) are verified with no caveats, with

$$\delta_A \chi^z_+ = -i\chi^z_+ \tag{4.25}$$
$$\delta_A \chi^{\bar{z}}_- = i\chi^{\bar{z}}_-. \tag{4.26}$$

V. OUTLOOK

Over the past decade there has been a fruitful revision on the subject of asymptotic symmetries in (asymptotically) flat spacetimes, driven by the discovery of their connection with soft theorems and memory effects [3]. Whereas asymptotic symmetries were originally tied to gauge symmetries that are non-trivial at infinity, their relation to soft theorems led to a broader perspective. Indeed, there is a growing list of soft theorems that admit an interpretation in terms of asymptotic symmetries with no obvious gauge origin.[12] Here we have enlarged such list, by presenting an asymptotic symmetry associated to soft electrons in tree-level massless QED.[13] Following what is by now a standard procedure, we identified the form of the asymptotic charge and initiated the study of its symmetry action.

To our knowledge, this is the first example of a fermionic asymptotic symmetry in a theory with no *standard* supersymmetry. The existence of such fermionic symmetry, however, suggest that tree-level massless QED possess an *asymptotic* supersymmetry algebra. Unfortunately the present work remains agnostic as to what such algebra should be. Below we describe this and others shortcomings of our analysis as well as possible strategies to overcome them.

We followed what may be referred to as a canonical approach to asymptotic symmetries, in which the classical phase space at null infinity provides the bridge between symmetries and charges. A difficulty with this approach is that it often requires an enlarged version of the radiative phase space, with no simple recipe to determine it.[14] Whereas we showed that an enlargement is implied by soft electrons, we were not able to characterize it beyond linear level. This in turn impeded us to trustfully evaluate the commutator between fermionic asymptotic symmetries.

There is however a second approach in which symmetries are described in terms of 2d conserved currents of a dual CFT [49–51]. There, the algebraic structure is read off from collinear factorization theorems [52–59] without the need of phase-space considerations.[15] It would be very interesting to study the fermionic symmetries presented here from this

---

[12] What appears as a non-gauge asymptotic symmetry may sometimes be realized as large gauge, either by allowing divergent gauge transformations [43] or by performing a change of field variables [44]. It is however not clear whether such reinterpretation is always possible.

[13] Although we have set our discussion in the context of QED, it should admit a straightforward generalization to the case of non-abelian gauge fields as well as chiral fermions.

[14] Recent developments on so called corner symmetries [45–48] may in fact yield such recipe (at least in the case of large gauge symmetries).

[15] The price to pay, however, is that so far such approach only captures symmetry relations among generators of the same helicity sign. See [61–65] for discussions on how the canonical and CFT approaches relate to each other.

perspective, as it could allow us to extract information on the fermionic symmetry algebra.

Regardless of the approach, it has been found in the context of gauge and gravitational theories that completeness of the asymptotic symmetry algebra requires charges associated to subleading soft theorems of arbitrarily high order, see e.g. [60, 61]. It would be interesting to explore the existence of higher order soft electron theorems and their connection with the soft photon ones. A hint that a nontrivial interplay may occur is provided by a naive evaluation of the commutator between the fermionic variations presented here, leading to an expression that is reminiscent to the subleading soft photon charge action (see appendix C).

A final pressing open problem regards the fate of the symmetries beyond tree-level. The structure of their modification or breaking due to loop corrections should be worth studying. In particular, it would be interesting to explore any possible connection with the chiral anomaly, whose consequences at null infinity were recently analyzed in [66].

Appendix A: Asymptotic vs. momentum-space Fock operators

In this appendix we work out the relation between the Fock operators introduced in section II D and the standard momentum-space Fock operators. In particular we verify they yield equivalent (anti) commutation relations.

We start with the momentum expansion of asymptotic free fields:

$$
\begin{aligned}
\mathcal{A}_\mu(X) &= \sum_{s=\pm} \int \widetilde{dp} \left(a_s(p)\varepsilon_\mu^{s*}(p)e^{ip\cdot X} + a_s^\dagger(p)\varepsilon_\mu^s(p)e^{-ip\cdot X} \right) \\
\Psi(X) &= \sum_{s=\pm} \int \widetilde{dp} \left(b_s(p)u_s(p)e^{ip\cdot X} + c_s^\dagger(p)v_s(p)e^{-ip\cdot X} \right)
\end{aligned}
\tag{A1}
$$

where $\widetilde{dp} = d^3p/((2\pi)^3 2|p|)$ and the photon/electron polarization vectors are described below. To simplify notation we do not yet include additional labels to the Fock operators, but it should be kept in mind that they have different normalization from those introduced in section II D.

To study the null infinity limit of these expressions, we parametrize X^μ in terms of (r, u, x) as in (2.11) and write the null momentum p^μ as

$$
p^\mu = \omega q^\mu(x')
\tag{A2}
$$

with q^μ as in (2.9) with $x' = (z', \bar{z}')$. In this parametrization the momentum measure takes the form

$$\widetilde{dp} = \frac{\omega}{2(2\pi)^3} d^2x' d\omega \tag{A3}$$

and the plane-wave phase becomes

$$p \cdot X = r\omega q(x) \cdot q(x') + u\omega k \cdot q(x') \tag{A4}$$

where

$$q(x) \cdot q(x') = -|z - z'|^2. \tag{A5}$$

In the $r \to \infty$ limit, the d^2x' integral can be evaluated by a saddle point analysis. In the simplest case of a positive-frequency scalar one gets

$$\lim_{r \to \infty} \int \widetilde{dp}\, \phi(p) e^{ip \cdot X} = \frac{1}{4\pi i r} \int_0^\infty \frac{d\omega}{2\pi} \phi(\omega q(x)) e^{-i\omega u} + O(1/r^2). \tag{A6}$$

Before extending this formula to the photon and fermion fields, let us recall the form of the corresponding momentum wave-functions. The polarization vector for a photon with momentum $p = \omega q^\mu(x)$ can be taken to be [6]

$$\varepsilon^{+\mu} = \partial_z q^\mu = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \quad \varepsilon^{-\mu} = \partial_{\bar{z}} q^\mu = \frac{1}{\sqrt{2}}(z, 1, i, -z) \tag{A7}$$

which satisfy

$$\varepsilon^\pm \cdot q = 0, \quad \varepsilon^\pm \cdot \varepsilon^\pm = 0, \quad \varepsilon^+ \cdot \varepsilon^- = 1, \quad \varepsilon_\mu^+ \varepsilon_\nu^- + \varepsilon_\mu^- \varepsilon_\nu^+ + \frac{q_\mu k_\nu + k_\mu q_\nu}{q \cdot k} = \eta_{\mu\nu} \tag{A8}$$

with k^μ a null vector with non-zero dot product with q^μ, as for instance the one given in (2.10). For the fermion we have (see e.g. [8, 67])

$$u_+ = v_- = 2^{1/4}\sqrt{\omega} \begin{pmatrix} 0 \\ 0 \\ 1 \\ z \end{pmatrix}, \quad u_- = v_+ = 2^{1/4}\sqrt{\omega} \begin{pmatrix} -\bar{z} \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{A9}$$

which satisfy

$$\slashed{q} u_\pm = 0, \quad \bar{u}_s \gamma^\mu u_{s'} = 2p^\mu \delta_{ss'}, \quad \sum_s u_s \bar{u}_s = -\slashed{p}, \tag{A10}$$

and similarly for $v_\pm$. Expressions (A7), (A9) can be obtained from those used in the spinor-helicity formalism [67] with the choice of spinor $2^{1/4}\sqrt{\omega}(1, z)$ for the momentum $p^\mu = \omega q^\mu(z, \bar{z})$.[16]

Using the above expressions, one finds

$$\mathcal{A}_z = \frac{1}{4\pi i} \int_0^\infty \frac{d\omega}{2\pi} \left(a_+ e^{-i\omega u} - a_-^\dagger e^{i\omega u} + \right) + O(1/r) \tag{A11}$$

---

[16] The photon polarization in (A7) corresponds to a choice of "reference spinor" of the form $(1, \infty)$.

$$\Psi = \frac{1}{4\pi i r} \int_0^\infty \frac{d\omega}{2\pi} 2^{1/4} \sqrt{\omega} \begin{pmatrix} -\bar{z} \ (b_- e^{-i\omega u} - c_+^\dagger e^{i\omega u}) \\ 1 \\ 1 \\ z \ (b_+ e^{-i\omega u} - c_-^\dagger e^{i\omega u}) \end{pmatrix}. \tag{A12}$$

Comparing with (2.23), (2.25) we see that the momentum space Fock operators are related to the Fock operators of section II D by

$$a_s = \frac{1}{4\pi i} a_s^{standard}, \quad b_s = \frac{\sqrt{2\omega}}{4\pi i} b_s^{standard}, \quad c_s = \frac{\sqrt{2\omega}}{4\pi i} c_s^{standard}, \tag{A13}$$

where we have added the label "standard" to the operators of this section to distinguish them from those of section II D. The commutation relations (2.26) then imply the standard momentum-space (anti) commutators

$$[d_s^{standard}(p), d_{s'}^{standard\,\dagger}(p')] = 2|p|(2\pi)^3 \delta_{ss'}\delta^{(3)}(p - p'), \quad d = \{a, b, c\}, \tag{A14}$$

where we used that $\delta^{(3)}(p - p') = \frac{1}{\omega^2}\delta(\omega - \omega')\delta^{(2)}(x, x')$ for $p = \omega q(x)$ and $p' = \omega q(x')$.

Appendix B: Soft photons and soft electrons

In this appendix we derive the tree-level formulas for the emission of soft particles. We treat all particles as outgoing.

Since the formulas are dictated by the QED 3-point interaction, it will be useful to start the discussion by recalling the structure of general 3-point amplitudes. Consider then a process involving an electron, positron and photon of momenta

$$\begin{aligned} p_e &= \omega_e q(z_e, \bar{z}_e) \\ p_{\bar{e}} &= \omega_{\bar{e}} q(z_{\bar{e}}, \bar{z}_{\bar{e}}) \\ p_\gamma &= \omega_\gamma q(z_\gamma, \bar{z}_\gamma) \end{aligned} \tag{B1}$$
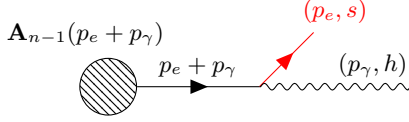
and helicities $s, \bar{s}$ and h. The (momentum conserving delta function stripped) amplitude is given by

$$A_3((p_e, s), (p_{\bar{e}}, \bar{s}), (p_\gamma, h)) = \qquad\qquad = -ie\bar{u}_s(p_e)\gamma^\mu v_{\bar{s}}(p_{\bar{e}})\varepsilon_\mu^h(p_\gamma). \tag{B2}$$

In order for the amplitude to be non-zero we must have $\bar{s} = -s$ whereas h can be arbitrary. Writing the 4 posibilites in holomorphic coordinates (using the expressions for polarization vectors given in the previous section) one finds

$$\begin{aligned} A_3((p_e, +), (p_{\bar{e}}, -), (p_\gamma, +)) &= 2ie\sqrt{\omega_e\omega_{\bar{e}}}(\bar{z}_\gamma - \bar{z}_e) \\ A_3((p_e, +), (p_{\bar{e}}, -), (p_\gamma, -)) &= 2ie\sqrt{\omega_e\omega_{\bar{e}}}(z_\gamma - z_{\bar{e}}) \\ A_3((p_e, -), (p_{\bar{e}}, +), (p_\gamma, +)) &= 2ie\sqrt{\omega_e\omega_{\bar{e}}}(\bar{z}_\gamma - \bar{z}_{\bar{e}}) \\ A_3((p_e, -), (p_{\bar{e}}, +), (p_\gamma, -)) &= 2ie\sqrt{\omega_e\omega_{\bar{e}}}(z_\gamma - z_e). \end{aligned} \tag{B3}$$

We have not yet imposed momentum conservation. This typically leads to trivial amplitudes (unless one allows for complex momenta [67]). An exception however is when one of the particles goes soft with the remaining two becoming coincident:[17]

$$
\begin{aligned}
\omega_\gamma \to 0 &\implies z_{\bar{e}} \to z_e, \quad \omega_{\bar{e}} \to \omega_e \\
\omega_e \to 0 &\implies z_{\bar{e}} \to z_\gamma, \quad \omega_{\bar{e}} \to \omega_\gamma \\
\omega_{\bar{e}} \to 0 &\implies z_e \to z_\gamma, \quad \omega_e \to \omega_\gamma.
\end{aligned}
\tag{B4}
$$

Notice that in the soft electron/positron limit, two out of the four amplitudes in (B3) vanish.

We next describe the behavior of general amplitudes when one particle goes soft. As is well known, the leading contribution comes from diagrams where the soft particle is attached to an external hard particle. In the following we discuss the relevant diagrams for each type of soft particle. We consider processes involving n hard particles and 1 soft particle.

1. Soft photon

The amplitude for a process in which a photon is attached to an external electron is given by
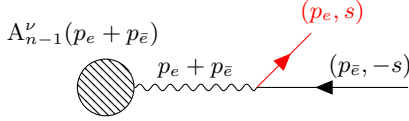

$$
\begin{aligned}
&= -ie\varepsilon_\mu^h(p_\gamma)\bar{u}_s(p_e)\gamma^\mu i \frac{\not{p}_e + \not{p}_\gamma}{(p_e + p_\gamma)^2}\mathbf{A}_{n-1}(p_e + p_\gamma) \\
&\overset{\omega_\gamma \to 0}{=} e\varepsilon_\mu^h(p_\gamma)\bar{u}_s(p_e)\gamma^\mu \frac{\not{p}_e}{2p_e \cdot p_\gamma}\mathbf{A}_{n-1}(p_e) + O(\omega_\gamma^0) \\
&= -e\frac{\varepsilon^h(p_\gamma) \cdot p_e}{p_\gamma \cdot p_e}A_n + O(\omega_\gamma^0)
\end{aligned}
\tag{B5}
$$

where $\mathbf{A}_{n-1}$ denotes the spinor-valued off-shell amplitude corresponding to the remaining $n-1$ hard particles. In going to the second line we kept the leading terms in the $\omega_\gamma \to 0$ limit, whereas in going to the third line we used the identities in (A10) and the fact that the fermion helicities in the vertex must match for the amplitude to be nonzero. A_n represents the amplitude for the n hard particles.

One can similarly compute the amplitude for the case where the photon is attached to an outgoing positron, resulting in an expression as the above modulo an overall sign. Summing over all external particles one obtains Weinberg's soft photon theorem [68].

2. Soft electron

We now consider processes in which an electron goes soft. From the structure of the QED vertex, we see that the electron can be attached either to an external photon or to an external positron (recall we are taking all particles to be outgoing).

---

[17] Since we are working with real momenta, $z \to z_0$ implies $\bar{z} \to \bar{z}_0$.

The contribution when the electron is attached to an external photon is

$$= -ie\varepsilon_\mu^h(p_\gamma)\bar{u}_s(p_e)\gamma^\mu i\frac{\slashed{p}_e + \slashed{p}_\gamma}{(p_e + p_\gamma)^2}\mathbf{A}_{n-1}(p_e + p_\gamma)$$

$$\stackrel{\omega_e \to 0}{=} e\varepsilon_\mu^h(p_\gamma)\bar{u}_s(p_e)\gamma^\mu\frac{\slashed{p}_\gamma}{2p_e \cdot p_\gamma}\mathbf{A}_{n-1}(p_\gamma) + O(\omega_e^0)$$

$$= -e\frac{\varepsilon_\mu^h(p_\gamma)\bar{u}_s(p_e)\gamma^\mu u_s(p_\gamma)}{2p_e \cdot p_\gamma}A_n((\gamma, p_\gamma, h) \to (e, p_\gamma, s)) + O(\omega_e^0).$$
(B6)

Notice that the starting point is the same as in the soft photon case discussed in (B5). The difference is that we are now taking the electron, rather than the photon, to be soft. Again one obtains an expression that is proportional to an n-point amplitude A_n. However, unlike the soft photon theorem, the external hard particle involved in the process is replaced by a different type of particle (in this case a photon is replaced by an electron, as indicated in the argument of A_n). The multiplicative factor in front of A_n can be simplified by writing the momenta and polarization vectors in holomorphic coordinates. The only non-trivial case is when h and s are of the same sign (this can be seen from Eqs. (B3) and (B4)) and one finds

$$-e\frac{\varepsilon_\mu^+(p_\gamma)\bar{u}_+(p_e)\gamma^\mu u_+(p_\gamma)}{2p_e \cdot p_\gamma} = \frac{e}{\sqrt{\omega_e\omega_\gamma}(z_e - z_\gamma)}, \tag{B7}$$

$$-e\frac{\varepsilon_\mu^-(p_\gamma)\bar{u}_-(p_e)\gamma^\mu u_-(p_\gamma)}{2p_e \cdot p_\gamma} = \frac{e}{\sqrt{\omega_e\omega_\gamma}(\bar{z}_e - \bar{z}_\gamma)}. \tag{B8}$$

The contribution when the electron is attached to an external positron is

$$= -ie\bar{u}_s(p_e)\gamma^\mu v_{-s}(p_{\bar{e}})\frac{-i\eta_{\mu\nu}}{(p_e + p_{\bar{e}})^2}A_{n-1}^\nu(p_e + p_{\bar{e}})$$

$$\stackrel{\omega_e \to 0}{=} -e\bar{u}_s(p_e)\gamma^\mu u_s(p_{\bar{e}})\frac{\sum_h \varepsilon_\mu^h(p_{\bar{e}})\varepsilon_\nu^{-h}(p_{\bar{e}})}{2p_e \cdot p_{\bar{e}}}A_{n-1}^\nu(p_{\bar{e}}) + O(\omega_e^0)$$

$$= -e\frac{\varepsilon_\mu^s(p_{\bar{e}})\bar{u}_s(p_e)\gamma^\mu u_s(p_{\bar{e}})}{2p_e \cdot p_{\bar{e}}}A_n((\bar{e}, p_{\bar{e}}, -s) \to (\gamma, p_{\bar{e}}, -s)) + O(\omega_e^0).$$
(B9)

Here we assumed the positron helicity is opposite to that of the soft electron, since otherwise the amplitude vanishes. A_{n-1}^ν is a vector-valued off-shell amplitude associated to the $n-1$ hard particles not involved in the process under consideration. In going to the second line we used the resolution of the identity (A8) together with $\slashed{p}u(p) = 0$ and $p_\nu A_{n-1}^\nu(p) = 0$. Finally, in the third line, we used the fact that only the $h = s$ photon polarization yields a non-zero contribution.

We can finally combine both type of contributions to obtain the "soft electron theorem" of Eq. (3.2).

Appendix C: Naive commutator of fermionic charges and subleading soft photons

In this appendix we present the asymptotic symmetry associated to Low's subleading photon theorem [69], as first introduced by Lysov, Pasterski and Strominger (LPS) [7]. We next evaluate the commutator of two fermionic symmetries and compare it with the LPS symmetry. Even though the two expression do not coincide, there are certain similarities that suggest the LPS symmetry may feature in an eventual (super) symmetry algebra involving the fermionic generators. We hope the calculation presented here will be of use in a more complete treatment.

1. Asymptotic symmetries for subleading soft photon

Soft photons obey a factorization theorem at subleading order [69] that can be understood in terms of LPS asymptotic symmetries [7]. The corresponding charges are parametrized by vector fields Y^a on the sphere and take the form,

$$Q_Y^{\text{LPS}} = \int_{\mathcal{I}} du d^2x \left(-2u\dot{A}_{\bar{z}}\partial_z^2 Y^z + u\partial_z Y^z j_u + Y^z j_z \right) + c.c., \tag{C1}$$

where j_u and j_a are the leading components of the asymptotic current,

$$\mathcal{J}_u = j_u/r^2 + \cdots \tag{C2}$$

$$\mathcal{J}_a = j_a/r^2 + \cdots . \tag{C3}$$

In order to compute the symmetry action on the spinor field, we need to express j_u and j_a in terms of ψ_s. This is achieved by substituting the asymptotic expansion of the Dirac spinor (2.14) in the current expression (2.3). For the u component one gets

$$j_u = -e \sum_s \bar{\psi}_s \psi_s. \tag{C4}$$

The evaluation of j_a requires one order further in the asymptotic expansion of the Dirac field,

$$\Psi = \frac{1}{r}\overset{0}{\Psi} + \frac{1}{r^2}\overset{1}{\Psi} + \cdots \tag{C5}$$

where $\overset{0}{\Psi}$ is the leading term displayed in (2.14). Substituting (C5) in the current expression one finds

$$\mathcal{J}_a = \frac{e}{r}\left(\overset{0}{\overline{\Psi}} + \frac{1}{r}\overset{1}{\overline{\Psi}} + \cdots \right)\partial_a\slashed{q}\left(\overset{0}{\Psi} + \frac{1}{r}\overset{1}{\Psi} + \cdots \right) \tag{C6}$$

$$= \frac{e}{r^2}\left(\overset{0}{\overline{\Psi}}\partial_a\slashed{q}\overset{1}{\Psi} + \overset{1}{\overline{\Psi}}\partial_a\slashed{q}\overset{0}{\Psi} \right) + \cdots \tag{C7}$$

where we used that

$$\overset{0}{\overline{\Psi}}\partial_a\slashed{q}\overset{0}{\Psi} = 0. \tag{C8}$$

We finally need to express $\overset{1}{\Psi}$ in terms of ψ_s. Imposing the asymptotic free-field equation on Ψ one finds[18]

$$0 = \slashed{\partial}\Psi = -\frac{1}{r}\slashed{q}\partial_u \overset{0}{\Psi} + \frac{1}{r^2}\left(-\slashed{q}\partial_u \overset{1}{\Psi} + \slashed{k}\overset{0}{\Psi} + \partial_a \slashed{q}\partial^a \overset{0}{\Psi}\right) + \cdots \tag{C9}$$

The vanishing of the $1/r$ term leads to (2.14), which we write as[19]

$$\overset{0}{\Psi} = \psi_+ \mathbf{u}_+ + \psi_- \mathbf{u}_- \tag{C10}$$

where

$$\mathbf{u}_+ = \frac{1}{2^{1/4}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ z \end{pmatrix}, \quad \mathbf{u}_- = \frac{1}{2^{1/4}}\begin{pmatrix} -\bar{z} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{C11}$$

span the kernel of the matrix $\slashed{q}$.

$\overset{1}{\Psi}$ is to be determined from the vanishing of the $1/r^2$ term in (C9). Due to the non-invertibility of $\slashed{q}$, it is not immediately obvious this equation can be solved. However, using that $k^\mu = 1/2\partial^a\partial_a q^\mu$, the equation can be brought into the form

$$0 = -\slashed{q}(\partial_u \overset{1}{\Psi} + \frac{1}{2}\partial^a\partial_a \overset{0}{\Psi}) + \frac{1}{2}\partial^a\partial_a(\slashed{q}\overset{0}{\Psi}). \tag{C12}$$

Since the last term vanishes due to (C10), the equation fixes $\overset{1}{\Psi}$ modulo elements in the kernel of $\slashed{q}$:[20]

$$\overset{1}{\Psi} = -\frac{1}{2}\partial_u^{-1}\partial^a\partial_a \overset{0}{\Psi} + \ker(\slashed{q}). \tag{C13}$$

$$= -\sum_s \partial_u^{-1}\partial_a\psi_s\partial^a \mathbf{u}_s + \ker(\slashed{q}). \tag{C14}$$

Substituting in (C6) we get

$$j_z = -e\left(\bar{\psi}_- \partial_z \partial_u^{-1}\psi_- + \partial_z \partial_u^{-1}\bar{\psi}_+ \psi_+\right), \tag{C15}$$

$$j_{\bar{z}} = -e\left(\bar{\psi}_+ \partial_{\bar{z}} \partial_u^{-1}\psi_+ + \partial_{\bar{z}} \partial_u^{-1}\bar{\psi}_- \psi_-\right), \tag{C16}$$

---

[18] Whereas this suffices for tree-level considerations, a more complete treatment should deal with the full non-linear asymptotic field equations. This in turn may modify the assumed fall-offs (C5) (and hence (C2), (C3)). See [21] for related discussions.

[19] In (2.14) and (C10) we are setting to zero a possible u-independent spinor that may not be in the kernel of $\slashed{q}$. It may be that such spinor is needed in an extended version of the radiative phase space.

[20] The first term in (C13) is what one gets from solving $\Box\Psi = 0$. The apparent indeterminacy in $\overset{1}{\Psi}$ gets fixed upon requiring integrability on the equation for $\overset{2}{\Psi}$. However, we do not need such terms for the present discussion.

where we used that $\bar{\mathbf{u}}_s \partial_a \slashed{q} \mathbf{u}_{s'} = 0$ (which ensures j_a is independent of the $\ker(\slashed{q})$ indeterminacy in $\overset{1}{\Psi}$), together with the fact that the only nonzero components of $\bar{\mathbf{u}}_s \partial_a \slashed{q} \partial_b \mathbf{u}_{s'}$ are

$$\bar{\mathbf{u}}_+ \partial_{\bar{z}} \slashed{q} \partial_z \mathbf{u}_+ = \bar{\mathbf{u}}_- \partial_z \slashed{q} \partial_{\bar{z}} \mathbf{u}_- = 1. \tag{C17}$$

Defining $\delta_Y^{\text{LPS}} = \{\cdot, Q_Y^{\text{LPS}}\}$ with the PBs (2.21) one obtains

$$\begin{aligned}
\delta_Y^{\text{LPS}} \psi_+ &= ie \left(Y^a \partial_a + u \partial_a Y^a \partial_u + \partial_z Y^z \right) \partial_u^{-1} \psi_+, \\
\delta_Y^{\text{LPS}} \psi_- &= ie \left(Y^a \partial_a + u \partial_a Y^a \partial_u + \partial_{\bar{z}} Y^{\bar{z}} \right) \partial_u^{-1} \psi_-, \\
\delta_Y^{\text{LPS}} \dot{A}_z &= -\partial_z^2 Y^z.
\end{aligned} \tag{C18}$$

2. Naive commutator of two fermionic generators

In this section we evaluate the (anti) commutator between two fermionic variations δ_χ defined in (4.2). As discussed there, one should not take this result too seriously, since the expressions do not incorporate "corner" fermionic degrees of freedom that are presumably needed to have a well defined phase space action.

To facilitate the computation, let us introduce the notation

$$\chi_+ = \chi_{z-}, \quad \chi_- = \chi_{\bar{z}+} \tag{C19}$$

so that (4.2) can be written more compactly as,

$$\begin{aligned}
\delta_\chi \psi_s &= D_s \chi_{-s}, \\
\delta_\chi \dot{A}_s &= \frac{e}{2} \left(\bar{\chi}_{-s} \psi_s - \chi_s \bar{\psi}_{-s} \right).
\end{aligned} \tag{C20}$$

We focus on the anticommutator $[\delta_\chi, \delta_\chi] = 2\delta_\chi \delta_\chi$, since the general case can be obtained from this basic one. Acting twice with (C20) one finds,

$$\begin{aligned}
[\delta_\chi, \delta_\chi] \psi_+ &= -ie^2 \left(\bar{\chi}_- \chi_- \partial_u^{-1} \psi_+ - \chi_+ \chi_- \partial_u^{-1} \bar{\psi}_- \right) \\
[\delta_\chi, \delta_\chi] \dot{A}_z &= -e \left(D_z \bar{\chi}_+ \chi_+ + \bar{\chi}_- D_z \chi_- \right)
\end{aligned} \tag{C21}$$

(the corresponding expressions for ψ_- and $\dot{A}_{\bar{z}}$ can be obtained from (C21) by interchanging $+ \leftrightarrow -$ and $z \leftrightarrow \bar{z}$). The result bears certain resemblance with δ_Y^{LPS} (C18) if

$$Y^z \sim e \partial_z^{-1} \bar{\chi}_s \chi_s. \tag{C22}$$

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