Radiative phase space extensions at all orders in r for self-dual Yang-Mills and gravity

Silvia Nagy[a,1] **and Javier Peraza**[b]

[a]*Department of Mathematical Sciences, Durham University,*
Stockton Road, DH1 3LE, Durham, U.K.

[b]*Facultad de Ciencias, Universidad de la República,*
Igúa 4225, Montevideo, Uruguay

E-mail: silvia.nagy@durham.ac.uk, jperaza@cmat.edu.uy

Abstract: Working in the self-dual sector for Yang-Mills and gravity, we show how to construct an extended phase space at null infinity, to all orders in the radial expansion. This formalises the symmetry origin of the infrared behaviour of these theories to all subleading orders. As a corollary, we also derive a double copy mapping from a subset of YM gauge transformations to a subset of diffeomorphisms to all orders in the transformation parameters, which to our knowledge has not been presented before in the literature.

---

[1]Corresponding author.

Contents

1 Introduction

Symmetries for the $S-$matrix have been proven to be a richer family than originally expected. In the seminal paper [1], the authors showed that asymptotic symmetries provide a Ward identity for the $S-$matrix via the soft photon theorem, later also proved in other contexts [2, 3]. This equivalence is expected to continue to be valid between sub^n-leading theorems[1] and new asymptotic symmetries with more general fall offs. These symmetries are not accommodated by the standard symplectic structure of the radiative phase space. In order to put this equivalence on a more solid footing, we are then required to construct an extended phase space with a well-defined variational algebra.

For gravity, in [4] it was suggested that the extension of the BMS group should be generalized by taking into account the group $\text{Diff}(S^2)$ for super-rotations, which is the so-called generalised BMS group (GBMS). In [5], one of the authors constructed charges for GBMS and defined a phase space, such that in this new phase space GBMS acts canonically. The extension is done by considering an extra field supported in the corners of null infinity.[2] Later, in [6] the same strategy was used to extend the radiative phase space of Yang-Mills theory, obtaining a closed algebra up to linear level in the extension. The charges used come from the leading and subleading charges from the soft theorems ([7, 8]) and the phase space was computed so that the charges act canonically. An alternative approach, the corner symmetry construction, was presented originally in [9] and later explored in e.g. [10–18]. In contrast with the previous approach, gauge symmetries with any fall off are required to have a vanishing associated charge (pure gauge). The non-trivial charges now come from the corner symmetry.

In the Yang-Mills case, as well as in gravity, it is natural to consider the question of whether it is possible to extend the phase space to account for all the symmetries coming form the sub^n-leading/Ward identity equivalences. These symmetries will correspond to symmetry parameters with increasingly divergent behaviour at null infinity coming form the radial expansion. The present work aims to answer this question, at least in a formal approach, by restricting to the self-dual sector (SD) of the theory.

On the other hand, there is a proposed connection between gravity and (two copies of) YM theory, which goes under the name of the double copy.[3] In the self-dual sector this takes a particularly elegant form [22–33]. In the search for a more general double copy map, a valuable criterion is the robustness under local transformations, i.e. the requirement that diffeomorphism transformations in gravity follow from gauge transformations in YM [34–45]. In [24] one of the authors constructed the first non-perturbative such map for a subset of the local symmetries, looking at *infinitessimal* transformations. Here, we will be able to extend this to *finite* transformations. We note that this does not follow immediately from the infinitesimal statement, as the exponentiation is different for the

---

[1]The term sub^n-leading refers to the energy expansion for the factor that relates the amplitude with a soft gauge boson with the same amplitude without the gauge boson.

[2]Here *corner* denotes the boundary of the hypersurface on which the symplectic structure is defined.

[3]See [19–21] for some reviews.

gauge field and the graviton. We will also define the map at null infinity,[4] extending the results in [24] to all orders in r.

The paper is structured as follows: in section 2 we give an overview of the self-dual sectors of gauge theory and gravity in the light-cone formulation. In section 3 we review the linear phase space extension of [6], adapted to the self-dual sector. In section 4 we give a general procedure for constructing this to all orders in the expansion coordinate for YM, which we adapt in section 5 to gravity as well. As a by-product this gives us a double copy for finite local transformations. In section 6 we take our results to null infinity. In section 7 we comment on the connection to the infinite towers of symmetries present in the self-dual sectors, the various links between them, and their manifestation at null infinity. We conclude in section 8.

2 Light-cone gauge in the self-dual sectors of YM and gravity

We will follow the conventions of [24] and use light-cone coordinates $(U, V, Z, \bar{Z})$, related to Cartesian coordinates X^μ via:

$$U = \frac{X^0 - X^3}{\sqrt{2}}, \quad V = \frac{X^0 + X^3}{\sqrt{2}}, \quad Z = \frac{X^1 + iX^2}{\sqrt{2}}, \quad \bar{Z} = \frac{X^1 - iX^2}{\sqrt{2}}. \qquad (2.1)$$

It is useful to introduce the notation:

$$x^i := (U, \bar{Z}), \quad y^\alpha := (V, Z). \qquad (2.2)$$

which splits space-time into two $2d$ subspaces. The Minkowski metric is then

$$ds^2 = 2\eta_{i\alpha} dx^i dy^\alpha = -2dUdV + 2dZd\bar{Z}. \qquad (2.3)$$

and we introduce the anti-symmetric "area elements" of the $2d$ subspaces:

$$\Omega_{ij} dx^i \wedge dx^j = dU \wedge d\bar{Z} - d\bar{Z} \wedge dU \qquad (2.4)$$

$$\Pi_{\alpha\beta} dy^\alpha \wedge dy^\beta = dV \wedge dZ - dZ \wedge dV \qquad (2.5)$$

which act as inverses of each other:

$$\Omega_i{}^\alpha \Pi_\alpha{}^j = \delta_i^j \qquad (2.6)$$

$$\Pi_\alpha{}^i \Omega_i{}^\beta = \delta_\alpha^\beta. \qquad (2.7)$$

2.1 Self-dual Yang-Mills

Consider a YM field $\mathcal{A}_\mu$ with field strength

$$F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu], \qquad (2.8)$$

The dual is defined as

$$\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma} F_{\rho\sigma}. \qquad (2.9)$$

---

[4]For other formulations of the double copy at null infinity, including relations to soft theorems, celestial holography and solution-generating transformations see e.g. [46–68].

The field strength is self-dual provided

$$\tilde{F}_{\mu\nu} = iF_{\mu\nu}. \tag{2.10}$$

Then, working in the light-cone gauge $\mathcal{A}_U = 0$, the self-dual sector can be described via (see [22, 24, 69–71])

$$\mathcal{A}_i = 0, \quad \mathcal{A}_\alpha = \Pi_\alpha{}^i \partial_i \Phi \tag{2.11}$$

with $\Pi_{\alpha\beta}$ as in (2.5) and where the Lie algebra valued scalar satisfies

$$\Box \Phi = -i\Pi^{ij}[\partial_i \Phi, \partial_j \Phi] \tag{2.12}$$

It is easy to see that the Lorenz gauge is satisfied automatically

$$\partial^\mu \mathcal{A}_\mu = \eta^{i\alpha}\partial_i \mathcal{A}_\alpha = \Pi^{ij}\partial_i\partial_j\Phi = 0 \tag{2.13}$$

making use of the anti-symmetry of Π. It also follows that

$$D^\mu \mathcal{A}_\mu = 0, \qquad \text{with} \quad D_\mu \mathcal{A}_\nu = \partial_\mu \mathcal{A}_\nu + i[\mathcal{A}_\nu, \mathcal{A}_\mu]. \tag{2.14}$$

In this paper, we will be be considering the family of gauge transformations

$$\delta_\Lambda \mathcal{A}_\mu = D_\mu \Lambda = \partial_\mu \Lambda + i[\Lambda, \mathcal{A}_\mu], \tag{2.15}$$

which preserve the light-cone gauge condition (2.11).[5] It is easy to see that these are parametrised by

$$\partial_i \Lambda = 0 \implies \Lambda = \Lambda(y) \tag{2.16}$$

We note that this subset of the gauge transformations also automatically preserves the gauge condition (2.14). This is one of the crucial simplifications that follow from starting with the more restrictive light-cone gauge, instead of (2.14). Had we taken (2.14) as our starting point and attempted to solve for Λ (for example as a perturbation series in the coordinate V), we would obtain a field-dependent expression for Λ (see [6] for details), which would have significantly increased the difficulty of calculations in subsequent sections. We leave this extension for future work.

2.1.1 Phase space

In this section, we construct a phase space which will later be extended, first formally in section 4 and section 5. Then, in section 6 we will see the implications of such extension in the radiative case, when taking the limit to $\mathcal{I}$.

First, we should restrict to the subset of relevant solutions. This step is in general clear when dealing with boundaries: we take solutions that satisfy certain boundary conditions. In the case of radiative fields, the condition is a sufficiently strong decay of the fields, such that the energy reaching infinity is finite.

---

[5]We are working at linear order in the gauge parameter for now, but we will be extending to all orders in section 4.

In our case, we take the coordinate V as the expansion parameter, and impose some conditions on the fields which will restrict the phase space. Consider the following expansion for the scalar field,[6]

$$\Phi = \Phi^{(0)}(U, Z, \bar{Z}) + \frac{\Phi^{(-1)}(U, Z, \bar{Z})}{V} + \frac{\Phi^{(-2)}(U, Z, \bar{Z})}{V^2} + \ldots, \qquad (2.17)$$

where the first term is a general term coming from the formula relating $\mathcal{A}$ with Φ, (2.11).

We impose the self-duality condition on Φ,[7] given by (2.12), which, using (2.3) and (2.4) can be written as

$$2(\partial_U \partial_V - \partial_Z \partial_{\bar{Z}})\Phi = -2i[\partial_U \Phi, \partial_{\bar{Z}} \Phi], \qquad (2.18)$$

from which we obtain a sequence of equations for each term in the expansion (following from (2.17) and (2.18)),

$$\partial_Z \partial_{\bar{Z}} \Phi^{(0)} = i[\partial_U \Phi^{(0)}, \partial_{\bar{Z}} \Phi^{(0)}], \qquad (2.19)$$

$$\partial_Z \partial_{\bar{Z}} \Phi^{(-1)} = i[\partial_U \Phi^{(-1)}, \partial_{\bar{Z}} \Phi^{(0)}] + i[\partial_U \Phi^{(0)}, \partial_{\bar{Z}} \Phi^{(-1)}], \qquad (2.20)$$

$$\partial_Z \partial_{\bar{Z}} \Phi^{(-2)} = -\partial_U \Phi^{(-1)} + i[\partial_U \Phi^{(-1)}, \partial_{\bar{Z}} \Phi^{(-1)}] + i[\partial_U \Phi^{(0)}, \partial_{\bar{Z}} \Phi^{(-2)}]$$
$$+ i[\partial_U \Phi^{(-2)}, \partial_{\bar{Z}} \Phi^{(0)}], \qquad (2.21)$$

$$\ldots \qquad (2.22)$$

This set of equations indicates that $\Phi^{(0)}$ can be a candidate for initial data if we take it to be holomorphic,

$$\Phi^{(0)} = \Phi^{(0)}(U, Z), \qquad (2.23)$$

while the coefficients $\Phi^{(-1)}, \Phi^{(-2)}, \ldots$ remain to be solved, using (2.20), (2.21), (2.22).

From the fall-off (2.17), together with (2.11) we have

$$\mathcal{A}_Z = \partial_U \Phi = \mathcal{A}_Z^{(0)} + \frac{\mathcal{A}_Z^{(-1)}}{V} + \frac{\mathcal{A}_Z^{(-2)}}{V^2} + \ldots \qquad (2.24)$$

$$\mathcal{A}_V = \partial_{\bar{Z}} \Phi = \frac{\mathcal{A}_V^{(-1)}}{V} + \frac{\mathcal{A}_V^{(-2)}}{V^2} + \ldots \qquad (2.25)$$

where the coefficients in the expansions are functions of $(U, Z, \bar{Z})$. Finally, from (2.18) we have several restrictions on the form of the functions in the expansions. We will define our *initial phase space* as the set,

$$\Gamma_{\text{YM}}^0 = \{\Phi^{(0)} \mid \mathcal{A}_i = 0, \mathcal{A}_\alpha = \Pi_\alpha{}^i \partial_i \Phi, \text{ with } \Phi^{(0)} \text{ as initial data for } \Phi.\}. \qquad (2.26)$$

---

[6]For simplicity, in this work we do not include logarithmic terms - in our set-up, this is a consistent choice. However, they might appear in more general set-ups, such as in [6], where they are required by the consistency of the equations.

[7]Since we are working in the SD sector, we are identifying the Bianchi identity with the equation of motion. Therefore, if we have a potential for $\mathcal{A}$, the equations of motion are trivially satisfied.

Next, we study the space of variations and its action on Γ_{YM}^0. Take Λ to be the parameter of gauge transformations. Its V expansion is

$$\Lambda = \sum_{n=-\infty}^{+\infty} \Lambda^{(n)}(Z) V^n, \tag{2.27}$$

where $\Lambda^{(n)} = \Lambda^{(n)}(Z)$, since the variations must be gauge preserving (see (2.16)). To ensure convergence, for the rest of the paper we will assume that only finitely many of the terms with power $n > 0$ are non-zero. Then the variation of $\mathcal{A}_Z$ and $\mathcal{A}_V$ is

$$\sum_{n=-\infty}^{\infty} \delta_\Lambda \mathcal{A}_Z^{(n)} V^n = \sum_{n=-\infty}^{+\infty} \left(\partial_Z \Lambda^{(n)} - i \sum_{k=0}^{\infty} \left[\mathcal{A}_Z^{(n-k)}, \Lambda^{(k)} \right] \right) V^n \tag{2.28}$$

$$\sum_{n=-\infty}^{\infty} \delta_\Lambda \mathcal{A}_V^{(n)} V^n = \sum_{n=-\infty}^{+\infty} \left(n \Lambda^{(n)} V^{-1} - i \sum_{k=0}^{\infty} \left[\mathcal{A}_V^{(n-k)}, \Lambda^{(k)} \right] \right) V^n \tag{2.29}$$

Therefore, if we want the gauge transformations to preserve some chosen decay in the fields, we must impose certain conditions on Λ. The fall-offs for $\mathcal{A}_Z$ and $\mathcal{A}_V$ given in (2.24) and (2.25) are simultaneously preserved by any Λ of the form,

$$\Lambda = \sum_{n=-\infty}^{0} \Lambda^{(n)} V^n, \tag{2.30}$$

where for $\Lambda^{(0)} \neq 0$ corresponds to the large gauge transformations, [1]. Unlike in the general case, [6], the coefficients $\Lambda^{(n)}$ are field-independent. The action of Λ on the scalar is given by [24]

$$\delta_\Lambda \Phi = \Omega_i{}^\alpha x^i \partial_\alpha \Lambda + i[\Lambda, \Phi], \tag{2.31}$$

which can be seen to be compatible with the fall off, if Λ is of the form (2.30).

2.2 Self-dual gravity

Next we turn our attention to gravity. Here, we define the dual of the Riemann tensor as

$$\tilde{R}_{\mu\nu\rho}{}^\sigma := \frac{1}{2} \epsilon_{\mu\nu}{}^{\eta\lambda} R_{\eta\lambda\rho}{}^\sigma \tag{2.32}$$

The self-duality condition is then

$$\tilde{R}_{\mu\nu\rho}{}^\sigma = i R_{\mu\nu\rho}{}^\sigma. \tag{2.33}$$

It is convenient to split the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.34}$$

We emphasize that, despite the notation, we are considering the full non-perturbative metric here. In the light-cone gauge self-dual gravity is then described via

$$h_{i\mu} = 0, \quad h_{\alpha\beta} = \Pi_\alpha{}^i \Pi_\beta{}^j \partial_i \partial_j \phi, \tag{2.35}$$

with the scalar ϕ satisfying

$$\Box\phi = \frac{1}{2}\Pi^{ij}\Pi^{kl}\partial_i\partial_k\phi\partial_j\partial_l\phi = \frac{1}{2}\Pi^{ij}\{\partial_i\phi, \partial_j\phi\}, \tag{2.36}$$

where we introduced the Poisson bracket

$$\{f, g\} := \Pi^{ij}\partial_i f \partial_j g. \tag{2.37}$$

The equation of motion for ϕ (2.36) is derived in appendix A of [24] in this notation.

The graviton automatically satisfies a transverse-traceless gauge condition

$$\partial^\mu h_{\mu\nu} = 0, \quad \eta^{\mu\nu}h_{\mu\nu} = 0 \tag{2.38}$$

We are interested in general coordinate transformations which preserve (2.35). At linear order in the parameter, these will be a subset of the diffeomorphism transformations

$$\delta_\xi h_{\mu\nu} := \mathcal{L}_\xi \eta_{\mu\nu} + \mathcal{L}_\xi h_{\mu\nu} \tag{2.39}$$

parametrised by

$$\xi_i = 0, \quad \xi_\alpha = b_\alpha(y) = \partial_\alpha b(y). \tag{2.40}$$

where the final equality ensures that it will also be a symmetry of equation (2.36).We remark that these diffeomorphisms automatically preserve the transverse-traceless gauge condition (2.38). We note that, unlike in the YM case, this is not the most general symmetry that preserves (2.35). These exists a second family of diffeomorphisms, which can be obtained recursively from (2.40) (see [24]), and is in fact part of an infinite tower of symmetries arising as a consequence of the integrability of the self-dual sector. We will revisit these in section 7.

2.2.1 Phase space

We take the same fall-off as (2.17) for the gravity scalar,

$$\phi = \phi^{(0)}(U, Z, \bar{Z}) + \frac{\phi^{(-1)}(U, Z, \bar{Z})}{V} + \dots, \tag{2.41}$$

which has to satisfy (from (2.36)):

$$-2(\partial_U\partial_V - \partial_Z\partial_{\bar{Z}})\phi = \frac{1}{2}\Pi^{ij}\Pi^{kl}\partial_k\partial_i\phi\partial_l\partial_j\phi. \tag{2.42}$$

For $\phi^{(0)}$ this reads

$$\partial_Z\partial_{\bar{Z}}\phi^{(0)} = \frac{1}{4}\Pi^{ij}\Pi^{kl}\partial_k\partial_i\phi^{(0)}\partial_l\partial_j\phi^{(0)}. \tag{2.43}$$

Equation (2.43) is trivially satisfied if we assume that $\phi^{(0)}$ is holomorphic, and we can take it as a candidate for the initial data, with the other components calculated from (2.42). Therefore, we define the initial phase space as,

$$\Gamma^0_{\text{grav}} = \{\phi^{(0)} \mid h_{\alpha\beta} = \Pi_\alpha{}^i\Pi_\beta{}^j\partial_i\partial_j\phi, \text{ with } \phi^{(0)} \text{ the initial data for } \phi\}. \tag{2.44}$$

The diffeomorphism with parameter (2.40) will act on the scalar ϕ as

$$\delta_b\phi = \Omega_i{}^\alpha \Omega_j{}^\beta x^i x^j \partial_\alpha \partial_\beta b + \eta^{i\alpha} \partial_\alpha b \partial_i \phi, \tag{2.45}$$

By defining the "Hamiltonian" of the vector field ξ as [24],

$$\lambda := 2\Omega_i{}^\alpha x^i \xi_\alpha = 2\Omega_i{}^\alpha x^i \partial_\alpha b(y), \tag{2.46}$$

we can write the variation of ϕ as

$$\delta_b\phi = \frac{1}{2}\Omega_i{}^\alpha x^i \partial_\alpha \lambda - \frac{1}{2}\{\lambda, \phi\}. \tag{2.47}$$

We note the useful identity

$$\xi^i \partial_i = -\frac{1}{2}\{\lambda(\xi), \cdot\} \tag{2.48}$$

when ξ is defined as in (2.40). The fall-off of the diffeomorphism has to be consistent with the definition of $\Gamma^{(0)}_{\mathrm{grav}}$, as in the SDYM case above. It can be shown that the diffeomorphism generator is given by

$$\lambda = \sum_{n=-\infty}^{0} V^n \lambda^{(n)}. \tag{2.49}$$

2.2.2 Double copy

The similarities between the scalar equations (2.12) and (2.36) suggest simple double-copy replacement rules [22]:

$$\Phi \to \phi, \qquad -i[\,,\,] \to \frac{1}{2}\{\,,\,\} \tag{2.50}$$

and the fields $\mathcal{A}_\alpha$ and $h_{\alpha\beta}$ are also directly mapped via the operator $\Pi_\beta{}^j \partial_j$. The residual symmetry transformations also obey a double copy relation. The graviton transformation can be written in terms of the Hamiltonian (2.46) as

$$\delta_\lambda h_{\alpha\beta} = \Pi_{(\alpha}{}^i \partial_i \left(\partial_{\beta)}\lambda - \frac{1}{2}\Pi_{\beta)}{}^j \partial_j \{\lambda, \phi\}\right), \tag{2.51}$$

This is immediately obtained from the YM transformation, written below in terms of the scalar Φ

$$\delta_\Lambda \mathcal{A}_\alpha = \partial_\alpha \Lambda + i\Pi_\alpha{}^i \partial_i [\Lambda, \Phi] \tag{2.52}$$

by making use of the double copy rules (2.50), supplemented by

$$\Lambda \to \lambda \tag{2.53}$$

In the next few sections, we will see that these double copy relations continue to hold for the extended phase spaces that we are constructing.

3 Linear extended phase space

3.1 Yang-Mills

Here we review the linear extended phase space construction of [6], specialising to the subset of symmetries in the self-dual sector which we described in section 2. In [6], one of the authors implemented the null infinity limit in Bondi retarded coordinates, $(\bar{r}, \bar{u}, z, \bar{z})$, and took the order $\bar{r}^0$ and $\bar{r}^1$ large gauge transformations (LGT) in a linearly extended phase space $\Gamma_{\rm YM}^1$.

In our case, we will initially work in light-cone coordinates and consider the fall-off in the coordinate V. As we will see in section 6, this expansion is analogous to the fall-off in r when going to null infinity, see (6.1). As explained in section 1, we want to allow for a gauge generator with arbitrary higher powers of V in the fall-off,

$$\Lambda = \sum_{n=-\infty}^{+\infty} V^n \Lambda^{(n)}, \tag{3.1}$$

where $\Lambda^{(n)} = \Lambda^{(n)}(Z)$. This of course violates the fall-offs in (2.24) and (2.25). The first step in resolving this issue was taken in [6], where a linear extension of the field was given. To show how one arrives at this result, we first attempt to define the extended phase space candidate to all orders

$$\Gamma_{\rm YM}^{\rm ext} := \Gamma_{\rm YM}^0 \times \left\{ \Psi \;\middle|\; \quad \Psi = \sum_{n=1}^{+\infty} V^n \Psi^{(n)}(Z) \right\}, \tag{3.2}$$

with $\Gamma_{\rm YM}^0$ as in (2.26), and each pair $(\mathcal{A}_\alpha(\Phi^{(0)}), \Psi)$ in the Cartesian product above defines a *new* vector potential in the linear extension,

$$\tilde{\mathcal{A}}_\alpha := \mathcal{A}_\alpha + D_\alpha \Psi \tag{3.3}$$

with $\mathcal{A}_\alpha$ as in (2.11). In this way, we are splitting the original full phase space in the bulk (where $\mathcal{A}_\alpha$ has any arbitrary V^n order), into two contributions, the $\Gamma_{\rm YM}^0$ part plus a LGT generated part.[8] We will show below that (3.3) forces us to set $\Psi^{(n)}(Z) = 0$ for $n \geq 2$.

We consider any LGT, as in (3.1), which can be decomposed as,

$$\Lambda = \sum_{n=-\infty}^{0} V^n \Lambda^{(n)} + \sum_{n=1}^{+\infty} V^n \Lambda^{(n)} =: \Lambda_- + \Lambda_+. \tag{3.4}$$

We will derive an expression for $\delta_\Lambda \Psi$ using the consistency of the gauge condition, i.e., *the extended gauge field $\tilde{\mathcal{A}}_\alpha$ transforms as a gauge field in the extended space*,

$$\delta_\Lambda \tilde{\mathcal{A}}_\alpha = \tilde{D}_\alpha \Lambda, \tag{3.5}$$

where $\tilde{D}$ is the covariant derivative associated to $\tilde{\mathcal{A}}$. This is the main identity that must be preserved in any extension supported on the bulk [6].

---

[8]These types of subsystems and extensions have been introduced in [9] for the case of Yang-Mills and gravity in the presence of boundaries.

By direct computation, (3.5) gives

$$\delta_\Lambda \mathcal{A}_\alpha - i[\delta_\Lambda \mathcal{A}_\alpha, \Psi] + D_\alpha \delta_\Lambda \Psi = \partial_\alpha \Lambda - i[\mathcal{A}_\alpha, \Lambda] - i[\partial_\alpha \Psi, \Lambda] - [[\mathcal{A}_\alpha, \Psi], \Lambda]$$
$$= -i[D_\alpha \Lambda, \Psi] + D_\alpha (\Lambda - i[\Psi, \Lambda]). \tag{3.6}$$

In order to solve it, we have to prescribe the action of $\Lambda_\pm$ on $\mathcal{A}$, and deduce its action on Ψ. We present the prescription in analogy with what was done in [6].

At this stage we can focus on $A_Z^{(0)}$, which will correspond to the non-vanishing component when going to null infinity. The covariant derivative at $\mathcal{O}(V^0)$ acts as,

$$D_Z^{(0)} X = \partial_Z X - i[A_Z^{(0)}, X]. \tag{3.7}$$

We *define* the action of $\Lambda_\pm$ on $\mathcal{A}_Z$ as,

$$\delta_{\Lambda_-} \mathcal{A}_Z = D_Z \Lambda_-, \quad \delta_{\Lambda_+} \mathcal{A}_Z = 0, \tag{3.8}$$

where the variation of $\mathcal{A}_Z^{(0)}$ w.r.t. Λ can be read off as

$$\delta_\Lambda A_Z^{(0)} = \partial_Z \Lambda^{(0)} - i[\mathcal{A}_Z^{(0)}, \Lambda^{(0)}]. \tag{3.9}$$

Here the expected extra terms $i \sum_{k \geq 1} [A_Z^{(-k)}, \Lambda^{(k)}]$ do not enter the definition, due to the projection (3.8). As we will see below, we can incorporate them in the variation $\delta_\Lambda \Psi$.

From (3.6), focusing on $\alpha = Z$, splitting Λ as in (3.4), and using the definition (3.8), we arrive at

$$D_Z \delta_\Lambda \Psi = -i[D_Z \Lambda_+, \Psi] + D_Z (\Lambda_+ - i[\Psi, \Lambda]). \tag{3.10}$$

We show in subsection A.1 that the above can be solved when working to linear order in V (and in the fluctuations $\Psi^{(n)}, \Lambda_+$), to give:

$$(\delta_\Lambda \Psi)^{(1)} = \Lambda_+^{(1)} - i[\Psi^{(1)}, \Lambda_-^{(0)}] \tag{3.11}$$

We will see in section 4 that in order to accommodate a fall-off of $\mathcal{O}(V^n)$ (corresponding to $\mathcal{O}(r^n)$ at null infinity), we will need to further extend the phase space (3.2)–(3.3) to $\tilde{\mathcal{A}}_\alpha = \tilde{\mathcal{A}}_\alpha(\mathcal{A}_\alpha, \Psi, \ldots, \Psi^n)$.

For now, we have to reduce the phase space extension, and only consider Ψ fields of $\mathcal{O}(V^1)$, i.e.,

$$\Gamma_{\text{lin,YM}}^{\text{ext}} = \Gamma_{\text{YM}}^0 \times \left\{ \Psi \mid \quad \Psi = V \Psi^{(1)}(Z) \right\}, \tag{3.12}$$

and therefore the extension can only be linear. Regarding the algebra of variations, since the linearized extension is valid only up to order $O(V^1)$, we have to check the commutators of $[\delta_{\Lambda_{1,\pm}}, \delta_{\Lambda_{2,\pm}}]$ on the phase space, for Λ_+ of order V and Λ_- acting on $A_Z^{(0)}$. To simplify notation, we will call Λ_+ as Λ^1 and Λ_- as Λ^0, in analogy with [6],

$$\left[\delta_{\Lambda_1^0}, \delta_{\Lambda_2^0} \right] = -i \delta_{[\Lambda_1^0, \Lambda_2^0]}, \tag{3.13}$$

$$\left[\delta_{\Lambda_1^1}, \delta_{\Lambda_2^0} \right] = -i \delta_{[\Lambda_1^1, \Lambda_2^0]}, \tag{3.14}$$

$$\left[\delta_{\Lambda_1^1}, \delta_{\Lambda_2^1} \right] = 0, \tag{3.15}$$

where the last bracket vanishes in the linearized (in Λ^1) approximation. The commutators above are inherited from the commutators corresponding to the transformation (3.5), using (3.3), indeed

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] \tilde{\mathcal{A}}_\alpha = i\delta_{[\Lambda_1, \Lambda_2]} \tilde{\mathcal{A}}_\alpha \tag{3.16}$$

This construction is related to the introduction of the "extra" field in [9], which is a local trivialization of the bundle around the corner $S = \partial\Sigma$, where Σ is the hypersurface on which the symplectic structure is computed. This new field takes into account the dynamics of the boundary. In [9, 10, 15] there are similar formulas for the embedding map in the case of gravity.

It is of course not a coincidence that $\tilde{\mathcal{A}}_\alpha$ has the functional form of a gauge transformation of $\mathcal{A}_\alpha$, with the gauge parameter replaced by the field Ψ. This is reminiscent of the Stückelberg procedure,[9] which reinstates a broken local symmetry in the action of some field theory via the introduction of additional fields transforming non-linearly (similar to our Ψ). The Stückelberg procedure consists of performing a transformation of the non-invariant action (for example a gauge theory with a mass term), and then promoting the parameters to new fields.

If we wish to extend our phase space to $\mathcal{O}(V^n)$ (corresponding to $\mathcal{O}(\Psi^n)$), the natural guess is then that $\tilde{\mathcal{A}}_\alpha$ should functionally look like the full gauge transformation of $\mathcal{A}_\alpha$ (i.e. the exponentiated group action), truncated to order n. As we will see in section 4, this indeed turns out to be the case.

3.2 Gravity

We will proceed along the same lines as in the YM case. Completely analogous arguments to the YM case prohibit the construction of the phase space beyond $\mathcal{O}(V^1)$ in this approach, so in this section we will just look for a transformation parameter with fall-off

$$\lambda = \sum_{n=-\infty}^{1} V^n \lambda^{(n)} \tag{3.17}$$

where λ is the "Hamiltonian" (2.46) built out of the diffeomophism parameter ξ^μ given in (2.40). This will violate the fall-off in (2.41). To rectify this, we construct the linear extension of the phase space as

$$\Gamma_{\text{grav}}^{\text{ext}} := \Gamma_{\text{grav}}^0 \times \left\{ \psi \mid \quad \psi = V\psi^{(1)}(Z) \right\}, \tag{3.18}$$

with Γ_{grav}^0 as in (2.44) and each pair $(h_{\alpha\beta}, \psi)$ in the Cartesian product above defining a *new* graviton in the linear extension:

$$\tilde{h}_{\alpha\beta} = h_{\alpha\beta} + \Pi_{(\alpha}^i \partial_i \partial_{\beta)} \psi - \frac{1}{2}\{\psi, h_{\alpha\beta}\} \tag{3.19}$$

We note that $\tilde{h}_{\alpha\beta}$ is related to $h_{\alpha\beta}$ via a diffeomorphism transformation (see subsection B.1), where the "Hamiltonian" transformation parameter λ has been replaced by

---

[9]Initially introduced in the context of electromagnetism [72], also extended to (super)gravity [73, 74].

ψ. This is of course in complete analogy with the YM construction in subsection 3.1. The form of ψ is then inherited from that of λ (2.46), and we have

$$\psi^{(1)}(x^i, Z) = 2\Omega_i{}^\alpha x^i \partial_\alpha \beta^{(1)}, \tag{3.20}$$

with $\beta^{(1)} = \beta^{(1)}(Z)$. The consistency condition for the gravity extended phase space is[10]

$$\delta_\lambda \tilde{h}_{\alpha\beta} = \mathcal{L}_{\lambda(\xi)} \tilde{g}_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2} \{\lambda, \tilde{h}_{\alpha\beta}\}, \tag{3.21}$$

i.e. we want $\tilde{h}$ to transform as a graviton in the extended space. We will now focus on the $\alpha\beta = ZZ$ component of the equation above (this will give the asymptotic shear C_{zz} which captures the free data at null infinity). Then, working perturbatively in V and, as in the YM case, requiring that $\lambda^{(1)}$ does not transform h_{ZZ}, we have

$$\delta_\lambda h^0_{ZZ} = \Pi^i_Z \partial_i \partial_Z \lambda^{(0)} - \frac{1}{2} \{\lambda^{(0)}, h^{(0)}_{ZZ}\} \tag{3.22}$$

and

$$\delta_\lambda \psi^{(1)} = \lambda^{(1)} + \frac{1}{2} \{\psi^{(1)}, \lambda^{(0)}\}. \tag{3.23}$$

Finally, the linearised commutators (3.13)–(3.15) will now have completely analogous forms in gravity in terms of the Poisson brackets.

As mentioned above, higher orders in V will not work in this approach, so one has to extend the phase space to higher orders in ψ. We will do this by exponentiating the action of the Lie derivative in section 5.

Double copy relations. The double copy for the self-dual fields and their symmetry transformations has already been established [22, 24], as described towards the end of section 2. It remains to extend it to the new scalars Ψ and ψ, and their symmetry transformations (3.11) and (3.23). This is very straightforward, as we notice that the double copy replacement rules (2.50) and (2.53) simply need to be supplemented with

$$\Psi^{(1)} \to \psi^{(1)} \tag{3.24}$$

For now, we are only able to map the coefficients of V^1 in the respective expansions of Ψ and ψ. We will extend this to all orders in subsection 5.1.

4 Yang-Mills extension to all orders

We propose an extension of (3.12) and (3.3) to all orders in Ψ, given by

$$\Gamma^{\text{ext}}_{\infty,\text{YM}} := \Gamma^0_{\text{YM}} \times \{\Psi \mid \Psi = \sum_{n=1}^{+\infty} V^n \Psi^{(n)}(Z)\}. \tag{4.1}$$

such that

$$\hat{\mathcal{A}}_\alpha = e^{i\Psi} \mathcal{A}_\alpha e^{-i\Psi} + i e^{i\Psi} \partial_\alpha e^{-i\Psi}, \tag{4.2}$$

---

[10]Following the same logic as in subsection B.1, with $h_{\alpha\beta} \to \tilde{h}_{\alpha\beta}$.

where $\mathcal{A}_\alpha \in \Gamma^0_{\text{YM}}$, and the coefficients $\{\Psi^{(n)}(Z)\}_{n \geq 1}$ are taken such that only finitely many of them are non-zero. This is the natural generalisation of (3.3), as it takes the functional form of a full gauge transformation, with the parameter replaced by the field Ψ.

The consistency condition (3.5) now becomes

$$\delta_\Lambda \hat{\mathcal{A}}_\alpha = \hat{D}_\alpha \Lambda. \tag{4.3}$$

Note that it is still sufficient to work to linear order in the transformation parameter Λ.[11] Then, making use of

$$\delta e^X = e^X \mathcal{O}_X(\delta X) \tag{4.4}$$

with

$$\mathcal{O}_X := \frac{1 - e^{-ad_X}}{ad_X} = \sum_{k=0}^\infty \frac{(-1)^k}{(k+1)!} \left(ad_X\right)^k, \tag{4.5}$$

where $ad_X(Y) = [X, Y]$, we can rewrite (4.3) as (see subsection A.2 for details)

$$e^{i\Psi} \left[\delta A_\alpha + D_\alpha \left(e^{-i\Psi} \mathcal{O}_{-i\Psi}(\delta_\Lambda \Psi) e^{i\Psi} \right) \right] e^{-i\Psi} = e^{i\Psi} \left[D_\alpha(e^{-i\Psi} \Lambda e^{i\Psi}) \right] e^{-i\Psi}. \tag{4.6}$$

This equation, as it stands, is valid everywhere. As in subsection 3.1, we need to write (4.6) in the phase space variables, and prescribe the action of the variation on $A_Z^{(0)}$. We will define the same transformation rule for $A_Z^{(0)}$ as in equation (3.9):

$$\delta_\Lambda A_Z^{(0)} = D_Z^{(0)} \Lambda^{(0)} \tag{4.7}$$

where the expansion of Λ is given in (3.1). The interesting part follows when considering the variation of Ψ, which turns out to satisfy a remarkably simple constraint, at all orders in V, as can be read off directly from (4.6), upon substracting (4.7):

$$\mathcal{O}_{-i\Psi}(\delta_\Lambda \Psi) = \Lambda - e^{i\Psi} \Lambda^{(0)} e^{-i\Psi}. \tag{4.8}$$

Formally, we can write

$$[\delta_\Lambda \Psi]^{(n)} = \left(\mathcal{O}_{-i\Psi}^{-1}(\Lambda - e^{i\Psi} \Lambda^{(0)} e^{-i\Psi}) \right)^{(n)}. \tag{4.9}$$

where $^{(n)}$ denotes the coefficient of V^n in the V-expansion. We show below that the operator $\mathcal{O}$ is indeed invertible, working order by order in Ψ. Let $\delta^{[m]} X$ denote the variation of X at order m in Ψ, and $T^{[m]}$ the component of order Ψ^m in the expression T.[12] Then, in equation (4.9) we have two expansions, one in V and on in Ψ, where each equation reads,

$$\left[\delta_\Lambda^{[m]} \Psi \right]^{(n)} = \left(\mathcal{O}_{-i\Psi}^{-1}(\Lambda - e^{i\Psi} \Lambda^{(0)} e^{-i\Psi}) \right)^{[m],(n)}. \tag{4.10}$$

First, observe that since Ψ is of orders V^1 and higher, then the equations in (4.10) are meaningful only when $n \geq m$, since a Ψ^m term starts at order V^m. Then, we have a

---

[11]In principle, we could consider these transformations to all orders in Λ, however this is not necessary for applications to soft theorems.

[12]This is to distinguish it from the expansion in V, which we have denoted by $T^{(n)}$.

natural "induction step" to solve the inverse operator both in V and in Ψ. By the definition of $\mathcal{O}_{-i\Psi}$ we have

$$
\begin{aligned}
\mathcal{O}_{-i\Psi}(\delta_\Lambda \Psi) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \left(ad_{-i\Psi}\right)^k (\delta_\Lambda \Psi) \\
&= \delta_\Lambda \Psi + \frac{i}{2}[\Psi, \delta_\Lambda \Psi] - \frac{1}{6}[\Psi, [\Psi, \delta_\Lambda \Psi]] - \frac{i}{24}[\Psi, [\Psi, [\Psi, \delta_\Lambda \Psi]]] + \dots,
\end{aligned}
\tag{4.11}
$$

Then for $[n] = 0$

$$
\delta_\Lambda^{[0]} \Psi = \Lambda - \Lambda^{(0)}
\tag{4.12}
$$

This is true to all orders in V. When restricting to linear order in V, we find agreement with the first term in (3.11), as expected, using the expansions for Λ and Ψ given in (3.1) and (4.1). For $[n] = 1$, we get

$$
\delta_\Lambda^{[1]} \Psi + \frac{i}{2}[\Psi, \delta_\Lambda^{[0]} \Psi] = -i[\Psi, \Lambda^{(0)}]
\tag{4.13}
$$

Making use of (4.12), this becomes

$$
\delta_\Lambda^{[1]} \Psi = -\frac{i}{2}[\Psi, \Lambda + \Lambda^{(0)}]
\tag{4.14}
$$

which, to linear order in V, agrees with the second term in (3.11) as expected, again using (3.1) and (3.12). We introduce the notation $\overset{n}{\delta}$ to denote the variation *up to* order n in Ψ. Then

$$
\overset{1}{\delta}\Psi := \delta_\Lambda^{[0]} \Psi + \delta_\Lambda^{[1]} \Psi = \Lambda - \Lambda^{(0)} - \frac{i}{2}[\Psi, \Lambda + \Lambda^{(0)}]
\tag{4.15}
$$

Now we can press on to higher orders. For example, at $[n] = 2$ we have

$$
\delta_\Lambda^{[2]} \Psi + \frac{i}{2}[\Psi, \delta_\Lambda^{[1]} \Psi] - \frac{1}{6}[\Psi, [\Psi, \delta_\Lambda^{[0]} \Psi]] = \frac{1}{2}[\Psi, [\Psi, \Lambda^{(0)}]]
\tag{4.16}
$$

Then, plugging in (4.12) and (4.14), we get

$$
\delta_\Lambda^{[2]} \Psi = -\frac{1}{12}[\Psi, [\Psi, \Lambda - \Lambda^{(0)}]]
\tag{4.17}
$$

This will allow us to extract the transformation of Ψ at $\mathcal{O}(V^2)$.

It turns out that we can explicitly solve (4.10) in order to give an expression for the variation of Ψ at order n. We use the expansion

$$
\frac{x}{1 - e^{-x}} = \sum_{m=0}^{\infty} \frac{B_m^+ x^m}{m!},
\tag{4.18}
$$

where B_n^+ are the Bernoulli numbers

$$
B_0^+ = 1, \ B_1^+ = \frac{1}{2}, \ B_2^+ = \frac{1}{6}, \ \dots
\tag{4.19}
$$

One can then show that (see subsection A.3 for details):

$$
\delta_\Lambda^{[m]} \Psi = \frac{B_m^+}{m!} \left(ad_{-i\Psi}\right)^m \left[\Lambda + (-1 + 2\delta_{m,1})\Lambda^{(0)}\right]
\tag{4.20}
$$

We can now use this to extract the transformation at any order in V.

5 Full gravity extension

In order to proceed to the full gravity extension, we first recall that the diffeomorphism transformation to all orders in the parameter can be written as

$$g'_{\mu\nu} = \left(e^{\mathcal{L}_\xi} g\right)_{\mu\nu} \tag{5.1}$$

with the Lie derivative

$$(\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + 2\partial_{(\mu} \xi^\rho g_{\nu)\rho} \tag{5.2}$$

However, recall that we are only interested in a subset of diffeomorphisms parametrised by (2.40). Additionally, we have found it useful to introduce the "Hamiltonian" quantity $\lambda = \lambda(\xi)$ (2.46), which allowed us to write (see subsection B.1):

$$(\mathcal{L}_\xi g)_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2} \{\lambda, h_{\alpha\beta}\} \tag{5.3}$$

Let us denote the Poisson bracket (2.37) by

$$\mathfrak{ad}_\theta := \{\theta, \cdot\} \tag{5.4}$$

Then one can show that (see subsection B.2)

$$(\mathcal{L}_\xi)^n g_{\alpha\beta} = \frac{1}{(-2)^n} \mathfrak{ad}^n_\lambda (h_{\alpha\beta}), \qquad n > 1 \tag{5.5}$$

which allows us to write the all orders transformation rule as

$$g'_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda + e^{-\frac{1}{2}\mathfrak{ad}_\lambda} g_{\alpha\beta} \tag{5.6}$$

We now define our extended phase space as

$$\Gamma^{\text{ext}}_{\infty,\text{grav}} := \Gamma^0_{\text{grav}} \times \{\psi \mid \psi = \sum_{n=1}^{+\infty} V^n \psi^{(n)}(Z)\}. \tag{5.7}$$

such that the extended graviton is defined as

$$\hat{g}_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \psi + e^{-\frac{1}{2}\mathfrak{ad}_\psi} g_{\alpha\beta} \tag{5.8}$$

where $g_{\alpha\beta} \in \Gamma^0_{\text{grav}}$ and the extended phase space fields are of the form

$$\psi^{(n)}(x^i, Z) = 2\Omega_i{}^\alpha x^i \partial_\alpha \beta^{(n)}(Z). \tag{5.9}$$

Note that the definition above still preserves the splitting (2.34) and we have

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu} \tag{5.10}$$

with $\hat{h}_{i\mu} = 0$ and

$$\hat{h}_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \psi + e^{-\frac{1}{2}\mathfrak{ad}_\psi} h_{\alpha\beta} \tag{5.11}$$

The consistency condition (3.21) is generalised to

$$\delta \hat{h}_{\alpha\beta} = \mathcal{L}_{\lambda(\xi)} \hat{g}_{\alpha\beta} = \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2} \mathfrak{ad}_\lambda \hat{h}_{\alpha\beta} \tag{5.12}$$

where, as in the YM case, we found it sufficient to work to linear order in the transformation parameter. At this stage, we could proceed as in the YM case and solve the consistency condition above in order to read off the transformation rule for ψ. However, we find it both simpler and more instructive to obtain this from the double copy rules in the next section.

5.1 Double copy for all order extensions

We will show that, just like in the case of the linear extensions, the fully extended phase spaces for YM and gravity are related by the double copy rules described at the end of section 2. To see this, we first note that, using (A.18) and (4.4) we can rewrite the extended vector potential (4.2) as

$$\hat{\mathcal{A}}_\alpha = e^{iad_\Psi} A_\alpha(\Phi) + \mathcal{O}_{-i\Psi}(\partial_\alpha\Psi), \tag{5.13}$$

We will now make use of the double copy rules in subsubsection 2.2.2. With our new notation, the second term in (2.50) becomes

$$-iad \to \frac{1}{2}\mathfrak{ad} \tag{5.14}$$

We will also need (2.53), as well as the natural extension of (3.24) to all orders in V,

$$\Psi^{(n)} \to \psi^{(n)}, \quad n \geq 1 \ . \tag{5.15}$$

We propose that the double copy of $\hat{A}_\alpha$ is[13]

$$\begin{aligned}
\hat{H}_{\alpha\beta} &= \Pi^i_{(\alpha}\partial_i \left(e^{-\frac{1}{2}\mathfrak{ad}_\psi} A_{\beta)}(\phi) + \mathcal{W}_{\frac{1}{2}\psi}(\partial_{\beta)}\psi) \right) \\
&= e^{-\frac{1}{2}\mathfrak{ad}_\psi} h_{\alpha\beta} + \Pi^i_{(\alpha}\partial_i \left(\mathcal{W}_{\frac{1}{2}\psi}(\partial_{\beta)}\psi) \right)
\end{aligned} \tag{5.16}$$

where $\mathcal{W}_{\frac{1}{2}\psi}$ is the double copy of $\mathcal{O}_{-i\Psi}$ under the rules (2.50), i.e.

$$\mathcal{O}_{-i\Psi} := \frac{1 - e^{-ad_{-i\Psi}}}{ad_{-i\Psi}} \quad \to \quad \mathcal{W}_{\frac{1}{2}\psi} := \frac{1 - e^{-\mathfrak{ad}_{\frac{1}{2}\psi}}}{\mathfrak{ad}_{\frac{1}{2}\psi}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \left(\mathfrak{ad}_{\frac{1}{2}\psi} \right)^k \tag{5.17}$$

Plugging this back into (5.16) and using the commutativity of the partial derivatives, we get

$$\begin{aligned}
\hat{H}_{\alpha\beta} &= e^{-\frac{1}{2}\mathfrak{ad}_\psi} h_{\alpha\beta} + \mathcal{W}_{\frac{1}{2}\psi}(\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\psi) \\
&= e^{-\frac{1}{2}\mathfrak{ad}_\psi} h_{\alpha\beta} + \Pi^i_{(\alpha}\partial_i\partial_{\beta)}\psi \\
&= \hat{h}_{\alpha\beta}
\end{aligned} \tag{5.18}$$

thus establishing the double copy — see (5.11)! In the above, to get to the second line, we used the fact that only the $k = 0$ term in (5.17) contributes, due to the fact that ψ is linear in x^i ((5.9)) and remembering the definition of the Poisson bracket (5.4) and (2.37).

Recall that $\hat{h}_{\alpha\beta}$ and $\hat{A}_\alpha$ formally look like diffeomorphism transformations and, respectively, gauge transformations to all orders, with parameters replaced by ψ and Ψ. We thus have a double copy for (a subset of) local symmetries to all orders in the parameters, which, to our knowledge, has not been previously presented in the literature (the results in [24], while non-perturbative in the fields, were still linear in the transformation parameters). To summarise our results in the more standard language, we have found that

$$\mathcal{A}'_\alpha = e^{i\Lambda}\mathcal{A}_\alpha e^{-i\Lambda} + ie^{i\Lambda}\partial_\alpha e^{-i\Lambda}, \tag{5.19}$$

---

[13]We have used the notation $A_\alpha(\phi) = \Pi^i_\alpha\partial_i\phi$. In order to commute $\Pi^i_{(\alpha}\partial_i$ and $e^{-\frac{1}{2}\mathfrak{ad}_\psi}$, we used the fact that ψ is linear in x^i, and hence $e^{-\frac{1}{2}\mathfrak{ad}_\psi}$ is independent of x^i.

with

$$\Lambda = \Lambda(y) \tag{5.20}$$

double copies to

$$g'_{\mu\nu} = \left(e^{\mathcal{L}_\xi} g\right)_{\mu\nu} \tag{5.21}$$

with

$$\xi_i = 0, \quad \xi_\alpha = b_\alpha(y) = \partial_\alpha b(y). \tag{5.22}$$

Finally we can also use the double copy to read off the transformation of ψ to arbitrary orders in V and ψ. Starting with (4.20) and applying the double copy rules (specifically, we need (2.53), (5.14), and (5.15)), we get

$$\delta_\lambda^{[m]} \psi = \frac{B_m^+}{m!} \left(\frac{1}{2}\mathfrak{a}\mathfrak{d}_\psi\right)^m \left[\lambda + (-1 + 2\delta_{m,1})\lambda^{(0)}\right] \tag{5.23}$$

where $[m]$ denotes the order in ψ, as before. At this point, we recall again that both λ and ψ are linear in x^i ((2.46) and (5.9)), which, in conjunction with the definition of the Poisson bracket $\mathfrak{a}\mathfrak{d}$ ((5.4) and (2.37)) implies that only the $m = 0$ and $m = 1$ terms will contribute, so we have the remarkably simple expression

$$\delta_\lambda \psi = \lambda - \lambda^{(0)} + \frac{1}{4}\mathfrak{a}\mathfrak{d}_\psi \left[\lambda + \lambda^{(0)}\right] \tag{5.24}$$

In subsection B.3 we show that this satisfies the consistency condition (5.12), as needed, thus completing the proof for the double copy relations.

6 Extension in Bondi coordinates

We will study fields near future null infinity, $\mathcal{I}^+$, so we are switching to coordinates adapted to it. The natural choice are Bondi-type coordinates $(r, u, z, \bar{z})$ given by

$$r = V, \quad z = \frac{Z}{V}, \quad \bar{z} = \frac{\bar{Z}}{V}, \quad u = U - \frac{Z\bar{Z}}{V}. \tag{6.1}$$

In these coordinates, the Minkowski metric reads

$$ds^2 = -2dudr + 2r^2 dzd\bar{z}, \tag{6.2}$$

where we see that by taking the conformally rescaled metric $\frac{1}{r^2}ds^2$, we have a well defined (degenerate) metric on $\mathcal{I}^+$ given by $2dzd\bar{z}$.[14]

The radiative data of a massless scalar field is given by

$$\varphi(r, u, z, \bar{z}) = \frac{\varphi_\mathcal{I}(u, z, \bar{z})}{r} + \mathcal{O}(r^{-2}) \tag{6.3}$$

as $r \to +\infty$.

---

[14]See e.g. section 4 of [75] for a derivation of this metric from general assumptions about asymptotic flatness. We remark that these coordinates were chosen due to the simplicity of the map to the light-cone coordinates. One could also perform the analysis in standard Bondi coordinates, resulting in slightly more complicated expressions.

6.1 Phase space for SDYM and fields near infinity

Let us use the notation

$$\mathcal{A}_z(r,u,z,\bar{z}) \overset{r\to\infty}{=} A_z(u,z,\bar{z}) + \mathcal{O}(r^{-1}) \tag{6.4}$$

Then, considering equations (2.11) in Bondi coordinates (6.1), we have

$$\mathcal{A}_z = r\partial_u\Phi, \quad \mathcal{A}_{\bar{z}} = 0, \tag{6.5}$$

The *radiative phase space* will then be given by $\Phi_\mathcal{I}$ as free data,

$$\Gamma^{\mathrm{rad}} = \{\Phi_\mathcal{I}(u,z,\bar{z}) \mid A_z = \partial_u\Phi_\mathcal{I}, \ A_{\bar{z}} = 0\}. \tag{6.6}$$

where we have used (6.3)–(6.5).

Recall that the gauge parameters depend only on y coordinates, $\Lambda(y) = \Lambda(V,Z)$, which in this case are (r,z). Therefore, we can assume

$$\Lambda = \sum_{n=-\infty}^{+\infty} \Lambda^{(n)} r^n, \tag{6.7}$$

as an $r-$expansion for the gauge parameter, with $\Lambda^{(n)} = \Lambda^{(n)}(z)$. The first residual gauge symmetry is given by $O(r^0)$,

$$\Lambda = \Lambda^{(0)}(z) + \ldots \tag{6.8}$$

The induced transformation on the phase space,

$$\delta_\Lambda\Phi_\mathcal{I} = u\partial_z\Lambda^{(0)} + i[\Lambda^{(0)}, \Phi_\mathcal{I}] \tag{6.9}$$

induces the leading asymptotic symmetry. The extended phase space is now:

$$\Gamma^{\mathrm{ext}}_{\infty,\mathrm{YM}} := \Gamma^{\mathrm{rad}}_{\mathrm{YM}} \times \{\Psi \mid \Psi = \sum_{n=1}^{+\infty} r^n \Psi^{(n)}(z)\}. \tag{6.10}$$

with the extended gauge field constructed as in (4.2), with the approprite coordinate change. The transformation of Ψ to all orders (equation (4.20)) then immediately translates to our Bondi coordinates

$$\left[\delta_\Lambda^{[m]}\Psi\right]^{(n)} = \frac{B_m^+}{m!}\left\{(ad_{-i\Psi})^m\left[\Lambda + (-1 + 2\delta_{m,1})\Lambda^{(0)}\right]\right\}^{(n)} \tag{6.11}$$

with Λ as in (6.7), Ψ as in (6.10), $[m]$ denoting the order in field fluctuations as before, and (n) now denoting the order in r.

We expect these LGTs to be associated with sub$^{(n)}$-leading charges [76, 77], whose construction we leave for future work. The linear extension was already done in [6], while it remains to be shown in general. In this work, we propose a first step towards this general result, in particular we look at how the variation algebra can be computed when considering SDYM.

Take two large gauge transformations with parameters,

$$\Lambda_1 = \sum_{n=-\infty}^{N_1} \Lambda_1^{(n)} r^n, \quad \Lambda_2 = \sum_{n=-\infty}^{N_2} \Lambda_2^{(n)} r^n, \tag{6.12}$$

then

$$[\Lambda_1, \Lambda_2] = r^{N_1+N_2}[\Lambda_1^{(N_1)}, \Lambda_2^{(N_2)}] + O(r^{N_1+N_2-1}). \tag{6.13}$$

Then, truncating $\delta_\lambda \Psi$ up to order $O(\Psi^{(N_1+N_2)})$, one can verify

$$[\overset{N_1+N_2}{\delta}_{\Lambda_1}, \overset{N_1+N_2}{\delta}_{\Lambda_2}]\Psi = -i\overset{N_1+N_2}{\delta}_{[\Lambda_1,\Lambda_2]}\Psi. \tag{6.14}$$

thus showing that this is a consistent truncation.

6.2 Phase space for SDG

For gravity, we will use the notation

$$h_{zz}(r, u, z, \bar{z}) \overset{r\to\infty}{=} rC_{zz}(u, z, \bar{z}) + \mathcal{O}(r^0) \tag{6.15}$$

where C_{zz} is the *shear tensor*. The scalar potential $\phi_\mathcal{I}$ defines a decay for h_{zz} as follows,

$$h_{zz}(r, u, z, \bar{z}) = r\partial_u^2\phi_\mathcal{I} + \mathcal{O}(r), \quad h_{\bar{z}\bar{z}} = 0 \tag{6.16}$$

The phase space is defined analogously with the YM case

$$\Gamma^{\text{rad}} = \{\phi_\mathcal{I}(u, z, \bar{z}) \mid C_{zz} = \partial_u^2\phi_\mathcal{I}, \ C_{\bar{z}\bar{z}} = 0\}. \tag{6.17}$$

where we used (6.3) and (6.15). The action of the residual symmetry (2.40) on the radiative phase space corresponds to the fall-off

$$b(r, z) = -rf(z). \tag{6.18}$$

and then

$$\delta_b\phi_\mathcal{I} = -u^2 f''(z) + f(z)\partial_u\phi_\mathcal{I} \tag{6.19}$$

corresponding to a holomorphic supertranslation with parameter $f(z)$. To allow the Hamiltonian λ given in (2.46) to go to to all orders in r, we introduce the extended phase space as in (5.7), resulting in the transformation rules (5.24), translated into Bondi coordinates.

For the family of LGT that we are taking, (2.40), we can construct directly the variation algebra from the double copy prescription given in the last section. In terms of the charges, it is not clear whether sub^n-leading soft charges correspond to higher order LGT, although it was conjectured that certain $O(r)$ diffeomorphisms could be associated to the charge corresponding to the sub-sub-leading soft graviton theorem, [78]. It would be interesting to explore further this direction.

Finally we note that the transformations discussed in this section correspond to ($\mathcal{O}(r^n)$ extensions of) supertranslations. It turns out that the self-dual sector also accommodates superrotations within its rich collection of residual symmetries. This will be further detailed in the next section.

7 Discussion: an infinite cube of symmetries

Self-dual YM and gravity, being integrable, have an in finite number of symmetries [79–85]. These can be defined recursively as follows. Assume $\delta_1\Phi$ is a symmetry of the SDYM equation (2.12), i.e.

$$\Box\delta_1\Phi = -2i\Pi^{ij}[\partial_i\Phi,\partial_j\delta_1\Phi] \tag{7.1}$$

Then we can define a new symmetry $\delta_2\Phi$ via

$$\partial_i\delta_2\Phi = \Omega_i{}^\alpha\partial_\alpha\delta_1\Phi - i[\partial_i\Phi,\delta_1\Phi] \tag{7.2}$$

Thus we can start with $\delta_1\Phi$ as defined in (2.15) and (2.16) and proceed indefinitely to construct an infinite tower of symmetries. The method works identically for gravity, and the double copy relations continue to hold at every order [24]:

$$
\begin{array}{cccc}
\textbf{gauge} & \delta_1\Phi \xrightarrow{\text{DC}} \delta_1\phi & \textbf{diffeo}_1 \\
& \downarrow \qquad \downarrow & \\
\textit{extra} & \delta_2\Phi \xrightarrow{\text{DC}} \delta_2\phi & \textbf{diffeo}_2 \\
& \downarrow \qquad \downarrow & \\
& \vdots \qquad \vdots & \\
\textit{extra} & \delta_n\Phi \xrightarrow{\text{DC}} \delta_n\phi & \textit{extra} \\
& \vdots \qquad \vdots &
\end{array}
\tag{7.3}
$$

Note that $\delta_1\Phi$ and $\delta_1\phi$ are subsets of the standard gauge and diffeomorphism transformations respectively. $\delta_2\phi$ turns out to be a different subset of diffeomorphisms [24].[15] Intriguingly, $\delta_2\Phi$, its YM origin in the double copy, *is not a subset of gauge transformations*. This might suggest that a broader framework is needed in order to recover gravitational symmetries via the double copy. Conversely, there is an interesting programme aiming to reveal the existence of a wider (perhaps infinite) set of symmetries in gravity and YM [86]. In this context, one might speculate that the *extra* symmetries in (7.3) could be revealed as genuine symmetries of the full (i.e. non-self-dual) theories.

How does this relate to the subject of the present article? To see this, we first go to null infinity, working in flat Bondi coordinates described in the previous section, to summarise the results of [24] in figure 1.

In figure 1, LGT stands for non-abelian large gauge transformations, ST and SR for supertranslations and superrotations. Horizontal lines represent the double copy relations. Vertical lines represent the symmetry raising map (7.2) between different families of symmetries. The diagonal arrows connecting different symmetries in the 2nd family denote an increase in the power of r in the "seed" symmetries belonging to family 1 that need to be plugged into (7.2) in order to generate them. For example the $\sqrt{SR}$ symmetry with

---

[15]Specifically, the transformation parameter is given by

$$\xi_i = -c_i(y), \quad \xi_\alpha = \partial_\alpha c_j(y)x^j. \tag{7.4}$$
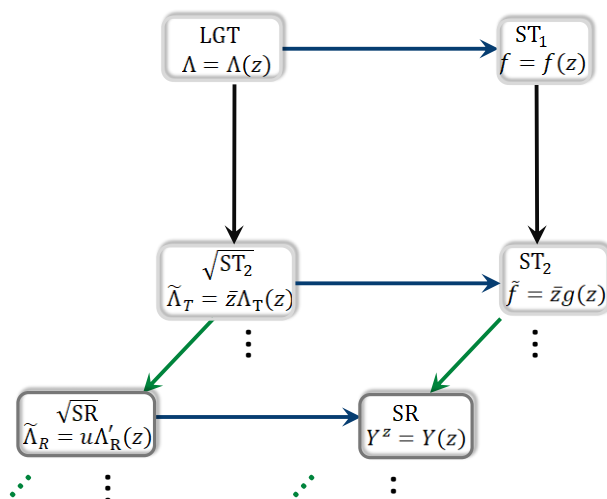
Figure 1. Asymptotic symmetries in self-dual YM and gravity, linked by the double copy and the symmetry raising map.
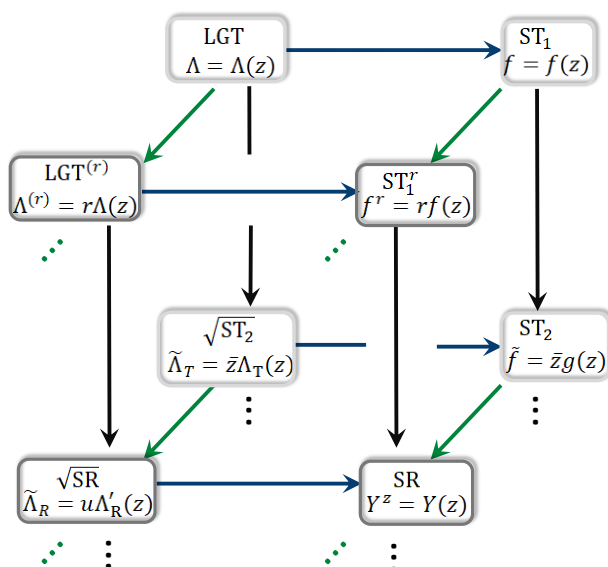
Figure 2. Asymptotic symmetries in self-dual YM and gravity, at all orders in r.

parameter $u\Lambda'_r(z)$ was obtained by plugging a family 1 gauge symmetry with parameter $r\Lambda(z)$ into the recursion relation (7.2) (see [24]). However, the transformation with parameter $r\Lambda(z)$ was not actually allowed, as it violated the fall-off! The situation is rectified in the current work, via the definition of the extended phase space, which reinterprets the violating transformation as a legitimate symmetry of the extended phase space, acting on Ψ (and analogously for gravity). This means that we are now allowed to add diagonal rungs to the figure above, corresponding to increasing powers of r, as they correspond to legitimate symmetries in our interpretation — see figure 2. This completes the construction

of the various relations between the infinite tower of symmetries in SDYM and gravity. We summarise these in figure 2 below, which is understood to extend infinitely in the vertical and diagonal directions.

8 Conclusions

In this paper we showed how to construct an extended phase space to all orders in the expansion coordinate. This is important for studying the subleading (to all orders) behaviour of soft gluons and gravitons. Our analysis was restricted to the self-dual sectors of gravity and YM. This allowed us to additionally establish a double copy map for a subset of finite local transformations in these sectors.

We recall that the self-dual sectors of gravity and YM are integrable, thus possessing infinite towers of symmetries [79–82, 87–91], defined recursively, as described in the previous section (see (7.3)). In the present work, we have restricted ourselves to only the first family of symmetries, however to all orders in r. The various interconnections between the symmetries are sketched in section 7 (see also figure 2 above). In would be interesting to connect with the celestial holography programme, where the sub^n-leading effects in amplitudes are naturally incorporated, and a similarly rich algebraic structure emerges [92–107].

On the other hand, we are looking at extending our procedure for the all-orders phase space beyond the self-dual sectors.[16] Here a natural starting point will be to work in the light-cone gauge. Even for the full theories, this is more restrictive than the Lorenz and De Donder gauge respectively - this gives a simpler set of residual symmetries at null infinity and is thus a good starting point for constructing extended phase spaces to all orders.

Acknowledgments

A YM calculations

A.1 Linear extended phase space transformation

Consider (3.10), reproduced below for convenience

$$D_\alpha \delta_\Lambda \Psi = -i[D_\alpha \Lambda_+, \Psi] + D_\alpha \left(\Lambda_+ - i[\Psi, \Lambda]\right). \tag{A.1}$$

---

[16]See [108] for a novel way to connect between the self-dual sector and the full theory, in the Hamiltonian formalism.

At order V^0, for $\alpha = Z$,

$$\partial_Z(\delta_\Lambda\Psi)^{(0)} - i\sum_{n=0}^\infty [A_Z^{(-n)}, (\delta_\Lambda\Psi)^{(n)}] = -i([\partial_Z\Lambda_+, \Psi])^{(0)} - \sum_{k=0}^\infty \left[\sum_{n=0}^\infty [A_Z^{(-n)}, \Lambda_+^{(n-k)}], \Psi^{(k)}\right]$$
$$+ \partial_Z(\Lambda_+ - i[\Psi, \Lambda])^{(0)} - i\sum_{n=0}^\infty [A_Z^{(-n)}, \Lambda_+^{(n)}]$$
$$- \sum_{k=0}^\infty\sum_{n=0}^\infty [A_Z^{(-n)}, [\Psi^{(k)}, \Lambda^{n-k}]] \tag{A.2}$$

Note that several terms in the above vanish at $\mathcal{O}(V^0)$, specifically those involving only Ψ and Λ_+, due to (3.2) and (3.4). Then,

$$-i\sum_{n=1}^\infty [A_Z^{(-n)}, (\delta_\Lambda\Psi)^{(n)}] = \sum_{k=1}^\infty\sum_{n=1}^\infty [A_Z^{(-n-k)}, \Lambda_+^{(n)}], \Psi^{(k)}]$$
$$- i\partial_Z([\Psi, \Lambda_-])^{(0)} - i\sum_{n=1}^\infty [A_Z^{(-n)}, \Lambda_+^{(n)}]$$
$$- \sum_{k=1}^\infty\sum_{n=0}^\infty [A_Z^{(-n)}, [\Psi^{(k)}, \Lambda^{(n-k)}]] \tag{A.3}$$

Remembering that $\Psi^{(n)}$ and $\Lambda_+^{(n)}$ were arbitrary we can, in a first instance, choose $\Psi^{(n)} = 0$, $n > 1$, and same for Λ_+. We will also neglect small gauge transformations, thus only keeping Λ_-^0 and Λ_+^1. Then

$$-i[A_Z^{(-1)}, (\delta_\Lambda\Psi)^{(1)}] = [[A_Z^{(-2)}, \Lambda_+^{(1)}], \Psi^{(1)}] - i[A_Z^{(-1)}, \Lambda_+^{(1)}]$$
$$- [A_Z^{(-2)}, [\Psi^{(1)}, \Lambda_+^{(1)}]] - [A_Z^{(-1)}, [\Psi^{(1)}, \Lambda_-^{(0)}]] \tag{A.4}$$

At this stage, as in [6], we are forced to employ an additional simplification, and require that we only work to linear order in the new fields and tranformations, i.e. Ψ and Λ_+. The above then reduces to

$$-i[A_Z^{(-1)}, (\delta_\Lambda\Psi)^{(1)}] = -i[A_Z^{(-1)}, \Lambda_+^{(1)} - i[\Psi^{(1)}, \Lambda_-^{(0)}]] \tag{A.5}$$

finally allowing us to read off

$$(\delta_\Lambda\Psi)^{(1)} = \Lambda_+^{(1)} - i[\Psi^{(1)}, \Lambda_-^{(0)}] \tag{A.6}$$

As a sanity check, we can also attempt to look at order V^1 in (A.1). With the restrictions above, this reduces to:

$$D_Z^{(0)}\delta_\Lambda\Psi^{(1)} = D_Z^{(0)}(\Lambda_+ - i[\Psi, \Lambda])^{(1)}. \tag{A.7}$$

which is in agreement with (A.6).

Now let us attempt to extend these results to second order in V, i.e. consider Ψ^1, Ψ^2 and Λ_-^0, Λ_+^1, Λ_+^2. In this case, (A.1) should hold at order V^2. We get

$$D_Z^{(0)}\delta_\Lambda\Psi^{(2)} = -i[D_Z^{(0)}\Lambda^{(1)}, \Psi^{(1)}] + D_Z^{(0)}\left(\Lambda^{(2)} - i[\Psi^{(1)}, \Lambda^{(1)}] - i[\Psi^{(2)}, \Lambda^{(0)}]\right) \tag{A.8}$$

If we apply the linearization with respect to LGT's and ψ, again the terms $[\Psi, \Lambda_+]$ are thrown away, obtaining

$$(\delta_\Lambda \Psi)^{(2)} = \Lambda_+^{(2)} - i[\Psi^{(2)}, \Lambda_-^{(0)}]. \tag{A.9}$$

Nevertheless, we will show that is not the desirable result when considering higher order LGTs. The linearisation w.r.t. Ψ should be preserved by the algebra, i.e. we want the variations to have the correct representation on the phase space functions,[17]

$$[\overset{1}{\delta}_{\Lambda_1}, \overset{1}{\delta}_{\Lambda_2}]\Psi = -i\overset{1}{\delta}_{[\Lambda_1,\Lambda_2]}\Psi. \tag{A.10}$$

Take two LGTs, let say Λ_1 and Λ_2. For simplicity, let us set $\Lambda_+^{(n)} = 0$, $n \geq 2$ for both of them. Then,

$$[\Lambda_1, \Lambda_2] = V^2[\Lambda_1^{(1)}, \Lambda_2^{(1)}] + V([\Lambda_1^{(1)}, \Lambda_2^{(0)}] + [\Lambda_1^{(0)}, \Lambda_2^{(1)}]) + [\Lambda_0^{(0)}, \Lambda_2^{(1)}], \tag{A.11}$$

Our linearisation assumptions would now dictate that we should throw away the first term in the equation above. But this would now not reflect the non-abelian nature of the YM theory. The problem will persist for higher order LGTs, i.e. $[\Lambda_1^{(n)}, \Lambda_2^{(m)}] = 0$ for all $n, m > 0$.

For $[\Lambda_1^{(1)}, \Lambda_2^{(1)}]$ to appear we now have to relax our linearisation assumptions, in other words we are seeking a solution to (A.8) up to second order in LGT's and ψ. It is straightforward to see that this does not admit a field-independent, local solution.

A.2 Equation (4.6)

We start with (4.3), reproduced below for convenience

$$\delta_\Lambda \hat{\mathcal{A}}_\alpha = \hat{D}_\alpha \Lambda \tag{A.12}$$

The r.h.s. can be rearranged as

$$\begin{aligned}
\hat{D}_\alpha \Lambda &= \partial_\alpha \Lambda - i[\hat{\mathcal{A}}_\alpha, \Lambda] \\
&= \partial_\alpha \Lambda - i[e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi} + ie^{i\Psi}\partial_\alpha e^{-i\Psi}, \Lambda] \\
&= \partial_\alpha \Lambda + [e^{i\Psi}\partial_\alpha e^{-i\Psi}, \Lambda] - i[e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi}, \Lambda] \\
&= e^{i\Psi}\partial_\alpha\left(e^{-i\Psi}\Lambda e^{i\Psi}\right)e^{-i\Psi} - ie^{i\Psi}[\mathcal{A}_\alpha, e^{-i\Psi}\Lambda e^{i\Psi}]e^{-i\Psi} \\
&= e^{i\Psi}D_\alpha\left(e^{-i\Psi}\Lambda e^{i\Psi}\right)e^{-i\Psi}
\end{aligned} \tag{A.13}$$

and the l.h.s. :

$$\begin{aligned}
\delta_\Lambda \hat{\mathcal{A}}_\alpha =& \delta_\Lambda(e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi} + ie^{i\Psi}\partial_\alpha e^{-i\Psi}) \\
=& -\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi} + e^{i\Psi}(\delta_\Lambda\mathcal{A}_\alpha)e^{-i\Psi} + e^{i\Psi}\mathcal{A}_\alpha e^{-i\Psi}\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi) \\
& - i\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)e^{i\Psi}\partial_\alpha e^{-i\Psi} + ie^{i\Psi}\partial_\alpha\left(e^{-i\Psi}\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)\right) \\
=& e^{i\Psi}\Big\{\delta_\Lambda\mathcal{A}_\alpha + \mathcal{A}_\alpha e^{-i\Psi}\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)e^{i\Psi} - e^{-i\Psi}\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)e^{i\Psi}\mathcal{A}_\alpha \\
& + ie^{-i\Psi}\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)\partial_\alpha e^{i\Psi} + i\partial_\alpha\left(e^{-i\Psi}\mathcal{O}_{-i\Psi}(-i\delta_\Lambda\Psi)\right)e^{i\Psi}\Big\}e^{-i\Psi} \\
=& e^{i\Psi}\Big\{\delta_\Lambda\mathcal{A}_\alpha - i\left[\mathcal{A}_\alpha, e^{-i\Psi}\mathcal{O}_{-i\Psi}(\delta_\Lambda\Psi)e^{i\Psi}\right] + \partial_\alpha\left(e^{-i\Psi}\mathcal{O}_{-i\Psi}(\delta_\Lambda\Psi)e^{i\Psi}\right)\Big\}e^{-i\Psi} \\
=& e^{i\Psi}\Big[\delta_\Lambda\mathcal{A}_\alpha + D_\alpha\left(e^{-i\Psi}\mathcal{O}_{-i\Psi}(\delta_\Lambda\Psi)e^{i\Psi}\right)\Big]e^{-i\Psi}
\end{aligned} \tag{A.14}$$

---

[17]Recall that $\overset{n}{\delta}$ denotes the variation up to order n in Ψ.

where we used

$$\delta e^X = e^X \mathcal{O}_X(\delta X) \quad \Rightarrow \quad \delta e^{-X} = -\mathcal{O}_X(\delta X)e^{-X} \tag{A.15}$$

with $\mathcal{O}_X$ as defined in (4.5) and we set $X = -i\Psi$. Finally, equation (4.6) follows from (A.12), (A.14) and (A.13).

A.3 Extended phase space transformation at all orders

We wish to find an explicit perturbative expansion of equation (4.9), reproduced below for convenience, working at some arbitrary order m in ψ:

$$\delta_\Lambda^{[m]}\Psi = \left(\mathcal{O}_{-i\Psi}^{-1}(\Lambda - e^{i\Psi}\Lambda^{(0)}e^{-i\Psi})\right)^{[m]}. \tag{A.16}$$

with

$$\mathcal{O}_{-i\Psi} := \frac{1 - e^{-ad_{-i\Psi}}}{ad_{-i\Psi}} \tag{A.17}$$

Using

$$e^B A e^{-B} = e^{ad_B} A, \tag{A.18}$$

we can rewrite (A.16) as

$$\delta^{[m]}\delta_\Lambda\Psi = \left(\mathcal{O}_{-i\Psi}^{-1}\Lambda - \mathcal{O}_{-i\Psi}^{-1}e^{ad_{i\Psi}}\Lambda^{(0)}\right)^{[m]} \tag{A.19}$$

At this point, to simplify calculations, we introduce the notation

$$Y \equiv ad_{-i\Psi} \tag{A.20}$$

Then we can perturbatively invert the operator

$$\mathcal{O}_{-i\Psi} = \frac{1 - e^{-Y}}{Y} \equiv Y^{-1}(1 - e^{-Y}) \tag{A.21}$$

by making use of the expansion

$$(1 - e^{-Y})^{-1}Y \equiv \frac{Y}{1 - e^{-Y}} = \sum_{m=0}^{\infty} \frac{B_m^+ Y^m}{m!} \tag{A.22}$$

where B_n^+ are the Bernoulli numbers $B_0^+ = 1$, $B_1^+ = \frac{1}{2}$, $B_2^+ = \frac{1}{6}$, ...

The first term in (A.19) is then simply

$$\left(\mathcal{O}_{-i\Psi}^{-1}\Lambda\right)^{[m]} = \frac{B_m^+}{m!}Y^m(\Lambda) \tag{A.23}$$

The second term requres a little more work. We have

$$
\begin{aligned}
-\left(\mathcal{O}_{-i\Psi}^{-1}e^{-Y}\Lambda^{(0)}\right)^{[m]} &= -\sum_{k=0}^{m} \left(\mathcal{O}_{-i\Psi}^{-1}\right)^{[k]}\left(e^{-Y}\Lambda^{(0)}\right)^{[m-k]} \\
&= -\sum_{k=0}^{m} \frac{B_k^+ Y^k}{k!}(-1)^{m-k}\frac{Y^{m-k}}{(m-k)!}\Lambda^{(0)} \\
&= -\sum_{k=0}^{m}(-1)^{m-k}B_k^+\frac{1}{k!(m-k)!}Y^m\Lambda^{(0)}
\end{aligned}
\tag{A.24}
$$

Now we use $(-1)^k B_k^+ = B_k^-$ (this follows from the fact that all odd Bernoulli numbers vanish, except $B_1^- = -B_1^+$) to write

$$
-\left(\mathcal{O}_{-i\Psi}^{-1} e^{-Y}\Lambda^{(0)}\right)^{[m]} = -(-1)^m \sum_{k=0}^{m} B_k^- \frac{1}{k!(m-k)!} Y^m \Lambda^{(0)}
$$

$$
= \left[-(-1)^m \frac{B_m^-}{m!} - (-1)^m \sum_{k=0}^{m-1} B_k^- \frac{1}{k!(m-k)!}\right] Y^m \Lambda^{(0)} \qquad (A.25)
$$

$$
= \left[-\frac{B_m^+}{m!} - \frac{(-1)^m}{m!} \sum_{k=0}^{m-1} B_k^- \binom{m}{k}\right] Y^m \Lambda^{(0)}
$$

Using the sum formula for the Bernoulli numbers

$$
\sum_{k=0}^{n} \binom{n+1}{k} B_k^- = \delta_{n,0} \qquad (A.26)
$$

the equation above finally becomes

$$
-\left(\mathcal{O}_{-i\Psi}^{-1} e^{-Y}\Lambda^{(0)}\right)^{[m]} = \left[-\frac{B_m^+}{m!} - \frac{(-1)^m}{m!}\delta_{m-1,0}\right] Y^m \Lambda^{(0)}
$$

$$
= \left[-\frac{B_m^+}{m!} + \delta_{m,1}\right] Y^m \Lambda^{(0)}
$$

$$
= \left[-\frac{B_m^+}{m!} + \frac{2B_m^+}{m!}\delta_{m,1}\right] Y^m \Lambda^{(0)} \qquad (A.27)
$$

$$
= \frac{B_m^+}{m!}(-1 + 2\delta_{m,1}) Y^m \Lambda^{(0)}
$$

where to get to the third line we used $B_1^+ = \frac{1}{2}$. Finally, putting together (A.23) and (A.27), and using (A.20), we get

$$
\delta_\Lambda^{[m]}\Psi = \frac{B_m^+}{m!} (ad_{-i\Psi})^m \left[\Lambda + (-1 + 2\delta_{m,1})\Lambda^{(0)}\right] \qquad (A.28)
$$

B Gravity calculations

B.1 Restricted diffeomorphism transformation

As explained in section 2, we are working with a subset of diffeomorphisms parametrised by $\xi_i = 0$ and $\xi_\alpha = b_\alpha(y)$. The transformation of the metric $h_{\alpha\beta} = \Pi_\alpha^{\ i}\Pi_\beta^{\ j}\partial_i\partial_j\phi$ can then be written in terms of the "Hamiltonian" $\lambda = 2\Omega_i^{\ \alpha}x^i b_\alpha$ as

$$
\delta h_{\alpha\beta} = \mathcal{L}_\xi g_{\alpha\beta} = \mathcal{L}_\xi \eta_{\alpha\beta} + \mathcal{L}_\xi h_{\alpha\beta}
$$

$$
= \Pi_{(\alpha}^{\ i}\partial_i \left(\partial_{\beta)}\lambda - \frac{1}{2}\Pi_{\beta)}^{\ j}\partial_j\{\lambda,\phi\}\right)
$$

$$
= \Pi_{(\alpha}^i\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\left(\{\Pi_{(\alpha}^i\partial_i\Pi_{\beta)}^j\partial_j\lambda,\phi\} + \{\lambda,\Pi_{(\alpha}^i\partial_i\Pi_{\beta)}^j\partial_j\phi\} + 2\{\Pi_{(\alpha}^i\partial_i\lambda,\Pi_{\beta)}^j\partial_j\phi\}\right)
$$

$$
= \Pi_{(\alpha}^i\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\{\lambda,h_{\alpha\beta}\},
$$

$$
(B.1)
$$

where we used the fact that λ is at most linear in x^i, together with the definition of the Poisson bracket (2.37).

B.2 Poisson bracket exponentiation

Start with (see subsection B.1)

$$\mathcal{L}_\xi g_{\alpha\beta} = \Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\{\lambda, h_{\alpha\beta}\}. \tag{B.2}$$

Next, we compute:

$$\mathcal{L}_\xi(\mathcal{L}_\xi(g_{\alpha\beta})) = \mathcal{L}_\xi(\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\{\lambda, h_{\alpha\beta}\}) \tag{B.3}$$

$$= \xi^k\partial_k\left(\Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\{\lambda, h_{\alpha\beta}\}\right) \tag{B.4}$$

$$= -\frac{1}{2}\xi^k\partial_k\{\lambda, h_{\alpha\beta}\} \tag{B.5}$$

$$= \frac{1}{4}\{\lambda, \{\lambda, h_{\alpha\beta}\}\}. \tag{B.6}$$

In the above, to get (B.4), we used the definition of the Lie derivative, together with our restricted diffeomorphism parameter (2.40) and the fact that $(\mathcal{L}_\xi g)_{\alpha i} = 0$ by construction. Then, (B.5) follows if we recall that λ is at most linear in x^i (see (2.46)). Finally, we get (B.6) by using (2.48).

We can immediately see that for $n > 1$,

$$(\mathcal{L}_\xi)^n g_{\alpha\beta} = \frac{1}{(-2)^n}\mathfrak{ad}^n_\lambda(h_{\alpha\beta}), \tag{B.7}$$

which follows by the considerations above and repeated use of (2.48).

B.3 Consistency check

We wish to check the consistency condition (5.12), copied below for convenience

$$\delta\hat{h}_{\alpha\beta} = \Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda - \frac{1}{2}\mathfrak{ad}_\lambda\hat{h}_{\alpha\beta}, \tag{B.8}$$

making use of (5.11), together with the transformation rule for $h_{\alpha\beta}$, which is unchanged by the extension of λ to higher orders in V:

$$\delta h_{\alpha\beta} = \Pi^i_{(\alpha}\partial_i\partial_{\beta)}\lambda^{(0)} - \frac{1}{2}\mathfrak{ad}_{\lambda^{(0)}}h_{\alpha\beta}, \tag{B.9}$$

and the transfomation rule for ψ at all orders in V (5.24). We have

$$
\begin{aligned}
\delta \hat{h}_{\alpha\beta} &= \delta \left(\Pi^i_{(\alpha} \partial_i \partial_{\beta)} \psi + e^{-\frac{1}{2}\mathfrak{ad}_\psi} h_{\alpha\beta} \right) \\
&= \delta \left(\Pi^i_{(\alpha} \partial_i \partial_{\beta)} \psi + \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!} \mathfrak{ad}_\psi^n h_{\alpha\beta} \right) \\
&= \Pi^i_{(\alpha} \partial_i \partial_{\beta)} (\lambda - \lambda^{(0)}) + \mathfrak{ad}_{(\lambda - \lambda^{(0)})} \sum_{n=1}^{\infty} \frac{1}{(-2)^n (n-1)!} \mathfrak{ad}_\psi^{(n-1)} h_{\alpha\beta} \\
&\quad + \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda^{(0)} + \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!} \mathfrak{ad}_\psi^n \left(-\frac{1}{2} \mathfrak{ad}_{\lambda^{(0)}} h_{\alpha\beta} \right) \\
&= \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2} \mathfrak{ad}_{(\lambda - \lambda^{(0)})} \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!} \mathfrak{ad}_\psi^n h_{\alpha\beta} - \frac{1}{2} \mathfrak{ad}_{\lambda^{(0)}} \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!} \mathfrak{ad}_\psi^n h_{\alpha\beta} \\
&= \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2} \mathfrak{ad}_\lambda \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!} \mathfrak{ad}_\psi^n h_{\alpha\beta} \\
&= \Pi^i_{(\alpha} \partial_i \partial_{\beta)} \lambda - \frac{1}{2} \mathfrak{ad}_\lambda \hat{h}_{\alpha\beta}
\end{aligned}
\tag{B.10}
$$

as needed. In the above, we repeatedly used the fact that both λ and ψ are linear in x^i, together with the definition of the Poisson bracket ((5.4) and (2.37)) in order to set a number of terms to 0 as well as to commute $\mathfrak{ad}_\psi$ and $\mathfrak{ad}_\lambda$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

[1] A. Strominger, *Asymptotic Symmetries of Yang-Mills Theory*, *JHEP* **07** (2014) 151 [arXiv:1308.0589] [INSPIRE].

[2] A. Strominger, *On BMS Invariance of Gravitational Scattering*, *JHEP* **07** (2014) 152 [arXiv:1312.2229] [INSPIRE].

[3] T. He, P. Mitra, A.P. Porfyriadis and A. Strominger, *New Symmetries of Massless QED*, *JHEP* **10** (2014) 112 [arXiv:1407.3789] [INSPIRE].

[4] M. Campiglia and A. Laddha, *New symmetries for the Gravitational S-matrix*, *JHEP* **04** (2015) 076 [arXiv:1502.02318] [INSPIRE].

[5] M. Campiglia and J. Peraza, *Generalized BMS charge algebra*, *Phys. Rev. D* **101** (2020) 104039 [arXiv:2002.06691] [INSPIRE].

[6] M. Campiglia and J. Peraza, *Charge algebra for non-abelian large gauge symmetries at* $O(r)$, *JHEP* **12** (2021) 058 [arXiv:2111.00973] [INSPIRE].

[7] A. Strominger, *On BMS Invariance of Gravitational Scattering*, *JHEP* **07** (2014) 152 [arXiv:1312.2229] [INSPIRE].

[8] V. Lysov, S. Pasterski and A. Strominger, *Low's Subleading Soft Theorem as a Symmetry of QED*, *Phys. Rev. Lett.* **113** (2014) 111601 [arXiv:1407.3814] [INSPIRE].

[9] W. Donnelly and L. Freidel, *Local subsystems in gauge theory and gravity*, *JHEP* **09** (2016) 102 [arXiv:1601.04744] [INSPIRE].

[10] A.J. Speranza, *Local phase space and edge modes for diffeomorphism-invariant theories*, *JHEP* **02** (2018) 021 [arXiv:1706.05061] [INSPIRE].

[11] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part III. Corner simplicity constraints*, *JHEP* **01** (2021) 100 [arXiv:2007.12635] [INSPIRE].

[12] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part II. Corner metric and Lorentz charges*, *JHEP* **11** (2020) 027 [arXiv:2007.03563] [INSPIRE].

[13] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part I. Corner potentials and charges*, *JHEP* **11** (2020) 026 [arXiv:2006.12527] [INSPIRE].

[14] L. Freidel, *A canonical bracket for open gravitational system*, arXiv:2111.14747 [INSPIRE].

[15] L. Ciambelli, R.G. Leigh and P.-C. Pai, *Embeddings and Integrable Charges for Extended Corner Symmetry*, *Phys. Rev. Lett.* **128** (2022) [arXiv:2111.13181] [INSPIRE].

[16] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, *Extended corner symmetry, charge bracket and Einstein's equations*, *JHEP* **09** (2021) 083 [arXiv:2104.12881] [INSPIRE].

[17] L. Freidel, D. Pranzetti and A.-M. Raclariu, *Sub-subleading soft graviton theorem from asymptotic Einstein's equations*, *JHEP* **05** (2022) 186 [arXiv:2111.15607] [INSPIRE].

[18] L. Ciambelli and R.G. Leigh, *Isolated surfaces and symmetries of gravity*, *Phys. Rev. D* **104** (2021) 046005 [arXiv:2104.07643] [INSPIRE].

[19] Z. Bern, J.J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, *The Duality Between Color and Kinematics and its Applications*, Tech. Rep. CERN-TH-2019-135, UCLA/TEP/2019/104, NUHEP-TH/19-11, UUITP-35/19, NORDITA 2019-079 [arXiv:1909.01358] [INSPIRE].

[20] L. Borsten, *Gravity as the square of gauge theory: a review*, *Riv. Nuovo Cim.* **43** (2020) 97 [INSPIRE].

[21] T. Adamo et al., *Snowmass White Paper: the Double Copy and its Applications*, in *2022 Snowmass Summer Study*, Seattle, U.S.A. (2022) [arXiv:2204.06547] [INSPIRE].

[22] R. Monteiro and D. O'Connell, *The Kinematic Algebra From the Self-Dual Sector*, *JHEP* **07** (2011) 007 [arXiv:1105.2565] [INSPIRE].

[23] E. Chacón, H. García-Compeán, A. Luna, R. Monteiro and C.D. White, *New heavenly double copies*, *JHEP* **03** (2021) 247 [arXiv:2008.09603] [INSPIRE].

[24] M. Campiglia and S. Nagy, *A double copy for asymptotic symmetries in the self-dual sector*, *JHEP* **03** (2021) 262 [arXiv:2102.01680] [INSPIRE].

[25] K. Armstrong-Williams, C.D. White and S. Wikeley, *Non-perturbative aspects of the self-dual double copy*, *JHEP* **08** (2022) 160 [arXiv:2205.02136] [INSPIRE].

[26] K. Krasnov and E. Skvortsov, *Flat self-dual gravity*, *JHEP* **08** (2021) 082 [arXiv:2106.01397] [INSPIRE].

[27] G. Elor, K. Farnsworth, M.L. Graesser and G. Herczeg, *The Newman-Penrose Map and the Classical Double Copy*, *JHEP* **12** (2020) 121 [arXiv:2006.08630] [INSPIRE].

[28] K. Farnsworth, M.L. Graesser and G. Herczeg, *Twistor space origins of the Newman-Penrose map*, *SciPost Phys.* **13** (2022) 099 [arXiv:2104.09525] [INSPIRE].

[29] E. Skvortsov and R. Van Dongen, *Minimal models of field theories: SDYM and SDGR*, *JHEP* **08** (2022) 083 [arXiv:2204.09313] [ɪɴSPIRE].

[30] R.H. Boels, R.S. Isermann, R. Monteiro and D. O'Connell, *Colour-Kinematics Duality for One-Loop Rational Amplitudes*, *JHEP* **04** (2013) 107 [arXiv:1301.4165] [ɪɴSPIRE].

[31] S. He, R. Monteiro and O. Schlotterer, *String-inspired BCJ numerators for one-loop MHV amplitudes*, *JHEP* **01** (2016) 171 [arXiv:1507.06288] [ɪɴSPIRE].

[32] D.S. Berman, E. Chacón, A. Luna and C.D. White, *The self-dual classical double copy, and the Eguchi-Hanson instanton*, *JHEP* **01** (2019) 107 [arXiv:1809.04063] [ɪɴSPIRE].

[33] R. Monteiro, R. Stark-Muchão and S. Wikeley, *Anomaly and double copy in quantum self-dual Yang-Mills and gravity*, Tech. Rep. QMUL-PH-22-36 (2022) [arXiv:2211.12407] [ɪɴSPIRE].

[34] L. Borsten, I. Jubb, V. Makwana and S. Nagy, *Gauge × gauge = gravity on homogeneous spaces using tensor convolutions*, *JHEP* **06** (2021) 117 [arXiv:2104.01135] [ɪɴSPIRE].

[35] L. Borsten and S. Nagy, *The pure BRST Einstein-Hilbert Lagrangian from the double-copy to cubic order*, *JHEP* **07** (2020) 093 [arXiv:2004.14945] [ɪɴSPIRE].

[36] A. Luna, S. Nagy and C. White, *The convolutional double copy: a case study with a point*, *JHEP* **09** (2020) 062 [arXiv:2004.11254] [ɪɴSPIRE].

[37] L. Borsten, I. Jubb, V. Makwana and S. Nagy, *Gauge × gauge on spheres*, *JHEP* **06** (2020) 096 [arXiv:1911.12324] [ɪɴSPIRE].

[38] A. Anastasiou, L. Borsten, M.J. Duff, S. Nagy and M. Zoccali, *Gravity as Gauge Theory Squared: A Ghost Story*, *Phys. Rev. Lett.* **121** (2018) 211601 [arXiv:1807.02486] [ɪɴSPIRE].

[39] G. Lopes Cardoso, G. Inverso, S. Nagy and S. Nampuri, *Comments on the double copy construction for gravitational theories*, *PoS* **CORFU2017** (2018) 177 [arXiv:1803.07670] [ɪɴSPIRE].

[40] A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes and S. Nagy, *Yang-Mills origin of gravitational symmetries*, *Phys. Rev. Lett.* **113** (2014) 231606 [arXiv:1408.4434] [ɪɴSPIRE].

[41] L. Borsten, H. Kim, B. Jurčo, T. Macrelli, C. Saemann and M. Wolf, *Double Copy from Homotopy Algebras*, *Fortsch. Phys.* **69** (2021) 2100075 [arXiv:2102.11390] [ɪɴSPIRE].

[42] L. Borsten, B. Jurčo, H. Kim, T. Macrelli, C. Saemann and M. Wolf, *Becchi-Rouet-Stora-Tyutin-Lagrangian Double Copy of Yang-Mills Theory*, *Phys. Rev. Lett.* **126** (2021) 191601 [arXiv:2007.13803] [ɪɴSPIRE].

[43] R. Bonezzi, F. Diaz-Jaramillo and O. Hohm, *The gauge structure of double field theory follows from Yang-Mills theory*, *Phys. Rev. D* **106** (2022) 026004 [arXiv:2203.07397] [ɪɴSPIRE].

[44] P. Ferrero and D. Francia, *On the Lagrangian formulation of the double copy to cubic order*, *JHEP* **02** (2021) 213 [arXiv:2012.00713] [ɪɴSPIRE].

[45] M. Godazgar, C.N. Pope, A. Saha and H. Zhang, *BRST symmetry and the convolutional double copy*, *JHEP* **11** (2022) 038 [arXiv:2208.06903] [ɪɴSPIRE].

[46] N.E.J. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [arXiv:1010.3933] [ɪɴSPIRE].

[47] S. Oxburgh and C.D. White, *BCJ duality and the double copy in the soft limit*, *JHEP* **02** (2013) 127 [arXiv:1210.1110] [ɪɴSPIRE].

[48] S.G. Naculich, H. Nastase and H.J. Schnitzer, *All-loop infrared-divergent behavior of most-subleading-color gauge-theory amplitudes*, *JHEP* **04** (2013) 114 [arXiv:1301.2234] [INSPIRE].

[49] S. He, Y.-t. Huang and C. Wen, *Loop Corrections to Soft Theorems in Gauge Theories and Gravity*, *JHEP* **12** (2014) 115 [arXiv:1405.1410] [INSPIRE].

[50] C.D. White, *Diagrammatic insights into next-to-soft corrections*, *Phys. Lett. B* **737** (2014) 216 [arXiv:1406.7184] [INSPIRE].

[51] A. Sabio Vera and M.A. Vazquez-Mozo, *The Double Copy Structure of Soft Gravitons*, *JHEP* **03** (2015) 070 [arXiv:1412.3699] [INSPIRE].

[52] A. Luna, S. Melville, S.G. Naculich and C.D. White, *Next-to-soft corrections to high energy scattering in QCD and gravity*, *JHEP* **01** (2017) 052 [arXiv:1611.02172] [INSPIRE].

[53] A. P. V. and A. Manu, *Classical double copy from Color Kinematics duality: A proof in the soft limit*, *Phys. Rev. D* **101** (2020) 046014 [arXiv:1907.10021] [INSPIRE].

[54] R. Monteiro, *Celestial chiral algebras, colour-kinematics duality and integrability*, *JHEP* **01** (2023) 092 [arXiv:2208.11179] [INSPIRE].

[55] T. Adamo, E. Casali, L. Mason and S. Nekovar, *Scattering on plane waves and the double copy*, *Class. Quant. Grav.* **35** (2018) 015004 [arXiv:1706.08925] [INSPIRE].

[56] E. Casali and A. Puhm, *Double Copy for Celestial Amplitudes*, *Phys. Rev. Lett.* **126** (2021) 101602 [arXiv:2007.15027] [INSPIRE].

[57] E. Casali and A. Sharma, *Celestial double copy from the worldsheet*, *JHEP* **05** (2021) 157 [arXiv:2011.10052] [INSPIRE].

[58] N. Kalyanapuram, *Soft Gravity by Squaring Soft QED on the Celestial Sphere*, *Phys. Rev. D* **103** (2021) 085016 [arXiv:2011.11412] [INSPIRE].

[59] N. Kalyanapuram, *Gauge and Gravity Amplitudes on the Celestial Sphere*, *Phys. Rev. D* **103** (2021) 085015 [arXiv:2012.04579] [INSPIRE].

[60] S. Pasterski and A. Puhm, *Shifting spin on the celestial sphere*, *Phys. Rev. D* **104** (2021) 086020 [arXiv:2012.15694] [INSPIRE].

[61] Y.-T. Huang, U. Kol and D. O'Connell, *Double copy of electric-magnetic duality*, *Phys. Rev. D* **102** (2020) 046005 [arXiv:1911.06318] [INSPIRE].

[62] W.T. Emond, Y.-T. Huang, U. Kol, N. Moynihan and D. O'Connell, *Amplitudes from Coulomb to Kerr-Taub-NUT*, *JHEP* **05** (2022) 055 [arXiv:2010.07861] [INSPIRE].

[63] R. Alawadhi, D.S. Berman, B. Spence and D. Peinador Veiga, *S-duality and the double copy*, *JHEP* **03** (2020) 059 [arXiv:1911.06797] [INSPIRE].

[64] A. Banerjee, E.O. Colgáin, J.A. Rosabal and H. Yavartanoo, *Ehlers as EM duality in the double copy*, *Phys. Rev. D* **102** (2020) 126017 [arXiv:1912.02597] [INSPIRE].

[65] A. Guevara, *Towards Gravity From a Color Symmetry*, arXiv:2209.00696 [INSPIRE].

[66] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga and C.N. Pope, *Asymptotic Weyl double copy*, *JHEP* **11** (2021) 126 [arXiv:2109.07866] [INSPIRE].

[67] T. Adamo and U. Kol, *Classical double copy at null infinity*, *Class. Quant. Grav.* **39** (2022) 105007 [arXiv:2109.07832] [INSPIRE].

[68] P. Mao and W. Zhao, *Note on the asymptotic structure of Kerr-Schild form*, *JHEP* **01** (2022) 030 [`arXiv:2109.09676`] [ɪɴSPIRE].

[69] D. Cangemi, *Selfdual Yang-Mills theory and one-loop maximally helicity violating multi-gluon amplitudes*, *Nucl. Phys. B* **484** (1997) 521 [`hep-th/9605208`] [ɪɴSPIRE].

[70] G. Chalmers and W. Siegel, *The Selfdual sector of QCD amplitudes*, *Phys. Rev. D* **54** (1996) 7628 [`hep-th/9606061`] [ɪɴSPIRE].

[71] J.F. Plebanski, *Some solutions of complex Einstein equations*, *J. Math. Phys.* **16** (1975) 2395 [ɪɴSPIRE].

[72] E.C.G. Stueckelberg, *Interaction energy in electrodynamics and in the field theory of nuclear forces*, *Helv. Phys. Acta* **11** (1938) 225 [ɪɴSPIRE].

[73] K.V. Kuchar, *Does an unspecified cosmological constant solve the problem of time in quantum gravity?*, *Phys. Rev. D* **43** (1991) 3332 [ɪɴSPIRE].

[74] S. Nagy, A. Padilla and I. Zavala, *The Super-Stückelberg procedure and dS in pure supergravity*, *Proc. Roy. Soc. Lond. A* **476** (2020) 20200035 [`arXiv:1910.14349`] [ɪɴSPIRE].

[75] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, *JHEP* **05** (2010) 062 [`arXiv:1001.1541`] [ɪɴSPIRE].

[76] Y. Hamada and G. Shiu, *Infinite Set of Soft Theorems in Gauge-Gravity Theories as Ward-Takahashi Identities*, *Phys. Rev. Lett.* **120** (2018) 201601 [`arXiv:1801.05528`] [ɪɴSPIRE].

[77] M. Campiglia and A. Laddha, *Asymptotic charges in massless QED revisited: A view from Spatial Infinity*, *JHEP* **05** (2019) 207 [`arXiv:1810.04619`] [ɪɴSPIRE].

[78] M. Campiglia and A. Laddha, *Sub-subleading soft gravitons and large diffeomorphisms*, *JHEP* **01** (2017) 036 [`arXiv:1608.00685`] [ɪɴSPIRE].

[79] A.D. Popov, M. Bordemann and H. Romer, *Symmetries, currents and conservation laws of selfdual gravity*, *Phys. Lett. B* **385** (1996) 63 [`hep-th/9606077`] [ɪɴSPIRE].

[80] A.D. Popov, *Selfdual Yang-Mills: Symmetries and moduli space*, *Rev. Math. Phys.* **11** (1999) 1091 [`hep-th/9803183`] [ɪɴSPIRE].

[81] L. Dolan, *Kac-moody Algebras and Exact Solvability in Hadronic Physics*, *Phys. Rept.* **109** (1984) 1 [ɪɴSPIRE].

[82] M.K. Prasad, A. Sinha and L.-L. Wang, *Nonlocal Continuity Equations for Selfdual* SU(N) *Yang-Mills Fields*, *Phys. Lett. B* **87** (1979) 237 [ɪɴSPIRE].

[83] Q.-H. Park, *Selfdual Gravity as a Large N Limit of the Two-dimensional Nonlinear σ Model*, *Phys. Lett. B* **238** (1990) 287 [ɪɴSPIRE].

[84] V. Husain, *Selfdual gravity as a two-dimensional theory and conservation laws*, *Class. Quant. Grav.* **11** (1994) 927 [`gr-qc/9310003`] [ɪɴSPIRE].

[85] V. Husain, *The Affine symmetry of selfdual gravity*, *J. Math. Phys.* **36** (1995) 6897 [`hep-th/9410072`] [ɪɴSPIRE].

[86] L. Freidel, D. Pranzetti and A.-M. Raclariu, *Higher spin dynamics in gravity and $w_{1+\infty}$ celestial symmetries*, *Phys. Rev. D* **106** (2022) 086013 [`arXiv:2112.15573`] [ɪɴSPIRE].

[87] A. Parkes, *A Cubic action for selfdual Yang-Mills*, *Phys. Lett. B* **286** (1992) 265 [`hep-th/9203074`] [ɪɴSPIRE].

[88] V. Husain, *Selfdual gravity as a two-dimensional theory and conservation laws*, *Class. Quant. Grav.* **11** (1994) 927 [`gr-qc/9310003`] [INSPIRE].

[89] V. Husain, *The Affine symmetry of selfdual gravity*, *J. Math. Phys.* **36** (1995) 6897 [`hep-th/9410072`] [INSPIRE].

[90] M. Wolf, *On hidden symmetries of a super gauge theory and twistor string theory*, *JHEP* **02** (2005) 018 [`hep-th/0412163`] [INSPIRE].

[91] A.D. Popov and M. Wolf, *Hidden symmetries and integrable hierarchy of the N=4 supersymmetric Yang-Mills equations*, *Commun. Math. Phys.* **275** (2007) 685 [`hep-th/0608225`] [INSPIRE].

[92] C. Cheung, A. de la Fuente and R. Sundrum, *4D scattering amplitudes and asymptotic symmetries from 2D CFT*, *JHEP* **01** (2017) 112 [`arXiv:1609.00732`] [INSPIRE].

[93] A. Nande, M. Pate and A. Strominger, *Soft Factorization in QED from 2D Kac-Moody Symmetry*, *JHEP* **02** (2018) 079 [`arXiv:1705.00608`] [INSPIRE].

[94] T. He, P. Mitra and A. Strominger, *2D Kac-Moody Symmetry of 4D Yang-Mills Theory*, *JHEP* **10** (2016) 137 [`arXiv:1503.02663`] [INSPIRE].

[95] E. Himwich and A. Strominger, *Celestial current algebra from Low's subleading soft theorem*, *Phys. Rev. D* **100** (2019) 065001 [`arXiv:1901.01622`] [INSPIRE].

[96] W. Fan, A. Fotopoulos and T.R. Taylor, *Soft Limits of Yang-Mills Amplitudes and Conformal Correlators*, *JHEP* **05** (2019) 121 [`arXiv:1903.01676`] [INSPIRE].

[97] N. Kalyanapuram, *Infrared and Holographic Aspects of the S-Matrix in Gauge Theory and Gravity*, `arXiv:2107.06660` [INSPIRE].

[98] A.-M. Raclariu, *Lectures on Celestial Holography*, `arXiv:2107.02075` [INSPIRE].

[99] S. Pasterski, *Lectures on celestial amplitudes*, *Eur. Phys. J. C* **81** (2021) 1062 [`arXiv:2108.04801`] [INSPIRE].

[100] A. Guevara, E. Himwich, M. Pate and A. Strominger, *Holographic symmetry algebras for gauge theory and gravity*, *JHEP* **11** (2021) 152 [`arXiv:2103.03961`] [INSPIRE].

[101] H. Jiang, *Holographic chiral algebra: supersymmetry, infinite Ward identities, and EFTs*, *JHEP* **01** (2022) 113 [`arXiv:2108.08799`] [INSPIRE].

[102] J.H. Schwarz, *Diffeomorphism Symmetry in Two Dimensions and Celestial Holography*, Tech. Rep. CALT-TH-2022-030 (2022) [INSPIRE].

[103] S. Pasterski, *A Shorter Path to Celestial Currents*, `arXiv:2201.06805` [INSPIRE].

[104] S. Albayrak, C. Chowdhury and S. Kharel, *On loop celestial amplitudes for gauge theory and gravity*, *Phys. Rev. D* **102** (2020) 126020 [`arXiv:2007.09338`] [INSPIRE].

[105] H.A. González, A. Puhm and F. Rojas, *Loop corrections to celestial amplitudes*, *Phys. Rev. D* **102** (2020) 126027 [`arXiv:2009.07290`] [INSPIRE].

[106] H.A. González and F. Rojas, *The structure of IR divergences in celestial gluon amplitudes*, *JHEP* **2021** (2021) 171 [`arXiv:2104.12979`] [INSPIRE].

[107] L. Magnea, *Non-abelian infrared divergences on the celestial sphere*, *JHEP* **05** (2021) 282 [`arXiv:2104.10254`] [INSPIRE].

[108] A. Ashtekar and M. Varadarajan, *Gravitational Dynamics — A Novel Shift in the Hamiltonian Paradigm*, *Universe* **7** (2021) 13 [`arXiv:2012.12094`] [INSPIRE].