Classical axisymmetric gravity in real Ashtekar variables

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We formulate axisymmetric general relativity in terms of real Ashtekar–Barbero variables. We study the constraints and equations of motion and show how the Kerr, Schwarzschild and Minkowski solutions arise. We also discuss boundary conditions. This opens the possibility of a midisuperspace quantization using loop quantum gravity techniques for spacetimes with axial symmetry and time dependence.

I. INTRODUCTION

Due to the complexities of the quantized version of the Einstein equations in loop quantum gravity, the study of mini and midisuperspaces has proved a valuable tool to gain insights into the physics of the theory. The study first started with homogeneous cosmologies, giving rise to loop quantum cosmology (see [1] and references therein). It was later expanded to include spherically symmetric spacetimes (see [2] and references therein), including charged black holes [3]. In both cases interesting physical insights, like the elimination of singularities due to quantum effects, were found. It is natural to try to extend these studies to situations with less symmetry, like the case of axisymmetric spacetimes, which include physically important situations, like the Kerr geometry. There is virtually no literature on the subject. An exception is the work on isolated horizons and black hole entropy [**? ? ? ? ? ? ? ?**]. An early study of spacetimes with one Killing vector field made some progress, partially addressing the situation of axial symmetry in complex connection variables [4]. Some progress was also made in planar space-times ([5] and references therein, [6]) and the case of two spatial Killing vector fields was also discussed for the Gowdy models [7], including the use of hybrid quantizations (see [8] for a review). Some of these studies were in terms of the early form of the Ashtekar variables which were complex.

Here we would like to discuss the case of axisymmetric space-time using real Ashtekar–Barbero variables. We introduce a suitable Killing vector field and coordinates adapted to it. We will also show how the Kerr, Schwarzschild and Minkowski solutions arise. Besides, we will make some remarks on boundary conditions. This completes a classical setup suitable to perform a loop quantization, which we will discuss in a subsequent paper. This is the first example of a system with only one Killing vector field to be formulated with the real Ashtekar–Barbero variables.

The organization of this paper is as follows. In section 2 we discuss a set of symmetry adapted variables and set up the kinematics of the problem. In section 3 we introduce the constraints of general relativity in terms of the reduced axisymmetric variables introduced. In section 4 we work out the equations of motion. In section 5 we check that some particular solutions of interest including the Kerr, Schwarzschild and Minkowski space-times solve the equations we present. In section 6 we discuss boundary conditions. We end with a discussion.

II. KINEMATICS: SYMMETRY ADAPTED VARIABLES

Here we will impose a symmetry reduction due to a spatial Killing field with orbits tangent to S^1. Let us consider a choice of fiducial coordinates $\{x, y, \phi\}$, where $\phi \in S^1$ and $x, y \in \mathbb{R}$. The Killing field will be then

$$K^a = (\partial_\phi)^a \,. \tag{2.1}$$

We will be following the typical reduction procedure adopted for connection variables. Namely, a connection $A = A_a^i \tau_i \mathrm{d}x^a$ will be invariant under the Killing symmetries if it satisfies the condition

$$\mathcal{L}_{\tilde{K}} A_a^i = \epsilon_{ijk} \lambda^j A_a^k, \tag{2.2}$$

where $\tilde{K} = \lambda_i \partial^i = \lambda_3 \partial_\phi$ and $\lambda_1 = 0 = \lambda_2$. The previous equation amounts to

$$\partial_\phi A_a^i = \epsilon_{i3k} A_a^k. \tag{2.3}$$

Notice that we are imposing that the Lie derivative be proportional to a constant $O(2)$ gauge transformation [9]. We have found this to be the simplest choice that is general enough to recover all solutions with axisymmetry. In other situations one may need to consider λ^i that are more general, perhaps including spatial dependence.

The same equation is valid for the densitized triad E_i^a. The most general solution (see Appendix A) to these equations are

$$\begin{aligned} A &= A_a^i \tau_i \mathrm{d}x^a = \left((\cos(\phi)\tau_1 + \sin(\phi)\tau_2)\mathbf{a}_a^1 + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2)\mathbf{a}_a^2 + \mathbf{a}_a^3 \tau_3 \right) \mathrm{d}x^a & (2.4)\\ E &= E_i^a \tau^i \partial_a = \left((\cos(\phi)\tau^1 + \sin(\phi)\tau^2)\mathbf{e}_1^a + (-\sin(\phi)\tau^1 + \cos(\phi)\tau^2)\mathbf{e}_2^a + \mathbf{e}_3^a \tau^3 \right) \partial_a, & (2.5)\end{aligned}$$

where the symmetry adapted variables $(\mathbf{a}_a^i, \mathbf{e}_j^b)$ do not depend on the angular coordinate ϕ, i.e. only on (x, y), and are canonically conjugate. In order to prove this, it is very easy to verify that

$$\Omega = \frac{1}{8\pi G\beta} \int dxdyd\phi\, \delta E_i^a \wedge \delta A_a^i = \frac{1}{4G\beta} \int dxdy\, \delta\mathbf{e}_i^a \wedge \delta\mathbf{a}_a^i, \tag{2.6}$$

with β the Immirzi parameter. In other words,

$$\{\mathbf{a}_a^i(\vec{x}), \mathbf{e}_j^b(\vec{x}')\} = 4G\beta\, \delta_j^i \delta_a^b \delta^{(2)}(\vec{x} - \vec{x}'). \tag{2.7}$$

Another geometrical quantity that will be useful and can be computed now is the determinant of the symmetry-reduced densitized triad

$$E = \det(E) = \frac{1}{3!} \varepsilon_{abc} \varepsilon^{ijk} E_i^a E_j^b E_k^c = \frac{1}{3!} \varepsilon_{abc} \varepsilon^{ijk} \mathbf{e}_i^a \mathbf{e}_j^b \mathbf{e}_k^c = \det(\mathbf{e}) = \mathbf{e}. \tag{2.8}$$

The inverse of the densitized triad, E_a^i, takes a similar form as E_i^a, but replacing $\mathbf{e}_i^a$ by $\mathbf{e}_a^i$. One can easily see that $\mathbf{e}_a^i$ fulfills $\mathbf{e}_a^i \mathbf{e}_j^a = \delta_j^i$ and $\mathbf{e}_a^i \mathbf{e}_i^b = \delta_b^a$, i.e. it is the inverse of $\mathbf{e}_i^a$ (and therefore it can be written in terms of $\mathbf{e}_i^a$). Then, the symmetry-reduced spatial metric can be written as

$$q_{ab} = E E_a^i E_b^i = \mathbf{e}\,\mathbf{e}_a^i \mathbf{e}_b^i, \tag{2.9}$$

and it only depends on $\mathbf{e}_i^a$.

Similarly, the same reduction process can be applied to the extrinsic curvature $K = K_a^i \tau_i \mathrm{d}x^a$, in its triadic form, and the spin connection $\Gamma = \Gamma_a^i \tau_i \mathrm{d}x^a$, namely

$$K = \big((\cos(\phi)\tau_1 + \sin(\phi)\tau_2)\mathbf{k}_a^1 + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2)\mathbf{k}_a^2 + \mathbf{k}_a^3\tau_3\big)\,\mathrm{d}x^a, \tag{2.10}$$

$$\Gamma = \big((\cos(\phi)\tau_1 + \sin(\phi)\tau_2)\gamma_a^1 + (-\sin(\phi)\tau_1 + \cos(\phi)\tau_2)\gamma_a^2 + \gamma_a^3\tau_3\big)\,\mathrm{d}x^a. \tag{2.11}$$

Actually, the components of the symmetry-reduced spin connection can be written in terms of the components of the symmetry-reduced triads as

$$\gamma_a^i = \frac{1}{2}\epsilon_{ijk}\mathbf{e}_j^b\left(\mathbf{e}_{a,b}^k - \mathbf{e}_{b,a}^k + \mathbf{e}_k^c\mathbf{e}_a^l\mathbf{e}_{c,b}^l + \mathbf{e}_a^k\mathbf{e}_c^l\mathbf{e}_{l,b}^c\right) - \delta_3^i\delta_a^\phi = \boldsymbol{\gamma}_a^i - \delta_3^i\delta_a^\phi, \tag{2.12}$$

where $\boldsymbol{\gamma}_a^i$ is the spin connection compatible with $\mathbf{e}_i^a$. This means that $\{\mathbf{e}_i^a, \boldsymbol{\gamma}_b^j\} = 0$, and therefore $\{\mathbf{e}_i^a, \gamma_b^j\} = 0$.

The relation between the components of the symmetry-reduced extrinsic curvature with the ones of the symmetry-reduced Ashtekar-Barbero and the spin connections is

$$\beta\mathbf{k}_a^i = \mathbf{a}_a^i - \gamma_a^i = \mathbf{a}_a^i - \boldsymbol{\gamma}_a^i + \delta_3^i\delta_a^\phi. \tag{2.13}$$

Finally, we will conclude this section by introducing some identities that can be useful for the calculations in the next sections. The first identity is the Poisson bracket of the connection with the inverse triad,

$$\{\mathbf{a}_a^i(\vec{x}), \mathbf{e}_b^j(\vec{x}')\} = -4G\beta\delta^{(2)}(\vec{x} - \vec{x}')\,\mathbf{e}_b^i\mathbf{e}_a^j. \tag{2.14}$$

and it is easy to prove. The second identity,

$$\{\mathbf{a}_a^i(\vec{x}), \mathbf{e}(\vec{x}')\} = 4G\beta\delta^{(2)}(\vec{x} - \vec{x}')\,\mathbf{e}\,\mathbf{e}_a^i, \tag{2.15}$$

is based on $\delta\mathbf{e} = \mathbf{e}\,\mathbf{e}_b^i\delta\mathbf{e}_i^b$, for any given variation $\delta\mathbf{e}$ (in particular it is therefore valid for $\partial_a\mathbf{e}$). For the next identity, we should first notice that, given any phase space tensor of the form T_i^a with density weight one, we can define

$$\boldsymbol{\gamma}(T) = \int dxdy\, T_i^a \boldsymbol{\gamma}_a^i, \tag{2.16}$$

at least formally (we are not taking into account boundary terms neither fall-off conditions). These types of expressions appear, for instance, in the Lorentzian part of the symmetry-reduced Hamiltonian constraint. After a lengthy but simple calculation, one can prove that

$$\{\mathbf{a}_a^i(\vec{x}), \boldsymbol{\gamma}(T)\} = 4G\beta\; P^{ijb}{}_{ac}\,{}^{\mathbf{e}}D_b T_j^c\big|_{\vec{x}} + \int dx'dy'\,\{\mathbf{a}_a^i(\vec{x}), T_j^b(\vec{x}')\}\boldsymbol{\gamma}_b^j(\vec{x}'), \tag{2.17}$$

where

$$P^{ijb}{}_{ac} = \frac{1}{2}\left[\epsilon_{ijk}\mathbf{e}_c^l\left(\mathbf{e}_a^l\mathbf{e}_k^b + \mathbf{e}_l^b\mathbf{e}_a^k\right) + \epsilon_{ljk}\mathbf{e}_k^b\left(\mathbf{e}_c^i\mathbf{e}_a^l - \mathbf{e}_a^i\mathbf{e}_c^l\right)\right] \tag{2.18}$$

and ${}^{\mathbf{e}}D_b$ is the covariant derivative compatible with $\mathbf{e}^a_i$, namely ${}^{\mathbf{e}}D_b\mathbf{e}^a_i = 0$.

Similar identities also hold in the full theory, namely, for the original phase space variables A^i_a and E^b_j, and the spin connection Γ^i_a.

III. THE CONSTRAINTS

The total Hamiltonian of the full theory is a combination of 7 constraints: 3 Gauss constraints, 3 vector constraints and the Hamiltonian constraint. Concretely,

$$H_T = \frac{1}{16\pi G}\left[G(\vec{\Lambda}) + D(\vec{N}) + C(N)\right], \tag{3.1}$$

where

$$G(\vec{\Lambda}) = \frac{2}{\beta}\int d^3x\Lambda^i\left(\partial_a E^a_i + \varepsilon_{ijk}A^j_a E^a_k\right), \tag{3.2}$$

$$D(\vec{N}) = \frac{2}{\beta}\int d^3x N^a\left(E^b_i\partial_a A^i_b - \partial_b(E^b_i A^i_a)\right), \tag{3.3}$$

$$C(N) = H_E(N) + H_L(N), \tag{3.4}$$

and where $H_E(N)$ and $H_L(N)$ are the Euclidean and Lorentzian parts of the Hamiltonian constraint, given, respectively, by

$$H_E(N) = -\int d^3x N e^{-1}(A^i_{b,a} - A^i_{a,b} + \epsilon_{ilm}A^l_a A^m_b)\epsilon_{ijk}E^a_j E^b_k, \tag{3.5}$$

$$H_L(N) = \int d^3x N(1+\beta^2)e^{-1}\epsilon_{ijk}\epsilon_{ilm}E^a_j E^b_k K^l_a K^m_b. \tag{3.6}$$

Now, we replace the symmetry-reduced connection A^i_a and the densitized triad E^a_i in the previous expressions. The symmetry-reduced Hamiltonian will be

$$h_T = \frac{1}{8G}\left[g(\vec{\Lambda}) + d(\vec{N}) + c(N)\right], \tag{3.7}$$

and the (smeared) constraints are,

$$g(\vec{\lambda}) = \frac{2}{\beta}\int dxdy\lambda^i\left(\partial_a\mathbf{e}^a_i + \varepsilon_{ijk}\mathbf{a}^j_a\mathbf{e}^a_k + \varepsilon_{ijk}\delta^j_3\delta^\phi_a\mathbf{e}^a_k\right), \tag{3.8}$$

$$d(\vec{N}) = \frac{2}{\beta}\int dxdy N^a\left(\mathbf{e}^b_i\partial_a\mathbf{a}^i_b - \partial_b(\mathbf{e}^b_i\mathbf{a}^i_a) + \delta^\phi_a\delta^i_3\varepsilon_{ijk}\mathbf{a}^j_b\mathbf{e}^b_k\right), \tag{3.9}$$

$$h_E(N) = -\int dxdy\frac{N}{\sqrt{\mathbf{e}}}\left[(\mathbf{a}^i_{b,a} - \mathbf{a}^i_{a,b} + \epsilon_{ilm}\mathbf{a}^l_a\mathbf{a}^m_b)\epsilon_{ijk}\mathbf{e}^a_j\mathbf{e}^b_k + 2\delta^j_3\delta^\phi_b\left(\mathbf{a}^i_a\mathbf{e}^a_i\mathbf{e}^b_j - \mathbf{a}^i_a\mathbf{e}^b_i\mathbf{e}^a_j\right)\right], \tag{3.10}$$

$$h_L(N) = \int dxdy\frac{N}{\sqrt{\mathbf{e}}}(1+\beta^2)\epsilon_{ijk}\epsilon_{ilm}\mathbf{e}^a_j\mathbf{e}^b_k\mathbf{k}^l_a\mathbf{k}^m_b, \tag{3.11}$$

where, as before, we have written the scalar constraint as $c(N) = h_E(N) + h_L(N)$.

IV. EQUATIONS OF MOTION

In this section we will provide the Poisson brackets of the components of the symmetry-reduced connection and densitized triad with the constraints. We will perform a local analysis, assuming suitable boundary terms have been chosen. We will discuss the issue of boundary terms for the asymptotically flat case in section 5, similar analyses can be carried out for other asymptotic behaviors. Let us start with the Gauss constraint. One can easily see that

$$\{\mathbf{e}^a_i(\vec{x}), g(\vec{\lambda})\} = \left(\epsilon_{ijk}\lambda^j\mathbf{e}^a_k\right)\Big|_{\vec{x}} \tag{4.1}$$

$$\{\mathbf{a}^i_a(\vec{x}), g(\vec{\lambda})\} = \left(-\lambda^i_{,a} + \epsilon_{ijk}\lambda^j\mathbf{a}^k_a + \varepsilon_{ijk}\lambda^j\delta^k_3\delta^\phi_a\right)\Big|_{\vec{x}}. \tag{4.2}$$

We see that $\mathbf{e}^a_i$ transforms as a tensor and that $\mathbf{a}^k_a + \delta^k_3\delta^\phi_a$ transforms as a connection.

The vector constraint yields

$$\{\mathbf{e}^a_i(\vec{x}), d(\vec{N})\} = \left(\left(N^b\mathbf{e}^a_i\right)_{,b} - \mathbf{e}^b_i N^a_{,b} - \varepsilon_{ijk}N^d\delta^\phi_d\delta^k_3\mathbf{e}^a_j\right)\Big|_{\vec{x}} \tag{4.3}$$

$$\{\mathbf{a}^i_a(\vec{x}), d(\vec{N})\} = \left(\mathbf{a}^i_b N^b_{,a} + N^b\mathbf{a}^i_{a,b} + \varepsilon_{ijk}N^d\delta^\phi_d\delta^j_3\mathbf{a}^k_a\right)\Big|_{\vec{x}}. \tag{4.4}$$

The Poisson brackets with the Euclidean and Lorentzian parts of the Hamiltonian constraint are,

$$\{\mathbf{e}^a_i(\vec{x}), h_E(N)\} = -\frac{\beta}{2}\left[2\left(\frac{N}{\sqrt{\mathbf{e}}}\varepsilon_{ijk}\mathbf{e}^b_j\mathbf{e}^a_k\right)_{,b} - 2\frac{N}{\sqrt{\mathbf{e}}}\varepsilon_{ilm}\varepsilon_{mjk}\mathbf{a}^l_b\mathbf{e}^a_j\mathbf{e}^b_k - 2\frac{N}{\sqrt{\mathbf{e}}}\delta^j_3\delta^\phi_b\left(\mathbf{e}^a_i\mathbf{e}^b_j - \mathbf{e}^b_i\mathbf{e}^a_j\right)\right]\Bigg|_{\vec{x}} \tag{4.5}$$

$$\{\mathbf{e}^a_i(\vec{x}), h_L(N)\} = \frac{\beta}{2}\left(\frac{2}{\beta}(1+\beta^2)\frac{N}{\sqrt{\mathbf{e}}}\varepsilon_{ilm}\varepsilon_{jkm}\mathbf{k}^l_b\mathbf{e}^b_j\mathbf{e}^a_k\right)\Bigg|_{\vec{x}} \tag{4.6}$$

Finally, we will provide the Poisson brackets of the components of the symmetry-reduced connection with the symmetry-reduced Hamiltonian constraint. The Euclidean part is simply given by

$$\begin{aligned}\{\mathbf{a}^i_a(\vec{x}), h_E(N)\} = -\frac{\beta}{2}\Bigg[&-\frac{N}{2}\mathbf{C}_E\mathbf{e}^i_a + 2\frac{N}{\sqrt{\mathbf{e}}}\epsilon_{ijk}\mathbf{F}^k_{ab}\mathbf{e}^b_j \\ &+\frac{2N}{\sqrt{\mathbf{e}}}\left(\delta^j_3\delta^\phi_b\mathbf{a}^i_a\mathbf{e}^b_j + \delta^i_3\delta^\phi_a\mathbf{a}^j_b\mathbf{e}^b_j - \delta^j_3\delta^\phi_a\mathbf{a}^i_b\mathbf{e}^b_j - \delta^i_3\delta^\phi_b\mathbf{a}^j_a\mathbf{e}^b_j\right)\Bigg]\Bigg|_{\vec{x}},\end{aligned} \tag{4.7}$$

where $\mathbf{F}^i_{ab} = \mathbf{a}^i_{b,a} - \mathbf{a}^i_{a,b} + \epsilon_{ilm}\mathbf{a}^l_a\mathbf{a}^m_b$. Here we have introduced

$$\mathbf{C}_E = \frac{1}{\sqrt{\mathbf{e}}}\left[\mathbf{F}^i_{ab}\epsilon_{ijk}\mathbf{e}^a_j\mathbf{e}^b_k + 2\delta^j_3\delta^\phi_b\left(\mathbf{a}^i_a\mathbf{e}^a_i\mathbf{e}^b_j - \mathbf{a}^i_a\mathbf{e}^b_i\mathbf{e}^a_j\right)\right], \tag{4.8}$$

that is nothing but the Euclidean part of the local Hamiltonian constraint $\mathbf{C}$. On the other hand, for the Lorentzian part we must notice that $\{\mathbf{a}^i_a, \mathbf{k}^j_b\} = -\beta^{-1}\{\mathbf{a}^i_a, \boldsymbol{\gamma}^j_b\}$ and also remember the identity in (2.17) and the definition (2.18). After some manipulations, one gets,

$$\{\mathbf{a}_a^i(\vec{x}), h_L(N)\} = -\frac{\beta}{2}\Bigg\{-\frac{N}{2}\mathbf{C}_L\mathbf{e}_a^i - 2(1+\beta^2)\frac{N}{\sqrt{\mathbf{e}}}\varepsilon_{ijk}\varepsilon_{lmk}\mathbf{k}_a^l\mathbf{k}_b^m\mathbf{e}_j^b + \frac{1}{\beta}(1+\beta^2)\left(\varepsilon_{ijk}\mathbf{e}_a^m - \frac{1}{2}\varepsilon_{mjk}\mathbf{e}_a^i\right)\mathbf{e}_j^b\mathbf{e}_k^c\left[{}^{\mathbf{e}}D_b\left(\frac{N\mathbf{k}_c^m}{\sqrt{\mathbf{e}}}\right) - {}^{\mathbf{e}}D_c\left(\frac{N\mathbf{k}_b^m}{\sqrt{\mathbf{e}}}\right)\right]\Bigg\}\Bigg|_{\vec{x}}. \tag{4.9}$$

Similarly,

$$\mathbf{C}_L = \frac{1}{\sqrt{\mathbf{e}}}(1+\beta^2)\epsilon_{ijk}\epsilon_{ilm}\mathbf{e}_j^a\mathbf{e}_k^b\mathbf{k}_a^l\mathbf{k}_b^m, \tag{4.10}$$

is the Lorentzian part of the Hamiltonian constraint C.

V. THE KERR, SCHWARZSCHILD AND MINKOWSKI SOLUTIONS

We have explicitly checked that the symmetry-reduced model admits the well-known solution of the full theory given by the Kerr metric.

In the usual spherical coordinates (r, θ, ϕ), the densitized triad for Kerr in a diagonal gauge, identifying the internal directions with the coordinates, takes the following form:

$$\begin{aligned}
\mathbf{e}_3^r &= \sin\theta\sqrt{(r^2+a^2)(r^2+a^2\cos^2\theta) + a^2\ r\ r_s\sin^2\theta},\\
\mathbf{e}_1^\theta &= \frac{\sin\theta\sqrt{(r^2+a^2)(r^2+a^2\cos^2\theta) + a^2\ r\ r_s\sin^2\theta}}{\sqrt{a^2+r^2-r\ r_s}},\\
\mathbf{e}_2^\phi &= \frac{r^2+a^2\cos^2\theta}{a^2+r^2-r\ r_s},
\end{aligned} \tag{5.1}$$

where $a = J/r_s$, while the rest of its components vanish. Together with the well-known choices of lapse and shift (e.g. [10, 11])

$$N = \sqrt{\frac{(a^2+r(r-r_s))(a^2+2r^2+a^2\cos(2\theta))}{2(a^2+r^2)(r^2+a^2\cos^2\theta+2a^2\ r\ r_s\sin^2\theta)}}, \tag{5.2}$$

$$N_\phi = -\frac{rr_s a\sin^2\theta}{r^2+a^2\cos^2\theta}, \quad N_r = 0, \quad N_\theta = 0, \tag{5.3}$$

and one can easily verify that this solution corresponds to the Kerr metric in Boyer–Lindquist coordinates. Here, r_s is the Schwarzschild radius and a the angular momentum per unit mass. The spin connection can be computed out of the densitized triad. Finally, the connection components can be easily computed provided the extrinsic curvature in triadic form. Concretely,

$$k_a^i = \delta^{ij}\frac{\mathbf{e}_j^b}{\sqrt{\mathbf{e}}}K_{ab} \tag{5.4}$$

where we obtain the extrinsic curvature from

$$K_{ab} = \frac{1}{2N}\left(-\dot{h}_{ab} + \nabla_a N_b + \nabla_b N_a\right), \tag{5.5}$$

keeping in mind that for our stationary solution $\dot{h}_{ab} = 0$. The expressions of the components of the connection are rather lengthy, as well as those of the Langrange multipliers λ^i of the Gauss constraint. We give them in the appendix. To determine the Lagrange multipliers we insert the above expressions for the triad, connection, lapse and shift in the equations of motion.

It is also very easy to check that in the limit $a \to 0$ we recover the Schwarzschild (static) solution. Concretely, the densitized triad reduces to

$$\begin{aligned} \mathbf{e}_3^r &= r^2 \sin\theta, \\ \mathbf{e}_1^\theta &= \frac{r\sin\theta}{\sqrt{1-\frac{r_s}{r}}}, \\ \mathbf{e}_2^\phi &= \frac{r}{\sqrt{1-\frac{r_s}{r}}}. \end{aligned} \tag{5.6}$$

the lapse now takes the familiar form

$$N = \sqrt{1-\frac{r_s}{r}}, \tag{5.7}$$

while the shift vanishes, namely $N_\phi = 0$. Besides, we also have that $K_{ab} = 0$. Therefore, the connection is completely determined by the spin connection.

The Lagrange multipliers for the Gauss constraint take the simple form (for the Schwarzschild case, for the Kerr case see the appendix):

$$\begin{aligned} \lambda^1 &= 0 \\ \lambda^2 &= 0 \\ \lambda^3 &= -\beta\frac{r_s}{r^2} \end{aligned} \tag{5.8}$$

Finally, the Minkowski solution can be recovered from the limit $a \to 0$ and $r_s \to 0$. The densitized triad reduces to

$$\mathbf{e}_3^r = r^2 \sin\theta, \tag{5.9}$$

$$\mathbf{e}_1^\theta = r\sin\theta, \tag{5.10}$$

$$\mathbf{e}_2^\phi = r. \tag{5.11}$$

The lapse function $N = 1$ becomes the usual one in flat spacetimes. As in the previous case, the spin connection completely determines the connection, and any other Lagrange multiplier (shift and λ^i) vanish.

VI. BOUNDARY TERMS

Up to now the analysis we made has been local. When one is in asymptotically flat space-times one needs to be mindful about falloff rates and integrations by parts. In particular, in addition to the constraints, one has a true Hamiltonian associated to the generators of the Lorentz group at infinity. In this section we will identify the boundary contributions needed to make the action differentiable in the asymptotically flat case for the Ashtekar–Barbero variables with axial symmetry. We will review individually each set of constraints to see if boundary terms are needed. We will follow closely [12, 13].

A. Diffeomorphism constraint

We start this section by writing the portion of the action that corresponds to the diffeomorphism constraints in ADM-variables:

$$D[\vec{N}] = -2 \int_\Sigma \mathrm{d}^3x \, N^a \, \nabla_b P^b{}_a, \tag{6.1}$$

∇ being the covariant derivative compatible with the metric g_{ab} and P^{ab} the ADM-momentum, given by:

$$P_{ab} = -\frac{1}{16\pi G}\sqrt{q}\left(K_{ab} - q^{ab}K\right), \tag{6.2}$$

where K_{ab} is the extrinsic curvature

$$K_{ab} = \frac{1}{2N}\left(-\dot{q}_{ab} + \nabla_b N_a + \nabla_a N_b\right). \tag{6.3}$$

Taking variations of that term of the action with respect to the canonical variables yields the following boundary contribution,

$$2\int_\Sigma \mathrm{d}^3x \nabla_b \left(N_a \delta P^{ab}\right), \tag{6.4}$$

which must be canceled at infinity. We must thus add to the action the following surface term:

$$\mathcal{P} = -2\oint_{\delta\Sigma} \mathrm{d}S^b N^a P_{ab}. \tag{6.5}$$

As we see, this boundary term not only depends on the phase space variables, but also on the Lagrange multipliers (the shift functions N^a in this case). In the following, we will assume that the latter will be prescribed functions at spatial infinity (determined by the asymptotic form of the Kerr metric given below in Eq. (6.6)). Therefore, we will not consider variations of these functions on the boundary.

The boundary term in Eq. (6.5) can be easily evaluated for the Kerr metric at spatial infinity. In spherical coordinates, its asymptotic form is given by

$$\mathrm{d}S^2 = -\left[1-\frac{2m}{r}\right]\mathrm{d}t^2 - \frac{2J\sin^2\theta}{r}\left(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t\right) + \left[1+\frac{2m}{r}\right]\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \quad (6.6)$$

Moreover,

$$\mathrm{d}S^b = \mathrm{d}Sn^b = \mathrm{d}S\delta^b_r, \quad (6.7)$$

and $\mathrm{d}S = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$. From the metric in Eq. (6.6), the only contributions to the integral are

$$\mathcal{P} = -2\oint_{\delta\Sigma}\mathrm{d}SN^\phi P_{\phi r} = -2\oint_{\delta\Sigma}\mathrm{d}\theta\mathrm{d}\phi r^2\sin\theta N^\phi\left(\frac{\sqrt{q}}{16\pi G}(K_{\phi r} - q_{\phi r}K)\right), \quad (6.8)$$

where q_{ab} is the spatial metric and K^{ab} the extrinsic curvature. At the boundary $\delta\Sigma$, we have

$$\begin{aligned} q_{\phi r} &= 0, \\ \sqrt{q} &= r^{3/2}\sqrt{r+r_s}\sin\theta, \\ N_\phi &= -\frac{2J\sin^2\theta}{r} \quad \to \quad N^\phi = -\frac{2J}{r^3}, \\ K_{\phi r} &= \frac{3J\sin^2\theta}{\sqrt{r^3(r-r_s)+4J^2\sin^2\theta}}, \\ K &= 0. \end{aligned}$$

Performing the integral, the leading term in the expansion in $1/r$ is

$$\mathcal{P} = \lim_{r\to\infty}\frac{9\pi}{64G}\frac{J^2}{r} = 0. \quad (6.9)$$

Now, in Ashtekar-Barbero variables, the diffeomorphism constraint of the full theory takes the form

$$\frac{1}{16\pi G}D(\vec{N}) = \frac{1}{8\pi G\beta}\int_\Sigma N^a\left(E^b_i\partial_a A^i_b - \partial_b(E^b_i A^i_a)\right). \quad (6.10)$$

Its boundary term (for asymptotically flat spacetimes) takes the form

$$\mathcal{P} = \frac{1}{8\pi G\beta}\oint_{\delta\Sigma}N^a E^b_i\ A^i_a\ \mathrm{d}S_b. \quad (6.11)$$

On the other hand, in our reduced theory, the reduced diffeomorphism constraint is given by

$$\frac{1}{8\pi G}d(\vec{N}) = \frac{1}{4\pi G\beta}\int_\sigma N^a\left(\mathbf{e}^b_i\partial_a\mathbf{a}^i_b - \partial_b(\mathbf{e}^b_i\mathbf{a}^i_a) + \delta^\phi_a\delta^i_3\varepsilon_{ijk}\mathbf{a}^j_b\mathbf{e}^b_k\right), \quad (6.12)$$

where σ are the r,θ 2D spatial sections of our reduced theory, with topology $\mathbb{R}^2$. The corresponding boundary term is

$$\mathfrak{p} = \frac{1}{4G\beta} \oint_{\delta\sigma} N^a \mathbf{e}^b_i \ \mathbf{a}^i_a \ \mathrm{d}s_b, \tag{6.13}$$

where

$$\mathrm{d}s^b = \mathrm{d}sn^b = \mathrm{d}s\delta^b_r, \tag{6.14}$$

$$\tag{6.15}$$

with $\mathrm{d}s = r^2 \sin\theta \mathrm{d}\theta$. Since the only non-vanishing component of the shift is N^ϕ, the required boundary term is

$$\mathfrak{p} = \frac{1}{4G\beta} \oint_{\delta\sigma} N^\phi \ \mathbf{e}^r_3 \mathbf{a}^3_\phi \ q_{rr} \ r^2 \sin\theta \ \mathrm{d}\theta. \tag{6.16}$$

Where we have chosen the triad as in (5.1). Again, we have at infinity:

$$\begin{aligned} N^\phi &= -\frac{2J}{r^3}, \\ e^r_3 &= r^2 \sin\theta, \\ q_{rr} &= \frac{1}{1+\frac{r_s}{r}}, \\ a^3_\phi &= \cos\theta + \gamma \frac{3Jr\sin^2\theta}{\sqrt{r(r+r_s)(r^3(r-r_s)+4J^2\sin^2\theta)}}. \end{aligned}$$

The term proportional to $\cos\theta$ integrates to zero (since it contains the integral of $\cos\theta\sin\theta$ between 0 and π) while the other term is easily seen to yield the same result as in (6.9).

B. Hamiltonian constraint

Now we turn to the portion of the action involving the Hamiltonian constraint. In ADM variables:

$$\frac{1}{16\pi G} C[N] = \frac{1}{16\pi G} \int_\Sigma N \left(q^{-1/2} \left(P_{ab} P^{ab} - \frac{1}{2} P \right) - q^{1/2} R \right). \tag{6.17}$$

Where R is the Ricci scalar. In order for the variations with respect to the dynamical variables to be well defined, it is necessary to add to the action the surface term (see e.g. [12]):

$$\mathcal{E} = \frac{1}{16\pi G} 2 \oint_{\delta\Sigma} \mathrm{d}S_d N \sqrt{q} q^{ac} q^{bd} \bar{\nabla}_{[c} q_{b]a}. \tag{6.18}$$

Where $\bar{\nabla}$ is the covariant derivative compatible with the order zero of expansion in $1/r$ of the spatial metric at infinity. This term corresponds to time translations at infinity. The surface term actually has another contribution coming from the fact that the Ricci tensor has second derivatives, requiring two integration by parts. That contribution corresponds to boosts at infinity, but due to our choice of adapted coordinates we do not allow such boosts.

We will evaluate the previous boundary term at spatial infinity in order to show that it is finite. For convenience, we will introduce an asymptotically Cartesian coordinate system with coordinates $\{x^a\}$. We then expand our metric asymptotically as $g_{ab} = \eta_{ab} + h_{ab}$, with η_{ab} the flat space metric and h_{ab} a small perturbation around η_{ab}. We also expand the lapse as $N = 1 + \mathcal{O}(1/r)$. Then, in the limit $r \to \infty$, the leading contribution to the boundary term takes the form

$$\mathcal{E} = \frac{1}{16\pi G} \oint_{\delta\Sigma} \left(\frac{\partial h^b_a}{\partial x^b} - \frac{\partial h^b_b}{\partial x^a} \right) \mathrm{d}S^a = \frac{r_s}{2G}. \tag{6.19}$$

Now, a direct calculation shows that, in Ashtekar variables, the equivalent boundary term to (6.18) takes the following form:

$$\mathcal{E} = -\frac{1}{8\pi G\beta} \oint_{\delta\Sigma} \mathrm{d}S_a \frac{N}{\sqrt{E}} \left(E^a_i \bar{D}_b E^b_i + E^b_i \bar{D}_b E^a_i \right), \tag{6.20}$$

where, similar to the derivative $\bar{\nabla}$ defined earlier, $\bar{D}$ is the covariant derivative compatible with the order zero component of the triad at infinity.

In our reduced theory, the boundary term is given by

$$\mathfrak{e} = -\frac{1}{4G\beta} \oint_{\delta\sigma} \mathrm{d}s_a \frac{N}{\sqrt{e}} \left(\mathbf{e}^a_i \bar{D}_b \mathbf{e}^b_i + \mathbf{e}^b_i \bar{D}_b \mathbf{e}^a_i \right). \tag{6.21}$$

Its evaluation in Cartesian coordinates agrees with the result given in (6.19).

C. Gauss constraint

The contribution to the action of the Gauss constraint in the full theory is given by

$$\frac{1}{16\pi G} G(\vec{\lambda}) = \frac{1}{8\pi G\beta} \int_\Sigma \lambda^i \left(\partial_a E^a_i + \varepsilon_{ijk} A^j_a E^a_k \right). \tag{6.22}$$

The variation of this contribution also requires another boundary term in order to make the full variational problem well defined. It is given by

$$\mathcal{Q} = \frac{1}{8\pi G\beta} \oint_{\delta\Sigma} dS_a (E^a_i - \bar{E}^a_i) \lambda^i, \tag{6.23}$$

where $\bar{E}^a_i$ is the densitized triad at spatial infinity. Without this term, inserting the asymptotic form of the triad, the result is divergent. Since at spatial infinity the metric is flat, $\bar{E}^a_i$ is independent of M and J. Therefore, any variational derivative of this term will be zero. Similar boundary terms have been suggested in previous treatments [12, 13].

In our symmetry reduced theory, the reduced Gauss constraint is given by:

$$\frac{1}{8G} g(\vec{\lambda}) = \frac{1}{4G\beta} \int_\sigma \lambda^i \left(\partial_a \mathbf{e}^a_i + \varepsilon_{ijk} \mathbf{a}^j_a \mathbf{e}^a_k + \varepsilon_{ijk} \delta^j_3 \delta^\phi_a \mathbf{e}^a_k \right). \tag{6.24}$$

The boundary term takes the same form in terms of the reduced densitized triad, namely,

$$\mathfrak{q} = \frac{1}{4G\beta} \oint_{\delta\sigma} dS_a (\mathbf{e}^a_i - \bar{\mathbf{e}}^a_i)\lambda^i, \tag{6.25}$$

where $\bar{\mathbf{e}}^a_i$ is the reduced triad at spatial infinity. After evaluation at spatial infinity, one gets

$$\mathfrak{q} = \frac{5\pi^2}{64G} \frac{J^2}{r_S}. \tag{6.26}$$

We note that it only depends on M and J. Therefore, no new observables appear. This is due to our choice of diagonal triads, which ties spatial rotations generated by J to internal rotations.

VII. CONCLUSIONS

We have developed the Ashtekar–Barbero framework for axisymmetric spacetimes. We found triads and connections adapted to the symmetry and wrote the Gauss law, vector and Hamiltonian constraints. We showed that the Kerr solution indeed solves the constraints and the evolution equations. We also discussed the boundary terms needed to make the action differentiable in a canonical treatment. This lays out a framework to attempt a loop quantization of axially symmetric spacetimes. These represent the most complex midisuperspaces considered up to date. The strategy we intend to follow for quantization is similar to the one we pursued in spherical symmetry ([2, 3]). We will build spin network states based on the reduced connection. The third component of the connection will be represented by a point holonomy and the first two with genuine holonomies in the two dimensional reduced space adapted to the symmetry (for instance r, θ if one were to consider spherical coordinates). On such states the fluxes of the triads will act naturally. We will use these basic operators to construct the Hamiltonian of the theory. We will discuss details in a future publication.

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Appendix A: Axisymmetric classical reduction

From the components of Eq. (2.3) we get the set of differential equations

$$\partial_\phi A^1_a = -A^2_a, \quad \partial_\phi A^2_a = A^1_a, \quad \partial_\phi A^3_a = 0. \tag{A1}$$

From $\partial_\phi A_a^3 = 0$ we conclude that A_a^3 must be equal to a function $\mathbf{a}_a^3$ independent of ϕ. On the other hand, the most general solutions for the differential equations of A_a^1 and A_a^2 are

$$A_a^1 = \cos\phi\, \mathbf{a}_a^1 - \sin\phi\, \mathbf{a}_a^2, \quad A_a^2 = \sin\phi\, \mathbf{a}_a^1 + \cos\phi\, \mathbf{a}_a^2, \tag{A2}$$

with $\mathbf{a}_a^1$ and $\mathbf{a}_a^2$ independent of ϕ. These solutions yield to Eq. (2.4). The symmetry reduction of the densitized triad E_i^a (despite being a tensor density of weight one) is similar to the one of A_a^i for axisymmetric spacetimes (for $\lambda_3 = \text{const}$). The reduced components of E_i^a take a similar form as those of A_a^i but replacing $\mathbf{a}_a^i$ by $\mathbf{e}_a^i$. One then obtains Eq. (2.5).

Appendix B: Kerr solution: Lagrange multipliers and connexion components

Gauss' law Lagrange multipliers:

$$\lambda^1 = 0,$$

$$\lambda^2 = \frac{\sqrt{2}a\sqrt{a^2+r(r-r_s)}r_s(a^4-3a^2r^2-6r^4+4ar(a^2+r^2)\beta\cos(\theta)+a^2(a^2-r^2)\cos(2\theta))\sin(\theta)}{((a^2+2r^2+a^2\cos(2\theta))(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))^{3/2})},$$

$$\begin{aligned}
\lambda^3 = {} & \left(\frac{a^2+r(r-r_s)}{(a^2+r^2)(r^2+a^2\cos^2(\theta))+a^2rr_s\sin^2(\theta)}\right)^{1/2} \Bigg\{\frac{(2r-r_s)\sin(\theta)}{(a^2+r(r-r_s))^{1/2}} \\
& + \frac{8a^3r(a^2+r(r-r_s))^{1/2}r_s(1+\beta^2)\cos(\theta)\sin^3(\theta)}{\beta\ (a^2+2r^2+a^2\cos(2\theta))(a^4+2r^4+a^2r(3r+r_s)+\ a^2(a^2+r(r-r_s))\cos(2\theta))} \\
& - \Bigg[\frac{4(r^2+a^2\cos^2(\theta))((a^2+r^2)(r^2+a^2\cos^2(\theta))+a^2rr_s(\sin(\theta))^2)^{1/2}}{a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta)} \\
& \Bigg(\frac{(16r^5+4a^2r^2(4r-r_s)+a^4(6r+r_s)+4a^2r(2a^2+r(4r+r_s))\cos(2\theta)+a^4(2r-r_s)\cos(4\theta))\sin(\theta)}{8\sqrt{2}(r^2+a^2\cos^2(\theta))^2\left(\frac{a^4+2r^4+a^2r(3r+r_s)}{a^2+r(r-r_s)}+a^2\cos(2\theta)\right)^{1/2}} \\
& + \frac{8a^3rr_s\beta\cos(\theta)\sin^3(\theta)((a^2+r(r-r_s))((a^2+r^2)(r^2+a^2\cos^2(\theta))+a^2rr_s\sin^2(\theta)))^{1/2}}{(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))}\Bigg)\Bigg]\Bigg\}.
\end{aligned}$$

Connection components:

$$a_r^1 = \frac{-2ar_s\beta(r^2+a^2\cos^2(\theta))(a^4-3a^2r^2-6r^4+a^2(a-r)(a+r)\cos(2\theta))\sin(\theta)}{(a^2+r(r-r_s))^{1/2}(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))}$$
$$+\frac{a^2\sin(2\theta)}{(a^2+r(r-r_s))^{1/2}(a^2+2r^2+a^2\cos(2\theta))},$$
$$a_r^2 = 0,$$
$$a_r^3 = 0,$$
$$a_\theta^1 = \frac{r(a^2+r(r-r_s))^{1/2}}{r^2+a^2\cos^2(\theta)}$$
$$-\frac{8a^3r(a^2+r(r-r_s))^{1/2}r_s\beta\cos(\theta)(r^2+a^2\cos^2(\theta))\sin^2(\theta)}{(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))},$$
$$a_\theta^2 = 0,$$
$$a_\theta^3 = 0,$$
$$a_\phi^1 = 0,$$
$$a_\phi^2 = \frac{(16r^5+4a^2r^2(4r-r_s)+a^4(6r+r_s)+4a^2r(2a^2+r(4r+r_s))\cos(2\theta)+a^4(2r-r_s)\cos(4\theta))\sin(\theta)}{8\sqrt{2}(r^2+a^2\cos^2(\theta))^2(\frac{a^4+2r^4+a^2r(3r+r_s)}{a^2+r(r-r_s)}+a^2\cos(2\theta))^{1/2}}$$
$$+\frac{8a^3rr_s\beta\cos(\theta)\sin^3(\theta)((a^2+r(r-r_s))((a^2+r^2)(r^2+a^2\cos^2(\theta))+a^2rr_s\sin^2(\theta)))^{1/2}}{(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))},$$

$$a_\phi^3 = \frac{2(a^2(a^2+r(r-r_s))((5a^2+8r^2)\cos(3\theta)+a^2\cos(5\theta))}{2\sqrt{2}(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))^{1/2}}$$
$$+\frac{2(5a^6+8r^6+4a^2r^3(5r+r_s)+a^4r(17r+3r_s))\cos(\theta)}{2\sqrt{2}(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))^{1/2}}$$
$$+\frac{\sqrt{2}ar_s\beta(-a^4+3a^2r^2+6r^4+a^2(-a^2+r^2)\cos(2\theta))\sin^2(\theta)}{(a^2+2r^2+a^2\cos(2\theta))^2(a^4+2r^4+a^2r(3r+r_s)+a^2(a^2+r(r-r_s))\cos(2\theta))^{1/2}}.$$

[1] A. Ashtekar and P. Singh, Class. Quant. Grav. **28**, 213001 (2011) [arXiv:1108.0893 [gr-qc]].
[2] R. Gambini, J. Olmedo and J. Pullin, Class. Quant. Grav. **31**, 095009 (2014) [arXiv:1310.5996 [gr-qc]].
[3] R. Gambini, E. M. Capurro and J. Pullin, Phys. Rev. D **91**, no. 8, 084006 (2015) [arXiv:1412.6055 [gr-qc]].
[4] V. Husain and J. Pullin, Mod. Phys. Lett. A **5**, 733 (1990).
[5] D. E. Neville, Phys. Rev. D **92**, no. 4, 044006 (2015) [arXiv:1305.2599 [gr-qc]].
[6] F. Hinterleitner and S. Major, Class. Quant. Grav. **29**, 065019 (2012) [arXiv:1106.1448 [gr-qc]].
[7] V. Husain and L. Smolin, Nucl. Phys. B **327**, 205 (1989).
[8] B. Elizaga Navascus, M. Martn-Benito and G. A. Mena Marugn, Int. J. Mod. Phys. D **25**, no. 08, 1642007 (2016) [arXiv:1608.05947 [gr-qc]].
[9] R. Benguria, P. Cordero, C. Teitelboim, Nuc. Phys. B122, 61 (1977).

[10] D. Wiltshire, M. Visser, S. Scott, "The Kerr spacetime: rotation black holes in general relativity", Cambridge University Press, Cambridge (2009).
[11] S. Chandrasekhar, "The mathematical theory of black holes" Oxford University Press, Oxford (1998).
[12] T. Thiemann, Class. Quant. Grav. 12, 181198 (1995) [arXiv:9910008 [gr-qc]]
[13] M. Campiglia, Class. Quant. Grav. 32, 145011 (2015) [arXiv:1412.5531 [gr-qc]]