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MODELOS BIDIMENSIONALES EN GRAVEDAD CUÁNTICA DE LAZOS

Autor:
Saeed Rastgoo

Director de Tesis:
Prof. Rodolfo Gambini

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TWO DIMENSIONAL MODELS IN LOOP QUANTUM GRAVITY

by:
Saeed Rastgoo

Advisor:
Prof. Rodolfo Gambini

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Abstract

In this work we study a generic class of two dimensional systems in loop quantum gravity. It is the most general diffeomorphism invariant action yielding second order differential equations for the metric and a scalar dilaton field. We introduce a Hamiltonian formulation of the generic case in the tetrad variables, and then focus on two specific cases of this general action, the two dimensional CGHS dilatonic model and the spherically symmetric reduced 3+1 dimensional model, both coupled to matter. Many of these two dimensional models, are simpler to analysis compared to the full four dimensional systems but they include many of the interesting characteristics of the full model, like black hole solutions and Hawking radiation and related issues. This makes these two dimensional models worth studying and there is hope that their analysis will give us clues and insights on how the full theory behaves and should be handled.

For the 3+1 case, we complete the Hamiltonian formulation and introduce canonical transformations that takes us from our formalism to the common Hamiltonian derived by others. In the next step, we try to find a vacuum state for this model by using the variational techniques and by utilizing master constraint program and uniform discretization method. In this process, we quantize the gravitational degrees of freedom using loop quantum gravity techniques but quantize the matter degrees of freedom using Fock space quantization. This is to allow us to compare our result with the ordinary quantum field theory. As a result of this analysis we find a ground state in the form of a direct product of a Fock vacuum for the scalar field and Gaussian states centered around flat space-time for the gravitational variables.

In the next step, we try to not only quantize the gravitational variables by loop quantum gravity methods, but also the matter degrees of freedom. Then we show that the correction terms that represent the difference between two types of quantization of matter field are small enough for our result to be accurate for our purpose. Then we proceed to find the propagator of the scalar matter field on the vacuum state that we derived in previous part. This gives us a modified dispersion relation that signals a kind of Lorentz invariance violation. We discuss this issue and show that this stems from both discretization and holonomization, where the part coming from holonomization can be made as small as needed.

In the last chapter, we apply our Hamiltonian formulation to the two dimensional CGHS model. This model has been studied in a conformally transformed context but we try to analyze it directly without such a transformation. One reason is that the pure gravitational part of the conformally transformed CGHS is trivial and also it is better to work with the variables that have direct geometric meaning so that we do not need to transform everything back from the non-physical geometry to the physical one at the end. This will allow to quantize the gravitational magnitudes in the loop formalism and study if this quantization would allow to have a singularity free theory. Thus with this method, we derive Ashtekar variables for the CGHS from our generic formalism and make a complete classical analysis of the system, including boundary conditions and asymptotic behavior of the variables. This opens the door for future works in quantizing this rich model.

Resumen

En este trabajo se estudia una clase genérica de sistemas de dos dimensiones en gravedad cuántica de lazos. Es la acción más general invariante bajo difeomorfismo que resultan en ecuaciones diferenciales de segundo orden para la métrica y un campo escalar dilatón. Se introduce una formulación Hamiltoniana en el caso genérico de las variables de tétrada. A continuación, consideramos dos casos específicos de esta acción general, el modelo de dos dimensiones CGHS dilatonico y el modelo reducido 3+1 dimensional con simetría esférica, ambos acoplados a la materia. Muchos de estos modelos, son más simples de analizar en comparación con los sistemas de cuatro dimensiones completos, ellos incluyen muchas de las características interesantes del modelo completo, como las soluciones con agujeros negros y la radiación de Hawking y otros temas relacionados. Esto hace que estos dos modelos bidimensionales sean importantes de estudiar ya que hay esperanza de que su análisis nos dará pistas y puntos de vista sobre cómo se comporta la teoría completa y como debe ser manejada.

Para el caso de 3+1, completamos la formulación Hamiltoniana e introducimos transformaciones canónicas que conducen de nuestro nos lleva desde nuestro formalismo al Hamiltoniano conocido derivado por otros colegas. Luego tratamos de encontrar un estado vacío para este modelo mediante el uso de las técnicas variacionales y utilizando el programa “master constraint” y método de “uniform discretization”. En este proceso, cuantizamos los grados de libertad gravitatorios utilizando las técnicas de la gravedad cuántica de lazos, pero cuantizamos los grados de libertad de la materia usando el motodo de cuantizacion de Fock. Esto es para que podamos comparar nuestros resultados con la teoría cuántica de campo ordinaria. Como resultado de este análisis nos encontramos con un estado base en la forma de un producto directo entre un vacío de Fock para el campo escalar y estados Gaussianos centrados alrededor de espacio-tiempo plano para las variables gravitatorias.

En el siguiente paso, se intenta no sólo la cuantizacion de las variables gravitatorias con los métodos de gravedad cuántica de lazos, sino también los grados libertad de la materia . Se muestra que los términos de corrección, que representan la diferencia entre dos tipos de cuantización del campo de la materia son lo suficientemente pequeños para que los resultados sean precisos para nuestro propósito. Despues se procede a encontrar el propagador del campo escalar de la materia en el estado de vacío que se deriva en parte anterior. Esto nos da una relación de dispersión modificada que señala un tipo de violación de la invariancia de Lorentz. Trataremos este tema y demostramos que esto se deriva debido la discretización y holonomización. Se muestra que la parte procedente de la holonomización puede ser tan pequeña como sea necesario.

En el último capítulo, aplicamos nuestra formulación Hamiltoniana al modelo de dos dimensiones CGHS. Este modelo ha sido estudiado en un contexto con transformación conforme, pero tratamos de analizarlo directamente sin necesidad de esta transformación. Una de las razones es que la parte pura gravitatoria de CGHS con transformación conforme es trivial y es mejor trabajar con las variables que tienen un significado geométrico directo. De este forma no necesitamos transformar todo de la geometría no-física a la geometría física al final del proceso. Esto permitirá cuantizar las magnitudes gravitacionales en el formalismo de lazos y estudiar si esta cuantiza-

ción permite tener una teoría libre de las singularidades. Así, con este método, se deriva las variables de Ashtekar para el CGHS con nuestro formalismo genérico y se hace un análisis clásico completo del sistema, incluyendo las condiciones de frontera y el comportamiento asintótico de las variables. Esto abre la puerta para futuros trabajos en cuantizacion de este interesante modelo.

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Los abajo firmantes certificamos que hemos leído el presente trabajo titulado “Two Dimensional Models in Loop Quantum Gravity” hecho por Saeed Rastgoo y encontramos que el mismo satisface los requerimientos curriculares que la Facultad de Ciencias exige para la tesis del título de Doctor en Física.

Dr. Raul Donangelo ______________________________

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To My Parents...

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Contents

1 Introduction

1.1 The unfinished revolution

In the beginning and mid years of the 20th century, two great discoveries happened in theoretical physics. The formulation of general theory of relativity and the quantum theory. Although being extremely successful in explaining various phenomena in the universe, many physicist agree that these two are part of an unfinished revolution. We will see the reason in a moment.

General relativity (GR) has fundamentally changed our understanding of spacetime. It has wiped out the notion of a unique time and a fixed background space that affects everything but is not affected (an unphysical and unreal notion). This can be viewed in two ways [1]: either there is no gravitational field or "force", just dynamic "spacetime" that affects and gets affected by all objects, and its dynamic geometry appears as a gravitational field, or there is no notion of spacetime at all but there is just gravitational field that couples to everything the same way (interacts with everything independent of their masses) and because of this property it appears to us as a dynamic spacetime. In the latter picture of reality, we don't have fields on spacetime but fields on fields [1]. Because of tight interconnection of the notion of spacetime and gravitational field, this field is (in the latter case) responsible for and contains all the information about the causal structure of the universe.

On the other hand, quantum theory (QT) has vastly and deeply changed our knowledge about the world of very small objects and has introduced a fundamental uncertainty of nature in physics that can not be avoided. It has shown us how quantum systems interact and has been successful in explaining the phenomena that happen in microcosmos.

Despite their astonishing success, these theories have one big problem: It seems that they are incompatible. This is very important because as long as we use each of these theories in their own domain of dominance, we will have no problem because we can practically neglect the effect of the other theory. But when it comes to domains in which both theories are equally important (such as big bang, singularities, etc.) so that one can not neglect effects of neither of them, the problem shows itself.

Basically this problem can be seen as following: for the phenomena for which the effects of both theories are important, one should use a combination of both theories which we call Quantum Gravity (QGR). But as we said these seem to be incompatible and thus we can not explain those phenomena because we do not have the theory. The incompatibility can be seen as follows: As we mentioned, the gravitational field is responsible for the causal structure of the universe. On the other hand, QT introduces a fundamental uncertainty in the phenomena. Therefore by bringing both theories together, we are essentially introducing a fundamental uncertainty in the causal structure of events! This does not really seem to be easy to comprehend or implement, because causality is the very notion that physics (or any other branch of science for that matter) is built upon. But physicists have learned that not everything in the physical world goes by their intuitive expectations and thus we should keep looking and thinking. By this attitude, we actually have gone far and there have been a lot of attempt to quantize gravity and some of them seem to be partly successful, but we are yet far from a full theory.

Another motivation for this fundamental search is the hope that by combining these two theories, the famous problems each of those theories (GR and QT) face on its own, namely singularities in GR and infinities in perturbative expansions in QT, disappear.

1.2 The two main approaches to quantum gravity and their differences

As mentioned in the last section, there has been several attempts to attack the problem of combining GR with QT. Two of them are probably more famous: string theory and loop quantum gravity (LQG). String theory, which has more of its basis on perturbative QT is dealing first and foremost with unification of fields and is not primarily concerned about the nature of spacetime. It assumes that QT is a fundamental theory of nature while GR is only an approximation that arises through the quantum interaction of the fundamental objects of the theory which are one dimensional entities called strings. It comes with several new additions to physics such as strings themselves, higher dimensions, branes, supersymmetry etc. The theory is able to calculate the probability amplitude of different phenomena among particles etc. but is not a background-independent theory.

On the other hand, LQG being a background-independent non-perturbative theory, primarily based on GR, is firstly concerned about the nature of spacetime itself. It assumes that, at least for the time being, GR and QT themselves without any new additional structure should be taken seriously on the same footing and the theory should be based on them. At least until there is some theoretical or experimental evidence found, telling us to add some additional structures to the theory. This claim is based on the lessons that we have learned from development of many parts of classical and modern physics previously [1, 2].

1.3 The motivations and structure of this work

1.3.1 General motivation

The loop quantization of the full four dimensional gravity is currently not complete and it seems that this is mostly due to mathematical technical issues. For that reason, we can use at least two workarounds for this problem. The first workaround is to try to look at some toy models that are mathematically more easier to handle but at the same time, since they acquire many interesting aspects of the full theory, will give us valuable clues about the full theory and how to treat it. The second workaround is to use four dimensional models that posses enough symmetries which makes their analysis easier, like the famous spherical symmetric model.

1.3.2 The problems we address

In this work we will first study the most general two dimensional model of a gravitational system minimally coupled to a scalar matter field. One of the reasons to do this is that many of the important models of quantum gravity are included in this general formalism. Among these models are string-theory inspired dilatonic systems coupled to matter field such as the two dimensional CGHS model [3], and the four dimensional spherically symmetric model coupled to matter which looks like a two dimensional model upon reduction by spherical symmetry.

The CGHS model is interesting because not only, being two dimensional, it is classically completely solvable and easier to handle in quantization, but also it includes black hole solutions and Hawking radiation. Thus one can study the quantum gravitational effects in black holes using this model in an easier way. For example it is suggested by Hawking that in the process of its evolution, a black hole loses its mass and energy through an evaporation process in which it emits a black body radiation called the Hawking radiation. It is not possible yet to strictly calculate this evaporation process in the full theory of quantum gravity because we do not have that theory yet. The next best thing is to use a semiclassical approximation where the matter field is quantized but the gravitational field (or the spacetime geometry) is treated classically. In four dimensions, using the semiclassical approximation, one is only able to calculate the Hawking radiation for a fixed background metric. Little success has been made so far to include the effect of the back-reaction of black hole emission on the geometry. One way to get an idea of what back-reaction might look like, is the use of two dimensional models since they are easier to analyze and have the important property that are conformally flat. Thus by analyzing those two dimensional models like the CGHS, one can hope for a better understanding of some interesting questions such as the back-reaction, the unitarity of the quantum evolution, the asymptotic fate of the spacetime and the information paradox . Our analysis of the CGHS model is meant to open a door for further more detailed analyses of these subjects.

The 3+1 model is also very interesting. First of all it was the first system for which the classical solutions to Einstein equations were found. It also shares the same interesting properties mentioned about the CGHS such as having black hole solutions and Hawking radiation. Especially we are going to study the state of the vacuum of the theory and the propagator of the matter field. This is important because we can compare our results with the ordinary quantum field theory results and look into their differences such as Lorentz invariance etc. Also the method of finding the vacuum state can be applied to black hole solutions in future to study the Hawking radiation and the related issues in this model.

1.3.3 The structure of this thesis

We start in chapter 2 by giving a brief overview of LQG and necessary background material that we use in the following chapters. The Dirac procedure will be explained in more detail since it is the basis of our classical analysis.

In chapter 3, we present a partially new work, which is the Hamiltonian formulation of the most general two dimensional model of a gravitational system minimally coupled to a scalar matter field in tetrad formulation. This generic model includes the 3+1 spherically symmetric and the two dimensional CGHS models, both of which we will analyze in chapters that come afterward. This chapter is based on [4] and an unpublished work.

Chapter 4 is dedicated to the study of the 3+1 spherically symmetric model coupled to matter field. It is an example of the second workaround in section 1.3.1. It includes mostly new work (from section 4.4 on). There we try to quantize the 3+1 model using LQG techniques. We holonomize the gravitational degrees of freedom while using Fock representation for the matter field. This is to make contact with the ordinary quantum field theory. Using variational technique, We find a lowest eigenstate of the master constraint that has the form of a direct product of a Fock vacuum for the scalar field and Gaussian states centered around flat space-time for the gravitational variables, which we will use in the later chapters. This chapter is mostly based on [5].

In chapter 5, which is a completely new work, we derive the propagator of the scalar field by holonomizing both gravitational and matter degrees of freedom and using the state we found in chapter 4. We will find out that the propagator in this case implies the violation of Lorentz invariance by introducing an alternative non-Lorentz-invariant dispersion relation. The work in this chapter is based on [6] and [7].

Finally, chapter 6, which also includes a completely new work, is on the Hamiltonian treatment of the two dimensional CGHS model without conformal transformation. It is based on [4] and an unpublished work. Here we are actually using the first workaround we mentioned in section 1.3.1 above. We analyze the system completely in classical regime. This analysis opens the door for future work on quantization of the model with matter field included to understand about the semiclassical effects in the CGHS black hole such as Hawking radiation and related issues.

2 An overview of loop quantum gravity

2.1 Constrained systems and their treatment

In this section we are going to review the Dirac treatment of Hamiltonian systems. Since this is the backbone of our classical analysis, we will give a rather thorough review of this subject.

2.1.1 Lagrangian singular systems

Consider a classical system which is described in terms of N canonical variables $q^n(t)$ where $n = 1, \ldots, N$ and t is the evolution parameter which is generally taken to be time. From the action principle we know that the classical motions of the system are those that make the action

$$S = \int_{t_1}^{t_2} L(q, \dot{q})\, dt. \tag{2.1}$$

stationary, i.e. the ones for which $\delta S = 0$. This condition leads to the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^n}\right) - \frac{\partial L}{\partial q^n} = 0. \tag{2.2}$$

This equation (2.2) can be written as

$$\ddot{q}^n \frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^m} = \frac{\partial L}{\partial q^m} - \dot{q}^n \frac{\partial^2 L}{\partial q^n \partial \dot{q}^m}. \tag{2.3}$$

From this it is clear that in order to be able to find the accelerations $\ddot{q}^n$ in terms of positions q^n and velocities $\dot{q}^n$, the matrix

$$\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^m}. \tag{2.4}$$

needs to be invertible. This means that we should have

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^n \partial \dot{q}^m}\right) \neq 0. \tag{2.5}$$

Thus if the matrix (2.4) is not invertible, the accelerations will not be uniquely determined by the positions and velocities and the solution of the equations of motion could then contain arbitrary functions of time. Such a system is called a **singular system**.

2.1.2 The Hamiltonian analysis of a constrained system

2.1.2.1 Hamiltonian and primary constraints

Let's turn to the Hamiltonian. The Hamiltonian is derived from the Lagrangian in two steps. First we introduce the **canonical momenta** p_n conjugate to the canonical variables q^n

$$p_n = \frac{\partial L}{\partial \dot{q}^n}. \tag{2.6}$$

Then we make a Legendre transformation to get the **Hamiltonian**

$$H(q,p) = \dot{q}^n p_n - L. \tag{2.7}$$

The important fact about the Hamiltonian is that it is really a function of q's and p's, and $\dot{q}$'s only enter through the combination $p(q,\dot{q})$. In order to see this explicitly, we derive the infinitesimal change δH in the Hamiltonian as follows

$$\delta H = \dot{q}^n \delta p_n + \left(\delta \dot{q}^n\right) p_n - \frac{\partial L}{\partial q^n}\delta q^n - \frac{\partial L}{\partial \dot{q}^n}\delta \dot{q}^n. \tag{2.8}$$

But using equations of motion (2.2) and the definition of momenta (2.6), this can be rewritten as

$$\delta H = \dot{q}^n \delta p_n - \dot{p}_n \delta q^n. \tag{2.9}$$

This means that H is a function of just q's and p's.

As a part of the second step, one needs to write (2.7) explicitly in terms of q's and p's. It is obvious that in order to do this, we need to invert the velocities $\dot{q}^n$ in terms of the momenta p_n and the canonical variables q^n. This should be done using the definition of momenta (2.6). Looking at (2.6), we see that the condition to be able to invert it to get $\dot{q}^n\left(p_n, q^n\right)$ is precisely the same as the condition(2.5), i.e. the matrix (2.4) must be invertible.

If we can not invert all the velocities in terms of q's and p's, then the momenta(2.6) are not all independent and there are some relations of the form

$$\phi_m(q,p) = 0, \qquad m = 1,\ldots,M \tag{2.10}$$

between q's and p's. The conditions (2.10) are called **primary constraints**. The name stems from the fact that these constraints are solely the result of introducing the momenta and that the equations of motion are not used to obtain them. A system that has constraints, is called a **constrained or a gauge system**.

2.1.2.2 Equations of motion

Now a question arises: in case there are primary constraints, how can we obtain the equations of motion? Comparing

$$\delta H(q,p) = \frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p, \tag{2.11}$$

with equation (2.9), one can infer

$$\left(\frac{\partial H}{\partial q^n} + \dot{p}_n\right)\delta q^n + \left(\frac{\partial H}{\partial p^n} - \dot{q}_n\right)\delta p^n = 0. \tag{2.12}$$

But based on a theorem (see [8] for example), this means we can write

$$\dot{q}^n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}, \tag{2.13}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - u^m \frac{\partial \phi_m}{\partial q_n}, \tag{2.14}$$

for some u^m. Note the importance of the first equation. It now enables us to find velocities $\dot{q}^n$ in terms of momenta p_n and the extra parameters u^m. We can also derive the equations of motion of a function $F(q,p)$ of canonical pairs using the above equations as

$$\begin{aligned}
\dot{F}(q,p) &= \frac{dF(q,p)}{dt} \\
&= \frac{\partial F}{\partial q^n}\dot{q}^n + \frac{\partial F}{\partial p_n}\dot{p}_n \\
&= \frac{\partial F}{\partial q^n}\left(\frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n}\right) - \frac{\partial F}{\partial p_n}\left(\frac{\partial H}{\partial q_n} + u^m \frac{\partial \phi_m}{\partial q_n}\right) \\
&= \{F, H\} + u^m \{F, \phi_m\},
\end{aligned} \tag{2.15}$$

where $\{,\}$ stands for the **Poisson bracket** and is defined by

$$\{F, G\} = \frac{\partial F}{\partial q^n}\frac{\partial G}{\partial p_n} - \frac{\partial G}{\partial q^n}\frac{\partial F}{\partial p_n}. \tag{2.16}$$

2.1.2.3 Secondary constraints

Constraints of a system are relations between canonical variables and canonical momenta which hold at all times. This means that the constraints should not change over time or

$$\dot{\phi}_m = 0. \tag{2.17}$$

These are what we call the consistency conditions. These consistency conditions give rise to new constraints or to some restrictions on the coefficients u^m. To see this, we evaluate these conditions by evaluating the time evolution of the primary constraints using the formula for equations of motion introduced in (2.15):

$$\{\phi_m, H\} + u^{m'}\{\phi_m, \phi_{m'}\} = 0. \tag{2.18}$$

For a given m, this equation either gives us a restrictions on the Lagrange multipliers $u^{m'}$, or it yields a relation between canonical variables and momenta independent of $u^{m'}$'s. In the latter case, those relations which are of the form $\rho_\ell(q,p) = 0$ are new constraints called the **secondary constraints**. The reason for the calling them "secondary" is that equations of motion (2.18) were used to derive them.

This process of evaluating the consistency conditions should be done for all of the primary constraints, and at the end this will leave of with a set of restrictions on u^m's and/or a set of new secondary constraints. After that, since the secondary constraints $\rho_\ell(q,p) = 0$ should also obey the consistency conditions, $\dot{\rho}_\ell = 0$, we need to follow the same procedure for them as well, which is to evaluate

$$\{\rho_\ell, H\} + u^m\{\rho_\ell, \phi_m\} = 0. \tag{2.19}$$

This again will leave us either with new restrictions on u^m's and/or with new constraints which are also called secondary constraints for which we should repeat the consistency conditions. After we go through the consistency conditions for all primary and secondary constraints and there is no more constraint left for which the consistency condition has not been evaluated, we are left with some restrictions on u^m's and a total number of M primary constraints ϕ_m and K secondary constraints ρ_k which together form $J = M + K$ constraints ϕ_j:

$$\phi_j(q,p) = 0, \qquad\qquad j = 1,\ldots,M+K. \tag{2.20}$$

2.1.2.4 Weak and strong equalities

We introduce the **weak equality** symbol "$\approx$" such that by writing $F \approx G$, we mean the quantity F is restricted to be equal to quantity G "on the constraint surface[1]", but this equality does not hold identically throughout all of the phase space. Equivalently we could say that F and G are equal if constraints are implemented, or they are equal "on shell", or they are weakly equal. On the other hand, the ordinary equality "=" means an exact equality or **strong equality**, i.e. the one that holds throughout all of the phase space.

Specifically if $G = 0$ (identically) in what we said above, then we have $F \approx 0$ or F is weakly vanishing. This is the case for the constraints and in fact equation (2.20) should be written in this notation as

$$\phi_j(q,p) \approx 0, \qquad\qquad j = 1,\ldots,M+K. \tag{2.21}$$

Note that if $F \approx 0$, then it can have nonzero Poisson brackets with canonical variables.

If two phase space quantities F and G are weakly equal, $F \approx G$, then they can be written in terms of constraints and vice versa [8]:

$$F \approx G \Leftrightarrow F - G = c^j(q,p)\phi_j(q,p). \tag{2.22}$$

[1] A constraint surface is a surface in phase space on which the constraints vanish, i.e. a surface on which $C_i = 0$ identically, where C_i's are the constraints.

2.1.2.5 The total Hamiltonian

Following what we said above, the restrictions on Lagrange multipliers u^m are some of the equations that we derive from the consistency conditions for all the primary and secondary constraints

$$\dot{\phi}_j \approx 0 \Rightarrow \{\phi_j, H\} + u^m \{\phi_j, \phi_m\} \approx 0, \tag{2.23}$$

that involve those multipliers (or, that are not constraints). The above set of equations is a set of J nonhomogeneous linear equations (J being the total number of primary and secondary constraints) in the $M < J$ unknowns Lagrange multipliers u^m, with coefficients that are functions of the q's and the p's. The general solution to this set of equations is

$$u^m \approx U^m + v^a V_a{}^m. \tag{2.24}$$

Here U^m is a particular solution of the inhomogeneous equation (2.23) and $V^m = v^a V_a{}^m$ is the most general solution of the associated homogeneous system

$$V^m \{\phi_j, \phi_m\} \approx 0, \tag{2.25}$$

where v^a's are totally arbitrary coefficients, $a = 1, \ldots, A$, and A is the number of independent solutions of the above homogeneous equation.

Now if we substitute (2.24) for u^m in (2.15), we get the **equations of motion**

$$\dot{F} \approx \{F, H_{\mathrm{T}}\}, \tag{2.26}$$

where the function

$$H_{\mathrm{T}} = H' + v^a \phi_a = H + U^m \phi_m + v^a V_a{}^m \phi_m, \tag{2.27}$$

is called the **total Hamiltonian** of the system and

$$H' = U^m \phi_m, \tag{2.28}$$

$$\phi_a = V_a{}^m \phi_m. \tag{2.29}$$

Thus H_{T} provides the evolution of the system.

2.1.2.6 First class and second class functions

In previous subsections, we introduced one type of classification of constraints by dividing them into primary and secondary ones. This classification is not of much importance, however there is another classification which is important, namely the property of being first class or second class. It is a classification of the functions on phase space in general, but it is especially important when applied to constraints.

A **first class function** on phase space, $F(q, p)$, is called first class if its Poisson bracket with every constraint vanishes weakly

$$\{F, \phi_j\} \approx 0, \qquad j = 1, \ldots, J. \tag{2.30}$$

A function of phase space that is not first class is said to be a **second class function**. If those functions are constraints, then we call them **first class and second class constraints**. By using (2.22), the above equation is equivalent to

$$\{F, \phi_j\} = c_j{}^{\ell} \phi_{\ell}. \tag{2.31}$$

Using this classification, we see that the H_{T},H' and ϕ_a defined in (2.27), (2.28) and (2.29) respectively, are all first class. This can be seen from (2.23), (2.24) and (2.25). To show this explicitly for H_{T}, we note that from (2.27) and (2.24) we can write

$$H_{\mathrm{T}} \approx H + u^m \phi_m. \tag{2.32}$$

Thus from this and (2.23) it can be seen that

$$\begin{aligned} \left\{H_{\mathrm{T}}, \phi_j\right\} &\approx \left\{H + u^m \phi_m, \phi_j\right\} \\ &\approx \left\{H, \phi_j\right\} + u^m \left\{\phi_m, \phi_j\right\} + \left\{u^m, \phi_j\right\} \underbrace{\phi_m}_{\approx 0} \\ &\approx 0, \end{aligned} \tag{2.33}$$

where in the second line we used the fact that primary constraints vanish weakly, and the remaining terms in the second line are precisely (the negative of) (2.23). To see that ϕ_a is first class, we have

$$\begin{aligned} v^a\{\phi_a, \phi_j\} &= v^a \left\{V_a{}^m \phi_m, \phi_j\right\} \\ &= v^a V_a{}^m \left\{\phi_m, \phi_j\right\} + \left\{V_a{}^m, \phi_j\right\} \underbrace{\phi_m}_{\approx 0} \\ &\approx V^m \left\{\phi_m, \phi_j\right\} \\ &\approx 0, \end{aligned} \tag{2.34}$$

where in the third line we used (2.25) and the fact that v^a are completely arbitrary and thus the equation holds independent of v^a.

Finally to show that H' is first class, we can write

$$\begin{aligned} \{H', \phi_j\} &= \left\{H_{\mathrm{T}} - v^a \phi_a, \phi_j\right\} \\ &= \underbrace{\left\{H_{\mathrm{T}}, \phi_j\right\}}_{\approx 0} - v^a \underbrace{\left\{\phi_a, \phi_j\right\}}_{\approx 0} \\ &\approx 0, \end{aligned} \tag{2.35}$$

where we have used (2.33) and (2.34). Thus **the evolution is generated by a first class total Hamiltonian** which is the sum of a first class Hamiltonian H' and the linear combination of the first class primary constraints.

2.1.3 First class constraints, gauge systems and gauge transformations

2.1.3.1 The arbitrariness in the evolution and gauge transformations

As can be seen from the equations of motion (2.26), which by using (2.27) can be written as

$$\dot{F} \approx \left\{F, H'\right\} + v^a \left\{F, \phi_a\right\}, \tag{2.36}$$

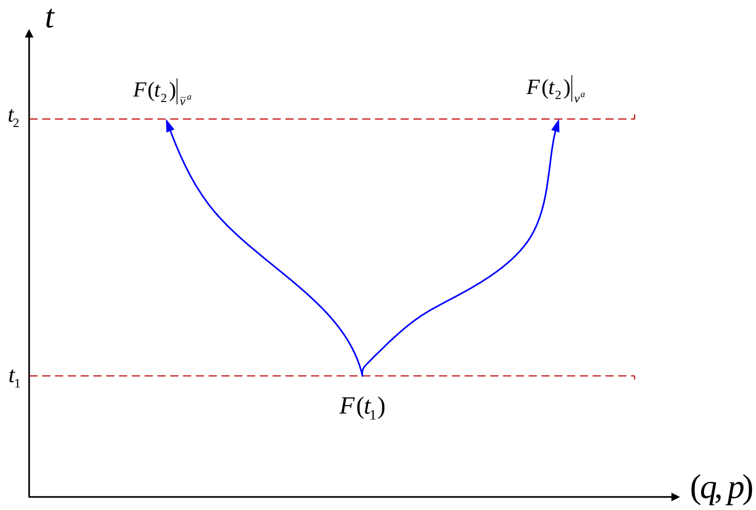


Figure 2.1: The evolution of a phase space function $F(q,p)$ under two different arbitrary variables v^a and $\bar{v}^a$ from the same initial state.

the solutions to the equation of motion contain a set of totally arbitrary parameters v^a. As a consequence, with the same initial conditions, i.e. a set of q's and p's at say t_0, one can get different solutions in a future time t. This means that **although a set of q's and p's at some time t_1 define a state of the system uniquely, a state of the system does not uniquely single out a set of q's and p's**. As can be seen from figure (2.1), the reason is that given a set of $(q^n(t_1), p_n(t_1))$ at a given time t_1, we can evolve those variables by the equations of motion (2.36) to get $(q^n(t_2), p_n(t_2))$ at a time t_2. But these values at t_2 not only depend on the H' and ϕ_a, but also on the totally arbitrary variables v^a. Therefore the same evolution but by two or more different variables v^a's, takes us to a same new unique state and thus this state is defined by different sets of $(q(t_2), p(t_2))$ (which depend of v^a's). In other words although the future state is also unique, but there are more than one set of q's and p's that correspond to it.

This can be seen more explicitly as follows. Suppose we have a variable $F(q,p)$ and we evolve it from an initial condition by v^a as

$$\dot{F} \approx \{F, H'\} + v^a \{F, \phi_a\}, \tag{2.37}$$

which by Taylor expansion to first order in δt can be written as

$$F(t+\delta t)\Big|_{v^a} - F(t) = \{F(t), H'\}\delta t + v^a \delta t \{F(t), \phi_a\}. \tag{2.38}$$

Now suppose that we evolve F again from the same initial condition, but this time with $\bar{v}^a$ and similar to above we would get

$$F(t+\delta t)\Big|_{\bar{v}^a} - F(t) = \{F(t), H'\}\delta t + \bar{v}^a \delta t \{F(t), \phi_a\}. \tag{2.39}$$

Now the difference between the two future values of F is

$$\delta F = F(t+\delta t)\Big|_{\bar{v}^a} - F(t+\delta t)\Big|_{v^a} = \left(\bar{v}^a - v^a\right)\delta t\left\{F(t),\phi_a\right\} = \delta v^a\left\{F(t),\phi_a\right\}. \qquad (2.40)$$

This shows that with the same evolution and the same initial condition, future values of F can be different by choosing different v^a's. The transformation (2.40) and similar ones that are generated by first class constraints are called **gauge transformations.**

One can see that the gauge transformation (2.40) is actually generated by linear combination of constraints ϕ_a which from (2.29) is clear that are *primary* first class constraints. But the most general physically permissible motion should allow for transformations generated by *all* first class constraints, being primary or secondary. In fact there is a conjecture called the **Dirac conjecture** which states: **in general, all first class constraints generate gauge transformations**. Thus to have the most general motion, one that includes all possible gauge transformations, one should add to H_{T}, the linear combination of all *secondary* first class constraints too. The result is which is a first class function generating the most general evolution is called the extended Hamiltonian

$$H_E = H' + u^a\gamma_a, \qquad (2.41)$$

where γ_a are all the primary and secondary first class constraints and u^a are arbitrary Lagrange multipliers. A system possessing first class constraints is called a **gauge system**.

We expect that the physical degrees of freedom, i.e. the observables, be independent of u^a. This means that for physical observables, δF should vanish, or in other word, observables, as well as the state of the system should be invariant under gauge transformations.

There is an important note we should mention. What we have discussed until now might implicitly suggest that the evolution variable t which we call time, is an observable. However in reality, especially for example in general relativity, there is no such notion of a universal observable time. In fact in those types of theories which are called generally covariant theories, the parameter t is just a non-observable parameter which we choose to write the evolution in terms of it. It could be one of our coordinates, in which case it is called the coordinate time. It is a variable with which we parametrize the evolution. It can be transformed to another variable $t' = f(t)$ such that the theory remains invariant. Thus some part of the gauge arbitrariness of the theory is due to the freedom of the choice of the parametrization variable t. We will discuss a bit about this in section 2.1.5.2.

2.1.4 Second class constraints and the Dirac bracket

2.1.4.1 The Dirac bracket

In many cases, including for example the CGHS case we are going to study, we are dealing with a theory which constrains second class constraints, which we call χ_α, as well as the first class ones. Then how should we treat such a system?

It turns out [8] that we can set the second class constraints strongly equal to zero, $\chi_\alpha = 0$, provided that instead of using the ordinary Poisson bracket, we use a modified version called the **Dirac bracket**

$$\{F,G\}_D = \{F,G\} - \{F,\chi_\alpha\}C^{\alpha\beta}\{\chi_\beta,G\}, \tag{2.42}$$

where $C^{\alpha\beta}$ are the elements of the matrix $\mathbf{C}^{-1}$ which is the inverse of the matrix $\mathbf{C}$ with elements $C_{\alpha\beta}$ defined by

$$C_{\alpha\beta} = \{\chi_\alpha,\chi_\beta\}. \tag{2.43}$$

Thus $\mathbf{C}$ is the the matrix of the Poisson brackets of the secondary constraints among themselves and we have

$$\mathbf{CC}^{-1} = \mathbf{1} \Rightarrow C_{\alpha\beta}C^{\beta\rho} = \delta_\alpha{}^\rho. \tag{2.44}$$

One of the properties of the Dirac bracket [8] is that for an arbitrary F and a first class G we have

$$\{F,G\}_D \approx \{F,G\}. \tag{2.45}$$

Thus we can infer that even after switching to the Dirac bracket, the first class extended Hamiltonian (2.41) still generates the correct equations of motion since

$$\dot{F} \approx \{F,H_E\} \approx \{F,H_E\}_D. \tag{2.46}$$

The effect of the gauge transformations will remain unchanged under Dirac bracket since

$$\{F,\gamma_a\} \approx \{F,\gamma_a\}_D. \tag{2.47}$$

Thus after using the Poisson bracket to find the secondary constraints, restrictions on Lagrange multipliers, and classifying the constraints into first class and second class, if there is any second class constraints, then the Poisson bracket has served its purpose and should be replaced by Dirac bracket in all of the analyses that come afterward. At the same time that we switch to the Dirac bracket, we put all of the second class constraints strongly equal to zero and use these strong equalities to substitute some canonical variables and momenta in terms of some other ones. For example if a theory has a second class constraint $p_1 + p_2 \approx 0$, we substitute P_2 for p_1 in the theory by using the strong equality $p_1 + p_2 = 0 \Rightarrow p_2 = -p_1$.

2.1.4.2 Gauge fixing

As we saw, the presence of first class constraints induces arbitrary degrees of freedom in the theory such that the more than one set of canonical variables correspond to a given state. Sometime we need to eliminate this arbitrariness and make a one-to-one correspondence between the canonical variables and states. This is done by **gauge fixing** which is introduction of **gauge conditions,** $C_a(q,p) = 0$**,** which are **ad-hoc relations between some of the canonical variables** brought in from outside to eliminate those arbitrary freedoms in the theory. After complete gauge fixing there should not be any first class constraint left. We can also partially gauge-fix the theory to allow for simpler calculations although this removes some of the arbitrariness of the theory not all of it.

2.1.5 Classical observables, generally covariant systems

2.1.5.1 Classical observables

From the discussion we have, it is somehow clear that we can define a classical observable as a function F on the constraint surface that is gauge invariant, i.e. has a weakly vanishing Dirac bracket with first class constraints

$$\{F, \gamma_a\}_D \approx 0. \tag{2.48}$$

It should be mentioned that some classical observable functions might not be quantum observables after quantization.

2.1.5.2 Generally covariant systems

As we mentioned above, there are some theories in which time is just a variable in terms of which the evolution of the theory is parametrized, and its reparametrization as $t' = f(t)$ will leave the theory invariant. This happens for example in general relativity where the coordinate time is not an observable variable. Thus in these theories, the time should be treated on an equal footing as other canonical variables. The theories that are invariant under reparametrization of time (the evolution parameter) are called **generally covariant** theories.

As an example consider the action of a relativistic point particle in Minkowski space

$$S = -m \int dt \left(-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{\frac{1}{2}}. \tag{2.49}$$

This action is invariant under reparametrization $t \to t'(t)$:

$$\begin{aligned} S' &= -m \int dt' \left(-\eta_{\mu\nu} \frac{dx^\mu}{dt'} \frac{dx^\nu}{dt'} \right)^{\frac{1}{2}} \\ &= -m \int \frac{dt'}{d\tau} d\tau \left(-\left(\frac{dt}{dt'} \right)^2 \eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{\frac{1}{2}} \\ &= -m \int dt \left(-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \right)^{\frac{1}{2}} \\ &= S. \end{aligned} \tag{2.50}$$

In many cases in the generally covariant system, including general relativity, it happens that the H' in the extended Hamiltonian (2.41) is zero. Thus the evolution of the system is only unfolding of gauge transformation since the generator of the evolution,

$$H_{\mathrm{T}} = u^a \gamma_a, \tag{2.51}$$

is just a first class constraint. Such a Hamiltonian for which $H' = 0$ and is just sum of the first class constraints is called a zero Hamiltonian. The action above can also include second class constraints but since we use Dirac bracket and use strong vanishing conditions for the secondary constraints, the evolution remains the same.

2.2 ADM decomposition

The first step in non-perturbative canonical quantization of general relativity is to develop a Hamiltonian formulation. This has first been done by Arnowitt, Deser and Misner (ADM) [9]. The idea is that a hyperbolic [2] spacetime manifold M with metric g_{ab} on it, is diffeomorphic to a manifold $\Sigma \times \mathbb{R}$ with Σ spatial hypersurfaces on which a function of spacetime say $t(X^\mu)$ =constant, where X^μ is the spacetime coordinate and $t \in \mathbb{R}$ [10]. Thus one can foliate the spacetime manifold by spatial hypersurfaces Σ_t on which a parameter which we call "the coordinate time t" is constant. On each of these hypersurfaces we introduce a set of spatial coordinates $x^i(X^\nu)$. Next we introduce a congruence of curves in spacetime parametrized by t with unit tangent vectors

$$t^a = \left(\frac{\partial}{\partial t}\right)^a. \tag{2.52}$$

t^a is called the time flow vector field. Now we tie these hypersurfaces together by those integral curves of t^a in a way that points of hypersurfaces intersected by the same curve be given the same spatial coordinate x^i. The above procedure introduces a valid four dimensional coordinate in spacetime manifold which we call $x^\mu(t, x^i)$ where as we said before, x^i are coordinates on the spatial hypersurfaces.

Next let n^a be the unit timelike vector field normal to the spatial hypersurfaces, i.e.

$$g_{ab}n^a n^b = -1. \tag{2.53}$$

This vector field should not be confused with the time flow vector t^a which is also timelike but generally is not normal to the hypersurfaces and obeys

$$g_{ab}t^a t^b = g_{00}. \tag{2.54}$$

Now the spacetime metric g_{ab} induces a spatial metric q_{ab} on each Σ_t as

$$q_{ab} = g_{ab} + n_a n_b. \tag{2.55}$$

This metric can be considered as a projection operator on Σ_t

$$q_a{}^b n_b = 0, \tag{2.56}$$

$$q_a{}^b q_b{}^c = q_a{}^c. \tag{2.57}$$

Note that we lower and raise the indices with the spacetime metric g_{ab}. The situation can be seen in figure (2.2). The time flow vector t^a can be decomposed as

$$t^a = Nn^a + N^a, \tag{2.58}$$

where N is called the lapse function which measure the component of t^a in the direction of n^a, N^a is the shift vector which is the projection of t^a into Σ_t. The "old" coordinates X^μ on spacetime provide four basis vectors at each spacetime point

$$e_\mu{}^a = \left(\frac{\partial}{\partial X^\mu}\right)^a. \tag{2.59}$$

[2]A hyperbolic spacetime manifold is one that has a Cauchy surface. A Cauchy surface is a closed achronal surface which its domain of dependence is the entire manifold; i.e. from information given on a Cauchy surface, one can predict what happens throughout all of spacetime manifold.

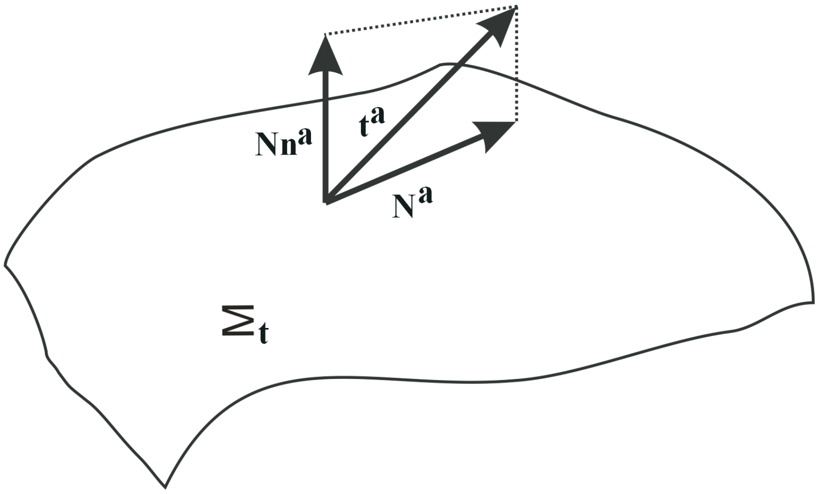


Figure 2.2: The decomposition of spacetime into spacelike hypersurfaces Σ_t. The time flow vector t^a, lapse function N and shift vector N^a are shown.

After the foliation which is given by the function $X^\mu(t, x^i)$, the coordinates x^i provide three basis vectors on Σ_t

$$E_i{}^a = \left(\frac{\partial}{\partial x^i}\right)^a. \tag{2.60}$$

These are the projections of $e_\mu{}^a$ into Σ_t

$$\begin{aligned} E_a{}^I &= q_a{}^b e_b{}^I \\ &= \left(g_a{}^b + n_a n^b\right) e_b{}^I \\ &= e_b{}^I + n_a n^I, \end{aligned} \tag{2.61}$$

where I can be seen as a generic index, for example an index in the Lorentz Lie algebra $\mathfrak{su}(2)$ (in which case, the basis vectors are tetrads). Since the shift vector lies in the tangent space of Σ_t, it can be written in terms of $E^a{}_i$ basis as

$$N^a = N^i E^a{}_i. \tag{2.62}$$

Then the full metric will take the following form [11, 12]

$$g_{\mu\nu} dx^\mu dx^\nu = \left(-N^2 + q_{ij} N^i N^j\right) dt^2 + q_{ij} dx^i dx^j - q_{ij} N^j dx^i dt - q_{ij} N^i dt dx^j, \tag{2.63}$$

and hence the components of the spacetime metric will be

$$g_{00} = -N^2 + q_{ij} N^i N^j, \tag{2.64}$$

$$g_{ij} = q_{ij}, \tag{2.65}$$

$$g_{i0} = g_{0i} = -q_{ij} N^j. \tag{2.66}$$

The metric components can be written as

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + q_{ij} N^i N^j & \left[-q_{ij} N^j\right]_{1\times 3} \\ \left[-q_{ij} N^j\right]_{3\times 1} & \left[q_{ij}\right]_{3\times 3} \end{pmatrix}, \tag{2.67}$$

where the terms in [,] are block matrices of dimensions that are written for them as subscripts. Using the formula for the determinant of block matrices we have for the determinant of this matrix

$$\det\left(g_{\mu\nu}\right) = -N^2 \det\left(q_{ij}\right), \tag{2.68}$$

which means

$$\sqrt{-g} = N\sqrt{q}, \tag{2.69}$$

where we have used the notation $\det\left(g_{\mu\nu}\right) = g$ and $\det\left(q_{ij}\right) = q$.

2.3 Geometrodynamics

The idea here is to write general relativity in the Hamiltonian form by utilizing the ADM decomposition and using the spatial metric and its canonical conjugate momentum, which is related to the extrinsic curvature of Σ_t, as the degrees of freedom of the theory.

Consider the Einstein-Hilbert action

$$S = \int \mathscr{L} d^4x, \tag{2.70}$$

with $\mathscr{L}$, the Lagrangian density

$$\mathscr{L} = \sqrt{-g}R, \tag{2.71}$$

where g is the determinant of the spacetime metric and R is its curvature. For simplicity of the discussion, we only consider the pure gravitational case without matter fields. Using the ADM decomposition which we saw in section 2.2, we can write the Lagrangian density as

$$\mathscr{L} = \sqrt{q}N\left[{}^3R + K_{ab}K^{ab} - K^2\right], \tag{2.72}$$

where q is the determinant of the spatial metric, N is the lapse, ${}^3R$ is the scalar curvature of the spatial metric (or Σ_t), K_{ab} is the extrinsic curvature of Σ_t and $K = g_{ab}K^{ab} = q_{ab}K^{ab}$ is the trace of the extrinsic curvature. The extrinsic curvature can be expressed as the lie derivative of the spatial metric q_{ab} along the integral curves of the normal vector field n^a

$$K_{ab} = \frac{1}{2}\mathfrak{L}_{\vec{n}} q_{ab}. \tag{2.73}$$

Hence K_{ab} measures the rate of change of the spatial metric (or the change in the amount of bending of the geometry of Σ_t) as one moves along the congruences defined by (or integral curves of) the vector field n^a.

The canonical variables or the variables whose time derivatives appear in the Lagrangian are q_{ab} for which

$$\dot{q}_{ab} = \mathfrak{L}_{\vec{t}} q_{ab}, \tag{2.74}$$

and its canonically conjugate momentum

$$p^{ab} = \frac{\partial \mathscr{L}}{\partial \dot{q}_{ab}} = \sqrt{q}\left(K^{ab} - Kq^{ab}\right), \tag{2.75}$$

and since the time derivative of N and N^a does not appear, they are just Lagrange multiplier, not canonical variables. Making a Legendre transform, we arrive at the Hamiltonian density

$$H = N\mathscr{C} + N^a\mathscr{C}_a, \tag{2.76}$$

where $\mathscr{C}$ and $\mathscr{C}_a$ are constraints called Hamiltonian constraint

$$\mathscr{C} = -{}^3R + q^{-1}p^{ab}p_{ab} - \frac{1}{2}q^{-1}p^2, \tag{2.77}$$

and diffeomorphism constraint

$$\mathscr{C}^b = -2D_a\left(q^{-\frac{1}{2}}p^{ab}\right). \tag{2.78}$$

Here $p = q_{ab}p^{ab}$ and D_a is the derivative operator on Σ_t which is compatible with q_{ab}, i.e. $D_a q_{bc} = 0$. Notice that (2.77) is quadratic in momentum. Such constraints seem unavoidable in a Hamiltonian formulation of general relativity and this makes the formulation of a quantum theory rather difficult.

2.4 Introduction of self-dual connection: Ashtekar variables

In the Geometrodynamics approach to quantum gravity, the canonical variables are the metric q_{ab} on the spatial hypersurface Σ_t with t =constant and the conjugate momenta are related to the extrinsic curvature of this hypersurface. But in loop quantization, the canonical variables are the $\mathfrak{su}(2)$-valued Ashtekar-Barbero connection ${A_a}^I$ and the conjugate momenta are the $\mathfrak{su}(2)$-valued densitized triad vector fields ${E^a}_I$, where $\mathfrak{su}(2)$ is the algebra of $SU(2)$ group. These densitized triads encodes all the information of the intrinsic geometry of the hypersurface Σ_t. The spatial 3d metric q_{ab} is constructed from it as

$$q_{ab} = \frac{1}{q}{E_a}^I E_{bI}, \tag{2.79}$$

where $q = \det(q_{ab})$ appears here because as we said ${E_a}^I$'s are densities. Since these variables are $\mathfrak{su}(2)$-valued, we can write them as

$$A_a = {A_a}^I \tau_I, \tag{2.80}$$

$$E^a = {E^a}_I \tau^I, \tag{2.81}$$

where $\{\tau^I\}_{I=1,2,3}$ is a basis of the Lie algebra of the $SU(2)$ group. As a convention, we take this basis to be

$$\tau^I = \tau_I = -\frac{i}{2}\sigma_I, \quad I = 1,2,3, \tag{2.82}$$

and σ_I are the Pauli matrices. The connection ${A_a}^I\tau_I$ is related to the extrinsic curvature ${K_a}^I$ and the triad-compatible spin connection ${\Gamma_a}^I$ by

$$A_a{}^I = {\Gamma_a}^I + \gamma {K_a}^I, \tag{2.83}$$

where

$$K_a{}^I = \det\left(E_a{}^I\right)^{-\frac{1}{2}} K_{ab} E^{bI}, \tag{2.84}$$

and γ is called the Barbero-Immirzi parameter that classically does not play an important rule since it can be changed by canonical transformation. On the quantum level, however, this parameter becomes important since the transformation that can change γ are not represented in unitary manner.

As it is expected from a generally covariant theory such as gravity, the dynamics of the system is purely gauge transformation and thus the Hamiltonian becomes the sum of three constraints

$$H = \int d^3x \left(N\mathscr{H} + N^a \mathscr{D}_a + \lambda^I \mathscr{G}_I\right), \tag{2.85}$$

which are the Hamiltonian constraint

$$\mathscr{H} = \frac{1}{16\pi G} \frac{E^a{}_I E^b{}_J}{\sqrt{\det\left(E_a{}^I\right)}} \left[\epsilon^{IJ}{}_K F^K{}_{ab} - 2\left(1+\gamma^2\right) K_{[a}{}^I K_{b]}{}^J\right], \tag{2.86}$$

the diffeomorphism constraint

$$\mathscr{D}_a = \frac{1}{8\pi G\gamma} \left[2\partial_{[a} A_{b]}{}^I E^b{}_I - A_a{}^J \partial_b E^b{}_J\right], \tag{2.87}$$

and the Gauss constraint

$$\mathscr{G}_I = \frac{1}{8\pi G\gamma} \left[\partial_a E^a{}_I + \epsilon_{IJ}{}^K A_a{}^J E^a{}_K\right], \tag{2.88}$$

which are seven constraints (three Gauss, three diffeomorphism and one Hamiltonian constraints) per point of the hypersurface. Here G is the gravitational constant, the lapse function N, shift vector N^a and λ^I are Lagrange multipliers and we have used the curvature of the Ashtekar connection

$$F^K{}_{ab} = 2\partial_{[a} A_{b]}{}^K + \epsilon_{IJ}{}^K A_a{}^I A_b{}^J. \tag{2.89}$$

The Hamiltonian constraint generates the coordinate time evolution, the three components of the diffeomorphism constraint generate spatial diffeomorphisms and the three components of the Gauss constraint are the generators of the $SU(2)$ transformation. The symplectic structure is given by the Poisson brackets

$$\{A_a{}^I(x), E^b{}_J(y)\} = 8\pi G\gamma \delta_a^b \delta_I^J \delta(x-y), \qquad x, y \in \Sigma_t. \tag{2.90}$$

The commutator of the smeared version of the constraints above involve structure functions instead of structure constants and thus form an open algebra instead of a Lie algebra. This is known as the problem of dynamics.

2.5 Quantization: connection and loop representations

2.5.1 Connection representation

In order to quantize the theory, one needs to introduce a vector space of states. In the so called connection representation, these states which are the members of the kinematical Hilbert space are holomorphic functions $\Psi(A)$ of the connection ${A_a}^I$. The elementary quantum operators in this representation are

$$\hat{A}_a{}^I \Psi(A) = A_a{}^I \Psi(A), \tag{2.91}$$

$$\hat{E}_I{}^a \Psi(A) = i\hbar \frac{\delta}{\delta A_a{}^I} \Psi(A). \tag{2.92}$$

Now to find the physical Hilbert space, one should find the subset of these state functions that are the kernel of constrains. This means that one should write the constraints in connection representation and find the states such that

$$\hat{\mathscr{H}} \Psi(A) = 0, \tag{2.93}$$

$$\hat{\mathscr{D}} \Psi(A) = 0, \tag{2.94}$$

$$\hat{\mathscr{G}} \Psi(A) = 0. \tag{2.95}$$

For technical reasons, this representation is not the best suitable representation to quantize the theory and there is a better representation called the loop representation.

2.5.2 Loop representation

The idea behind this representation is to use functions of holonomy. Given a curve $\gamma^a(t)$ on hypersurface Σ , we can define a holonomy or parallel transport as

$$U_\gamma(t_1, t_2) = P\exp \int_{\gamma(t_1)}^{\gamma(t_2)} ds A_a(\gamma(s)) \dot{\gamma}^a(s), \tag{2.96}$$

where $P\exp$ is the path ordered exponential and we have not written the $\mathfrak{su}(2)$ indices. Now a loop on Σ is obtained by identifying both ends of the curve $\gamma(t)$ and the related holonomy is

$$U_\gamma(t) \equiv U_\gamma(t,t) = P\exp \oint_t ds A_a(\gamma(s)) \dot{\gamma}^a(s). \tag{2.97}$$

This quantity transforms like a local covariant quantity under a gauge transformation. Its trace

$$\mathrm{tr}\left[U_\gamma(t)\right] = \mathrm{tr}\left[P\exp \oint_t ds A_a(\gamma(s)) \dot{\gamma}^a(s)\right], \tag{2.98}$$

is a gauge invariant quantity. Remember that in the Dirac sense, gauge invariant quantities were observables. Now the functionals of $\mathrm{tr}\left[U_\gamma(t)\right]$ can be used to construct the phase space of general relativity. Then in order to quantize, we can introduce operators that are parametrized by the closed loops on Σ and the states would be functions on the loop space.

2.6 Master constraint program

The quantum dynamics is perhaps the hardest problem quantum gravity faces. Although a mathematically well-defined Hamiltonian constraint operator has been proposed for loop quantum gravity (e.g. [13]), there has been questions regarding the algebra of Hamiltonian constraint with itself that should be answered in order to be able to advance in LQG. The problem roughly is that although the algebra of commutators among smeared Hamiltonian constraint operators is anomaly free in the mathematical sense, it does not manifestly reproduce the classical Poisson algebra among the smeared Hamiltonian constraint functions.

It has been proposed that this and related problems would disappear if instead of Hamiltonian constraint, we use the master constraint

$$\mathbf{M} = \int_\Sigma d^3x \frac{[\mathscr{H}(x)]^2}{\sqrt{q}}, \tag{2.99}$$

which is a continuous sum of all of the Hamiltonian constraints at all of the points of the hypersurface Σ. We will use this quantity to quantize the 3+1 case in the following chapters.

2.7 Uniform discretization technique

The reason to introduce uniform discretization [14] is to address some of problems in the dynamics of loop quantum gravity. It consists of discretizing the variables on a lattice such that the discrete theory is unconstrained. Then one can proceed to quantize the resulting discrete theory. In order to go back to the continuum limit two cases may happen: One case is that it will be possible to take the continuum limit in the quantum theory and this completes the quantization of the original continuum theory one started with satisfactorily.

The other case, the continuum limit cannot be taken in the quantum theory. In this case, because the classical discrete theory approximates the continuum theory well, one expects that the quantum discrete theory also approximates well the quantum continuum theory, even in cases where we cannot construct the latter exactly via this method, at least for certain states. One would therefore end with a quantum theory that approximates semiclassically the classical theory one started with, plus corrections. We will use this method in quantization of the 3+1 model.

3

Analysis of the generic 1+1 model: Lagrangian and Hamiltonian

3.1 Introduction

In this chapter, we are going to develop a general Hamiltonian for the generic two dimensional models of gravity coupled to a scalar field. We will write this Hamiltonian in tetrad formulation because this kind of formulation of the theory is an essential first step towards further formulation of the theory in Ashtekar variables [15].

To do this, first we write the action of the four dimensional spherically symmetric and the CGHS models in metric formulation in a specific form and show that they are specific examples of the most general diffeomorphism invariant action in two dimensions. Then we will write this action in tetrad form and proceed with the ADM decomposition and Dirac procedure to arrive at the related Hamiltonian.

3.2 The generic action

It is well known that in two dimensions, the Einstein-Hilbert action, $\int d^2x\sqrt{-|g|}R$ is actually a topological or boundary term and does not provide any field equations. Therefore one can not use it to study the 1+1 dimensional gravity. On the other hand, there is a generic class of two dimensional actions that are not just topological terms and contain useful information about 1+1 gravity and its equations of motion. Among these models are the CGHS model and the 3+1 spherically symmetric model. Below we take a look at them in brief and then show that they belong to this generic class of two dimensional actions.

3.2.1 The 3+1 spherically symmetric model

Consider a system containing gravitational field and matter in four dimensions. The action will have the form

$$S = S_g + S_m, \tag{3.1}$$

where

$$S_g = \frac{1}{16\pi} \int d^4x \sqrt{-|\bar{g}|} R, \tag{3.2}$$

is the Einstein-Hilbert action in four dimensions, $\bar{g}_{ab}$ the full metric in four dimensions and S_m is the action of the matter field f which we take to be the minimal coupling action

$$S_\mathrm{m} = -\frac{1}{4\pi} \int d^4x \sqrt{-|\bar{g}|} \bar{g}^{ab} \partial_a f \partial_b f. \tag{3.3}$$

The Latin indices are abstract indices. In order to reduce the action to a two dimensional one and get rid of the angular coordinates, we use spherical symmetry. Spherical symmetry means that we can assume that the spatial hypersurface have the topology $\Sigma = \mathbb{R}^+ \times S^2$ and thus we can use spherical symmetry ansatz to write the metric $\bar{g}_{ab}$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \Phi^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \qquad (\mu, \nu = 0, 1)$$

where x^0, x^1, θ and ϕ are some coordinates adapted to the spherical symmetry and $g_{\mu\nu}$ is the metric on the x^0, x^1 plane. Substituting $(\mu, \nu = 0, 1)$ into (3.2) and integrating over angular coordinates θ and ϕ, yields the reduced two dimensional action

$$S_\text{g-spher} = \int d^2x \sqrt{-|g|} \left(\frac{1}{4} \Phi^2 R(g) + \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{2} \right), \tag{3.4}$$

where R is the Ricci tensor of the two dimensional metric g_{ab} and $|g|$ is its determinant. Using the ansatz $(\mu, \nu = 0, 1)$ for the matter action (3.3) and integrating over angular variables, gives

$$S_\text{m-spher} = -\frac{1}{2} \int d^2x \sqrt{-|g|} \Phi^2 g^{ab} \partial_a f \partial_b f. \tag{3.5}$$

Thus the full reduced spherically symmetric action is then

$$\begin{aligned} S_\text{spher} = &\int d^2x \sqrt{-|g|} \left(\frac{1}{4} \Phi^2 R(g) + \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{2} \right) \\ &- \frac{1}{2} \int d^2x \sqrt{-|g|} \Phi^2 g^{ab} \partial_a f \partial_b f. \end{aligned} \tag{3.6}$$

3.2.2 The CGHS model

The dilatonic or CGHS model [3] (which we will describe in more detail in chapter 6) can be thought as a genuine two dimensional model with a dilatonic or gravitational part,

$$S_\text{g-CGHS} = \int d^2x \sqrt{-|g|} e^{-2\phi} \left(R + 4 g^{ab} \partial_a \phi \partial_b \phi + 4\lambda^2 \right), \tag{3.7}$$

where ϕ (not to be confused with the coordinate ϕ in 3+1 model) corresponds to the dilaton field and λ^2 is the cosmological constant. The matter action is again the standard minimal coupling but in two dimensions,

$$S_\text{m-CGHS} = -\frac{1}{2} \int d^2x \sqrt{-|g|} g^{ab} \partial_a f \partial_b f. \tag{3.8}$$

In order to change the form of the CGHS action into a more similar one to the spherically symmetric action in 3.4, we redefine the field ϕ as

$$\Phi = 2\sqrt{2}e^{-\phi}, \tag{3.9}$$
$$4\lambda^2 = -\Lambda, \tag{3.10}$$

so that

$$e^{-2\phi} = \frac{\Phi^2}{8}, \tag{3.11}$$
$$\partial_a \phi = -\frac{1}{\Phi}\partial_a \Phi. \tag{3.12}$$

Then the gravitational and matter parts of CGHS model, (3.7) and (3.8), will become

$$\begin{aligned} S_{\text{CGHS}} = & \int d^2x\sqrt{-|g|}\left\{\frac{1}{8}\Phi^2 R + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi - \frac{1}{8}\Phi^2\Lambda\right\} \\ & -\frac{1}{2}\int d^2x\sqrt{-|g|}g^{ab}\partial_a f\partial_b f. \end{aligned} \tag{3.13}$$

It can be seen now that the CGHS and spherically symmetric actions, (3.13) and (3.6), show much similarity.

3.2.3 General 1+1 action

3.2.3.1 General action

One can see that the gravitational part of both of the above actions are special cases of a general action

$$S_{g\text{-dil}} = \int d^2x\sqrt{-|g|}\left(D(\Phi)R(g) + \frac{1}{2}g^{ab}\partial_a\Phi\partial_b\Phi + U(\Phi)\right). \tag{3.14}$$

This is the most general diffeomorphism invariant action yielding second order differential equations for the metric g and a scalar (dilaton) field Φ [16]. It is called the generalized dilaton action and was first suggested in [17, 18].

For the next step we want to eliminate the kinetic term $g^{ab}\partial_a\Phi\partial_b\Phi$. For this, we apply a conformal transformation

$$\tilde{g}_{ab} = \Omega^2(\Phi)g_{ab}. \tag{3.15}$$

Under this conformal transformation, the variables in the action (3.14) transform as

$$\begin{aligned} |g| &= \epsilon^{ab}\epsilon_{cd}g_a{}^c g_b{}^d = \Omega^{-4}(\Phi)\epsilon^{ab}\epsilon_{cd}\tilde{g}_a{}^c\tilde{g}_b{}^d = \Omega^{-4}(\Phi)|\tilde{g}|, & (3.16) \\ \tilde{R} &= \Omega^{-2}(\Phi)(R - 2g^{ab}\partial_a\partial_b \ln\Omega(\Phi)), & (3.17) \end{aligned}$$

where the second equation implies

$$R = \Omega^2(\Phi)\tilde{R} + 2\Omega^2(\Phi)\tilde{g}^{ab}\partial_a\partial_b\Omega(\Phi). \tag{3.18}$$

Substituting these back into the action (3.14) yields the conformal transformed action

$$\begin{aligned}S_{g\text{-dil}} &= \int d^2x\sqrt{-|\tilde{g}|}\Omega^{-2}(\Phi)\Big(D(\Phi)\Omega^2(\Phi)\tilde{R}(g) + \frac{1}{2}\Omega^2(\Phi)\tilde{g}^{ab}\partial_a\Phi\partial_b\Phi \\ &\qquad + 2D(\Phi)\Omega^2(\Phi)\tilde{g}^{ab}\partial_a\partial_b\ln\Omega(\Phi) + U(\Phi)\Big) \\ &= \int d^2x\sqrt{-|\tilde{g}|}\Big(D(\Phi)\tilde{R}(g) + \frac{1}{2}\tilde{g}^{ab}\partial_a\Phi\partial_b\Phi \\ &\qquad + 2D(\Phi)\tilde{g}^{ab}\partial_a\partial_b\ln\Omega(\Phi) + \Omega^{-2}(\Phi)U(\Phi)\Big).\end{aligned} \tag{3.19}$$

In order to eliminate the kinetic term we need

$$\frac{1}{2}\tilde{g}^{ab}\partial_a\Phi\partial_b\Phi + 2D(\Phi)\tilde{g}^{ab}\partial_a\partial_b\ln\Omega(\Phi) = 0, \tag{3.20}$$

and thus

$$\begin{aligned}0 &= \frac{1}{2}\partial_a\Phi\partial_b\Phi + 2D(\Phi)\partial_a\partial_b\ln\Omega(\Phi) \\ &= \frac{1}{2}\partial_a\Phi\partial_b\Phi + 2\partial_a\left(D(\Phi)\partial_b\ln\Omega(\Phi)\right) - 2\partial_a D(\Phi)\partial_b\ln\Omega(\Phi) \\ &= \frac{1}{2}\partial_a\Phi\partial_b\Phi + 2\partial_a\left(D(\Phi)\partial_b\ln\Omega(\Phi)\right) - 2\frac{\partial D(\Phi)}{\partial\Phi}\partial_a\Phi\frac{\partial\ln\Omega(\Phi)}{\partial\Phi}\partial_b\Phi.\end{aligned} \tag{3.21}$$

The second term in the last line above is a total derivative (that will appear in the action) and may be put away. So we get

$$\frac{1}{2} - 2\frac{dD(\Phi)}{d\Phi}\frac{d\ln\Omega(\Phi)}{d\Phi} = 0, \tag{3.22}$$

as the condition for eliminating the kinetic term in (3.14). Computing $\Omega(\Phi)$ from (3.22), yields

$$\Omega(\Phi) = C\exp\left(\frac{1}{4}\int d\Phi\frac{1}{\frac{dD(\Phi)}{d\Phi}}\right), \tag{3.23}$$

where C is constant to be determined. So now by using the condition (3.20) or equivalently (3.22), the generic dilaton action(3.19) becomes

$$S_{g\text{-dil}} = \int d^2x\sqrt{-|\tilde{g}|}\left(D(\Phi)\tilde{R}(g) + \Omega^{-2}(\Phi)U(\Phi)\right). \tag{3.24}$$

Looking at this, we see that we can write it in a simpler form by introducing a new field variable

$$X^3 \equiv D(\Phi). \tag{3.25}$$

and a new "potential" term

$$V(X^3) \equiv \Omega^{-2}U(D^{-1}(X^3)), \tag{3.26}$$

where it is understood as $\Omega^{-2}U$ is a function of $D^{-1}(X^3)$. According to (3.23)-(3.26), we restrict ourselves to the case that D has an inverse D^{-1} everywhere on its domain and, for simplicity, we assume that D, D^{-1}, and U are C^∞.

Using (3.24), (3.25), (3.26), one can write the pure gravitational part of the generic 1+1 action as

$$S_{\text{g-gen}} = \int d^2x\sqrt{-|\tilde{g}|}\left\{X^3\tilde{R} + V(X^3)\right\}. \tag{3.27}$$

For the matter parts, one can use (3.5), (3.8), the conformal transformation (3.15) and equations (3.16) and (3.25) to rewrite the matter part of the spherically symmetric action as

$$S_{\text{m-spher}} = -\int d^2x\sqrt{-|\tilde{g}|}\frac{1}{2}\left[D^{-1}(X^3)\right]^2\tilde{g}^{ab}\partial_a f\partial_b f, \tag{3.28}$$

$$S_{\text{m-CGHS}} = -\int d^2x\sqrt{-|\tilde{g}|}\frac{1}{2}\tilde{g}^{ab}\partial_a f\partial_b f. \tag{3.29}$$

Using (3.28) and (3.29), one can write a general action for both cases as

$$S_{\text{m-gen}} = -\int d^2x\sqrt{-|\tilde{g}|}W(X^3)\tilde{g}^{ab}\partial_a f\partial_b f. \tag{3.30}$$

Now by virtue of equations (3.27) and (3.30), we can write the full generic $1+1$ action with minimally coupled matter as

$$S_{1+1} = \int d^2x\sqrt{-|\tilde{g}|}\left\{X^3\tilde{R} + V(X^3)\right\} - \int d^2x\sqrt{-|\tilde{g}|}W(X^3)\tilde{g}^{ab}\partial_a f\partial_b f. \tag{3.31}$$

3.2.3.2 Identifying variables in spherically symmetric action

In order to be able to go back to the specific actions we need to identify $V(X^3)$ and $W(X^3)$ for both cases. Comparing (3.14) and (3.4) one can see for the spherically symmetric case

$$X^3 \equiv D(\Phi) \equiv \frac{1}{4}\Phi^2, \quad U(\Phi) \equiv \frac{1}{2}. \tag{3.32}$$

Using (3.23) and (3.32) and putting $C = 1$, one gets

$$\Omega(\Phi) = \sqrt{\Phi}. \tag{3.33}$$

Using (3.26), (3.32) and (3.33) yields

$$V(X^3) = \frac{1}{2}\Phi^{-1}(X^3) = \frac{1}{4\sqrt{X^3}}. \tag{3.34}$$

And finally comparing (3.30) and (3.28) and using (3.32), one will find

$$W(X^3) = \frac{1}{2}\Phi^2(X^3) = 2X^3. \tag{3.35}$$

3.2.3.3 Identifying variables in CGHS action

For the CGHS model one can find, by comparing (3.14) and the gravitational part of (i.e. the first integral in) the equation (3.13)

$$X^3 \equiv D(\Phi) \equiv \frac{1}{8}\Phi^2, \quad U(\Phi) \equiv -\frac{1}{8}\Phi^2\Lambda = -D(\Phi)\Lambda. \tag{3.36}$$

Using (3.23) and (3.36) one gets

$$\Omega(\Phi) = C\Phi. \tag{3.37}$$

Using (3.26), (3.36) and (3.37) and choosing $C = \sqrt{\frac{1}{8}}$ yields

$$V(X^3) = -\frac{1}{8}C^{-2}\Lambda = -\Lambda. \tag{3.38}$$

And finally comparing (3.30) and (3.29), gives

$$W(X^3) = \frac{1}{2}. \tag{3.39}$$

3.3 The Lagrangians in tetrad formulation

3.3.1 The choice of the generic action

Before we continue we should make an important note: for the 3+1 case, we will use the conformally transformed action (3.31) with (3.32), (3.34) and (3.35) since the analysis will coincide with other independent analyses with Ashtekar variables. But for the CGHS we will use its non-transformed form in (3.13) because in this case, we think that it is better to work with the variables that have direct geometric meaning so that we do not need to transform everything back from the non-physical geometry to the physical one at the end. Thus we redefine both actions and introduce a new action that (in a formal way) contains both, the most general action (3.31) and the specific CGHS action (3.13), in it as

$$S_{1+1} = \int d^2x\sqrt{-|g|}\left\{YR + \frac{1}{2}Zg^{ab}\partial_a\Phi\partial_b\Phi + V\right\} - \int d^2x\sqrt{-|g|}Wg^{ab}\partial_a f\partial_b f, \tag{3.40}$$

with the Lagrangian densities

$$L_{\mathrm{g}} = \sqrt{-|g|}\left\{YR + \frac{1}{2}Zg^{ab}\partial_a\Phi\partial_b\Phi + V\right\}, \tag{3.41}$$

$$L_{\mathrm{m}} = -\sqrt{-|g|}Wg^{ab}\partial_a f\partial_b f, \tag{3.42}$$

where it is understood that

- For the specific physical CGHS case, g_{ab} is the physical metric while for the generic case (including the 3+1 and CGHS conformally transformed cases) it is the conformally transformed metric.

- For the specific physical CGHS case we have

$$Y = \frac{1}{8}\Phi^2, \tag{3.43}$$
$$Z = 1, \tag{3.44}$$
$$V = -\frac{1}{8}\Phi^2\Lambda, \tag{3.45}$$
$$W = \frac{1}{2}. \tag{3.46}$$

- For the generic case we have

$$Y = X^3, \tag{3.47}$$
$$Z = 0, \tag{3.48}$$
$$V = V(X^3), \tag{3.49}$$
$$W = W(X^3). \tag{3.50}$$

- For the spherically symmetric case we have from generic case above and (3.32), (3.34) and (3.35)

$$Y = X^3 = \frac{1}{4}\Phi^2, \tag{3.51}$$
$$Z = 0, \tag{3.52}$$
$$V = \frac{1}{2}\Phi^{-1} = \frac{1}{4\sqrt{X^3}}, \tag{3.53}$$
$$W = \frac{1}{2}\Phi^2 = 2X^3. \tag{3.54}$$

3.4 Tetrad formulation of the generic Lagrangian

The first step towards writing the theory in Ashtekar's variables is to write the Lagrangian densities (3.41) and (3.42) in terms of tetrads and the spin connection instead of the metric. To start, we write the determinant of the metric in terms of the determinant of the tetrad. For this, we note that

$$g_{ab} = \eta_{IJ} e^I{}_a e^J{}_b, \tag{3.55}$$

where I, J are Lorentz or internal indices and a, b are abstract indices. Taking the determinant of both sides yields

$$\begin{aligned} g &= \eta e^2 \\ &= -e^2, \end{aligned} \tag{3.56}$$

where $g = \det(g_{ab})$, $\eta = \det(\eta_{ab}) = -1$ and $e = \det\left(e_a{}^I\right)$. Thus we have

$$\sqrt{-g} = e. \tag{3.57}$$

The curvature can be written in terms of curvature of the spin connection and ultimately in terms of the spin connection ${\omega_a}^{IJ}$ itself as

$$\begin{aligned} R &= {R_{ab}}^{IJ} {e^a}_I {e^b}_J \\ &= \left(2\partial_{[a}\omega_{a]}{}^{IJ} + [\omega_a, \omega_b]^{IJ}\right) {e^a}_I {e^b}_J, \end{aligned} \tag{3.58}$$

where $[,]$ stands for the Lie commutator in the Lorentz Lie algebra and the indices I, J take value in this algebra. Since the spin connection is antisymmetric in I, J, we can write it as

$${\omega_a}^{IJ} = \omega_a \epsilon^{IJ}. \tag{3.59}$$

This way, the curvature (3.58) becomes

$$\begin{aligned} R &= \left(2\partial_{[a}\omega_{a]}\epsilon^{IJ} + \omega_{[a}\omega_{b]}\eta^{IJ}\right) {e^a}_I {e^b}_J \\ &= 2\partial_{[a}\omega_{a]}\epsilon^{IJ} {e^a}_I {e^b}_J, \end{aligned} \tag{3.60}$$

where we have used the following fact about the Lie commutator in this case

$$\begin{aligned} [\omega_a, \omega_b]^{IJ} &= \omega_a {\epsilon^I}_K \omega_b \epsilon^{KJ} - \omega_b {\epsilon^I}_K \omega_a \epsilon^{KJ} \\ &= \omega_{[a}\omega_{b]} \underbrace{{\epsilon^I}_K \epsilon^{KJ}}_{\eta^{IJ}}, \end{aligned} \tag{3.61}$$

and also the fact that $\eta^{IJ} {e^a}_I {e^b}_J$ is symmetric in a, b while $\omega_{[a}\omega_{b]}$ is antisymmetric and thus

$$\omega_{[a}\omega_{b]}\eta^{IJ} {e^a}_I {e^b}_J = 0. \tag{3.62}$$

So the first term in (3.41) can be written as

$$\begin{aligned} eYR &= eY\left(2\partial_{[a}\omega_{a]}\epsilon^{IJ} {e^a}_I {e^b}_J\right) \\ &= 2Y\partial_{[a}\omega_{b]} e\epsilon^{IJ} e^a_I e^b_J. \end{aligned} \tag{3.63}$$

The second term in (3.41) can simply be written as

$$\frac{1}{2} Z e g^{ab} \partial_a \Phi \partial_b \Phi = \frac{1}{2} Z \eta^{IJ} e {e_I}^a {e_J}^b \partial_a \Phi \partial_b \Phi. \tag{3.64}$$

We also would like to add the torsion free condition (contracted by a Lagrange multiplier) to this action. This condition reads

$$0 = de^I + {\epsilon^I}_J \omega \wedge e^J = 2\partial_{[a} e_{b]}{}^I + 2{\epsilon^I}_J \omega_{[a} e_{b]}{}^J. \tag{3.65}$$

Since this is a 2-form, we need to contract it with ϵ^{ab} to get an scalar to be able to add it to the Lagrangian density. Doing so and also contracting it with a Lagrange multiplier $-X_I$ and substituting it in (3.41), together with (3.63) and (3.64), we get

$$L_{\rm g} = -2X_I\epsilon^{ab}(\partial_{[a} e_{b]}{}^I + {\epsilon^I}_J \omega_{[a} e_{b]}{}^J) + 2Y\partial_{[a}\omega_{b]} e\epsilon^{IJ} {e^a}_I {e^b}_J + \frac{1}{2} Z \eta^{IJ} e {e_I}^a {e_J}^b \partial_a \Phi \partial_b \Phi + eV. \tag{3.66}$$

We can write ϵ^{ab} in terms of ϵ^{IJ}, $e_a{}^I$ and its determinant $\det(e_a{}^I)$:

$$e = \det\left(e_a{}^I\right) = \frac{1}{2}\epsilon^{ab}\epsilon_{KL}e_a{}^K e_b{}^L. \tag{3.67}$$

Contracting both sides by $e_I{}^a e_J{}^b$ yields

$$\begin{aligned}
ee_I{}^a e_J{}^b &= \frac{1}{2}\epsilon^{ab}\epsilon_{KL}\delta_I^K\delta_J^L \\
ee_I{}^a e_J{}^b &= \frac{1}{2}\epsilon^{ab}\epsilon_{IJ} \\
ee_I{}^a e_J{}^b\epsilon^{IJ} &= \frac{1}{2}\epsilon^{ab}\underbrace{\epsilon_{IJ}\epsilon^{IJ}}_{-2!} \\
ee_I{}^a e_J{}^b\epsilon^{IJ} &= -\epsilon^{ab},
\end{aligned} \tag{3.68}$$

and thus

$$\epsilon^{ab} = -ee_I{}^a e_J{}^b\epsilon^{IJ}. \tag{3.69}$$

If we integrate by parts in the first term in Lagrangian density (3.66) to bring the partial derivative to act on X_I and then use the above result for ϵ^{ab} in both first two terms of this Lagrangian, we will get

$$\begin{aligned}
L_{\mathrm{g}} =& -2\partial_a(X_I)ee_K{}^a\underbrace{e_L{}^b e_b{}^I}_{\delta_L^I}\epsilon^{KL} + 2X_I ee_K{}^a\underbrace{e_L{}^b e_b{}^J}_{\delta_L^J}\epsilon^{KL}\epsilon^I{}_J\omega_a + 2Y\partial_a\omega_b e\epsilon^{IJ}e^a{}_I e^b{}_J \\
&+\frac{1}{2}Z\eta^{IJ}ee_I{}^a e_J{}^b\partial_a\Phi\partial_b\Phi + eV \\
=& -2\partial_a(X_I)ee_K{}^a\epsilon^{KI} + 2X_I ee_K{}^a\underbrace{\epsilon^{KJ}\epsilon^I{}_J}_{-\eta^{KI}}\omega_a \\
&+2Y\partial_a\omega_b e\epsilon^{IJ}e^a{}_I e^b{}_J + \frac{1}{2}Z\eta^{IJ}ee_I{}^a e_J{}^b\partial_a\Phi\partial_b\Phi + eV.
\end{aligned} \tag{3.70}$$

So the pure gravitational Lagrangian density can be written as

$$L_{\mathrm{g}} = e\left(-2\partial_a(X_I)e_K{}^a\epsilon^{KI} - 2X_I e^{Ia}\omega_a + 2Y\partial_a\omega_b\epsilon^{IJ}e_I{}^a e_J{}^b + \frac{1}{2}Z\eta^{IJ}e_I{}^a e_J{}^b\partial_a\Phi\partial_b\Phi + V\right). \tag{3.71}$$

From (3.42),the matter Lagrangian density can also simply be written as

$$L_{\mathrm{m}} = -W\eta^{IJ}ee_I{}^a e_J{}^b\partial_a f\partial_b f. \tag{3.72}$$

3.5 The 1+1 ADM decomposition of the generic Lagrangian

3.5.1 The pure gravitational part

We begin by foliating the spacetime manifold into spatial hypersurface Σ_t on which t =constant as we have seen in section 2.2. The difference here is that on the hypersurface Σ_t, now use the projection of tetrads on this hypersurface as (non-coordinate) basis vectors. From (2.69) and (3.57), the determinant of the induced metric q_{ab} is

$$e = N\sqrt{q}. \tag{3.73}$$

As we have mentioned there, ${q_a}^b$ can be seen as a projection operator which projects onto the spatial hypersurface Σ_t and can be used to project tetrads ${e_a}^I$ onto Σ_t as in (2.61). Using this and also the the decomposition of the timelike vector $t^a = (\partial/\partial t)^a$ as in (2.58) we can cast ${e_a}^I$ in the following form

$$\begin{aligned} {e^a}_I &= {E^a}_I - n^a n_I \qquad (3.74)\\ &= {E^a}_I - \left(\frac{t^a - N^a}{N}\right) n_I. \qquad (3.75) \end{aligned}$$

If we substitute (3.75) and (3.73) into the pure gravitational Lagrangian density (3.71) we arrive at

$$\begin{aligned} L_{\rm g} =& N\sqrt{q}\Big(-2\partial_a(X_I)\epsilon^{KI}{E^a}_K + \frac{2}{N}t^a\partial_a(X_I)n_K\epsilon^{KI} - \frac{2}{N}N^a\partial_a(X_I)n_K\epsilon^{KI} \\ & -2X_IE^{aI}\omega_a + \frac{2}{N}X_In^It^a\omega_a - \frac{2}{N}X_In^IN^a\omega_a \\ & +2Y\partial_a\omega_b\epsilon^{IJ}{E^a}_I{E^b}_J + \frac{1}{2}Z\eta^{IJ}{E^a}_I{E^b}_J\partial_a\Phi\partial_b\Phi \\ & + \frac{2Y}{N}\left[-t^b\partial_a\omega_b + N^b\partial_a\omega_b + t^b\partial_b\omega_a - N^b\partial_b\omega_a\right]\epsilon^{IJ}{E^a}_In_J \\ & + \left[\left(\frac{2}{N^2}Y\partial_a\omega_bt^a - \frac{2}{N^2}Y\partial_a\omega_bN^a\right)\left(t^b - N^b\right)\right]\epsilon^{IJ}n_In_J \\ & + \frac{1}{2N}\left[-t^b\partial_a\Phi\partial_b\Phi + N^b\partial_a\Phi\partial_b\Phi - t^b\partial_b\Phi\partial_a\Phi + N^b\partial_b\Phi\partial_a\Phi\right]Z\eta^{IJ}{E^a}_In_J \\ & + \frac{1}{2N^2}\left[t^at^b\partial_a\Phi\partial_b\Phi - t^aN^b\partial_a\Phi\partial_b\Phi - N^at^b\partial_a\Phi\partial_b\Phi + N^aN^b\partial_a\Phi\partial_b\Phi\right]Z\eta^{IJ}n_In_J + V\Big). \end{aligned}$$
$$(3.76)$$

Using the Lie derivative of ω_a in the direction of t^a, we have

$$\begin{aligned} \mathscr{L}_t\omega_a &= t^b\partial_b\omega_a + \omega_b\partial_at^b \\ &= \partial_a\left(t^b\omega_b\right) + t^b\partial_b\omega_a - t^b\partial_a\omega_b. \qquad (3.77) \end{aligned}$$

So in the fourth line in (3.76) above, we can substitute

$$t^b\partial_b\omega_a - t^b\partial_a\omega_b = \mathscr{L}_t\omega_a - \partial_a\left(t^b\omega_b\right). \qquad (3.78)$$

We can also simplify things further by noting that

$$\epsilon^{IJ}n_In_J = 0, \qquad (3.79)$$

because of the antisymmetry of ϵ^{IJ} and symmetry of n_In_J. This makes the fifth line of (3.76) vanish. Furthermore since n^a is normal to Σ_t and thus to the basis vectors ${E_J}^a$

in the tangent space of Σ_t, we will have

$$\begin{aligned}\eta^{IJ} n_I E_J{}^c &= g^{ab} e_a{}^I e_b{}^J n_I E_J{}^c \\ &= g^{ab} e_b{}^J n_a E_J{}^c \\ &= g^{ab} e_b{}^J n_a q^c{}_d e^d{}_J \\ &= g^{ab} n_a q^c{}_d \delta_b{}^d \\ &= g^{ab} q^c{}_b n_a \\ &= q^{ac} n_a \\ &= 0,\end{aligned} \tag{3.80}$$

which implies that the sixth line of (3.76) vanishes too. For the seventh line, we can use

$$\begin{aligned}-1 &= g_{ab} n^a n^b \\ &= \eta_{IJ} e^I{}_a e^J{}_b n^a n^b \\ &= \eta_{IJ} n^I n^J,\end{aligned} \tag{3.81}$$

since n^a or n^I is a unit timelike vector field. Thus until now we have

$$\begin{aligned}L_{\mathrm{g}} =& N\sqrt{q}\bigg(-2\partial_a(X_I)\epsilon^{KI} E^a{}_K + \frac{2}{N} t^a \partial_a(X_I) n_K \epsilon^{KI} - \frac{2}{N} N^a \partial_a (X_I) n_K \epsilon^{KI} \\ & -2X_I E^{aI}\omega_a + \frac{2}{N} X_I n^I t^a \omega_a - \frac{2}{N} X_I n^I N^a \omega_a \\ & + 2Y\partial_a \omega_b \epsilon^{IJ} E^a{}_I E^b{}_J + \frac{1}{2} Z \eta^{IJ} E^a{}_I E^b{}_J \partial_a \Phi \partial_b \Phi \\ & + \frac{2Y}{N}\left[\mathscr{L}_t \omega_a - \partial_a\left(t^b\omega_b\right) + N^b \partial_a \omega_b - N^b \partial_b \omega_a\right] \epsilon^{IJ} E^a{}_I n_J \\ & - \frac{Z}{2N^2}\left[t^a t^b \partial_a \Phi \partial_b \Phi - t^a N^b \partial_a \Phi \partial_b \Phi - N^a t^b \partial_a \Phi \partial_b \Phi + N^a N^b \partial_a \Phi \partial_b \Phi\right] + V\bigg).\end{aligned} \tag{3.82}$$

Since we have adapted the (t, x) coordinate system on spacetime and t^a is one of our coordinate basis vectors, we can write

$$t^a = \left(\frac{\partial}{\partial t}\right)^a = \left(\frac{\partial}{\partial x^0}\right)^a = e_0{}^a. \tag{3.83}$$

On the other hand, we can expand N^a in terms of $E^a{}_I$ as in (2.62). The tetrad basis vector field $E^a{}_I$ itself can be expressed in terms of the spacelike sector of the coordinate basis vectors $e_\mu{}^a$ which would be only $e_1{}^a$. This is because our theory is two dimensional and we have only one timelike and one spacelike coordinate basis vectors i.e. $e_\mu{}^a = \{e_0{}^a, e_1{}^a\}$. This means that we can write

$$E^a{}_I = E^1{}_I e^a{}_1, \tag{3.84}$$

$$N^a = N^I E^a{}_I = N^I E^1{}_I e^a{}_1. \tag{3.85}$$

Considering the above simplifications, the Lagrangian density (3.82) takes the form

$$\begin{aligned}L_{\rm g} =N\sqrt{q}\Big(&-2\partial_1(X_I)\epsilon^{JI}E^1{}_J+\frac{2}{N}\partial_0(X_I)n_J\epsilon^{JI}-\frac{2}{N}N^JE^1{}_J\partial_1(X_I)n_K\epsilon^{KI}\\&-2X_IE^{1I}\omega_1+\frac{2}{N}X_In^I\omega_0-\frac{2}{N}X_In^IN^JE^1{}_J\omega_1\\&+2Y\partial_1\omega_1\epsilon^{IJ}E^1{}_IE^1{}_J+\frac{1}{2}Z\eta^{IJ}E^1{}_IE^1{}_J\partial_1\Phi\partial_1\Phi\\&+\frac{2Y}{N}\left[\dot\omega_1-\partial_1\omega_0\right]\epsilon^{IJ}E^1{}_In_J\\&-\frac{Z}{2N^2}\left[\partial_0\Phi\partial_0\Phi-N^IE^1{}_I\partial_0\Phi\partial_1\Phi-N^IE^1{}_I\partial_1\Phi\partial_0\Phi+N^IN^JE^1{}_IE^1{}_J\partial_1\Phi\partial_1\Phi\right]+V\Big).\end{aligned} \tag{3.86}$$

The term with $\epsilon^{IJ}E^1{}_IE^1{}_J$ will vanish due to the symmetry of $E^1{}_IE^1{}_J$ and antisymmetry of ϵ^{IJ}. One can also identify the Hodge dual of X^I as

$$X_I=\epsilon_{IJ}{}^*X^J, \tag{3.87}$$

and its Lie derivative

$$\mathscr{L}_t{}^*X^I=t^a\partial_a{}^*X^I=e^a{}_0\partial_a{}^*X^I=\partial_0{}^*X^I={}^*\dot{X}^I. \tag{3.88}$$

Another thing we note is that the spin connection $\omega_{aI}{}^J=\omega_a\epsilon_I{}^J$ defines a covariant derivative on the spatial hypersurface which acts on internal (Lorentz) indices as for example

$$D_1{}^*X^I=\partial_1{}^*X^I+\omega_1\epsilon^{IJ*}X_J=\partial_1{}^*X^I+\omega_1X^I. \tag{3.89}$$

Using all the above substitutions in the Lagrangian density (3.86) yields

$$\begin{aligned}L_{\rm g} =N\sqrt{q}\Big(&-2E^1{}_KD_1{}^*X^K-\frac{2}{N}N^1n_KD_1{}^*X^K+\frac{2}{N}n_K{}^*\dot X^K+\frac{2}{N}\epsilon_{IJ}{}^*X^Jn^I\omega_0\\&+\frac{1}{2}Z\eta^{IJ}E^1{}_IE^1{}_J\Phi'^2+\frac{2Y}{N}\left[\dot\omega_1-\omega_0'\right]\epsilon^{IJ}E^1{}_In_J\\&-\frac{Z}{2N^2}\left[\dot\Phi^2-2N^1\Phi'\dot\Phi+\left(N^1\right)^2\Phi'^2\right]+V\Big),\end{aligned} \tag{3.90}$$

where we have used (3.85) to write $N^IE^1{}_I=N^1$ and also used prime to denote the partial derivative with respect to $x^1=x$ and dot to denote the partial derivative with respect to $x^0=t$. There is one important step remaining which is about the tetrad basis vectors in Σ_t. We know that by use of (3.74) and (2.56), we can write

$$\begin{aligned}q^{ab}&=\eta^{IJ}e^a{}_Ie^b{}_J\\&=\eta^{IJ}\left(E^a{}_I-n^an_I\right)\left(E^b{}_J-n^bn_J\right)\\&=\eta^{IJ}E^a{}_IE^b{}_J.\end{aligned} \tag{3.91}$$

This means that

$$\eta^{IJ}E^1{}_IE^1{}_J=q^{11}=\frac{1}{q^{11}}=\frac{1}{q}, \tag{3.92}$$

where q is the determinant of the spatial metric q_{ab} and we have used the fact that the spatial hypersurface is one dimensional and thus its only independent component $q_{11} = \frac{1}{q^{11}}$ is at the same time its determinant. This shows that $E^1{}_I$ is a density of weight -1 because (based on conventions in canonical quantum gravity), determinant of the metric is a scalar of density $+2$. We can use an alternative set of non-coordinate basis vectors which are absolute tensors (of weight 0) by noting from (3.92) that

$$\eta^{IJ}\sqrt{q}E^1{}_I\sqrt{q}E^1{}_J = \eta^{IJ}\tilde{E}^1{}_I\tilde{E}^1{}_J = 1, \tag{3.93}$$

where

$$\tilde{E}^1{}_I = \sqrt{q}E^1{}_I, \tag{3.94}$$

is now an absolute vector since it is the product of a vector density of weight -1 and the square of determinant of spatial metric (a scalar density of weight $+1$). We can write the basis vector field $\tilde{E}^1{}_I$ in terms of a spacelike unit vector field. The dual field of ${}^*n_I = \epsilon_{IJ}n^J$ serves the purpose since

$$\begin{aligned}\eta^{IJ*}n_I{}^*n_J &= \eta^{IJ}\epsilon_{IK}n^K\epsilon_{JL}n^L\\ &= \underbrace{\epsilon_{IK}\epsilon^{IL}}_{-\delta_K{}^L}n^K n_L\\ &= -n^K n_K\\ &= 1,\end{aligned} \tag{3.95}$$

where we have used the fact that n_I is timelike unit vector. Thus we have

$$\tilde{E}^1{}_I = \tilde{E}^1{}_{||}{}^*n_I, \tag{3.96}$$

where $\tilde{E}^1{}_{||}$ are the components of $\tilde{E}^1{}_I$. Using (3.93) we can find the value of the components

$$\begin{aligned}1 &= \eta^{IJ}\tilde{E}^1{}_I\tilde{E}^1{}_J\\ &= \eta^{IJ}\left(\tilde{E}^1{}_{||}\right)^2{}^*n_I{}^*n_J\\ &= \left(\tilde{E}^1{}_{||}\right)^2,\end{aligned} \tag{3.97}$$

where we have used the fact that ${}^*n_I$ is spacelike unit vector field. Thus we find out that $\tilde{E}^1{}_{||} = \pm 1$ and we choose

$$\tilde{E}^1{}_{||} = +1. \tag{3.98}$$

Putting this back into (3.96) leads to

$$\tilde{E}^1{}_I = {}^*n_I, \tag{3.99}$$

and using this and (3.94) yields

$$E^1{}_I = \frac{{}^*n_I}{\sqrt{q}}. \tag{3.100}$$

From this and (3.95) we find

$$\eta^{IJ}E^1{}_IE^1{}_J = \eta^{IJ}\frac{{}^*n_I}{\sqrt{q}}\frac{{}^*n_I}{\sqrt{q}} = \frac{1}{q}, \tag{3.101}$$

which is precisely (3.92). Substituting (3.100) and(3.92) in the Lagrangian density (3.90) leads to

$$\begin{aligned} L_{\rm g} = &- 2N^* n_I D_1{}^* X^I - 2\sqrt{q} N^1 n_I D_1{}^* X^I + 2\sqrt{q} n_I{}^* \dot{X}^I \\ &- 2\sqrt{q}^* n_I{}^* X^I \omega_0 + \frac{NZ}{2\sqrt{q}} \Phi'^2 + 2Y \left[\dot{\omega}_1 - \omega_0'\right] \epsilon^{IJ*} n_I n_J \\ &- \frac{Z\sqrt{q}}{2N} \left[\dot{\Phi}^2 - 2N^1 \Phi' \dot{\Phi} + \left(N^1\right)^2 \Phi'^2\right] + N\sqrt{q} V, \end{aligned} \tag{3.102}$$

where we have used

$$\epsilon_{IJ}{}^* X^J n^I = -\epsilon_{JI} n^{I*} X^J = -^* n_J{}^* X^J. \tag{3.103}$$

Finally substituting

$$\begin{aligned} \epsilon^{IJ*} n_I n_J &= \epsilon^{IJ} \epsilon_{IK} n^K n_J \\ &= -\delta^J{}_K n^K n_J \\ &= -n^K n_K \\ &= 1, \end{aligned} \tag{3.104}$$

into (3.102) yields

$$\begin{aligned} L_{\rm g} = &- 2N^* n_I D_1{}^* X^I - 2\sqrt{q} N^1 n_I D_1{}^* X^I + 2\sqrt{q} n_I{}^* \dot{X}^I \\ &- 2\sqrt{q}^* n_I{}^* X^I \omega_0 + 2Y \dot{\omega}_1 - 2Y \omega_0' \\ &+ \frac{Z}{2} \left[\frac{N}{\sqrt{q}} \Phi'^2 - \frac{\sqrt{q}}{N} \dot{\Phi}^2 + \frac{\sqrt{q}}{N} 2N^1 \Phi' \dot{\Phi} - \frac{\sqrt{q}}{N} \left(N^1\right)^2 \Phi'^2\right] \\ &+ N\sqrt{q} V. \end{aligned} \tag{3.105}$$

This is the tetrad formulations of the most generic form of a pure gravitational Lagrangian density in two dimensions. We can see that by putting $Z = 0$, we get the 3+1 or other generic cases and by $Z = 1$, we get the Lagrangian of the physical geometry of the CGHS model (in both cases with appropriate Y and V).

3.5.2 The matter part

Following the same steps as above for the matter Lagrangian (3.72) leads to

$$\begin{aligned} L_{\rm m} = &- N\sqrt{q} W \eta^{IJ} \left[E^a{}_I - \left(\frac{t^a - N^a}{N}\right) n_I\right] \left[E^b{}_J - \left(\frac{t^b - N^b}{N}\right)\right] n_J \partial_a f \partial_b f \\ = &- N\sqrt{q} W \left[\eta^{IJ} E^a{}_I E^b{}_J \partial_a f \partial_b f - \left(\frac{t^b - N^b}{N}\right) \underbrace{\eta^{IJ} n_J E^a{}_I}_{0} \partial_a f \partial_b f \right. \\ &\left. - \left(\frac{t^a - N^a}{N}\right) \underbrace{\eta^{IJ} n_I E^b{}_J}_{0} \partial_a f \partial_b f + \left(\frac{t^a - N^a}{N}\right) \left(\frac{t^b - N^b}{N}\right) \eta^{IJ} n_I n_J \partial_a f \partial_b f \right] \\ = &- N\sqrt{q} W \left[\frac{1}{q} \partial_1 f \partial_1 f - \frac{1}{N^2} \left(t^a t^b - t^a N^b - N^a t^b + N^a N^b\right) \partial_a f \partial_b f\right] \end{aligned}$$

$$
\begin{aligned}
&= -N\sqrt{q}W\left[\frac{1}{q}f'^2 - \frac{1}{N^2}\left(\dot{f}^2 - 2N^1\dot{f}f' + \left(N^1\right)^2 f'^2\right)\right] \\
&= -W\left(\frac{Nf'^2}{\sqrt{q}} + \frac{-\sqrt{q}\dot{f}^2 + 2\sqrt{q}N^1\dot{f}f' - \sqrt{q}\left(N^1\right)^2 f'^2}{N}\right).
\end{aligned}
\tag{3.106}
$$

So we can write the full Lagrangian density of the most general form of a two dimensional model as

$$
\begin{aligned}
L = &-2N^* n_I D_1{}^* X^I - 2\sqrt{q}N^1 n_I D_1{}^* X^I + 2\sqrt{q}n_I{}^* \dot{X}^I \\
&- 2\sqrt{q}^* n_I{}^* X^I \omega_0 + 2Y\dot{\omega}_1 - 2Y\omega_0' \\
&+ \frac{Z}{2}\left[\frac{N}{\sqrt{q}}\Phi'^2 - \frac{\sqrt{q}}{N}\dot{\Phi}^2 + \frac{2\sqrt{q}}{N}N^1\Phi'\dot{\Phi} - \frac{\sqrt{q}}{N}\left(N^1\right)^2\Phi'^2\right] \\
&+ N\sqrt{q}V - W\left(\frac{Nf'^2}{\sqrt{q}} + \frac{-\sqrt{q}\dot{f}^2 + 2\sqrt{q}N^1\dot{f}f' - \sqrt{q}\left(N^1\right)^2 f'^2}{N}\right).
\end{aligned}
\tag{3.107}
$$

3.6 Canonical variables and momenta and general Hamiltonian for the conformally transformed and the non-transformed versions

Looking at the full Lagrangian density (3.107), we notice a few things:

- **For non-transformed theory** ($Z \neq 0$) **(physical CGHS for example):**
  - The canonical variables of the theory, i.e. those for which their time derivative is present in the Lagrangian density, are

$$
{}^* X^I, \omega_1, \Phi, f. \tag{3.108}
$$

  - The canonical momenta corresponding to these variables are

$$
\begin{aligned}
P_I &= \frac{\partial L}{\partial^* \dot{X}^I} = 2\sqrt{q}n_I, && (3.109) \\
P_\omega &= \frac{\partial L}{\partial \dot{\omega}_1} = 2Y, && (3.110) \\
P_\Phi &= \frac{\partial L}{\partial \dot{\Phi}} = \frac{Z\sqrt{q}}{N}\left(N^1\Phi' - \dot{\Phi}\right), && (3.111) \\
P_f &= \frac{\partial L}{\partial \dot{f}} = -\frac{2W\sqrt{q}}{N}\left(N^1 f' - \dot{f}\right). && (3.112)
\end{aligned}
$$

  - If Y involves one of the canonical variables, the second equation (3.110) will be a new primary constraint. This happens in physical CGHS where $Z = 1$ and $Y = {}^1\!/_8\Phi^2$. The equation (3.110) is a primary constraint in the physical CGHS case which we should later on, add to the general Hamiltonian (3.124) below, to get the total Hamiltonian for that model.

- **For conformally transformed theory** ($Z = 0$) **(generic case including 3+1 spherically symmetric):**

  – The canonical variables are

$$^*X^I, \omega_1, f \tag{3.113}$$

thus the dilation field Φ is not a canonical variable and so its conjugate momentum will not appear in the theory as can be seen by vanishing of Z in (3.107) and (3.111). Therefore the canonical momenta are just

$$P_I, P_\omega, P_f \tag{3.114}$$

and although there is a relation between P_ω and Φ through (3.110) and $Y = 1/4\Phi^2$ in the 3+1 case, equation (3.110) does not define a constraint since Φ is not a canonical variable.

So we continue to work with the Lagrangian (3.107), the variables (3.108) and the momenta (3.109)-(3.112), but bearing in mind that at the end, the general Hamiltonian can be cast into the one suitable for the physical CGHS by using $Z = 1$ and for other theories by $Z = 0$ and $P_\Phi = 0$ or equivalently no Φ canonical variable.

Solving (3.111) and (3.112) for $\dot{\Phi}$ and $\dot{f}$, yields

$$\dot{\Phi} = N^1\Phi' - \frac{N}{Z\sqrt{q}}P_\Phi, \tag{3.115}$$

$$\dot{f} = N^1 f' + \frac{N}{2W\sqrt{q}}P_f. \tag{3.116}$$

Applying a Legendre transformation to the Lagrangian (3.107) and using equations (3.109) to (3.116), yields

$$\begin{aligned} H = &N\left[2^*n_I D_1{}^*X^I - \frac{Z}{2\sqrt{q}}\Phi'^2 - \frac{P_\Phi^2}{2Z\sqrt{q}} + \frac{Wf'^2}{\sqrt{q}} + \frac{P_f^2}{4W\sqrt{q}} - \sqrt{q}V\right] \\ &+ N^1\left[P_\Phi\Phi' + P_f f' + P_I D_1{}^*X^I\right] \\ &+ \omega_0\left[\epsilon_{IJ}P^{J*}X^I - (2Y)'\right], \end{aligned} \tag{3.117}$$

where we have integrated by parts over the term $\frac{1}{4}\Phi^2\omega_0'$ to make the derivative act on Φ and used (3.109) to substitute for $2\sqrt{q}n_I$. Now using the definition of the covariant derivative (3.89), we can rewrite the above Hamiltonian as

$$\begin{aligned} H = &N\left[2\epsilon_{IJ}n^J\partial_1{}^*X^I - 2n_I\omega_1{}^*X^I - \frac{Z}{2\sqrt{q}}\Phi'^2 - \frac{P_\Phi^2}{2Z\sqrt{q}} + \frac{Wf'^2}{\sqrt{q}} + \frac{P_f^2}{4W\sqrt{q}} - \sqrt{q}V\right] \\ &+ N^1\left[P_\Phi\Phi' + P_f f' + P_I\partial_1{}^*X^I + \epsilon^{IJ}P_I\omega_1{}^*X_J\right] \\ &+ \omega_0\left[\epsilon_{IJ}P^{J*}X^I - (2Y)'\right]. \end{aligned} \tag{3.118}$$

Substituting for n_I from (3.109) yields

$$
\begin{aligned}
H = & N\left[\epsilon_{IJ}\frac{P^J}{\sqrt{q}}\partial_1{}^* X^I - \frac{P_I}{\sqrt{q}}\omega_1{}^* X^I - \frac{Z}{2\sqrt{q}}\Phi'^2 - \frac{P_\Phi^2}{2Z\sqrt{q}} + \frac{Wf'^2}{\sqrt{q}} + \frac{P_f^2}{4W\sqrt{q}} - \sqrt{q}V\right] \\
& + N^1\left[P_\Phi \Phi' + P_f f' + P_I \partial_1{}^* X^I + \epsilon_{IJ}P^I \omega_1{}^* X^J\right] \\
& + \omega_0\left[\epsilon_{IJ}P^{J*} X^I - (2Y)'\right].
\end{aligned} \tag{3.119}
$$

We can also infer from (3.109) that

$$
\begin{aligned}
-\eta^{IJ}P_I P_J &= -4q\eta^{IJ}n_I n_J, && (3.120)\\
|P|^2 &= 4q, && (3.121)\\
\frac{|P|}{2} &= \sqrt{q}. && (3.122)
\end{aligned}
$$

where we have used $-\eta^{IJ}P_I P_J = |P|^2$ since P_I is a timelike vector by virtue of (3.109). Substituting $\sqrt{q}$ from above into (3.119) yields

$$
\begin{aligned}
H = & N\left[2\epsilon_{IJ}\frac{P^J}{|P|}\partial_1{}^* X^I - 2\frac{P_I}{|P|}\omega_1{}^* X^I - \frac{Z}{|P|}\Phi'^2 - \frac{P_\Phi^2}{Z|P|} + \frac{2Wf'^2}{|P|} + \frac{P_f^2}{2W|P|} - \frac{|P|}{2}V\right] \\
& + N^1\left[P_\Phi \Phi' + P_f f' + P_I \partial_1{}^* X^I + \epsilon_{IJ}P^I \omega_1{}^* X^J\right] \\
& + \omega_0\left[\epsilon_{IJ}P^{J*} X^I - (2Y)'\right].
\end{aligned} \tag{3.123}
$$

Expanding the above total Hamiltonian in its components in tetrad basis yields

$$
\begin{aligned}
H = & N\Bigg[2\frac{P_2}{|P|}\partial_1{}^* X^1 + 2\frac{P_1}{|P|}\partial_1{}^* X^2 - 2\frac{P_1}{|P|}\omega_1{}^* X^1 - 2\frac{P_2}{|P|}\omega_1{}^* X^2 \\
& - \frac{Z}{|P|}\Phi'^2 - \frac{P_\Phi^2}{Z|P|} + \frac{2Wf'^2}{|P|} + \frac{P_f^2}{2W|P|} - \frac{|P|}{2}V\Bigg] \\
& + N^1\left[P_\Phi \Phi' + P_f f' + P_1 \partial_1{}^* X^1 + P_2 \partial_1{}^* X^2 - P_1 \omega_1{}^* X^2 - P_2 \omega_1{}^* X^1\right] \\
& + \omega_0\left[P_1{}^* X^2 + P_2{}^* X^1 - (2Y)'\right],
\end{aligned} \tag{3.124}
$$

where we used the fact that $P_1 = -P^1$ and $P_2 = P^2$ and used the convention $\epsilon_{IJ} = +1$. From this point on, which we are going to study the Hamiltonian formulation of the theory in Ashtekar variables, we will need the specific form of the functions Y, Z, V and W to proceed. Thus we study each of the two systems in a separate chapter.

4

Hamiltonian analysis of the 3+1 spherically symmetric model

4.1 Reduction using spherical symmetry

Spherically symmetric manifold are the ones that can be foliated by 2-spheres or have the symmetries of S^2. More technically, in the language of group theory, it means that if the rotation group $SO(3)$ acts on points of the spatial hypersurface Σ_t, the symmetry orbits or the integral curves of the Killing vectors of the symmetry group, are 2-spheres. Furthermore, the spatial manifold Σ_t has three metric Killing vector fields that in a coordinate dependent way, in coordinates (θ,ϕ), read [19]

$$R^a = \left(\partial_\phi\right)^a, \tag{4.1}$$

$$S^a = \cos(\phi)\,(\partial_\theta)^a - \cot(\theta)\sin(\phi)\left(\partial_\phi\right)^a, \tag{4.2}$$

$$T^a = -\sin(\phi)\,(\partial_\theta)^a - \cot(\theta)\cos(\phi)\left(\partial_\phi\right)^a. \tag{4.3}$$

or in coordinate independent way have the algebra

$$[R,S]^a = T^a, \tag{4.4}$$

$$[S,T]^a = R^a, \tag{4.5}$$

$$[T,R]^a = S^a. \tag{4.6}$$

In addition, the Killing vectors of the symmetry group $SO(3)$ are also the Killing vectors of the (spatial) metric q_{ab} and thus the metric is invariant under the action of the rotation group $SO(3)$. In other words, we have a diffeomorphism [20]

$$\Upsilon_{C(s)} : \Sigma_t \to \Sigma_t, \tag{4.7}$$

which is generated by the orbits in $SO(3)$

$$C(s) : \mathbb{R} \to SO(3), \tag{4.8}$$

such that $\Upsilon_{C(s)}$ is an isometry, i.e.

$$\Upsilon^*_{C(s)} q_{ab} = q_{ab}, \tag{4.9}$$

where $\Upsilon^*_{C(s)} q_{ab}$ is the pullback of the metric by $\Upsilon_{C(s)}$.

It can be shown that from the spherical symmetry ansatz ($\mu, \nu = 0, 1$) and purely geometrical arguments that the metric in a spherically symmetric manifold can be written as [19]

$$ds^2 = T(x,t)dt^2 + L(x,t)^2 dx^2 + R(x,t)^2 \left(d\theta^2 + \sin^2(\theta) d\phi^2\right), \tag{4.10}$$

with coordinates (t, x, θ, ϕ), and T, R, L functions of the coordinates t and x, which should not to be confused with the Killing vectors R^a, S^a, T^a. Under action of the gauge group of $SU(2)$-valued functions h on Σ_t, the metric g_{ab} and the phase space variables, the connection $A_a = A_\mu dx^\mu{}_a$ and the triad $E^a = E^\mu \partial_\mu{}^a$, transform as usual as [21, 22]

$$g \to g, \tag{4.11}$$
$$A \to h^{-1} A h + h^{-1} dh, \tag{4.12}$$
$$E \to h^{-1} E h, \tag{4.13}$$

and all the pairs related together by gauge transformations are equivalent. Thus A and E need not to be exactly invariant under the action of the rotation group like the metric and can be invariant up to a gauge transformation. Using these properties it can be shown that the connection can be written in (x, θ, ϕ) coordinates as

$$A = A_x(x)\tau_3 dx + (A_1(x)\tau_1 + A_2(x)\tau_2)\, d\theta + ((A_1(x)\tau_2 - A_2(x)\tau_1)\sin\theta + \tau_3 \cos\theta)\, d\phi, \tag{4.14}$$

where A_x, A_1 and A_2 are real arbitrary functions on $\mathbb{R}^+$, the τ_I are generators of $\mathfrak{su}(2)$, which as we said before, can be conventionally taken to be $\tau_I = \tau^I = -i\sigma_I/2$ where σ_I are the Pauli matrices. The invariant triad takes the form

$$E = E^x(x)\tau_3 \sin\theta \frac{\partial}{\partial x} + \left(E^1(x)\tau_1 + E^2(x)\tau_2\right)\sin\theta \frac{\partial}{\partial \theta} + \left(E^1(x)\tau_2 - E^2(x)\tau_1\right)\frac{\partial}{\partial \phi}, \tag{4.15}$$

where again, E^x, E^1 and E^2 are functions on $\mathbb{R}^+$. In terms of these quantities, the spatial part of the (4.10) reads [23]

$$ds_q^2 = \frac{E^{\varphi 2}}{|E^x|} dx^2 + |E^x| \left(d\theta^2 + \sin^2(\theta) d\phi^2\right), \tag{4.16}$$

where $E^\varphi = \sqrt{\left(E^1\right)^2 + \left(E^2\right)^2}$ is a canonical momenta conjugate to the canonical variable K^φ which is the angular part of the extrinsic curvature $K = K_\mu{}^I \tau_I dx^\mu = \det(E)^{-1/2} K_{\mu\nu} E^{\nu I} \tau_I dx^\mu$. In terms of these variables the diffeomorphism and Hamiltonian constraints for gravity minimally coupled to a massless scalar field are[21, 22]

$$\mathscr{D} = (|E^x|)' K_x - E^\varphi (K_\varphi)' - P_f f', \tag{4.17}$$

$$\begin{aligned}\mathscr{H} = &- \frac{E^\varphi}{2\sqrt{|E^x|}} - 2K_\varphi \sqrt{|E^x|} K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{|E^x|}} + \frac{\left((|E^x|)'\right)^2}{8\sqrt{|E^x|}E^\varphi} \\ &- \frac{\sqrt{|E^x|}(|E^x|)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{|E^x|}(E^x)'' \mathrm{sgn}(E^x)}{2E^\varphi} \\ &+ \frac{P_f^2}{2\sqrt{|E^x|}E^\varphi} + \frac{(|E^x|)^{3/2}(f')^2}{2E^\varphi},\end{aligned} \tag{4.18}$$

where K_x is the radial component of the extrinsic curvature of the spatial hypersurface which is also related to the radial component of the Ashtekar connection A_x.

This is the work that has been done before. Now in the coming section we would like to show that our generic formalism leads to the same result provided that we use suitable canonical transformations on our generic Hamiltonian (3.124).

4.2 The Hamiltonian density

Since we we mentioned in (3.6), Φ is not a canonical variable in the 3+1 case, and also looking at the form of (3.31), we would like to substitute Φ for X^3 in the theory. Thus our canonical variable in this case will be

$$ {}^*X^I, \omega_1, f \tag{4.19}$$

and using (3.51)-(3.54) and (3.109)-(3.112), we will have for Φ and for canonical momenta

$$\begin{aligned}
\Phi =& 2\sqrt{X^3}, && (4.20)\\
\Phi' =& \frac{X^{3\prime}}{\sqrt{X^3}}, && (4.21)\\
P_I =& \frac{\partial L}{\partial^* \dot{X}^I} = 2\sqrt{q}\, n_I, && (4.22)\\
P_\omega =& P_3 = \frac{\partial L}{\partial \dot{\omega}_1} = 2X^3, && (4.23)\\
P_f =& \frac{\partial L}{\partial \dot{f}} = -\frac{4X^3\sqrt{q}}{N}\left(N^1 f' - \dot{f}\right). && (4.24)
\end{aligned}$$

There is no further constraint to add to the general Hamiltonian (3.124) and by substituting (3.51)-(3.54), the above values together with (3.116) and $Z = P_\Phi = 0$ in that Hamiltonian we get

$$\begin{aligned}
H =& N\left[2\frac{P_2}{|P|}\partial_1{}^*X^1 + 2\frac{P_1}{|P|}\partial_1{}^*X^2 - 2\frac{P_1}{|P|}\omega_1{}^*X^1\right.\\
&\left. - 2\frac{P_2}{|P|}\omega_1{}^*X^2 + \frac{4X^3 f'^2}{|P|} + \frac{P_f^2}{4X^3|P|} - \frac{|P|}{8\sqrt{X^3}}\right]\\
&+ N^1\left[P_f f' + P_1\partial_1{}^*X^1 + P_2\partial_1{}^*X^2 - P_1\omega_1{}^*X^2 - P_2\omega_1{}^*X^1\right]\\
&+ \omega_0\left[P_1{}^*X^2 + P_2{}^*X^1 - \left(2X^3\right)'\right].
\end{aligned} \tag{4.25}$$

Now we would like to make a suitable canonical transformation to go from the present canonical pairs to the one used in the literature for the 3+1 case, namely the pairs (K_x, E^x) and (K_φ, E^φ).

We begin by observing from $(\mu, \nu = 0, 1)$ and (4.16) that

$$\Phi^2 = |E^x|. \tag{4.26}$$

This means by using the above and (3.51) we can write

$$X^3 = \frac{1}{4}\Phi^2 = \frac{E^x}{4}. \tag{4.27}$$

Now note that we are currently working with the conformally transformed metric(3.15) in the 3+1 case. This means that if we want to find the determinant of the spatial metric q_{ab} (remember that our spatial hypersurface in our method is one dimensional), we can write it as

$$\sqrt{q} = \sqrt{\tilde{g}_{xx}} = \Omega\sqrt{g_{xx}} = \Omega\frac{E^\varphi}{\sqrt{|E^x|}}, \tag{4.28}$$

where g_{xx} is the xx-component of the original physical metric and we have substituted its value from (4.16), i.e.

$$g_{xx} = \frac{E^{\varphi 2}}{|E^x|}. \tag{4.29}$$

But from (3.33) and (4.27) we can see that

$$\Omega = \sqrt{\Phi} = \left(E^x\right)^{1/4}, \tag{4.30}$$

and this together with (4.28) and (3.122) gives

$$\sqrt{q} = \frac{E^\varphi}{(E^x)^{\frac{1}{4}}} = \frac{|P|}{2}, \tag{4.31}$$

or

$$|P| = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}. \tag{4.32}$$

On the other hand, since n_I is unit timelike vector field we can write

$$n_1 = \cosh(\eta), \tag{4.33}$$

$$n_2 = \sinh(\eta), \tag{4.34}$$

for some gauge angle η. Using these together with (4.22) and (4.31) one can write

$$P_1 = n_1|P| = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\cosh(\eta), \tag{4.35}$$

$$P_2 = n_1|P| = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\sinh(\eta). \tag{4.36}$$

Substituting (4.27), (4.32), (4.35) and (4.36) in the Hamiltonian (4.25) yields

$$\begin{aligned}
H =& N\Bigg[2\sinh(\eta)\partial_1{}^*X^1 + 2\cosh(\eta)\partial_1{}^*X^2 - 2\cosh(\eta)\omega_1{}^*X^1 \\
&- 2\sinh(\eta)\omega_1{}^*X^2 + \frac{(E^x)^{\frac{5}{4}}f'^2}{2E^\varphi} + \frac{P_f^2}{2(E^x)^{\frac{3}{4}}E^\varphi} - \frac{2E^\varphi}{4(E^x)^{\frac{3}{4}}}\Bigg] \\
&+ N^1\Bigg[P_f f' + \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\cosh(\eta)\partial_1{}^*X^1 + \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\sinh(\eta)\partial_1{}^*X^2 \\
&- \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\cosh(\eta)\omega_1{}^*X^2 - \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\sinh(\eta)\omega_1{}^*X^1\Bigg] \\
&+ \omega_0\left[\frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\cosh(\eta)^*X^2 + \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\sinh(\eta)^*X^1 - \left(\frac{E^x}{2}\right)'\right].
\end{aligned} \tag{4.37}$$

At this stage, we see that we have the new momenta E^x and E^φ but are working with the "old" canonical variables X^1 and X^2 and the "old" connection ω_1 as a conjugate to E^x. So we can set up a generating function of a canonical transformation of the form

$$F(q,P) = {}^*X^1 \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\cosh(\eta) + {}^*X^2 \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}\sinh(\eta) + \omega_1 \frac{E^x}{2}, \tag{4.38}$$

and derive the new canonical variables as follows

$$Q_\eta = \frac{\partial F}{\partial \eta} = \frac{2E^\varphi}{(E^x)^{\frac{1}{4}}}({}^*X^1\sinh(\eta) + {}^*X^2\cosh(\eta)) = {}^*X^1P_2 + {}^*X^2P_1, \tag{4.39}$$

$$K_\varphi = \frac{\partial F}{\partial E^\varphi} = \frac{2{}^*X^1\cosh(\eta) + 2{}^*X^2\sinh(\eta)}{(E^x)^{\frac{1}{4}}} = \frac{{}^*X^1P_1 + {}^*X^2P_2}{E^\varphi}, \tag{4.40}$$

$$\begin{aligned} A_x = \frac{\partial F}{\partial E^x} &= -\frac{E^\varphi}{2(E^x)^{\frac{5}{4}}}({}^*X^1\cosh(\eta) + {}^*X^2\sinh(\eta)) + \frac{\omega_1}{2} \\ &= \frac{-{}^*X^1P_1 - {}^*X^2P_2}{4E^x} + \frac{\omega_1}{2} \qquad (4.41) \\ &= -\frac{E^\varphi K_\varphi}{4E^x} + \frac{\omega_1}{2}. \qquad (4.42) \end{aligned}$$

From the last equation above we get

$$\omega_1 = 2A_x + \frac{E^\varphi K_\varphi}{2E^x}. \tag{4.43}$$

In order to express everything in terms of the new variables, we need to find ${}^*X^1$ and ${}^*X^2$ in terms of these new variables. One can see from (4.39) and (4.40) that

$$E^\varphi K_\varphi P_1 - Q_\eta P_2 = {}^*X^1\left(P_1^2 - P_2^2\right), \tag{4.44}$$

which upon using (4.35) and (4.36) becomes

$${}^*X^1 = \frac{K_\varphi (E^x)^{\frac{1}{4}}\cosh(\eta)}{2} - \frac{Q_\eta (E^x)^{\frac{1}{4}}\sinh(\eta)}{2E^\varphi}. \tag{4.45}$$

Also it can be seen from the same two equation (4.39) and (4.40) that

$$E^\varphi K_\varphi P_2 - Q_\eta P_1 = {}^*X^2\left(P_2^2 - P_1^2\right), \tag{4.46}$$

which again by using (4.35) and (4.36) gives

$${}^*X^2 = \frac{Q_\eta (E^x)^{\frac{1}{4}}\cosh(\eta)}{2E^\varphi} - \frac{K_\varphi (E^x)^{\frac{1}{4}}\sinh(\eta)}{2}. \tag{4.47}$$

Substituting ${}^*X_1, {}^*X_2$ and ω_1 from (4.43), (4.45) and (4.47) into (4.37) leads to

$$\begin{aligned}H =&N\Bigg[\frac{Q_\eta'(E^x)^{\frac{1}{4}}}{E^\varphi}+\frac{Q_\eta(E^x)^{-\frac{3}{4}}E^{x\prime}}{4E^\varphi}-\frac{Q_\eta(E^x)^{\frac{1}{4}}E^{\varphi\prime}}{E^{\varphi 2}}-2K_\varphi(E^x)^{\frac{1}{4}}K_x\\&-\frac{E^\varphi}{2}K_\varphi^2(E^x)^{-\frac{3}{4}}+\frac{(E^x)^{\frac{5}{4}}f'^2}{2E^\varphi}+\frac{P_f^2}{2(E^x)^{-\frac{3}{4}}E^\varphi}-\frac{2E^\varphi}{4(E^x)^{\frac{3}{4}}}\Bigg]\\&+N^1\left[P_f f'-2Q_\eta K_x+E^\varphi K_\varphi'+\frac{E^\varphi K_\varphi E^{x\prime}}{4E^x}-\frac{E^\varphi K_\varphi}{2E^x}Q_\eta\right]\\&+\omega_0\left[Q_\eta-\left(\frac{E^x}{2}\right)'\right],\end{aligned}\tag{4.48}$$

where we have defined a new variable $K_x=\frac{1}{2}\eta'+A_x$. Solving the Gauss constraint for Q_η

$$Q_\eta=\frac{1}{2}E^{x\prime},\tag{4.49}$$

and substituting this back into the Hamiltonian yields

$$\begin{aligned}H =&N\Bigg[\frac{E^{x\prime\prime}(E^x)^{\frac{1}{4}}}{2E^\varphi}+\frac{\frac{1}{2}E^{x\prime}(E^x)^{-\frac{3}{4}}E^{x\prime}}{4E^\varphi}-\frac{\frac{1}{2}E^{x\prime}(E^x)^{\frac{1}{4}}E^{\varphi\prime}}{E^{\varphi 2}}-2K_\varphi(E^x)^{\frac{1}{4}}K_x\\&-\frac{E^\varphi}{2}K_\varphi^2(E^x)^{-\frac{3}{4}}+\frac{(E^x)^{\frac{5}{4}}f'^2}{2E^\varphi}+\frac{P_f^2}{2(E^x)^{\frac{3}{4}}E^\varphi}-\frac{2E^\varphi}{4(E^x)^{\frac{3}{4}}}\Bigg]\\&+N^1\left[P_f f'-E^{x\prime}K_x+E^\varphi K_\varphi'\right],\end{aligned}\tag{4.50}$$

where it is clearly seen that the Hamiltonian just consists of a Hamiltonian constraint and a diffeomorphism constraint. Rescaling the Hamiltonian constraint by multiplying it by a factor of $(E^x)^{1/4}$ and the diffeomorphism constraint by multiplying it by -1, and noting that we only work with strictly positive values of E^x, yields

$$\begin{aligned}H =&N\Bigg[\frac{E^{x\prime\prime}\sqrt{E^x}}{2E^\varphi}+\frac{E^{x\prime}E^{x\prime}}{8E^\varphi\sqrt{E^x}}-\frac{\frac{1}{2}E^{x\prime}\sqrt{E^x}E^{\varphi\prime}}{E^{\varphi 2}}-2K_\varphi\sqrt{E^x}K_x\\&-\frac{E^\varphi K_\varphi^2}{2\sqrt{E^x}}+\frac{(E^x)^{\frac{3}{2}}f'^2}{2E^\varphi}+\frac{P_f^2}{2\sqrt{E^x}E^\varphi}-\frac{E^\varphi}{2\sqrt{E^x}}\Bigg]\\&+N^1\left[E^{x\prime}K_x-E^\varphi K_\varphi'-P_f f'\right].\end{aligned}\tag{4.51}$$

This Hamiltonian which we derived by our new generic method, is precisely the Hamiltonian of the spherically symmetric case which had been introduced before by other methods.

4.3 Partially gauge fixing the theory and the boundary term

Loop quantum gravity is being explored in model situations of increasing complexity. There has been steady advance in treating homogeneous cosmologies [24], an area of activity that has come to be known as loop quantum cosmology. There has also been progress in spherical symmetry in vacuum [25]. However, in all these cases one did not have to face the "problem of dynamics", i.e. dealing with the non-Lie algebra of constraints of general relativity. In homogeneous cosmologies there is only one constraint and it therefore has a trivial algebra. In spherical symmetry, special gauges were chosen that resulted in an Abelian algebra. In this chapter we would like to study spherically symmetric gravity coupled to a spherically symmetric scalar field using loop quantum gravity techniques. It is not known in this situation how to formulate the problem in a way that one ends up with a Lie algebra of constraints. A total gauge fixing was introduced by Unruh [26], but it leads to a non-local expression for the Hamiltonian. Here we will fix partially the gauge to eliminate the diffeomorphism constraint in order to simplify things. This still leads to a Hamiltonian constraint that has a non-Lie Poisson bracket with itself, involving structure functions instead of structure constants. To treat this problem we will use the "uniform discretization" technique [14]. We will introduce a variational technique adapted to the minimization of the master constraint (in the context of uniform discretizations one should probably refer to it as "master operator" since it only vanishes in the continuum limit). In the case that zero is in the kernel of the master constraint the technique yields the correct physical state in model situations.

The inclusion of scalar fields in spherical symmetry opens a rich set of possibilities to be studied including the formation of black holes, critical collapse, the emergence of Hawking radiation, among others. Here we will have much more modest goals: to see how the complete theory approximates the vacuum state of the scalar field living on a flat space-time. An outstanding problem in a full quantum gravity treatment involving matter fields is the emergence of a vacuum state for the fields and what relation it may have to the ordinary Fock vacuum of quantum field theory in curved space-time. We will apply the variational technique in the case of spherically symmetric gravity coupled to a scalar field and show that it yields a vacuum state that is closely related to the Fock one.

As we mentioned above in 4.17 and 4.18, the diffeomorphism and Hamiltonian constraints for gravity minimally coupled to a massless scalar field are

$$\mathscr{D} = (|E^x|)' K_x - E^\varphi (K_\varphi)' - P_f f', \tag{4.52}$$

$$\begin{aligned}\mathscr{H} = &-\frac{E^\varphi}{2\sqrt{|E^x|}} - 2K_\varphi\sqrt{|E^x|}K_x - \frac{E^\varphi K_\varphi^2}{2\sqrt{|E^x|}} + \frac{\left((|E^x|)'\right)^2}{8\sqrt{|E^x|}E^\varphi} \\ &- \frac{\sqrt{|E^x|}(|E^x|)'(E^\varphi)'}{2(E^\varphi)^2} + \frac{\sqrt{|E^x|}(E^x)''\mathrm{sgn}(E^x)}{2E^\varphi} \\ &+ \frac{P_f^2}{2\sqrt{|E^x|}E^\varphi} + \frac{(|E^x|)^{3/2}(f')^2}{2E^\varphi},\end{aligned} \tag{4.53}$$

and since the variables are gauge invariant there is no Gauss law and we have been able to solve it. We have taken the Immirzi parameter equal to one. We now proceed to partially fix the gauge by choosing $E^x = x^2$ (the motivation can be seen from $\mu, \nu = 0, 1$

and 4.26). One can solve the diffeomorphism constraint for K_x,

$$K_x = \frac{E^\varphi (K_\varphi)' + P_f f'}{2x}, \tag{4.54}$$

which yields the Hamiltonian constraint for the partially gauge fixed model as,

$$\begin{aligned} H =& \frac{1}{G}\left[-\frac{E^\varphi}{2x} - \frac{E^\varphi K_\varphi^2}{2x} + \frac{3x}{2E^\varphi} - \frac{x^2 (E^\varphi)'}{(E^\varphi)^2} - E^\varphi K_\varphi (K_\varphi)' \right] \\ &+ \frac{P_f^2}{2xE^\varphi} + \frac{x^3 (f')^2}{2E^\varphi} - K_\varphi P_f f'. \end{aligned} \tag{4.55}$$

We now rescale the Lagrange multiplier $N_{\text{old}} = N_{\text{new}} G E^{x\prime}/E^\varphi$, the rescaled Hamiltonian constraint is,

$$H = H_{\text{vac}} + 2GH_{\text{matt}}, \tag{4.56}$$

where

$$H_{\text{vac}} = \left(-x - xK_\varphi^2 + \frac{x^3}{(E^\varphi)^2} \right)' = \frac{\partial H_v(x)}{\partial x}, \tag{4.57}$$

$$H_{\text{matt}} = \frac{P_f^2}{2(E^\varphi)^2} + \frac{x^4 (f')^2}{2(E^\varphi)^2} - \frac{x K_\varphi P_f f'}{E^\varphi}. \tag{4.58}$$

This form of the Hamiltonian constraint allows an easy identification of the required boundary term if one assumes asymptotically flat conditions. The total Hamiltonian is given by,

$$H_T = \int_0^{x^+} dx N(x) (H_{\text{vac}}(x) + 2GH_{\text{matt}}(x)) + H_B, \tag{4.59}$$

where $N(x)$ is the rescaled lapse N_{new} and H_B is the boundary term at the asymptotic region x^+. Integrating by parts we get

$$\begin{aligned} H_T =& -\int_0^{x^+} dx \frac{dN(x)}{dx} \left(H_v(x) + 2G \int_0^x dy H_{\text{matt}}(y) \right) \\ &+ N(x^+) \left(-2GM + 2G \int_0^{x^+} dy H_{\text{matt}}(y) \right) + H_B \\ =& -\int_0^{x^+} dx \frac{dN(x)}{dx} \left(H_v(x) - 2G \int_x^{x^+} dy H_{\text{matt}}(y) + 2GM \right) - 2GM\dot{\tau}. \end{aligned} \tag{4.60}$$

The boundary term $H_B = -2GM\dot{\tau}$ has been introduced in order to ensure that M is a constant and τ the proper time in the asymptotic region. This is the standard boundary term in the spherically symmetric case. M is the spacetime mass while the Schwarzschild radius is given by $R_S = 2G(M - \int_0^{x^+} dy H_{\text{matt}}(y)))$. In the case of a spacetime with a black hole the radial coordinate is given by $R = x + R_S$. M is a Dirac observable. In the case of weak fields therefore, so is the integral from 0 to ∞ of H_{matt} that we shall call H_M. Even in presence of black holes H_M is an observable if the black hole is isolated. We will treat H_M as an energy in order to define the vacuum and the excited states of the theory.

4.4 Quantization of the matter field on a fixed flat background

Since we wish to understand in which way loop quantum gravity recovers results from ordinary quantum field theory in curved spacetime, we would like to outline some of those results for later comparison. If the space-time is flat, it is convenient to fix the gauge $K_\varphi = 0$ to obtain explicitly the background metric in the usual spherical coordinates. In this case one solves $H_{\rm vac} = 0$ one gets that $E^\varphi = x$. Solving the evolution equation yields the Lagrange multiplier and one recovers the full flat space-time metric. With this choice of K_φ and E^φ, the matter portion of the Hamiltonian constraint 4.58 becomes,

$$H_{\rm matt} = \frac{P_f^2}{2x^2} + \frac{x^2(f')^2}{2}. \tag{4.61}$$

The evolution equation obtained from this Hamiltonian

$$\dot{f} = \{f, H_{\rm matt}\} = \frac{P_f}{x^2}, \tag{4.62}$$

$$\dot{P}_f = \{P_f, H_{\rm matt}\} = 2xf' + x^2 f'', \tag{4.63}$$

corresponds to spherical waves,

$$f'' - \ddot{f} + 2\frac{f'}{x} = 0. \tag{4.64}$$

This can be solved by Fourier decomposition,

$$f(x,t) = \int_0^\infty d\omega \frac{\left(C(\omega)\exp(-i\omega t) + \bar{C}(\omega)\exp(i\omega t)\right)\sin(\omega x)}{\sqrt{\pi\omega}x}, \tag{4.65}$$

which corresponds to spherical waves that are regular at the origin. From Hamilton's equation 4.63 we can get an expression for P_ϕ,

$$P_f(x,t) = \int_0^\infty d\omega \frac{\left(-iC(\omega)\omega\exp(-i\omega t) + i\bar{C}(\omega)\omega\exp(i\omega t)\right)x\sin(\omega x)}{\sqrt{\pi\omega}}. \tag{4.66}$$

Finding the coefficients $C(\omega)$ and $\bar{C}(\omega)$ from the two equations above and using the standard commutation relations, $[\hat{f}(x,t), \hat{P}_f(y,t)] = i\delta(x-y)$, one gets the $[\hat{C}(\omega), \hat{\bar{C}}(\omega')] = \delta(\omega-\omega')$. One can proceed to define a vacuum state $|0\rangle$ as the state that is annihilated by $\hat{C}$. If one evaluates the expectation value of $H_{\rm matt}$ on the vacuum state one finds that it has an ultraviolet divergence. The usual resolution of this problem is to introduce a cutoff. It should be noted that when one treats this problem in loop quantum gravity this type of divergence does not appear because the well defined objects are holonomies associated with finite paths. In our treatment this aspect is lost since we have gauge fixed the radial variable which therefore becomes a c-number. As we usually proceed when we use the uniform discretization technique, we regularize the expression by placing it on a lattice. We will discuss later on the issue of taking the lattice spacing to zero.

We will assume that the radial direction is bounded with a spatial extent L and consists of discrete points x_i separated by a coordinate distance ϵ, and in particular we take x_i as ϵ times an integer. We reinterpret the integrals as sums, Dirac deltas as Kronecker deltas, functional derivatives as partial derivatives, and partial derivatives in the radial directions as finite differences. Specifically [27]

$$\int dx \to \epsilon \sum_x, \tag{4.67}$$

$$\delta(x-y) \to \frac{\delta_{x,y}}{\epsilon}, \tag{4.68}$$

$$\frac{\delta}{\delta f(x)} \to \frac{1}{\epsilon}\frac{\partial}{\partial f}, \tag{4.69}$$

$$f(x)' \to \frac{f(x_{i+1}) - f(x_i)}{\epsilon}, \tag{4.70}$$

$$(\omega)^2 \to \frac{\sum_i (2 - 2\cos(\epsilon\omega_i))}{\epsilon^2}. \tag{4.71}$$

If the spatial direction is discrete, the associated momentum space is bounded with extent $2\pi/\epsilon$. To the first nontrivial order in epsilon, all formulae involving momenta ω are unchanged except that momentum integrals are now sums over a momentum space of finite extent.

The expectation value of $\hat{H}_{\rm matt}$ can be computed replacing the quantum version of the expressions given above in 4.65 and 4.66 for $f(x,t)$ and $P_f(x,t)$ in $\hat{H}_{\rm matt}$ equation 4.61. Computing the expectation value on the vacuum state, one is only left with contributions proportional to $\hat{C}\hat{\hat{C}}$. On the lattice the result may be approximated in the limit of large L by the integral,

$$\langle 0|\hat{H}_{\rm matt}(x)|0\rangle = \int_0^{2\pi/\epsilon} d\omega \frac{\omega^2 x^2 - 2x\omega\cos(\omega x)\sin(\omega x) + \sin^2(\omega x)}{2x^2\pi\omega}. \tag{4.72}$$

The integral can be computed in closed form in terms of integrals of cosine functions. It is more useful to give an approximation for its value as an expansion in ϵ,

$$\langle 0|\hat{H}_{\rm matt}(x)|0\rangle = \frac{\pi}{\epsilon^2} - \frac{\sin^2(2\pi x/\epsilon)}{\pi x^2} + \frac{\ln(x/\epsilon)}{4x^2\pi} + O(\epsilon^0). \tag{4.73}$$

The leading order in the energy density expansion is π/ϵ^2 which has the correct dimensions for an energy density in one spatial dimension, since we are only considering the radial mode of the scalar field.

As in four dimensions, the energy of the vacuum gives rise to a cosmological constant if one allows the field to back-react on gravity. The nature of this constant is different, however in two dimensions [28]. First of all, notice that if one had started from four dimensional gravity with a cosmological constant and imposed spherical symmetry, one can view the model as a $1+1$ dimensional theory with a dilaton with a mass given by the four dimensional cosmological constant (as we showed in chapter 3 but without a cosmological constant). That is, it does not produce a term that behaves like a cosmological constant in $1+1$ dimensions. The vacuum energy, by contrast produces a constant term in the Hamiltonian constraint. Second, notice that even in vacuum, $H_{\rm vac}$ already has a constant term in it. So the energy of the vacuum essentially operates as a rescaling of that constant term, which in turn can be absorbed by a rescaling of the radial coordinate. In four dimensions, if one chooses a Planck scale cutoff it implies that the radius of curvature of space-time becomes of the order of Planck length, which is clearly unphysical. In spherical symmetry the presence of the constant can be reabsorbed in a redefinition of the coordinates. This redefinition however, has consequences when one wishes to reinterpret the model as an approximation to a four dimensional space-time. The redefinition of the radial coordinate implies that the spheres do not have $4\pi r^2$ area anymore. The four dimensional universe modeled contains a topological defect, a "global texture" [29]. Notice that this immediately precludes taking the lattice spacing to zero, since already when the lattice spacing is of the order of $\ell_{\rm Planck}$ one will have a solid deficit angle that exceeds 4π and does not allow to interpret the model as a four dimensional space-time.

There are two avenues to handle the situation: either one rescales the radial variable and accepts that the model approximates four dimensional space-times with (large) topological defects, or one can modify the two dimensional model by adding a constant to the Hamiltonian constraint (a cosmological constant in $1+1$ dimensional gravity). Such a model will not stem from a dimensional reduction of four dimensional gravity, but upon quantization will turn out to approximate four dimensional spherical gravity around a flat background without a topological defect.

We will take the first point of view and write the Hamiltonian constraint, equations 4.57 and 4.58 as, $H = H_{\rm vac} + GH_{\rm matt}$, with

$$H_{\rm vac} = \left(-x(1-2\Lambda) - xK_\varphi^2 + \frac{x^3}{(E^\varphi)^2}\right)', \tag{4.74}$$

$$H_{\rm matt} = \frac{P_f^2}{(E^\varphi)^2} + \frac{x^4(f')^2}{(E^\varphi)^2} - 2\frac{xK_\varphi P_f f'}{E^\varphi} - \rho_{\rm vac}, \tag{4.75}$$

where $\Lambda = \frac{G}{2}\rho_{\rm vac}$ and $\rho_{\rm vac}$ is the vacuum energy density. We choose $\hbar = c = 1$ units. This rewriting of the constraint has the property that the expectation value of $H_{\rm matt}$ will be zero in the vacuum.

4.5 Full quantization of the model

We would like to write the master constraint based on the Hamiltonian constraint of the model we introduced in the last section. Although the discrete Hamiltonian constraint fails to close a first class algebra, it has been shown in [30] that with the uniform discretization technique, one can consistently treat the problem by minimizing the resulting master constraint. To write the master constraint at a quantum level we will polymerize [31] the expression of the gravitational part of the constraint. We will not use a polymer representation in the scalar sector in this chapter for simplicity and because we want to make contact with the usual treatments based on a Fock quantization. It is known that the Fock quantization for fields can be recovered from the polymer quantization [31, 32]. In the next chapter we will polymerize the scalar matter field as well as the gravitational degrees of freedom.

4.5.1 Variational technique to study the expectation value of the master constraint

Here we will introduce a variational technique to minimize the master constraint. The technique is general and is not restricted to the model we study in this paper. We start by considering a fiducial Hilbert space $\mathscr{H}_{\text{aux}}$ in which the master constraint is a well defined self-adjoint operator. We will then use a variational technique to find approximations to the minimum value of the expectation value of the master constraint within this space. In many cases of interest, the minimum expectation value will not be zero, but will be small (the master constraint has units of action squared, so normally one would require it to be much smaller than $\hbar^2$, in order to have a good approximation of the physical space, and in our units that translates into much smaller than one). As we will see in the examples, the resulting quantum theory will therefore not reproduce exactly the symmetries of the continuum theory but it will approximate them, even at the quantum level. We will see that if zero is in the spectrum of the master constraint operator, the corresponding eigenstates in many cases will be distributional with respect to the fiducial space we are considering.

To implement the variational method, we consider trial states in $\mathscr{H}_{\text{aux}}$ that are Gaussians centered around the classical solution of the model of interest in phase space. That means that as functions of $\mathscr{H}_{\text{aux}}$, these will generically be Gaussians times phase factors such that the resulting state is centered around the classical solution in both configuration variables and momenta. The states are parametrized by the values of the standard deviations of the Gaussians in either configuration or momentum space. A caveat is that in gauge theories one may choose to work with a classical solution that is not in a completely determined gauge. Such a solution will be a trajectory in phase space. Such a trajectory will determine some of the canonical variables as functions of others, which will remain free. In that case one has to allow such variables to be free in the trial solution by considering Gaussians centered around a value that is a free parameter. If one chooses to work with a classical solution in a completely specified gauge one just considers Gaussians around the point in phase space represented by the classical solution of interest and extremizes the expectation value of the master constraint with respect to the standard deviations of the Gaussians. It can happen that the extremum occurs as a limit in the parameter space, in which case the resulting state does not belong in $\mathscr{H}_{\text{aux}}$ but in its dual (after a suitable rescaling, it becomes a distribution).

Before attacking the problem of interest, it is useful to see the technique we just described in action in a couple of simple examples. The first example we choose is a system with two degrees of freedom q_1, p_1 and q_2, p_2, and two constraints $p_1 = 0$ and $p_2 = 0$. The total Hamiltonian for the system is $H_T = N_1 p_1 + N_2 p_2$ with $N_{1,2}$ Lagrange multipliers. The states annihilated by the constraints are trivial and given by the distribution $\delta(p_1)\delta(p_2)$. We fix a gauge $q_1 - q_2 = 0$. Fixing the gauge is not needed in a simple model like this, but may be a necessity to simplify things in more complicated models. So we will choose a gauge fixing here to show that in the end, the process loses all information about the gauge fixing and recovers the correct physical state. This requires fixing the Lagrange multipliers so there is only one (N) is left at the end and the total Hamiltonian becomes $H_T = N(p_1 + p_2)$. The conjugate variable to the gauge fixing, $p_1 - p_2$ is strongly zero. We start with a two parameter family of states in $\mathscr{H}_{\text{aux}}$ choosing as configuration variables $q_1 - q_2$ and $p_1 + p_2$,

$$\psi_{\sigma_\pm,\beta} = \frac{1}{\sqrt{\pi\sqrt{\sigma_+\sigma_-}}}\exp\left(-\frac{(q_1-q_2)^2}{2\sigma_-}\right)\exp\left(-\frac{(p_1+p_2)^2}{2\sigma_+}\right)\exp\left(i\beta\left(p_1+p_2\right)\right), \tag{4.76}$$

with β an arbitrary parameter associated with the fact that the variable $q_1 + q_2$ is a pure gauge. One could choose to work in a completely gauge fixed solution in which $q_1 + q_2$ is zero, in that case there is no need to introduce the parameter β. The choice of this family of states is based on the fact that they describe wave-packets centered around the classical solutions of the constraints, $q_1 - q_2 = 0$, $p_1 - p_2 = 0$ and $p_1 + p_2 = 0$. We now define the master constraint

$$\mathbb{H} = p_1^2 + p_2^2, \tag{4.77}$$

and act on this space of states. The expectation value is,

$$\langle\psi_{\sigma_\pm,\beta}|\mathbb{H}|\psi_{\sigma_\pm,\beta}\rangle = \frac{1}{4\sigma_-} + \frac{1}{4}\sigma_+, \tag{4.78}$$

where $\sqrt{\sigma_\pm}$ are the standard deviations of the Gaussians, $\sigma_\pm$ is taken to be positive. One therefore sees that the expectation value cannot be zero for any finite value of the σ's. However, if one takes $\sigma_- = \frac{1}{2\epsilon^2}$ and $\sigma_+ = 2\epsilon^2$, then in the limit $\epsilon \to 0$ we get $\langle\mathbb{H}\rangle = O(\epsilon^2)$. The states $|\psi_\epsilon\rangle$ become

$$\langle q_1 - q_2, p_1 + p_2|\psi_\epsilon\rangle = \frac{1}{\sqrt{\pi}}\exp\left(-\left(q_1-q_2\right)^2\epsilon^2\right)\exp\left(-\frac{(p_1+p_2)^2}{4\epsilon^2}\right)\exp\left(i\beta\left(p_1+p_2\right)\right), \tag{4.79}$$

And their Fourier transform becomes

$$\langle p_1 - p_2, p_1 + p_2|\psi_\epsilon\rangle = \frac{1}{\epsilon\sqrt{2\pi}}\exp\left(-\frac{(p_1-p_2)^2}{4\epsilon^2}\right)\exp\left(-\frac{(p_1+p_2)^2}{4\epsilon^2}\right)\exp\left(i\beta\left(p_1+p_2\right)\right). \tag{4.80}$$

These states are normalized in $\mathscr{H}_{\text{aux}}$ but they vanish (in the sense of distributions) in the limit $\epsilon \to 0$. They need to be rescaled in order to end up with well defined distribution on some suitable subspace of $\mathscr{H}_{\text{aux}}$.

So the physical states would be

$$\begin{aligned}\langle p_1 - p_2, p_1 + p_2|\psi\rangle_{\text{ph}} &\equiv \lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi\epsilon}}\langle p_1 - p_2, p_1 + p_2|\psi_\epsilon\rangle\\ &= 2\delta(p_1+p_2)\delta(p_1-p_2)\\ &= \delta(p_1)\delta(p_2).\end{aligned} \tag{4.81}$$

Notice that the parameter β is free at the end of the process since it corresponds to the value of a variable that is pure gauge in this model.

There is an additional element that the above example does not capture and we would like to discuss. When we apply this technique in situations of interest, we will be discretizing the theories we analyze. Usually, discretization turns first class constraints into second class ones. The uniform discretization procedure tells us that we do not need to concern ourselves with the second class nature of the constraints (for a discussion see [30]). We can still consider the master constraint and seek the minimization of its eigenvalues, but the presence of second class constraints in the discrete theory usually implies that the minimum eigenvalue of the master constraint will not be zero. The best one can hope for, is that it will be small and the resulting quantum theory will approximate the symmetries of the theory one started with. This is a point of view that has been held as natural for some time in the context of quantum gravity, where one expects that some level of fundamental discreteness will emerge. We would like to illustrate this with a modification of the previous example. Instead of taking $p_1 = 0$ and $p_2 = 0$ as the constraints we will take $p_1 + \alpha q_2 = 0$ and $p_2 = 0$ with α a small parameter (in realistic theories the small parameter is related to the lattice spacing in the discretization). We will still take the same set of $\psi_{\sigma_\pm,\beta}$ as before, that is, for the trial solution we have chosen, Gaussians centered around classical solutions of the gauge theory where the anomalous term vanishes. We do this because one usually knows solutions to the continuum theory one wishes to approximate (e.g. flat space or the Schwarzschild solution in the case of gravity) whereas the discrete theories have complicated solutions that usually cannot be treated in analytic form. The master constraint now becomes,

$$\mathbb{H} = p_1^2 + p_2^2 + 2\alpha p_1 q_2 + \alpha^2 q_2^2, \tag{4.82}$$

and using the same ansatz (4.76) for the states, one finds that

$$\langle\psi_{\sigma_\pm,\beta}|\mathbb{H}|\psi_{\sigma_\pm,\beta}\rangle = \alpha^2\beta^2 + \frac{1}{4\sigma_-} + \frac{1}{4}\sigma_+ + \frac{\alpha^2}{2\sigma_+} + \frac{\alpha^2\sigma_-}{2}. \tag{4.83}$$

We would like to identify a limit in the variables $\sigma_\pm$ such that this quantity vanishes. As was to be expected, this is not possible. We can attempt to find values of the parameters $\sigma_\pm$ and β that minimize this expression. The result is $\beta = 0$, $\sigma_+ = \sqrt{2}\alpha$ and $\sigma_- = \frac{1}{\sqrt{2}\alpha}$. which yields $\langle\psi_{\min}|\mathbb{H}|\psi_{\min}\rangle = \sqrt{2}\alpha$. The state is,

$$\langle p_1, p_2|\psi_{\min}\rangle = \exp\left(-\frac{\left(p_1^2 + p_2^2\right)\sqrt{2}}{\alpha}\right)\sqrt{\frac{\sqrt{2}}{\alpha\pi}}. \tag{4.84}$$

It is interesting to compare this state and the corresponding expectation value of $\mathbb{H}$ obtained from our variational technique with the exact minimum of this model. A naive analysis would tell us that the minimum corresponds to an exact eigenstate with zero eigenvalue for $\mathbb{H}$. However, that solution is not well behaved. It is known that one can find solutions of the master constraint that do not solve the constraints if one does not impose regularity in the solutions found [33]. The master constraint is an operator in the Hilbert space and one can analyze its spectral resolution. The spurious solutions do not belong in the spectral resolution of the master constraint. In this case one can solve exactly the eigenvalue problem $\mathbb{H}|\psi\rangle = E|\psi\rangle$. The solutions with minimum eigenvalue are of the form $\delta(p_1)\psi_0(p_2)$ where $\psi_0(p_2)$ is the fundamental state of the Hamiltonian of a harmonic oscillator in the momentum representation. The minimum eigenvalue for such exact solution is α (compare with the variational one, in which the eigenvalue was slightly higher $\sqrt{2}\alpha$). It is also interesting to note that if instead of choosing the gauge $q_1 - q_2 = 0$ we had chosen $q_1 = 0$ and proceeded with the variational technique, we would have obtained the exact state directly. This illustrates that the method approximates well the state of interest in situations where zero is not in the kernel of the master constraint. The solution that minimizes the master constraint admits a very simple interpretation that shows the uniform discretization of the theory with the anomalous term α, small but non-vanishing, approximately reproduces the invariances of the theory with first class constraints $p_1 = p_2 = 0$. In fact q_1 and q_2 are gauge variables and the physical space is independent of these variables. The physical state is constant in q_1 and q_2. For a small but non vanishing α, the physical states are independent of q_1 and weakly dependent on q_2. A final comment is that in this case the parameter β, which was not determined in the case with first class constraints, gets determined here. That is, in the case where β was associated with an exact gauge symmetry, the minimization of the master constraint was insensitive to the value of β. In the case where the constraints are second class and we do not get zero as minimum of the master constraint, there is some dependence on β, but it is weak, since the term in the master constraint is $\beta^2\alpha^2$ and α is small (in the quantum state one has approximately $\delta(p)\exp(ip\beta)$). The theory where one does not exactly annihilate the master constraint only has approximate gauge symmetries and therefore has slightly "preferred" gauges from the point of view of minimizing the master constraint.

4.5.2 The discrete master constraint

Let us now consider the complete Hamiltonian constraint. We wish to discretize it and to polymerize the gravitational variables. The Hamiltonian gets rescaled in the discretization $H(x_i) \to H(i)/\epsilon$. We also rescale the expression multiplying the continuum Hamiltonian constraint times G. The resulting discrete expression is,

$$\begin{aligned}
H(i) = &-(1-2\Lambda)\epsilon - x(i+1)\frac{\sin^2\left(\rho K_\varphi(i+1)\right)}{\rho^2} \\
&+ x(i)\frac{\sin^2\left(\rho K_\varphi(i)\right)}{\rho^2} + \frac{x(i+1)^3\epsilon^2}{(E^\varphi(i+1))^2} - \frac{x(i)^3\epsilon^2}{(E^\varphi(i))^2} \\
&+ G\Bigg(\epsilon\frac{(P_f(i))^2}{(E^\varphi(i))^2} + \epsilon\frac{x(i)^4\left[f(i+1)-f(i)\right]^2}{(E^\varphi(i))^2} \\
&- 2x(i)\frac{\sin\left(\rho K_\varphi(i)\right)}{E^\varphi(i)\rho}\left[f(i+1)-f(i)\right]P_f(i) - \rho_{\text{vac}}\epsilon\Bigg).
\end{aligned} \tag{4.85}$$

We need to construct the master constraint. Since the Hamiltonian is a density of weight one, we define the master constraint associated with the Hamiltonian constraint in the full theory as,

$$\mathbb{H} = \frac{1}{2}\int dx \frac{H(x)^2}{\sqrt{g}}\ell_{\rm P}, \tag{4.86}$$

or, in terms of the variables of the model, (4.16), up to a constant factor,

$$\mathbb{H} = \frac{1}{2}\int dx \frac{H(x)^2}{(E^\varphi)\sqrt{E^x}}\ell_{\rm P}, \tag{4.87}$$

and in the discretized theory

$$\mathbb{H}^\epsilon = \sum_i \mathbb{H}(i), \tag{4.88}$$

with

$$\mathbb{H}(i) = \frac{1}{2}\frac{H(i)^2\ell_{\rm P}}{\sqrt{E^x(i)}E^\varphi(i)}. \tag{4.89}$$

The constant $\ell_{\rm P}$ must be introduced so that $\mathbb{H}$ is dimensionless with $\hbar = c = 1$, one could use $\sqrt{G}$ instead of it. It is convenient to rescale the Hamiltonian constraint by $\sqrt{E^\varphi/(E^x)'}$. This does not change the density weight. If one does not rescale things it turns out $\mathbb{H}$ is proportional to $1/E^\varphi$. In the polymer representation this implies that the vacuum is the “zero loop” state, which is degenerate (it corresponds to zero volume space-times). To eliminate this unphysical possibility one exploits the fact that the Hamiltonian constraint is defined up to a factor given by a scalar function of the canonical variables without changing the first class nature of the classical constraint algebra. The rescaling factor in the discrete theory after the gauge fixing is $\sqrt{E^\varphi(i)/(2x(i)\epsilon)}$. So (4.85) has to be multiplied times that factor when constructing the master constraint (4.89).

Let us focus on the matter portion of the Hamiltonian, we will write it as,

$$H_{\rm matt}(i) = \frac{H^{(1)}_{\rm matt}(i)}{(E^\varphi)^2(i)} + \frac{H^{(2)}_{\rm matt}(i)\sin\left(\rho K_\varphi(i)\right)}{\rho E^\varphi(i)} - H^{(3)}_{\rm matt}(i). \tag{4.90}$$

The master constraint can be written as,

$$\begin{aligned}\mathbb{H}(i) = \ell_{\rm P}\Big[& c_{11}(i)\left(H^{(1)}_{\rm matt}(i)\right)^2 + c_{22}(i)\left(H^{(2)}_{\rm matt}(i)\right)^2 + c_{33}(i)\left(H^{(3)}_{\rm matt}(i)\right)^2 \\ & + c_1(i)H^{(1)}_{\rm matt}(i) + c_2(i)H^{(2)}_{\rm matt}(i) + c_3(i)H^{(1)}_{\rm matt}(i) + c_{00}(i) \\ & + c_{12}(i)H^{(1)}_{\rm matt}(i)H^{(2)}_{\rm matt}(i) + c_{13}(i)H^{(1)}_{\rm matt}(i)H^{(3)}_{\rm matt}(i) + c_{23}(i)H^{(2)}_{\rm matt}(i)H^{(3)}_{\rm matt}(i)\Big],\end{aligned} \tag{4.91}$$

where

$$H^{(1)}_{\rm matt}(i) = \left(\epsilon\left(P_f(i)\right)^2 + \epsilon x(i)^4\left(f(i+1)-f(i)\right)^2\right)\ell_P^2, \tag{4.92}$$

$$H^{(2)}_{\rm matt}(i) = \left(-2x(i)\left(f(i+1)-f(i)\right)P_f(i)\right)\ell_P^2, \tag{4.93}$$

$$H^{(3)}_{\rm matt}(i) = 2\rho_{\rm vac}\epsilon\ell_P^2. \tag{4.94}$$

We will not give the classical expressions for the coefficients, since they can be readily obtained from the quantum expressions below.

In order to quantize the master constraint we need to choose a factor ordering. The expression of the master constraint is a sum of symmetric operators consisting of polynomials in $\hat{E}^\varphi$ and $\sin(\rho\hat{K}_\varphi)$, $\hat{P}_f$ and $\hat{f}$. We choose a factor ordering with the factors of $\hat{E}^\varphi$ are distributed symmetrically to the right and the left of the factors of $\sin(\rho\hat{K}_\varphi)$. For the factors $\hat{P}_f$ and $\hat{f}$ we follow a similar strategy, putting the $\hat{P}_f$ symmetrically to the left and to the right of $\hat{f}$'s. The coefficients in the above expression of the master constraint with this factor ordering are,

$$\hat{c}_{11}(i) = \frac{1}{4x(i)^2\epsilon\hat{E}^\varphi(i)^4}, \tag{4.95}$$

$$\hat{c}_{12}(i) = \frac{1}{2x(i)^2\rho\epsilon}\frac{1}{\hat{E}^\varphi(i)^{3/2}}\sin(\rho K_\varphi(i))\frac{1}{\hat{E}^\varphi(i)^{3/2}}, \tag{4.96}$$

$$\hat{c}_{13}(i) = -\frac{1}{2x(i)^2\epsilon}\frac{1}{\hat{E}^\varphi(i)^2}, \tag{4.97}$$

$$\hat{c}_{22}(i) = \frac{1}{8x(i)^2\rho^2\epsilon}\left(\frac{1}{\hat{E}^\varphi(i)^2} - \frac{1}{\hat{E}^\varphi(i)}\cos(2\rho K_\varphi(i))\frac{1}{\hat{E}^\varphi(i)}\right), \tag{4.98}$$

$$\hat{c}_{23}(i) = -\frac{1}{2x(i)^2\rho\epsilon}\frac{1}{\sqrt{\hat{E}^\varphi(i)}}\sin(\rho K_\varphi(i))\frac{1}{\sqrt{\hat{E}^\varphi(i)}}, \tag{4.99}$$

$$\hat{c}_{33}(i) = \frac{1}{4x(i)^2\epsilon}, \tag{4.100}$$

$$\begin{aligned}\hat{c}_1(i) = &-\frac{x(i)\epsilon}{2\hat{E}^\varphi(i)^4} + \frac{1}{4\epsilon x(i)^2\hat{E}^\varphi(i)}\left(-2\epsilon(1-2\Lambda) + \frac{x(i+1)\cos(2\rho K_\varphi(i))}{\rho^2} - \frac{x(i+1)}{\rho^2}\right)\frac{1}{\hat{E}^\varphi(i)}\\ &-\frac{1}{\hat{E}^\varphi(i)}\frac{\cos(2\rho K_\varphi(i))}{4x(i)\rho^2\epsilon}\frac{1}{\hat{E}^\varphi(i)} + \frac{1}{4x(i)\rho^2\epsilon\hat{E}^\varphi(i)^2} + \frac{\epsilon x(i+1)^3}{2x(i)^2}\frac{1}{\left(\hat{E}^\varphi(i)\hat{E}^\varphi(i+1)\right)^2},\end{aligned} \tag{4.101}$$

$$\begin{aligned}\hat{c}_2(i) = &\Bigg[-\frac{1}{2\rho x(i)^2}(1-2\Lambda) + \frac{x(i+1)}{4\rho^3x(i)^2\epsilon}\left(\cos(2\rho K_\varphi(i+1)) - 1\right) + \frac{3}{8\rho^3x(i)\epsilon}\\ &+\frac{\epsilon x(i+1)^3}{2\rho x(i)^2\hat{E}^\varphi(i+1)^2}\Bigg] \times \frac{1}{\sqrt{\hat{E}^\varphi(i)}}\sin(\rho K_\varphi(i))\frac{1}{\sqrt{\hat{E}^\varphi(i)}}\\ &-\frac{1}{8\rho^3x(i)\epsilon}\frac{1}{\sqrt{\hat{E}^\varphi(i)}}\sin(3\rho K_\varphi(i))\frac{1}{\sqrt{\hat{E}^\varphi(i)}}\\ &-\frac{x(i)\epsilon}{2\rho}\frac{1}{\hat{E}^\varphi(i)^{3/2}}\sin(\rho K_\varphi(i))\frac{1}{\hat{E}^\varphi(i)^{3/2}},\end{aligned} \tag{4.102}$$

$$\begin{aligned}\hat{c}_3(i) = &\frac{1}{2x(i)^2}(1-2\Lambda) + \frac{x(i+1)}{4x(i)^2\epsilon\rho^2}\left[1-\cos(2\rho K_\varphi(i+1))\right]\\ &-\frac{1}{4x(i)\epsilon\rho^2}\left[1-\cos(2\rho K_\varphi(i))\right] + \frac{x(i)\epsilon}{2\hat{E}^\varphi(i)^2} - \frac{\epsilon x(i+1)^3}{2x(i)^2\hat{E}^\varphi(i+1)^2},\end{aligned} \tag{4.103}$$

$$\begin{aligned}\hat{c}_{00}(i) = &\frac{1}{32\epsilon\rho^4}\left(3 - 4\cos(2\rho K_\varphi(i)) + \cos(4\rho K_\varphi(i))\right)\\ &+\frac{\epsilon}{4x(i)^2} + \frac{x(i+1)}{4x(i)^2\rho^2}\left[1-\cos(2\rho K_\varphi(i+1))\right]\\ &-\frac{x(i+1)}{8x(i)\epsilon\rho^4}\left(1-\cos(2\rho K_\varphi(i)) - \cos(2\rho K_\varphi(i+1)) + \cos(2\rho K_\varphi(i))\cos(2\rho K_\varphi(i+1))\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{x(i+1)^2}{32\epsilon\rho^4 x(i)^2}\left(3+\cos(4\rho K_\varphi(i+1))-4\cos(2\rho K_\varphi(i+1))\right)\\
&-\frac{\Lambda x(i+1)}{2x(i)^2\rho^2}\left(1-\cos(2\rho K_\varphi(i+1))\right)\\
&-\frac{1}{4x(i)\rho^2}(1-2\Lambda)\left[1-\cos(2\rho K_\varphi(i))\right]-\frac{\epsilon\Lambda}{x(i)^2}(1-\Lambda)\\
&+\frac{\epsilon x(i+1)^3}{4x(i)\rho^2}\left(\frac{1}{\hat{E}^\varphi(i+1)^2}-\frac{1}{\hat{E}^\varphi(i+1)}\cos(2\rho K_\varphi(i))\frac{1}{\hat{E}^\varphi(i+1)}\right)\\
&-\frac{x(i)\epsilon}{4\rho^2}\left(x(i)\left(\frac{1}{\hat{E}^\varphi(i)^2}-\frac{1}{\hat{E}^\varphi(i)}\cos(2\rho K_\varphi(i))\frac{1}{\hat{E}^\varphi(i)}\right)\right.\\
&\left.-x(i+1)\left(\frac{1}{\hat{E}^\varphi(i)^2}-\frac{1}{\hat{E}^\varphi(i)}\cos(2\rho K_\varphi(i+1))\frac{1}{\hat{E}^\varphi(i)}\right)\right)\\
&-\frac{\epsilon x(i+1)^4}{4x(i)^2\rho^2}\left(\frac{1}{\hat{E}^\varphi(i+1)^2}-\frac{1}{\hat{E}^\varphi(i+1)}\cos(2\rho K_\varphi(i+1))\frac{1}{\hat{E}^\varphi(i+1)}\right)\\
&+\frac{x(i)\epsilon^2}{2\hat{E}^\varphi(i)^2}(1-2\Lambda)-\frac{\epsilon^2 x(i+1)^3}{2x(i)^2\hat{E}^\varphi(i+1)^2}(1-2\Lambda)+\frac{\epsilon^3 x(i+1)^6}{4x(i)^2\hat{E}^\varphi(i+1)^4}\\
&-\frac{x(i)\epsilon^3 x(i+1)^3}{2\left(\hat{E}^\varphi(i+1)\hat{E}^\varphi(i)\right)^2}+\frac{x(i)^4\epsilon^3}{4\hat{E}^\varphi(i)^4},
\end{aligned}
\tag{4.104}
$$

and it should be noted that the coefficients commute with $H^{(1)}_{\text{matt}}, H^{(2)}_{\text{matt}}$ and $H^{(3)}_{\text{matt}}$ so there are no ordering issues with them.

4.5.3 Construction of the trial states

Since we are interested in the vacuum solution, that classically corresponds to vanishing scalar fields, $f = P_f = 0$, we will therefore ignore H_{matt} (4.75) and only consider the gravitational part (4.74) in order to construct the classical solution used to build the ansatz states for the variational technique,

$$H_{\text{vac}} = \left(-x(1-2\Lambda) - xK_\varphi^2 + \frac{x^3}{(E^\varphi)^2}\right)'. \tag{4.105}$$

As we discussed in (4.5.1), we will choose a definite gauge to work in. Our choice is $K_\varphi = 0$, and this implies

$$E^\varphi = \frac{x}{\sqrt{1-2\Lambda}} \tag{4.106}$$

from above (4.105). As we claimed before, the presence of the cosmological constant rescales the radial variable (recall that without the constant the solution was $E^\varphi = x$). The resulting four dimensional space-time will be locally flat with a solid deficit angle and described in spherical coordinates.

We construct a polymer representation. As it has been described in [23], one sets up a lattice of points $j = 0 \ldots N$ in the radial direction and writes a "point holonomy" for the K_φ variable at each lattice site,

$$T_{\vec{\mu}} = \exp\left(i\sum_j \mu_j K_\varphi(j)\right) = \langle K_\varphi|\vec{\mu}\rangle. \tag{4.107}$$

In this expression, the quantities μ_i play the role of the "loop" in this one dimensional context. They are also proportional to the eigenvalues of the triad operator $\hat{E}^\varphi(i)$. The quantum state we will choose for the variational method will be centered around the classical solution and therefore we will choose to have the variable μ_i centered at the classical value of $E^\varphi(i) = \epsilon x_1(i) \equiv \epsilon x(i)/\sqrt{1-2\Lambda}$,

$$\langle\vec{\mu}|\psi_{\vec{\sigma}}\rangle = \prod_i \sqrt[4]{\frac{2}{\pi\sigma(i)}} \exp\left(-\frac{1}{\sigma(i)}\left(\mu_i - \frac{x_1(i)\epsilon}{\ell_\mathrm{P}^2}\right)^2\right). \tag{4.108}$$

On this state $\langle E^\varphi(i)\rangle = \epsilon x_1(i)$ and $\langle K_\varphi(i)\rangle = 0$. Notice that this type of ansatz in general will be too restrictive: we have ignored possible correlations among neighboring points by assuming a Gaussian at each point. This could potentially be problematic when studying excited states and computing propagators. We will study those problems in the following chapters, so we will continue with the restrictive ansatz for the moment being.

We will now compute the expectation value of the matter portion of the Hamiltonian constraint in (4.90) on the above state. The result will be an operator acting on the matter fields. We will then construct the vacuum for the resulting operator. What we are doing is to construct a quantum field theory living on the geometry given by the expectation values of the triad and extrinsic curvature on the above state. We proceed in this way for expediency since this is our first approach to the problem. In the next chapters, we revisit the problem treating all the variables in a polymerized representation, both gravitational and material ones, with the variational technique.

For the matter field one would start by considering a coherent state centered at zero values for the field and then will obtain the vacuum as a limit. This would yield valuable insights into the relation of the usual Fock quantization with the loop quantum gravity techniques, especially when one gets to discuss physical elements like the propagators of fields.

In order to take the expectation value of the matter portion of the Hamiltonian constraint, (4.90) on the state (4.108), we need to realize two quantum operators. The first one is,

$$\frac{1}{\left(\hat{E}^\varphi(i)\right)^2}\langle\mu(i)|\psi_{\sigma(i)}\rangle = \left(\frac{2}{3}\right)^{12} |\mu(i)| \left(\left(|\mu(i)+\rho|\right)^{3/4} - \left(|\mu(i)-\rho|\right)^{3/4}\right)^{12}$$
$$\times \sqrt[4]{\frac{2}{\pi\sigma(i)}} \exp\left(-\frac{\left(\mu(i) - \frac{\epsilon x_1(i)}{\ell_\mathrm{P}^2}\right)^2}{\sigma(i)}\right), \tag{4.109}$$

where we have considered the action on one of the factors of (4.108). To derive this expression we consider $\left(\hat{E}^\varphi\right)^{-3/2}\hat{E}^\varphi\left(\hat{E}^\varphi\right)^{-3/2}$ and use the realization of $\left(\hat{E}^\varphi\right)^{-3/2}$ that was discussed in the context of loop quantum cosmology in [34]. The reason we can use the loop quantum cosmology results is that our Hilbert space is a direct product of loop quantum cosmology Hilbert spaces each at one of the lattice sites in the radial direction. With the above result one can compute the expectation value,

$$\langle\psi_{\vec{\sigma}}|\frac{1}{\left(\hat{E}^{\varphi}(i)\right)^2}|\psi_{\vec{\sigma}}\rangle = \frac{1-2\Lambda}{\epsilon^2 x(i)^2} + \frac{5}{8}\frac{\ell_{\rm P}^4(1-2\Lambda)^2\rho^2}{\epsilon^4 x(i)^4} + \frac{3}{4}\frac{\sigma\ell_{\rm P}^4(1-2\Lambda)^2}{\epsilon^4 x(i)^4}. \tag{4.110}$$

The calculation is done by integrating in $\vec{\mu}$ and the result is lengthy, here we just show it in the approximation $\epsilon > \ell_{\rm P}$. The first term is the classical value. The rest are quantum corrections in which the first one comes from the polymerization, the second from fluctuations in $\vec{\mu}$. The second operator we need is the one arising in the second term of the Hamiltonian (4.90), i.e.

$$\langle\psi_{\vec{\sigma}}|\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}\frac{\sin(\rho\hat{K}_{\varphi}(i))}{\rho}\frac{1}{\sqrt{\hat{E}^{\varphi}(i)}}|\psi_{\vec{\sigma}}\rangle = 0. \tag{4.111}$$

To quickly see why this is zero keep in mind that the state is a Gaussian centered at $K_{\varphi}=0$ and the sine is an odd function. With these results the expectation value of the Hamiltonian (4.90) (the "effective Hamiltonian") is,

$$\hat{H}^{\rm eff}_{\rm matt} = \langle\psi_{\vec{\sigma}}|\hat{H}_{\rm matt}(x,t)|\psi_{\vec{\sigma}}\rangle = \frac{(1-2\Lambda)\left(\hat{P}_f(x,t)\right)^2}{x^2 g(x)^2} + \frac{x^2(1-2\Lambda)\left(\hat{f}'(x,t)\right)^2}{g(x)^2} - \rho_{\rm vac}. \tag{4.112}$$

In this equation we have pursued the unusual approach of taking the continuum limit in the terms that involve derivatives and the terms that involve the momenta of the scalar field. This simplifies calculations since we will be dealing with differential equations rather than difference equations. The idea is that the solutions to the differential equations, suitably discretized, will be a good approximation (at least to $\mathcal{O}(\epsilon)$ corrections) to the solutions of the difference equations. In the above expression the quantity $g(x)$ is given by,

$$g(x) = 1 - \frac{5}{16}\frac{\ell_{\rm P}^4\rho^2(1-2\Lambda)}{x^2\epsilon^2} - \frac{3}{8}\frac{\sigma(x)\ell_{\rm P}^4(1-2\Lambda)}{x^2\epsilon^2}. \tag{4.113}$$

By finding $\dot{f}$ and $\dot{P}_f$ from the effective Hamiltonian, we get the "wave equation" for the fields living on the curved semiclassical background,

$$\frac{2}{x}\frac{\partial f(x,t)}{\partial x} - \frac{2}{g(x)}\frac{\partial f(x,t)}{\partial x}\frac{\partial g(x)}{\partial x} + \frac{\partial^2 f(x,t)}{\partial x^2} - \frac{1}{4}\frac{g(x)^4}{(1-2\Lambda)^2}\frac{\partial^2 f(x,t)}{\partial t^2} = 0. \tag{4.114}$$

Since the background is time-independent, positive and negative frequency modes can be introduced by going to Fourier space in t. The resulting equation can be cast in Sturm–Liouville form as,

$$\left(2B(x)f'(x,\omega)\right)' + \frac{\omega^2}{2}f(x,\omega)A(x) = 0, \tag{4.115}$$

where

$$A(x) = \frac{x^2}{1-2\Lambda} - \frac{5}{8}\frac{\ell_{\rm P}^4\rho^2}{\epsilon^2} - \frac{3}{4}\frac{\sigma\ell_{\rm P}}{\epsilon^2}, \tag{4.116}$$

$$B(x) = x^2(1-2\Lambda) + \frac{5}{8}\frac{\ell_{\rm P}^4\rho^2}{\epsilon^2} + \frac{3}{4}\frac{\sigma\ell_{\rm P}^4}{\epsilon^2} \tag{4.117}$$

The solution to this Sturm–Liouville problem is

$$
\begin{aligned}
f(x,\omega) =& \frac{1}{x}\sin\left(\frac{\omega x}{2(1-2\Lambda)}\right)\\
&-\frac{1}{3x^3}\left[x^2\omega^2\cos\left(\frac{\omega x}{2}\right)\mathrm{Si}(\omega x)-\frac{x}{2}\omega\cos\left(\frac{\omega x}{2}\right)-x^2\omega^2\sin\left(\frac{\omega x}{2}\right)\mathrm{Ci}(\omega x)+\sin\left(\frac{\omega x}{2}\right)\right]\\
&\times\frac{\ell_{\mathrm{P}}^4}{4\epsilon^2}\left[\frac{5\rho^2}{2}+3\sigma\right],
\end{aligned}
\tag{4.118}
$$

where $\mathrm{Si}(x) \equiv \int_0^x dt \sin(t)/t$, and $\mathrm{Ci}(x) \equiv \gamma + \ln(x) + \int_0^x dt(\cos(t) - 1)/t$ are the sine integral and cosine integral functions respectively and Euler's Gamma is given by $\gamma = 0.5772156649$. This solution neglects terms with higher powers than $\ell_{\mathrm{P}}^4/(\epsilon x)^2$. The first term in the bracket in (4.118) corresponds to the standard spherical mode decomposition in (locally) flat space-time. The next parenthesis includes two terms that are corrections, the first, involving ρ, due to polymerization and the next, involving σ, is a quantum correction. These terms would not be present in a treatment of quantum field theory on a classical space-time. Using the Hamilton equations from (4.112) we can compute P_f,

$$
P_f(x,t) = \frac{x^2 g(x)^2}{2\sqrt{\omega}(1-2\Lambda)}\frac{\partial f(x,t)}{\partial t},
\tag{4.119}
$$

and use it back to compute the effective Hamiltonian (4.112),

$$
\hat{H}^{\mathrm{eff}}_{\mathrm{matt}} = (1-2\Lambda)\int_0^{2\pi/\epsilon} d\omega\omega\hat{\bar{C}}(\omega)\hat{C}(\omega).
\tag{4.120}
$$

To obtain this expression we note that the solution (4.118) can be written as $f(x,t) = \int_0^\infty d\omega u(x,\omega)h(\omega,t)$ where $h(\omega,t)$ is the last parenthesis in (4.118). Notice that we have introduced a lattice cutoff for the frequency $2\pi/\epsilon$. Then one uses the lattice version of the closure relation $\int_0^\infty d\omega u(x,\omega)u(x',\omega) = 2\delta(x-x')/A(x)$ and the orthogonality relation $\int_0^\infty dx A(x)u(x,\omega)u(x,\omega')/2 = \delta(\omega-\omega')$.

We have therefore concluded the computation of the state that we will use as a trial in the variational method. It will be given by a direct product of the vacuum of the matter part of the Hamiltonian (4.120) and the Gaussian (4.108) on the gravitational variables.

$$
|\psi^{\mathrm{trial}}_{\vec{\sigma}}\rangle = |\psi_{\vec{\sigma}}\rangle \otimes |0\rangle.
\tag{4.121}
$$

The parameters $\vec{\sigma}$ will be varied to minimize the master constraint. Notice that the state is a direct product because we are considering the vacuum. If we were to consider excitations then there might be entanglement between the matter and gravitational variables [35].

4.5.4 Minimizing the master constraint

The realization of the master constraint (4.91) as a quantum operator depends on the realization of six key operators as can be seen from the coefficients (4.95) to (4.104). We proceed to present their expectation values here. We start by the operators involving the cosine of $\hat{K}_\varphi$,

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\cos\left(2\rho\hat{K}_\varphi(i)\right)|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = \exp\left(-\frac{2\rho^2}{\sigma(i)}\right), \tag{4.122}$$

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\cos\left(4\rho\hat{K}_\varphi(i)\right)|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = \exp\left(-\frac{8\rho^2}{\sigma(i)}\right). \tag{4.123}$$

We then consider the powers of the inverse of $\hat{E}^\varphi$. We already computed the expectation value of the square in (4.110). Here we list the other needed powers,

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\frac{1}{\left(\hat{E}^\varphi(i)\right)^4}|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = \frac{(1-2\Lambda)^2}{\epsilon^4 x(i)^4} + \frac{5}{4}\frac{\ell_{\text{P}}^4(1-2\Lambda)^3\rho^2}{\epsilon^6 x(i)^6} + \frac{5}{2}\frac{\sigma\ell_{\text{P}}^4(1-2\Lambda)^3}{\epsilon^6 x(i)^6}, \tag{4.124}$$

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\frac{1}{\hat{E}^\varphi(i)}\cos\left(2\rho\hat{K}_\varphi(i)\right)\frac{1}{\hat{E}^\varphi(i)}|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = \frac{1-2\Lambda}{\epsilon^2 x(i)^2\exp(\frac{2\rho^2}{\sigma})}\left(1+\frac{5}{2}\frac{\rho^2 l_p^4}{\epsilon^2 x(i)^2}+\frac{3}{4}\frac{\sigma l_p^4}{\epsilon^2 x(i)^2}\right), \tag{4.125}$$

and

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\frac{1}{\left(\hat{E}^\varphi(i)\right)^{3/2}}\sin\left(\rho\hat{K}_\varphi(i)\right)\frac{1}{\left(\hat{E}^\varphi(i)\right)^{3/2}}|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = 0, \tag{4.126}$$

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\frac{1}{\sqrt{\hat{E}^\varphi(i)}}\sin\left(\rho\hat{K}_\varphi(i)\right)\frac{1}{\sqrt{\hat{E}^\varphi(i)}}|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = 0, \tag{4.127}$$

$$\langle\psi^{\text{trial}}_{\vec{\sigma}}|\frac{1}{\sqrt{\hat{E}^\varphi(i)}}\sin\left(3\rho\hat{K}_\varphi(i)\right)\frac{1}{\sqrt{\hat{E}^\varphi(i)}}|\psi^{\text{trial}}_{\vec{\sigma}}\rangle = 0. \tag{4.128}$$

With these results we can proceed to compute the expectation value of the master constraint on the gravitational state. The result will be an operator acting on the matter part. The calculation of the expectation values of the coefficients $\hat{c}_i$ and $\hat{c}_{ij}$ (4.95)-(4.104) is straightforward, but lengthy. We will not list the results here. What is more challenging is the computation of the expectation value of the matter part of the expansion of (4.91). It helps that some of the coefficients vanish. The non-vanishing contributions are,

$$\begin{aligned}\langle\psi_{\vec{\sigma}}|\hat{\mathbb{H}}(i)|\psi_{\vec{\sigma}}\rangle = \ell_{\text{P}}\Big[&\langle\hat{c}_{11}(i)\rangle\left(\widehat{H^{(1)}_{\text{matt}}(i)}\right)^2+\langle\hat{c}_{22}(i)\rangle\left(\widehat{H^{(2)}_{\text{matt}}(i)}\right)^2\\&+\langle\hat{c}_{33}(i)\rangle\left(\widehat{H^{(3)}_{\text{matt}}(i)}\right)^2+\langle\hat{c}_{13}(i)\rangle\widehat{H^{(1)}_{\text{matt}}(i)}\,\widehat{H^{(3)}_{\text{matt}}(i)}\\&+\langle\hat{c}_1(i)\rangle\widehat{H^{(1)}_{\text{matt}}(i)}+\langle\hat{c}_3(i)\rangle\widehat{H^{(1)}_{\text{matt}}(i)}+\langle\hat{c}_{00}(i)\rangle\Big].\end{aligned} \tag{4.129}$$

We now need to compute the expectation value of this operator on the matter vacuum. To do this we again use the procedure of going to the continuum limit in the matter terms involving derivatives and momenta and integrating in the frequencies with an ultraviolet cutoff. Let us start with $H^{(1)}_{\text{matt}}(i)$. The continuum limit expression is

$$H^{(1)}_{\text{matt}}(x,t) = \ell_{\text{P}}^2\left(\left(P_f(x,t)\right)^2 + x^4\left(f'(x,t)\right)^2\right). \tag{4.130}$$

We now substitute P_f and f by their mode decomposition and get a quadratic expression in the $\hat{C}$'s and $u's$. The expectation value only gets contributions from the $\hat{C}\hat{\hat{C}}$ terms. The result is,

$$\langle 0|\hat{H}^{(1)}_{\text{matt}}|0\rangle = l_p^2\int_0^{\frac{2\pi}{\epsilon}} d\omega \frac{1}{8\omega(1-2\Lambda)}[A(x)^2u^2(x,\omega)\omega^2(1-2\Lambda)^2+4x^4(\partial_x u(x,\omega))^2], \tag{4.131}$$

and substituting $u(\omega,x)$ and $A(x)$ we obtain,

$$\begin{aligned}\langle 0|\hat{H}^{(1)}_{\text{matt}}(x)|0\rangle = {}& l_p^2(1-2\Lambda)A(x)^2\left(\frac{\pi^2}{8x^2\epsilon^2}+\frac{1}{8x^4}-\frac{\cos^2\left(\frac{\pi x}{\epsilon}\right)}{8x^4}-\frac{\pi\sin\left(\frac{\pi x}{\epsilon}\right)\cos\left(\frac{\pi x}{\epsilon}\right)}{4x^3\epsilon}\right)\\ &+\frac{l_p^2}{(1-2\Lambda)}\left(\frac{\pi^2x^2}{8\epsilon^2}+\frac{\ln(2)}{4}+\frac{x\pi\cos\left(\frac{\pi x}{\epsilon}\right)\sin\left(\frac{\pi x}{\epsilon}\right)}{4\epsilon}-\frac{5}{8}\sin^2\left(\frac{\pi x}{\epsilon}\right)+\frac{1}{4}\text{Cin}\left(\frac{\pi x}{\epsilon}\right)\right),\end{aligned} \tag{4.132}$$

where $\text{Cin}(x)=\gamma+\ln x-\text{Ci}(x)$. One can get a more manageable expression, which we will use in the rest of the paper by ignoring corrections of ℓ_{P}^4 and neglecting the highly oscillating terms that involve $\sin(\pi x/\epsilon)$ or cosines and the integral cosines. The result is,

$$\langle 0|\hat{H}^{(1)}_{\text{matt}}(x)|0\rangle = \frac{l_p^2}{4(1-2\Lambda)}\left(-2+\frac{\pi^2x^2}{\epsilon^2}+\ln(2)+\gamma+\ln\left(\frac{\pi x}{\epsilon}\right)\right), \tag{4.133}$$

and the dominant term is π^2x^2/ϵ^2. Reverting to the discrete theory, it reads,

$$\langle 0|\hat{H}^{(1)}_{\text{matt}}(i)|0\rangle = \frac{l_p^2\epsilon^3}{4(1-2\Lambda)}\left(-2+\frac{\pi^2x(i)^2}{\epsilon^2}+\ln(2)+\gamma+\ln\left(\frac{\pi x}{\epsilon}\right)\right). \tag{4.134}$$

The procedure to compute the expectation value of the other terms in (4.129) is exactly the same, but the size of the expressions involved is quite larger so we will not write them down here.

The result for the expectation value of the integrand of the master constraint is,

$$\begin{aligned}\langle\hat{\mathbb{H}}(x)\rangle ={}&\frac{\sigma_0\ell_{\text{P}}^3}{\epsilon x^2}+\left(8\frac{\pi^2}{\epsilon^3x^2}+\frac{32}{\epsilon x^4}\ln\left(\frac{L}{\epsilon}\right)-\frac{\left(\gamma-2+\ln\left(\frac{2\pi x}{\epsilon}\right)\right)\pi}{\epsilon x^4(1-2\Lambda)}\right.\\ &+\frac{1}{96}\frac{\pi^3}{\epsilon^5x^2\sigma_0(1-2\Lambda)^2}-\frac{1}{48}\frac{\Lambda\pi^3}{\epsilon^5x^2\sigma_0(1-2\Lambda)^2}-\frac{43}{128}\frac{\Lambda\pi}{\epsilon^3x^4\sigma_0(1-2\Lambda)^2}\\ &+\frac{\epsilon(\gamma-2+\ln\left(\frac{2\pi x}{\epsilon}\right))\pi}{x^4(1-2\Lambda)L^2}+8\frac{\epsilon\pi^2}{x^2L^4}-\frac{2\pi}{\epsilon x^4(1-2\Lambda)}\ln\left(\frac{L}{\epsilon}\right)-16\frac{\pi^2}{\epsilon x^2L^2}\\ &+\frac{1}{32}\frac{\pi}{\epsilon x^4(1-2\Lambda)^2}+\frac{1}{48}\frac{\pi^3}{\epsilon^3x^2(1-2\Lambda)^2}+\frac{32\epsilon}{x^6\pi^2}\left(\ln\left(\frac{L}{\epsilon}\right)\right)^2-\frac{\pi^3}{\epsilon^3x^2(1-2\Lambda)}\\ &+4\frac{\pi}{\sigma_0\epsilon^3x^2}-\frac{32}{x^4L^2}\epsilon\ln\left(\frac{L}{\epsilon}\right)-3\frac{\sigma_0\pi}{x^3L^2}+\frac{43}{256}\frac{\pi}{\epsilon^3x^4\sigma_0(1-2\Lambda)^2}\\ &+\frac{8}{\sigma_0\epsilon x^4\pi}\ln\left(\frac{L}{\epsilon}\right)-\frac{1}{4(1-2\Lambda)}\frac{\left(4x^2\epsilon\sigma_0+4x\sigma_0^3\epsilon^2+4\sigma_0\epsilon^2x+7\sigma_0^3\epsilon^3\right)}{(x+\epsilon)\sigma_0^2x^5\epsilon^4}\\ &\times\left(\frac{1}{4}\pi^2x^2+\frac{1}{4}\epsilon^2\left(\gamma-2+\ln\left(\frac{2\pi x}{\epsilon}\right)\right)\right)-3\frac{\sigma_0\pi}{(x+\epsilon)\,x^2\epsilon^2}\end{aligned}$$

$$+3\frac{\sigma_0\pi}{(x+\epsilon)\,x^2L^2}-\frac{6\sigma_0}{(x+\epsilon)x^4\pi}\ln\left(\frac{L}{\epsilon}\right)-4\frac{\pi}{x^2\epsilon\sigma_0L^2}+\frac{\pi^3}{\epsilon x^2(1-2\Lambda)L^2}$$
$$-2\frac{\epsilon}{x^6(1-2\Lambda)\pi}\left(\gamma-2+\ln\left(\frac{2\pi x}{\epsilon}\right)\right)\ln\left(\frac{L}{\epsilon}\right)+3\frac{\sigma_0\pi}{x^3\epsilon^2}$$
$$+6\frac{\sigma_0}{x^5\pi}\ln\left(\frac{L}{\epsilon}\right)-\frac{1}{16}\frac{\epsilon}{x^6(1-2\Lambda)^2\pi}\Bigg)\ell_{\rm P}^5. \qquad (4.135)$$

We have assumed

$$\sigma=\sigma_0\epsilon^2/\ell_{\rm P}^2, \qquad (4.136)$$

with σ_0 of order unity and we have neglected the terms of order $\mathcal{O}(\ell_{\rm P}^7)$ and higher. Also we have assumed σ to be independent of x in order to simplify the above expression, which otherwise becomes too large. Experiments we have carried out suggest that allowing variations in x leads to the same minimum value of σ approximately independent of x.

We would like to study the minimum of the master constraint as a function of σ_0 for different choices of $\epsilon/\ell_{\rm P}$. Notice that we have assumed σ_0 to be of order one. One can change that by varying the ansatz for σ above, including other powers $\epsilon/\ell_{\rm P}$ different than 2. We have carried out such experiments. The results can be summarized as follows.

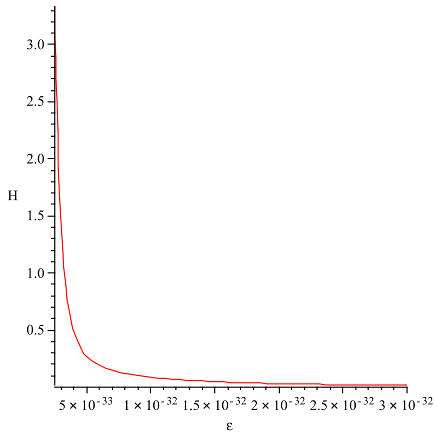


Figure 4.1: The expectation value of the master constraint as a function of the lattice spacing. We see that the value of the master constraint is small unless one chooses lattice separations of the order of the Planck length, 10^{-33} cm. The figure does not show it, but for separations of the order of 10^{-23}cm the master constraint is very small, of the order of 10^{-20} (we are using units in which $\hbar$ is one and therefore the master constraint is dimensionless).

In figure (4.1) we show the value of the master constraint as a function of ϵ (in centimeters) and for $\sigma_0=10$ and $\sigma=\sigma_0\epsilon^3/\ell_{\rm P}^3$. Varying σ_0 while keeping it of order one changes little the shape of the curve. We see that in the approximation studied the theory does not appear to have a continuum limit, but we see that the master constraint quickly drops to zero for lattice spacings larger than the Planck scale. Although the figure suggests that the master constraint drops even further for larger lattice spacings, the approximation in which we have handled expressions (in which we have neglected higher powers of $\epsilon/\ell_{\rm P}$) is inadequate for large values of ϵ and the master constraint very likely will increase its value for large values of ϵ. So there exists a genuine preferred value of ϵ that minimizes the master constraint. Even so, the approximation should be reliable up to values of $\epsilon \sim 10^{-23}$cm and for such values the master constraint is of the order of 10^{-20}, so one sees that this is a regime where one approximates the continuum theory very well.

We have explored other ranges of σ's (with different powers of $\epsilon/\ell_{\rm P}$). The observation is the following. For lower powers than three we get a curve that looks similar to the one shown in the figure, but that grows faster as one approaches smaller lattice spacings and therefore the minimum occurs farther away from the Planck scale. For powers higher than 10/3 one violates the approximation that $\ell_{\rm P}/\epsilon$ is small and the expressions we derived are not valid. From these considerations and an analysis of the powers involved, we conclude that the minimum for the master constraint is achieved for a power of $\epsilon/\ell_{\rm P}$ in σ close to two and $\epsilon \sim 10^{13}\ell_{\rm P}$.

An interesting speculation is that if the minimum of the master constraint happens in the range mentioned, the cosmological constant, which goes as $\Lambda \sim \ell_{\rm P}^2/\epsilon^2$ would not be of Planck scale but several orders of magnitude smaller.

Another observation of interest is to note what would have happened if instead of choosing the state peaked around the flat metric (with a topological defect) one would have chosen the "loop quantum gravity vacuum", i.e. a state with zero loops which corresponds to a degenerate metric $|\mu(i)=0\rangle$. Such a state annihilates the matter Hamiltonian in the loop representation and has zero volume. It would be disturbing if this state yielded a lower value for the master constraint than the state we constructed, since it would imply that degenerate geometries dominate. This is not the case, as can be easily seen. For such a state all expectation values (4.122)-(4.128) vanish. One can check that the expectation value of the master constraint is,

$$\langle\hat{\mathbb{H}}\rangle = \frac{1}{8}\frac{L\ell_{\rm P}}{\epsilon^2\rho}. \tag{4.137}$$

That is, the result is very large. For $\epsilon \sim \ell_{\rm P}$ it goes as $L/\ell_{\rm P}$, the size of the universe in Planck lengths. Therefore these degenerate states are heavily suppressed.

4.6 Summary and discussion

We have studied spherically symmetric gravity coupled to a spherically symmetric scalar field using loop quantum gravity techniques. The problem has a non-Lie algebra of constraints and we used the "uniform discretization" technique to treat the dynamics. We used a variational technique to minimize the discrete master constraint. With the trial states proposed, we were not able to reach a zero eigenvalue for the master constraint, that is, the theory does not seem to have a quantum continuum limit. The lowest eigenstate of the master constraint has the form of a direct product of a Fock vacuum for the scalar field and Gaussian states centered around flat space-time for the gravitational variables. Although the theory does not have a continuum limit, it approximates general relativity well for small values of the lattice separation, which in turn regularizes the cosmological constant. The lattice treatment we have performed diverges when one takes the continuum limit. The reader may wonder why loop quantum gravity has failed to act as the "natural regulator of matter quantum field theories" as claimed, for instance in [13]. The problem arises with the gauge fixing of the diffeomorphism constraint that we performed at the classical level. This leads us to variables that have the structure of a Bohr compactification in the "transverse" φ direction, but the variable in the radial direction is a c-number and therefore is not dynamical and has continuous character. There is no chance therefore that loop quantum gravity based on this gauge fixing could regulate the short distance behavior, which is responsible for the emergence of the cosmological constant. To tackle this issue one would have to allow both the diffeomorphism and Hamiltonian constraint to remain in the theory. The computational complexity would increase importantly, since one will have to regulate the master constraint in such a way that the resulting states have remnants of diffeomorphism invariance in the discrete theory. This has been successfully accomplished with uniform discretizations in the Husain–Kuchař model [30], but the complexity there was considerably reduced by the lack of a Hamiltonian constraint. It is worthwhile noticing that even if one allowed loop quantum gravity to regulate matter in the proposed way, the resulting cosmological constant is likely to be finite but still very large with respect to the current observed value.

In this chapter we made a first exploration of a difficult problem, carried out with several assumptions and limitations that we have outlined in the text. In the next chapter, we will relax the assumption that one has a Fock vacuum for the scalar field and will treat both the gravitational and scalar variables on the same footing with the variational technique for the master constraint. We will study the excited states of matter and study the modifications in dispersion relations for the matter fields due to the quantum geometry. One should also relax the gauge fixing of diffeomorphisms to see if the cosmological constant problem becomes better under control. Other future directions would be to consider solutions centered around non-flat geometries, for instance, including a black hole with the aim of studying if the scalar field states involve Hawking radiation.

5 The propagator in the 3+1 model

5.1 Introduction

In previous chapter, in order to study the spherically symmetric case in loop quantum gravity, we used a "polymer" representation for the gravitational variables but followed a regular Fock quantization for the scalar field, both for simplicity and to make connections with the ordinary field theory. We found that the state has the form of a direct product of Gaussians for the gravitational variables at each lattice site times a modified Fock vacuum for the scalar field variables (the modification is due to the fact that the background is not globally flat, in 1+1 dimensions the zero point energy of the vacuum generates a deficit angle, and also that we incorporate quantum corrections to the background geometry).

In this chapter we will study the following: First we will verify that the vacuum state we derived in the last chapter discussed is a good vacuum for the polymerized theory, at least in the case in which the polymerization parameter is small. We will compute the expectation value of the master constraint for the fully polymerized theory (both gravitational and matter degrees of freedom) in the vacuum state to leading order in the polymerization parameter, and show that the resulting terms are very small. Second, we will study the low energy propagator for the scalar field on the above discussed quantum state. We will see that one has different options for polymerizing the scalar field and this will lead to different types of propagators. Generically they fall within the class of propagators considered by Hořava [36]. We will again work in the limit in which the polymerization parameter is small. The resulting propagators are not Lorentz invariant.

It is worthwhile mentioning related recent work. Husain and Kreienbuehl [37] consider the polymerization of a scalar field without assuming spherical symmetry and proceed to define creation and annihilation operators for the polymerized theory. More recently, Hossain, Husein and Seahra [38] have analyzed the propagator in that context and have found Lorentz violations. Their work cannot be directly compared to ours for reasons we will discuss in section 5.3.2. Laddha and Varadarajan [39] consider a scalar field in $1+1$ dimensions but parametrized, including the embedding variables in their treatment. They are apparently able to recover Lorentz invariance exactly so the connection to our work is at the moment unclear.

5.2 Appropriateness of using the Fock vacuum for the scalar field

To get the state that we derived in the last chapter, we used a Fock vacuum for the scalar field. The use of the Fock vacuum appeared compelling in part due to the fact that we were not polymerizing the scalar field in our treatment. Since in this chapter we will be polymerizing the scalar field, it begs the question of the appropriateness of continuing to use the Fock vacuum. In this section we would like to show that the Fock vacuum still yields a very small value for the master constraint even if one polymerizes the scalar field variables. In other words, the corrections to the Fock representation due to polymerization are small enough for our purpose.

We start by considering the Hamiltonian of gravity coupled to a scalar field in spherical symmetry we considered in (4.56), (4.57) and (4.58). We will now rescale the variables,

$$P_f^{\text{orig}} = xP_f^{\text{new}}, \tag{5.1}$$

$$f^{\text{orig}} = f^{\text{new}}/x, \tag{5.2}$$

and will drop the "new" superscript from now on to economize in the notation. The matter Hamiltonian then becomes,

$$H_{\text{matt}} = \frac{H^{(1)}}{(E^\varphi)^2} + \frac{H^{(2)}K_\varphi}{E^\varphi}, \tag{5.3}$$

where

$$H^{(1)} = \frac{1}{2}P_f^2x^2 + \frac{1}{2}f^2 - xf'f + \frac{1}{2}x^2(f')^2, \tag{5.4}$$

$$H^{(2)} = fP_f - xf'P_f. \tag{5.5}$$

We now proceed to discretize and polymerize the matter Hamiltonian,

$$H_{\text{matt}}(i) = \frac{H^{(1)}(i)}{(E^\varphi(i))^2} + \frac{H^{(2)}(i)\sin\left(\rho K_\varphi(i)\right)}{\rho E^\varphi(i)}, \tag{5.6}$$

where,

$$\begin{aligned} H^{(1)}(i) = {} & \frac{\epsilon}{2}P_f^2(i)x(i)^2 + \frac{\epsilon^3\sin^2\left(\beta f(i)\right)}{2\beta^2} - \frac{\epsilon^2 x(i)}{\beta^2}\sin\left(\beta f(i)\right)\sin\left(\beta\left(f(i+1)-f(i)\right)\right) \\ & + \frac{\epsilon x(i)^2}{2\beta^2}\sin^2\left(\beta\left(f(i+1)-f(i)\right)\right), \end{aligned} \tag{5.7}$$

$$H^{(2)}(i) = \frac{P_f(i)}{\beta}\left(\epsilon\sin\left(\beta f(i)\right) - x(i)\sin\left(\beta\left(f(i+1)-f(i)\right)\right)\right). \tag{5.8}$$

We now write the complete Hamiltonian but expand the trigonometric functions in β and keep the two lowest orders, e.g. $\sin(\beta f)/\beta \sim f - \beta^2 f^3/6$, we do this so it is clear that to leading order one will have the same results as in last chapter, and the next order will be the corrections introduced by the polymerization and we can analyze their influence. We get for (5.7) and (5.8),

$$H^{(1)}(i) = H^{(1)}_{\text{lead}}(i) + H^{(1)}_{\text{corr}}(i), \tag{5.9}$$
$$H^{(2)}(i) = H^{(2)}_{\text{lead}}(i) + H^{(2)}_{\text{corr}}(i), \tag{5.10}$$

for which "lead" refers to leading order and "corr" refers to the correction terms based on the above expansion of matter Hamiltonian in β and

$$\begin{aligned} H^{(1)}_{\text{lead}}(i) =& \frac{\epsilon}{2} P_f(i)^2 x(i)^2 + \frac{1}{2}\epsilon^3 f(i)^2 \\ &+ \frac{1}{2}\epsilon x(i)^2 \left(f(i+1) - f(i)\right)^2 - \epsilon^2 x(i) f(i) \left(f(i+1) - f(i)\right), \end{aligned} \tag{5.11}$$

$$\begin{aligned} H^{(1)}_{\text{corr}}(i) =& \frac{\epsilon^3 \beta^2}{6}\Bigg(- x(i)^2 \frac{\left(f(i+1) - f(i)\right)^4}{\epsilon^2} + \frac{x(i)\left(f(i+1) - f(i)\right) f(i)^3}{\epsilon} \\ &+ \frac{x(i)\left(f(i+1) - f(i)\right)^3 f(i)}{\epsilon} - f(i)^4\Bigg), \end{aligned} \tag{5.12}$$

$$H^{(2)}_{\text{lead}}(i) = \epsilon\left(- \frac{x(i) P_f(i)\left(f(i+1) - f(i)\right)}{\epsilon} + P_f(i) f(i)\right), \tag{5.13}$$

$$H^{(2)}_{\text{corr}}(i) = \frac{\epsilon\beta^2}{6}\left(\frac{x(i) P_f(i)\left(f(i+1) - f(i)\right)^3}{\epsilon} - P_f(i) f(i)^3\right). \tag{5.14}$$

These should be substituted into (5.6) to give the matter Hamiltonian expanded in β. We are now going to focus on the master constraint. It can be written as,

$$\begin{aligned} \mathbb{H}(i) =& c_{11}(i)\left(H^{(1)}(i)\right)^2 + c_1(i) H^{(1)}(i) + c_{12}(i) H^{(1)}(i) H^{(2)}(i) \\ &+ c_{22}(i)\left(H^{(2)}(i)\right)^2 + c_2(i) H^{(2)}(i), \end{aligned} \tag{5.15}$$

where the c coefficients depend only on the gravitational variables. Substituting the leading order terms of (5.11) and (5.13) into (5.15), yields the results of last chapter (taking into account the re-scalings (5.1) and (5.2)). What we want to show now is that substituting the correction terms into the master constraint (5.15) and taking its expectation value with respect to the trial vacuum state of the previous paper, yields corrective terms which are very small.

For this, we observe that the contribution of the correction terms to the master constraint can be written as,

$$\begin{aligned} \mathbb{H}_{\text{corr}}(i) =& c_{11}(i)\left(H^{(1)}_{\text{lead}}(i) H^{(1)}_{\text{corr}}(i)\right) + c_1(i) H^{(1)}_{\text{corr}}(i) \\ &+ c_{12}(i)\left(H^{(1)}_{\text{lead}}(i) H^{(2)}_{\text{corr}}(i) + H^{(2)}_{\text{lead}}(i) H^{(1)}_{\text{corr}}(i)\right) \\ &+ c_{22}(i)\left(H^{(2)}_{\text{lead}}(i) H^{(2)}_{\text{corr}}(i)\right) + c_2(i) H^{(2)}_{\text{corr}}(i) + c_{00}(i), \end{aligned} \tag{5.16}$$

and we want to show that $\langle\psi^{\text{trial}}_{\vec{\sigma}}|\mathbb{H}_{\text{corr}}(i)|\psi^{\text{trial}}_{\vec{\sigma}}\rangle$ is very small.

Our strategy is the following: We compute the dominant terms by first going to the continuum limit and writing (5.11)-(5.14) in their continuum limit form by using,

$$P_f(i) = \epsilon P_f(x), \tag{5.17}$$
$$E^{\varphi}(i) = \epsilon E^{\varphi}(x), \tag{5.18}$$
$$f(i) = f(x), \tag{5.19}$$
$$\frac{f(i+1)-f(i)}{\epsilon} = \frac{\partial f(x)}{\partial x}. \tag{5.20}$$

We then substitute in the result the continuum form of (5.11)-(5.14) that we just calculated and also the Fourier expansions of the $f(x)$ and its conjugate momentum $P_f(x)$ fields, which are,

$$f(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega \frac{\left(C(\omega)e^{-i\omega t} + \bar{C}(\omega)e^{i\omega t}\right)\sin(\omega x)}{\sqrt{\pi\omega}}, \tag{5.21}$$

and

$$P_f(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} d\omega \frac{-i\omega\left(C(\omega)e^{-i\omega t} - \bar{C}(\omega)e^{i\omega t}\right)\sin(\omega x)}{\sqrt{\pi\omega}}. \tag{5.22}$$

Next, using the expanded version of the terms (5.11)-(5.14) resulting from the substitution of the Fourier expansion of the fields, we find the individual terms that constitute (5.16), meaning the terms that appear multiplied by c_m and c_{kl}'s in (5.16) (i.e. $H^{(1)}_{\text{lead}}(x)H^{(1)}_{\text{corr}}(x)$ etc).

From here on, let us focus on one of the individual terms that build up (5.16) (an arbitrary one). We can then repeat the process for all the other terms. We proceed to find the portions of the individual terms that do not have vanishing expectation values, taking into account that $C(\omega)$ and $\bar{C}(\omega)$ are annihilation and creation operators. It turns out that we encounter terms with four $C(\omega)$ and/or $\bar{C}(\omega)$ operators for non-cross terms like for example $H^{(1)}_{\text{corr}}(x)$ and six of them in cross terms like $H^{(1)}_{\text{lead}}(x)H^{(1)}_{\text{corr}}(x)$. Then the parts of the non-cross terms with four operators with non-vanishing expectation values just include the terms with

$$C_4\bar{C}_3C_2\bar{C}_1, \tag{5.23}$$
$$C_4C_3\bar{C}_2\bar{C}_1, \tag{5.24}$$

where we wrote $C(\omega_1)$ as C_1, etc. for the sake of brevity. Also the parts of the cross terms with six operators with non-vanishing expectation values turn out to include only the terms with

$$C_6C_5C_4\bar{C}_3\bar{C}_2\bar{C}_1, \tag{5.25}$$
$$C_6C_5\bar{C}_4C_3\bar{C}_2\bar{C}_1, \tag{5.26}$$
$$C_6\bar{C}_5C_4C_3\bar{C}_2\bar{C}_1, \tag{5.27}$$
$$C_6C_5\bar{C}_4\bar{C}_3C_2\bar{C}_1, \tag{5.28}$$
$$C_6\bar{C}_5C_4\bar{C}_3C_2\bar{C}_1. \tag{5.29}$$

Using the commutation relation

$$\left[\hat{C}(\omega_1), \hat{\bar{C}}(\omega_2)\right] = \delta(\omega_1 - \omega_2), \tag{5.30}$$

we evaluate the expectation values of the relevant parts (of the individual term) we are working with. We will have

$$
\begin{aligned}
\langle C_4 \bar{C}_3 C_2 \bar{C}_1 \rangle &= \delta(\omega_4 - \omega_3)\delta(\omega_2 - \omega_1), && (5.31)\\
\langle C_4 C_3 \bar{C}_2 \bar{C}_1 \rangle &= \delta(\omega_4 - \omega_2)\delta(\omega_3 - \omega_1) + \delta(\omega_4 - \omega_1)\delta(\omega_3 - \omega_2), && (5.32)\\
\langle C_6 C_5 C_4 \bar{C}_3 \bar{C}_2 \bar{C}_1 \rangle &= \delta(\omega_6 - \omega_3)[\delta(\omega_5 - \omega_2)\delta(\omega_4 - \omega_1) + \delta(\omega_5 - \omega_1)\delta(\omega_4 - \omega_2)] \\
&\quad + \delta(\omega_6 - \omega_2)[\delta(\omega_5 - \omega_3)\delta(\omega_4 - \omega_1) + \delta(\omega_5 - \omega_1)\delta(\omega_4 - \omega_3)] \\
&\quad + \delta(\omega_6 - \omega_1)[\delta(\omega_5 - \omega_3)\delta(\omega_4 - \omega_2) + \delta(\omega_5 - \omega_2)\delta(\omega_4 - \omega_3)], && (5.33)\\
\langle C_6 C_5 \bar{C}_4 C_3 \bar{C}_2 \bar{C}_1 \rangle &= \delta(\omega_6 - \omega_4)[\delta(\omega_5 - \omega_2)\delta(\omega_3 - \omega_1) + \delta(\omega_5 - \omega_1)\delta(\omega_3 - \omega_2)] \\
&\quad + \delta(\omega_5 - \omega_4)[\delta(\omega_6 - \omega_2)\delta(\omega_3 - \omega_1) + \delta(\omega_6 - \omega_1)\delta(\omega_3 - \omega_2)], && (5.34)\\
\langle C_6 \bar{C}_5 C_4 C_3 \bar{C}_2 \bar{C}_1 \rangle &= \delta(\omega_6 - \omega_5)[\delta(\omega_4 - \omega_2)\delta(\omega_3 - \omega_1) + \delta(\omega_4 - \omega_1)\delta(\omega_3 - \omega_2)], && (5.35)\\
\langle C_6 C_5 \bar{C}_4 \bar{C}_3 C_2 \bar{C}_1 \rangle &= \delta(\omega_2 - \omega_1)[\delta(\omega_6 - \omega_4)\delta(\omega_5 - \omega_3) + \delta(\omega_6 - \omega_3)\delta(\omega_5 - \omega_4)], && (5.36)\\
\langle C_6 \bar{C}_5 C_4 \bar{C}_3 C_2 \bar{C}_1 \rangle &= \delta(\omega_6 - \omega_5)\delta(\omega_4 - \omega_3)\delta(\omega_2 - \omega_1). && (5.37)
\end{aligned}
$$

Finally, we add up the expectation values of the relevant parts resulting from the previous step (these results are the non-vanishing expectation-values parts of the individual term), to get the complete expectation value of the individual term we chose. We now repeat the procedure to get the complete expectation value of all the other individual terms that build up (5.16) and after that, add up all the results to get the expectation value of $\mathbb{H}_{\text{corr}}(x)$.

Next we convert the resulting expectation value $\langle \mathbb{H}_{\text{corr}}(x) \rangle$ back to its discrete form, $\langle \mathbb{H}_{\text{corr}}(i) \rangle$, by reversing the continuum limit and neglecting highly oscillating terms like $\sin(\frac{n\pi x}{\epsilon})$ and the similar cosine and Ci terms. Expanding the result in ℓ_p, collecting the terms of the order of β^2, and expanding it in ϵ, yields the leading term of corrections that are of order

$$\langle \mathbb{H}_{\text{corr}}(x) \rangle \sim \frac{\ell_p^5 \ln\left(\frac{\pi x}{\epsilon}\right)^2}{\epsilon \pi x^4} \beta^2. \tag{5.38}$$

This leading term is actually the expectation value of a master constraint density. In order to get the expectation value of the master constraint itself, we need to integrate the above term with respect to x which will yield relevant terms of order

$$\int_\epsilon^L \langle \mathbb{H}_{\text{corr}}(x) \rangle dx \sim \frac{\ell_p^5}{\epsilon^4} \beta^2. \tag{5.39}$$

But in the previous chapter, in equation (4.135), for the master constraint density we had the leading order result of the form

$$\langle \mathbb{H}_{\text{lead}}(x) \rangle \sim \frac{\ell_p^3}{\epsilon x^2}, \tag{5.40}$$

where here “lead” means not the leading term of the corrections but the leading term of the expectation value of the master constraint density. Thus integrating the above term with respect to x will give us the master constraint relevant terms of the order

$$\int_{\epsilon}^{L} \langle \mathbb{H}_{\text{lead}}(x) \rangle dx \sim \frac{\ell_p^3}{\epsilon^2}. \tag{5.41}$$

Thus we see that the corrections to the master constraint are indeed considerably smaller than the leading contributions, provided that the lattice spacing ϵ is large compared to the Planck length (but still small compared to particle physics scales, as we discussed in more detail in previous chapter).

5.3 Low energy propagators for the scalar field

5.3.1 The standard treatment

In the previous section we showed that the vacuum of the theory is well approximated by the tensor product state obtained variationally in chapter 4. For such a state the space-time metric is locally flat with a global deficit angle. We would like to study the propagator of the polymerized scalar field in such a metric and determine possible corrections to the usual propagator introduced by the polymerization. We will study the propagator perturbatively in β, the polymerization coefficient, assuming this parameter is small.

As we have seen before, the Hamiltonian for a scalar field in spherical symmetry on a locally flat background is given by,

$$H = \frac{P_f^2}{2x^2} + \frac{x^2 \left(f'\right)^2}{2}, \tag{5.42}$$

where x is the radial coordinate. For convenient we rescale the matter field and its conjugate momentum as in(5.1) and (5.2) in the above Hamiltonian and then drop the “new” superscript to simplify the notation. The Hamiltonian then becomes (ignoring boundary terms),

$$H = \frac{P_f^2}{2} + \frac{\left(f'\right)^2}{2}. \tag{5.43}$$

The resulting wave equation can be solved in Fourier space,

$$f(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} dk \frac{\left(C(\omega(k))e^{-i\omega(k)t} + \bar{C}(\omega(k))e^{i\omega(k)t}\right)\sin(|k|x)}{\sqrt{\pi\omega}}, \tag{5.44}$$

and in this case the dispersion relation is very simple $\omega = |k|$, and with this, one can easily reconstruct the solution to the original form of the wave equation before the rescaling.

Let us now consider the discretized version of the Hamiltonian,

$$H(i) = \frac{P_f(i)^2}{2\epsilon} + \frac{\left(f(i+1) - f(i)\right)^2}{2\epsilon}, \tag{5.45}$$

where the ϵ in the first term is a result of the fact that the momentum is a density. The resulting discrete wave equation can be solved in modes,

$$f(j) = \sum_{n=-N}^{N} \frac{1}{\sqrt{2N\omega(n)}} \left(C(\omega(n)) e^{-i\omega(n)t} + \bar{C}(\omega(n)) e^{i\omega(n)t} \right) \operatorname{sgn}(n) \sin\left(\frac{j\pi n}{N} \right), \tag{5.46}$$

where all the sums from $-N$ to N exclude zero since there is a minimum value for the momentum in a box. The frequencies are given by

$$\omega(n) = \left| \frac{2\sin\left(\frac{\pi n}{2N}\right)}{\epsilon} \right|. \tag{5.47}$$

For further computations it is useful to define

$$p(n) \equiv \pi n/L, \tag{5.48}$$

$$L = N\epsilon, \tag{5.49}$$

and

$$f(n,t) \equiv \frac{1}{\sqrt{\omega(n)}} \left(C(\omega(n)) e^{-i\omega(n)t} + \bar{C}(\omega(n)) e^{i\omega(n)t} \right) \operatorname{sgn}(n). \tag{5.50}$$

Using the equations of motion from the Hamiltonian (5.43), the momentum is given by,

$$P_f(j) = \sum_{n=-N}^{N} \frac{i}{\sqrt{2N\omega(n)}} \left(-\omega(n) C(\omega(n)) e^{-i\omega(n)t} + \omega(n)\bar{C}(\omega(n)) e^{i\omega(n)t} \right) \operatorname{sgn}(n) \times \sin\left(\frac{j\pi n}{N} \right) \epsilon, \tag{5.51}$$

and we define,

$$P_f(n,t) = \frac{i}{\sqrt{\omega(n)}} \left(-\omega(n) C(\omega(n)) e^{-i\omega(n)t} + \omega(n)\bar{C}(\omega(n)) e^{i\omega(n)t} \right) \operatorname{sgn}(n)\epsilon, \tag{5.52}$$

One can quantize the fields, with discrete commutation relations,

$$\left[\hat{f}(i), \hat{P}_f(j)\right] = i\delta_{i,j}, \tag{5.53}$$

which naturally lead to the introduction of the creation and annihilation operators,

$$\left[\hat{C}(\omega(n)), \hat{\bar{C}}(\omega(m))\right] = \frac{1}{2\epsilon}(\delta_{n,m} + \delta_{n,-m}). \tag{5.54}$$

With this one can compute the free propagators. The Feynman propagator is given by,

$$G^{(0)}(n,t,n',t') = \langle 0| T\left(f(n,t), f(n',t')\right) |0\rangle = D(n,t,t')\left(\delta_{n,n'} - \delta_{-n,n'}\right), \tag{5.55}$$

where T is the time ordered product and

$$D(n,t,t') = \left[\frac{\Theta(t-t')\exp\left(-i\omega(n)(t-t')\right)}{\epsilon\omega(n)} + \frac{\Theta(t'-t)\exp\left(-i\omega(n)(t'-t)\right)}{\epsilon\omega(n)} \right], \tag{5.56}$$

or, using the residue theorem,

$$D(n,t,t') = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\omega}{\epsilon}\frac{1}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t'-t)\right). \tag{5.57}$$

We can write the previous expressions, which were in Fourier space, in direct space as following,

$$G^{(0)}(j,t,k,t') = \sum_{n=-N}^{N}\sum_{n'=-N}^{N}\frac{1}{N}\sin\left(\frac{j\pi n}{N}\right)\sin\left(\frac{k\pi n'}{N}\right)G^{(0)}(n,t,n',t'). \tag{5.58}$$

which is free propagator we are going to polymerize.

5.3.2 Polymerizing the scalar field

Having computed the free propagator we now turn to study the polymerized propagator. We start by noticing that the Hamiltonian (5.45) can be rewritten (again ignoring boundary terms) as,

$$\begin{aligned} H &= \sum_i H(i) \\ &= \sum_i \frac{P_f(i)^2}{2\epsilon}+\frac{\left(f(i+1)-f(i)\right)^2}{2\epsilon} \\ &= \sum_i \frac{P_f(i)^2}{2\epsilon}-\frac{\left(f(i+1)+f(i-1)-2f(i)\right)f(i)}{2\epsilon}, \end{aligned} \tag{5.59}$$

and the rearrangement makes the expression appear more readily symmetric in $i+1$ and $i-1$. We proceed to polymerize as

$$H = \sum_i\left(\frac{P_f(i)^2}{2\epsilon}-\frac{\sin\left(\beta\left(f(i+1)+f(i-1)-2f(i)\right)\right)\sin(\beta f(i))}{2\epsilon\beta^2}\right). \tag{5.60}$$

At this point some comments are in order. There are many possible choices at the time of polymerizing the theory. For instance, we could have chosen to polymerize $f(i+1)+f(i-1)-2f(i)$ as we did or we could have polymerized each term in the sum individually. In the lattice one can also choose to polymerize the momentum $P_f(i)$ (in the continuum this may be more difficult since P is a density)[1]. Suppose we polymerize $P_f(i)$. In this case since the continuum momentum is $P_f(i)/\epsilon$, taking the continuum limit yields $\sin^2(\beta\epsilon P_f)/\beta^2\epsilon$ for the first term in the Hamiltonian. Thus in the limit $\epsilon\to 0$ we would recover a non-polymerized theory and therefore we would not be making contact with usual loop quantum gravity results. Polymerizing the fields as we have chosen yields in the continuum limit a term $f''(x)\sin(\beta f(x))/\beta$ showing that the continuum theory is polymerized. It is interesting to notice that spatial derivatives of fields

[1] This is the reason our work is not easily compared with that of Hossain, Husain and Seahra [38]. They polymerize the momentum in the continuum. The density nature of the momentum leads them to a polymerization parameter that is dimensionful, unlike our case.

are well defined in the Bohr compactification even if the field operators themselves are not. In this section we will work with a polymerization of the field rather than of the momentum. In a discrete theory, polymerizing either fields or momenta is possible, but it does not lead to equivalent theories. For completeness, in the next section we will discuss the theory that results from polymerizing the momenta. In previous treatments in the continuum [40], the scalar field has been polymerized, although in the case of the harmonic oscillator, which one can consider closely related to a scalar field, a polymerization of the momentum has been preferred [31].

We are going to work perturbatively, expanding in β. The Hamiltonian we will consider is $H = H_0 + H_{\text{int}}$ with

$$H_0 = \sum_i \left(\frac{P_f(i)^2}{2\epsilon} - \frac{f(i)\left(f(i+1)+f(i-1)-2f(i)\right)}{2\epsilon} \right), \tag{5.61}$$

and

$$H_{\text{int}} = \sum_i \frac{1}{2\epsilon}\left(\frac{1}{6} f(i)\left(f(i+1)+f(i-1)-2f(i)\right)^3 + \frac{1}{6} f(i)^3\left(f(i+1)+f(i-1)-2f(i)\right)\right)\beta^2. \tag{5.62}$$

This interaction Hamiltonian above comes from expansion in beta and keeping the first two leading terms. With it, we compute the interacting propagator to leading order,

$$\begin{aligned} G^{(2)}(j,t,k,t') =& G^{(0)}(j,t,k,t') + \frac{i^2}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \\ &\times \sum_{j'=-N}^{N}\sum_{k'=-N}^{N} \langle 0| T\left(f(j,t) f(k,t') H_{\text{int}}(j',t_1) H_{\text{int}}(k',t_2)\right)|0\rangle. \end{aligned} \tag{5.63}$$

To compute this expression it is convenient to rewrite the interaction Hamiltonian in momentum space (we use letters up to k for the field representation and letters starting with m for the momentum representation)

$$\begin{aligned} H_{\text{int}}(j',t_1) = \sum_{n,m,p,q=-N}^{N} & \left\{ \frac{1}{48N^2}\beta^2\epsilon^5 \sin\left(\frac{\pi j' n}{N}\right) f(n,t_1)\omega(m)^2 \sin\left(\frac{\pi j' m}{N}\right) f(m,t_1) \right. \\ & \times \omega(p)^2 \sin\left(\frac{\pi j' p}{N}\right) f(p,t_1)\omega(q)^2 \sin\left(\frac{\pi j' q}{N}\right) f(q,t_1) \\ & + \frac{1}{48N^2}\beta^2\epsilon \sin\left(\frac{\pi j' n}{N}\right) f(n,t_1) \sin\left(\frac{\pi j' m}{N}\right) f(m,t_1) \\ & \left. \times \sin\left(\frac{\pi j' p}{N}\right) f(p,t_1)\omega(q)^2 \sin\left(\frac{\pi j' q}{N}\right) f(q,t) \right\}. \end{aligned} \tag{5.64}$$

We can use the identity,

$$\begin{aligned} \Delta(n,m,p,q) \equiv & \sum_{j'=-N}^{N} \frac{4}{N^2} \sin\left(\frac{\pi j' n}{N}\right)\sin\left(\frac{\pi j' m}{N}\right)\sin\left(\frac{\pi j' p}{N}\right)\sin\left(\frac{\pi j' q}{N}\right) \\ =& \frac{1}{N}\Big[\delta_{n+m,p+q} + \delta_{n+p,m+q} + \delta_{n+q,m+p} + \delta_{n+m+p+q} \\ & - \delta_{n,m+p+q} - \delta_{m,n+p+q} - \delta_{p,n+m+q} - \delta_{q,n+m+p}\Big], \end{aligned} \tag{5.65}$$

in the above expression. Using this identity one can write,

$$\sum_{j'=-N}^{N} H_{\text{int}}(j', t_1) = \frac{1}{192} \sum_{n,m,p,q=-N}^{N} f(n,t_1) f(m,t_1) f(p,t_1) f(q,t_1) \\ \times \left[\left(\omega(m)^2 \omega(p)^2 \epsilon^4 + 1 \right) \epsilon \omega(q)^2 \right] \beta^2 \Delta(n,m,p,q), \tag{5.66}$$

where the definition

$$\zeta(m,p,q) = \left[\left(\omega(m)^2 \omega(p)^2 \epsilon^4 + 1 \right) \epsilon \omega(q)^2 \right], \tag{5.67}$$

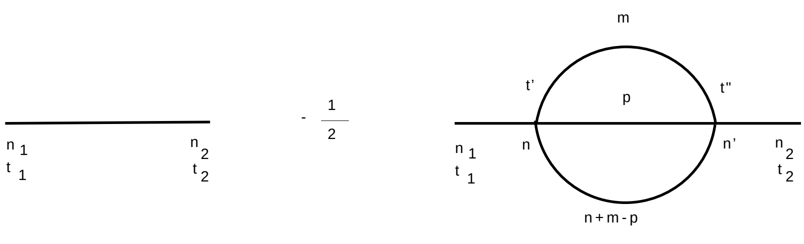
is introduced. Putting everything together we get,

$$\begin{aligned} G^{(2)}(n_1,t_1,n_2,t_2) = & G^{(0)}(n_1,t_1,n_2,t_2) + \frac{i^2}{2!} \langle 0 | T \big(f(n_1,t_1) f(n_2,t_2) \\ & \times \frac{1}{192} \int_{-\infty}^{\infty} dt' \sum_{n,m,p,q=-N}^{N} : f(n,t') f(m,t') f(p,t') f(q,t') : \\ & \times \zeta(n,m,p) \beta^2 \Delta(n,m,p,q) \\ & \times \frac{1}{192} \int_{-\infty}^{\infty} dt'' \sum_{n',m',p',q'=-N}^{N} : f(n',t'') f(m',t'') f(p',t'') f(q',t'') : \\ & \times \zeta(n',m',p') \beta^2 \Delta(n',m',p',q') \big) | 0 \rangle . \end{aligned} \tag{5.68}$$

Using Wick's theorem, the above expression can be rewritten as a sum of diagrams of the form,

$$\begin{aligned} G^{(2)}(n_1,t_1,n_2,t_2) = & G^{(0)}(n_1,t_1,n_2,t_2) - \frac{32}{3N^2} \sum_{m,p=-N}^{N} \int_{-\infty}^{\infty} dt' dt'' \left[D(n_1,t_1,t') D(m,t',t'') \right. \\ & \times \left. D(p,t',t'') D(n+m-p,t',t'') D(n_2,t'',t_2) \right] \\ & \times \zeta^2(m,p,n+m-p) \beta^4 \left(\delta_{n_1,n_2} - \delta_{n_1,-n_2} \right), \end{aligned} \tag{5.69}$$

or, graphically,



It is now convenient to Fourier transform the propagator in time,

$$
\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) =& \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-\omega(n_1)^2+i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)\\
&-\frac{32}{3}\frac{1}{2}\frac{4\pi i}{\epsilon\left(\omega_1^2-\omega(n_1)^2+i\sigma\right)}\sum_{m,p=-N}^{N}\int_{-\infty}^{\infty}d\omega' d\omega''\\
&\times\frac{4\pi i}{\epsilon\left((\omega')^2-\omega(m)^2+i\sigma\right)}\frac{4\pi i}{\epsilon\left((\omega'')^2-\omega(p)^2+i\sigma\right)}\\
&\times\frac{4\pi i}{\left((\omega_1-\omega'-\omega'')^2-\omega(n_1-m-p)^2+i\sigma\right)}\zeta^2(m,p,n_1-m-p)\beta^4\\
&\times\frac{4\pi i}{\left(\omega_2^2-\omega(n_2)^2+i\sigma\right)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right),
\end{aligned}
\tag{5.70}
$$

and the sums can be converted to integrals. Care should be taken not to allow the denominators to vanish, since in the original discrete expression the denominators did not vanish. Recalling (5.47),(5.48) and (5.49), we get

$$
\omega(n)=\frac{2\left|\sin\left(\frac{\epsilon p(n)}{2}\right)\right|}{\epsilon}\sim p(n). \tag{5.71}
$$

One then approximates,

$$
\sum_{m=1}^{N}\to\frac{L}{\pi}\int_{\pi/L}^{\pi/\epsilon}dp, \tag{5.72}
$$

and the sum from $-N$ to 1 takes an analogous form. The expression for the Green function up to second order is,

$$
\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) =& \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-p(n_1)^2+i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)\\
&-\frac{32}{3}\frac{1}{2}\frac{4\pi i}{\epsilon\left(\omega_1^2-p(n_1)^2+i\sigma\right)}\frac{1}{\pi^2}\\
&\times\int_{-\infty}^{\infty}d\omega' d\omega''\left[\int_{-\pi/\epsilon}^{-\pi/L}+\int_{\pi/L}^{\pi/\epsilon}\right]dp_1dp_2\\
&\times\frac{4\pi i}{\epsilon\left((\omega')^2-{p_1}^2+i\sigma\right)}\frac{4\pi i}{\epsilon\left((\omega'')^2-{p_2}^2+i\sigma\right)}\\
&\times\frac{4\pi i}{\left((\omega_1-\omega'-\omega'')^2-p(n_1-p_1-p_2)^2+i\sigma\right)}\\
&\times\tilde{\zeta}^2(p_1,p_2,p(n_1)-p_1-p_2)\beta^4\\
&\times\frac{4\pi i}{\left(\omega_2^2-\omega(n_2)^2+i\sigma\right)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right),
\end{aligned}
\tag{5.73}
$$

where we have introduced

$$
\tilde{\zeta}(p_1,p_2,p(n_1)+p_1-p_2)=\left(\epsilon^4\left({p_1}^2{p_2}^2\right)+1\right)\left(p(n_1)+p_1-p_2\right)^2\epsilon^2. \tag{5.74}
$$

The integrals can be computed by an analytic extension to the Euclidean theory and by carrying out an expansion in $p\epsilon$. The next expression is correct up to order $\mathcal{O}(\epsilon^4 p^4)$. That is, we are assuming the wavelength of the scalar field is much larger than the lattice spacing. If one takes into account powers higher than $p\epsilon$ one has higher corrections in powers of p. The result, not including those terms, is,

$$\begin{aligned} G^{(2)}(n_1,\omega_1,n_2,\omega_2) =& G^{(0)}(n_1,\omega_1,n_2,\omega_2) \\ &+ \left[\frac{\alpha_1\beta^4}{\epsilon^2}+\beta^4\alpha_2 p(n_1)^2\right]\frac{4\pi i}{\epsilon}\frac{\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)}{\left(\omega_1^2-p(n_1)^2+i\sigma\right)^2} \\ =& \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2-p(n_1)^2\left(1+\alpha_2\beta^4\right)-\frac{\alpha_1\beta^4}{\epsilon^2}+i\sigma}\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)\delta(\omega_1-\omega_2), \end{aligned} \tag{5.75}$$

where α_1 and α_2 are constants of order one, and the third line above has been derived from the first and second lines by expanding the right hand side of the first and the second lines, assuming β^4 is small.

5.3.3 Polymerizing the momentum of the field

We now discuss the choice of polymerizing the momentum. As before, we write the Hamiltonian as,

$$H=\sum_i \frac{P_f(i)^2}{2\epsilon}-\frac{\left(f(i+1)+f(i-1)-2f(i)\right)f(i)}{2\epsilon}, \tag{5.76}$$

and proceed to polymerize,

$$H=\sum_i \frac{\sin^2\left(\beta P_f(i)\right)}{2\beta^2\epsilon}-\frac{\left(f(i+1)+f(i-1)-2f(i)\right)f(i)}{2\epsilon}. \tag{5.77}$$

As before, we work perturbatively, expanding in β. The Hamiltonian we will consider is $H=H_0+H_{\rm int}$ with

$$H_{\rm int}(i)=-\frac{1}{6\epsilon}\beta^2 P_f(i)^4. \tag{5.78}$$

We can now write the Green function up to the second order,

$$\begin{aligned} G^{(2)}(j,t,k,t') =& G^{(0)}(j,t,k,t') \\ &+\frac{i^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\sum_{j'=-N}^{N}\sum_{k'=-N}^{N}\langle 0|T\left(f(j,t)f(k,t')H_{\rm int}(j',t_1)H_{\rm int}(k',t_2)\right)|0\rangle. \end{aligned} \tag{5.79}$$

Similar to the computation methods of the previous subsection, one can write,

$$\sum_{j'=-N}^{N} H_{\rm int}(j',t_1)=-\frac{1}{96\epsilon}\sum_{n,m,p,q=-N}^{N} P_f(n,t')P_f(m,t')P_f(p,t')P_f(q,t')\Delta(n,m,p,q)\beta^2. \tag{5.80}$$

Putting everything together we get,

$$\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) =& G^{(0)}(n_1,t_1,n_2,t_2) + \frac{i^2}{2!}\langle 0|T\big(f(n_1,t_1)f(n_2,t_2) \\
&\times \frac{1}{96}\int_{-\infty}^{\infty}\frac{dt'}{\epsilon}\sum_{n,m,p,q=-N}^{N} :P_f(n,t')P_f(m,t')P_f(p,t')P_f(q,t'): \\
&\times \beta^2\Delta(n,m,p,q) \\
&\times \frac{1}{96}\int_{-\infty}^{\infty}\frac{dt''}{\epsilon}\sum_{n',m',p',q'=-N}^{N} :P_f(n',t'')P_f(m',t'')P_f(p',t'')P_f(q',t''): \\
&\times \beta^2\Delta(n',m',p',q')\big)|0\rangle. \qquad (5.81)
\end{aligned}$$

If we now use Wick's theorem as we did before, there will appear contractions not only of f with itself, but also between f and P_f. Taking into account that the momentum is related to the derivative of the field by $P_f = \epsilon\dot{f}$, one can compute the expectation values of products of the field and momentum or products of the momenta by taking derivatives of (5.57) with respect to time.

$$\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) =& G^{(0)}(n_1,t_1,n_2,t_2) \\
&- 128\frac{\beta^4}{3N^2\epsilon^2}\sum_{m,p,q,m',p',q'=-N}^{N}\int_{-\infty}^{\infty} dt'dt'' D_{fP_f}(n_1,t_1,m,t') \\
&\times D_{P_fP_f}(p,t',p',t'')D_{P_fP_f}(q,t',q',t'') \\
&\times D_{P_fP_f}(m+p-q,t',m'+p'-q',t'')D_{P_ff}(m',t'',n_2,t_2), \qquad (5.82)
\end{aligned}$$

where

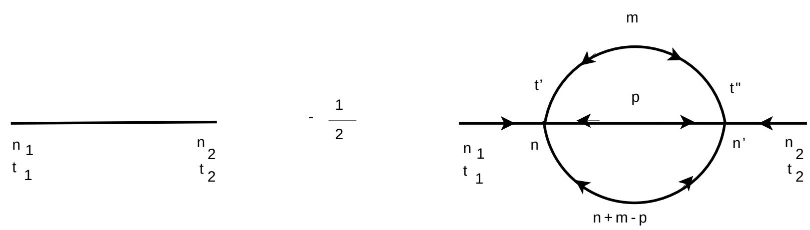
$$D_{ff}(n_1,t_1,n_2,t_2) = \frac{iL^2}{\pi\epsilon}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2-\omega(n_1)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right), \qquad (5.83)$$

$$D_{P_ff}(n_1,t_1,n_2,t_2) = -\frac{L^2}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)d\omega}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right), \qquad (5.84)$$

$$D_{fP_f}(n_1,t_1,n_2,t_2) = \frac{L^2}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)d\omega}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right), \qquad (5.85)$$

$$D_{P_fP_f}(n_1,t_1,n_2,t_2) = -\frac{iL^2\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)^2d\omega}{\omega^2-\omega(n_1)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right), \qquad (5.86)$$

or, graphically,



where the direction of the arrows depend on the order of appearance of f and P_f in their product, meaning an arrow to the right is fP_f, an arrow to the left $P_f f$, two arrows mean $P_f P_f$, and no arrow means ff.

We now Fourier transform in time and take the continuum approximation for the sums in p and q,

$$\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) = & \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2 - p(n_1)^2 + i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) + \\
& + \frac{128}{3}\frac{1}{2\pi^2}\frac{1}{\left(\omega_1^2 - p(n_1)^2 + i\sigma\right)} \\
& \times \frac{1}{\pi^2}\int_{-\infty}^{\infty} d\omega' d\omega'' \left[\int_{-\pi/\epsilon}^{-\pi/L} + \int_{\pi/L}^{\pi/\epsilon}\right] dp_1 dp_2 \left(\frac{i}{2\pi^3}\right)^3 \\
& \times \frac{\epsilon p_1^2}{\left((\omega')^2 - p_1{}^2 + i\sigma\right)}\frac{\epsilon p_2^2}{\left((\omega'')^2 - p_2{}^2 + i\sigma\right)} \\
& \times \frac{\left(p(n_1) - p_1 - p_2\right)^2}{\left((\omega_1-\omega'-\omega'')^2 - (p(n_1)-p_1-p_2)^2 + i\sigma\right)}\beta^4 \\
& \times \frac{4\pi i\omega(n_1)^2}{\left(\omega_2^2 - \omega(n_2)^2 + i\sigma\right)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right).
\end{aligned} \tag{5.87}$$

The integrals can be computed as before, expanding in $p\epsilon$ and analytically continuing to the Euclidean theory,

$$\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) &= G^{(0)}(n_1,\omega_1,n_2,\omega_2) + \beta^4\alpha_2 p(n_1)^2\frac{4\pi i}{\epsilon}\frac{\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)}{\left(\omega_1^2 - p(n_1)^2 + i\sigma\right)^2} \\
&= \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2 - p(n_1)^2\left(1+\alpha_2\beta^4\right) + i\sigma}\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)\delta(\omega_1-\omega_2),
\end{aligned} \tag{5.88}$$

where again the second line above has been derived from the first line by expanding the right hand side of the first line, assuming β^4 is small.

5.3.4 Lorentz invariance violation

Both of the propagators we derived in (5.75) and (5.88) violate the Lorentz invariance by modifying the dispersion relation. One can see that there are two distinct origins for the modifications of dispersion relation in (5.75) and (5.88). One is stemming from the polymerization and the other from the discreteness that is remnant from the uniform discretization procedure, since the state that minimizes the expectation value of the master constraint does so for a finite lattice spacing. This could be a temporary limitation until a better state is found, or it could well be that such a state actually does not exist.

The Lorentz violation due to polymerization can be made arbitrarily small by a suitable choice of the polymerization parameter β. This can be seen in terms like the dispersion relation implied by the denominator of (5.88),

$$\omega_1^2 - p(n_1)^2\left(1+\alpha_2\beta^4\right). \tag{5.89}$$

It should be noted that these terms depend on the value of the polymerization parameter β. The order in β at which these terms appear depends on choices made at the time of polymerization. To see this, let us write the polymerized momentum term in Hamiltonian (5.76) as

$$\frac{c\sin(\beta P_f(i))}{\sqrt{2\epsilon}\beta}+\frac{(1-c)\sin(3\beta P_f(i)}{3\sqrt{2\epsilon}\beta}, \tag{5.90}$$

and try to find c such that we are left only with the non-perturbative term in $P_f(i)$ and a perturbative term in β^4, thus neglecting the β^2 term. This way we can analyze just the effects of β^4 order term in the propagator. Expanding (5.90) in β we get

$$\frac{P_f(i)^2}{2\epsilon}+\left(\frac{4}{3}\frac{cP_f(i)^4}{\epsilon}-\frac{3}{2}\frac{P_f(i)^4}{\epsilon}\right)\beta^2+\left(\frac{8}{9}\frac{c^2P_f(i)^6}{\epsilon}-\frac{8}{3}\frac{cP_f(i)^6}{\epsilon}+\frac{9}{5}\frac{P_f(i)^6}{\epsilon}\right)\beta^4. \tag{5.91}$$

From the coefficient of β^2 we see that by setting $c=\frac{9}{8}$, the β^2 order term cancels and we are left only with a non-perturbative term and a perturbative term in β^4 which is

$$H_{\mathrm{int}}(i)=-\frac{3}{40\epsilon}P_f(i)^6\beta^4. \tag{5.92}$$

This would lead to corrections of order β^8 instead of β^4 in (5.88). We therefore see that the order in β at which corrections appear can be shifted arbitrarily by choosing suitable polymerizations of the theory and therefore, assuming that the polymerization parameter is small, one can make the corrections as small as desired.

This is because in the case of a scalar field this parameter is not obviously associated with an area and therefore not limited by the minimum area eigenvalue as is the case for gravitational variables. The order of the violation in the parameter can also be changed by choices in polymerization.

6

Hamiltonian analysis of the CGHS model

6.1 Introduction: the original CGHS formulation

As we mentioned before, there are interesting questions about black holes and their evolutions, Hawking radiation and related issues that are really hard to analyze in full four dimensions, even using semiclassical approximation in which the gravitational field is treated as a classical entity and only the matter field is quantized. Thus it is useful to use toy models that have many of those interesting properties but in which greater analytical control is possible.

The CGHS model [3] is such a theory. It is a renormalizable theory of quantum gravity coupled to matter in two dimensions having black hole solutions and Hawking radiation. It is classically completely solvable and because of this and its simplicity, one may hope for an exact quantum treatment of it, allowing to test the semiclassical considerations leading to the Hawking effect.

The action of CGHS is an effective action arising from the radial modes of the extremal dilatonic black holes in higher dimensions and is related to an specific action in noncritical strings [3]. Although the origin of this model is higher dimensional, it can be considered in its own right for the study of black hole formation and subsequent evaporation in a simplified way.

The original theory has the gravitational action (3.7) and the minimally coupled matter (3.8). The classical theory can be described in the conformal gauge where

$$g_{+-} = -\frac{1}{2}e^{2\rho}, \tag{6.1}$$

$$g_{--} = g_{++} = 0, \tag{6.2}$$

where we are using the null coordinates

$$x^{\pm} = x^0 \pm x^1. \tag{6.3}$$

The equations of motion for the variation of metric are

$$T_{++} = e^{-2\phi}(4\partial_+\rho\partial_+\phi - 2\partial_+^2\phi) + \frac{1}{2}(\partial_+ f)^2 = 0, \tag{6.4}$$

$$T_{--} = e^{-2\phi}(4\partial_-\rho\partial_-\phi - 2\partial_-^2\phi) + \frac{1}{2}(\partial_- f)^2 = 0, \tag{6.5}$$

$$T_{+-} = e^{-2\phi}(2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho}) = 0, \tag{6.6}$$

where T is the energy-momentum tensor, ϕ is the dilaton field and λ^2 the cosmological constant. The matter equation of motion is

$$\partial_+\partial_- f = 0, \tag{6.7}$$

and the dilaton equation of motion reads

$$-4\partial_+\partial_-\phi + 4\partial_+\phi\partial_-\phi + 2\partial_+\partial_-\rho + \lambda^2 e^{2\rho} = 0. \tag{6.8}$$

If there is no matter in the theory, $f = 0$, the solution is

$$e^{-2\rho} = e^{-2\phi} = \frac{M}{\lambda} - \lambda^2 x^+ x^-. \tag{6.9}$$

where M is a constant which is the mass of the black hole solution of the model (see below). Since in this case the scalar curvature is [41]

$$R = \frac{4M\lambda}{\frac{M}{\lambda} - \lambda^2 x^+ x^-}, \tag{6.10}$$

the solution (6.9) corresponds to a black hole of mass

$$M = \lambda^3 x^+ x^-. \tag{6.11}$$

The spacetime diagram of this black hole is shown in figure (6.1).

If we include matter, for example a shockwave of magnitude a of the form

$$\frac{1}{2}\partial_+ f\partial_+ f = a\delta\left(x^+ - x_0^+\right), \tag{6.12}$$

where x_0^+ is the position of the shockwave, the solution is [3]

$$e^{-2\rho} = e^{-2\phi} = -a\left(x^+ - x_0^+\right)\Theta\left(x^+ - x_0^+\right) - \lambda^2 x^+ x^-, \tag{6.13}$$

where Θ is the Heaviside step function. This shows that for $x^+ < x_0^+$, the solution is the linear dilaton vacuum

$$e^{-2\rho} = -\lambda^2 x^+ x^-. \tag{6.14}$$

But for $x^+ > x_0^+$ and by shifting $x^- \to x^- - a/\lambda^2$, the solution (6.13) becomes

$$e^{-2\rho} = e^{-2\phi} = a x_0^+ - \lambda^2 x^+ x^-, \tag{6.15}$$

which by comparing to (6.9) is evident that corresponds to a black hole of mass

$$M = a x_0^+ \lambda. \tag{6.16}$$

The conformal diagram of this case can be seen in figure (6.2).

In the following sections, we are going to analyze the Hamiltonian formulation of this model.

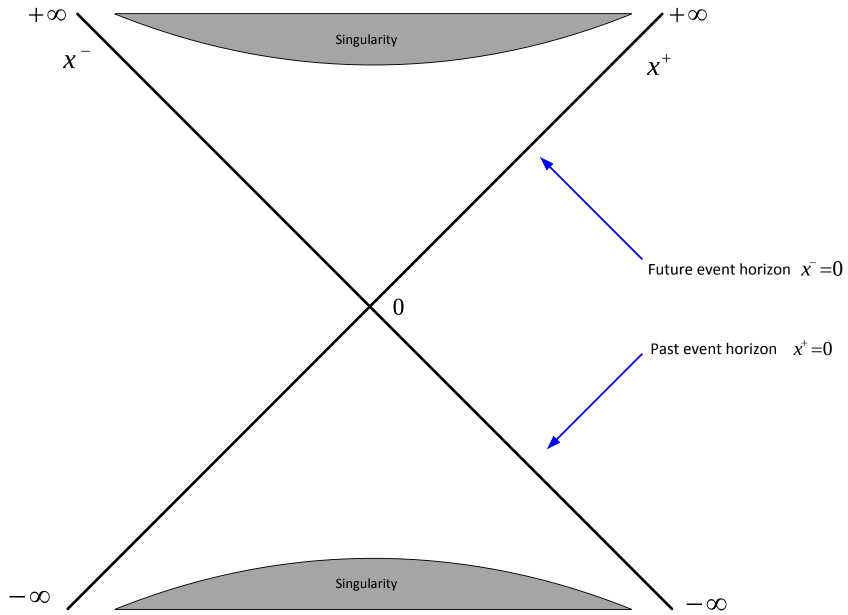


Figure 6.1: The spacetime diagram of the CGHS model without matter field.

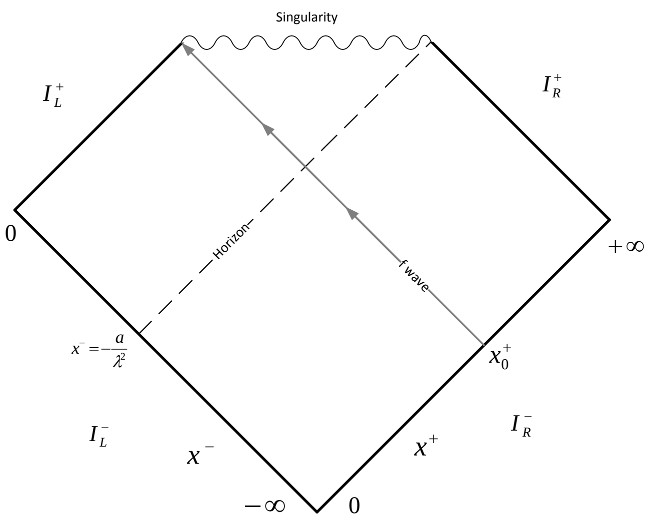


Figure 6.2: The conformal diagram of the CGHS spacetime with a shockwave matter field.

6.2 The Hamiltonian density in Ashtekar variables

The Hamiltonian analysis of the CGHS have been done in conformally transformed regime before (see for example [16]). Here we would like to analyze the CGHS model without conformal transformation. One reason is that the pure gravitational part of the conformally transformed CGHS is trivial and also as we mentioned before, it is better to work with the variables that have direct geometric meaning so that we do not need to transform everything back from the non-physical geometry to the physical one at the end. This will allow to quantize the gravitational magnitudes in the loop formalism and study if this quantization would allow to have a singularity free theory.

As we mentioned in section 3.6, in the physical CGHS case where the dilation field Φ is a canonical variable, the equation (3.110) is a primary constraint which should be added to the general Hamiltonian (3.124). Doing this and substituting the suitable parameters (3.43)-(3.46) in the resultant Hamiltonian yields

$$\begin{aligned}H =& N\left(\frac{2P_2}{|P|}\partial_1{}^*X^1 + \frac{2P_1}{|P|}\partial_1{}^*X^2 - \frac{2P_1}{|P|}\omega_1{}^*X^1 - \frac{2P_2}{|P|}\omega_1{}^*X^2 + \frac{|P|}{16}\Lambda\Phi^2\right.\\ &\left.- \frac{\Phi'^2}{|P|} - \frac{P_\Phi^2}{|P|} + \frac{(f')^2}{|P|} + \frac{P_f^2}{|P|}\right) \qquad (6.17)\\ &+ N^1\left(P_1\partial_1{}^*X^1 + P_2\partial_1{}^*X^2 - P_2{}^*X^1\omega_1 - P_1{}^*X^2\omega_1 + \Phi' P_\Phi + f' P_f\right)\\ &+ \omega_0\left(P_1{}^*X^2 + P_2{}^*X^1 - \left(\frac{1}{4}\Phi^2\right)'\right)\\ &+ M\left(P_\omega - \frac{1}{4}\Phi^2\right), \qquad (6.18)\end{aligned}$$

where M is a Lagrange multiplier (not to be confused with the mass of the black hole). In order to transform to the Ashtekar variables and following a similar pattern the form of Ashtekar variables in the 3+1 model, we introduce the following new momenta with a canonical transformation:

$$P_\omega = E^x, \qquad (6.19)$$

$$|P| = 2E^\varphi, \qquad (6.20)$$

$$P_1 = 2\cosh(\eta)E^\varphi, \qquad (6.21)$$

$$P_2 = 2\sinh(\eta)E^\varphi, \qquad (6.22)$$

where E^x, E^φ and η are the new momenta. This gives us the generating function

$$F(q,P) = 2^*X^1\cosh(\eta)E^\varphi + 2^*X^2\sinh(\eta)E^\varphi + \omega_1 E^x + \Phi P_\Phi + f P_f. \qquad (6.23)$$

Using $F(q,P)$, we can find the new canonical variables as

$$Q_\eta = \frac{\partial F}{\partial\eta} = 2^*X^1\sinh(\eta)E^\varphi + 2^*X^2\cosh(\eta)E^\varphi, \qquad (6.24)$$

$$K_\varphi = \frac{\partial F}{\partial E^\varphi} = 2^*X^1\cosh(\eta) + 2^*X^2\sinh(\eta), \qquad (6.25)$$

$$A_x = \frac{\partial F}{\partial E^x} = \omega_1, \qquad (6.26)$$

where Q_η, K_φ and A_x correspond to E^x, E^φ and η respectively. From the above equations, we can find ${}^*X^1$, ${}^*X^2$ and ω_1 as

$$
\begin{aligned}
{}^*X^1 =& \frac{1}{2}\left(K_\varphi \cosh(\eta) - \frac{Q_\eta \sinh(\eta)}{E^\varphi}\right), && (6.27)\\
{}^*X^2 =& -\frac{1}{2}\left(K_\varphi \sinh(\eta) - \frac{Q_\eta \cosh(\eta)}{E^\varphi}\right), && (6.28)\\
\omega_1 =& A_x. && (6.29)
\end{aligned}
$$

In order to write the Hamiltonian density 6.18 in these new variables, we substitute (6.19)-(6.22) and (6.27)-(6.28) in the total Hamiltonian (6.18), and then make a field redefinition

$$A_x = K_x - \eta',$$

to get rid of η in the Hamiltonian and get

$$
\begin{aligned}
H =& N\left(\frac{Q'_\eta}{E^\varphi} - \frac{Q_\eta E^{\varphi\prime}}{E^{\varphi 2}} + \frac{1}{8}E^\varphi \Lambda \Phi^2 - K_\varphi K_x - \frac{\Phi'^2}{2E^\varphi} - \frac{P_\Phi^2}{2E^\varphi} + \frac{(f')^2}{2E^\varphi} + \frac{P_f^2}{2E^\varphi}\right)\\
&+ N^1\left(E^\varphi K'_\varphi - Q_\eta K_x + \Phi' P_\Phi + f' P_f\right)\\
&+ \omega_0\left(Q_\eta - \left(\frac{1}{4}\Phi^2\right)'\right) + M\left(E^x - \frac{1}{4}\Phi^2\right). \qquad (6.30)
\end{aligned}
$$

We can see from here that the total Hamiltonian is just the sum of four constraints as is expected for a totally constrained system. The first constraint multiplied by the lapse function N is the Hamiltonian constraint. The one that is multiplied by the shift vector N^1 is the diffeomorphism constraint. The constraint that is multiplied by ω_0 is the Gauss constraint and the last one is the one we got from the definition of the momentum P_ω. Solving the Gauss constraint in the above Hamiltonian and substituting the resultant Q_η from it into the Hamiltonian yields

$$
\begin{aligned}
H =& N\left(\frac{\Phi\Phi''}{2E^\varphi} - \frac{\Phi\Phi' E^{\varphi\prime}}{2E^{\varphi 2}} + \frac{1}{8}E^\varphi \Lambda \Phi^2 - K_\varphi K_x - \frac{P_\Phi^2}{2E^\varphi} + \frac{(f')^2}{2E^\varphi} + \frac{P_f^2}{2E^\varphi}\right)\\
&+ N^1\left(E^\varphi K'_\varphi - \frac{1}{2}\Phi\Phi' K_x + \Phi' P_\Phi + f' P_f\right) + M\left(E^x - \frac{1}{4}\Phi^2\right). \qquad (6.31)
\end{aligned}
$$

Since we now know our canonical variables and momenta, we can write their Poisson brackets as

$$
\begin{aligned}
\{K_x(x), E^x(y)\} &= \delta(x-y), && (6.32)\\
\{K_\varphi(x), E^\varphi(y)\} &= \delta(x-y), && (6.33)\\
\{\Phi(x), P_\Phi(y)\} &= \delta(x-y), && (6.34)\\
\{f(x), P_f(y)\} &= \delta(x-y), && (6.35)
\end{aligned}
$$

and we have not written the Poisson bracket of (Q_η, η) pair because they no longer appear in the Hamiltonian. The rest of the Poisson brackets are strongly zero.

6.3 The consistency conditions on constraints

Following the Dirac procedure, we should check the preservation of the constraints to see if there are any new secondary constraints and/or to find the value of the Lagrange multipliers in terms of canonical variables. This means that the constraints $\mathscr{C}$, being the constants of motion, should remain weakly vanishing during the evolution

$$\dot{\mathscr{C}} = \{\mathscr{C}, H\} \approx 0. \tag{6.36}$$

This condition can also be seen from a different viewpoint: the Hamiltonian or the energy of the system should not change along the orbits produced by constraints in the phase space. The Poisson bracket of the Hamiltonian and diffeomorphism constraints with H vanishes weakly. Let's check the consistency of the constraint

$$\mu = E^x - \frac{1}{4}\Phi^2. \tag{6.37}$$

For this, we need (6.32) and (6.34). The preservation condition of μ constraint leads to a new, and by definition secondary, constraint which we call α:

$$\dot{\mu} = \{\mu, H\} \approx 0 \Rightarrow \alpha = K_\varphi + \frac{1}{2}\frac{P_\Phi \Phi}{E^\varphi} \approx 0. \tag{6.38}$$

We also need to check the preservation of the new α constraint. This leads to a relation between the Lagrange multipliers N, N^1 and M (and canonical variables). Finding M from this relation and substituting it into the total Hamiltonian (6.31) yields

$$\begin{aligned} H = N\big(&- K_\varphi K_x - \frac{2\Phi' E^{\varphi\prime} E^x}{\Phi E^{\varphi 2}} + \frac{2E^x \Phi''}{\Phi E^\varphi} - \frac{P_\Phi^2}{2E^\varphi} + \frac{1}{8}E^\varphi \Lambda \Phi^2 - \frac{1}{2}\frac{\Phi P_\Phi K_x}{E^\varphi} + \frac{2P_\Phi K_x E^x}{\Phi E^\varphi} \\ &+ \frac{2E^x P_f^2}{\Phi^2 E^\varphi} - \frac{2\Phi'^2 E^x}{\Phi^2 E^\varphi} + \frac{\Phi'^2}{2E^\varphi} + \frac{2E^x f'^2}{\Phi^2 E^\varphi}\big) \\ &+ N^1\big(-\frac{1}{2}\Phi\Phi' K_x + \Phi' P_\Phi + f' P_f + E^x K_x' + E^\varphi K_\varphi' - \frac{1}{4}\Phi^2 K_x'\big). \end{aligned} \tag{6.39}$$

Next step is to check if the constraints are first class or second class. Calculating the Poisson brackets of constraints among themselves shows that μ and α are second class and do not commute with each other. In other words, their Poisson bracket with each other does not vanish weakly. Now we can abandon the Poisson bracket and move on to the Dirac brackets. Also we should put the second class constraint strongly equal to zero and eliminate some of the variables in term of others. By doing this as you will see as following, we can get rid of the (Φ, P_Φ) pair in the Hamiltonian. Equating both the μ and α constraints strongly to zero yields

$$\mu = 0 \Rightarrow \Phi = 2\sqrt{E^x}, \tag{6.40}$$

$$\alpha = 0 \Rightarrow P_\Phi = -\frac{K_\varphi E^\varphi}{\sqrt{E^x}}. \tag{6.41}$$

Substituting these in the total Hamiltonian (6.39) will yield

$$
\begin{aligned}
H =& N\left(-K_\varphi K_x - \frac{E^{\varphi\prime} E^{x\prime}}{E^{\varphi 2}} - \frac{1}{2}\frac{E^{x\prime 2}}{E^\varphi E^x} + \frac{E^{x\prime\prime}}{E^\varphi} - \frac{1}{2}\frac{K_\varphi^2 E^\varphi}{E^x} + \frac{1}{2}E^\varphi E^x \Lambda + \frac{1}{2}\frac{P_f^2}{E^\varphi} + \frac{1}{2}\frac{f'^2}{E^\varphi}\right)\\
&+ N^1\left(-K_x E^{x\prime} + f' P_f - \frac{K_\varphi E^\varphi E^{x\prime}}{E^x} + E^\varphi K_\varphi'\right).
\end{aligned}
\tag{6.42}
$$

where now we are only left with one Hamiltonian constraint and one diffeomorphism constraint.

6.4 Dirac bracket and the algebra of canonical variables

In order to switch to the Dirac bracket, we need to find the general form of the Dirac bracket for our theory. As we saw in chapter 2, for a field theory (where the variables have continuous indices) the Dirac bracket is

$$
\{A(x), B(y)\}_D = \{A(x), B(y)\} - \int dw \int dz \left(\{A(x), \chi_\rho(w)\} C^{\rho\sigma}(w,z)\{\chi_\sigma(z), B(y)\}\right), \tag{6.43}
$$

where the $\{,\}_D$ refers to the Dirac bracket. Here χ's are the second class constraints and we define the the Poisson bracket between the ρ'th and σ'th second class constraint as

$$
C_{\rho\sigma}(w,z) = \{\chi_\rho(w), \chi_\sigma(z)\}. \tag{6.44}
$$

As we mentioned in chapter 2, these Poisson brackets define an invertible matrix $\mathbf{C}$. The inverse of this matrix is $\mathbf{C}^{-1}$ whose elements $C^{\rho\sigma}(w,z)$'s are the ones that appear in (6.43). In our model, there are only two second class constraints, μ and α. Thus the matrix of the Poisson bracket of the second class constraints will be

$$
\begin{aligned}
\mathbf{C} = C_{\rho\sigma}(x,y) &= \begin{pmatrix} \{\mu(x), \mu(y)\} & \{\mu(x), \alpha(y)\} \\ \{\alpha(x), \mu(y)\} & \{\alpha(x), \alpha(y)\} \end{pmatrix}\\
&= \begin{pmatrix} 0 & \{\mu(x), \alpha(y)\} \\ \{\alpha(x), \mu(y)\} & 0 \end{pmatrix}.
\end{aligned}
\tag{6.45}
$$

To compute the elements of this matrix we use (6.37) and (6.38) along with (6.34) to get

$$
\begin{aligned}
\{\mu(x), \alpha(y)\} &= \left\{E^x(x) - \frac{1}{4}\Phi(x)^2, K_\varphi(y) + \frac{1}{2}\frac{P_\Phi(y)\Phi(y)}{E^\varphi(y)}\right\}\\
&= -\frac{1}{8}\frac{\Phi(y)}{E^\varphi(y)}\left\{\Phi(x)^2, P_\Phi(y)\right\}\\
&= -\frac{1}{4}\frac{\Phi(y)^2}{E^\varphi(y)}\delta(x-y).
\end{aligned}
\tag{6.46}
$$

The same method of computations gives

$$
\{\mu(x), \alpha(y)\} = \frac{1}{4}\frac{\Phi(x)^2}{E^\varphi(x)}\delta(x-y). \tag{6.47}
$$

To calculate the elements of $\mathbf{C}^{-1}$, we use the property $\mathbf{C}\mathbf{C}^{-1} = \mathbf{1}$, or in terms of their elements

$$\int C_{\rho\sigma}(x,z) C^{\sigma\beta}(z,y) dz = \delta_\rho{}^\beta \delta(x-y), \tag{6.48}$$

which yields

$$\mathbf{C}^{-1} = C^{\rho\sigma}(x,y) = \begin{pmatrix} 0 & \frac{4E^\varphi(x)}{\Phi^2(x)} \\ -\frac{4E^\varphi(x)}{\Phi^2(x)} & 0 \end{pmatrix} \delta(x-y). \tag{6.49}$$

Using this and (6.43), the general form of the Dirac bracket for our theory will become

$$\begin{aligned} \{A(x), B(y)\}_D =& \{A(x), B(y)\} + \int dw \int dz \Big(\{A(x), \mu(w)\} \frac{4E^\varphi(w)}{\Phi^2(w)} \delta(w-z) \{\alpha(z), B(y)\} \Big) \\ & - \int dw \int dz \Big(\{A(x), \alpha(w)\} \frac{4E^\varphi(w)}{\Phi^2(w)} \delta(w-z) \{\mu(z), B(y)\} \Big). \end{aligned} \tag{6.50}$$

If we use this formula and the Poisson brackets (6.32)-(6.35), we can find the Dirac brackets of the canonical variables between each other as

$$\{K_x(x), E^x(y)\}_D = \{K_\varphi(x), E^\varphi(y)\}_D = \{f(x), P_f(y)\}_D = \delta(x-y), \tag{6.51}$$

$$\{K_x(x), K_\varphi(y)\}_D = \frac{K_\varphi}{E^x} \delta(x-y), \tag{6.52}$$

$$\{K_x, E^\varphi\}_D = -\frac{E^\varphi}{E^x} \delta(x-y), \tag{6.53}$$

$$\{E^x, K_\varphi\}_D = \{E^x, E^\varphi\}_D = \{f, \blacksquare\}_D = \{P_f, \bullet\}_D = 0, \tag{6.54}$$

where $\blacksquare$ means everything except P_f and the $\bullet$ means everything except f. Using these brackets we can evaluate the equations of motion as

$$\begin{aligned} \dot{K}_x =& \{K_x, H_\mathrm{T}\}_D \\ =& N\Big(-\frac{K_x K_\varphi}{E^x} + \frac{1}{2}\frac{f'^2}{E^x E^\varphi} + \frac{1}{2}\frac{P_f^2}{E^x E^\varphi} + \frac{E^{x\prime\prime}}{E^x E^\varphi} - \frac{E^{x\prime} E^{\varphi\prime}}{E^x E^{\varphi 2}} - \frac{E^{x\prime 2}}{E^{x2} E^\varphi} \Big) \\ & + N^1 K_x' + K_x N^{1\prime} + \frac{N''}{E^\varphi} - \frac{E^{\varphi\prime} N'}{E^{\varphi 2}}, \end{aligned} \tag{6.55}$$

$$\dot{E}_x = \{E_x, H_\mathrm{T}\}_D = N K_\varphi + N^1 E^{x\prime}, \tag{6.56}$$

$$\begin{aligned} \dot{K}_\varphi =& \{K_\varphi, H_\mathrm{T}\}_D \\ =& N\Big(-\frac{1}{2}\frac{f'^2}{E^{\varphi 2}} - \frac{1}{2}\frac{P_f^2}{E^{\varphi 2}} + \frac{1}{2} E^x \Lambda + \frac{1}{2}\frac{K_\varphi^2}{E^x} + \frac{1}{2}\frac{E^{x\prime 2}}{E^x E^{\varphi 2}} \Big) \\ & + \frac{N' E^{x\prime}}{E^{\varphi 2}} + N^1 K_\varphi', \end{aligned} \tag{6.57}$$

$$\dot{E}_\varphi = \{E_\varphi, H_\mathrm{T}\}_D = N K_x + N^{1\prime} E^\varphi + N^1 E^{\varphi\prime}, \tag{6.58}$$

$$\dot{f} = \{f, H_\mathrm{T}\}_D = \frac{N P_f}{E^\varphi} + N^1 f', \tag{6.59}$$

$$\dot{P}_f = \{P_f, H_\mathrm{T}\}_D = N\Big(\frac{f''}{E^\varphi} - \frac{f' E^{\varphi\prime}}{E^{\varphi 2}} \Big) + \frac{N' f'}{E^\varphi} + N^{1\prime} P_f + N^1 P_f', \tag{6.60}$$

where H_{T} here is not the density but the Hamiltonian itself, i.e. the spatial integral of (6.42)

$$H_{\mathrm{T}} = \int dx H. \tag{6.61}$$

6.5 Comparing with the original theory

We would like to get a better understanding of the model in our formulation by getting a more clear picture of the connection between the two formulations, the original CGHS Lagrangian formulation which we saw in section 6.1 and our Hamiltonian formalism. This is also a way of checking the consistency of our formulation. For these reasons we are going to transform our equations of motion into the null coordinates (6.3) and use the conformal gauge, (6.1) and (6.2), to make a connection between the two formalisms.

6.5.1 Metric and other variables in null coordinates

The first step to make a connection between two formulations is finding the relations between the form of the metric and other canonical variables in both models. As we mentioned before, the coordinates used in the original CGHS formulation are the null coordinates

$$x^+ = x^0 + x^1, \tag{6.62}$$

$$x^- = x^0 - x^1, \tag{6.63}$$

where $x^0 = t$ and $x^1 = x$ are the coordinates used in our model up to now. We can find the relation between the components of the null and non-null metrics using the general transformation

$$g_{ab} = \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b}\bar{g}_{a'b'}. \tag{6.64}$$

In the CGHS model, the metric components in the conformal gauge are

$$\bar{g}_{+-} = -\frac{1}{2}e^{2\rho}, \tag{6.65}$$

$$\bar{g}_{--} = \bar{g}_{++} = 0. \tag{6.66}$$

Thus the relations between the components in two coordinate systems are

$$g_{00} = 2\bar{g}_{+-} = -e^{2\rho}. \tag{6.67}$$

$$g_{11} = -2\bar{g}_{+-} = e^{2\rho}. \tag{6.68}$$

$$g_{01} = g_{10} = g_{++} - g_{--} = 0. \tag{6.69}$$

The relations between the partial derivatives in the two coordinates become

$$\frac{\partial}{\partial x^0} = \partial_+ + \partial_-, \tag{6.70}$$
$$\frac{\partial}{\partial x^1} = \partial_+ - \partial_-, \tag{6.71}$$
$$\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0} = \partial_+\partial_+ + \partial_-\partial_- + 2\partial_+\partial_-, \tag{6.72}$$
$$\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} = \partial_+\partial_+ + \partial_-\partial_- - 2\partial_+\partial_-, \tag{6.73}$$
$$\frac{\partial}{\partial x^0}\frac{\partial}{\partial x^1} = \partial_+\partial_+ - \partial_-\partial_-. \tag{6.74}$$

We can also write E^x in terms of dilaton field. Using the relation (3.9) between Φ and ϕ (the dilaton field in original CGHS paper) we get

$$\Phi = 2\sqrt{2}e^{-\phi}. \tag{6.75}$$

Substituting this into the μ constraint (6.37) and equating this second class constraint strongly to zero yields

$$E^x = \frac{1}{4}\Phi^2 = 2e^{-2\phi}. \tag{6.76}$$

The variable E^φ can also be written as

$$E^\varphi = \frac{|P|}{2} = \sqrt{q} = \sqrt{q_{11}} = \sqrt{g_{11}} = \sqrt{-2g_{+-}} = \sqrt{e^{2\rho}} = e^\rho, \tag{6.77}$$

where we have used the canonical transformation (6.20), (3.122), the fact that q_{ab} has only one independent component so that $q = q_{11}$ and (2.65). We can also find the lapse function and shift vector in a generic form from (2.64) and (2.66) as

$$g_{00} = -N^2 + q_{11}(N^1)^2, \tag{6.78}$$
$$g_{11} = q_{11}, \tag{6.79}$$
$$g_{01} = -q_{11}N^1. \tag{6.80}$$

From this, one can find N and N^1 in terms of metric components as

$$N^1 = -\frac{g_{01}}{q_{11}} = -\frac{g_{01}}{g_{11}}, \tag{6.81}$$
$$N = \sqrt{q_{11}(N^1)^2 - g_{00}} = \sqrt{\frac{g_{01}^2}{q_{11}} - g_{00}} = \sqrt{\frac{g_{01}^2}{g_{11}} - g_{00}}. \tag{6.82}$$

Substituting (6.67)-(6.69) in the above two equations and using (6.77) yields

$$N^1 = 0, \tag{6.83}$$
$$N = \sqrt{-g_{00}} = e^\rho = E^\varphi. \tag{6.84}$$

Now we are ready to compare the equations of motion.

6.5.2 The equations of motion in null coordinates

In order to compare our equations of motion with the original ones in the CGHS paper, we need to transform ours to second order form and then bring them into the null coordinates. Starting from the equations for matter field and its conjugates, if we find P_f from (6.59), substitute it into (6.60) and then use (6.70)-(6.74), (6.77), (6.83) and (6.84) we get

$$\partial_+\partial_- f = 0. \tag{6.85}$$

Next if we find K_φ from (6.56) and substitute it in (6.57) and again use (6.70)-(6.74), (6.76), (6.77), (6.83) and (6.84) we get

$$\begin{aligned} V_1 = e^{-\rho}e^{-2\phi}\big(& e^{2\phi}[(\partial_+ f)^2 + (\partial_- f)^2] + 4e^{2\rho}\lambda^2 - 4\partial_+^2\phi - 4\partial_-^2\phi \\ & - 8\partial_+\partial_-\phi + 16\partial_+\phi\partial_-\phi + 8\partial_+\rho\partial_+\phi + 8\partial_-\rho\partial_-\phi\big) = 0, \end{aligned} \tag{6.86}$$

where we have substituted $\Lambda = -4\lambda^2$. For the next second order equation, we find K_x from (6.58) and substitute it in (6.55). Then upon using (6.70)-(6.74), (6.76), (6.77), (6.83), (6.84) and the values of P_f and K_φ from (6.59) and (6.56) respectively, we get

$$\begin{aligned} V_2 = & -\frac{1}{2}e^{2\phi}[(\partial_+ f)^2 + (\partial_- f)^2] + 2\partial_+^2\phi + 2\partial_-^2\phi - 4\partial_+\partial_-\phi \\ & + 4\partial_+\partial_-\rho - 4\partial_+\rho\partial_+\phi - 4\partial_-\rho\partial_-\phi = 0. \end{aligned} \tag{6.87}$$

We can follow the same procedure and find the Hamiltonian and diffeomorphism constraints in (6.42) in the null coordinate as

$$\begin{aligned} \mathscr{H} = e^{-\rho}e^{-2\phi}\big(& e^{2\phi}[(\partial_+ f)^2 + (\partial_- f)^2] - 4e^{2\rho}\lambda^2 - 4\partial_+^2\phi - 4\partial_-^2\phi \\ & + 8\partial_+\partial_-\phi - 16\partial_+\phi\partial_-\phi + 8\partial_+\rho\partial_+\phi + 8\partial_-\rho\partial_-\phi\big) = 0, \end{aligned} \tag{6.88}$$

$$\mathscr{D} = e^{-2\phi}\big(8\partial_+\rho\partial_+\phi - 8\partial_-\rho\partial_-\phi - 4\partial_+^2\phi + 4\partial_-^2\phi\big) + (\partial_+ f)^2 - (\partial_- f)^2 = 0. \tag{6.89}$$

6.5.3 Identifying our equations of motion with those of the CGHS paper

If we compare the above equations with the equations of motion of the original CGHS model, (6.4)-(6.8), we can note the following:

The matter field equation (6.7) is identically the same in both methods and is given by (6.85). The $T_{++} = 0$ and $T_{--} = 0$ equations in the CGHS paper, (6.4) and (6.5), are combined in the diffeomorphism equation (6.89) as

$$\begin{aligned} T_{++} - T_{--} = & \frac{1}{2}\mathscr{D} \\ = & \left[e^{-2\phi}(4\partial_+\rho\partial_+\phi - 2\partial_+^2\phi) + \frac{1}{2}(\partial_+ f)^2\right] \\ & - \left[e^{-2\phi}(4\partial_-\rho\partial_-\phi - 2\partial_-^2\phi) + \frac{1}{2}(\partial_- f)^2\right] \\ = & 0. \end{aligned} \tag{6.90}$$

The $T_{+-} = 0$ equation in the CGHS paper, (6.6), can be obtained by combining the Hamiltonian constraint equation (6.88) and equation (6.86) as following:

$$T_{+-} = -\frac{e^{\rho}}{8}(V_1 - \mathscr{H}) = e^{-2\phi}(2\partial_+\partial_-\phi - 4\partial_+\phi\partial_-\phi - \lambda^2 e^{2\rho}) = 0, \tag{6.91}$$

and finally the dilaton field equation of motion in CGHS paper, (6.8), is obtained by combining V_1 and V_2 i.e. equation (6.87) and (6.86) as

$$\frac{e^{\rho}e^{2\phi}}{4}V_1 + \frac{1}{2}V_2 = -4\partial_+\partial_-\phi + 4\partial_+\phi\partial_-\phi + 2\partial_+\partial_-\rho + \lambda^2 e^{2\rho} = 0. \tag{6.92}$$

So we have a clear insight into the relation between the variables and equations of motion in both of the formulations.

6.6 Boundary conditions

It is important to be aware and take care of the boundary conditions in our theory [42]. One important reason is that energy in general relativity is related to the surface integral or boundary term at infinity[1].

We consider the variations of the action to see what surface terms are needed to be added to the action. After varying the action we get a term

$$\delta S = \int dt dx \left(\Xi_q \delta q + \Xi_p \delta p\right) + \delta S_{\text{surface}}, \tag{6.93}$$

where q (not to be confused with the determinant of the spatial metric) and p are shorthand notations for canonical variables and momenta and Ξ_q and Ξ_p are expressions involving the canonical variables and momenta and lapse and shift and their derivatives. The integral term gives the equations of motion. We use the solutions and conditions of [3] which is

$$e^{-2\rho} = e^{-2\phi}, \tag{6.94}$$

and thus

$$\rho = \phi. \tag{6.95}$$

This way we get

$$\begin{aligned} E^x(x,t) =& 2(E^{\varphi}(x,t))^{-2}, & (6.96)\\ E^{x\prime}(x,t) =& -4(E^{\varphi}(x,t))^{-3}E^{\varphi\prime}(x,t). & (6.97)\end{aligned}$$

[1] More precisely it is identified as the conserved quantity associated to the invariance of the action under time translations at infinity.

Using (6.42) and (6.83) and comparing it to (6.93), the surface term turns out to be

$$\begin{aligned}\delta S_{\text{surface}} &= \int dtdx\Big(-\frac{d}{dx}\left[N\frac{\delta E^{\varphi}E^{x\prime}}{E^{\varphi 2}}\right] - \frac{d}{dx}\left[N\frac{E^{\varphi\prime}\delta E^{x}}{E^{\varphi 2}}\right] - \frac{d}{dx}\left[N\frac{E^{x\prime}\delta E^{x}}{E^{\varphi}E^{x}}\right] \\ &\quad + \frac{d}{dx}\left[N\frac{\delta E^{x\prime}}{E^{\varphi}}\right] - \frac{d}{dx}\left[\frac{d}{dx}\left[\frac{N}{E^{\varphi}}\right]\delta E^{x}\right] + \frac{d}{dx}\left[\frac{f'}{E^{\varphi}}\delta f\right]\Big) \\ &= \int dt\Big[-N\frac{E^{x\prime}\delta E^{\varphi}}{E^{\varphi 2}} - N\frac{E^{\varphi\prime}\delta E^{x}}{E^{\varphi 2}} - N\frac{E^{x\prime}\delta E^{x}}{E^{\varphi}E^{x}} + N\frac{\delta E^{x\prime}}{E^{\varphi}} - \frac{d}{dx}\left[\frac{N}{E^{\varphi}}\right]\delta E^{x} \\ &\quad + \frac{f'}{E^{\varphi}}\delta f\Big]_{x=0}^{x=\infty}. \end{aligned} \tag{6.98}$$

Using the usual prescription at infinity for canonical variables (not the momenta) as

$$\delta f(x,t)\big|_{x=\infty} = \delta f(x,t)\big|_{x=0} = 0, \tag{6.99}$$

yields

$$\begin{aligned}\delta S_{\text{surface}} &= \int dt\Big[-N\frac{E^{x\prime}\delta E^{\varphi}}{E^{\varphi 2}} - N\frac{E^{\varphi\prime}\delta E^{x}}{E^{\varphi 2}} - N\frac{E^{x\prime}\delta E^{x}}{E^{\varphi}E^{x}} \\ &\quad + N\frac{\delta E^{x\prime}}{E^{\varphi}} - N'\frac{\delta E^{x}}{E^{\varphi}} + N\frac{E^{\varphi\prime}\delta E^{x}}{E^{\varphi 2}}\Big]_{x=0}^{x=\infty} \\ &= \int dt\left[-N\frac{E^{x\prime}\delta E^{\varphi}}{E^{\varphi 2}} - N\frac{E^{x\prime}\delta E^{x}}{E^{\varphi}E^{x}} + N\frac{\delta E^{x\prime}}{E^{\varphi}} - N'\frac{\delta E^{x}}{E^{\varphi}}\right]_{x=0}^{x=\infty}. \end{aligned} \tag{6.100}$$

Considering (6.84), (6.96) and (6.97), the above equation can be written as

$$\begin{aligned}\delta S_{\text{surface}} &= \int dt\left[-\frac{E^{x\prime}\delta E^{\varphi}}{E^{\varphi}} - \frac{E^{x\prime}\delta E^{x}}{E^{x}} + \delta E^{x\prime} - \frac{E^{\varphi\prime}\delta E^{x}}{E^{\varphi}}\right]_{x=0}^{x=\infty} \\ &= \int dt\Big[-\frac{\delta E^{\varphi}(-4(E^{\varphi})^{-3}E^{\varphi\prime})}{E^{\varphi}} - \frac{(-4(E^{\varphi})^{-3}E^{\varphi\prime})\delta(2(E^{\varphi})^{-2})}{2(E^{\varphi})^{-2}} \\ &\quad + \delta(-4(E^{\varphi})^{-3}E^{\varphi\prime}) - \frac{E^{\varphi\prime}\delta(2(E^{\varphi})^{-2})}{E^{\varphi}}\Big]_{x=0}^{x=\infty} \\ &= \int dt\Big[4\frac{E^{\varphi\prime}\delta E^{\varphi}}{E^{\varphi 4}} + 4\frac{E^{\varphi\prime}\delta E^{\varphi}}{E^{\varphi 4}} - 8\frac{E^{\varphi\prime}\delta E^{\varphi}}{(E^{\varphi})^{4}} \underbrace{-4(E^{\varphi})^{-3}\delta(E^{\varphi\prime}) - 4E^{\varphi\prime}\delta((E^{\varphi})^{-3})}_{-4\delta\left(\frac{E^{\varphi\prime}}{E^{\varphi 3}}\right)}\Big]_{x=0}^{x=\infty} \\ &= \int dt\left[-4\delta\left(\frac{E^{\varphi\prime}}{E^{\varphi 3}}\right)\right]_{x=0}^{x=\infty}. \end{aligned} \tag{6.101}$$

The solution without mass of the black hole and without matter field is the linear dilaton vacuum

$$e^{-2\rho} = e^{-2\phi} = (E^{\varphi})^{-2} = -\lambda^2 x^+ x^- = -\lambda^2(t^2 - x^2), \tag{6.102}$$

and the solution without matter field but with mass of the black hole (a preexisting black hole) is

$$e^{-2\rho} = e^{-2\phi} = (E^{\varphi})^{-2} = \frac{M}{\lambda} - \lambda^2 x^+ x^- = \frac{M}{\lambda} - \lambda^2(t^2 - x^2). \tag{6.103}$$

Substituting either of these into (6.101) gives

$$\begin{aligned}\delta S_{\text{surface}} &= \int dt \left[-4\delta\left(\frac{E^{\varphi\prime}}{E^{\varphi 3}}\right)\right]_{x=0}^{x=\infty} \\ &= -4\int dt\left[\delta\left(\lambda^2 x\right)\right]_{x=0}^{x=\infty},\end{aligned} \tag{6.104}$$

which does not include any dynamical variable and just consists of coordinates and cosmological constant. Hence there will be no surface terms in this case. On the other hand, if we use the solution

$$\begin{aligned}e^{-2\rho} = e^{-2\phi} &= -a(x^+ - x_0^+)\Theta(x^+ - x_0^+) - \lambda^2 x^+ x^- \\ &= -a(t + x - t_0 - x_0)\Theta(t + x - t_0 - x_0) - \lambda^2(t^2 - x^2),\end{aligned} \tag{6.105}$$

which corresponds to the situation where there is a matter field but no black hole mass initially, the scenario in which there is no black hole at first but matter field creates one with mass

$$M = a x_0^+ \lambda, \tag{6.106}$$

we get

$$\delta S_{\text{surface}} = -4\int dt \delta\left(\frac{E^{\varphi\prime}}{E^{\varphi 3}}\right), \tag{6.107}$$

where again we have used (6.77)i.e. $E^\varphi = e^\rho$. The term inside the integral inside the variation can be written in null coordinates as

$$\frac{(\partial_+ + \partial_-)E^\varphi}{E^{\varphi 3}} = \frac{1}{2}\left[a\Theta(x^+ - x_0^+) + a(x^+ - x_0^+)\delta(x^+ - x_0^+) - \lambda^2(x^+ - x^-)\right] \tag{6.108}$$

$$= \begin{cases} -\frac{1}{2}\lambda^2(x^+ - x^-) & , x^+ < x_0^+ \\ \frac{a}{2} - \frac{1}{2}\lambda^2(x^+ - x^-) & , x^+ > x_0^+ \end{cases}, \tag{6.109}$$

or in ordinary coordinates as

$$\frac{E^{\varphi\prime}}{E^{\varphi 3}} = \frac{1}{2}\left[a\Theta(t + x - t_0 - x_0) + a(t + x - t_0 - x_0)\delta(t + x - t_0 - x_0) - 2\lambda^2 x\right] \tag{6.110}$$

$$= \begin{cases} -\lambda^2 x & , t + x < t_0 + x_0 \\ \frac{a}{2} - \lambda^2 x & , t + x > t_0 + x_0 \end{cases}, \tag{6.111}$$

Now we can see that because from (6.106), we have

$$a = \frac{M}{\lambda x_0^+} = \frac{M}{\lambda(t_0 + x_0)}, \tag{6.112}$$

there is a surface term in this case which is

$$\delta S_{\text{surface}} = -4\int dt \delta\left(\frac{M}{2\lambda(t_0 + x_0)} - \lambda^2 x\right) \qquad \text{For } t + x > t_0 + x_0, \tag{6.113}$$

so we should add the term

$$S_{\text{surface}} = 2\int dt \frac{M}{\lambda(t_0 + x_0)} \qquad \text{For } t + x > t_0 + x_0, \tag{6.114}$$

to the action to cancel the offending surface term.

6.7 Behavior of phase space variables at the singularity and at infinity

6.7.1 The general form of the variables

In order to study the behavior of the phase space variables in Hamiltonian formulation at infinity and singularity, we first need to write those variables explicitly in terms of coordinates. for this, we substitute the definition of x^+ and x^-, (6.62), (6.63) into the CGHS solution (6.105) to get

$$e^{-2\rho} = e^{-2\phi} = -a(t+x-t_0-x_0)\Theta(t+x-t_0-x_0) - \lambda^2(t^2-x^2). \tag{6.115}$$

Using this and considering the relations (6.76), (6.77), the equations of motion (6.55)-(6.60) and the value of the shift and lapse from (6.83) and (6.84), we can write phase space variables in terms of E^φ and E^x and then in terms of ordinary coordinates as follows

$$\begin{aligned}
E^\varphi =& e^\rho = \frac{1}{\sqrt{-a(t+x-t_0-x_0)\Theta(t+x-t_0-x_0)-\lambda^2(t^2-x^2)}}, && (6.116)\\
E^x =& 2e^{-2\phi} = -2a(t+x-t_0-x_0)\Theta(t+x-t_0-x_0) - 2\lambda^2(t^2-x^2), && (6.117)\\
K_\varphi =& \frac{\dot{E}^x}{N} = \frac{\dot{E}^x}{E^\varphi}\\
=& \left[\sqrt{-a(t+x-t_0-x_0)-\lambda^2(t^2-x^2)}\left[a+2\lambda^2 t\right] - 2\lambda^2 t\sqrt{-\lambda^2(t^2-x^2)}\right]\\
& \times\Theta(t_0+x_0-t-x)\\
& - \sqrt{-a(t+x-t_0-x_0)-\lambda^2(t^2-x^2)}\left[a+2\lambda^2 t\right], && (6.118)\\
K_x =& \frac{\dot{E}^\varphi}{N} = \frac{\dot{E}^\varphi}{E^\varphi}\\
=& \frac{-\frac{a}{2}[(t+x)^2-2t(t_0+x_0)]\Theta(t_0+x_0-t-x) - (a-2\lambda^2 t)(t^2-x^2)}{\left[a(t+x-t_0-x_0)+\lambda^2(t^2-x^2)\right](t^2-x^2)}. && (6.119)
\end{aligned}$$

We also introduce a transformation from K_x to a new canonical variable U_x and we need its behavior too,

$$\begin{aligned}
U_x =& K_x + \frac{E^\varphi K_\varphi}{E^x}\\
=& \frac{\frac{a}{2}[(t+x)^2-2t(t_0+x_0)]\Theta(t_0+x_0-t-x) - +(a-2\lambda^2 t)(t^2-x^2)}{\left[a(t+x-t_0-x_0)+\lambda^2(t^2-x^2)\right](t^2-x^2)}. && (6.120)
\end{aligned}$$

6.7.2 The behavior of variables at the singularity

To understand the behavior of these variables at the singularity, we need to find the coordinates of the singularity in our coordinate system. The singularity happens when the curvature blows up and in the CGHS case it is equivalent to the case when the metric blows up. This corresponds to the condition that for $x^+ > x_0^+$, or equivalently at

$t+x>t_0+x_0$, the inverse of the metric component (6.65) vanishes. From (6.105), we can see that this condition means

$$-a(t+x-t_0-x_0)-\lambda^2(t^2-x^2)=0, \tag{6.121}$$

and using this, one can find the x coordinate of the singularity for any given time as

$$x_s=\frac{a+\sqrt{a^2+4\lambda^2[a(t-t_0-x_0)+\lambda^2 t^2]}}{2\lambda^2}. \tag{6.122}$$

So for $t+x>t_0+x_0$ and $x=x_s$, we get from (6.116)-(6.120)

$$K_\varphi\Big|_{\text{sing}} \to 0, \tag{6.123}$$

$$E^\varphi\Big|_{\text{sing}} \to \infty, \tag{6.124}$$

$$K_x\Big|_{\text{sing}} \to \infty, \tag{6.125}$$

$$U_x\Big|_{\text{sing}} \to -\infty, \tag{6.126}$$

$$E^x\Big|_{\text{sing}} \to 0. \tag{6.127}$$

6.7.3 The behavior of variables at infinity

Using the general form of the variables (6.116)-(6.120), we can find their value at infinity by taking the limit $x\to\infty$. This leads to

$$K_\varphi\Big|_{\infty} \to -\infty, \tag{6.128}$$

$$E^\varphi\Big|_{\infty} \to 0, \tag{6.129}$$

$$K_x\Big|_{\infty} \to 0, \tag{6.130}$$

$$U_x\Big|_{\infty} \to 0, \tag{6.131}$$

$$E^x\Big|_{\infty} \to \infty. \tag{6.132}$$

6.8 Behavior of terms of the Hamiltonian and diffeomorphism constraints at the singularity and at infinity

6.8.1 The terms of the Hamiltonian constraint

6.8.1.1 The behavior at the singularity

If we rescale $N\to\sqrt{E^x}N$, resulting in multiplication of all of the terms of Hamiltonian constraint in (6.42) by $\sqrt{E^x}$, and then substitute (6.116)-(6.120) in the terms and consider the singularity conditions with an approximation, meaning that evaluating the terms for $t+x>t_0+x_0$ and for

$$\begin{aligned} x_s &= x(t+\epsilon) \\ &= \frac{1}{2}\frac{a+\sqrt{4\lambda^2\left(a\left[t-t_0-x_0\right]+t^2\right)+a^2}}{\lambda^2}+\frac{a+2\lambda^2 t}{\sqrt{4\lambda^2\left(a\left[t-t_0-x_0\right]+t^2\right)+a^2}}+\mathcal{O}(\epsilon^2), \end{aligned} \tag{6.133}$$

which is an expansion of x around t for $t+\epsilon$ when $\epsilon\to 0$, then all of the terms remain finite or turn out to be zero. The terms that are finite then are seen to cancel each other as expected.

6.8.1.2 The behavior at infinity

Using the same method as above but without rescaling $N\to\sqrt{E^x}N$ and using $x=\frac{1}{l}$ and expand for $l\to 0$ instead of singularity conditions, and also noting that at infinity $f(x,t)=P_f(x,t)=0$, then some of the terms remain finite and some others become zero. Again the terms that have a non-vanishing expansion cancel each other.

6.8.2 The terms of diffeomorphism constraint

6.8.2.1 The behavior at the singularity

If we rescale $N^1\to E^xN^1$, resulting in multiplication of all of the terms of the diffeomorphism constraint in (6.42) by E^x, and then substitute (6.116)-(6.120) in the terms and consider the singularity conditions with an approximation just like the case of the Hamiltonian constraint, i.e. evaluating the terms for $t+x>t_0+x_0$ and $x_s=x(t+\epsilon)$ from (6.133), then all of the terms of the diffeomorphism constraint remain finite and cancel each other except for the term containing $f'(x,t)$ which will be zero.

6.8.2.2 The behavior at infinity

Using the same method as above but without rescaling $N^1\to E^xN^1$ and using $x\to\infty$ instead of singularity conditions and again noting that at infinity $f(x,t)=P_f(x,t)=0$, all the terms of the diffeomorphism constraint turn out to be zero at infinity.

7 Conclusion

7.1 Summary of the work

During this work we have studied the Hamiltonian formulation of a generic class of two dimensional gravitational systems coupled to matter containing local degrees of freedom. We have specifically studied two members of this class, the 3+1 spherically symmetric and the CGHS models. Both of them are rich models containing black hole solutions and Hawking radiation.

For the 3+1 case we have studied the holonomized and discretized master constraint and have tried to find a trial state for the system by using variational methods. We have assumed that the vacuum is a product of Gaussians for gravitational field around the classical flat solution and the Fock vacuum for the matter field. Then by minimizing the expectation value of the master constraint using this state, we showed that although the theory seems to has no continuum limit (corresponding to zero for the expectation value), but the expectation value of master constraint is very close to zero for the lattice spacing being very large compared to the Planck scale. It is also worth noting that our model regularizes the cosmological constant and gives a value several orders of magnitude smaller than the Planck scale.

Using the mentioned trial state, we calculated the propagator of the scalar field in two ways: one by polymerizing the matter field itself and the other by polymerizing the momentum of the matter field. It turns out that they do not lead to equivalent theories but both of them indicate violation of Lorentz invariance by introducing corrections to the dispersion relation. There are two sources for this violation: discretization and polymerization. We showed that the corrections due to polymerization can be shifted arbitrarily by changing the order of the polymerization variable that appears in the dispersion relation. So if this variable is small, the violations due to polymerization can be made as small as needed.

The other specific subsystem of the generic case we studied is the CGHS model. This is a dilatonic two dimensional model and comes from string theory. We have shown that the system is a second class one, derived the Hamiltonian of the system in terms of Ashtekar variables and have made a complete classical analysis of this model, including the form of the Dirac bracket, equations of motion, the asymptotic limits etc. By introducing a new variable we have brought the algebra of canonical pairs into the standard form such that the theory looks like a first class system. This classical study opens a way into the quantization of the theory by loop quantum gravity methods.

7.2 Future directions

In both cases the future aim of the work is to study the quantum effects in black hole solutions and the Hawking radiation. For example in 3+1 case, one can study solutions centered around non-flat geometries including a black hole solution and study several relevant quantum effects. Also in the CGHS case, the theory can be quantized and one in principle is able to study the back-reaction of the radiation on geometry, the unitarity of the evolution, the information paradox and the asymptotic fate of spacetime.

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