

On the geometry of positive cones in finitely generated groups

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Abstract

We study the geometry of positive cones of left-invariant total orders (left-order, for short) in finitely generated groups. We introduce the *Hucha property* and the *Prieto property* for left-orderable groups. We say that a group has the Hucha property if in any left-order the corresponding positive cone is not coarsely connected, and the Prieto property if in any left-order the corresponding positive cone is coarsely connected. We show that all left-orderable free products have the Hucha property, and that the Hucha property is stable under certain free products with amalgamation over Prieto subgroups. As an application we show that non-abelian limit groups in the sense of Z. Sela (e.g. free groups, fundamental group of hyperbolic surfaces, doubles of free groups and others) and non-abelian finitely generated subgroups of free \mathbb{Q} -groups in the sense of G. Baumslag have the Hucha property. In particular, this implies that these groups have empty BNS-invariant Σ^1 and that they do not have finitely generated positive cones.

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1 Introduction

Let G be a group. A subset P of G is a *positive cone* of G if it is a subsemigroup which, together with P^{-1} and the identity $\{1\}$, forms a partition of G . Elements from P and P^{-1} are called positive and negative respectively, and groups admitting positive cones are called *left-orderable* since from every positive cone P a total and left-multiplication-invariant order \prec (a *left-order*, for short) can be defined on G by setting $g \prec h$ whenever $g^{-1}h \in P$.

Suppose that G is finitely generated, and it is endowed with a word metric. In this paper, we study the geometry of the positive cones P of G , focusing on whether P is (coarsely) connected or not¹. As we will see below, this geometric information give us algebraic and formal-language complexity information about the positive cone P .

Our initial observation is that, although locally the geometry of a positive cone P and the ambient group G might be quite different, when G acts on a (Gromov) hyperbolic space, both the action of G and of P on the boundary look the same. To be more concrete, let us fix some notation. Suppose that G is acting by isometries on an hyperbolic space Γ . For H a subset of G and $x_0 \in \Gamma$, the set $Orb_H(x_0) = \{hx_0 \mid h \in H\}$

¹Recall that, given a metric space (X, d) , a subset $Y \subseteq X$ is coarsely connected if there is N such that $\{x \in X \mid d(x, Y) \leq N\}$, the N -neighborhood of Y , is connected

is called the orbit of x_0 under H and we denote by $\Lambda(H)$ the accumulation points of $Orb_H(x_0)$ in $\partial\Gamma$ (it is easy to see that this notion is independent of x_0). The action is called *non-elementary* if $|\Lambda(G)| \geq 3$. We will say that the action is *of general type* if it is non-elementary and not quasi-parabolic i.e. G does not fix a point of $\Lambda(G)$. (The literature about isometries of Gromov hyperbolic spaces is vast. We recommend [15] for an introduction.)

Proposition 1.1. Let Γ be a Gromov hyperbolic metric space and suppose that $G \curvearrowright \Gamma$ is an action by isometries of general type. Let P be a positive cone of G . Then, the P -orbits in Γ accumulate on every point of $\Lambda(G)$.

By a theorem of Osin [28, Theorem 1] every non-elementary acylindrical action on a hyperbolic space by isometries is of general type. In particular the theorem applies to hyperbolic groups acting on their Cayley graph, relatively hyperbolic groups acting on their relative Cayley graph [28, Proposition 5.2], or mapping class groups of surfaces of sufficiently high complexity acting on their curve complex, to name some classical examples. It is worth pointing out that Koberda and Kielak had already observed interactions between geometry and orderability. On the one hand, Koberda [24] note that the intersection of positive cones containing a given element of an hyperbolic group G yields a unique point in its Gromov boundary, and that the collection of all such points is a dense subset of ∂G . On the other hand, Kielak [23] exhibits an argument that recovers Proposition 1.1 in the case of a group with infinitely many ends acting on its set of ends.

Let us see, as an example, how to use Proposition 1.1 to prove that non-abelian free groups have no coarsely connected positive cones. Suppose that $G = F_n$ is a free group of rank $n \geq 2$ and suppose that Γ is the Cayley graph of G with respect to a basis. In this case, Γ is a tree whose vertices are in correspondence with the elements of G and G acts on Γ by left-multiplications. Fix a positive cone P with corresponding left-order \prec . Then for every $R > 0$, there is $g_R \in G$ which is larger (with respect to \prec) than every element from $\mathbb{B}(1_G, R)$, the ball of radius R in G . By left-invariance, every element of $g_R^{-1}\mathbb{B}(1, R) = \mathbb{B}(g_R^{-1}, R)$ is smaller than 1_G and we conclude that $\mathbb{B}(g_R^{-1}, R)$ is contained in P^{-1} . Finally, by Propostion 1.1, every component of $\Gamma \setminus \mathbb{B}(g_R^{-1}, R)$ contains at least one positive element, and so we can find two elements of P such that any path connecting them must go through $\mathbb{B}(g_R^{-1}, R)$. Since R is arbitrary, we conclude that P is not a coarsely connected subset of Γ .

It is easy to see that (up to adjusting constant) the previous argument is independent of generating sets. So we have proved the following.

Corollary 1.2. Positive cones of non-abelian free groups are not coarsely connected.

Corollary 1.2 easily follows from Kielak's work [23] and also from the analysis of the BNS invariant of the free group (see definition below). Yet, the previous proof exemplifies how we will exploit the interaction of the geometry at infinity of P , with the local geometry of G . We use a similar but more involved argument in Theorem 3.4, where we show that fundamental groups of hyperbolic surfaces do not have coarsely connected positive cones. In this case, G acts properly and co-compactly on its Cayley graph, a space which is quasi-isometric to the hyperbolic plane \mathbb{H}^2 . Finding sets of negative elements separating the boundary of \mathbb{H}^2 (we call such sets *negative swamps*) becomes more delicate, and we heavily relies on the "planarity" of \mathbb{H}^2 .

The situation for hyperbolic groups in general is much more complex, as there exists examples of groups with connected positive cones. To see this, recall that given a finitely generated group G , a non-trivial homomorphism $\phi: G \rightarrow \mathbb{R}$ belongs to $\Sigma^1(G)$, the Bieri-Neumann-Strebel invariant (BNS invariant for short), if and only if $\phi^{-1}((0, \infty))$ is connected. Moreover, the kernel of ϕ is finitely generated if and only if both ϕ and $-\phi$ belong to $\Sigma^1(G)$ (see [6]). In this light, it is easy to come up with examples of hyperbolic groups with non-trivial BNS invariant, for instance by considering (finitely generated free)-by- \mathbb{Z} groups that are hyperbolic (see [8] for a characterization) or, up to passing to finite index, any fundamental group of a closed hyperbolic 3-manifold (see [1]). Clearly, any (left-orderable)-by- \mathbb{Z} group is left-orderable (since left-orderability is stable under extensions, see for instance [13]), and we can show that all the previous examples support connected positive cones. Indeed, in Section 2 we provide an easy argument showing the following.

Proposition 1.3. Every finitely generated left-orderable group G with non-empty $\Sigma^1(G)$ admits connected positive cones. In particular, there are left-orderable hyperbolic groups with connected positive cones.

This may seem counter-intuitive in view of Proposition 1.1, yet, we will show that when positive cones of hyperbolic groups are coarsely connected, they have to be very distorted. More precisely, in Theorem 3.5, we show that if a positive cone in a non-elementary hyperbolic group is coarsely connected, then for every $\lambda \geq 1$, c , and $r \geq 0$ there are pairs of positive elements that *cannot* be joined by a (λ, c) -quasi-geodesic supported on the r -neighbourhood of the positive cone.

Remark 1.4. It is tempting to believe that the converse of Proposition 1.3 holds in general, namely that having a coarsely connected positive cone implies that the BNS invariant is not trivial. This is, however, not true in general. For instance there are perfect groups (i.e. groups with trivial abelianization) which supports only coarsely connected positive cones. A concrete example is $\tilde{T} = \langle a, b, c \mid a^2 = b^3 = c^7 = abc \rangle$, which is easily seen to be a perfect and left-orderable group such that all its positive cones are connected (see Corollary 4.16).

Coming back to applications of Proposition 1.1, we can show that many HNN extensions and amalgamated free products do not admit coarsely connected positive cones by exploiting their action on their associated Bass-Serre trees. Curiously, to show that certain groups acting on trees do not admit coarsely connected positive cones, we will need the opposite property for edge stabilizers, namely that for (certain) edge stabilizers all their positive cones are coarsely connected.

Before stating our main result, we need to set some terminology.

Definition 1.5. Let G be a finitely generated left-orderable group endowed with a word metric with respect to a finite generating set and let H be a subgroup of G .

1. We say that G is *Prieto* if all its positive cones are coarsely connected.
2. We say that G is *Hucha with respect to H* if for all $r > 0$ and all positive cones P of G , the r -neighbourhood of P is not connected and there are cosets g_1H and g_2H that cannot be connected by a path inside the r -neighbourhood of P .

We will say that a group is Hucha if it is Hucha relative to the trivial subgroup.²

Prototypical examples of Prieto groups are finitely-generated torsion-free abelian groups. It is well known that if one view \mathbb{Z}^n as the standard lattice in \mathbb{R}^n , positive cones essentially correspond to half-spaces defined by hyperplanes in \mathbb{R}^n going through the origin (see Section 4.2). Less obvious examples are braid groups and the group \tilde{T} from Remark 1.4, among others (see Corollary 4.16). On the other hand, free groups are *Hucha groups* with respect to all its infinite index finitely generated subgroups (see Proposition 5.4).

As mentioned before, there is a beautiful interplay between the opposite properties Prieto and Hucha, that allows to construct new Hucha groups. For instance we can show the following.

Theorem 1.6. Suppose that A has the Hucha property with respect to a subgroup C , and suppose that C is Prieto. Let G be either an HNN extension $A *_C t$ of an amalgamated free product of $A *_C B$, where B is any finitely generated group. If G is left-orderable, then G is Hucha.

Theorem 1.6 is really a corollary of our main technical result, Theorem 5.12. There, we conclude that HNN extensions and amalgamated free product are Hucha with respect to a family of non-trivial subgroups. Starting from free groups, we use Theorem 5.12 to inductively build an infinite family of Hucha groups that we denote by \mathfrak{H} .

Definition 1.7. Let \mathfrak{H} be the smallest family of groups containing non-abelian finitely generated free groups, that is closed under taking free products and taking non-abelian finitely generated subgroups, and such that for every $G \in \mathfrak{H}$ and any cyclic centralizer subgroup C of G , the group $G *_C A$, where A is finitely generated torsion-free abelian, lies in \mathfrak{H} .

It easily follows from a theorem of Howie [19] (see Proposition 6.1) that every group in \mathfrak{H} is locally indicable (i.e. every non-trivial finitely generated subgroups maps onto \mathbb{Z}), and hence left-orderable [9]. Further, there are two important families of groups closely related to our family \mathfrak{H} . One is the family of *limit groups* introduced by Sela (see [11] for a nice survey). Indeed, it follows from the hierarchical characterization given by Kharlampovich and Mysiakov [22] that *non-abelian limit groups* belong to \mathfrak{H} . The second, is the family of *free \mathbb{Q} -groups* introduced by G. Baumslag [5]. It turns out that free \mathbb{Q} -groups also admit a hierarchical construction starting from free groups, from which we can deduce that *finitely generated subgroups of free \mathbb{Q} -groups* are also contained in \mathfrak{H} . See Section 6 for details.

The main theorem of this paper is the following.

Theorem 1.8. Any group in \mathfrak{H} is Hucha. In particular, non-abelian limit groups and finitely generated subgroups of free \mathbb{Q} -groups are Hucha.

In light of Proposition 1.3, we obtain the following immediate consequence of Theorem 1.8.

²About the terminology. Prieto is an Spanish word which could be translate as tight. Whereas hucha refers to an old leather gadget who has an elongated slot which opens under a small pressure in order to insert and keep the coins.

Corollary 1.9. The BNS-invariant Σ^1 of every group in \mathfrak{H} is trivial. In particular, the BNS-invariants Σ^1 of non-abelian limit groups and finitely generated subgroups of free \mathbb{Q} -groups are trivial.

We point out that for the case of non-abelian limit groups this was already known by Kochloukova, who proved it with homological tools [25, Lemma E]. Also, the emptiness of Σ^1 for non-abelian limit groups can also be derived from a results of ℓ^2 -Betti numbers. Hillman [18, Theorem 6] proved that a necessary condition for a finitely presented group to have a non-empty Σ^1 is to have vanishing first ℓ^2 -Betti number equal to zero, however Pichot [29, Theorem 1] proved that these is not the case for non-abelian limit groups.

Another immediate consequence of the lack of coarse connectivity of a positive cone is that it can not be finitely generated as a semi-group (indeed see Remark 2.1). The search for groups supporting or not supporting finitely generated positive cones is an active line of research (see [13] and reference therein for some partial account). For instance, using dynamical techniques it was shown that free-products of groups [30] and fundamental groups of closed hyperbolic surfaces [2] do not support finitely generated positive cones (in fact, this method proves something stronger: it rules out the existence of *isolated* left-orders). This dynamical approach, roughly, consists of perturbing the dynamics of the generator of a group acting on the line while preserving its relations, a task that became impractical when dealing with complicated presentations (which is the case, for instance, of limit groups that are high in the hierarchy). In contrast, our method immediately yields the following.

Corollary 1.10. No positive cone of a group in \mathfrak{H} is finitely generated as a semi-group. In particular, no positive cone in a non-abelian limit group or in a finitely generated subgroup of free \mathbb{Q} -group is finitely generated as a semi-group.

In fact, something stronger holds for Hucha groups: they do not admit positive cones that can be described by a regular language over the generators (regular positive cones for short). See Corollary 7.5. Recall that, roughly, a subset of a finitely generated group is regular if it can be described by the set of paths in a finite labelled graph (see Section 7 for a precise definition). Certainly, finitely generated positive cones are regular, but being regular is a much more stable property. For instance, regularity passes to finite index subgroups [32] while finite generation does not³. We recommend [4] for an introduction. It is easy to show that finitely presented groups with a regular positive cone have solvable word problem (see [4]). A more refined criterium for not admitting regular positive cones was obtained by Hermiller and Šunić in [17] where they show that no positive cone in a free product of groups is regular. Their criterion is stated in a language-theoretical fashion and finding a geometrical interpretation of it was one of our initial motivations to pursue the present work.

The paper is organized as follows: we start in Section 2 proving some basic features of the geometry of positive cones. Among other things, we show that a positive cone naturally defines a special geodesic on the group with the property that *it goes deep* into the negative cone (see Proposition 2.2). In Section 3 we recall the basics of hyperbolic geometry to show Proposition 1.1. In that section we also show that the Hucha property holds for surface groups (Theorem 3.4) and that there is no quasi-geodesic combings for positive cones of hyperbolic groups (Theorem 3.5). We remark that results of Section

³An easy example is given by \mathbb{Z}^2 , which has no finitely generated positive cones but it is a finite index subgroup of $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$, which contains a positive cone generated as a semi-group by a and b .

3 are not needed elsewhere in the paper so readers interested on Theorem 1.8 and its applications might want to skip the section. In Section 4, we introduce the Prieto and Hucha properties, and show basic stability results such as the independence of the generating set that these properties passes to finite index overgroups. In Section 5, we will study left-orderable groups acting on trees and see how the geometry of the tree dominates the geometry of the group to show our combination theorems (Theorem 5.12 and Theorem 1.6). In Section 6, we will review the needed facts of limit groups and \mathbb{Q} -groups and show Theorem 1.8. Finally, in Section 7, we recall the definition of regular languages, regular positive cones, and observe that Hucha groups do not admit regular positive cones.

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2 Notation and basics facts

Let $\Gamma = (V\Gamma, E\Gamma)$ be a graph. A *path* or *1-sequence* in Γ , is a function $\alpha: \{0, 1, \dots, k\} \rightarrow V\Gamma$ where $\alpha(i) = \alpha(i+1)$ or there is an edge connecting $\alpha(i)$ and $\alpha(i+1)$. We also use the notation $\{\alpha(i)\}_{i=0}^k$ to denote α . We now can define the combinatorial metric on $V\Gamma$, setting $d_\Gamma(u, v)$ to be the minimum k such that there is a path $\{v_i\}_{i=0}^k$ with $v_0 = u$ and $v_k = v$. If no such path exists we set $d_\Gamma(u, v) = \infty$.

By $\mathbb{B}_\Gamma(v, r)$ we denote the set $\{u \in V\Gamma \mid d_\Gamma(u, v) \leq r\}$, the closed ball of radius r .

Let $r \geq 1$. An *r-path* or *r-sequence* $\{v_n\}_{n=1}^k$ is a sequence in Γ such that $d_\Gamma(v_i, v_{i+1}) \leq r$. Let $\lambda \geq 1, c \geq 0$. An *r-path* is a (λ, c) -*quasi-geodesic* if

$$\frac{|i-j|}{\lambda} - c \leq d_\Gamma(v_i, v_j) \quad \forall 1 \leq i, j \leq k.$$

A *geodesic* is a 1-path that is a $(1, 0)$ -quasi-geodesic.

For infinite r -paths we will use the term *r-rays*, *r-quasi-geodesics* or *geodesic rays*.

A subset S of $V\Gamma$ is *coarsely connected* if there is some $r \geq 1$ such that for all u, v in S there is an r -sequence supported at S starting at u and ending at v . Equivalently, S is coarsely connected if there is $r \geq 1$ such $\{v \in V\Gamma \mid d(v, S) \leq r\}$ spans a connected subgraph.

Let G be a group and X a generating set of G . Generating sets, unless otherwise stated, will assumed to be symmetric, that is $X = X^{-1}$. We denote by $\Gamma(G, X)$ the *Cayley graph* of G with respect to X . By identifying the group G with the vertices of $\Gamma(G, X)$, G is endowed with a metric $d_X = d_{\Gamma(G, X)}$. We also write $|g|_X$ for $d_X(1_G, g)$. By $\mathbb{B}_X(g, r)$ we denote the ball in $\Gamma(G, X)$ with center g and radius r . We will drop the subscripts X and Γ when the meaning is clear from the context.

2.1 General facts about the geometry of positive cones

Recall that $P \subseteq G$ is a *positive cone* of G , if P is a sub-semigroup and $G = P \sqcup P^{-1} \sqcup \{1\}$ where $P^{-1} = \{g^{-1} : g \in P\}$. Moreover, $a \prec b \Leftrightarrow a^{-1}b \in P$ defines a G -left-invariant total order on G . Conversely, if \prec is a G -left-invariant total order on G , then $P = \{g \in G \mid 1_G \prec g\}$ is a positive cone. Groups with positive cones must be torsion-free.

If \prec is a left-order on G , and X and Y are subsets of G we will write $X \prec Y$ if for all $x \in X$ and all $y \in Y$, $x \prec y$ holds. If $X = \{x\}$ a singleton, we will write simply $x \prec Y$ to denote $\{x\} \prec Y$.

Given $S \subseteq G$, we denote by $\langle S \rangle^+$ the sub-semigroup generated by S . A positive cone is *finitely generated* if there is a finite set S of G such that $P = \langle S \rangle^+$.

Remark 2.1. If P is a finitely generated positive cone of a group G , then it is coarsely connected subset of the Cayley graph of G . Indeed, let $P = \langle S \rangle^+$ with S finite and $G = \langle X \rangle$. Let $r = \max_{s \in S} |s|_X$, then there are r -paths in P from 1_G to $g \in P$ for all $g \in P$.

We show now that positive cones define preferred ways to go to infinity and that positive cones contain arbitrarily large balls. This will be essential for showing that fundamental groups of hyperbolic surfaces do not have coarsely connected positive cones.

Proposition 2.2. Let G be generated by a finite symmetric set X . Let P be a positive cone of G with associated order \prec . Let $g_n = \max_{\prec} \mathbb{B}_X(1_G, n)$. The following holds:

1. The map $\{g_n^{-1}\}_{n=0}^{\infty}$ is a geodesic ray in $\Gamma(G, X)$. In particular, $|g_n|_X = n$.
2. $\mathbb{B}_X(g_n^{-1}, n-1) \subseteq P^{-1}$.

In particular P^{-1} contains $\bigcup_n \mathbb{B}_X(g_n^{-1}, n-1)$.

Proof. To show 1, for $0 < i \leq n$ let $g_i = \max_{\prec} \mathbb{B}(1_G, i)$ and assume, by induction on n , that $|g_i|_X = i$ and $g_i = x_i g_{i-1}$ with $x_i \in X$. The base of induction, $n = 1$, $g_1 = \max_{\prec} X$ and $g_0 = \{1_G\}$, clearly holds. Let $g_{n+1} = \max_{\prec} \mathbb{B}(1_G, n+1)$. By definition $|g_{n+1}|_X \leq n+1$. So we need to show that $g_{n+1} \notin \mathbb{B}(1_G, n)$ and that there is some $x \in X$ such that $g_{n+1} = xg_n$.

By left-invariance, $xh \preceq xg_n$ for all $x \in X$ and all $h \in \mathbb{B}(1_G, n)$. Thus

$$\mathbb{B}(1_G, n+1) = \bigcup_{x \in X} x\mathbb{B}(1_G, n) \preceq \max_{\prec} \{xg_n \mid x \in X\}.$$

Thus $g_{n+1} \in \{xg_n \mid x \in X\}$. It remains to show that $g_{n+1} \notin \mathbb{B}(1_G, n)$. For that, by totality of the order, for each $x \in X$, either $xg_n \prec g_n$ or $g_n \prec xg_n$, which, by left-invariance it is equivalent to $g_n \prec xg_n$ or $g_n \prec x^{-1}g_n$. Thus $\mathbb{B}(1_G, n) \preceq g_n \prec \max_{\prec} \{xg_n \mid x \in X\}$. This completes the proof of the induction.

It follows that the map $n \mapsto g_n^{-1}$ is a geodesic ray, that is $d(g_n^{-1}, g_m^{-1}) = |m - n|$.

To show 2, we let $b \in \mathbb{B}(1_G, n-1)$. By definition we have that $g_n \succ b$ and so by left-invariance we get that $1_G \succ g_n^{-1}b$. \square

Corollary 2.3. Let $X = X^{-1}$ be a finite generating set of G and P a positive cone of G . Every connected component of P in $\Gamma(G, X)$ is infinite and contains a geodesic ray.

Proof. Applying Proposition 2.2 to P^{-1} , we get a geodesic ray $\{g_n\}_{i=0}^{\infty}$ starting at 1_G and contained (except from g_0) in P . Let $g \in P$. By left-invariance of the action of G on $\Gamma(G, X)$, $\{gg_n\}_{i=0}^{\infty}$ is a geodesic ray. Since P is a sub-semigroup $\{gg_n\}_{i=0}^{\infty} \subseteq P$. \square

We finish this section by showing that left-orderable groups with non-trivial BNS invariant enjoys connected positive cones.

Proof of Proposition 1.3. Let G be a left-orderable finitely generated by $X = X^{-1}$, and assume that $\Sigma^1(G)$ is non-empty. Then, there is a non-trivial homomorphism $\phi: G \rightarrow \mathbb{R}$ such that $\phi^{-1}((0, \infty))$ is connected. Let $x_0 \in X$ such that $\phi(x_0) > 0$. It is known that $\{g \in G \mid \phi(x_0) < \phi(g)\}$ is also connected [6].

Now let $K = \ker(\phi)$ and \preceq_K be any left-ordering on K . Using ϕ , we can lexicographically extend \preceq_K to produce a left-ordering of G . Precisely we set $1 \prec_G g$ if and only if $\phi(g) > 0$ or $\phi(g) = 0$ and $1 \prec_K g$ (checking that this is a left-order is left to the reader). Now, if g_1 and g_2 are positive in \preceq_G then $g_1 x_0$ and $g_2 x_0$ both belong to $\{g \in G \mid \phi(x_0) < \phi(g)\}$, which we already pointed out that is connected. In particular, there is a path made of positive elements connecting g_1 and g_2 . \square

3 Positive cones acting on hyperbolic spaces

In this section we prove Proposition 1.1. As stated, the proof works in general for not necessarily proper (i.e. locally compact) metric spaces. Yet, in this section it will only be applied to hyperbolic groups (i.e. proper spaces), thus the reader only interested in these applications can assume that the spaces involved are proper. We follow [16].

Let Γ be a graph. The Gromov product of a and b in $V\Gamma$ (with respect to v) is defined by

$$(a|b)_v = d(a, v) + d(b, v) - d(a, b).$$

Let $\delta \geq 0$. The graph Γ is δ -hyperbolic if for a given base-point $v \in V\Gamma$ we have

$$(a|b)_v \geq \min\{(a|c)_v, (b|c)_v\} - \delta$$

for all $a, b, c \in V\Gamma$. With this definition of hyperbolicity, every geodesic triangle in Γ is (4δ) -thin. Moreover, if α is a geodesic from v to a , and β is a geodesic from v to b then

$$d(\alpha(t), \beta(t)) \leq 2\delta \quad \forall t \in [0, (a|b)_v]. \quad (1)$$

Finally, if Γ is δ -hyperbolic with respect to some base-point v , then it is $\delta(u)$ -hyperbolic for any other base-point u .

Let $\epsilon > 0$ with $\epsilon' = \exp(\epsilon \cdot \delta) - 1 < \sqrt{2} - 1$. For $a, b \in V\Gamma$, define $\rho_\epsilon(a, b) = \exp(\epsilon \cdot (a|b)_v)$ if $a \neq b$, and $\rho_\epsilon(a, a) = 0$. The function ρ_ϵ is a pseudo-metric. However we can construct a metric on $V\Gamma$ by setting

$$d_\epsilon(a, b) = \inf_n \left\{ \sum_{i=1}^n \rho_\epsilon(z_i, z_{i+1}) \mid z_1 = a, z_{n+1} = b, z_i \in V\Gamma \right\}$$

and this metric satisfies that $(1 - 2\epsilon')\rho_\epsilon(a, b) \leq d_\epsilon(a, b) \leq \rho_\epsilon(a, b)$.

Let $\hat{\Gamma}$ be the metric completion of $V\Gamma$ with respect to d_ϵ . Then, one can define the Gromov boundary $\partial\Gamma$ of Γ as $\hat{\Gamma} \setminus V\Gamma$. The topology of the Gromov boundary is independent of the base point v and the parameter ϵ .

Let G be a group acting by isometries on Γ . This action naturally extends to a continuous action on $\hat{\Gamma}$ and hence on $\partial\Gamma$. Given a subset H of G and a vertex $v \in V\Gamma$, we denote by $\Lambda(H)$ the intersection of the closure of the orbit Hv in $\hat{\Gamma}$ (with respect to the metric d_ϵ) with $\partial\Gamma$. The set $\Lambda(H)$ is independent of v .

The following theorem classifies the actions of G on Γ in terms of the dynamics of G on $\Lambda(G)$.

Theorem 3.1. [16, Theorem 2.7] Suppose that G acts by isometries on an hyperbolic geodesic space Γ . Then one of the following holds.

- (i) (*elliptic action*) $\Lambda(G)$ is empty.
- (ii) (*parabolic action*) G fixes a unique point of $\Lambda(G)$.
- (iii) (*dihedral action*) $\Lambda(G)$ has two points.
- (iv) There are two elements $g, h \in G$ such that $|\Lambda(\langle g \rangle) \cup \Lambda(\langle h \rangle)| = 4$.

The action is called *non-elementary* if $|\Lambda(G)| > 2$ (in fact this is equivalent to $|\Lambda(G)| = \infty$). We will say that a non-elementary action is *of general type* if it lies on case (iv) of the previous theorem.

By a theorem of Osin [28, Theorem 1] every non-elementary acylindrical action on a hyperbolic space by isometries is of general type. Recall that the action of G on Γ is *acylindrical* if for every $r \geq 0$, there are N and R such that for any pair $a, b \in V\Gamma$ with $d(a, b) \geq R$, the set $\{g \in G \mid d(a, ga) \leq r \text{ and } d(b, gb) \leq r\}$ has at most N elements. This implies, although it is much easier to prove, that

Remark 3.2. non-elementary proper co-compact actions are of general type. See also [15, Chapitre 8, Théorème 30] for a direct proof.

An element $g \in G$ is called *loxodromic* if $|\Lambda(\langle g \rangle)| = 2$. Equivalently, an element $g \in G$ is loxodromic if $n \mapsto g^n v$, $n \in \mathbb{Z}$, defines a quasi-geodesic. The *loxodromic limit set* of G , denote $\mathcal{L}(G)$, is the set $\Lambda(\{g \in G \mid g \text{ loxodromic}\})$. The loxodromic limit set is *bilaterally dense in $\Lambda(G)$* if for every pair of disjoint non-empty open subsets A, B of $\Lambda(G)$, there is a loxodromic element $g \in G$ such that $\Lambda(\langle g \rangle)$ has non-trivial intersection with both A and B , in other words, the quasi-geodesic $g^n v$ goes from A to B .

Proposition 1.1 follows from the following.

Theorem 3.3. [16, Theorem 2.9.] Suppose that G acts non-elementary by isometries on an hyperbolic graph Γ . Then

1. $\Lambda(G)$ is closed without isolated points (i.e is a perfect set).
2. the hyperbolic limit set is bilaterally dense in $\Lambda(G)$ if and only if the action is of general type.

Proof of Proposition 1.1. Let $\eta \in \Lambda(G)$ and A an open neighbourhood of η in $\Lambda(G)$. Since the action is non-elementary, $\Lambda(G)$ is perfect and thus η is not isolated and we can assume that there is $\nu \neq \eta$ in A . Since the topology on $\partial\Gamma$ (and $\Lambda(G)$) is Hausdorff, there are disjoint open neighborhoods U and V of η and ν respectively lying in A . Since the action of G is of general type, the hyperbolic limit set is bilaterally dense in $\Lambda(G)$ and thus, there is a loxodromic element g such that $\Lambda(\langle g \rangle)$ meets U and V . Now, if P is positive cone of G , we have that either g or g^{-1} lies in P . And thus either $\Lambda(\langle g \rangle^+)$ or $\Lambda(\langle g^{-1} \rangle^+)$ lies in $\Lambda(P)$ and thus $\Lambda(P) \cap A \neq \emptyset$. \square

3.1 Positive cones in surface groups

As promised, we show how the ideas of the proof of Corollary 1.2 can be adapted to the case of surface groups. The proof uses strongly the topology of \mathbb{H}^2 . Specifically, that \mathbb{H}^2 can be cut into two infinite connected components using a curve.

Theorem 3.4. Suppose that G is the fundamental group of a closed hyperbolic surface. Then, no positive cone in G is coarsely connected.

Proof. Assume that X is the standard generating set of G , so that the Cayley graph, not only is quasi-isometric to \mathbb{H}^2 , but also it is embedded as a 1-dimensional topological subspace. In particular, any path in G connecting two different points of ∂G separates the Cayley graph into two infinite connected components. We fix a positive cone P and the corresponding left-order \prec . We also fix $N_0 > 1$. By Proposition 1.1, P accumulates on every point of ∂G , so to show that P is not N_0 -connected, it is enough to build a bi-infinite path $\gamma : \mathbb{Z} \rightarrow G$ such that

- i)* the N_0 -neighbourhood of γ is contained in P^{-1} i.e. $\mathbb{B}_X(\gamma(i), N_0) \subseteq P^{-1} \forall i \in \mathbb{Z}$,
- ii)* $\gamma(i)$ converges, as i tends to $\pm\infty$, to two different boundary points of ∂G .

We let $g_n = \max_{\prec} \mathbb{B}_X(1_G, n)$. By Proposition 2.2, $n \mapsto g_n^{-1}$ is a geodesic ray such that $\mathbb{B}(g_n^{-1}, n) \subseteq P^{-1}$. We let $\xi = \lim_{n \rightarrow \infty} g_n^{-1} \in \Lambda(G)$.

Given any $g \in G$ there exist some generator $x \in X^{\pm 1}$ such that gx does not belong to the stabilizer of ξ . Indeed, otherwise $\langle gX^{\pm 1} \rangle \supseteq \langle \{x_1x_2 \mid x_1, x_2 \in X^{\pm 1}\} \rangle$ is a finite index subgroup of G that stabilizes ξ , contradicting that the action of G on itself is of general type (Remark 3.2). Take $h \in g_{2N_0+1}^{-1}X^{\pm 1} \subseteq P^{-1}$ not belonging to the stabilizer of ξ and define $\gamma : \mathbb{Z} \mapsto G$ as:

$$\gamma(n) = g_{2N_0+1+n}^{-1} \text{ if } n \geq 0; \quad \gamma(n) = hg_{-(n+1)}^{-1} \text{ if } n < 0.$$

We claim that γ is the desired path. Notice that $\lim_{n \rightarrow +\infty} \gamma(n) = \xi$, $\lim_{n \rightarrow -\infty} \gamma(n) = h\xi$ and $h\xi$ differs from ξ by construction of h . Thus *ii)* holds.

To see condition *i)*, take $n \in \mathbb{Z}$. If $n \geq 0$ the condition follows from the definition of g_n and Proposition 2.2. If $n \in [-N_0, -1]$ we get that $\gamma(n) \in h\mathbb{B}(1_G, N_0) \subseteq \mathbb{B}(g_{2N_0+1}^{-1}, N_0 + 1)$ and therefore $\gamma(n)\mathbb{B}(1_G, N_0) \subseteq \mathbb{B}(g_{2N_0+1}^{-1}, 2N_0 + 1) \subseteq P^{-1}$. Finally, if $n < -N_0$ we get $\gamma(n) = hg_m^{-1}$ for some $m \geq N_0$ and therefore $\gamma(n)\mathbb{B}(1_G, N_0) = hg_m^{-1}\mathbb{B}(1_G, N_0) \subseteq P^{-1}$ since $h \in P^{-1}$ and $g_m^{-1}\mathbb{B}(1_G, N_0) \subseteq g_m^{-1}\mathbb{B}(1_G, m) \subseteq P^{-1}$. \square

3.2 Quasi-geodesic combings of positive cones

Let G be a non-elementary hyperbolic group and X a finite generating set. Let $\Gamma = \Gamma(G, X)$ be the Cayley graph. A (λ, c) -quasi-geodesic combing of P is a subset \mathcal{P} of (λ, c) -quasi-geodesics paths in Γ starting at 1_G and ending at vertices of P , and such that for each $g \in P$, there is at least a path in \mathcal{P} from 1_G to g . We say that \mathcal{P} is supported on the r -neighbourhood of P if every $p \in \mathcal{P}$ lies in $\cup_{g \in P} \mathbb{B}_X(g, r)$.

Theorem 3.5. Let G be a finitely generated and non-elementary hyperbolic group and P a positive cone of G . Then, there is no quasi-geodesic combing of P supported on a neighbourhood of P .

Proof. Let X be a finitely generating set of G and $\Gamma = \Gamma(G, X)$ its associated Cayley graph. Let P be a positive cone of G and suppose that there exists $\lambda \geq 1$, $c \geq 0$ and $r \geq 0$ and a (λ, c) -quasi-geodesic combing \mathcal{P} of P supported on the r -neighbourhood of P . Let \prec denote the associated order of P .

Let $g_n = \max_{\prec} \mathbb{B}(1_G, n)$. By Proposition 2.2, $n \mapsto g_n^{-1}$ is a geodesic ray. We denote by $\xi \in \partial G$ its limit as $n \rightarrow \infty$. Also by Proposition 2.2, the horoball $\cup_n \mathbb{B}(g_n^{-1}, n - 1)$ is contained in P^{-1} .

By Proposition 1.1, we know that there is a sequence $h_n \in P$ such that h_n converges to ξ , *i.e.* $(h_n|\xi)_{1_G} \rightarrow \infty$. Thus, for every $N > 0$, we can find $n(N), m(N)$ such that $(h_{n(N)}|g_{m(N)}^{-1})_1 > N$. Let α^N and β^N be geodesic paths from 1_G to $h_{n(N)}$ and $g_{m(N)}^{-1}$ respectively. By Proposition 2.2, we can assume that $\beta^N(t) = g_t^{-1}$. By equation (1) in the beginning of Section 3, $d(\alpha^N(t), \beta^N(t)) \leq 2\delta$ for all $t \in [0, N]$.

Denote by p_n a path in \mathcal{P} from 1_G to h_n . By the stability of quasi-geodesics, there is constant D (only depending on δ, λ and C) such that p_n and α_n are at Hausdorff distance at most D . Suppose that $N > D + r + 2\delta$. Since p_n is contained in the r -neighbourhood of P , there is $z \in P$ such that $d(z, \alpha^N(N)) \leq D + r$. Since $d(\alpha^N(N), g_N^{-1}) \leq 2\delta$, we have that $z \in \mathbb{B}(g_N^{-1}, D + r + 2\delta) \subseteq P^{-1}$, which is a contradiction. \square

4 Connectedness properties for positive cones

In this section we introduce two properties that essentially say that all positive cones have the same connectedness behaviour: either all positive cones are coarsely connected (Prieto property) or all positive cones are not coarsely connected (Hucha property).

4.1 The Hucha property

Let Γ be a graph. A subset S of $V\Gamma$ *r-disconnects* subsets H_1 and H_2 of Γ if any r -path from H_1 to H_2 has non-trivial intersection with S . We say that S *r-disconnects* a subset P , if P contains a subset $\{u, v\}$ so that S *r-disconnects* $\{u\}$ and $\{v\}$.

Definition 4.1. Let P be a positive cone of a group G , $H \leq G$ a subgroup and $\Gamma(G, X)$ a Cayley graph. A subset $S \subseteq P^{-1}$ is a *negative swamp of width r for H* in $\Gamma(G, X)$ if:

1. S *r-disconnects* P ,
2. there are $g_1, g_2 \in G$ such that S *r-disconnects* g_1H and g_2H .

Remark 4.2. If $S \subseteq P^{-1}$ is a negative swamp of with r for H , then the same holds for any S' with $S \subseteq S' \subseteq P^{-1}$. That is, S' is also a negative swamp of width r for H .

If S is a negative swamp of with r for H , then the same holds for any subgroup K of H . That is S is also a negative swamp of width r for K .

The Definition 4.1 is modeled after our proofs. We could just defined what it means for P^{-1} to be a negative swamp. However, when we establish that P^{-1} is a negative swamp, in many case we use a proper subset of P^{-1} to show that it disconnects the appropriate sets. For example, we just used balls in the argument of Corollary 1.2 in the introduction.

Definition 4.3. Let G be a left-orderable group, X a finite generating set and \mathcal{H} a family of subgroups. We say that the Cayley graph $\Gamma(G, X)$ is *Hucha with respect to \mathcal{H}* if for every positive cone P of G , for every $H \in \mathcal{H}$ and for every $r > 0$, there is a negative swamp S of width r for H .

If \mathcal{H} is the trivial subgroup we just say that $\Gamma(G, X)$ is Hucha.

Remark 4.4. By Remark 4.2, if $\Gamma(G, X)$ is Hucha with respect to \mathcal{H} , then it is also Hucha with respect to $\{K \leq H \mid H \in \mathcal{H}\}$. Thus one might assume that the family \mathcal{H} is closed under taking subgroups.

By Remark 4.4, if $\Gamma(G, X)$ is Hucha with respect to \mathcal{H} then $\Gamma(G, X)$ is Hucha. The following is worth noticing.

Proposition 4.5. If $\Gamma(G, X)$ is Hucha, then no positive cone P of G is coarsely connected.

Proof. Let P be a positive cone and r a positive integer. Since $\Gamma(G, X)$ is Hucha, there is a negative swamp $S \subseteq P^{-1}$ that r -disconnects P . In particular, there are $g_1, g_2 \in P$ such that every r -path connecting them must go through S and thus it can not be supported in P . Since r is arbitrary we deduce that P is not coarsely connected. \square

The following is easy from the definitions.

Lemma 4.6. Being Hucha with respect to \mathcal{H} is independent of the generating set.

Therefore, we will say that a group G is Hucha with respect to \mathcal{H} if some (any) Cayley graph is Hucha with respect to \mathcal{H} . By Remark 2.1 we also have the following:

Corollary 4.7. Hucha groups do not have finitely generated positive cones.

Although we will not need this, it is worth recording it. The proof is easy but will exemplify some of the ideas used later.

Lemma 4.8. Let G be a finitely generated left-orderable group, H a finite index subgroup such that H is Hucha with respect to a family \mathcal{H} of subgroups of H . Then G is Hucha with respect to the family \mathcal{H} .

Proof. Fix a finite generating set X of G . By Lemma 4.6, we can assume that $Y = X \cap H$ generates H . Thus $\Gamma(H, Y)$ is a subgraph of $\Gamma(G, X)$. Let R such that $H\mathbb{B}(1, R) = G$. Fix a positive cone P of G with corresponding order \prec . Let $t^{-1} = \max_{\prec} \mathbb{B}(1, R)$. Thus $t\mathbb{B}(1, R) \subseteq P^{-1} \cup \{1\}$.

Let $r > 0$ and $K \in \mathcal{H}$. We need to find a negative swamp $S \subseteq P^{-1}$ for K of width r . Notice that $P' = t^{-1}Pt \cap H$ is a positive cone for H . By hypothesis, for every $r' > 0$ there exists $S' \subseteq (P')^{-1}$ and $a_1, a_2 \in H$ such that S' r' -disconnects a_1K and a_2K in $\Gamma(H, Y)$. We take r' satisfying that $|h|_Y \leq r'$ for every $h \in H$ with $|h|_X \leq 2R + r$.

Note that $S' \subseteq t^{-1}P^{-1}t$. Thus $tS't^{-1} \subseteq P^{-1}$ and therefore $tS'\mathbb{B}(1, R) \subseteq P^{-1}$. We claim that $tS'\mathbb{B}(1, R)$ is a negative swamp of width r for K . Let $\{g_i\}_{i=0}^m$ be an r -sequence in G with $g_0 \in ta_1K$ and $g_m \in ta_2K$. Let $h_0, h_m \in H$ such that $th_0 = g_0$, $th_m = g_m$ and for $0 < i < m$, let $h_i \in H$ such that $d(g_i, th_i) \leq R$. Then $\{h_i\}_{i=0}^m$ is an r' -sequence in $\Gamma(H, Y)$ from a_1K to a_2K and hence there is some i such that $h_i \in S'$. Thus $th_i \in tS'$ and $g_i \in th_i\mathbb{B}(1, R)$.

It remains to show that $tS'\mathbb{B}(1, R)$ r -disconnects P . By hypothesis, there are $a_1, a_2 \in P'$ such that S' r' -disconnects them. Note that $ta_it^{-1} \in P$. Let $\{g_i\}_{i=0}^m$ be an r -sequence from ta_1t^{-1} to ta_2t^{-1} . Let $h_0 = a_1$, $h_m = a_2$, and for $1 < i < m$ let $h_i \in H$ such that $d(g_i, th_i) \leq R$. Arguing as before, since the r' -sequence $\{h_i\}_{i=0}^m$ meets S' , we conclude that $\{g_i\}_{i=0}^m$ meets $tS'\mathbb{B}(1, R)$. \square

We do not know if the Hucha property passes to finite index subgroups since it is not always the case that a left-order on a finite index subgroup can be extended to a left-order on the ambient group. We remark that in all the groups that we show that are Hucha, the property is inherited by its finite index subgroups.

However, we know that the Hucha property is not preserved under direct products.

Example 4.9. The free group of rank 2, F_2 has the Hucha property with respect to the trivial group. This follows from Corollary 1.2. It is known that the BNS-invariant $\Sigma^1(F_2 \times F_2)$ is non-empty. Concretely, the map that sends each standard generator of $F_2 \times F_2$ to $1 \in \mathbb{Z}$ has finitely generated kernel. Therefore, by Proposition 1.3, $F_2 \times F_2$ has a coarsely connected positive cone.

4.2 The Prieto property

We introduce a strong negation of the Hucha property.

Definition 4.10. Let G be a left-orderable group and X a finite generating set of G . We say that $\Gamma(G, X)$ is *Prieto* if every positive cone P of G is coarsely connected.

It is easy to see that Prieto is a geometric property of the group, and thus if a Cayley graph of G is Prieto, then every finitely generated Cayley graph of G is Prieto. We record this fact in the following lemma.

Lemma 4.11. Being Prieto is independent of the finite generating set.

We say that G is *Prieto* if there is some finite generating set Y of G such that $\Gamma(G, Y)$ is Prieto. The following is the analogous to Lemma 4.8 in the Prieto case. We omit the proof, since it is straightforward.

Lemma 4.12. Let G be a finitely generated left-orderable group, and H a finite index Prieto subgroup. Then G is Prieto.

The easiest example of a Prieto groups are finitely generated torsion free-abelian groups.

Proposition 4.13. Finitely generated free-abelian groups are Prieto.

Proof. We view the Cayley graph Γ of \mathbb{Z}^n with respect to a basis as a lattice in \mathbb{R}^n . It is well known (see [13, Section 1.2.1] and references therein) that for every positive cone P of \mathbb{Z}^n there exists a hyperplane π of \mathbb{R}^n passing through the origin, such that P consists on the lattice points in some connected component of $\mathbb{R}^n \setminus \pi$, and perhaps some points on π (if π contains lattice points). Therefore P is a connected subset of Γ . \square

To exhibit more involved examples, recall that given \prec a left-invariant order on a group G . A subgroup H of G is *cofinal* if for every $g \in G$, there are h_1, h_2 such that $h_1 \prec g \prec h_2$.

Lemma 4.14. Let (G, \prec) be a finitely generated left-ordered group. Suppose that there is a finitely generated, cofinal and central subgroup. Then $P_\prec = \{g \in G \mid 1 \prec g\}$ is coarsely connected.

Proof. Fix a finite generating set X of G . Let $g_0, g_m \in P_\prec$ and $\{g_i\}_{i=0}^m$ be a 1-sequence connecting them. Let Z be a finitely generated, cofinal and central subgroup of G . Since Z is cofinal, there is $z \in Z$ such that $z \prec g_i$ for $0 < i < m$. Without loss of generality, assume $z \prec 1$. Then $\{z^{-1}g_i\}_{i=0}^m$ is a 1-sequence contained in P . Since Z is finitely generated, it is free-abelian and hence Prieto. Thus there is some $r > 0$, such that $P_\prec \cap Z$ (with the induced metric) is r -connected. Thus there is an r -path $\{z_i\}_{i=0}^k$ supported in P_\prec from 1 to z^{-1} . Now the concatenation $\{g_0 z_i\}_{i=0}^k$ with $\{z^{-1}g_i\}_{i=0}^m$ followed by $\{z_i g_m\}_{i=k}^0$ is an r -sequence supported in P_\prec from g_0 to g_m . \square

Remark 4.15. The group $F_2 \times \mathbb{Z}$ is not a Hucha group neither a Prieto group. This will follow by considering different lexicographic orders. For showing that is not Hucha, we take the order on F_2 as the leading order in the lexicographic order. For showing that it is not Prieto, we consider the order on \mathbb{Z} as the leading one.

Indeed, from Corollary 1.2 we know that positive cones of F_2 are not coarsely connected, so, if we fix P a positive cone in F_2 , it is easy to check that $P' = (P \times \mathbb{Z}) \cup (\{1\} \times \mathbb{Z}_{\geq 1}) \subseteq F_2 \times \mathbb{Z}$ is positive cone for $F_2 \times \mathbb{Z}$ which not coarsely connected either.

If instead one takes $P' = (F_2 \times \mathbb{Z}_{\geq 1}) \cup (P \times \{0\})$ this is a positive cone for $F_2 \times \mathbb{Z}$ on which \mathbb{Z} is co-final, and by Lemma 4.14, P' is coarsely connected. It is worth noticing that not only $F_2 \times \mathbb{Z}$ has coarsely connected positive cones, but also it has finitely generated ones. This was recently showed by H. L. Su [32].

The braid group B_n can be defined as the mapping class group of a punctured disk with n punctures. E. Artin gave a presentation for B_n

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid (\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} : 1 \leq i \leq n-2), (\sigma_i \sigma_j = \sigma_j \sigma_i : |i-j| > 1) \rangle.$$

Iterated Torus knot groups are groups with presentation

$$T_{n_1, \dots, n_k} = \langle a_1, \dots, a_k \mid a_1^{n_1} = a_2^{n_2} = a_3^{n_3} = \dots = a_k^{n_k} \rangle.$$

Corollary 4.16. Braid groups, iterated Torus knot groups and $\tilde{T} = \langle a, b, c \mid a^2 = b^3 = c^7 = abc \rangle$ are Prieto groups.

Proof. The left-orderability of these groups is well-known. The orderability of B_n was first proved by P. Dehornoy (see [13, Section 1.2.6]). The observation that \tilde{T} is left-orderable is attributed to W. Thurston and rediscovered by G. Bergman (see [13, Discussion before Example 3.2.7]), while the left-orderability of T_{n_1, \dots, n_k} is discussed in [13, End of Section 2.2].

With the given presentation, the center of B_n is generated by $(\sigma_1 \sigma_2 \dots \sigma_{n-1})^2$. A. Clay showed that this cyclic group is cofinal in every left-order of B_n . See [13, Example 3.5.14.] for a proof. By Lemma 4.14 each positive cone is coarsely connected.

For iterated Torus knot groups the same holds: $z = a_1^{n_1}$ is always cofinal and central. The centrality is clear from the presentation. For the cofinality we argue as follows. Let \preceq be a left-order of T_{n_1, \dots, n_k} and set $U = \{t \in T_{n_1, \dots, n_k} \mid t \preceq z^m \text{ for some } m \in \mathbb{Z}\}$. We claim that $a_i U \subseteq U$ for all a_i . Indeed, let $t \in U$, $t \prec z^m$. If $a_i t \prec t \prec z^m$ then $a_i t \in U$ by definition. On the other hand if $t \prec a_i t$ then, inductively, we get that $t \prec a_i^{n_i} t = zt$, so $a_i t \prec z^{m+1}$. This implies that U is invariant under every generator, so $U = T_{n_1, \dots, n_k}$. In particular z is cofinal.

The preceding argument also shows that $a^2 \in \tilde{T}$ is always (central and) cofinal. \square

Another family of examples of groups with the Prieto property are left-orderable groups for which every order can be described by a regular language. This family is described in detail in [4].

In contrast with the Hucha property, we know that the Prieto property does not pass to finite index subgroups. Indeed, the previous corollary shows that B_3 is Prieto while the pure braid group PB_3 is a finite index subgroup isomorphic to $F_2 \times \mathbb{Z}$, which, by Remark 4.15, we know that is not Prieto.

We finish this section by showing that, unlike Hucha property (see Example 4.9), the Prieto property is stable under direct products.

Proposition 4.17. The direct product of Prieto groups is Prieto.

Proof. Let A and B be two finitely generated, left-orderable, Prieto groups. Fix a positive cone P of $G = A \times B$. Let X_A and X_B be finite generating sets for A and B respectively. We consider $X = X_A \times \{1\} \cup \{1\} \times X_B$ as generating set for G . Since A and B are Prieto, we assume that $P \cap A$ is an r -connected subset of $\Gamma(A, X_A)$. Similarly, we assume that $P \cap B$ is an r -connected subset of $\Gamma(B, X_B)$. Note that for simplicity we identify A with the subgroup $A \times \{1\}$ and similarly B with $\{1\} \times B$.

Let $g_1 = (a_1, b_1)$ and $g_2 = (a_2, b_2)$ be two positive elements. We will connect them with an r -path in $\Gamma(G, X)$ supported in P .

We can assume without loss of generality that $(a_1, 1_B)$ is in P . Indeed, if that was not the case, then $a_1^{-1} \in P \cup \{1\}$ and there is an r -path $\{h_i\}_{i=1}^n$ from 1_A to a_1^{-1} in $\Gamma(A, X_A)$ supported on $P \cap A$. Thus $\{(a_1 h_i, b_1)\}_{i=1}^n$ is an r -path in P from g_1 to $(1_A, b_1)$. Let $a' \in P \cap A$, $|a'|_X \leq r$. Then there is an r -path from g_1 to (a', b_1) supported in P and we can replace g_1 by (a', b_1) if necessary.

Repeating the argument, we assume that $a_1, a_2 \in P \cap A$ and $b_1, b_2 \in B \cap P$. Thus there are r -paths $\{y_i\}_{i=1}^m$ in $\Gamma(A, X_A)$ from a_1 to a_2 and $\{z_i\}_{i=1}^l$ in $\Gamma(B, X_B)$ from b_1 to b_2 both supported on positive elements. Thus $\{(y_i, b_1)\}_{i=1}^m$ concatenated with $\{(a_2, z_i)\}_{i=1}^l$ is an r -path from g_1 to g_2 . \square

5 Groups acting on trees

In this section we will show the main technical result of the paper, which is a combination theorem that produces Hucha groups. For that, we will first prove Proposition 5.3, which is a self contained proof of Proposition 1.1 in the special case of groups acting on trees. There are several reasons for giving this proof. Firstly, it is not lengthy and we produce a statement better adapted for our purposes, sparing some translation effort. Secondly, to show that certain groups are Hucha with respect to cyclic subgroups, we are using something slightly more general than what was proved on Proposition 1.1, namely that for any loxodromic isometry h of the tree, and any neighbourhood A of an end, there is a coset of $\langle h \rangle$ whose limit points lie in A .

We clarify that throughout this paper trees are simplicial trees, and actions on trees are by graph automorphisms that do not reverse edge orientations, which are the standard conventions when dealing with Bass-Serre theory.

The key of this section is the strong interplay between the geometries of T and of G , that arises from the action of G on T . We could say that the geometry of the tree dominates the geometry of the group. This allows us to construct negative swamps and to show that certain groups acting on trees are Hucha in Subsection 5.3.

5.1 Positive Cones of groups acting on trees

Let G be a group acting on a tree T . We recall the following classical definitions and well known facts about such an action. An element $g \in G$ that stabilizes a vertex of T is called T -elliptic, and if g is not T -elliptic then it is called T -loxodromic. If g is a T -loxodromic element, then there exists a unique subtree of T homeomorphic to \mathbb{R} on which g acts by translation and we denote it by $\text{axis}_T(g)$, see for example [14, Proposition I.4.11]. Then $\text{axis}_T(g)$ is the minimal $\langle g \rangle$ -invariant subtree of T . We will

drop the T from the notation when the action is clear from the context. We say that the action is *minimal* when T is the only G -invariant sub-tree.

A reduced path in T is a *geodesic* with respect to the combinatorial metric on T , that assigns length 1 to the edges. Given $v_1, v_2 \in T$ we denote $[v_1, v_2]_T$ the geodesic segment between them.

When a path in T is semi-infinite and reduced we will call it a *ray*. We say that two rays of T are *equivalent* if they have infinite intersection, and define an *end* of T as an equivalence class of rays. It is easy to see a correspondence between the ends of T and the Gromov boundary ∂T that comes from the combinatorial metric on T , though in this section we prefer to take the more specific approach of focusing on ends. We say that G *fixes an end of T* if there is a ray p in T such that $gp \cap p$ is infinite for all $g \in G$.

Recall from Section 3 that the action of a group G on a tree T is *of general type* if G does not fix neither a vertex nor an end of T , and the G -orbit of a vertex accumulates on at least 3 different ends of T . We see from Theorem 3.1 that a general type action on a tree always has loxodromic elements. A direct proof of this fact can be found in [14, Theorem I.4.12].

Example 5.1. For the purposes of the paper, the main examples of general type actions on trees are the following: Let G be either an amalgamated free product $A *_C B$, with C a proper subgroup of A and B , or an HNN extension $A *_C t$, with C a proper subgroup of A . Then the action of G on its associated Bass-Serre tree is of general type. If $G = A *_C B$, the Bass-Serre tree has vertex set $G/A \sqcup G/B$ and edge set G/C with gC adjacent to gA and gB . If $G = A *_C t$, the Bass-Serre tree has vertex set G/A and edge set G/C with gC adjacent to gA and gtA . The action of G on the Bass-Serre tree is induced by the left multiplication action of G on its subgroup's cosets. See [31] for a detailed discussion.

The following fact is well-known, we include the proof for completeness.

Lemma 5.2. Let $G \curvearrowright T$ be a general type, co-bounded action of a group G on a tree T . Let h be an element acting loxodromically on T . Then there are $h_1, h_2 \in h^G$, two G -conjugates of h , such that $\text{axis}_T(h_1) \cap \text{axis}_T(h_2)$ is empty.

Proof. Recall that for $g \in G$, one has that $g \text{axis}_T(h) = \text{axis}_T(ghg^{-1})$. Since T has more than two ends and the action of G is co-bounded, we see that there is some $g \in G$ such that $g \text{axis}_T(h) \neq \text{axis}_T(h)$. Moreover, since G does not fix an end of T , there must be g_1 and g_2 in G such that $g_1 \text{axis}_T(h) \cap g_2 \text{axis}_T(h) \cap \text{axis}_T(h)$ is finite (i.e. that $g_1 \text{axis}_T(h)$ and $g_2 \text{axis}_T(h)$ could only have infinite intersection with different ends of $\text{axis}_T(h)$). The action of $\langle h \rangle$ on $\text{axis}_T(g_1 h g_1^{-1})$ "slides" this axis along $\text{axis}_T(h)$, therefore we can find n such that $h^n \text{axis}_T(g_1 h g_1^{-1}) \cap \text{axis}_T(g_2 h g_2^{-1})$ is empty. Now we can take $h_1 = h^n g_1 h g_1^{-1} h^{-n}$ and $h_2 = g_2 h g_2^{-1}$. \square

We are ready to show that the orbit of a positive cone visits every neighbourhood of every end of T .

Proposition 5.3. Let $G \curvearrowright T$ be a co-bounded, general type action of a group G on a tree T . Let $h \in G$ be an element acting loxodromically on T , and v a vertex of T . Consider B a bounded subset of VT and \mathcal{C} an unbounded connected component of $T - B$.

Then there exists $k \in G$ such that $k\langle h \rangle v = \{kh^n v \mid n \in \mathbb{Z}\}$ lies in \mathcal{C} . Moreover, for every positive cone P of G the set Pv has non-trivial intersection with every unbounded connected component of $T - B$.

Proof. By Lemma 5.2 there exist loxodromic elements $h_1 = g_1 h g_1^{-1}$ and $h_2 = g_2 h g_2^{-1}$ such that $\text{axis}_T(h_1) \cap \text{axis}_T(h_2) = \emptyset$. Let \mathcal{O}_i be the orbit of $g_i v$ under $\langle h_i \rangle$, that is $\mathcal{O}_i = \{g_i h^n v \mid n \in \mathbb{Z}\} = g_i \langle h \rangle v$. Let L denote the minimal segment connecting \mathcal{O}_1 and \mathcal{O}_2 . Since the action of G on T is co-bounded, there is some $g \in G$ such that $gL \subseteq \mathcal{C}$. Moreover, we can assume that

$$d(gL, B) > D. \quad (2)$$

where D is an arbitrary constant, which we will choose later.

We claim that for a choice of g that makes D sufficiently large, either $g\mathcal{O}_1$ or $g\mathcal{O}_2$ is contained in \mathcal{C} . Suppose this is not the case, thus for $i = 1, 2$ there are $w_i \in B$ that lie in the geodesic between two consecutive vertices of $g\mathcal{O}_i$. To be more specific, for each $i = 1, 2$ there is some n such that $gg_i h^n v$ and $gg_i h^{n+1} v$ are in different components of $T - B$, thus the geodesic segment $[gg_i h^n v, gg_i h^{n+1} v]_T$ must meet B in at least a point w_i . Note that $d(gg_i h^n v, gg_i h^{n+1} v) = d(hv, v)$ for every $n \in \mathbb{Z}$, since G acts by isometries of the combinatorial metric. Therefore

$$d(g\mathcal{O}_1, g\mathcal{O}_2) \leq d(w_1, w_2) + 2d(v, hv) \leq \text{diam}(B) + 2d(v, hv).$$

On the other hand, we get from (2) that $d(w_1, w_2) \geq 2D + \text{diam}(L) - 2d(v, hv)$. Putting these two things together we get that $2D + \text{diam}(L) \leq \text{diam}(B) + 2d(v, hv)$. Since we can make D arbitrarily big, for an appropriate choice of g we get a contradiction. Thus either $g\mathcal{O}_1$ or $g\mathcal{O}_2$ in \mathcal{C} and this completes the proof of the first statement.

For the second statement, let P be a positive cone and \mathcal{C} an infinite component of $T - B$. Pick $k \in G$ so that $k\langle h \rangle v \subset \mathcal{C}$. We consider the geodesic segment $[v, kv]_T$ and act on it by $\langle khk^{-1} \rangle$, obtaining the segments $kh^n k^{-1}[v, kv]_T = [kh^n k^{-1} v, kh^n v]_T$. Since $kh^n v \in \mathcal{C}$ for all n , the set B is bounded and khk^{-1} is loxodromic, we see that all but finitely many of these segments lie inside \mathcal{C} . In particular, $kh^n k^{-1} v \in \mathcal{C}$ for all but finitely many $n \in \mathbb{Z}$.

Since either $khk^{-1} \in P$ or $kh^{-1}k^{-1} \in P$, we get that $\langle khk^{-1} \rangle v$ contains infinitely many elements of Pv . \square

The following is an strengthening of Corollary 1.2, and it will be used as the base case for showing that limit groups are Hucha.

Proposition 5.4. Finitely generated non-abelian free groups are Hucha with respect to the family of finitely generated, infinite-index subgroups.

Proof. Let F be a finitely generated non-abelian free group and X a finite generating set. Without loss of generality, by Lemma 4.6, we can assume that X is a basis and hence the Cayley graph of F is a tree T with infinitely many ends. The action of F on T is co-compact, free and of general type.

Fix a positive cone P , an infinite-index finitely generated subgroup H and $r \geq 0$. Denote by \prec the order on F corresponding to P . Let $x_n = \max_{\prec} \mathbb{B}(1_F, n)$ and recall that $x_n^{-1} \mathbb{B}(1_F, n-1) \subseteq P^{-1}$. Take $S = x_{r+1}^{-1} \mathbb{B}(1_F, r)$. Since T is a tree, there are two different components of $T - S$ so that every r -path joining them must intersect S . To be more specific, this happens when the geodesic joining these two complementary components passes through the ball's center x_{r+1}^{-1} . On the other hand, by Proposition

5.3 every connected component of $T - S$ contains positive elements. Thus P is not r -connected.

Every finitely generated subgroup of a free group is *quasi-convex* (geodesically, not with respect to the order \prec), therefore by [20, Theorem 4.8] we get that H has *bounded packing*, meaning that for every D there is some number $N = N(D)$ such that for any collection g_1H, \dots, g_NH of N distinct cosets of H , there are at least two of them separated by a distance of at least D .

So there exist cosets g_1H and g_2H at distance $> 2r$. Let γ be a geodesic path joining g_1H and g_2H and assume that v is a vertex in γ with $d(v, g_1H) > r$ and $d(v, g_2H) > r$. Since F acts transitively by isometries, we can assume that $v = x_{r+1}^{-1}$ and therefore, g_1H and g_2H are separated by $S = x_{r+1}^{-1}\mathbb{B}(1_F, r)$ where the geodesic connecting them passes through x_{r+1}^{-1} as we discussed before. Thus there is no r -path in P connecting g_1H and g_2H . \square

5.2 Cayley graphs of groups acting on trees

Consider a group G acting on a tree T . In this subsection we will show how to find generating sets for G which are *adapted* for the action on T (Lemma 5.5). These generating sets will be useful to relate the geometries of G and T .

For $v \in VT$ denote $\text{link}_T(v)$ the set of edges in ET which are adjacent to v . We say that a sub-tree $T_0 \subseteq T$ is a *fundamental domain* for the G -action on T if T_0 contains exactly one representative of every edge-orbit under the action. Also, given $v \in VT$ we denote G_v its stabilizer under the G -action, that is $G_v = \{g \in G : gv = v\}$. Analogously, for $e \in ET$, denote the edge stabilizer by G_e .

The following is the key definition for our discussion of adapted generating sets. Given X a generating set for G , $v \in VT$ and $E \subseteq \text{link}_T(v)$, we say that X has *v -reductions modulo E* if for every 1-path $\gamma: \{0, \dots, n\} \rightarrow \Gamma(G, X)$ with

- $\gamma(i)v \neq v$ for $0 < i < n$, and
- $\gamma(0)v = \gamma(n)v = v$,

we have that $\gamma(n) \in \gamma(0)G_e$ for some $e \in E$.

In other words, if a generating set X has v -reductions modulo E , then for every path in $\Gamma(G, X)$ leaving and then coming back to G_v there is an edge $e \in E \subseteq \text{link}_T(v)$ and an element $g \in G_v$ such that the path leaves from and comes back to (the same) coset gG_e . Informally, this is saying that cosets of edge stabilizers are check-points for paths in $\Gamma(G, X)$ passing through cosets of vertex stabilizers.

Next we show the existence of generators having v -reductions, as well as some other properties. The statement of the following lemma is implicit in Serre's book [31] but we include a full proof for completeness.

Lemma 5.5. Let G be a finitely generated group with a co-compact action on a tree T with finitely generated vertex stabilizers. Given a vertex $v \in VT$, there exists a compact fundamental domain T_0 containing v and a symmetric and finite generating set X satisfying:

- (1) $X \cap G_v$ generates G_v ,
- (2) $Gv \cap [v, sv]_T = \{v, sv\}$ for every $s \in X$,
- (3) $sT_0 \cap T_0$ is non-empty for every $s \in X$,

(4) There exists a finite subset $E \subseteq \text{link}_T(v)$ so that X has v -reduction modulo E .

Proof. First we construct the fundamental domain T_0 and the generating set X . After that we shall verify the properties (1)–(4).

Construction of T_0 : Let $A := G \backslash T$ be the quotient graph of T by the action of G . Let $\pi : T \rightarrow A$ be the associated quotient map, and consider $A_0 \subseteq A$ a maximal subtree of A . Let T'_0 be a lift of A_0 that contains v , and notice that it is a subtree $T'_0 \subseteq T$ so that π induces an isomorphism between T'_0 and A_0 .

Finally we extend T'_0 to a fundamental domain T_0 such that every edge in $ET_0 \setminus ET'_0$ has one endpoint in T'_0 . We do so by choosing lifts of the edges $\bar{e} \in EA \setminus EA_0$ that begin in a vertex of T'_0 . There are several choices on how to do this, and we can require one extra condition: that an edge $e \in ET_0$ is adjacent to v whenever $\pi(e)$ is adjacent $\pi(v)$, i.e. the edges $e \in ET_0$ are adjacent to v when that is possible. This is achieved by choosing a lift that begins at v for every $\bar{e} \in EA \setminus EA_0$ with $\pi(v)$ as an endpoint.

Note that T_0 is compact because, by hypothesis, the action is co-compact. It will also be important to notice that our construction yields that

$$E^* := \text{link}_{T_0}(v)$$

is a set of representatives for the orbits of edges that have some endpoint in the orbit of v .

Construction of X : For every $w \in VT'_0$ consider a finite and symmetric set X_w so that $\langle X_w \rangle = G_w$. We can do that because, by hypothesis, vertex stabilizers are finitely generated. Also, for every edge $e = [w_1, w_2]_T$ in $ET_0 \setminus ET'_0$ with $w_1 \in VT'_0$ we consider an element $g_e \in G$ so that $g_e^{-1}w_2 \in T'_0$. We define

$$X = (\cup_{w \in VT'_0} X_w) \cup \{g_e, g_e^{-1} : e \in ET_0 \setminus ET'_0\}.$$

Condition (1): It follows immediately from our definition of X , recalling that $v \in VT'_0$.

Conditions (2) and (3): Take $s \in X$. We distinguish two cases.

Case 1: $s \in G_w$ for some $w \in VT'_0$.

We write $[v, sv]_T \subseteq [v, w]_T \cup [w, sv]_T$ and notice that $[w, sv]_T = s[w, v]_T$. Since T'_0 meets every vertex orbit under G exactly once and $[v, w]_T \subseteq T'_0$, we can deduce that $[v, sv]_T \cap Gv = \{v, sv\}$, which proves condition (2) in this case. Since $w \in T_0 \cap sT_0$ condition (3) follows.

Case 2: $s \notin G_w$ for every $w \in VT'_0$.

Since the action is simplicial and T'_0 meets every vertex orbit exactly once, we see that if $sT'_0 \cap T'_0 \neq \emptyset$ then $s \in G_w$ for some $w \in VT'_0$. So we can assume that $sT'_0 \cap T'_0 = \emptyset$ for this case. We may assume without loss of generality that $s = g_e$ for some $e \in ET_0 \setminus ET'_0$, since the argument for $s = g_e^{-1}$ is analogous. Denote $e = [w_1, w_2]_T$ with $w_1 \in T'_0$. By construction of g_e we have that $w_2 \in g_e T'_0$, thus $w_2 \in T_0 \cap g_e T_0$ which gives condition (3).

To verify condition (2) note that, since e meets both T'_0 and $g_e T'_0$, we have that

$$[v, g_e v]_T \subseteq T'_0 \cup e \cup g_e T'_0.$$

So $Gv \cap [v, g_e v]_T \subset VT'_0 \cup Vg_e T'_0$. On the other hand, the orbit of the vertex v meets $VT'_0 \cup Vg_e T'_0$ exactly on v and $g_e v$, by construction of T'_0 . This gives condition (2) in this case.

Condition (4): We define the suitable set $E \subset \text{link}_T(v)$ by enlarging $E^* = \text{link}_{T_0}(v)$ as follows:

$$E = E^* \cup \{g_e^{-1}e : e \in E^* \text{ such that } g_e^{-1}e \in \text{link}_T(v)\}$$

By the choice of T_0 , this amounts to adding the edges of the form $g_e^{-1}e$ for $e \in ET'_0 \setminus ET'_0$ that have v as an endpoint. The added edges are always lifts of loops in $A = G \setminus T$ based at $\pi(v)$. It is clear that E is finite, since E^* is. The reason for this construction will become clear at the very end of the proof.

We will show that X has v -reduction modulo E . For this consider a 1-path $\{\gamma_i\}_{i=0}^n$ in $\Gamma(G, X)$ so that $\gamma_0 v = \gamma_n v = v$ and $\gamma_i v \neq v$ for $0 < i < n$. By condition (2) we have that $\{\gamma_i\}_{i=1}^{n-1}$ is supported on a single connected component of $T \setminus \{v\}$, whose closure (by adding v) we denote by T_* .

Let \bar{e} be the *unique* edge in ET_* that is adjacent to v . Recalling that E^* is a set of representatives of the edges with endpoints in Gv , we get that there is a unique $e \in E^*$ and some $g \in G$ with $\bar{e} = ge$.

In order to show that γ v -reduces modulo E , we will distinguish two cases. Let d_T be the combinatorial metric of the tree T .

Case 1: $d_T(v, \gamma_1 v) > 1$.

First we show that $\bar{e} = \gamma_0 e$. Since $d_T(v, \gamma_1 v) > 1$ we see that $\gamma_1 e$ is not adjacent to v , in particular $\gamma_1 e \neq \bar{e}$. Since T_0 is a fundamental domain, $\gamma_1 e$ is the only edge in the orbit of e that lies in $\gamma_1 T_0$, therefore \bar{e} does not belong to $\gamma_1 T_0$. On the other hand, condition (3) gives that $\gamma_0 T_0 \cup \gamma_1 T_0$ is connected and so it contains the geodesic segment $[v, \gamma_1 v]_T$. This segment is also contained in T_* , so its first edge is \bar{e} . Then we must have $\bar{e} \in \gamma_0 T_0$, which implies that $\bar{e} = \gamma_0 e$.

Now we point out that $d_T(\gamma_{n-1} v, v) > 1$. Indeed, if it was not the case we would have that $\gamma_{n-1} v$ would belong to the interior of $[\gamma_0 v, \gamma_1 v]_T$ contradicting condition (2). Since $d_T(\gamma_{n-1} v, v) > 1$, we can repeat the previous argument and show that $\bar{e} = \gamma_n e$. This implies that $\gamma_0^{-1} \gamma_n \in G_e$. Therefore, since $e \in E$, condition (4) holds in this case.

Case 2: $d_T(v, \gamma_1 v) = 1$.

Arguing as in the previous case we can use condition (2) to show that $d_T(\gamma_{n-1} v, v) = 1$. Since $\{\gamma_i\}_{i=0}^n$ is a 1-path in $\Gamma(G, X)$ there are $s_1, s_n \in X$ such that $\gamma_1 = \gamma_0 s_1$ and $\gamma_n = \gamma_{n-1} s_n$.

We claim that $s_1, s_n \in \{g_e, g_e^{-1}\}$. We prove it for s_1 , since the other case is analogous. For this, first notice that

$$d_T(v, s_1 v) = 1. \tag{3}$$

This tells us that s_1 is not elliptic, because whenever $h \in G$ is elliptic we have that $d_T(v, hv)$ is even. Therefore $s_1 = g_{e_0}^{\pm 1}$ for some $e_0 \in T_0 \setminus T'_0$. Moreover, the edge

$[v, s_1 v]_T$ is either e_0 or $g_{e_0}^{-1}e_0$, and in any case we have $e_0 \in E^*$. On the other hand $\bar{e} = [v, \gamma_1 v]_T = \gamma_0[v, s_1 v]_T = g_{e_0}$ for either $g = \gamma_0$ or $g = \gamma_0 g_{e_0}^{-1}$. This gives us $e_0 = e$, and therefore the claim.

It will be useful to record that either $\bar{e} = \gamma_0 e$ with $s_1 = g_e$, or $\bar{e} = \gamma_1 e$ with $s_1 = g_e^{-1}$. The same argument would yield that \bar{e} is either $\gamma_n e$ or $\gamma_{n-1} e$, according to whether s_n is g_e^{-1} or g_e .

Now, in order to finish the proof that condition (4) holds, we will distinguish in two sub-cases according to $s_1 = g_e$ or $s_1 = g_e^{-1}$.

Subcase 2A: $s_1 = g_e$.

In this case we have that $\bar{e} = \gamma_0 e$. We argue that $\bar{e} = \gamma_n e$ by contradiction: Assuming the contrary we would have $\bar{e} = \gamma_{n-1} e$, but then $\gamma_{n-1}^{-1} \gamma_0$ would fix the edge e without fixing the endpoint v . This is not possible, since the action has no edge inversions. Thus we get that $\bar{e} = \gamma_n e$ and so $\gamma_0^{-1} \gamma_n \in G_e$. Recall that $e \in E$, so this shows condition (4) in this instance.

Subcase 2B: $s_1 = g_e^{-1}$.

Now we have that $\bar{e} = \gamma_1 e$, and we can show that $\bar{e} = \gamma_{n-1} e$ by the same argument used in the previous case: Supposing that $\bar{e} = \gamma_n e$ would yield that $\gamma_n \gamma_1^{-1}$ induces an edge inversion on e . As previously mentioned, this allows us to deduce that $s_n = g_e$.

So we get that $\bar{e} = \gamma_{n-1} e = \gamma_n g_e^{-1} e$, and recalling that $\bar{e} = \gamma_1 e = \gamma_0 g_e^{-1} e$ we deduce that $\gamma_0^{-1} \gamma_n \in G_{g_e^{-1} e}$. Since $g_e^{-1} e \in E$ this finishes the proof of condition (4). \square

If T_0 is a sub-tree of a tree T , and v is a vertex of T_0 , we call the pair (T_0, v) a *pointed subtree* of T .

Definition 5.6. Let G be a group acting on a tree T with the hypothesis of Lemma 5.5. If a generating set X of G satisfies the conclusions (1)–(4) of Lemma 5.5 for a pointed sub-tree (T_0, v) , we will say that X is *adapted to* (T_0, v) . Lemma 5.5 shows that adapted generators always exist.

Remark 5.7. If X is a generating set of G adapted to (T_0, v) , then $\Gamma(G, X) \setminus G_v$ is not connected. Indeed, if $g_1 v$ and $g_2 v$ are in different components of $T \setminus \{v\}$, then for every path $\{\gamma_i\}_{i=0}^n$ in $\Gamma(G, X)$ joining g_1 and g_2 , by condition (2) in Lemma 5.5 there exist i such that $\gamma_i v = v$.

5.3 Creating Hucha groups

Given a geodesic segment $p = [w_1, w_2]_T$ in T , denote by $T_{w_1}^p$ the minimal subtree of T that contains w_1 and the connected components of $T \setminus \{w_1\}$ that do not contain w_2 . Clearly $T_{w_2}^p$ is defined as well. Given $v \in VT$ let $\mathcal{C}_{w_i}^p \subset G$ be the pre-image of $T_{w_i}^p$ under the orbit map of v . That is, for $i = 1, 2$, define

$$\mathcal{C}_{w_i}^p = \{g \in G : gv \in T_{w_i}^p\}.$$

The marked point v is absent from the notation, as it will be clear from the context. We may occasionally drop the p from the notation as well. Our next immediate objective is to reduce the problem of proving the Hucha property with respect to cyclic subgroups, to showing that negative cones r -disconnect subsets of the form $\mathcal{C}_{w_i}^p$. This is stated precisely in the following:

Lemma 5.8. Let $G \curvearrowright T$ be a co-compact, minimal, general type action of a left-orderable group G on a tree T . Let $v \in VT$ and X be a generating set of G . Assume that for every positive cone P and any $r > 0$ there exists a geodesic segment $p = [v_1, v_2]_T$ in T and a set $S \subset P^{-1}$ so that S r -disconnects $\mathcal{C}_{v_1}^p$ from $\mathcal{C}_{v_2}^p$ inside $\Gamma(G, X)$.

Then G has the Hucha property with respect to subgroups acting elliptically on T and cyclic subgroups acting loxodromically. In particular, G has the Hucha property with respect to all cyclic subgroups.

Proof. Let P be a positive cone in G , H be either an elliptic subgroup or a cyclic subgroup acting loxodromically, and $r > 0$. In order to prove the desired Hucha property, we need to find a negative swamp in P^{-1} of width r for H . For this we consider the geodesic segment $p = [v_1, v_2]_T$ and the subset $S \subseteq P^{-1}$ provided by the hypothesis, so that S r -disconnects \mathcal{C}_{v_1} from \mathcal{C}_{v_2} in $\Gamma(G, X)$.

We aim to show that S is our negative swamp. For this it is enough to show that:

1. $P \cap \mathcal{C}_{v_i}$ is non-empty for $i = 1, 2$,
2. there are cosets $h_i H$ for $i = 1, 2$, such that $h_i H$ is contained in \mathcal{C}_{v_i} .

Since T is an infinite tree with no leaves, we see that T_{v_i} are unbounded for $i = 1, 2$. On the other hand, the action is of general type, so there is some $h \in G$ acting loxodromically on T . Then we may apply Proposition 5.3 to conclude that Pv intersects both T_{v_1} and T_{v_2} . This implies that $P \cap \mathcal{C}_{v_i}$ is non-empty for $i = 1, 2$, proving point 1.

We will split the proof of point 2 into cases, according to the action of H on T .

If H is elliptic, there exists $w \in VT$ so that $H \subseteq G_w$. Let d_T be the combinatorial metric on T , take $R = d_T(v, w)$ and fix $i = 1, 2$. Notice that since T_{v_i} is unbounded and the action is co-compact, there is $h_i \in G$ so that $h_i w \in T_{v_i}$ and $d_T(h_i w, v_i) > R$. That is to say that the R -ball in T around $h_i w$ is contained in T_{w_i} . Since $h_i H w = h_i w$ we get that $h_i H v$, whose points lie at distance R from $h_i w$, is contained in T_{v_i} . Therefore $h_i H \subseteq \mathcal{C}_{v_i}$ as desired.

If H is infinite cyclic acting loxodromically, we are in condition to apply Proposition 5.3 to show that there exist $h_1, h_2 \in G$ so that $h_i H v \subseteq T_{v_i}$ for $i = 1, 2$. This implies that $h_i H \subseteq \mathcal{C}_{v_i}$ for $i = 1, 2$, proving point 2. \square

We proceed to prove our combination theorems, starting with the easier case where the action has an edge with trivial stabilizer. It will be useful to consider *geodesic interpolations* of r -paths in $\Gamma(G, X)$. Namely, given an r -path $\gamma: \{0, \dots, n\} \rightarrow \Gamma(G, X)$ and a 1-path $\bar{\gamma}: \{0, \dots, m\} \rightarrow \Gamma(G, X)$, we say that $\bar{\gamma}$ is a *geodesic interpolation* of γ if there exists a monotone map $\sigma: \{0, \dots, n\} \rightarrow \{0, \dots, m\}$ satisfying:

- $\gamma = \bar{\gamma} \circ \sigma$
- $\{\bar{\gamma}_j\}_{j=\sigma(i)}^{\sigma(i+1)}$ is a geodesic for $i = 0, \dots, n - 1$.

Proposition 5.9. Suppose that $G \curvearrowright T$ is a co-compact, minimal, general type action of a finitely generated left-orderable group G on a tree T . Assume there exists $e \in ET$ with trivial stabilizer and that vertex stabilizers are finitely generated. Then G has the Hucha property with respect to cyclic subgroups and subgroups acting elliptically on T .

Proof. By Lemma 5.5, we can consider a finite generating set X and a finite pointed sub-tree (T_0, v) so that X is adapted to (T_0, v) . Take \prec a left-order on G with positive cone P and $r > 0$. We will find a subset $S \subseteq P^{-1}$ and a geodesic segment $[w_1, w_2]_T$ so

that S r -disconnects \mathcal{C}_{w_1} from \mathcal{C}_{w_2} in $\Gamma(G, X)$. Then the proposition will follow from Lemma 5.8.

Take $g \in G$ such that $\max_{\prec} \mathbb{B}_X(1, r) \prec g^{-1}$, so we have that $g\mathbb{B}_X(1, r) \subseteq P^{-1}$. Since T_0 is a fundamental domain we have $e_0 \in ET_0$ with trivial stabilizer. We set $[w_1, w_2]_T := ge_0$ and $S := g\mathbb{B}_X(1, r)$, and claim that S r -disconnects \mathcal{C}_{w_1} from \mathcal{C}_{w_2} .

To see this, take $\gamma = \{\gamma_i\}_{i=0}^n$ an r -path between $h_1 \in \mathcal{C}_{w_1}$ and $h_2 \in \mathcal{C}_{w_2}$, and consider $\bar{\gamma} = \{\bar{\gamma}_i\}_{i=0}^m$ a geodesic interpolation of γ . By condition (3) in Lemma 5.5 we have that $\bar{\gamma}_{i+1}T_0 \cap \bar{\gamma}_iT_0 \neq \emptyset$ for every i , thus $ge_0 \subset \bar{\gamma}_iT_0$ for some i . Since e_0 has trivial stabilizer, this implies that $\bar{\gamma}_i = g$ and therefore γ intersects $g\mathbb{B}_X(1, r)$ as desired. \square

Corollary 5.10. Let A, B be non-trivial, finitely generated left-orderable groups. Then $G = A * B$ has the Hucha property with respect to A, B and all cyclic subgroups.

Proof. Consider $G \curvearrowright T$ the standard action of G on the Bass-Serre Tree associated to the free product. The edge stabilizers of this action are trivial, so we can apply Proposition 5.9 which gives the corollary. (Recall that A and B are elliptic in this action). \square

We now move towards the general case where the edge stabilizers are Prieto. The following is a preliminary technical lemma to that objective. If X is generating set adapted to (T_0, v) , we adopt the notation

$$X_v = X \cap G_v$$

Recall that this set generates G_v . It is illustrative to note that $\Gamma(G_v, X_v)$ is a connected subgraph of $\Gamma(G, X)$, and thus if $g_1, g_2 \in G_v$ we have $d_X(g_1, g_2) \leq d_{X_v}(g_1, g_2)$.

Lemma 5.11. Suppose that $G \curvearrowright T$ is a co-compact, minimal, general type action of a finitely generated left-orderable group G on a tree T . Let v be a vertex on a compact subtree T_0 , and X a finite generating set of G adapted to (T_0, v) with v -reductions modulo a finite set $E \subseteq \text{link}_T(v)$. Let P be a positive cone of G , and assume that there exists $r_0 > 0$ so that $P \cap G_e$ is r_0 -connected in $\Gamma(G_v, X_v)$ for every $e \in E$.

Then, given an r -path $\gamma = \{\gamma_i\}_{i=0}^n$ in $\Gamma(G, X)$ between g_1 and g_2 in G_v we have one of the following:

1. either γ meets $P^{-1}\mathbb{B}(1, r)$,
2. or there exists $\delta = \{\delta_i\}_{i=0}^m$ a r_0 -path in $\Gamma(G_v, X_v)$ supported on $P \cup \{1_G\}$ and joining g_1 with g_2 .

Proof. Take $\bar{\gamma}$ a geodesic interpolation of γ in $\Gamma(G, X)$. Since γ is an r -path, if $\bar{\gamma}$ meets P^{-1} then γ meets $P^{-1}\mathbb{B}(1, r)$ and we are done. Suppose it is not the case, so $\bar{\gamma}$ is supported on $P \cup \{1_G\}$. Then we will find a positive r_0 -path in $\Gamma(G_v, X_v)$ with the same endpoints as γ .

We say that an r_0 -sequence $\delta = \{\delta_i\}_{i=0}^n$ in $\Gamma(G, X)$ is admissible if it is a path supported on $P \cup \{1_G\}$ with $\delta_0v = \delta_nv = v$ and satisfying that if $d_X(\delta_i, \delta_{i+1}) > 1$ then δ_i and δ_{i+1} are in G_v and $d_{X_v}(\delta_i, \delta_{i+1}) \leq r_0$. Informally speaking, δ may only jump inside G_v and when it does, at most at distance r_0 with respect to X_v . Clearly $\bar{\gamma}$ is such an admissible path. We define the defect of δ as $D(\delta) = \#\{i : \delta_iv \neq v\}$. Notice that δ is supported in G_v if and only if $D(\delta) = 0$. Also notice that point (2) in the

statement can be rephrased as: there exists an admissible path δ joining g_1 with g_2 with $D(\delta) = 0$.

We will show a procedure to reduce the defect of admissible r_0 -paths with positive defect, while preserving the endpoints. Suppose that δ is admissible with positive defect. Since $D(\delta) > 0$ there must exist $0 \leq i < j \leq n$ such that $\delta_i v = \delta_j v = v$ and $\{\delta_k\}_{k=i+1}^{j-1}$ is a 1-path not meeting G_v . Since X has v -reductions modulo E , we deduce that $\delta_i^{-1} \delta_j \in G_e$ for some $e \in E$. Then either $\delta_i^{-1} \delta_j$ or $\delta_j^{-1} \delta_i$ belong to $G_e \cap P$ (it is clear we may assume $\delta_i \neq \delta_j$). Suppose that $\delta_i^{-1} \delta_j \in G_e \cap P$. Then, by hypothesis, we have that there exists a r_0 -path $\alpha = \{\alpha_k\}_{k=0}^m$ in $\Gamma(G_v, X_v)$ supported on $P \cup \{1_G\}$ and joining 1_G with $\delta_i^{-1} \delta_j$. Define β as the concatenation of the admissible r_0 -paths $\{\delta_k\}_{k=0}^i$ followed by $\{\delta_i \alpha_k\}_{k=0}^m$, and finally followed by $\{\delta_k\}_{k=j}^n$. It is straightforward to check that β is an admissible path with strictly smaller defect. It is also clear that the same argument works for the case when $\delta_j^{-1} \delta_i \in G_e \cap P$.

Starting with $\bar{\gamma}$ and repeating this procedure finitely many times, we obtain an admissible path with zero defect, thus finishing the proof. \square

We are now ready for our main Theorem.

Theorem 5.12. Suppose that $G \curvearrowright T$ is a co-compact, minimal, general type action of a finitely generated left-orderable group G on a tree T . Suppose further that all vertex stabilizers are finitely generated.

Consider $v \in VT$ and $H \in \mathcal{P} = \{G_e : e \in \text{link}_T(v)\}$, and assume that G_v is Hucha with respect to H and all the groups of \mathcal{P} are Prieto. Then G is Hucha with respect to the family of its cyclic subgroups, and of the subgroups acting elliptically on T .

Proof. By Lemma 5.5 we can consider a generating set X and a compact pointed tree (T_0, v) so that X is adapted to (T_0, v) . Consider P a positive cone of G with corresponding left-order \prec , and take $r > 0$. We will find a geodesic segment $[w_1, w_2]_T$ in T and a subset $S \subset P^{-1}$ so that S r -disconnects \mathcal{C}_{w_1} from \mathcal{C}_{w_2} in $\Gamma(G, X)$. Since P and r are arbitrary, we will be able to conclude the proof by applying Lemma 5.8.

Consider $h \in G$ so that $g \prec h$ for every $g \in \mathbb{B}(1, 2r)$. So we have $h^{-1} \mathbb{B}(1, 2r) \subset P^{-1}$, and we can also deduce that $h^{-1} (h P^{-1} h^{-1}) \mathbb{B}(1, 2r) \subseteq P^{-1}$. Denote \prec_* the left-order on G with positive cone $P_* := h P h^{-1}$, and re-write the last statement to get

$$h^{-1} P_*^{-1} \mathbb{B}(1, 2r) \subseteq P^{-1}$$

By definition of an adapted generating set, there exists a finite subset $E \subseteq \text{link}_T(v)$ so that X has v -reduction modulo E . Since E is finite and the subgroups in \mathcal{P} are Prieto, there exists $k_0 > 0$ so that $P_* \cap G_e$ is k_0 -connected in $\Gamma(G_v, X_v)$ for every $e \in E$. On the other hand, since G_v is Hucha with respect to H , we can find $a_1, a_2 \in G_v$ so that $P_*^{-1} \cap G_v$ k_0 -disconnects $a_1 H$ from $a_2 H$ inside $\Gamma(G_v, X_v)$.

Take an edge $e = [v, w]_T$ with $G_e = H$ and denote $a_1 e = [v, w_1]_T$ and $a_2 e = [v, w_2]_T$. Consider the geodesic segment $[w_1, w_2]_T$.

Claim: $P_*^{-1} \mathbb{B}(1, 2r)$ r -disconnects \mathcal{C}_{w_1} from \mathcal{C}_{w_2} .

To see this, take $h_i \in \mathcal{C}_{w_i}$ for $i = 1, 2$ and an r -path $\gamma = \{\gamma_i\}_{i=0}^m$ in $\Gamma(G, X)$ joining them. Consider $\bar{\gamma} = \{\bar{\gamma}_i\}_{i=0}^m$ a geodesic interpolation of γ . Let $\sigma: \{0, \dots, m\} \rightarrow \{0, \dots, n\}$ such that $\gamma_i = \bar{\gamma}_{\sigma(i)}$.

Since X is adapted to (T_0, v) we have that $\bar{\gamma}_{i+1} T_0 \cap \bar{\gamma}_i T_0 \neq \emptyset$ for $i = 0, \dots, m-1$. Then the definition of the \mathcal{C}_{w_i} implies that there exist $0 < i < j < n$ so that $a_1 e \in \bar{\gamma}_i T_0$

and $a_2e \in \bar{\gamma}_j T_0$. Note that this implies that $\bar{\gamma}_i \in a_1 H$ and $\bar{\gamma}_j \in a_2 H$. If either $\bar{\gamma}_i$ or $\bar{\gamma}_j$ are in $P_*^{-1}\mathbb{B}(1, r)$ then γ meets $P_*^{-1}\mathbb{B}(1, 2r)$ and the claim follows. Assume the contrary, and let p (resp. q) be the smallest (resp. greatest) integer such that $\sigma(p) \geq i$ (resp. $\sigma(q) \leq j$). Consider the r -path $\beta = \{\beta_k\}_{p-1}^{q+1}$ with $\beta_{p-1} = \bar{\gamma}_i$, $\beta_{q+1} = \bar{\gamma}_j$ and $\beta_k = \gamma_k$ for $p \leq k \leq q$. Note that β is an r -path from $\bar{\gamma}_i \in a_1 H$ to $\bar{\gamma}_j \in a_2 H$, which are both included in G_v . We shall apply Lemma 5.11 to the cone P_* and the r -path β , noticing that taking $r_0 = k_0$ satisfies the hypotheses, as we deduced from the Prieto property of the subgroups in \mathcal{P} . On the other hand, the choice of a_1 and a_2 coming from the Hucha property of G_v with respect to H , prevents the existence of a k_0 -path in $\Gamma(G_v, X_v)$ supported on $P_* \cup \{1\}$ joining $\bar{\gamma}_i \in a_1 H$ with $\bar{\gamma}_j \in a_2 H$. Thus point 1 in Lemma 5.11 holds, giving that β meets $P_*^{-1}\mathbb{B}(1, r)$. Since we are assuming the endpoints $\beta_{p-1} = \bar{\gamma}_i$ and $\beta_{q+1} = \bar{\gamma}_j$ are not in $P_*^{-1}\mathbb{B}(1, r)$, we deduce that γ meets $P_*^{-1}\mathbb{B}(1, r) \subset P_*^{-1}\mathbb{B}(1, 2r)$. This concludes the claim. \diamond

Finally, from the claim we have that $h^{-1}P_*^{-1}\mathbb{B}(1, 2r)$ r -disconnects the sets $\mathcal{C}_{h^{-1}w_1}$ and $\mathcal{C}_{h^{-1}w_2}$ associated to the segment $[h^{-1}w_1, h^{-1}w_2]_T$. Since $h^{-1}P_*^{-1}\mathbb{B}(1, r) \subseteq P_*^{-1}$ the theorem follows from Lemma 5.8. \square

Remark 5.13. Note that $F_2 \times \mathbb{Z}$ is the amalgamated free product of two groups isomorphic to \mathbb{Z}^2 along an infinite cyclic subgroup. We know that this group is not Hucha nor Prieto (Remark 4.15), thus the fact that at least one vertex stabilizer is Hucha is essential in Theorem 5.12.

Corollary 5.14. Let A be Hucha with respect to $C \leq A$, where C is a proper Prieto subgroup. For any group B such that $G = A *_C B$ is left-orderable, G is Hucha with respect to $\{A, B\}$ and the collection of the cyclic subgroups.

Proof. The proof is analogous to that of Corollary 5.10. \square

By a result of Bludov and Glass [7], the amalgamation of left-orderable groups over a cyclic subgroup is again left-orderable.

Corollary 5.15. Let A be Hucha with respect to $C \leq A$ with C cyclic and let B be a left-orderable group. Then $A *_C B$ is Hucha with respect to A and B and the collection of infinite cyclic subgroups.

A one-relator group G is called *cyclically pinched* if $G = F_1 *_{\langle a=b \rangle} F_2$ with F_1, F_2 finitely generated free groups, $a \in F_1 - \{1\}$ and $b \in F_2 - \{1\}$.

Corollary 5.16. Cyclically pinched (≥ 3)-generated one-relator groups are Hucha with respect to the collection of infinite cyclic subgroups.

6 A family of Hucha groups

In this section we prove Theorem 1.8. Let \mathfrak{H}_0 be the family of non-abelian finitely generated free groups. For $i > 0$, let \mathfrak{H}_i be the closure under free products of the family consisting of finitely generated non-abelian subgroups, of groups of the form $G *_C A$ where $G \in \mathfrak{H}_{i-1}$, C is a cyclic centralizer subgroup of G and A is finitely generated abelian.

Thus, the family \mathfrak{H} of Definition 1.7, is equal to $\cup_{i \geq 0} \mathfrak{H}_i$.

Proposition 6.1. If $G \in \mathfrak{H}$ then G is locally indicable (and hence left-orderable).

Proof. Since every finitely generated subgroup of a free group is free, the groups in \mathfrak{H}_0 are locally indicable. It is easy to see that local indicability is preserved under free products and taking subgroups. So it remains to show that if $G \in \mathfrak{H}_n$, $C = \langle c \rangle$ is the centralizer of some $c \in G$ and A is a finitely generated abelian group, then $G *_C A$ is locally indicable.

Let H, K be locally indicable groups and $r \in H * K$ an element acting loxodromically on the Bass-Serre tree of the free product. Howie [19] showed that a one-relator quotient $H * K / \langle\langle r \rangle\rangle$ is again locally indicable if and only if $r \in H * K$ is not a proper power (see [3, Appendix] for an alternative proof). Since $G *_C A = G * A / \langle\langle ca^{-1} \rangle\rangle$ for some $a \in A - \{1\}$, and the element ca^{-1} acts loxodromically (on the Bass-Serre tree of the free product) and it is not a proper power, we deduce that $G *_C A$ is locally indicable. \square

A group G is called *CSA* if all its maximal abelian subgroups are malnormal, that means that if A is a maximal abelian subgroup of G and $g \in G$, then $gHg^{-1} \cap A \neq \{1\}$ implies $g \in A$.

In [27] define the class CSA^* as the CSA groups with no elements of order 2. It is proved that CSA^* is closed under free products [27, Theorem 4], and the following construction: if L is CSA^* , C is a centralizer of an element of L and A is torsion-free finitely generated abelian, then $L *_C A$ is CSA^* [27, Theorem 5]. Since CSA clearly passes to subgroups and free groups are CSA we have.

Lemma 6.2. If $G \in \mathfrak{H}$ then G is CSA.

Recall that a simplicial G -tree is k -acylindrical if the fixed point set of any $g \in G$ has diameter at most k (i.e. any set of diameter $> k$ has trivial stabilizer).

Lemma 6.3. If T is the Bass-Serre tree of $G = A *_C B$ with C malnormal in A then the action is 2-acylindrical.

Proof. Indeed, the stabilizer of an edge $e \in ET$ is a conjugate of C (we can assume by G -equivariance that it is C). Let v be the vertex adjacent to e with stabilizer L . All edges adjacent to v have stabilizer lCl^{-1} with $l \in L \setminus C$. By malnormality of C in L , $C \cap lCl^{-1}$ is trivial. Thus C does not fix any other edge adjacent to v different from e . Now any subset of diameter ≥ 3 , up to G -equivariance, contains a path of length ≥ 2 that has v as an internal vertex and hence it has trivial stabilizer. \square

We need the following fact.

Lemma 6.4. [12, Lemma 2] Let G be a finitely generated group acting co-finitely on a tree T . If the stabilizers of edges adjacent to $v \in VT$ are finitely generated, then the stabilizer of v is finitely generated.

We can prove now our main theorem.

Proof of Theorem 1.8. We will prove by induction that groups in $\cup_{i=0}^n \mathfrak{H}_i$ are Hucha with respect to the family of cyclic centralizers. The base case of induction ($n = 0$) is a consequence of Proposition 5.4. Thus assume that the induction hypothesis hold for $n > 0$. Have to prove the case $n + 1$.

Let $L \in \mathfrak{H}_n$, $C \leq L$ a subgroup of a cyclic centralizer and A a finitely generated abelian group. Without loss of generality we assume that C is a proper subgroup of A (since if not $L *_C A = L$). Let T be the associated Bass-Serre tree to $L *_C A$. By

Lemma 6.2, C is malnormal in L , which implies (Lemma 6.3) that the action on the T is 2-acylindrical. In particular, if H is a subgroup $L *_C A$ with elements acting loxodromically on T and such that the minimal H -tree is homeomorphic to \mathbb{R} , then it must be virtually cyclic (see for example [28, Theorem 1.1.]), and since $L *_C A$ is torsion-free (Proposition 6.1), H must be cyclic.

Let $G \leq L *_C A$ be a non-abelian finitely generated subgroup. We have to show that G is Hucha with respect to its cyclic centralizers.

If G is, up to conjugation, a subgroup of L . Then, by induction hypothesis G is Hucha with respect to its cyclic centralizers. So we restrict to the case where G is not contained, up to conjugation, in L . Note that, since G is non-abelian, it is not contained, up to conjugation, in A .

Thus the action of G on the Bass-Serre tree of $L *_C A$ has no-global fixed point. By [14, Proposition I.4.11] G contains elements acting loxodromically on T . Consider $T' \subseteq T$ the minimal G -invariant subtree of T . This is the subtree consisting on the union of the axis of the elements of G acting loxodromically. Since G is finitely generated, $G \backslash T'$ is finite (see [14, Proposition I.4.13]). By the previous discussion, if T' has two ends, G must be cyclic, contradicting that G is non-abelian. In fact, since the action of G of T' is acylindrical, any element acting loxodromically on T has cyclic stabilizer. Finally, since $L *_C A$ does not fix any end of T , G does not fix any end of T' . Thus, the action of G on T' is of general type and co-compact, with edge stabilizers that are trivial or cyclic, and vertex stabilizers that are isomorphic to subgroups of A or subgroups of L . Moreover, by Lemma 6.4 the vertex groups are finitely generated.

If G acts on T' with trivial vertex stabilizers, then G is free, and thus $G \in \mathfrak{H}_0$ and there is nothing to prove.

Suppose that G -stabilizer of all vertices of T' are non Hucha. Thus, all G -stabilizers of all vertices of T' must be abelian, since by induction, all non Hucha finitely generated subgroups of L are the abelian ones. If some edge has trivial stabilizer, by Proposition 5.9, G is Hucha with respect to the subgroups acting loxodromically on T' and the subgroups of the vertex stabilizers. In particular it is Hucha with respect to the cyclic centralizers. Thus we can assume that all edge stabilizer are non-trivial and cyclic. Let e be an edge of T' adjacent to u , and suppose that the stabilizer of u is a conjugate of a subgroup of L . Since G_u is abelian and $G_e \leq G_u$ is a centralizer of some element (restricted to G_u) we get that $G_u = G_e$. We can G -equivariantly collapse the edge e and we will still have an action on a tree with the previous properties, and we could repeat this argument and continue G -equivariantly collapsing edges, obtaining that G is abelian, a contradiction.

Thus the only case remaining is when there is some $v \in T'$ with G_v an Hucha group. Thus G_v must be (up to conjugation) a finitely generated subgroup of L and thus, by induction, it is Hucha with respect to all its cyclic centralizers. Since the stabilizers of edges adjacent to v are either trivial or Prieto, we can use Theorem 5.12 to conclude that G is Hucha with respect to all its cyclic centralizers. \square

Application to limit groups. There are several equivalent definitions in the literature for *limit groups*. See for example [11] for a nice survey. The quickest definition is that G is a limit group if it is a finitely generated fully residually free group, meaning that for any finite set $S \subseteq G$ there is a homomorphism $\phi: G \rightarrow F$ where F is a free group and ϕ is injective restricted to S . However, to show that limit groups belong to \mathfrak{H} we need a different characterization.

Let H be a group and Z the centralizer of some non-trivial element of H . A *free extension of H by the centralizer Z* is a group of the form $H *_Z (Z \times A)$ for some finitely generated free abelian group A .

Let ICE_0 denote the class of finitely generated free groups. For $i > 0$, let ICE_i denote the groups that are free extensions by cyclic centralizers of groups in ICE_{i-1} . Finally let ICE denote the union of ICE_i $i \in \mathbb{Z}_{\geq 0}$.

An important and very useful result for us is the following.

Theorem 6.5. [22, Thm 4.]. The following are equivalent

1. G is a limit group
2. G is a finitely generated subgroup of a group in ICE

Clearly, in view of Theorem 6.5 we have that non-abelian limit groups are in \mathfrak{H} .

Application to free \mathbb{Q} -groups. Recall that H is a \mathbb{Q} -group if for every $h \in H$ and all $n \in \mathbb{N}$ there is a unique $g \in H$ such that $g^n = h$. This allows to define an action of \mathbb{Q} on H , and we denote the image of the action of $\alpha \in \mathbb{Q}$ on $h \in H$ by h^α . If $\alpha = p/q$ then h^α is g^p where g is the unique element such that $g^q = h$. It is easy to check that indeed this is an action.

A \mathbb{Q} -group $G^\mathbb{Q}$ together with an homomorphism $\phi: G \rightarrow G^\mathbb{Q}$ is called the *tensor \mathbb{Q} -completion of the group G* if it satisfy the following universal property: for any \mathbb{Q} -group H and a homomorphism $f: G \rightarrow H$ there exists a unique \mathbb{Q} -homomorphism $\psi: G^\mathbb{Q} \rightarrow H$ (a homomorphism that commutes with the \mathbb{Q} -action) such that $f = \psi \circ \phi$.

\mathbb{Q} -groups were introduced by G. Baumslag in [5] where he showed that \mathbb{Q} -tensor completions exist and are unique. A free \mathbb{Q} -group is the tensor \mathbb{Q} -completion of a free group F . Crucially, Baumslag also showed that $F^\mathbb{Q}$ can be obtained from a free group F by iteratively adding roots to F i.e. there is chain

$$F = F_0 < F_1 < F_2 < \dots$$

such that F_{n+1} is obtained from F_n by adding a some q -th root of an element generating its own centralizer i.e. if $c \in F_n$ and $\langle c \rangle = \{g \in F_n : gc = cg\}$ then $F_{n+1} = F_n *_c \langle t \mid \rangle$. There are some subtleties about how to construct this ascending chain (see [27, Section 8]) however, clearly any finitely generated subgroup of a \mathbb{Q} -free group is a subgroup of a finite iterative addition of roots to a free group F and hence it lies in \mathfrak{H} ⁴.

Example 6.6. An example of a group in \mathfrak{H} is $G = \langle a, b, c \mid a^2 b^2 c^2 \rangle$, the fundamental group of the connected sum of 3-projective planes. It follows from a theorem of Lyndon [26] that the equation $a^2 b^2 c^2 = 1$ in a free group implies that a, b, c commute. Hence G is not a limit group. On the other hand, G is obtained from the free group $\langle a, b \mid \rangle$, by adding a square root to $a^2 b^2$ and hence it is a finitely generated subgroup of a free \mathbb{Q} -group.

⁴Since finitely generated subgroups of free \mathbb{Q} -groups are in \mathfrak{H} , Proposition 6.1 then implies that free \mathbb{Q} -groups are left-orderable. In fact, G Baumslag asked whether these groups are residually torsion free nilpotent a fact recently proved by A. Jaikin [21]. In particular, free \mathbb{Q} -groups are bi-orderable, that is they admit left-orders which are also right-invariant.

7 Regular sets are coarsely connected

Let X be a set. Recall that X^* denotes the *free monoid generated by X* and consists on the set of finite words on X together with concatenation. If $X \subseteq G$ generates the group G , then there is a natural monoid epimorphism $\text{ev}: X^* \rightarrow G$, called *evaluation map*, that is induced by viewing each element of $x \in X$ as element of G .

Definition 7.1. A *finite state automaton* is a 5-tuple $(\mathcal{S}, \mathcal{A}, s_0, X, \tau)$, where \mathcal{S} is a set whose elements are called *states*, \mathcal{A} is a subset of \mathcal{S} of whose states are called *accepting states*, a distinguished element $s_0 \in \mathcal{S}$ called *initial state*, a finite set X called the *input alphabet* and a function $\tau: \mathcal{S} \times X \rightarrow \mathcal{S}$ called *the transition function*. We extend τ to a function $\tau: \mathcal{S} \times X^* \rightarrow \mathcal{S}$ recursively, by setting $\tau(s, wx) = \tau(\tau(s, w), x)$ where $w \in X^*$, $x \in X$ and $s \in \mathcal{S}$.

A *language \mathcal{L} over X* is a subset of X^* , and \mathcal{L} is *regular* if there is a finite state automaton $(\mathcal{S}, \mathcal{A}, s_0, X, \tau)$ such that

$$\mathcal{L} = \{w \in X^* \mid \tau(s_0, w) \in \mathcal{A}\}.$$

Proposition 7.2. Let G be a group generated by a finite set X . If $\mathcal{L} \subseteq X^*$ is a regular language, then $\text{ev}(\mathcal{L})$ is an r -connected subset of $\Gamma(G, X)$ for some r .

Proof. Assume that \mathcal{L} is a regular language over X and $(\mathcal{S}, \mathcal{A}, s_0, X, \tau)$ is a finite state automaton accepting \mathcal{L} . We can view the automaton as a directed graph with vertices \mathcal{S} and edges $\mathcal{S} \times X$ where (s, x) is an edge from s to $\tau(s, x)$. We call this graph, the *directed automaton graph*. Let $r = |\mathcal{S}|$. Let $w \equiv x_1x_2 \dots x_n \in \mathcal{L}$. We will show that we can go from 1 to $\text{ev}(w)$ with an $(2r + 1)$ -path $\{v_i\}_{i=0}^n$ where $v_i \in \text{ev}(\mathcal{L})$. Indeed, let w_i be the prefix of length i of w , i.e. $w_i = x_1x_2 \dots x_i$. It is enough to find $v_i \in \text{ev}(\mathcal{L})$ such that $d(\text{ev}(w_i), v_i) \leq r$. Note that w_i is a prefix of some word in \mathcal{L} , so we take a word of minimal length w'_i with the property that $w_iw'_i \in \mathcal{L}$. Now, w'_i gives a path in directed automaton graph from $\tau(s_0, w_i)$ to some vertex in \mathcal{A} . By minimality, this path can not repeat a vertex and thus $\ell(w'_i) \leq r$. Note that w'_i will be the label of some path in the Cayley graph from $\text{ev}(w_i)$ to $\text{ev}(w_iw'_i)$ of length at most r . Thus we can take v_i to be $\text{ev}(w_iw'_i)$. \square

Definition 7.3. A positive cone P of a finitely generated group G is *regular* if there is a finite generating set X of G and a regular language \mathcal{L} over X such that $P = \text{ev}(\mathcal{L})$.

Remark 7.4. Finitely generated (as subsemigroups) positive cones are regular.

Corollary 7.5. If G is Hucha, then G has no regular positive cone.

Proof. Let P be a positive cone for G and X a finite generating set of G . Suppose that there is a regular language $\mathcal{L} \subseteq X^*$ such that the evaluation of \mathcal{L} in G is P . By Proposition 7.2, there is some $r > 0$ such that P is an r -connected subset of $\Gamma(G, X)$. This contradicts Proposition 4.5. \square

Remark 7.6. Let G be a group and X a generating set. If \mathcal{L} is a regular language over X such that $\text{ev}(\mathcal{L})$ is a positive cone P , then the set of paths in $\Gamma(G, X)$ starting at 1_G and with label $w \in \mathcal{L}$ is a combing of P and by Proposition 7.2 this combing is supported in the r -neighbourhood of P for some r .

Suppose that X is a finite generating set of a group G . A language $\mathcal{L} \subseteq X^*$ is *quasi-geodesic* if there are $\lambda \geq 1$ and $c \geq 0$ such that each word $w \in \mathcal{L}$ labels a (λ, c) -quasi-geodesic path.

In [10, Question 8.7] Calegari sketched an argument for showing that on the fundamental group of an hyperbolic 3-manifold, no positive cone language is both regular and quasigeodesic. Recently H. L. Su [32] showed, using Calegari’s ideas, that in fact no positive in the (much larger) class of acylindrically hyperbolic group can be described by a regular and quasigeodesic language. With our tools we can easily recover Su’s theorem in restriction to the class of δ -hyperbolic groups.

Corollary 7.7. Non-elementary hyperbolic groups do not admit regular quasi-geodesic positive cones.

Proof. By Remark 7.6, if P is a regular quasi-geodesic positive cone, then it admits a quasi-geodesic combing supported on an r -neighbourhood of P for some r . This contradicts Theorem 3.5 □

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