SINGLE INTERCONNECTION OF KURAMOTO COUPLED OSCILLATORS

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Abstract In this work, we analyze the almost global synchronization property of sinusoidally coupled oscillators. In contrast with previous works, we introduce an approach that uses the strong basic facts of algebraic graph theory to prove dynamical properties of the standard symmetric Kuramoto model. We show how we can interconnect two (or several) systems via bridges, keeping the almost global synchronization. These ideas can be used to explore other kinds of interconnection

Keywords: Almost global stability, Kuramoto coupled oscillators, nonlinear systems

1. INTRODUCTION

A few decades ago, Y. Kuramoto introduced a mathematical model of weakly coupled oscillators that gave a formal framework to some of the works of A.T. Winfree on biological clocks (Kuramoto, 1975; Kuramoto, 1984; Winfree, 1980). The model proposes the idea that several oscillators can *interact* in a way such that the individual oscillation properties change, in order to achieve a global behavior for the interconnected system. The Kuramoto model serves as a good representation of many systems in several contexts such a biology, engineering, physics, mechanics, etc. (Ermentrout, 1985; York, 1993; Strogatz, 1994; Strogatz, 2000; Dussopt, 1999; Rogge, 2004; Marshall, 2004; Moshtagh and Jadbabaie and Daniilidis, 2005).

Recently, many works on the control community have focused on the analysis of the Kuramoto model, specially the one with sinusoidal coupling. The *consensus* or *collective synchronization* of the individuals is particularly important in many applications that need to represent coordination, cooperation, emerging behavior, etc. Local stability properties of the consensus have been initially explored in (Jadbabaie, 2004), while global or *almost global* dynamical properties were studied in (Monzón and Paganini, 2005; Monzón, 2006; Monzón and Paganini, 2006). In these works, the relevance of the underlying graph describing the interconnection of the system was hinted. In the present article, we go deeper on the analysis of the relationships between the dynamical properties of the system and the algebraic properties of the interconnection graph, exploiting the strong algebraic structure that every graph has.

In Section 2 we quickly review the relevant aspects of the algebraic graph theory. After that, we summarize the main results of different previous works on the analysis of Kuramoto coupled oscillators. Section 4 contains the contributions of this article, showing how we can interconnect synchronized systems keeping the synchronization property and introducing an analysis methodology. Finally, we present come conclusions.

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2. ALGEBRAIC GRAPH THEORY

In this Section, we review the basic facts on algebraic graph theory that will be used along the article. A more detailed introduction to this theory can be found in (Biggs, 1993; Cvetkovic and Doob and Sachs, 1979). A graph consists in a set of n *nodes* or *vertices* $\mathcal{V} = \{v_1, \dots, v_n\}$ and a set of m *links* or *edges* $\mathcal{E} = \{e_1, \dots, e_m\}$ that describes how the nodes are related to each other. We say that two nodes are neighbors or adjacent if there is a link in $\mathcal{E}$ between them. The graph is *directed* if every link has a starting node and a final node. The topology of a directed graph may be described by the *incidence* matrix B with n rows and m columns:

$$B_{ij} = \begin{cases} 1 & \text{if edge } j \text{ reaches node } i \\ -1 & \text{if edge } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

The rank r of B is also called the rank of the graph. Observe that [2] $B^T \mathbf{1}_n = 0$. By removing a link, say e_k, we obtain a new graph G_k and we use the notation $G_k = G \setminus \{e_k\}$. If we split G into two disjoints subsets G_1 and G_2, we define a *cut* of G as the sets of links of $\mathcal{E}$ with initial or endings nodes in G_1 or G_2. Every graph has associated some vector spaces that lead to an algebraic and spectral graph theory. The *cycle space* of the graph G is the kernel of the matrix B and it is related with the *cycles* (non-intersecting closed paths) of the graph. Its dimension is equal to $m - r$. The *cut space* is the orthogonal complement of the cycle space, with the euclidean inner product. Its dimension is equal to r. Given a cycle (cut) of G, there is a systematic way for constructing a vector in the cycle (cut) space. This will be used in Proposition 4.2.

3. ALMOST GLOBAL SYNCHRONIZATION

The state of an oscillator can be described by its phase angle θ. As is explained in (Jadbabaie, 2004), when we have n oscillators coupled in a sinusoidal way, the expression

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i) \quad i = 1, \dots, n$$

describes the system, where $\mathcal{N}_i$ is the set of *neighbors* of agent i. Each phase θ_i belongs to the interval $[0, 2\pi)$, so the system evolves on the compact n-dimensional torus $\mathcal{T}^n$. The value of a phase must be considered modulo 2π. Consider the graph G, with nodes $\{v_1, \dots, v_n\}$ and edges $\{e_1, \dots, e_m\}$, that describes how the individual oscillators, or *agents*, interact between each others. The node v_i represents the i-th oscillator, with phase θ_i. Consider an arbitrary orientation of the links of G and let the matrix B, with n rows and m columns, be an incidence matrix. We will work with symmetric interaction: if $i \in \mathcal{N}_j$ then $j \in \mathcal{N}_i$. In this case, the system can be compactly written as

$$\dot{\theta} = -B \sin\left(B^T \bar{\theta}\right) \tag{1}$$

Equation (1) does not depend on the choice of B (Jadbabaie, 2004). As was done by Kuramoto, we may represent the agents as running points on a circumference or as unit phasors (Kuramoto, 1975; Monzón, 2006).

Since the system dynamic depends only on the phase differences, it is invariant under translations parallel to vector $\mathbf{1}_n$. We say the system *synchronizes* or reaches *consensus* if the individuals phases converge to a state where all the phases are identical. Of course, a consensus point is an equilibrium point of the system and actually we have a synchronization set, due to the invariance property. This also applies to every equilibrium point. We will also work with *partial consensus* equilibria, when most of the phases take the value 0 (taking a suitable reference) and the remaining phases take the value π. Other equilibria will be referred as *non-synchronized*. If $\bar{\theta}$ is an equilibrium point of (1) with underlying graph G, we will use the expression: *$\bar{\theta}$ is an equilibrium of G.*

We are concerned on whether or not all the trajectories converge to the synchronization set. Since the system has many equilibria, we can only expect that *most of the trajectories* presents this property. Following the ideas of (Rantzer, 2001), we say that the system has the *almost global synchronization* property if the set of trajectories that do not converge to the synchronization set has zero Lebesgue measure on $\mathcal{T}^n$. If the system is described by a graph G, we will shortly say the *G is a.g.s.*. In (Jadbabaie, 2004), it was proved that the synchronization set is locally stable. First results on almost global properties were presented in (Monzón and Paganini, 2005; Monzón and Paganini, 2006). These results were proved in two steps: firstly, using LaSalle's result on asymptotical behavior of trajectories in a compact invariant set, it was shown that the only ω-limit sets are the equilibria of the system; secondly, Jacobian linearization was used to locally classify the equilibria (Khalil, 1996). At an equilibrium point $\bar{\theta}$, a first order approximation is given by the symmetric, $n \times n$, matrix

$$A_G = -B\, diag\left[\cos(B^T \bar{\theta})\right] B^T \tag{2}$$

Observe that A_G always has a single zero eigenvalue with corresponding eigenvector $\mathbf{1}$, due to the invariance property of the system.

[2] By $\mathbf{1}_p$ we denote the column vector in $\mathbb{R}^p$ with all its components equal to 1.

4. SYNCHRONIZING INTERCONNECTION

Consider two graphs G_1 and G_2. An interconnection between them will be done putting links between one or more agents of both graphs. We first start with the addition of one single link. Consider the following general result.

Proposition 4.1. Let G be a graph with nodes $\{v_1, \ldots, v_n\}$, e_k be a link between v_i and v_{i+1} and $\bar{\theta}$ an equilibrium point of G. Consider the graph $G_k = G \setminus \{e_k\}$. Then, $\bar{\theta}$ is an equilibrium point of G_k if and only if $\bar{\theta}_i = \bar{\theta}_{i+1}$ or $\bar{\theta}_i - \bar{\theta}_{i+1} = \pi$.

Proof: Let B and B_k be respectively the incidence matrices of G and G_k. B can be decomposed as

$$B = \left[\begin{array}{c|c|c} & 0 & \\ & \vdots & \\ B_1 & -1 & B_2 \\ & 1 & \\ & \vdots & \\ & 0 & \end{array}\right]$$

The size of B_1 is $n \times (k-1)$ and the size of B_2 is $n \times (m-k)$. The k-th column of B has 0 everywhere except at places i and $i+1$. With this notation,

$$B_k = \left[\, B_1 | B_2 \,\right]$$

Then,

$$B \sin\left(B^T \bar{\theta}\right) = B_k \sin\left(B_k^T \bar{\theta}\right) - \begin{bmatrix} 0 \\ \vdots \\ \sin(\bar{\theta}_{i+1} - \bar{\theta}_i) \\ \sin(\bar{\theta}_i - \bar{\theta}_{i+1}) \\ \vdots \\ 0 \end{bmatrix}$$

Since $\bar{\theta}$ is an equilibrium point,

$$B_k \sin\left(B_k^T \bar{\theta}\right) = \begin{bmatrix} 0 \\ \vdots \\ \sin(\bar{\theta}_{i+1} - \bar{\theta}_i) \\ \sin(\bar{\theta}_i - \bar{\theta}_{i+1}) \\ \vdots \\ 0 \end{bmatrix}$$

It is clear that in order to have an equilibrium point of G_k, it must be true that $\bar{\theta}_i = \bar{\theta}_{i+1}$ or $\bar{\theta}_i - \bar{\theta}_{i+1} = \pi$.

□

When the phase difference between two agents is zero, we say that they are *in phase*, while they are *in counterphase* when the pase difference is $\pm\pi$. If we have a graph G with an equilibrium point $\bar{\theta}$ with two agents in phase or in counterphase, if those agents are not related, we can add a link between them and $\bar{\theta}$ is an equilibrium of the new graph.

4.1 Algebraic analysis

The previous result is for a non particular link between v_i and v_k. Now, consider the case when the k-th link e_k, joining nodes v_i and v_{i+1}, is a *bridge* of G. This means that the graph $G_k = G \setminus \{e_k\}$ is not connected. We denote by G_1 and G_2 the connected components of G_k. Observe that, starting with two graphs G_1 and G_2, we can interconnect them via a single link, in order to have a bigger graph G with a bridge.

Proposition 4.2. Consider the graph G with a bridge e_k between nodes v_i and v_{i+1}. Then, for every $\bar{\theta}$ equilibrium point of G, it must be true that the interconnection between v_i and v_{i+1} is in phase or in counterphase.

Proof: Define the vector $w \in \mathbb{R}^m$ as follows:

$$w_k = \begin{cases} 1 \text{ if } h = k \\ \\ 0 \text{ if } h \neq k \end{cases}$$

This vector belongs to the cut space of G (Biggs, 1993) and so, it is orthogonal to every vector in the cycle space. Then, for every $\bar{\theta}$ equilibrium point of (1), we have the identity

$$w_k^T . \sin\left(B^T \bar{\theta}\right) = \pm \sin\left(\bar{\theta}_i - \bar{\theta}_{i+1}\right) = 0 \quad (3)$$

Then, for every $\bar{\theta}$ equilibrium point of G, the k-th component of vector $\sin\left(B^T \bar{\theta}\right)$ must vanish and the phase difference $\bar{\theta}_i - \bar{\theta}_{i+1}$ is either 0 or π.

□

Since e_k joins the subgraphs G_1 and G_2, we say that at the equilibrium point $\bar{\theta}$, these subgraphs are connected *in phase* if $\bar{\theta}_i = \bar{\theta}_{i+1}$ and *in counterphase* if $\bar{\theta}_i = \bar{\theta}_{i+1} \pm \pi$. This general result imposes an important restriction to every bridge of a given graph.

Proposition 4.3. Consider the graph G with a bridge e_k joining the nodes v_i and v_{i+1}. Then, if $\bar{\theta} = \left[\bar{\theta}_1^T \,,\, \bar{\theta}_2^T\right]^T \in \mathbb{R}^n$ is an equilibrium point of G, with $\bar{\theta}_1 \in \mathbb{R}^i$ and $\bar{\theta}_2 \in \mathbb{R}^{n-i}$, then $\bar{\theta}_1$ and $\bar{\theta}_2$ are equilibrium points of G_1 and G_2 respectively. Conversely, if $\bar{\theta}_1$ and $\bar{\theta}_2$ are equilibrium points of G_1 and G_2 respectively, there exists a real number α such that $\bar{\theta} = \left[\bar{\theta}_1^T \,,\, \bar{\theta}_2^T + \alpha.\mathbf{1_{n-i}}^T\right]^T$ is an equilibrium point of G.

Proof: Let B be the incidence matrix of G. Since the link e_k is the only connection between G_1 y

G_2, we can order the nodes and links such that B has the following particular form:

$$B = \left[\begin{array}{c|c|c} & 0 & \\ B_1 & \vdots & \mathbf{0_{i\times(m-k)}} \\ & -1 & \\ \hline & 1 & \\ \mathbf{0_{(n-i)\times(k-1)}} & \vdots & B_2 \\ & 0 & \end{array}\right] \qquad (4)$$

B_1 and B_2 are $i \times (k-1)$ and $(n-i) \times (m-k)$ matrices respectively. The k-th column of B has -1 at the row i and a 1 at the row $i+1$. Assume that $\bar{\theta}$ is an equilibrium point of G. Then, by Proposition 4.2, the k-th component of $\sin(B^T\bar{\theta})$ is 0 and a direct calculation of the product of $B\sin(B^T\bar{\theta})$ gives that

$$B\sin(B^T\bar{\theta}) = \begin{bmatrix} B_1\sin(B_1^T\bar{\theta}_1) \\ B_2\sin(B_2^T\bar{\theta}_2) \end{bmatrix} \qquad (5)$$

and we have proved the first part of the Proposition.

Now, assume that $\bar{\theta}_1$ and $\bar{\theta}_2$ are equilibrium points of G_1 and G_2 respectively. Let α be a real number such that the last component of $\bar{\theta}_1$ is α plus the first component of $\bar{\theta}_2$. Due to the invariance of the system we have remarked on Section 1, the vector $\bar{\theta}_2 + \alpha\mathbf{1_{n-i}}$ is also an equilibrium point of G_2. In this case, G_1 and G_2 have an *in phase* interconnection and we recover the identity (5) and $\bar{\theta} = [\bar{\theta}_1^T, \bar{\theta}_2^T + \alpha\mathbf{1_{n-i}^T}]^T$ is an equilibrium point of G. We could also have defined α such that the interconnection was *in counterphase.*

□

4.2 Stability analysis

We will relate the stability properties of the graph G with a bridge with the stability properties of the resulting subgraphs G_1 and G_2. Since every equilibrium of G defines an equilibria for G_1 and G_2, we wonder whether or not the dynamic characteristics of these equilibria are or not the same. We will use Jacobian linearization. The zero eigenvalue is always present due to the invariance of the system by translations parallel to $\mathbf{1_n}$. We always refer to the *transversal stability* of the equilibrium set. If the multiplicity of the zero eigenvalue is more than one, Jacobian linearization fails in classifying the equilibria. Due to space reasons, we present the study of this particular problem in a different article. So, in this work, we assume that we always have a single null eigenvalue.

Theorem 4.1. Consider the graph G, with a bridge e_k joining the nodes v_i and v_{i+1}, and let G_1 and G_2 be the connected components of the graph G_k. Let $\bar{\theta} = \left[\bar{\theta}_1\ ,\ \bar{\theta}_2\right]^T \in \mathbb{R}^n$ be an equilibrium point of G, with $\bar{\theta}_1 \in \mathbb{R}^i$ and $\bar{\theta}_2 \in \mathbb{R}^{n-i}$. Then, $\bar{\theta}$ is locally stable if and only if $\bar{\theta}_1$ and $\bar{\theta}_2$ are locally stable and G_1 and G_2 have an in phase interconnection.

Proof: Recall that the first order approximation of the system around an equilibrium point is given by

$$A_G = -B\,diag\left[\cos(B^T\bar{\theta})\right]B^T$$

Due to the bridge e_k, $\cos(\bar{\theta}_i - \bar{\theta}_{i+1}) = \pm 1$ and

$$\cos(B^T\bar{\theta}) = \begin{bmatrix} \cos(B_1^T\bar{\theta}_1) \\ \pm 1 \\ \cos(B_2^T\bar{\theta}_2) \end{bmatrix}$$

We introduce the auxiliary block diagonal matrix

$$C = \left[\begin{array}{c|cc|c} \mathbf{0_{(i-1)\times(i-1)}} & & & \\ \hline & 1 & -1 & \\ & -1 & 1 & \\ \hline & & & \mathbf{0_{(n-i-1)\times(n-i-1)}} \end{array}\right]$$

This matrix is symmetric and positive semidefinite, with a single eigenvalue 2 and the null eigenvalue with multiplicity $n-1$. A direct calculation gives

$$A_G = \left[\begin{array}{c|c} A_{G_1} & \mathbf{0_{i\times(n-i)}} \\ \hline \mathbf{0_{(n-i)\times(i)}} & A_{G_2} \end{array}\right] \mp C \qquad (6)$$

The - and + signs correspond to an in phase and counterphase interconnection respectively.

First of all, we consider the case with $\bar{\theta}_1$ and $\bar{\theta}_2$ stable. Then, A_{G_1} and A_{G_2} are stable. So, for an in phase interconnection, A_G is the sum of two semidefinite negative matrices (recall that the vectors parallel to $\mathbf{1_n}$ are in the kernel of A_G and also in the kernel of C). This proves the stability of A_G.

Assume now that we face an in counterphase interconnection. Then,

$$A_G = \left[\begin{array}{c|c} A_{G_1} & \mathbf{0_{i\times(n-i)}} \\ \hline \mathbf{0_{(n-i)\times(i)}} & A_{G_2} \end{array}\right] + C$$

Consider the vector

$$v = \begin{bmatrix} \mathbf{1_i} \\ -\mathbf{1_{n-i}} \end{bmatrix}$$

We have that

$$v^T A_G v = v^T \left(\left[\begin{array}{c|c} A_{G_1} & \mathbf{0_{i\times(n-i)}} \\ \hline \mathbf{0_{(n-i)\times(i)}} & A_{G_2} \end{array}\right) + C\right] v =$$

$$v^T A_G v = \begin{bmatrix} \mathbf{1_i} \\ -\mathbf{1_{n-i}} \end{bmatrix}^T \left[\begin{array}{c|c} A_{G_1} & \mathbf{0_{i\times(n-i)}} \\ \hline \mathbf{0_{(n-i)\times(i)}} & A_{G_2} \end{array}\right] \begin{bmatrix} \mathbf{1_i} \\ -\mathbf{1_{n-i}} \end{bmatrix}$$

$$+v^T C v$$

The first term at the right of the equality vanishes, since $A_{G_1}\mathbf{1_{n-i}} = 0$ and $-A_{G_2}\mathbf{1_{n-i}} = 0$. The second term can be easily evaluated, to obtain

$$v^T A_G v = 4$$

$\bar\theta_1$	$\bar\theta_2$	Interconnection	$\bar\theta$
stable	stable	phase	stable
stable	stable	counterphase	unstable
stable	unstable	phase	unstable
stable	unstable	counterphase	unstable
unstable	unstable	phase	unstable
unstable	unstable	counterphase	unstable

Table 1. Stability analysis of Theorem 4.1

Then, A_G must have a positive eigenvalue and then the equilibrium $\bar\theta$ is unstable.

Now, we focus on the case with $\bar\theta_1$ or $\bar\theta_2$ unstable. We analyze the first case, since the other is similar. Then, A_{G_1} has a positive eigenvalue with associated eigenvector v_1 such that

$$v_1^T A_{G_1} v_1 > 0$$

Define the vector

$$v = \begin{bmatrix} v_1 \\ \alpha \mathbf{1_{n-i}} \end{bmatrix}$$

with α chosen in a way that the components i and $i+1$ of v coincide. Then,

$$v^T A_G v = v_1^T A_{G_1} v_1 + \alpha^2 \mathbf{1_{n-i}}^T A_{G_2} \mathbf{1_{n-i}} + v^T C v$$

which actually is

$$v^T A_G v = v_1^T A_{G_1} v_1 > 0$$

Then, $\bar\theta$ is unstable.

□

Table 1 summarizes the results of Theorem 4.1. We are now ready to state and prove the main result of this article.

Theorem 4.2. Consider the graph G, with a bridge e_k joining the nodes v_i and v_{i+1}, and let G_1 and G_2 be the connected components of the graph $G \setminus \{e_k\}$. Then, G_1 and G_2 have the almost global synchronization property if and only if G does.

Proof: First of all, let $\bar\theta = [\bar\theta_1^T \bar\theta_2^T]^T$ be an equilibrium point of G. According to theorem 4.1, $\bar\theta$ is stable only if $\bar\theta_1$ and $\bar\theta_2$ are too and the interconnection is in phase.

If G_1 and G_2 are a.g.s., the only locally stable set is the consensus, and due to the in phase interconnection, the only locally stable equilibria of G is also the consensus and G is a.g.s.

In the other direction, if $\bar\theta_1$ is a locally stable equilibrium of G_1, we chose $\bar\theta_2 = \alpha \mathbf{1}$ such that the interconnection is in phase and we construct a stable equilibrium for G. Since G is a.g.s., $\bar\theta$ must be a consensus equilibrium point.

□

Theorem 4.2 has many direct consequences. We point out some of them, with a brief hint of the respective proofs.

Proposition 4.4. If G is a tree, it has the almost global synchronization property.

Observe that in a tree, every link is a bridge. The proof is based on the iterative application of Theorem 4.2. This result was previously proved in (Monzón and Paganini, 2006) using a different approach.

Corollary 4.1. If G is a graph with the structure shown in figure 1, then G is a.g.s. if and only if G_1 is.

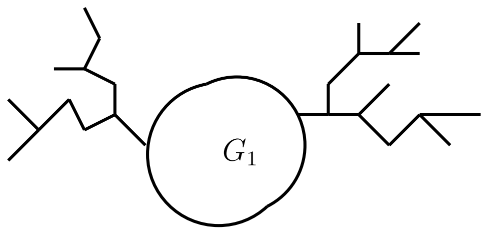

Figure 1. A graph with *arboricities.*

Corollary 4.2. If G consists of two subgraphs G_1 and G_2 connected through a tree, as in figure 2, G_1 and G_2 are a.g.s. is and only if G is.

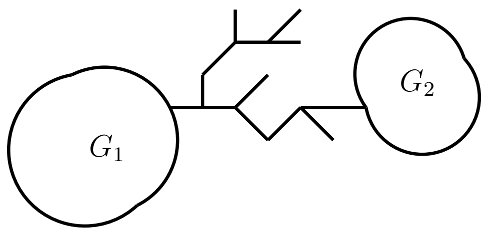

Figure 2. Two graphs connected by a tree

Summarizing, in the study of the a.g.s. property, every *arboricity* or interconnecting tree does not count.

Proposition 4.5. If G is a tree and we build a new graph K replacing every node of G by an a.g.s. graph, then K has the almost global synchronizing property.

The conclusion directly follows from the previous results. As an example, every vertex can be replaced by a complete graph, which always is a.g.s. (Monzón and Paganini, 2005). Observe that the interconnection can be done from any node of the a.g.s. graph.

Example 4.1. (Moshtagh and Jadbabaie and Daniilidis, 2005) Consider N agents $\{v_1, v_2, \ldots, v_n\}$ in the plane, that move around with constant unit velocity. The kinematic model of each agent is

$$\begin{aligned}\dot{x} &= \cos(\theta_i)\\ \dot{y} &= \sin(\theta_i)\\ \dot{\theta}_i &= \quad \omega_i\end{aligned}$$

where ω_i is the control input. In this context, the synchronization of the agents a *flocking* state. An interconnection graph G defines the interaction between agents. If we apply the control feedback

$$\omega_i = \sum_{j \in \mathcal{N}_i} v_j^T X_{\theta_i}$$

with $X_{\theta_i} = [-\sin(\theta_i), \cos(\theta_i)]^T$, the feedback systems has the following description:

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i)$$

In order to see if the feedback control almost always leads to a flocking, it is enough to verify this property on the blocks of G. For example, if G is a tree, we have flocking. Moreover, if we can design the interconnection, we can do it ensuring the property.

△

5. CONCLUSIONS

In this work we have studied how some algebraic properties of the underlying graph describing the interconnection of a symmetric Kuramoto model impose restrictions on the dynamical behavior. We focus on the particular case of the existence of a bridge between two agents. We proved that the interconnection by a bridge of almost global synchronized systems preserves that property. We think that the ideas we have presented here can be used to analyze other types of interconnection or graph interaction.

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