Gluing Kuramoto coupled oscillators networks

Eduardo Canale and Pablo Monzón

***Abstract*— In this work we prove that the problem of almost global synchronization of the Kuramoto model of sinusoidally symmetric coupled oscillators with a given topology could be reduced to the analysis of the blocks of the underlying interconnection graph.**

I. Introduction

A few decades ago, Y. Kuramoto introduced a mathematical model of weakly coupled oscillators that gave a formal framework to some of the works of A.T. Winfree on biological clocks [1], [2], [3]. The model proposes the idea that several oscillators can *interact* in a way such that the individual oscillation properties change in order to achieve a global behavior for the interconnected system. The Kuramoto model serves a a good representation of many systems in several contexts such a biology, engineering, physics, mechanics, etc. [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14].

Recently, many works on the control community have focused on the analysis of the Kuramoto model, specially the one with sinusoidal coupling. The *consensus* or *collective synchronization* of the individuals is particularly important in many applications that need to represent coordination, cooperation, emerging behavior, etc. Local stability properties of the consensus have been initially explored in [15], while global or *almost global* dynamical properties were studied in [16], [17], [18]. In these works, the relevance of the underlying graph describing the interconnection of the system was hinted. In the present article, we go deeper on the analysis of the relationships between the dynamical properties of the system and the algebraic properties of the interconnection graph, exploiting the strong algebraic structure that every graph has.

In Section II we quickly review the relevant aspects of the algebraic graph theory. After that, we summarize the main results of different previous works on the analysis of Kuramoto coupled oscillators. Section IV contains the contributions of this article, showing how we can interconnect synchronized systems keeping the synchronization property and introducing an analysis procedure for a kind of graphs. Finally, we present some conclusions.

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Pablo Monzón is with the Electrical Engineering Institute, Faculty of Engineering, Universidad de la Republica, Uruguay, monzon@fing.edu.uy

Eduardo Canale is with the Mathematics and Statistics Institute (IMERL), Faculty of Engineering, Universidad de la Republica, Uruguay, canale@fing.edu.uy

II. Algebraic graph theory

In this Section we review the basic facts on algebraic graph theory that will be used along the article. A more detailed introduction to this theory can be found in [19], [20]. A graph consists in a set of n *nodes* or *vertices* $VG = \{v_1, \ldots, v_n\}$ and a set of m *links* or *edges* $EG = \{e_1, \ldots, e_m\}$ that describes how the nodes are related to each other. If $n = 1$ the graph is called *trivial*. We say that two nodes are neighbors or adjacent if there is a link in EG between them. If all the vertices are pairwise adjacent the graph is called *complete* and written K_n. A *walk* is a sequence $v_0, \ldots, v_n$ of adjacent vertices. If the vertices are different except the first and the last which are equal ($v_i \neq v_j$ for $0 < i < j$ and $v_0 = v_n$) the walk is called a *cycle*. A graph with no cycle is called *acyclic*. The graph is *connected* if there is a walk between any given pair of vertices. A *tree* is an acyclic connected graph and has $m = n - 1$ edges. The graph is *oriented* if every link has a starting node and a final node. The topology of a oriented graph may be described by the *incidence* matrix B with n rows and m columns:

$$B_{ij} = \begin{cases} 1 & \text{if edge } j \text{ reaches node } i \\ -1 & \text{if edge } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

Observe that[1] $B^T \mathbf{1}_n = 0$. The *vertex space* and the *edge space* of G are the sets of real functions with domain VG and EG respectively, which we sometimes will identify, respectively, with the vectors sets $\mathbb{R}^n$ and $\mathbb{R}^m$. Thus, the incidence matrix B can be seen as a linear transformation from the edge space to the vertex space. The kernel of B is the *cycle space* of the graph G and its elements are called *flows*. Every *flow* can be thought as a vector of weights assigned to every link in a way that the total algebraic sum at each node is zero. The cycle space is spanned by the flows determined by the cycles: given a cycle $v_0, \ldots, v_n = v_0$, its associated flow $f_C(e)$ is ± 1 if e leaves some v_i and reaches $v_{i\pm 1}$ and 0 otherwise.

If the graph G is the union of two nontrivial graphs G_1 and G_2 with one and only one node v_i in common, then v_i is called a *cut-vertex* of G. A connected graph with more than two vertices and no cut-vertex is called *2-connected* and it follows that for every pair of nodes, there are at least two different paths between them. Given a subset $V_1 \subset VG$, its *induced subgraph* is $\langle V_1 \rangle$ with vertex set V_1 and edge set $\{e \in EG : e \text{ joins vertices of } V_1\}$. The maximal induced

[1] By $\mathbf{1}_p$ we denote the column vector in $\mathbb{R}^p$ with all its components equal to 1.

subgraphs of G with no cut-vertex, are called the *blocks* of G. Every graph has the form of figure 1: a collection of blocks joined by cut-vertices. For a complete graph, there is only one block, the graph itself.

We will use the following vector notation: given

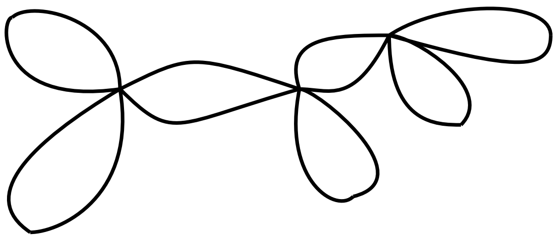


Fig. 1. Representation of a graph as a union of blocks.

a n-dimensional vector $\bar{\theta} = [\theta_1, \ldots, \theta_n]$, then $\bar{\theta}(i:j) = [\theta_i, \ldots, \theta_j]$ and $\bar{\theta}(i) = \theta_i$. Firstly, we present some basic results. We include two different proofs for Lemma 2.1, in order to show two distinct interpretations of the same facts: one based on linear algebra, the other using graph theory elements.

Lemma 2.1: Consider a graph G, with v a cut-vertex between G_1 and G_2. Then, an edge space element $f : EG \to \mathbb{R}$ is a flow on G, if and only if $f|_{EG_1}$ and $f|_{EG_2}$ are a flows on G_1 and G_2 respectively.

Proof 1: Suppose that the i vertices of G_1 and its k edges come first in the chosen labelling. Suppose, also, that $v = v_i$, then B has the following form:

$$B = \left[\begin{array}{c|c} W_1 & \mathbf{0}_{(\mathbf{i-1})\times(\mathbf{m-k})} \\ \hline w_1^T & w_2^T \\ \hline \mathbf{0}_{(\mathbf{n-i})\times\mathbf{k}} & W_2 \end{array}\right]$$

Where w_1 and w_2 are column vectors with appropriate dimensions. With this notation, the incidence matrices of G_1 and G_2 are, respectively

$$B_1 = \left[\begin{array}{c} W_1 \\ \hline w_1^T \end{array}\right], \qquad B_2 = \left[\begin{array}{c} w_2^T \\ \hline W_2 \end{array}\right].$$

Besides, B_1 as incidence matrix, verifies $\mathbf{1}_i^T B_1 = 0$, thus $\mathbf{1}_{i-1}^T W_1 + w_1^T = \mathbf{0}_k$, so

$$w_1^T = -\mathbf{1}_{i-1}^T W_1. \qquad (1)$$

Let f be a flow on G. Then, in order to prove that $f_1 = f|_{EG_1}$ is a flow on G_1, we must prove that $B_1 f_1 = \mathbf{0}_i$, i.e. $W_1 f_1 = \mathbf{0}_{i-1}$ and $w_1^T f_1 = 0$. The former is true because f is a flow on G, thus $Bf = \mathbf{0}_n$, but $W_1 f_1 = (Bf)(1:i-1)$. While for the last, we have that, by (1), $w_1^T f_1 = (-\mathbf{1}_{i-1}^T W_1) f_1 = -\mathbf{1}_{i-1}^T (W_1 f_1) = \mathbf{1}_{i-1}^T \mathbf{0}_{i-1} = 0$. Exchanging G_1 by G_2 we obtain that $f|_{EG_2}$ is a flow on G_2.
Conversely, if f_1 and f_2 are flows on G_1 and G_2 respectively, we have that $(Bf)(1:i-1) = B_1 f_1 = \mathbf{0}_i$, $(Bf)(i+1:n) = B_2 f_2 = \mathbf{0}_{n-i+1}$ and $(Bf)(i) = w_1^T f_1 + w_2^T f_2 = 0 + 0 = 0$.

□

Proof 2: Following [19] [Lemma 5.1, Theorem 5.2], given a spanning tree T of G, we obtain a basis of the cycle space in the following form: for each edge $e \in E' = EG \setminus ET$ we have an unique cycle $cyc(T, e)$ which determines a flow $f_{T,e}$. The set $\mathcal{B}$ of these flows is a basis of the cycle-space. However, since v is a cut-vertex, any cycle is included either in G_1 or in G_2, so its associated flow is null either in EG_1 or in EG_2. If we regard a flow on G which is null in EG_1 as a flow on G_2, we can split $\mathcal{B}$ into two sets $\mathcal{B}_1$ and $\mathcal{B}_2$ cycle-space basis of G_1 and G_2 respectively. Thus the cycle-space of G is the direct sum of the cycle-spaces of G_1 and G_2.

□

Lemma 2.2: Let G be a graph, $V_1 \subset VG$ and $G_1 = \langle V_1 \rangle$ the subgraph of G induced by the vertices V_1 with incidence matrix B_1. Let $H : \mathbb{R} \to \mathbb{R}$ be any real function, $\bar{\theta} : V \to \mathbb{R}$ an element of the vertex-space of G and $f = H(B^T \bar{\theta})$ then, if

$$f_1 = f|_{EG_1}, \qquad \bar{\theta}_1 = \bar{\theta}|_{VG_1}$$

it is true that

$$f_1 = H(B_1^T \bar{\theta}_1).$$

Proof: Suppose that the i vertices and k edges of G_1 come first in the chosen labelling. Then, for some B', B'' and $\bar{\theta}_2$, we have that

$$B^T \bar{\theta} = \left[\begin{array}{c|c} B_1^T & \mathbf{0}_{\mathbf{i}\times\mathbf{k}} \\ \hline B' & B'' \end{array}\right] \left[\begin{array}{c} \bar{\theta}_1 \\ \bar{\theta}_2 \end{array}\right] = \left[\begin{array}{c} B_1^T \bar{\theta}_1 \\ B'\bar{\theta}_1 + B''\bar{\theta}_2 \end{array}\right].$$

Thus, $(B^T\bar{\theta})(1:k) = B_1^T \bar{\theta}_1$, and $f_1 = f(1:k) = H(B^T\bar{\theta})(1:k) = H((B^T\bar{\theta})(1:k)) = H(B_1^T\bar{\theta}_1)$.

□

III. Almost global synchronization

Oscillators have been studied by engineers for a long time [21]. The state of an oscillator can be described by its phase angle θ. Consider now the Kuramoto model of n sinusoidally coupled oscillators [6]

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i) \quad i = 1, \ldots, n \qquad (2)$$

where $\mathcal{N}_i$ is the set of *neighbors* of agent i. Each phase θ_i belongs to the interval $[0, 2\pi)$, so the system evolves on the compact n-dimensional torus $\mathcal{T}^n$. The value of a phase must be considered modulo 2π. Consider the graph G, with nodes $\{v_1, \ldots, v_n\}$ and edges $\{e_1, \ldots, e_m\}$, that describes how the individual oscillators, or *agents*, interact between each other. The node v_i represents the i-th oscillator, with phase θ_i. Consider an arbitrary orientation of the links of G and let the matrix B, with n rows and m columns, be an incidence matrix for G. We will work with symmetric interaction: if

$i \in \mathcal{N}_j$ then $j \in \mathcal{N}_i$. In this case, as is explained in [15], the expression (2) can be compactly written as

$$\dot{\theta} = -B \sin\left(B^T \bar{\theta}\right) \tag{3}$$

Equation (3) does not depend on the choice of B. As was done by Kuramoto, we may represent the agents as running points on a circumference or as unit phasors, as in Example 3.1 [1], [17].

Since the system dynamic depends only on the phase differences, it is invariant under translations parallel to vector $\mathbf{1}_n$. We say the system *synchronizes* or reaches *consensus* if the individual phases converge to a state where all the phases are identical. Of course, a consensus point is an equilibrium point of the system and actually we have a synchronization set, due to the invariance property. This also applies to every equilibrium point. We will also work with *partial consensus* equilibria, when most of the phases take the value 0 (taking a suitable reference) and the remaining phases take the value π. Other equilibria will be referred as *non-synchronized*. If $\bar{\theta}$ is an equilibrium point of (3) with underlying graph G, we will use the expression: *$\bar{\theta}$ is an equilibrium of G*.

Example 3.1: Consider the graph shown at the left of Figure 2. A non-synchronized equilibrium point is given by

$$\bar{\theta} = \begin{bmatrix} 169.04 \\ 59.96 \\ -49.13 \\ 130.87 \\ -120.04 \\ -10.96 \\ -30.04 \\ 149.96 \end{bmatrix}$$

The angles are measured in degrees.

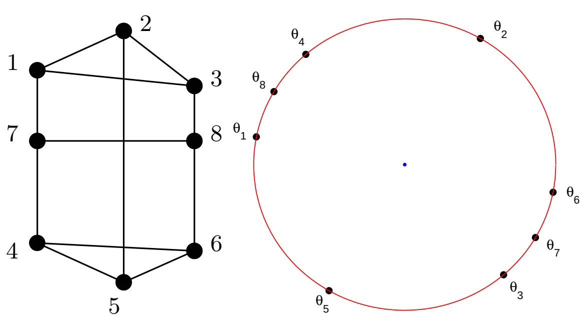


Fig. 2. *Phasor representation of the equilibrium point $\bar{\theta}$ of system (3) of Example 3.1. The underlying graph is shown at the left.*

△

We are concerned on whether or not all the trajectories converge to the synchronization set. Since the system has many equilibria, we can only expect that *most of the trajectories* presents this property. Following the ideas of [22], we say that the system has the *almost global synchronization* property if the set of trajectories that do not converge to the synchronization set has zero Lebesgue measure on $\mathcal{T}^n$. If the system is described by a graph G, we will shortly say the G *is a.g.s.*. In [15], it was proved that the synchronization set is locally stable. First results on almost global properties were presented in [16], [18]. There, it was proved that the complete graph K_n and the tree graphs always are a.g.s. These results were proved in two steps: firstly, using LaSalle's result on asymptotical behavior of trajectories in a compact invariant set, it was shown that the only ω-limit sets are the equilibria of the system; secondly, Jacobian linearization was used to locally classify the equilibria[2] [23]. At an equilibrium point $\bar{\theta}$, a first order approximation is given by the symmetric, $n \times n$, matrix

$$A_G = -B\, diag\left[\cos(B^T \bar{\theta})\right] B^T \tag{4}$$

Observe that A_G always has a zero eigenvalue with corresponding eigenvector $\mathbf{1}$, due to the invariance property of the system. In this work, we try to extend our knowledge of the family of a.g.s. graphs.

IV. Synchronizing interconnection

From equation (3) we see that a phase angle θ is an equilibrium point if and only if $\sin(B^T\theta)$ is a flow on G. Thus it should be possible that this equilibrium points could be obtained from the equilibrium points of the blocks of the graphs. In fact, this is exactly what happens. Furthermore, the stability of these equilibrium points depends only on the stability of the associated equilibrium points of the blocks. First, we study the existence problem, which will follow directly from Lemma 2.1, and then we study the stability properties.

A. Existence

If $\theta_1 : VG_1 \to \mathbb{R}$ is in the vector space of a subgraph G_1 of G, we will regard it also as its unique extension to the vector space of G which is null elsewhere of G_1. The same for an element of the edge space.

Proposition 4.1: Consider the graph G with a cut-vertex v between G_1 and G_2. If $\bar{\theta}$ is an equilibrium point of G, then $\bar{\theta}_1 = \bar{\theta}|_{VG_1}$ and $\bar{\theta}_2 = \bar{\theta}|_{VG_2}$ are equilibrium points of G_1 and G_2 respectively. Conversely, if $\bar{\theta}_1$ and $\bar{\theta}_2$ are equilibrium points of G_1 and G_2 respectively, there exists a real number α, such that $\bar{\theta}'_2 = \bar{\theta}_2 + \alpha$ is an equilibrium point of G_2 and $\bar{\theta} = \bar{\theta}_1 + \bar{\theta}'_2$ is an equilibrium point of G.

Proof: Let B, B_1, B_2, etc. like in Lemma 2.1. If $\bar{\theta}$ is an equilibrium point of G, then $f = \sin(B^T\bar{\theta})$ is a flow on G, thus, by Lemma 2.1, $f_1 = f|_{VG_1}$ is a flow on G_1. Thus, it is enough to prove that $f_1 = \sin(B_1^T\bar{\theta}_1)$, which follows from Lemma 2.2, taking $H(x) = \sin(x)$ and noticing that G_1 is an induced subgraph of G. The case for G_2 follows by the symmetry between G_1 and G_2.

[2] Almost global synchronization could be proved via density functions only for the cases of 2 and 3 agents [17].

Now, assume that $\bar{\theta}_1$ and $\bar{\theta}_2$ are equilibrium points of G_1 and G_2 respectively. Let $\alpha = \bar{\theta}_1(v) - \bar{\theta}_2(v)$, $\bar{\theta}'_2 = \bar{\theta}_2 + \alpha$, $\bar{\theta} = \bar{\theta}_1 + \bar{\theta}'_2$ and $f = \sin(B^T\bar{\theta})$. Then, by Lemma 2.2, $f_1 = f|_{EG_1} = \sin(B_1^T\bar{\theta}_1)$ and $f_2 = f|_{EG_2} = \sin(B_2^T\bar{\theta}'_2)$. On the other hand, due to the invariance of the system we have remarked on Section III, the vector $\bar{\theta}'_2$ is also an equilibrium point of G_2, and then, f_1 and f_2 are flows in G_1 and G_2 respectively. Therefore, by Lemma 2.1, $f_1 + f_2$ is a flow on G, but $f = f_1 + f_2$, because $EG_1 \cap EG_2 = \emptyset$.

□

B. Stability analysis

We will relate the stability properties of the graph G with a cut-vertex with the stability properties of the subgraphs G_1 and G_2 joined by it. Since every equilibrium of G defines an equilibria for G_1 and G_2, we wonder whether or not the dynamical characteristics of these equilibria are or not the same. We will use Jacobian linearization. The zero eigenvalue is always present due to the invariance of the system by translations parallel to $\mathbf{1_n}$. We always refer to the *transversal stability* of the equilibrium set. If the multiplicity of the zero eigenvalue is more than one, Jacobian linearization may fail in classifying the equilibria. Due to space reasons, we present the study of this particular problem in a different article. So, in this work, we assume that we always have a single null eigenvalue.

Theorem 4.1: Consider the graph G, with a cut-vertex v joining the subgraphs G_1 and G_2 of graph G. Let $\bar{\theta} \in \mathbb{R}^n$ be an equilibrium point of G. Then, $\bar{\theta}$ is locally stable if and only if $\bar{\theta}_1 = \bar{\theta}|_{VG_1}$ and $\bar{\theta}_2 = \bar{\theta}|_{VG_2}$ are locally stable and coincide in v $(= VG_1 \cap VG_2)$.

Proof: Recall that the first order approximation of the system around an equilibrium point is given by

$$A_G = -B\, diag\left[\cos(B^T\bar{\theta})\right] B^T.$$

Suppose that G_1 has i vertices, that they come first in the chosen labelling and that v is the last of them ($v = v_i$). Then, a direct calculation gives

$$A_G = A_1 + A_2, \tag{5}$$

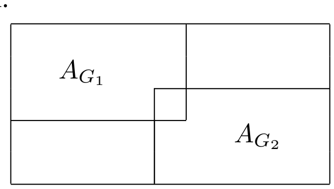
with

$$A_1 = \left[\begin{array}{c|c} A_{G_1} & \mathbf{0_{i\times(n-i)}} \\ \hline \mathbf{0_{(n-i)\times i}} & \mathbf{0_{(n-i)\times(n-i)}} \end{array}\right]$$

and

$$A_2 = \left[\begin{array}{c|c} \mathbf{0_{(i-1)\times(i-1)}} & \mathbf{0_{(i-1)\times(n-i+1)}} \\ \hline \mathbf{0_{(n-i+1)\times(i-1)}} & A_{G_2} \end{array}\right].$$

Observe that these matrices partially *overlap*, so the matrix A takes the form:

$A =$ 

First of all, we consider the case with $\bar{\theta}_1$ and $\bar{\theta}_2$ stable and $\bar{\theta}_1(i) = \bar{\theta}_2(i)$. Then, A_{G_1} and A_{G_2} are stable and equation (5) holds for $\bar{\theta} = [\bar{\theta}_1, \bar{\theta}_2(2 : n-i)]$. So, A_G is the sum of two semidefinite negative matrices which gives rise a semidefinite negative one. Besides, the kernel of A_G has dimension 1, since if $A_G w = 0$ then $w^T A_G w = 0$, thus, $w^T A_1 w + w^T A_2 w = 0$. But, $w^T A_1 w = w_1^T A_{G_1} w_1$ and $w^T A_2 w = w_2^T A_{G_2} w_2$ for $w_1 = w|_{VG_1}$ and $w_2 = w|_{VG_2}$. Then $w_1^T A_{G_1} w_1 + w_2^T A_{G_2} w_2 = 0$. That can happen if only if $w_1^T A_{G_1} w_1 = 0$ and $w_2^T A_{G_2} w_2 = 0$. But the kernels of A_{G_1} and A_{G_2} are spanned by $\mathbf{1_i}$ and $\mathbf{1_{n-i+1}}$ respectively, thus $w_1 = \alpha\mathbf{1_i}$ and $w_2 = \beta\mathbf{1_{n-i}}$. But $w_1(i) = w_2(1) = w(i)$, thus $\alpha = \beta$ and $w = \alpha\mathbf{1}_n$. This proves the stability of A_G.

Now, we focus on the case with $\bar{\theta}_1$ or $\bar{\theta}_2$ unstable. We analyze the first case, since the other is similar. Suppose that A_{G_1} has a positive eigenvalue with associated eigenvector w_1, thus

$$w_1^T A_{G_1} w_1 > 0.$$

Define the vector

$$w = \left[\begin{array}{c} w_1 \\ w_1(i)\mathbf{1_{n-i}} \end{array}\right] = \left[\begin{array}{c} w_1(1\!:\!i-1) \\ w_1(i)\mathbf{1_{n-i+1}} \end{array}\right].$$

Then,

$$w^T A_G w = w_1^T A_{G_1} w_1 + w_1(i)^2 \mathbf{1_{n-i+1}}^T A_{G_2} \mathbf{1_{n-i+1}}$$

which actually is $w_1^T A_{G_1} w_1 > 0$ since $A_{G_2}\mathbf{1_{n-i+1}} = 0$. Then, $\bar{\theta}$ is unstable.

□

We are now ready to state and prove the main result of this article.

Theorem 4.2: Consider the graph G, with a cut-vertex v_i joining the subgraphs G_1 and G_2. Then, G_1 and G_2 have the almost global synchronization property if and only if G does.

Proof: First of all, let $\bar{\theta}$ be an equilibrium point of G. According to Theorem 4.1, $\bar{\theta}$ is stable only if $\bar{\theta}_1 = \bar{\theta}|_{VG_1}$ and $\bar{\theta}_2 = \bar{\theta}|_{VG_2}$ are too.

If G_1 and G_2 are a.g.s., the only locally stable set is the consensus, and since they have a vertex in common, the only locally stable equilibria of G is also the consensus and G is a.g.s.

In the other direction, if $\bar{\theta}_1$ is a locally stable equilibrium of G_1, we chose $\bar{\theta} = [\bar{\theta}_1, \bar{\theta}_1(i)\mathbf{1_{n-i}}]$ and we construct a stable equilibrium for G (as we have mentioned before, a consensus equilibrium is always locally stable [15]). Since G is a.g.s., $\bar{\theta}$, and so $\bar{\theta}_1$, must be consensus equilibrium points.

□

Theorem 4.2 has many direct consequences. We point out some of them, with a brief hint of the respective proofs.

Proposition 4.2: Consider a graph G with a bridge e_k between the nodes v_i and v_j and let G_1 and G_2 be the connected components of $G \setminus \{e_k\}$. Then, G is a.g.s. if and only if G_1 and G_2 are.

If a graph has a *bridge*, i.e., an edge whose removal disconnect the graph, the behavior of the system depends only on the parts connected by the bridge. Indeed, the bridge together with its ends vertices form a block, which is in fact a complete graph and its vertices are cut-vertices of the graph, as is shown in figure 3. Since any complete graph is a.g.s., the a.g.s. character of the original graph depends on the other blocks.

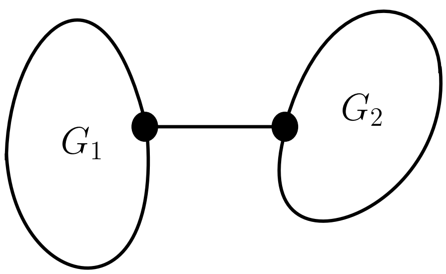


Fig. 3. A graph with a bridge.

Theorem 4.3: A graph G is a.g.s. if and only if every block of G is a.g.s.

The graph G can be partitioned into its blocks. Then, G can be thought as a collection of subgraphs connected by cut-vertices. An iterative use of Theorem 4.2 leads us to the result. Observe that Theorem 4.3 reduces the characterization of the family of a.g.s. graphs to the analysis of 2-connected graphs. As an application, consider the case where we connect two a.g.s. graphs through another a.g.s. graph. In this way, we construct a new a.g.s. graph. Figures 4 and 5 illustrate the situation.

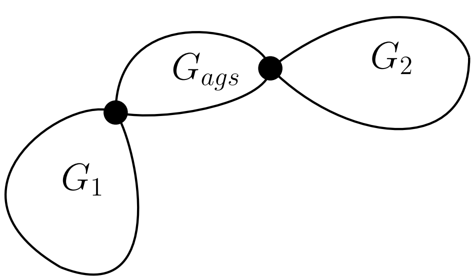


Fig. 4. Two graphs connected by an a.g.s. graph.

In [16], it was proved the next result

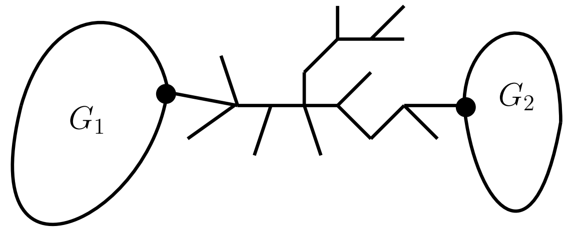


Fig. 5. Two graphs connected by a tree

Proposition 4.3: If G is a tree, it is always a.g.s.

The proof was done using a colouring technique at all the equilibria. Now, we have two alternatives proofs. The first one using Theorem 4.3. We observe that a the blocks of a tree are all K_2, and then, they are a.g.s. The second one is applying iteratively Proposition 4.2, since every link of a tree is a bridge.

If we have a graph with *arboricities*, like the one shown in figure 6, we can neglect the trees in order to prove the a.g.s. property.

Corollary 4.1: A graph with the structure shown in figure 6 is a.g.s. if and only if G_1 is.

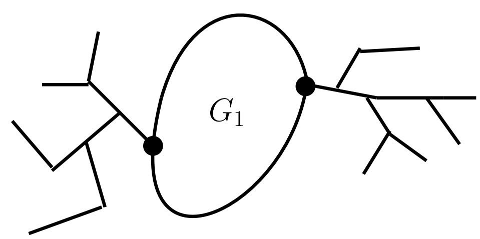


Fig. 6. A graph with *arboricities*.

To conclude this section, we present two general methods for constructing a.g.s. systems.

Proposition 4.4: If G is a tree and we build a new graph K replacing some (or every) edges of G by an a.g.s. graph, then K has the almost global synchronizing property.

Proposition 4.5: If G is a tree and we build a new graph K replacing some (or every) nodes of G by an a.g.s. graph, then K has the almost global synchronizing property.

These conclusions directly follow from the previous results and are illustrated in figure 7. In [16] it was proved that the complete and the tree graphs are a.g.s., while non a.g.s. graphs, like the cycles with more than 4 nodes, were found. Using this fact, we can prove the following sufficient condition for a.g.s. that partially characterizes the family of all a.g.s. graphs.

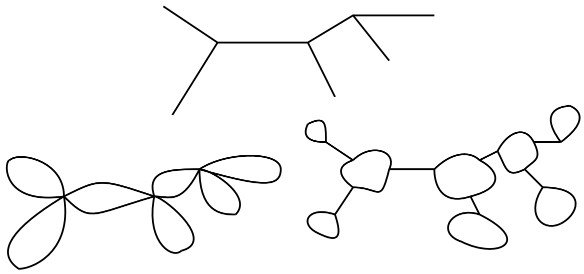


Fig. 7. Situation of Proposition 4.5.

Proposition 4.6: If G is a graph such that all its blocks are complete graphs, then G is a.g.s.

V. Example

Consider two Kuramoto systems with complete underlying interconnection graphs $G_1 = K_3$ and $G_2 = K_5$ (both a.g.s.). Starting from arbitrary initial conditions, each system quickly reaches a consensus state. At time $T = 3$ seconds, we connect the two systems through a bridge between an arbitrary pair of agents. Now, the whole systems reaches a new consensus state. Observe that this convergency is slower than the previous. Figure 8 shows the results obtained from the simulation. They perfectly agree with Proposition 4.2.

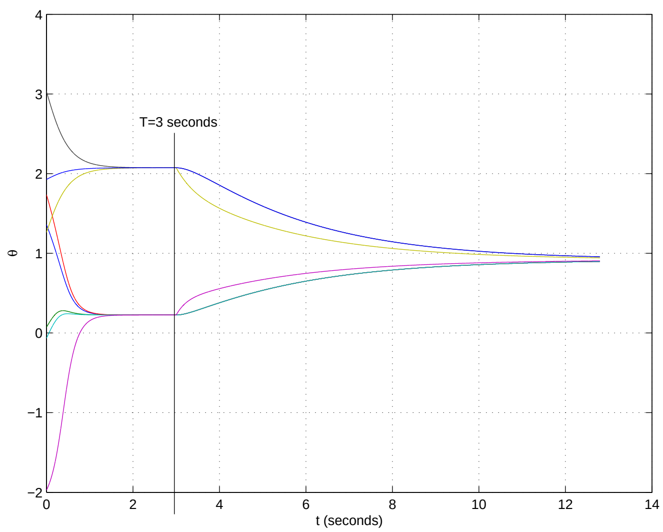


Fig. 8. Two systems connected by a bridge. The connection takes place at time $T = 3$ seconds.

VI. Conclusions

In this work we have studied how some algebraic properties of the underlying graph describing the interconnection of a symmetric Kuramoto model impose restrictions on the dynamical behavior. In particular, we have tried to advance toward a characterization of the a.g.s. graphs. We focus on the particular case of the existence of a cut-vertex between two subgraphs. We proved that the interconnection by a cut-vertex of almost global synchronized systems preserves that property. In particular, we have established that the almost global synchronization analysis of a system with a given interconnection graph G can be reduced to the analysis of the blocks of G. In other words, the general a.g.s. problem may be restricted to the analysis of 2-connected graph topologies. This reduction procedure can be also used to glue synchronized systems in order to get a bigger synchronized system. The gluing can be done using cut-vertices or bridges. We have built a family of a.g.s. graphs that includes both the trees and the complete graphs: all whose blocks are complete graphs. We will try to find more a.g.s. classes of graphs and extended the results to Kuramoto models with non sinusoidal interaction functions.

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