

GLOBAL PROPERTIES OF SYMMETRIC COUPLED OSCILLATORS WITH NON COMPLETE ASSOCIATED INTERCONNECTION GRAPH

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Abstract— : *In a previous work, we have found that almost global synchronization of sinusoidally coupled oscillators is always present when the interconnection is all to all. In this work we remove this hypothesis and investigate several cases where almost global synchronization is still present. We include several examples and counterexamples.*

Keywords— Nonlinear systems, coupled oscillators, almost global stability

1 Preliminaries

In the last years, part of the control theory research has been focused on the analysis of systems which involve many agents that interact in some way, in order to achieve a collective behavior. Cooperative, coordination, formation and synchronization are the keywords in this area. The idea is to exploit the individual capabilities to reach a new state where all the agents act in a collective way, like vehicle formation, flocking, swarming, locking, etc. Of course, distributed sensors and control are present here and this lead to unmanned vehicles, robots and so on. The reader is referred to (Jadbababie, 2003; Marshall, 2004) and references there in.

An aspect of this area is the synchronization of coupled oscillators, since several systems are of or can be reduced to this form (Moshtagh and Jadbabaie and Daniilidis, 2005). Coming from the biology field, through the works of Winfree and Kuramoto on collective synchronization of cells and insects (Strogatz, 2000), the ideas are also suitable for several physical and electrical systems like laser arrays or semiconductor junctions (York, 1993; Strogatz, 1994), wave polarization and arrays of antennas (Dussopt, 1999) and the very old electric oscillators (Van der Pol, 1934; Le Corbeiller, 1935). In view of the broad spectrum of applications, we will refer to the oscillators as agents. In a previous work, we explored global properties of a particular class of coupled oscillators: the ones with sinusoidal coupling (Monzón and Paganini, 2005). As in (Jadbababie, 2003; Jadbabaie, 2004), we used elements from graph theory to describe the interaction. We proved that for an *all to all* interconnection, almost all the trajectories converge to a synchronized state, in the sense that the set of initial conditions that not lead to synchronization has zero Lebesgue measure

and, from an engineering point of view, can be neglected (Rantzer, 2001). In this work, we remove the full interconnection hypothesis and analyze several graph topologies that still have the almost global synchronization property. We present examples and counterexamples to clarify the ideas.

The paper is organized as follows. In Section 2, we recall some elements from graph theory, we present the Kuramoto sinusoidal model and we review some known facts from previous works. In Section 3, we remove the full interconnection property and present some general results. After that, we analyze the particular case of regular graphs. Finally, we present some conclusions.

2 The Kuramoto model

2.1 The model

Consider a group of N oscillators working near their limit cycles. Each oscillator can be described by its phase θ_i , $i = 1, \dots, N$. Without coupling, we can write

$$\dot{\theta} = \omega_i$$

where ω_i is the natural frequency of the i -th oscillator. Kuramoto modelled the interaction between oscillators as follows (Kuramoto, 1984; Kuramoto, 1975):

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^N \Gamma_{ij}(\theta_j - \theta_i) \quad , \quad i = 1, \dots, N$$

where Γ_{ij} are the *interaction* functions. Since $\theta \in [0, 2\pi)$, the corresponding state space is the N -dimensional torus \mathcal{T}^N . In this work, we consider the case with mutual or reciprocal influence between agents and sinusoidal interaction functions:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \cdot \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i) \quad (1)$$

where \mathcal{N}_i refers to the set of index of agents that affect the behavior of agent i (the *neighbors* of i) and K is a the strength of the coupling. We will assume that all the agents have the same natural frequency and we can *shift* and normalize the time, in order to get the simplified expression

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i) \quad (2)$$

Observe that the dynamic depends only on the phase difference between agents. This imply that several properties may be invariant under translations on the torus (that is, if $\bar{\theta}$ have a property, so does $\bar{\theta} + c \cdot \mathbf{1}_N$ for every $c \in [0, 2\pi)$)¹. We denote by **consensus** or **synchronization** the state where all the phase differences are zero, ie., the diagonal of the state space. Every consensus state is of the form $\bar{\theta} = c \cdot \mathbf{1}_N$, with $c \in [0, 2\pi)$. We have a closed curve of consensus points. Observe that at a consensus point, all the associated phases coincide. When most of the phases takes the value 0 (taking a suitable reference), but there are m agents with phase $\pm\pi$, for some $0 < 2m \leq N$, we have a *partial synchronization* state. The other equilibria will be referred as *non-synchronized* states.

As we have mentioned, we will focus on the particular case where influence between agents is symmetric², that is:

$$\text{if } i \in \mathcal{N}_j \Rightarrow j \in \mathcal{N}_i$$

The *all to all* or *complete* case is the one with $\mathcal{N}_i = \{1, 2, \dots, N\}$ for all the agents.

As was done by Kuramoto, we associate to each oscillator a phasor $V_i = e^{j\theta_i}$. An immediate property is that at an equilibrium point $\bar{\theta}$ the numbers

$$\alpha_i = \sum_{j \in \mathcal{N}_i} \frac{V_j}{V_i} = \sum_{j \in \mathcal{N}_i} \cos(\bar{\theta}_j - \bar{\theta}_i)$$

are all real, for $i = 1, \dots, N$ (Monzón and Paganini, 2005).

2.2 Graph theory elements

The interaction between oscillators can be described by a graph G , with each node associated to each agent. A graph has a set of vertex $\mathcal{V} = \{v_1, \dots, v_N\}$ and a sets of links or edges \mathcal{E} . In our model, each vertex represents an oscillator and there is a link between two nodes if they influence each other. As in (Jadbabaie, 2004), we

¹ $\mathbf{1}_N$ denotes the column vector in \mathcal{R}^N with all the elements equal to one

²For some aspects of non-mutual interaction, see (Monzón and Paganini, 2005; Rogge, 2004) and references there in.

give to the associated graph G an arbitrary orientation, so we get a directed graph (digraph). Let us denote by $B = ((b_{ij}))$ *incidence* matrix with N rows and e columns, where e is the number of links of the graph. Then

$$B_{ij} = \begin{cases} 1 & \text{if edge } j \text{ reaches node } i \\ -1 & \text{if edge } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

The *adjacency* matrix $C_G = ((c_{ij}))$ of the graph G is an $N \times N$ symmetric matrix with

$$c_{ij} = \begin{cases} 1 & \text{if } i \text{ is connected to } j \\ 0 & \text{otherwise} \end{cases}$$

Observe that there are zeros at the diagonal of C_G . The *valence* matrix $D = ((d_{ij}))$ of a graph G is a $N \times N$ diagonal matrix with d_{ii} equal to the number of neighbors of the agent i ($d_{ii} = \#\mathcal{N}_i$). If $D = d \cdot I_N$, the graph G is called *regular* of degree d . We denote by \bar{G} the complement of the graph G . It is a new graph with the same vertices and there is a link between two nodes in \bar{G} if there is no link between them in G . Finally, the *laplacian* L is the square matrix defined by $L = D - C_G$. In this article, we will work with connected graph (there is always a path between any two agents). Let I_N be the identity $n \times n$ matrix and $J = \mathbf{1} \cdot \mathbf{1}^T$. We recall the following properties (Biggs, 1993; Cvetkovic and Doob and Sachs, 1979):

- $C_G \cdot \mathbf{1} = D \cdot \mathbf{1}$. So, $L \cdot \mathbf{1} = 0$.
- If G is complete, $C_G = J - I_N$.
- $C_{\bar{G}} = J - I - C_G$.
- The laplacian can be written as $L = BB^T$. So, it is a semidefinite matrix. If G is connected, 0 is a single eigenvalue of L .

If G is regular of degree d :

- d is an eigenvalue of C_G with eigenvector $\mathbf{1}$;
- $C_{\bar{G}}$ is regular with degree $N - 1 - d$;

We can re-write equation (2) in the compact form

$$\dot{\theta} = -B \cdot \sin(B^T \theta) \quad (3)$$

which will be used through the rest of the article. We also assume that all involved graphs are connected.

2.3 General results

Equation (3) has many other equilibria besides the consensus set. In (Jadbabaie, 2004), the local Lyapunov function

$$U(\theta) = e - \mathbf{1}_e^T \cos(B^T \theta) \quad (4)$$

was used to prove local stability of the consensus set. It is clear that $U \equiv 0$ at the synchronized set. The system can be written in the gradient form

$$\dot{\theta} = -\nabla U;$$

In particular this implies that

$$\dot{U}(\theta) = -\|\dot{\theta}\|^2,$$

Hence the function is non-increasing along the trajectories. If we start near enough to consensus set, we will converge to it. Since there are many equilibria, we can not expect that this property holds globally. Moreover, La Salle's result (Khalil, 1996) must be invoked in order to ensure that every trajectory goes to an equilibrium point. Our goal is to state conditions on the graph topology to ensure that the previous convergence property holds for almost all the trajectories of the system. This is equivalent to prove that the consensus set is the only attractor. In (Monzón and Paganini, 2005), it was proved that this is true for a complete graph G .

Theorem 2.1 Consider the system (3) with associated graph G complete. Then, the consensus set is the only attractor and the system has the almost global synchronization property.

The proof combines graph theory with Jacobian linearization and the center manifold analysis³. At an equilibrium point $\bar{\theta}$, the Jacobian matrix A of the system is given by

$$\begin{cases} a_{ii} = -\sum_{k \in \mathcal{N}_i} \cos(\bar{\theta}_k - \bar{\theta}_i) = -\alpha_i \\ a_{hi} = \begin{cases} \cos(\bar{\theta}_h - \bar{\theta}_i) & , h \in \mathcal{N}_i \\ 0 & , h \notin \mathcal{N}_i \end{cases} \end{cases}$$

where α_i are the numbers introduced in the previous Section. The matrix can be also written as $A = -B \cdot \text{diag} [\cos(B^T \bar{\theta})] \cdot B^T$. Completeness of the graph is crucial, as the following Example reflects.

Example 2.1 Consider the case with $N = 6$ in which the dynamics of the agents are as follows:

$$\dot{\theta}_i = [\sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i)]$$

Here the configuration is circular; we identify θ_7 with θ_1 and θ_0 with θ_6 . Consider the equilibrium point showed in Figure 1. Using an approach that will be presented later, it can be shown that this configuration is locally attractive.

△

³The Jacobian matrix always has the zero eigenvalue, which is simple for a connected graph G . We have the so-called transversal stability.

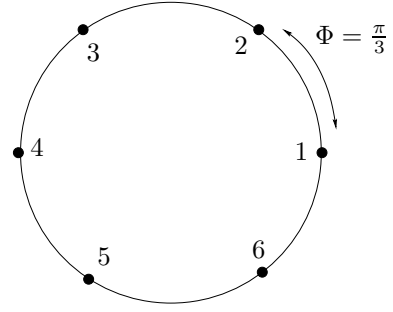


Figure 1: Stable non-consensus equilibrium for the Kuramoto model of Example 2.1.

3 Non-complete systems

In this Section we remove the hypothesis of completeness of the associated graph. The following results are true for general graph topologies.

Proposition 3.1 Let $\bar{\theta}$ be an equilibrium point of (3), such that at least one $\alpha_i < 0$. Then, $\bar{\theta}$ is unstable.

Proof: The thesis follows from the fact the number $-\alpha_i$ appears in the diagonal of the Jacobian matrix.

Proposition 3.2 Let $\bar{\theta}$ be an equilibrium point of (3), such that $\cos(\bar{\theta}_k - \bar{\theta}_i) > 0$ for every $k \in \mathcal{N}_i$, $i = 1, \dots, N$. Then, $\bar{\theta}$ is stable.

Proof: Recall that

$$A = -B \cdot \text{diag} [\cos(B^T \bar{\theta})] \cdot B^T$$

Since $\text{diag} [\cos(B^T \bar{\theta})]$ is positive definite, $\bar{\theta}$ is a local attractor.

Proposition 3.3 Let $\bar{\theta}$ be a partial consensus equilibrium point of (3). Then $\bar{\theta}$ is unstable.

Proof: Since we are dealing with a partial consensus equilibrium point, we can split the agents in two groups. Taking an appropriate reference, we only have phases 0 and π . Define the vector

$$v = \cos(\bar{\theta})$$

Then, v contains only the numbers 1 and -1 . Consider the product $B^T v$. Since each row of B^T refers to an specific link of G , a component of this vector is 0 if the respective link connects two

agents with the same phase, and is ± 2 if the link connects agents with different phases. The matrix

$$\text{diag} [\cos(B^T \bar{\theta})]$$

also has the value -1 at place (l, l) if the link related to the l -th row of B^T joins agents from different groups. Putting all these things together we have the identity

$$v^T A v = -v^T B \cdot \text{diag} [\cos(B^T \bar{\theta})] B^T v = 4 \times c$$

where c is the positive number of links that join agents with different phases. Then, A must have a positive eigenvalue and $\bar{\theta}$ is an unstable equilibrium point. ■

Finally, consider a graph G and its complement \bar{G} . The sum graph, the graph with the same vertices and the union of the links, is a complete graph, regular with degree $N - 1$. Given a non consensus equilibrium point $\bar{\theta}$ of (3), denote by A_G the Jacobian matrix of the system with associated graph G at $\bar{\theta}$. Note that A_G is always symmetric. If $\bar{\theta}$ is also an equilibrium point for the system with graph \bar{G} , then,

$$A_{G+\bar{G}} = A_G + A_{\bar{G}}$$

Since $G + \bar{G}$ is complete and $\bar{\theta}$ is a non consensus equilibrium point, we know that $A_{G+\bar{G}}$ has at least one positive eigenvalue. Then, the following result follows.

Proposition 3.4 *Let $\bar{\theta}$ be a non consensus equilibrium point of (3) with associated graph G and also for the system with associated graph \bar{G} . If $\bar{\theta}$ is a local attractor for the first system, then it is an unstable equilibrium point of the second one.*

Proof: Since A_G and $A_{\bar{G}}$ are symmetric, if they are both stable they would be negative definite and thus so would $A_G + A_{\bar{G}}$, which is a contradiction. ■

The following examples show that an equilibrium point of (3) with graph G may not be an equilibrium point for the graph \bar{G} and even when both systems share an equilibrium point, it can be unstable for both systems.

Example 3.1 *This Example shows that the an equilibrium point of a system with associated graph G may not be an equilibrium for the complement*

system. Consider the non-complete system described by

$$\begin{aligned} \dot{\theta}_1 &= \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1) \\ \dot{\theta}_2 &= \sin(\theta_1 - \theta_2) + \sin(\theta_3 - \theta_2) + \sin(\theta_4 - \theta_2) \\ \dot{\theta}_3 &= \sin(\theta_1 - \theta_3) + \sin(\theta_2 - \theta_3) \\ \dot{\theta}_4 &= \sin(\theta_2 - \theta_4) \end{aligned}$$

If we focus on the equilibrium point given by

$$\bar{\theta}_1 = 0, \quad \bar{\theta}_2 = \frac{2\pi}{3}, \quad \bar{\theta}_3 = \frac{4\pi}{3}, \quad \bar{\theta}_4 = \frac{5\pi}{3}$$

which is shown in figure 2. It is straightforward to show that $\bar{\theta}$ is not an equilibrium point for the system with graph \bar{G} .

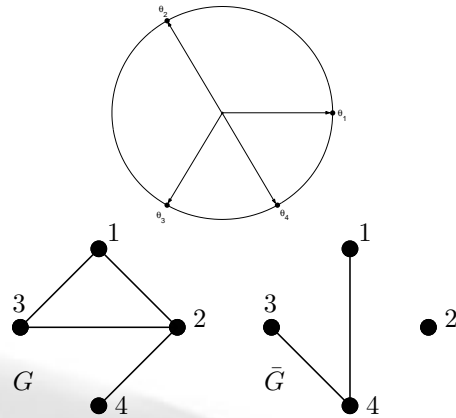


Figure 2: Non-complete system of Example 3.1. △

Next two Examples show an equilibrium point $\bar{\theta}$ which is unstable for both systems with G and \bar{G} . In particular, Example 3.3 illustrates the fact that even when G is connected, \bar{G} may be not connected and this can make things more complicated.

Example 3.2 *Consider the non-complete system with associated graph G shown in figure 3, where it is also shown its complementary graph \bar{G} . We focus on the equilibrium point given by*

$$\bar{\theta}_1 = 0, \bar{\theta}_2 = \frac{\pi}{3}, \bar{\theta}_3 = \frac{2\pi}{3}, \bar{\theta}_4 = \pi, \bar{\theta}_5 = \frac{4\pi}{3}, \bar{\theta}_6 = \frac{5\pi}{3}$$

Then, $\bar{\theta}$ is unstable for both systems. △

Example 3.3 *Figure 4 shows a regular and connected graph and its complement, which is also regular but it is not connected. The equilibrium point*

$$\bar{\theta}_1 = 0, \bar{\theta}_2 = \frac{\pi}{3}, \bar{\theta}_3 = \frac{2\pi}{3}, \bar{\theta}_4 = \pi, \bar{\theta}_5 = \frac{4\pi}{3}, \bar{\theta}_6 = \frac{5\pi}{3}$$

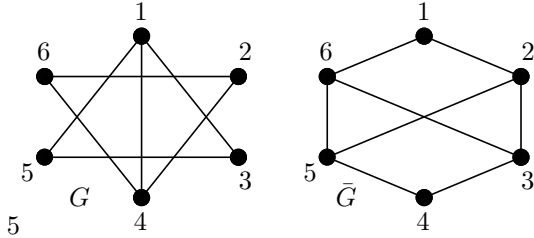


Figure 3: Non-complete system of Example 3.2.

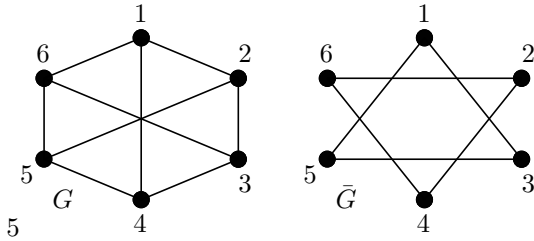


Figure 4: Non-complete system of Example 3.3.

is unstable for both systems.

△

The previous results apply to any associated graph. In order to go on with our analysis, we restrict the class of graphs we are dealing with. We have many possible ways to do that. The next result considers tree graphs.

Theorem 3.1 Consider the system (3) with associated graph G . If G is a connected tree with no cycles, the consensus set is an almost global attractor.

Proof: We prove that the only equilibria correspond to partial or total consensus. A (partial or total) consensus state $\bar{\theta}$ is such that

$$\sin(B^T \bar{\theta}) = 0$$

In order to have only partial or total consensus equilibria, 0 must be the only solution of the equation

$$0 = B.u$$

Observe that for a connected graph, the matrix B , with N rows and e columns, has always rank $N - 1$. Then, the previous equation has only the trivial solution when $e = N - 1$, that is, it has full column rank. The only connected graphs with $N - 1$ links are the trees without cycles. ■

A direct consequence of Theorem 3.1 is that we can interconnect systems with tree graphs in a way such that the almost global synchronization property holds for the augmented system (a kind

of *robust interconnection*). If we only add a single link to join both trees, the resulting graph is still a tree. The next Example shows a particular case of what we have already mentioned.

Example 3.4 A star graph is a connected tree that has a particular node, called the hub, which is related with all of the rest of the nodes, while all the rest of the nodes are related to the hub only. The graph can be drawn as a star and it models several examples of centralized interactions between agents. It is a particular case of Theorem 3.1. The synchronized state is an almost global attractor. Moreover, if we have two star graphs and we couple them through their hubs (or through any pair of agents), we obtain a new almost globally stable system (see figure 5). If we add one more link to a connected tree without cycles, we must have a cycle, and we may lose the almost global attraction property, as in Example 2.1.

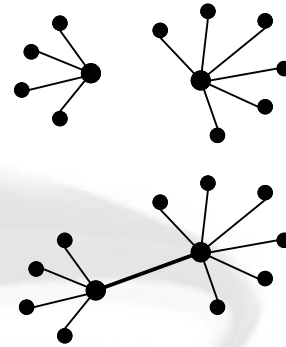


Figure 5: Two star graphs coupled through their hubs (Example 3.4).

△

4 Conclusions and future works

In an earlier work we have shown that sinusoidally coupled oscillators with non complete associated graph present almost global synchronization, in the sense that except for a zero measure set of the state space, almost every initial condition leads the system to a synchronized state. In this work we have explored what happens when we remove the completeness hypothesis on the graph structure of the system. We have introduced some general local and global results for non-complete systems. We think that the next step is to focus on particular classes of associated graphs, relating the topological graph characteristics with the dynamical properties of the system. In this direction, we have proved that systems with associated tree graphs have the almost global synchronization property. As a future work, we have seen that a complete graph, which is a connected regular graph with maximum degree $d = N - 1$, has

the almost global synchronization property. On the other side, a cycle graph like the one of Example 2.1, is a connected regular graph of minimum order $d = 2$ and does not have the property, since there are other attractors than the synchronization set. We wonder if there is a minimum degree for which a regular graph always has the desired property.

5 Acknowledgements

The first author would like to thank the CIC-FING and CSIC-UDELAR for the financial support.

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