ALMOST GLOBAL STABILITY OF TIME-VARYING SYSTEMS

PABLO MONZÓN*

*Instituto de Ingeniería Eléctrica, Facultad de Ingeniería, Universidad de la República, Uruguay.

Email: monzon@fing.edu.uy

Abstract— : *In this work we investigate how the ideas of almost global stability and density functions, introduced in nonlinear control theory few years ago, can be extended to time-varying systems. We present some direct extensions and formulate some questions that can guide the research.*

Keywords— Nonlinear systems, density functions, almost global stability, time-varying systems.

1 Introduction

In 2001, almost global stability (a.g.s.) and density functions were presented to the control community by A. Rantzer in (Rantzer, 2001a). It is said that the origin is an almost global attractor if almost all the trajectories[1] converge to it. Is a concept weaker than global asymptotical stability but that can fit well in nonlinear control applications, specially when it is combined with local asymptotical stability and when global properties can no be stated, for example, when there are multiple isolated equilibria, a typical nonlinear situation. For a.g.s., there is a class of functions that play a role similar of Lyapunov functions for asymptotical stability. For a system $\dot{x} = f(x)$, a density function ρ is a function in $C^1(\mathcal{R}^n \setminus \{0\}, \mathcal{R}^n)$, integrable outside an arbitrary ball centered at the origin and satisfying the first order condition

$$\nabla. [\rho.f](x) > 0 \quad a.e.$$

The key result states that the existence of a density functions ensures the almost global stability of the origin (Rantzer, 2001a). This result stand as a starting point to several research directions in control theory. Converse results were presented in (Rantzer, 2002) and (Monzón, 2003), theoretical properties were explored in (Angeli, 2003), (Rantzer and Prajna, 2003), (Monzón, 2004b), (Monzón, 2004a) and (Monzón, 2005) and control applications were analyzed in (Rantzer and Ceraggioli, 2001b), (Prajna and Rantzer and Parrilo, 2004) and (Angeli, 2004). Several advantages and drawbacks were found, leading to nice applications in some cases and hard restrictions in others.

In this work, we try to extend this new ideas to time-varying systems of the form

$$\dot{x} = f(t, x) \tag{1}$$

with $f \in C^1(\mathcal{R}\times\mathcal{R}^n, \mathcal{R}^n)$, $f(t, 0) = 0$ for all t. We use *time-dependent* density functions and define a time-varying concept of almost global stability. We presents some results and finally we formulate some questions that may help the research.

[1] I.e., all the trajectories but a zero Lebesgue measure set

2 Time-varying systems

Consider the time-varying system of (1). By $\Phi(t, t_0, x)$, $t \geq t_0$, we denote the time t of the trajectory solution of the system which at time t_0 starts at x. For a given set $Z \subset \mathcal{R}^n$, $\Phi(t, t_0, Z)$ will refer to the time t of all the trajectories that at t_0 start at Z.

In order to extend dynamical properties from time-invariant to time-variant systems, one must take care of the influence of the initial time t_0. Usually, the word *uniform* denotes the case when a property holds for every initial time.

In this section we introduce some results on almost global stability of time-varying dynamical systems.

2.1 A Liouville-like result for time-varying systems

Consider a function $\rho : \mathcal{R}\times\mathcal{R}^n \to [0, +\infty)$ of class C^1. Let Z be a Borelian set of $\mathcal{R}^n$ and $t > t_0$. We want to measure the growth of the volume of Z along the flux, weighted by ρ, i.e.„

$$\int_{\Phi(t,t_0,Z)} \rho(t, x)dx - \int_Z \rho(t_0, x)$$

(we assume that both integrals are well defined). Introduce the notation

$$\rho_{t,t_0}(x) = \rho\left[t, \Phi(t, t_o, x)\right].\left|\frac{\partial}{\partial x}\Phi(t, t_0, x)\right|$$

Observe that

$$\int_{\Phi(t,t_0,Z)} \rho(t, x)dx = \int_Z \rho_{t,t_0}(x)dx$$

Let us investigate the dependence of $\rho_{t,t_0}(x)$ with t.

$$\frac{\partial}{\partial t}\rho_{t,t_0}(x) = \lim_{h\to 0}\frac{1}{h}.\left[\rho_{t+h,t_0}(x) - \rho_{t,t_0}(x)\right] =$$

$$\lim_{h\to 0}\frac{1}{h}.\left[\rho\left[t+h,\Phi(t+h,t_o,x)\right].\left|\frac{\partial}{\partial x}\Phi(t+h,t_0,x)\right|\right.$$
$$\left.-\ \rho\left[t,\Phi(t,t_o,x)\right].\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|\right]$$

Since $\Phi(t+h,t_0,x)=\Phi\left[t+h,t,\Phi(t,t_o,x)\right]$ and

$$\left|\frac{\partial}{\partial x}\Phi(t+h,t_0,x)\right|=$$
$$\left|\frac{\partial}{\partial x}\Phi\left[t+h,t,\Phi(t,t_o,x)\right]\right|.\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|$$

we have

$$\frac{\partial}{\partial t}\rho_{t,t_0}(x)=\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|.\lim_{h\to 0}\frac{1}{h}.$$
$$\left\{\rho\left[t+h,\Phi(t+h,t,z)\right].\left|\frac{\partial}{\partial x}\Phi(t+h,t,z)\right|-\rho\left[t,z\right]\right\}$$

where $z=\Phi(t,t_0,x)$. Simplifying the notation we get

$$\frac{\partial}{\partial t}\rho_{t,t_0}(x) \quad = \quad \lim_{h\to 0}\frac{1}{h}.\left\{\rho_{t+h,t}(z)-\rho_{t,t}(z)\right\}$$
$$.\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right| \qquad (2)$$

Then, we only need to calculate the expression

$$\left.\frac{\partial}{\partial t}\rho_{t,t_0}(x)\right|_{t_0}=\lim_{h\to 0}\left\{\rho_{t_0+h,t_0}(z)-\rho_{t_0,t_0}(z)\right\} \quad (3)$$

By the chain rule we know that

$$\frac{\partial}{\partial t}\left.\left\{\rho(t,x).\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|\right\}\right|_{t_0}=$$
$$\left\{\frac{\partial}{\partial t}\rho(t,x).\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|\right.$$
$$+\nabla\rho(t,x).f(t,x).\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|$$
$$\left.\left.+\ \rho(t,x).\frac{\partial}{\partial t}\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|\right\}\right|_{t_0}$$

Then

$$\left.\frac{\partial}{\partial t}\rho(t,x).\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|\right|_{t_0}=$$
$$\frac{\partial}{\partial t}\rho(t_0,x)+\nabla\rho(t_0,x).f(t_0,x)+\rho(t_0,x).\nabla.f(t_0,x)=$$
$$\frac{\partial}{\partial t}\rho(t_0,x)+\nabla.\left[\rho.f\right](t_0,x)$$

and

$$\left.\frac{\partial}{\partial t}\rho(t,x).\left|\frac{\partial}{\partial x}\Phi(t,t_0,x)\right|\right|_{\tau}=$$
$$\left[\frac{\partial}{\partial t}\rho(\tau,z)+\nabla.\left[\rho.f\right](t_0,z)\right].\left|\frac{\partial}{\partial x}\Phi(\tau,t_0,x)\right|$$

with $z=\Phi(\tau,t_0,x)$. We have the following chain of identities

$$\int_{\Phi(t,t_0,Z)}\rho(t,x)dx-\int_Z\rho(t_0,x)=$$
$$\int_Z\rho_{t,t_0}(z)-\rho_{t_0,t_0}(z)dz=\int_Z dz\int_{t_0}^{t}\frac{\partial}{\partial\tau}\rho_{\tau,t_0}(z)d\tau=$$
$$\int_Z dz\int_{t_0}^{t}\left\{\left[\frac{\partial}{\partial t}\rho(\tau,z)+\nabla.\left[\rho.f\right](\tau,z)\right]\right.$$
$$\left..\left|\frac{\partial}{\partial x}\Phi(\tau,t_0,z)\right|\right\}d\tau=$$
$$\int_{t_0}^{t}d\tau\int_{\Phi(\tau,t_0,Z)}\left[\frac{\partial}{\partial\tau}\rho(\tau,x)+\nabla.\left[\rho.f\right](\tau,x)\right]dx$$

The following Proposition resumes the previous result.

Proposition 2.1 *Consider the dynamical system* $\dot{x}=f(t,x)$ *and a function* $\rho:\mathcal{R}\times\mathcal{R}^n\to[0,+\infty)$ *of class* C^1*. Let* Z *be a Borelian set of* $\mathcal{R}^n$ *and* $t>t_0$ *such that* ρ *is integrable in* Z *and* $\Phi(\tau,t_0,Z)$, $t_0\le\tau\le t$*. Then*

$$\int_{\Phi(t,t_0,Z)}\rho(t,x)dx-\int_Z\rho(t_0,x)=$$
$$\int_{t_0}^{t}\int_{\Phi(\tau,t_0,Z)}\left[\frac{\partial}{\partial t}\rho(\tau,x)+\nabla.\left(\rho.f\right)(\tau,x)\right]dxd\tau$$

■

For the particular case of an autonomous system and a constant ρ we recover the classical Liouville result (Arnold, 1974); for an autonomous system and a function ρ independent of time, we have the result of (Rantzer, 2001a); for a control system of the form

$$\dot{x}=f(x,u)$$

and a function ρ independent of time, we obtain the result of (Angeli, 2004).

2.2 Time-varying a.g.s.

The result of Proposition 2.1 has many interesting direct consequences when the expression

$$\frac{\partial}{\partial t}\rho(\tau,x)+\nabla.\left(\rho.f\right)(\tau,x)$$

has positive sign almost everywhere. First of all, observe that if ρ and a given Borelian set Z satisfy the hypothesis of Proposition 2.1 and

$$\Phi(t,t_0,Z)\subset Z$$

then Z has zero Lebesgue measure.

Proposition 2.2 *Consider the system* $\dot{x}=f(t,x)$ *such that* $f(t,0)=0$ *for all* t *and 0 is a locally stable*[2] *equilibrium point. Let* $\rho:\mathcal{R}^n\setminus\{0\}\to[0,+\infty)$ *of class* C^1 *such that*

$$\frac{\partial}{\partial t}\rho(\tau,x)+\nabla.\left(\rho.f\right)(\tau,x)>0 \qquad (4)$$

[2]Not necessarily uniformly stable.

for almost every point $x \in \mathcal{R}^n$, $t \in \mathcal{R}$. Moreover, assume that $\rho(t,x)$ is integrable for fixed t over $\{\|x\| > \epsilon\}$ for every $\epsilon > 0$. Then, the origin is almost globally stable in the sense that for every initial time t_0, the set of points that are not asymptotically attracted by the origin has zero Lebesgue measure.

Proof:

Consider an arbitrary initial time t_0, and arbitrary positive number r and the set

$$Z = \{z \in \mathcal{R}^n \mid \exists x \in \mathcal{R}^n \; and \; t \geq t_0, \; with$$

$$z = \Phi(t,t_0,x) \; and \; \limsup_{\tau \geq t_0} \|\Phi(\tau,t_0,x)\| \geq r\}$$

By construction the set Z contains all the trajectories that starting at time t_0 have superior limit greater or equal than r. Then, if $x \notin Z$, we know that

$$\|\Phi(t,t_0,x)\| < r \quad t \to +\infty$$

We will show that Z has zero Lebesgue measure. By construction we have

$$\Phi(t,t_0,Z) \subset Z$$

From the local stability hypothesis, we obtain a positive δ satisfying

$$\|\Phi(t,t_0,x)\| < r \quad \forall \|x\| < \delta \;\; \forall t \geq t_0$$

Then $Z \cap B(0,\delta) = \emptyset$, $\rho(t,x)$ is integrable on Z for fixed t and then Z has zero Lebesgue measure.

■

Consider the linear time-varying system

$$\dot{x} = A(t).x \quad A : \mathcal{R} \to \mathcal{R}^{n \times n} \tag{5}$$

It is well known that the solution of this system is given by

$$\Phi(t,t_0,x) = \Phi(t,t_0).x$$

where the *transition matrix* $\Phi(t,t_0)$ satisfy the linear differential equation (Zadeh, 1963)

$$\frac{d}{dt}\Phi(t,t_0) = A(t).\Phi(t,t_0) \quad \Phi(t_0,t_0) = I_{n\times n}$$

and it is also true the identity

$$\frac{d}{dt}\left|\Phi(t,t_0)\right| = tr\left[A(t)\right].\left|\Phi(t,t_0)\right|$$

Then the asymptotical behavior of the trajectories is totally determined by $\Phi(t,t_0)$. For the particular case of a piecewise continuous bounded $A(t)$ we have that the almost global stability of the origin is equivalent to global asymptotical uniform stability (Zadeh, 1963) and it is also equivalent to the existence of a quadratic Lyapunov function $V(t,x) = x^T P(t) x$, with $P(t)$ a C^1 matrix function, uniform definite positive[3] and such that

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) = -Q(t)$$

where $Q(t)$ is a continuous bounded uniform positive definite matrix function (Khalil, 1996).

Proposition 2.3 *Let the system*

$$\dot{x} = A(t).x$$

with $A(t)$ piecewise continuous bounded and assume that the origin is a globally asymptotically stable equilibrium point. Then, there exists a C^1 function $\rho : \mathcal{R} \times [\mathcal{R}^n \setminus \{0\}] \to [0,+\infty)$ such that for all real t

$$\frac{\partial}{\partial t}\rho(t,x) + \nabla.\left(\rho.f\right)(t,x) > 0 \;\; a.e.$$

Proof:

The proof follows the same ideas of the invariant case (Rantzer, 2001a). From the asymptotical properties of the system, we know the existence of a quadratic time-varying Lyapunov function $V(t,x) = x^T P(t) x$, with $P(t)$ continuous, differentiable, bounded, such that

$$\dot{P}(t) + A(t)^T P(t) + P(t)A(t) = -Q(t)$$

with $x^T Q(t) x \geq \gamma\left(|x\|\right)$, and

$$0 \leq \alpha\left(|x\|\right) \leq V(t,x) \leq \beta\left(|x\|\right)$$

α, β and γ class K functions. We affirm that

$$\rho(t,x) = \left[V(t,x)\right]^{-\alpha}$$

solves our problem, for big enough α. Note that

$$\nabla\rho(t,x) = -\alpha.\left[V(t,x)\right]^{-(\alpha+1)}\nabla V(t,x)$$

$$\nabla.\left[A(t).x\right] = tr\left[A(t)\right]$$

$$\nabla V(t,x).A(t).x = x^T\left[A(t)^T P(t) + P(t)A(t)\right]x$$

Then

$$\begin{aligned}&\tfrac{\partial}{\partial t}\rho(t,x) + \nabla.\left[\rho(t,x).A(t)x\right] = \\ &\left[V(t,x)\right]^{-(\alpha+1)} x^T\left[\alpha.Q(t) \; + tr\left[A(t)\right].P\right]x\end{aligned} \tag{6}$$

Since $tr\,A(t)$ is bounded and $P(t)$ and $Q(t)$ are bounded positive definite functions, we always can take α so big in order to obtain positive definition of (6) and the integrability of $\rho(t,x)$ in the domain $\|x\| \geq 1$.

■

[3]By this we mean that there are class K functions α, β such that(Khalil, 1996)

$$0 \leq \alpha(\|x\|) \leq x^T P(t) x \leq \beta(\|x\|)$$

.

3 Conclusions

In this work we have introduced the concept of monotone Borel measures in the context of almost global stability of dynamical systems. We have shown that for an almost global asymptotical system we can find a monotone Borel measure (increasing or decreasing) and this idea complements the direct result of (Rantzer, 2001a) and the particular converse result of (Monzón, 2003). We think that this approach can guide us to a new set of results, like the ones presented in (Monzón, 2004a).

4 Acknowledgements

I want to specially thank Professor Jorge Lewowicz for his constant dedication and advice. This work is partially supported by CSIC-UDELAR and CIC-FING.

References

Angeli, D. (2003). Some remarks on density functions for dual Lyapunov methods, *Proceedings of the 42nd IEEE Conference on Decision and Control*, Hawaii, pp. 5080–5082.

Angeli, D. (2004). An almost global notion of input-to-state stability, *Trans. on Automatic Control* **47**(3): 410–421.

Arnold, V. (1974). *Équations Differentielles Ordinaires*, MIR, Moscu.

Khalil, H. (1996). *Nonlinear Systems*, Prentice-Hall.

Monzón, P. (2003). On necessary conditions for almost global stability, *Trans. on Automatic Control* **48**(4): 631–634.

Monzón, P. (2004a). Monotone measures and almost global stability of dynamical systems, *Sixteenth International Symposium on Mathematical Theory of Networks and Systems (MTNS2004)*, Katholieke Universiteit Leuven, Leuven, Belgium, TP3-3.

Monzón, P. (2004b). Monotone measures for dynamical systems, *Congresso Brasileiro de Automática*, Gramado, Brasil.

Monzón, P (2005). Almost global attraction in planar systems., *Systems and Control Letters* 54 (8): 753–758.

Prajna, S.; Parrilo, P. Rantzer, A. (2004). Nonlinear control synthesis by convex optimization, *Trans. on Automatic Control* **41**(5): 310–314.

Rantzer, A. (2001a). A dual to Lyapunov's stability theorem, *Systems and Control Letters* **42**(3): 161–168.

Rantzer, A. Ceraggioli F. (2001b). Smooth blending of nonlinear controllers using density functions, In *Proceedings of the European Control Conference*.

Rantzer, A. (2002). A converse Theorem for density functions, *Proceedings of the 41st IEEE Conference on Decision and Control*, Las Vegas. pp. 1890-1891.

Rantzer, A.; Prajna S. (2003). On homogeneous density functions, *in* A. Rantzer and C. Byrnes (eds), *Directions in Mathematical Systems Theory and Control*, Springer Verlag, Berlin Heidelberg, pp. 261–274. ISBN 3-540-00065-8.

Zadeh, L. A.; Desoer, C. A. (1963). *Linear System Theory*, McGraw-Hill, NY.