On the Complexity of the Classification of Synchronizing Graphs

Eduardo Canale*, Pablo Monzón, and Franco Robledo

IMERL-IIE-Facultad de Ingeniería, Udelar, Montevideo, Uruguay
{canale,monzon,frobledo}@fing.edu.uy
http://www.fing.edu.uy/imerl,
http://iie.fing.edu.uy/

Abstract. This article deals with the general ideas of almost global synchronization of Kuramoto coupled oscillators and synchronizing graphs. It reviews the main existing results and gives some new results about the complexity of the problem. It is proved that any connected graph can be transformed into a synchronized one by making suitable groups of twin vertices. As a corollary it is deduced that any connected graph is the induced subgraph of a synchronizing graph. This implies a big structural complexity of synchronizability. Finally the former is applied to find a two integer parameter family $G(a, b)$ of connected graphs such that if b is the k-th power of 10, the synchronizability of $G(a, b)$ is equivalent to find the k-th digit in the expansion in base 10 of the square root of 2. Thus, the complexity of classify $G(a, b)$ is of the same order than the computation of square root of 2. This is the first result so far about the computational complexity of the synchronizability problem.

Keywords: Network synchronization, coupled oscillators, synchronizing graphs, graph complexity.

1 Introduction

Kuramoto's mathematical model for coupled oscillator represents a wide variety of real systems, from biology to engineering, in which the coordination of several individuals is relevant [1–3]. The i-th oscillator or *agent* is described by its phase $\theta_i : \mathbb{R} \to \mathbb{R}$ and the differential equation

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i) \quad , \quad i = 1, \ldots, n \tag{1}$$

models the variation of this phase as a result of the influences of the other oscillators[1]. $\mathcal{N}_i$ denotes the *neighbors* of agent i. These sets can be described by a graph. The natural state space is the n-dimensional torus $\mathbb{T}^n$, since the function sin is 2π-periodic. Thus, we will consider $\theta_i \in [0, 2\pi)$, $i = 1, \ldots, n$.

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[1] Equation (1) describes the Kuramoto model for weakly coupled identical oscillators; see [1] for more general descriptions.

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We work with mutual influence between agents, i.e., $i \in \mathcal{N}_j$ if and only if $j \in \mathcal{N}_i$. Synchronization of oscillators is a classical topic in electrical engineering [4]. In the last years, several works in the control community have addressed the problem of analyzing local stability properties of the Kuramoto model [5–8]. Global properties were explored in [9–13]. When a system has the almost global stability, the origin of the system attracts all the initial conditions, with the possible exception of a zero Lebesgue measure set [9]. From an engineering point of view, it is a desirable situation, specially when the system has several equilibrium points. For a Kuramoto model, the origin represents the *consensus* or *synchronization* of all the agents. *Almost global synchronization* (AGS) means that almost every initial condition leads to synchronization [9]. We denote by $\theta \in \mathbb{T}^n$ the vector of phases. Interaction between agents is modeled by using a non-oriented graph G, with nodes representing the agents and links showing the interconnections [14]. If we endow G with an arbitrary orientation, we obtain the respective incidence matrix B and the compact description [5]

$$\dot{\theta} = -B.\sin(B^T\theta) \tag{2}$$

for equation (1). We can write $V(\theta) = m - \mathbf{1}_m.\cos(B^T\theta)$ and $V''(\theta) = -B.diag[\cos(B^T\theta)].B^T$. So, (2) is a *gradient* system (see [5, 7]). Since the *Laplacian* of G is the square matrix $L = B.B^T$ [14], $V''(\theta)$ can be thought as a weighted laplacian. This fact gives a direct link between dynamical properties of system (2) and algebraic properties of graph G. When system (2) has the AGS property, we say that the graph G *synchronizes* or is *synchronizing.* As Kuramoto suggested, each oscillator can be thought as a running particle on the unit circumference, where the respective phase defines its position. So, each oscillator may be represented by a phasor $V_i = e^{j\theta_i}$, $i = 1, \ldots, n$, where $j = \sqrt{-1}$. For each agent i, we introduce the complex numbers

$$\alpha_i(\theta) = \frac{1}{V_i}\sum_{k\in\mathcal{N}_i} V_k = \sum_{k\in\mathcal{N}_i}\frac{V_k}{V_i} = \sum_{k\in\mathcal{N}_i} e^{j(\theta_k-\theta_i)} \tag{3}$$

As we have mentioned,

$$\dot{U}(\theta) = -\|\dot{\theta}\|^2 \tag{4}$$

So, U decreases along the trajectories of system (2). For $U_0 = m$, U is non negative and generalizes in some sense the square of the *order parameter* defined by Kuramoto ([5]). Since, in this case, $U(c\mathbf{1}_n) = 0$ for every $c \in [0, 2\pi)$, we may see U as a local Lyapunov function that proves the local stability of the consensus (see [5, 15]). Moving towards global properties, we wonder what happens when we start far from the consensus curve. Equation (4) says that the potential is non increasing along the trajectories of the system. Since we are working on a compact state space, we may apply LaSalle result and conclude that every trajectory converge to an equilibrium point of (2) (see [15]). Moreover, the only attractors of the system are the equilibria. So, in order to have the AGS property, the consensus must be the only attractor of the state space or, equivalently, every non consensus equilibria must be unstable. Our main tool for classifying the equilibria is Jacobian linearization. At an equilibrium point $\bar{\theta}$, the Jacobian matrix $M_{n\times n}$ of system (2) is symmetric and takes the explicit form

$$m_{ii} = -\sum_{k\in\mathcal{N}_i} \cos(\bar{\theta}_k - \bar{\theta}_i) = -\alpha_i(\bar{\theta}) \quad , \quad m_{hi} = \begin{cases} \cos(\bar{\theta}_h - \bar{\theta}_i) & , h \in \mathcal{N}_i \\ 0 & , h \notin \mathcal{N}_i \end{cases} \tag{5}$$

or, in a compact notation: $M = -B.diag\left[\cos(B^T\bar{\theta})\right].B^T$, which can be thought as a *weighted* laplacian. Observe that always $M.\mathbf{1}_n = 0$ and M has a null eigenvalue. This is related to the fact that the system is invariant under translations parallel to vector $\mathbf{1} \in \mathbb{T}^n$. Due to this invariance, we only care about the rest of the eigenvalues. If M has a positive eigenvalue, then $\bar{\theta}$ is unstable; if M has $n-1$ negative eigenvalues, $\bar{\theta}$ is stable. If the null eigenvalue is not simple, then Jacobian linearization is not enough for proving stability of $\bar{\theta}$. Without looking directly to the eigenvalues, we can work with the quadratic form induced by M. Let $x \in \mathbb{R}^n$ and denote by ik the link between nodes i and k, when it exists. Then,

$$x^T M x = -\sum_{ik\in E} (x_k - x_i)^2 \cos(\theta_k - \theta_i) \tag{6}$$

In Section 2, we quickly review the main properties of *synchronizing* graphs. In Section 3, we introduce the idea of twin vertices and later, in Section 4, we analyze the classification of a given family of graphs and we introduce some findings about its complexity. Finally, we present some conclusions.

2 Main Properties

In this Section, we will focus on the general properties of equation (2), its equilibria and its relationships with the underlying interconnection graph. We have the following general results.

Lemma 1. *$\bar{\theta} \in \mathbb{T}^n$ is an equilibrium point if and only if $\alpha_i(\bar{\theta})$ is real, for $i = 1, \ldots, n$.*

Proof: From equation (3), is clear that $\alpha_i(\theta) = \sum_{k\in\mathcal{N}_i} \cos(\theta_k - \theta_i) + j\sum_{k\in\mathcal{N}_i} \sin(\theta_k - \theta_i)$. So, at an equilibrium point, the imaginary part vanishes and the numbers α_i are all real. ∎

Lemma 2. *Let $\bar{\theta} \in \mathbb{T}^n$ be an equilibrium point.*

i) If $\cos(\bar{\theta}_i - \bar{\theta}_j) > 0$ for every connected pair of nodes i, j, then $\bar{\theta}$ is stable.
ii) If for some i, the number $\alpha_i(\bar{\theta})$ is negative, then $\bar{\theta}$ is unstable.
iii) If for some i, the number $\alpha_i(\bar{\theta})$ is null, then $\bar{\theta}$ is unstable.
iv) If for a suitable reference, some $\bar{\theta}_i \in (0, \frac{\pi}{2})$ and the rest of the agents' phases are in $(\pi, \frac{3\pi}{2})$, then $\bar{\theta}$ is unstable.

v) *If all the agents' phases are located inside a semi circumference, then $\bar{\theta}$ is a consensus equilibria.*
vi) *If $\bar{\theta}$ is a partial synchronized equilibrium point, then $\bar{\theta}$ is unstable.*
vii) *Consider an agent i with only two neighbors: $\mathcal{N}_i = \{h, k\}$. Let $\bar{\theta}$ be a stable equilibrium point and define the angles $\varphi_{ik} = |\bar{\theta}_i - \bar{\theta}_k|$ and $\varphi_{ih} = |\bar{\theta}_i - \bar{\theta}_h|$. Then, $\varphi_{ik} = \varphi_{ih}$.*

Proof: We only prove item *vii)*; the rest of the proofs can be found in [16]. The equilibrium condition at agent i implies that $\sin(\varphi_{ik}) = \sin(\varphi_{ih})$. Then, it must be true that either $\varphi_{ik} = \varphi_{ih}$ or $\varphi_{ik} + \varphi_{ih} = \pi$. But in the last case, we have $\alpha_i(\bar{\theta}) = 0$ and $\bar{\theta}$ should be unstable. ■

3 Twin Vertices

In this Section we introduce the idea of *twin vertices*, together with its main properties.

Definition 1. *Consider two nodes u and v of a graph G. We say they are twins if the have the same set of neighbors: $\mathcal{N}_u = \mathcal{N}_v$.* ♢

Slightly modifying previous definition, we also say that two vertices are *adjacent twins* if they are adjacent and $\mathcal{N}_u \setminus \{v\} = \mathcal{N}_v \setminus \{u\}$. Concerning synchronization, twins vertices act as a *team* in order to get equilibrium in equation (2). The following result concerns the necessary behavior of twins and will be useful for the complexity analysis we will perform.

Lemma 3. *Consider the system (2) with graph G. Let $\bar{\theta}$ be an equilibrium point of the system and v a vertex of G, with associated phasor V_v. Let T be the set of twins of v and $\mathcal{N}$ the set of common neighbors. If the real number $\alpha_v = \frac{1}{V_v}\sum_{w\in\mathcal{N}} V_w$ is nonzero, the twins of v are partially or fully coordinated with it, that is, the phasors V_h, with $h \in \mathcal{N}_v$ are all parallel to V_v. Moreover, if $\bar{\theta}$ is* ***stable****, the agents in T are fully coordinated.*

Proof: See [12]. ■

We may define an equivalence relationship in the node set of a graph: two nodes are *equivalent* if they are twins. So, we can obtain a *quotient* graph by direct identification of equivalent nodes. The quotient graph can be seen as an induced subgraph of the original one and reciprocally, we can *cover* a given graph by a larger graph, obtained by the addition of twins. These lead us to the following results.

Theorem 1. *Any connected graph G admits an AGS twin cover.*

Proof: Let us suppose that the set of vertices of G is $\{1, \dots, n\}$ and that we have constructed the cover $\bar{G}$ by splitting each vertex i of G in a number a_i of adjacent twins vertices. Then, if θ is a non linearly stable equilibrium of $\bar{G}$, by Lemma 3, the twins must be synchronized and for a given vertex i, we have: $\sum_{j \in N_i} a_j \sin(\theta_j - \theta_i) = 0$, since all the twins of vertex i have phase θ_i. Then, for any $k \in N_i$:

$$a_k \sin(\theta_k - \theta_i) = - \sum_{j \in N_i, j \neq k} a_j \sin(\theta_j - \theta_i) \quad , \quad |\sin(\theta_k - \theta_i)| \leq \frac{\sum_{j \in N_i, j \neq k} a_j}{a_k}$$

Then, if the second member is small enough to ensure that the sine in the first member is small we are done. Indeed, small sines, say smaller than $\sqrt{2}/D$ where D is the diameter of G, implies phasors in opposite quarter circumference, but by Lemma 2-*iv)* this implies unstability. Then the phasor must be in a quarter circumference, which implies by Lemma 2-*v)* that the equilibrium is a consensus. We can in fact do something weaker, bounding the sines of adjacent vertices in a spanning tree of G by $\epsilon = \sqrt{2}/H$, where H is the height of the tree. In order to do this, we will construct a rooted directed spanning tree T of $G = (V, E)$ and then for each i we will take as k to be the father of i in this tree. Let S_h be the vertices at distance h from vertex 1. Then sort each set S_h with an order $<_h$. We consider the following "lexicographical" order on V: given two vertices $u \in S_i$ and $w \in S_j$ we say that $u < w$ if $i > j$ or if $i = j$ and $u <_i w$. The order defined in this way is total, therefore we can label the vertices following this order so that $1 = v_1 > v_2 > \dots > v_n$. Next, set $a_i = \lceil (\Delta/\epsilon)^{|V|-i} \rceil$, where Δ is the maximum degree of a vertex in G. Then the arcs of T will be those $v_k v_i$ such that $ki \in E$ and $a_k = \max_{j \in N_i} a_j$. Notice that $v_k > v_i$. We must prove that T is a tree. Indeed, it is acyclic, because for each $i > 1$ any vertex in S_i is adjacent from exactly one vertex in S_{i-1}. Besides v_1 reaches every vertex, thus T is connected as well. Let us now find an upper bound for the sine of the difference between adjacent vertices of T. Let v_i and v_k be adjacent vertices of T with $i > k$, then

$$|\sin(\theta_k - \theta_i)| \leq \frac{1}{a_k} \sum_{j \in N_i, j \neq k} a_j \leq \frac{(\Delta - 1)\left[(\Delta/\epsilon)^{n-k-1} + 1\right]}{(\Delta/\epsilon)^{n-k}} < \epsilon$$

for any $\epsilon < \Delta$. ■

This result tells that the class of synchronizing graphs could be quite large, but not necessarily complex, since its complement could be small. Nevertheless, we will prove that any 2-connected graph is homeomorphic to a non AGS graph telling us that the complement to AGS graphs is also quite big and complex. In order to do it we will prove a stronger result, namely, that doing an enough amount of elementary subdivision in any edge produces a non AGS graph. We conjecture that it is enough to do three subdivisions.

Theorem 2. *If e is a non bridge edge of a graph G then there is an integer n_0 such that the graph obtained from G by making $n > n_0$ elementary subdivisions of e is non AGS.*

Proof: The idea is the following: since the cycle C_n is non AGS for $n \geq 5$, we can replace one of its edges, say uv by $G-e$ identifying the extremes of e with u and v. If n is large enough the "force" induced by C_n will be weak enough to change a consensus of $G-e$ to another still stable equilibrium. Let us denote by $v_1, \ldots,$ v_m be the vertices of G and let $e = v_1 v_2$. Since e is not a bridge, $G' = G - e$ is connected and then the consensus $\theta = (0, 0, \ldots, 0)$ is stable in $G' - e$. Now, connect the vertices v_1 and v_2 of G' through a path $P_n : v_1 = w_1, \ldots, w_n = v_2$ to obtain a graph $\tilde{G}$ with vertices $\tilde{V} = \{w_1, \ldots, w_n, v_3, \ldots, v_m\}$. We want to prove that for n large enough there exist an $\epsilon > 0$ and phases θ_i^ϵ for $3 \leq i \leq m$ such that $\theta^\epsilon : \tilde{V} \to \mathbb{R}$ defined by:

$$\theta_x^\epsilon = \begin{cases} i\epsilon, \text{ if } x = w_i; \\ \theta_i^\epsilon, \text{ if } x = v_i, \end{cases}$$

is a stable equilibrium point of $\tilde{G}$. In order θ^ϵ to be an equilibrium it must satisfies $\sum_{y \in \tilde{N}_x} \sin(\theta_y^\epsilon - \theta_x^\epsilon) = 0$ for all $x \in \tilde{V}$. These equations are trivially fulfilled for $x = w_2, \ldots, w_{n-1}$. Thus, it remains the following equations:

$$\begin{cases} \displaystyle\sum_{y \in N'_{v_1}} \sin(\theta_y^\epsilon - \theta_{v_1}^\epsilon) + \sin(\epsilon) = 0, \\ \displaystyle\sum_{y \in N'_{v_2}} \sin(\theta_y^\epsilon - \theta_{v_2}^\epsilon) - \sin(\epsilon) = 0, \\ \displaystyle\sum_{y \in N_x} \sin(\theta_y^\epsilon - \theta_x^\epsilon) = 0 \qquad x \in V \setminus \{v_1, v_2\}, \end{cases}$$

where N and N' denote neighbors in G and G' respectively. This system can be thought as an ϵ–perturbation of the system that defines the equilibrium of G'. Moreover, if we add an adequate equation, e.g. $\theta_{v_1} = 0$, the system verifies the hypothesis of the implicit function theorem for $\theta = \mathbf{0}_m$ and $\epsilon = 0$. Thus it implicitly defines the angles θ_x^ϵ as a function of ϵ, for each node of G, in a neighborhood $(-\epsilon_0, \epsilon_0)$ of 0. Moreover, we will have that θ^ϵ is a C^∞–curve in $\mathbb{R}^m$ passing through $\mathbf{0}_m$ for $\epsilon = 0$. Finally, in order to prove the stability, we notice that the cosines $\cos(i\epsilon - (i-1)\epsilon) > 0$ for ϵ small enough. Besides, when $\epsilon = 0$ all the cosines $\cos(\theta_i^\epsilon - \theta_j^\epsilon)$ are positive and since ϵ is small enough, by the continuity of θ_i^ϵ as a function of ϵ, the cosines $\cos(\theta_i^\epsilon - \theta_j^\epsilon)$ will remain positive, thus, by Lemma 2-*i)*, the equilibrium is stable. Therefore, it suffices to take $n_0 > 2\pi/\epsilon_0$. ∎

4 Complexity Analysis for a Graph Family

In this Section we use the previous ideas in order to analyze some complexity issues of the classification of AGS graphs. It is clear that the classification of a given graph requires some knowledge of the structure of the set of equilibrium points of (2). In some cases, this set is quite simple or has a well understood behavior[2]. But in general, this set is very complicated and its study can be

[2] For example, when G is a tree, a cycle or a complete graph. See [9, 17].

hard. In order to illustrate the complexity of the problem, we present a result concerning a class of graphs whose classification is as hard as the computation of $\sqrt{2}$. Let n, a and b be natural numbers and consider the cycle C_{n+2}. We build a new graph $G_n(a,b)$, adding a twins to agent 1 and b twins to the rest of the agents. We will explore the existence of other equilibrium points besides consensus. We will also impose conditions on n, a and b in order to have or not the AGS property.

Theorem 3. *Given positive integers a, b and n, the graph $G_n(a,b)$ is AGS if*

$$n > 3,\ \ a < \sin(\frac{\pi}{n})b \ \ , \ \ n = 3,\ \ \frac{\sqrt{3}}{2}b < a < \sqrt{2}b \tag{7}$$

Proof: from Lemma 2-*vii)* and Lemma 3, a stable non-consensus equilibrium candidate configuration can be easily derived, as is shown in Fig. 1. We retain the reference nodes 1 to $n+2$ from the original graph. We denote by α and β the two involved angles. Equilibrium conditions for node 1 and nodes i, $i = 3, \ldots, n+1$ are trivially satisfied. Equilibrium conditions for nodes 2 and $n+2$ results in the following equations for α and β: $a\sin(\alpha) = b\sin(\beta)$ and $2\alpha + n\beta = 2\pi$. So, since $\alpha = \pi - n\frac{\beta}{2}$ and $\sin\left(\pi - n\frac{\beta}{2}\right) = \sin\left(n\frac{\beta}{2}\right)$, we have this implicit equation for β:

$$\frac{a}{b}\sin\left(n\frac{\beta}{2}\right) = \sin(\beta)$$

We define the auxiliary function $f_n(\beta) = \frac{a}{b}\sin\left(n\frac{\beta}{2}\right)$. Notice we have the trivial solution $\beta = 0$, and the respective $\alpha = \pi$, which correspond to a partial consensus equilibrium configuration, which is unstable. So, we rule out this solution. In Figure 2, we show many distinct possibilities of the relative position of the curves $\sin(\beta)$ and $f_n(\beta)$, for the case $n = 3$. We will only look for solutions $\beta \in [0,\pi]$ and $n \geq 1$. A β greater than π is only possible for $n = 1$ but the respective number $\alpha = \pi - \frac{\beta}{2}$ will be greater than $\frac{\pi}{2}$ and will have a negative cosine, given an unstable equilibrium point. Note that the number n affects the frequency of function $f_n(\beta)$. An immediate condition for the existence of a non-consensus equilibrium is $f'(0) > 1$:

$$f'(0) = \frac{na}{2b} > 1 \Leftrightarrow a > \frac{2}{n}b \tag{8}$$

If it does not happens, $G_n(a,b)$ is synchronizing. Now, we assume that condition (8) is fulfilled and we have at least one solution $\beta^* \in [0,\pi]$ (and its corresponding α^*). We perform a stability analysis of the corresponding equilibrium point. According Lemma 2, a sufficient condition for stability is that $\cos(\beta^*) > 0$ and $\cos(\alpha^*) > 0$, while a sufficient condition for unstability is that either $\cos(\beta^*) < 0$ or $\cos(\alpha^*) < 0$, since this numbers directly defines the sign of all the involved numbers $\alpha_i(\bar{\theta})$. Consider first $\cos(\beta^*)$. Then, $\cos(\beta^*) > 0$ if and only if $n > 3$ or

$n = 3$ and $f(\pi/2) < 1$. This implies that $n > 3$ or $n = 3$ and $\frac{a}{b}\sin(n\pi/4) < 1$. So, we have the following condition for synchronizability: $n > 3$ or $n = 3$ and $a < \sqrt{2}b$. Consider now $\cos(\alpha^*)$. Then, $\cos(\alpha^*) > 0$ if and only if $\pi - n\frac{\beta}{2} < \frac{\pi}{2}$, which is equivalent to $\beta > \pi/n$. This implies the inequality $f(\pi/n) > \sin(\pi/n)$, since f attains its maximum at π/n (see Fig. 2 for the case $n = 3$). So, we obtain the following condition for synchronizability: $a < b\sin(\pi/n)$. Therefore, putting all things together, and considering that $\sin(\pi/n) > \frac{2}{n}$ and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, we have that the candidate equilibrium is stable if

$$\begin{aligned} &n > 3\ ,\ a < b\sin(\pi/n), \\ &n = 3\ ,\ \tfrac{\sqrt{3}}{2}b < a < \sqrt{2}b \end{aligned} \tag{9}$$

Conditions (9) are summarized in Figs. 3 and 4. ■

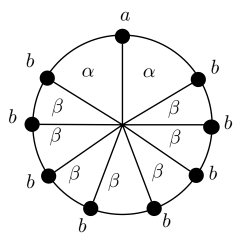

Fig. 1. A candidate equilibrium point for C_{n+2}. At each position, there are a or b twins.

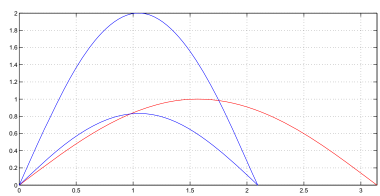

Fig. 2. Situation for $n = 3$

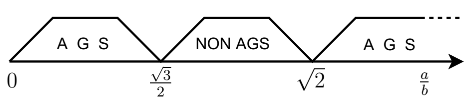

Fig. 3. Synchronizability condition on $\frac{\mathbf{a}}{\mathbf{b}}$

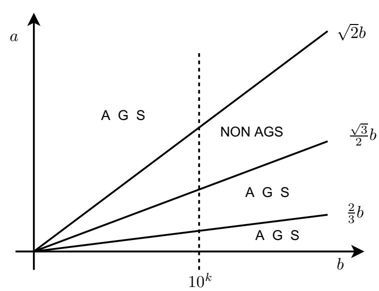

Fig. 4. $(\mathbf{a}, \mathbf{b})$ diagram for synchronizability $(n = 3)$

As an immediate consequence, we have the following result on the complexity of the classification of AGS graphs.

Theorem 4. *The classification of the family of graphs*

$$\mathcal{G} = \{G_3(a, 10^k)\ ,\ a > 0\ , k \geq 0\}$$

has the same order of complexity than the computation of $\sqrt{2}$.

Proof: from Theorem 3, we know that $G_3(a, 10^k)$ is AGS if $\frac{\sqrt{3}}{2}.10^k < a < \sqrt{2}.10^k$. Suppose we know c_k and d_k, the k-th elements of the decimal expansion of $\sqrt{3}$ and $\sqrt{2}$ respectively. Then, a sufficient condition for synchronizability is $\frac{c_k}{2} < a < d_k$. ■

5 Conclusions

In this work we present evidence about the hardness of the classification of graph as synchronizing or not. On one hand, the class $\mathbb{SG}$ of synchronizing graphs is a combinatorial one, but on the other hand, its definition is made in terms of the differential equations associated with them throughout the Kuramoto model of coupled oscillators. This does not make easy to answer questions like those about the computational complexity of the decision problem of classification. In other contexts, where similar non combinatorial definitions have been made, like for planar graphs, an structural theorem arises that enable the study of the computational complexity. However, in our case, Theorem 1 and 2 (1=every connected graph can be made synchronizing; 2= every graph with a non bridge edge can be made not synchronizing) tell that the class of synchronizing graph is quite complex from an structural point of view, and not such theorems like the Kuratowski one, can be given.

Nevertheless, in this work we present an infinite family whose classification is computational equivalent to find the n-th digit of square root of two. This result seems to tell that there is no a combinatorial definition of synchronizing graphs. However, there are many combinatorial problems where square of root appears, but they are of a enumerations nature. Therefore, if a combinatorial classification theorem exists, it seems to be in terms of amount of structures in the graphs.

As further works, we should find new families whose classification give new light over the computational complexity of the general classification problem. However, this could be a difficult task, since our present result rest over the well known equilibrium points of cycles, while we do not have a complete knowledge of the equilibriums of other 2-connected graphs. Indeed, if we identify the twins vertices of $G(n, m)$, we obtain a cycle, and that is why the equilibriums of $G(n, m)$ are quite similar to those of cycles. However, for other known 2-connected synchronizing graphs, named, complete graphs and wheels, the former are invariant over twins operations, while for the latter we have no idea of its equilibria.

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