Global Synchronization Properties for Different Classes of Underlying Interconnection Graphs for Kuramoto Coupled Oscillators

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Abstract. This article deals with the general ideas of almost global synchronization of Kuramoto coupled oscillators and synchronizing graphs. We review the main existing results and introduce new results for some classes of graphs that are important in network optimization: complete k-partite graphs and what we have called Monma graphs.

Keywords: Nonlinear systems, Network synchronization, coupled oscillators, synchronizing graphs.

1 Introduction

With the purpose of develope a formal mathematical theory for the analysis of synchronization problems, Y. Kuramoto derived a mathematical model, where each individual agent is represented by a dynamical oscillator [1]. It has several applications and has been widely studied from different points of view [2, 3]. Control community has applied classical, and has derived new, control techniques for the analysis of many synchronization properties that explain natural phenomena, like flocking, formation, self-organization, collective behavior, swarming, etc., and helps to design artificial networks of autonomous agents (see [4] and references there in). Most of these works deal with local stability properties of the synchronization. Recently, we have focused on global properties, trying to establish conditions for *almost global synchronization*, i.e., convergence to synchronization for almost all possible initial conditions. When the oscillator are all identical, the dynamical properties rely on the interconnection graph. We look for necessary and sufficient conditions for this graph, in order to have almost global synchronization of the agents (*synchronizing* graphs). In [5, 6], we derived some general properties and proved that a graph synchronizes if and only if its bi-connected components do. Following this idea, we concentrate our analysis on the particular, and huge, class of bi-connected graphs. In this work, we focus on two big subfamilies of bi-connected graphs: complete k-partite graphs

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and Minimum-weight two-connected spanning graphs, since these subfamilies have important applications on networks analysis and design.

In the next Section we review some basic facts. In Section 3 we prove synchronizability of complete k-partite graphs. In Section 4, we analyze necessary and sufficient conditions for synchronizability of Minimum-weight two-connected spanning graphs.

2 Kuramoto Model and Synchronizing Graphs

Consider n identical oscillators, described by their phases θ_i, $i = 1, \ldots, n$. Each oscillator has a set of *neighbors* $\mathcal{N}_i$, which influences its velocity in this way [1]: $\dot{\theta}_i = \omega + \sum_{k \in \mathcal{N}_i} \sin(\theta_k - \theta_i)$. The natural state space is the n-dimensional torus $\mathcal{T}^n$, since $\theta_i \in [0, 2\pi)$. We assume mutual influence: $k \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_k$. The interaction is given by a graph $G = (V, E)$, with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, associated to the phases $\theta_1, \theta_2, \ldots, \theta_n$ and edge set $E = e_1, e_2, \ldots, e_m$[1]. If we endow G with an arbitrary orientation, define the vector $\theta = (\theta_1, \ldots, \theta_n) \in \mathcal{T}^n$, introduce the incidence matrix B and reparameterize time, we obtain the compact expression [4]

$$\dot{\theta} = -B. \sin\left(B^T \theta\right). \tag{1}$$

The dynamical properties rely only on the interconnection graph. We observe that the results obtained for identical oscillators may be extended for *quasi-identical* oscillators using standard perturbation techniques. We say the systems reaches *(full) synchronization* or *consensus* when all the agents have the same phase: $\theta_i = \bar{\theta}_0$, $i = 1, 2, \ldots, n$. *Partial synchronization* describes a state with some agents at one phase $\bar{\theta}_0$ and the rest of the agents at $\bar{\theta}_0 \pm \pi$. Since system (1) depends only of the phase differences between agents, we may always assume $\bar{\theta}_0 = 0$. States with phases distributed along $[0, 2\pi)$ will be referred as *non synchronized*. Observe that for every $c \in \mathcal{R}$, we have that $-B. \sin(B^T\theta) = -B. \sin[B^T(\theta + c.\mathbf{1}_n)]$ and $\mathbf{1}_n^T.B. \sin(B^T\theta) = 0$ where $\mathbf{1}_n$ denote the n-dimensional column of ones. This means that the system is invariant under translations on the state space parallel to vector $\mathbf{1}_n$ and that the dynamics develops over hyperplanes orthogonal to this direction. Frequently, we will forget this aspect and say that a property holds for one state, meaning that the property holds for this state and for every state obtained from this one by a $\mathbf{1}_n$ translation. Equation (1) describes is a *gradient system*, with potential function [11]

$$U(\theta) = m - \sum_{ik \in E} \cos(\theta_i - \theta_k) = U_0 - \mathbf{1}_m^T. \cos(B^T\theta), \tag{2}$$

in the sense that (1) can be rewritten as $\dot{\theta} = -\nabla U(\theta)$. Since the state space is compact, we may apply LaSalle's invariant result [11] and conclude that every trajectory of the system will converge to an equilibrium point.

[1] We will use standard graph theory concepts. For those readers who are not familiar with this topic, we refer to [8].

Definition 1. *We say the system (1) has the almost global synchronization property if the set of initial conditions that do not lead to full synchronization of the agents has zero Lebesgue measure.*

Definition 2. *We say the graph $G = (V, E)$ associated to system (1) is* synchronizing *if system (1) has the almost global synchronization property.*

Clearly, we will only deal with connected graphs. Let us consider the function U around the synchronized state $\bar{\theta} = 0 \in \mathcal{T}^n$. Then, U takes non negative values around $\bar{\theta}$ and attains its minimum, which is 0, at $\bar{\theta}$. Thus, U is a local Lyapunov function and we have local stability of the consensus. In order to have the almost global synchronization property, we must prove that the consensus is the only equilibrium point with a non-zero measure of his basin of attraction. We need to find all the equilibrium points of (1) and analyze its local stability. The Jacobian matrix around an equilibrium point $\bar{\theta} \in \mathcal{T}^n$ is $J(\bar{\theta})$ with elements can be written as $J(\bar{\theta}) = -B.diag(\cos(B^T\bar{\theta})).B^T$. Observe that is a symmetric matrix, due to the mutual influence of the agents and it always has 0 as an eigenvalue, since $B^T.\mathbf{1}_n = 0$. For local stability, we will require negativity of the remaining $n-1$ eigenvalues of $J(\bar{\theta})$ (*transversal stability* [5]). As was done by Kuramoto, we introduce the phasors $V_i, \dots, V_n$ associated to every agent: $V_i = e^{j\theta_i}$. Then, we may think the system as particles running in the unit circumference. Consensus means that all the particles are together. These phasors will be very useful for the stability analysis.

We recall some previous results that will be useful in our present analysis. We do not include the proofs here. They can be found in [5, 6, 7]. For a given equilibrium point $\bar{\theta}$, we define the numbers $\alpha_i = \frac{1}{V_i}\sum_{k\in\mathcal{N}_i} V_k$. They all are real and if there is some negative, then $\bar{\theta}$ is unstable. This fact is extended in Lemma 1. Concerning stability of an equilibrium point, we may affirm it if all the phase differences belongs to $[-\frac{\pi}{2}, +\frac{\pi}{2}]$. Another important result is that *a graph synchronizes if and only if its blocks do* [6]. A block of a graph is either a single vertex, a *bridge* or a *bi-connected component* (a subgraph which has always two distinct paths between any pair of nodes). The result exploits the fact that every cycle of a graph belongs to only one block. This result can be seen either as a reduction procedure or a synchronization-preserving algorithm for networks construction. So, in order to state the synchronizability of a given graph, we only have to focus on its blocks. It reduces the characterization of synchronizing graphs to bi-connected families. Applying those previous results, we have found the first two families of synchronizing graphs: **complete** graphs and **trees** [6]. We have also proved that a cycle synchronizes if and only if it has less than five nodes [7]. In the next Section, we investigate the synchronizability of two important bi-connected families.

3 Complete *k*-Partite Graphs

Bi-partite graphs, and its generalization, k-partite graphs, conforms a graph family with multiple applications to several problems where natural clusters can be identified. k-partiteness allows the *coloring* of the graph and models a lot of *matching problems*, where a set of resources must be assigned to a set of users, with special applications to the development of codes. They have been widely studied and there are a lot necessary and sufficient conditions, testing algorithms and spectral characterization [8].

Definition 3. *A graph $G = (V, E)$ is* bi-partite *if there is a partition of it set of vertices $V = V_1 \cup V_2$ such that every link of E joins only an element of V_1 with an element of V_2.*

The idea generalizes to *k-partites graphs*, when we split V as the disjoint union of *clusters* $V_1 \cup V_2 \cup \ldots \cup V_k$. In a *complete k-partite graph* $G = (V, E)$, with $V = V_1 \cup \ldots \cup V_k$, every element of V_i is connected to all the elements of V_j, $j \neq i$. Observe that all the agents in the same cluster *share* the same neighbors. We introduce the following idea.

Definition 4. *Consider two nodes u and v of a graph G. We say they are twins if the have the same set of neighbors: $\mathcal{N}_u = \mathcal{N}_v$.*

Slightly modifying previous definition, we also say that two vertices are *adjacent twins* if they are adjacent and $\mathcal{N}_u \setminus \{v\} = \mathcal{N}_v \setminus \{u\}$. Concerning synchronization, twins vertices act as a *team* in order to get equilibrium in equation (1). Observe that for a complete k-partite graph, all the vertices of the same cluster are twins. The following Lemma extends a previous result and is very important for the main result of this Section.

Lemma 1. *Let $\bar{\theta} \in \mathcal{T}^n$ be an equilibrium point of (1). If for some i, the number α_i is non positive, then $\bar{\theta}$ is unstable.*

Proof. The case α_i negative was already proved in [5]. If we have a null α_k, the matrix $J(\bar{\theta})$ may have a multiple null eigenvalues. Looking carefully at equation (2), we observe that we can rewrite $U = U_0 - \frac{1}{2}\sum_{i=1}^{n} \alpha_i$. We chose U_0 such that $U(\bar{\theta}) = 0$. Consider the k-th element of the canonical base e_k, a small positive number δ and a perturbation $\tilde{\theta} = \bar{\theta} + \delta.e_k$. Then $U(\tilde{\theta}) = U_0 - \frac{1}{2}\sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{h\in\mathcal{N}_i} \cos\left(\bar{\theta}_h - \bar{\theta}_i\right) - \frac{1}{2}\sum_{h\in\mathcal{N}_k} \cos\left(\bar{\theta}_h - \bar{\theta}_k - \delta\right)$. After some calculations, we may write

$$U(\tilde{\theta}) = U_0 - \frac{1}{2}\sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{\substack{h \in \mathcal{N}_i \\ h \neq k}} \cos\left(\bar{\theta}_h - \bar{\theta}_i\right) - \sum_{h\in\mathcal{N}_k} \left[\cos\left(\bar{\theta}_k + \delta - \bar{\theta}_h\right)\right]$$

We have that $\cos\left(\bar{\theta}_k + \delta - \bar{\theta}_h\right) = \cos(\delta)\Re e\left[\alpha_k\right] + \sin(\delta)\mathcal{I}m\left[\alpha_k\right] = 0$. Then, it turns out that $U(\tilde{\theta}) = U(\bar{\theta})$ for all δ. We have proved that arbitrarily close to $\bar{\theta}$, we can find non equilibrium points with the same potential value. This implies that, arbitrarily close to $\bar{\theta}$, there are points with more or less potential value. So, $\bar{\theta}$ must be unstable[2]. □

[2] Actually, function U is in the hypothesis of Cetaev's instability theorem (see [11]).

We recall the following result, which will be crucial for the analysis of bi-partite graphs. More results on twins may be found in [7].

Lemma 2. *Consider the system (1) with graph G. Let $\bar{\theta}$ be an equilibrium point of the system and v a vertex of G, with associated phasor V_v. Let T be the set of twins of v and $\mathcal{N}$ the set of common neighbors. If the real number $\alpha_v = \frac{1}{V_v}\sum_{w\in\mathcal{N}} V_w$ is nonzero, the twins of v are partially or fully coordinated with it, that is, the phasors V_h, with $h \in \mathcal{N}_v$ are all parallel to V_v. Moreover, if $\bar{\theta}$ is* ***stable****, the agents in T are fully coordinated.*

Proof. Let $u \in T$ and consider the real numbers $\alpha_v = \frac{1}{V_v}\sum_{w\in\mathcal{N}} V_w$, $\alpha_u = \frac{1}{V_u}\sum_{w\in\mathcal{N}} V_w$. Then, it follows that $\alpha_v.V_v = \alpha_u.V_u$, for all $u \in T$. If there are u_1, $u_2 \in T$ linearly independent, their respective α_{u_1} and α_{u_2} must be zero, and so are all numbers α in T. Then, if there is some $\alpha_u \neq 0$, $u \in T$, all phasors in T are parallel. So, all the nodes in T are partially or fully coordinated. Now suppose that $\bar{\theta}$ is stable and that there are u_1, $u_2 \in T$ such that $u_1 = -u_2$. Then $\alpha_{u_1} = \frac{1}{V_{u_1}}\sum_{w\in\mathcal{N}} V_w = -\frac{1}{V_{u_2}}\sum_{w\in\mathcal{N}} V_w = -\alpha_{u_2}$ and we have at least one negative number α and $\bar{\theta}$ should be unstable, by Lemma 1. □

We are now ready to state and prove one of the main results of this article.

Theorem 1. *A complete k-partite graph synchronizes.*

Proof. We follow the same steps as before. Let $G = (V, E)$ and $V_1, V_2, \ldots, V_k$ be a partition of V such that all the agents in V_i are twins, $i = 1, 2, \ldots, k$. For $k = 2$, we have the particular case of bi-partite graph. Let $\bar{\theta}$ be an equilibrium point of (1). We have several cases, regarding the numbers α.

Case 1: There exists some null number α. Once again, according to Lemma 1, $\bar{\theta}$ should be unstable.

Case 2: For every $u \in V$, $\alpha_u \neq 0$. Then, by Lemma 2, in order to have a stable equilibrium, all agents in V_i are fully coordinated, $i = 1, \ldots, k$. Now, consider $u_i \in V_i$ and $u_h \in V_h$, with $\alpha_{u_i} \neq 0$ and $\alpha_{u_h} \neq 0$. Denote by R the sum of all phasors in V: $R = \sum_{w\in V} V_w$. Then, since the graph is complete k-partite, $R = \sum_{w\in V_i} V_w + \sum_{w\in V\setminus V_i} V_w = |V_i|.V_{u_i} + \alpha_{u_i}.V_{u_i}$, where $|V_i|$ is the cardinality of V_i. A similar expression is also valid for u_h. So, $[|V_i| + \alpha_{u_i}].V_{u_i} = [|V_h| + \alpha_{u_h}].V_{u_h}$. If V_{u_i} and V_{u_h} are not parallel, we obtain a negative number α and by Lemma 1, $\bar{\theta}$ is unstable. So, in order to have a stable equilibrium, all the agents in all the clusters should be parallel. Then, $\bar{\theta}$ should be a full or a partial synchronized equilibrium. Suppose we have $u_i \in V_i$ and $u_h \in V_h$ with $u_i = -u_h$. Then, $\alpha_{u_i} = -\alpha_{u_h}$ and $\bar{\theta}$ is unstable. Then, the only stable equilibrium point is the synchronization □

Previous Theorem also gives a new proof for the synchronization of complete graphs.

4 M_{lmn} Graphs

Typically, in the design of large optical fiber networks, the need of adding communication redundancy to the network comes up in order to increase its

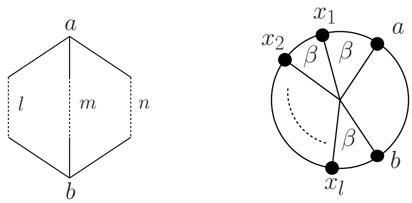

Fig. 1. Left: a M_{lmn} network topology. Right: necessary position of the phases of the path with l degree 2 nodes for a stable equilibrium configuration of M_{lmn}.

survivability. A commonly planning requirement is to ensure the existence of at least two-node-disjoint-paths between pairs of distinguished nodes. In this way, when occurring a failure (link or node), the network will remain in operational state, i.e. the resulting network is connected. To find a network topology verifying this restriction is known as the Steiner Node-Survivable Network Problem (STNSNP) [9]. A very important particular case of the STNSNP is the construction of a minimum-weight two-connected network spanning all the points in a set of nodes V. For this problem, in [9] Monma-Munson-Pulleyblanck introduce a characterization of optimal solutions. They prove that there exits an optimal two-connected solution whose nodes all have degree 2 or 3, and such that the removal of any edge or pair of edges leaves a bridge in the resulting connected components. Moreover, they prove that optimal solutions that are not a cycle contains, as a node induced subgraph, a graph like the one shown in figure 1-Left, with two nodes of degree 3, joined by three paths with l, m and n degree 2 nodes, that we will denote by M_{lmn}. For these graphs we will study the synchronization property. In what follows, call this class of graphs as Monma graphs [10]. Let a and b be the nodes with degree 3. We will denote by x_i, $i = 1, \dots, l$, the nodes of the first path between nodes a and b, y_i, $i = 1, \dots, m$, the nodes of the second path and z_i, $i = 1, \dots, n$, the nodes of the third path. We assume that $l \geq m \geq n \geq 1$. Firstly, we consider the particular constrains imposed to nodes with degree 2.

Lemma 3. *Let $\bar{\theta}$ be a stable equilibrium point of (1) with associated graph M_{lmn}. Let x_i, be a degree 2 node of M_{lmn}. Consider the set $\mathcal{N}_{x_i} = \{u, v\}$ of neighbors of x_i and define the angles $\varphi_u = \bar{\theta}_u - \bar{\theta}_{x_i}$, $\varphi_v = \bar{\theta}_v - \bar{\theta}_{x_i}$. Then, $\varphi_u = -\varphi_v$.*

Proof. Since $\bar{\theta}$ is an equilibrium, then $\sin(\varphi_u) + \sin(\varphi_v) = 0$ and it must be either $\varphi_u = -\varphi_v$ or $\varphi_u = \pi + \varphi_v$. But if it was the last case, we would have that $\alpha_{x_i} = \cos(\varphi_u) + \cos(\varphi_v) = 0$ and $\bar{\theta}$ should be unstable. $\square$

So, if we consider an equilibrium point $\bar{\theta}$ and its representation as particles on the unit circumference, the ones associated to degree 2 nodes of one of the paths between degree 3 nodes a and b, must be equally distributed over one of the arc of circumference between the particles associated to a and b, as is shown in Figure 1-Right. Now, we characterize synchronizability of M_{lmn}.

Theorem 2. *Consider the graph M_{lmn}. Then, M_{lmn} is synchronizing iff $max\{l, m, n\} \leq 2$.*

Proof. Without lost of generality, we assume that $l \geq m \geq n \geq 1$. First of all, we will prove that if $l \geq 3$, we can find non synchronized stable equilibrium points. Since the agents associated to the three paths from a to b must be equally spaced on the arcs of circumference between $\bar{\theta}_a$ and $\bar{\theta}_b$, we look for equilibrium configurations with the agents $\bar{\theta}_{x_i}$ on one arc and agents $\bar{\theta}_{y_i}$ and $\bar{\theta}_{z_i}$ on the other. Let be $\varphi = \bar{\theta}_a - \bar{\theta}_b$ (or viceversa), defined such that $\varphi \in (0, \pi)$ (we explicitly exclude partial or full synchronized equilibrium points). Observe that we must prove the existence of a suitable angle φ. The equilibrium condition at node a gives $f(\varphi) = -\sin\left(\frac{2\pi-\varphi}{l+1}\right) + \sin\left(\frac{\varphi}{m+1}\right) + \sin\left(\frac{\varphi}{n+1}\right) = 0$, where we have introduced the auxiliar function f, which is well defined and C^∞. Observe that $f(0) = -\sin\left(\frac{2\pi}{l+1}\right)$ and $f(\pi) = -\sin\left(\frac{\pi}{l+1}\right) + \sin\left(\frac{\pi}{m+1}\right) + \sin\left(\frac{\pi}{n+1}\right)$. If $l \geq 2$, $f(0) < 0$. On the other hand, since the sine is an increasing function in $(0, \pi/2)$ and $l \geq m \geq n$, it follows that $\sin\left(\frac{\pi}{l+1}\right) \leq \sin\left(\frac{\pi}{m+1}\right) \leq \sin\left(\frac{\pi}{n+1}\right)$. If $n \geq 1$, $f(\pi) > 0$. Then, in this conditions, there exists an angle $\varphi \in (0, \pi)$ such that we have a non synchronized equilibrium point. Now, we care about the stability of this equilibrium. A sufficient condition is that all involved phase differences belongs to $(-\pi/2, \pi/2)$ (see [5])). But all the phase differences are $\frac{2\pi-\varphi}{l+1}$, $\frac{\varphi}{m+1}$, $\frac{\varphi}{n+1}$. So, as $\varphi \in (0, \pi)$, it is enough to require $l \geq 3$. At this point, we must prove that the graphs M_{lmn} with $l \leq 3$ (M_{111}, M_{211}, M_{221} ,M_{222}) are synchronizing. We only present the analysis of one of these cases, since the proofs are all quite similar. Let us consider the graph M_{221}. Then, at an equilibrium point $\bar{\theta}$, the phases must correspond to one of the situations showed in figure 2. In case (I), the equilibrium condition for node a gives $-\sin(\delta) + 2\sin(\beta) = 0$, with the associated constrain $2\delta + 3\beta = 2\pi$. We eliminate $\delta = \pi - \frac{3\beta}{2}$ and get the equation $2\sin(\beta) - \sin\left(\frac{3\beta}{2}\right) = 0$. Since $\delta \geq 0$, the solutions must belong to the interval $[0, \frac{2\pi}{3}]$ and we obtain the only roots $\beta = 0$ and $\delta = \pi$, which corresponds to a partial synchronized (unstable) equilibrium point. The same ideas apply to case (II). The final case has the only solution $\delta = 0$, which corresponds to a full synchronized equilibrium. Then, M_{221} is synchronizing. □

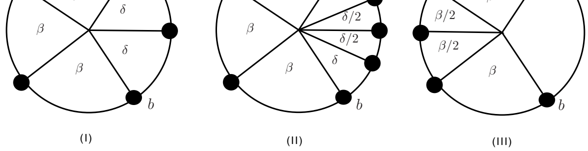

Fig. 2. Possible equilibrium configurations of graph M_{221}

5 Concluding Remarks

In this work we have analyzed the synchronizability property for two important families of bi-connected graphs. We have shown that complete bi-partite graphs, and, more generally, complete k-partite graphs always synchronizes. On the other hand, we have proved that Monma graphs (the M_{lmn} topologies) only synchronize if every degree 2 node has at most one degree 2 node as a neighbor. The techniques we have presented here show how ideas from graph and control theory can be combined in order to obtain more insight about the dynamics and the graph structure.

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