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On the Characterization of Families of Synchronizing Graphs for Kuramoto Coupled Oscillators

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Abstract: Kuramoto model of coupled oscillators represents situations where several individual agents interact and reach a collective behavior. The interaction is naturally described by a interconnection graph. Frequently, the desired performance is the synchronization of all the agents. Almost global synchronization means that the desire objective is reached for every initial conditions, with the possible exception of a zero Lebesgue measure set. This is a useful concept, specially when global synchronization can not be stated, due, for example, to the existence of multiple equilibria. In this survey article, we give an analysis of the influence of the interconnection graph on this dynamical property. We present in a ordered way several known and new results that help on the characterization of what we have called synchronizing topologies.

Keywords: nonlinear analysis, oscillators, global stability, coupled devices.

1. INTRODUCTION

The Kuramoto model of coupled oscillators represents several physical phenomena in which synchronization plays a crucial role. The mathematical model consists of a number of oscillators that influence each other in a way such that a collective behavior emerges. This mathematical model, derived in the middle 1970's, applies to several biological examples like circadian or cardiac pacemaker cells, brain cells'synchronization, fireflies flashing synchronously, physical phenomena like the Josephson junction for superconduction and laser arrays, and engineering problems like microwave antenna arrays, robots, etc. (see Kuramoto [1975], Strogatz [1994], Acebrón et al. [2005], Li [2008], Lin and Lin [2009]) and references there in.

In the last decade, the control community have studied synchronization problems in many different contexts. See, for example, the survey article of Olfati-Saber et al. [2007]. Other works describe different contexts and approaches that show the relevance of the underlying interconnection graph (Jadbabaie et al. [2004], Marshall et al. [2004], Rogge and Aeyels [2004], Verwoerd and Mason [2007], Wang and Ghosh [2007], Smet and Aeyels [2008], Aeyels and Smet [2008], Carareto et al. [2009], Chopra and Spong [2009]). For the Kuramoto model, when all the oscillators are identical, the dynamical properties of the system relay totally on the algebraic and topological properties of this graph. In (Monzón and Paganini [2005], Canale and Monzón [2007, 2008]), this fact and its properties were analyzed. It has motivated us to use the expression *synchronizing graph* for describing an interconnection that ensures synchronization for almost every initial condition, in a sense that we will explain later. In this work, we present a set of results on the road to characterize synchronizing graphs. We show some general dynamical and algebraic properties that evidence the complexity of the characterization problem. We introduce a reduction method and find some synchronizing families. Several results included here have been presented before in different conferences from control theory and graph theory. We collect them in a unified presentation, together with new results. We emphasize that most of the proofs make use of classical arguments of both theories. Nevertheless, since standard graph operations do not preserve the synchronization property, we have derived new theory in order to step forward towards a characterization of synchronizing graphs families.

In Section 2, we describe the Kuramoto model, we state the necessary graph notations and properties and we introduce the idea of synchronizing graphs. In Section 3, we present several results that will help us to advance towards the characterization of synchronizing graphs. In Section 4, we derive a reduction method for graph analysis. In Section 6, we present several synchronizing families. Finally, we state some conclusions and some directions of future research.

2. THE KURAMOTO MODEL

In 1975, Kuramoto introduced a nice mathematical model for describing synchronization phenomena in nature, following the works of A. Winfree on biological clocks (see Kuramoto [1975, 1984], Winfree [1980]). When several agents work together, they influence each other and new behaviors may emerge. At the mathematical model level, each agent is represented by an oscillator and it is described by a *phase*, an angle $\theta \in [0, 2\pi)$ (a particle *running* at the natural angular velocity). Interactions between agents enter the model as perturbations of the natural

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angular velocity of a given agent. The standard Kuramoto model is (Kuramoto [1975])

$$\dot{\theta}_i = \omega_i + \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i) \quad , \; i = 1, 2, \ldots, n \qquad (1)$$

where $\mathcal{N}_i$ denotes the set of agents that interact with agent i; we call them the *neighbors* of i. Define the phase vector $\theta = [\theta_1, \ldots, \theta_n]^T$ and the angular velocity vector $\omega = [\omega_1, \ldots, \omega_n]^T$. The natural state space is the n-dimensional torus $\mathbb{T}^n$. As was shown in (Jadbabaie et al. [2004]), equation (1) may be re-written as

$$\dot{\theta} = \omega - B. \sin\left(B^T \theta\right) \qquad (2)$$

where B is an incidence matrix of the graph G -with an arbitrary orientation- that describes the way the different agents influence each other [1]. This notation makes explicit the relevance of the underlying graph. As was remarked by Kuramoto, we can conceive the system as a bunch of particles running on a unit circumference; the interactions are like *nonlinear forces* that speed up or slow down the particles. In this model, *synchronization*, or *consensus*, means that all the agents move keeping constant their phase differences. At the circumference, all the particles are running, keeping constant their distances. From the dynamical point of view, questions like convergence to the synchronized state and its stability become natural. *Almost global synchronization* (a.g.s.) denotes the situation where almost every trajectory, with the possible exception of a zero Lebesgue measure set, converges to a synchronized state. This concept is based on the works of (Rantzer [2001]) and is particularly useful when the system have multiple equilibria and global convergence can not be achieved. When all the agents have the same angular velocity, we may perform a change of variables and get the general expression

$$\dot{\theta} = -B. \sin\left(B^T \theta\right) \qquad (3)$$

This is the equation we will work with. As can be seen, synchronization depends only on the underlying interconection graph. When the system has the (a.g.s.) property, we say the graph *synchronizes* or is *synchronizing*. The main objective of our research is the characterization of the synchronizing graphs.

We just recall that an incidence matrix $B_{n \times m}$ of a graph $G = (V, E)$ (with vertices set V and edges set E) is constructed as follows: $b_{ij} = 1$ if edge j reaches node i, -1 if edge j leaves vertex i and 0 otherwise. Given a non directed graph G, if we endow it with an arbitrary orientation and construct an incidence matrix B, it follows that the *laplacian* of G is $L = BB^T$.

3. MAIN PROPERTIES

In this Section, we will focus on the general properties of equation (3), its equilibria and its relationships with the underlying interconnection graph. Each oscillator may be represented by a phasor $V_i = e^{j\theta_i}$, $i = 1, \ldots, n$. For each agent i, we introduce the complex numbers $\alpha_i = \frac{1}{V_i} \sum_{k \in \mathcal{N}_i} V_k$. It is straightforward to see that system (3) is a *gradient* system (see Jadbabaie et al. [2004], Lin and Lin [2009]). We assume we have an underlying graph $G = (V, E)$. We have the *potential function*

$$U(\theta) = U_0 - \mathbf{1}_m^T . \cos(B^T \theta) \qquad (4)$$

with U_0 a real number. Observe that $U(\theta) = U(\theta + k.\mathbf{1}_n)$ for every real k. A direct calculation gives

$$\dot{U}(\theta) = -\|\dot{\theta}\|^2 \qquad (5)$$

So, U decreases along the trajectories of system (3). For $U_0 = m$, U is non negative and generalizes in some sense the square of the *order parameter* defined by Kuramoto (Kuramoto [1984], Jadbabaie et al. [2004]). Since, in this case, $U(c\mathbf{1}_n) = 0$ for every $c \in [0, 2\pi)$, we may see U as a local Lyapunov function that proves the local stability of the consensus (see Khalil [1996], Jadbabaie et al. [2004]). Moving towards global properties, we wonder what happens when we start far from the consensus curve. Equation (5) says that the potential is non increasing along the trajectories of the system. Since we are working on a compact state space, we may apply LaSalle result and conclude that every trajectory converge to an equilibrium point of (3) (see Khalil [1996]). Moreover, the only attractors of the system are the equilibria. So, in order to have the a.g.s. property, the consensus must be the only attractor of the state space or, equivalently, every non consensus equilibria must be unstable. Our main tool for classifying the equilibria is Jacobian linearization. At an equilibrium point $\bar{\theta}$, the Jacobian matrix $M_{n \times n}$ of system (3) is symmetric and takes the explicit form

$$\begin{cases} m_{ii} = -\displaystyle\sum_{k \in \mathcal{N}_i} \cos(\bar{\theta}_k - \bar{\theta}_i) = -\alpha_i \\ m_{hi} = \begin{cases} \cos(\bar{\theta}_h - \bar{\theta}_i) & , \; h \in \mathcal{N}_i \\ 0 & , \; h \notin \mathcal{N}_i \end{cases} \end{cases} \qquad (6)$$

or, in a compact notation: $M = -B.diag\left[\cos(B^T \bar{\theta})\right].B^T$, which can be thought as a *weighted* laplacian. Observe that always $M.\mathbf{1}_n = 0$ and M has a null eigenvalue. This is related to the existence of *equilibrium curves* (actually, we are trying to prove *transversal stability* of these curves). If M has a positive eigenvalue, then $\bar{\theta}$ is unstable; if M has $n-1$ negative eigenvalues, $\bar{\theta}$ is stable. If the null eigenvalue is not simple, then Jacobian linearization is not enough for proving stability of $\bar{\theta}$. Without looking directly to the eigenvalues, we can work with the quadratic form induced by M. Let $x \in \mathbb{R}^n$ and denote by ik the link between nodes i and k, when it exists. Then,

$$x^T M x = -\sum_{ik \in E} (x_k - x_i)^2 \cos(\theta_k - \theta_i) \qquad (7)$$

We have the following general results.

Lemma 3.1. $\bar{\theta} \in \mathbb{T}^n$ is an equilibrium point if and only if α_i is real, for $i = 1, \ldots, n$.

Proof: From its definition, the imaginary part of α_i is $\sum_{k \in \mathcal{N}_i} \sin(\theta_k - \theta_i)$. So, at an equilibrium point, the imaginary part vanishes and the numbers α_i are all real. ■

Lemma 3.2. Let $\bar{\theta} \in \mathbb{T}^n$ is an equilibrium point.

i) If $\cos(\bar{\theta}_i - \bar{\theta}_j) > 0$ for every connected pair of nodes i, j, then $\bar{\theta}$ is stable.

[1] We will make use of several standard graph theory concepts, results and techniques. For an introduction to this field or for more details,see (Biggs [1993], Godsil and Royle [2001]).

ii) If for some i, the number $\alpha_i(\bar{\theta})$ is negative, then $\bar{\theta}$ is unstable.
iii) If for some i, the number $\alpha_i(\bar{\theta})$ is null, then $\bar{\theta}$ is unstable.
iv) If for a suitable reference, some $\bar{\theta}_i \in (-\frac{\pi}{4}, \frac{\pi}{4})$ and the rest of the agents' phases are in $(-\frac{3\pi}{4}, \frac{5\pi}{4})$, then $\bar{\theta}$ is unstable.
v) If all the agents' phases are located inside a semi circumference, then $\bar{\theta}$ is a consensus equilibria.
vi) If $\bar{\theta}$ is a partial synchronized equilibrium point, then $\bar{\theta}$ is unstable.

Proof: The first assertion comes from the direct observation of the Jacobian matrix M in (6). It is a weighted laplacian, with a positive definite weight. Since the laplacian has $n-1$ non zero and positive eigenvalues, we obtain the stability of $\bar{\theta}$.

Now, let us consider the case where there is a negative α_i. Since this number appears at the diagonal of the symmetric matrix M, it implies that M has a positive eigenvalue.

We have a different situation when there is a null α_k, since the matrix M may have a multiple null eigenvalue. Looking carefully at equation (4), we observe that we can re write U as follows

$$U(\theta) = U_0 - \frac{1}{2}\sum_{i=1}^{n}\sum_{k\in\mathcal{N}_i}\cos(\theta_k - \theta_i) = m - \frac{1}{2}\sum_{i=1}^{n}\alpha_i(\theta) \quad (8)$$

Consider the k-th element of the canonical base e_k, a small positive number δ and a perturbation $\tilde{\theta} = \bar{\theta} + \delta.e_k$. Then $U(\tilde{\theta}) = U_0 - \frac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{h\in\mathcal{N}_i}\cos\left(\bar{\theta}_h - \bar{\theta}_i\right) - \frac{1}{2}\sum_{h\in\mathcal{N}_k}\cos\left(\bar{\theta}_h - \bar{\theta}_k - \delta\right)$. After some calculations, we may write

$$U(\tilde{\theta}) = U_0 - \frac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{h\in\mathcal{N}_i\\ h\neq k}}\cos\left(\bar{\theta}_h - \bar{\theta}_i\right)$$
$$-\sum_{h\in\mathcal{N}_k}\left[\cos\left(\bar{\theta}_k + \delta - \bar{\theta}_h\right)\right]$$

Using the identity: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, we have that $\cos\left(\bar{\theta}_k + \delta - \bar{\theta}_h\right) = \cos(\delta)\Re e\,[\alpha_k] + \sin(\delta)\mathcal{I}m\,[\alpha_k] = 0$. Then, it turns out that $U(\tilde{\theta}) = U(\bar{\theta})$ for all δ. We have proved that arbitrarily close to $\bar{\theta}$, we can find non equilibrium points with the same potential value. This implies that, arbitrarily close to $\bar{\theta}$, there are points with more or less potential value. So, $\bar{\theta}$ must be unstable [2].

Now, we prove iv). It follows by taking, in equation (7), x_i to be zero if $\theta_i \in (-\pi/4, \pi/4)$ and to be one otherwise. With this choice, $x^T M x > 0$ and M has a positive eigenvalue.

In order to prove v), suppose, by contradiction that exists unsynchronized agents. Then $\min_i\{\bar{\theta}_i\} < \max_i\{\bar{\theta}_i\}$. Let m and M be such that $\bar{\theta}_m = \min_i\{\bar{\theta}_i\}$ and $\bar{\theta}_M = \max_i\{\bar{\theta}_i$. We claim there should exist an index k achieving the minimum but unsynchronized with at leas one of its neighbors. Indeed, it suffices to consider a walk $v_m = v_0, v_1, \ldots, v_l = v_M$ from vertex v_m to vertex v_M, with respective phases $\bar{\theta}_m = \bar{\theta}_0, \bar{\theta}_1, \ldots, \bar{\theta}_l = \bar{\theta}_M$. Let $k = \max\{i \mid \bar{\theta}_0 = \bar{\theta}_1 = \ldots = \bar{\theta}_i\}$. Then $k < l$; otherwise, it would be $\bar{\theta}_m = \bar{\theta}_M$. Besides, $\bar{\theta}_k \neq \bar{\theta}_{k+1}$. Since the angles $\bar{\theta}_i$ are all in a semi circumference, $\sin(\bar{\theta}_i - \bar{\theta}_k)$ have the same sign of $(\bar{\theta}_i - \bar{\theta}_k)$ for all i and then $\sum_{i\in\mathcal{N}_k}\sin(\bar{\theta}_i - \bar{\theta}_k) > 0$. But it contradicts the equilibrium condition of $\bar{\theta}$.

Finally, we prove the last affirmation. We observe that at a partially synchronized equilibrium point, we may recognize two sets of agents, located at opposite sides on the circumference, and we can *color* the graph. We define a vector $x \in \mathcal{R}^n$ as follows: we assign 1 to the elements of x related with one set and 0 to the other components. Then, in equation (7), $x^T M x$ is positive and then, $\bar{\theta}$ is unstable. ■

Result in item v) is a particular case of Theorem 2 in (Jadbabaie et al. [2004]). The proof we present here is more intuitive and only involves graph theory elements. Property iii) is quite relevant, because a situation when a null α appears could be when the phase differences of angles at an equilibrium point are $\pm\frac{\pi}{2}$. In this case, the multiplicity of the null eigenvalue of the Jacobian M may be more than one and so, linearization does not allow to establish transversal stability and we need more complex tools.

Now, we focus again on function U as defined in (4). The following results will be useful. The proofs follow the lines of Lyapunov and Cetaev Theorems (Khalil [1996]).

Lemma 3.3. Consider a stable equilibrium point $\bar{\theta}$ of (3). Then, if we choose U_0 such that $U(\bar{\theta}) = 0$, the function U is positive in a neighborhood of $\bar{\theta}$ *transversal* [3] to direction $\mathbf{1}$ and U is a Lyapunov function for this equilibrium point.

Lemma 3.4. Consider an unstable equilibrium point $\bar{\theta}$ of (3) and an arbitrary number $\epsilon > 0$. Then, if we choose U_0 such that $U(\bar{\theta}) = 0$, the function U must take negative values in $B_{\bar{\theta}}(\epsilon)$.

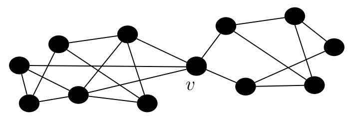

Fig. 1. A graph with a cut vertex v.

4. INTERCONNECTING SYSTEMS

In this Section we focus on how the synchronizing properties are affected when we interconnect systems. The main result we derive here can be thought also as a reduction procedure for analyze graph topologies.

If a graph G can be split into two non trivial sub graphs G_1 and G_2, such that: i) they only have one vertex v in common and ii) there is no edge between elements of G_1

[2] Actually, function U is in the hypothesis of Cetaev's instability theorem (see Khalil [1996]).

[3] Recall that $U(\theta) = U(\theta + c.\mathbf{1})$.

and G_2, we say that v is a *cut vertex* or an *articulation point* of G. Fig. 1 shows a graph with a cut vertex v. Cut vertices help us to decompose a given graph G into a set of *blocks*. Each block is either a single vertex, a *bridge* (an edge whose removal disconnect the graph or a *bi-connected component* (a subgraph which has always two distinct paths between any pair of nodes). We have the following result. A first proof based only on Jacobian linearization was presented in (Canale and Monzón [2007]).

Theorem 4.1. Let G a graph, v a cut vertex and G_1 and G_2 be the respective split of G, i.e., v is the only common vertex of G_1 and G_2. G synchronizes if and only if G_1 and G_2 do.

In order to prove this result, we introduce an auxiliary system. Let us denote by $f(\theta)$ the field of system (3): $f(\theta) = -B.\sin(B^T\theta)$. Consider the new system,

$$\dot{\theta} = \tilde{f}(\theta) \tag{9}$$

with field $\tilde{f}(\theta)$, such that $\tilde{f}_1 = 0$ and $\tilde{f}_i = f_i$, $i = 2, \ldots, n$, whose dynamic develops on an horizontal hyperplane. If we choose $\theta_1(0) = 0$, we have that the trajectories are in $\Pi = \{\theta \in \mathbb{T}^n \mid \theta_1 = 0\}$. Vector field $\tilde{f}$ can be also seen as the *orthogonal projection* of f onto Π. Let $g : \mathbb{T}^n \to \mathbb{T}^n$ be the translation parallel to vector $\mathbf{1}_n$ up to hyperplane Π: $g(\theta) = \theta - \theta_1.\mathbf{1}_n \in \Pi$. Due to the invariance property, it follows that $f(\theta) = f(g(\theta))$. So, for a given state θ, we will consider two dynamics: the one associated to the field f acting on θ and the one associated to the vector field $\tilde{f}$ at $g(\theta)$. We will analyze how this two dynamics are related. First of all, consider an equilibrium point $\bar{\theta}$ of (3). Since $f(\bar{\theta}) = 0 = f(g(\bar{\theta}))$, we have that $\tilde{f}(g(\bar{\theta})) = 0$ and $g(\bar{\theta})$ is an equilibrium of (9). Conversely, consider an equilibrium point $\bar{\theta}$ of (9). Then, since $\mathbf{1}_n.f(\theta) = 0$ for every θ, $f(\bar{\theta} + c.\mathbf{1}_n) = 0$ for every real c and we have a whole line of equilibria of (3). The following result is the key for Theorem 4.1. We do not include its proof here, for space reasons.

Proposition 4.1. Let $\bar{\theta} \in \mathbb{T}^n$ be an equilibrium point of (3) and consider the respective equilibrium $g(\bar{\theta})$ of (9). Then, $\bar{\theta}$ is stable (unstable) if and only if $g(\bar{\theta})$ is stable (unstable).

Proof of Theorem 4.1: Relabel the nodes of graph G such that the cut vertex v be the first node v_1. Then, by Proposition 4.1, the dynamic of (3) with graph G is related with the dynamic of (9) with the forced equation for the first state. Since this first state is the cut vertex, the dynamic of (9) consists actually in two uncoupled dynamics defined by G_1 and G_2. A stable equilibrium point of (3) can be split in two stable equilibrium for G_1 and G_2. So, if G synchronizes, that is, if the only stable equilibria for (3) is the consensus, G_1 and G_2 must be synchronizing as well. On the other hand, if G_1 and G_2 are synchronizing graphs, the only stable equilibrium point they have is the consensus. When we move from system (9) to system (3), we obtain either a partial consensus (unstable) or a full consensus (stable) and then G synchronizes. ■

This result is quite important since gives us a reduction method for analyzing graphs topology: we only need to focus on the blocks of the given graph. It also has several direct corollaries, like the following.

Corollary 4.1. Consider a graph G with a bridge. Let us denote by G_1 and G_2 the two subgraphs joined by the bridge. Then, G synchronizes if and only if G_1 and G_2 do.

One of the most important result we have obtained so far is the following.

Theorem 4.2. A graph synchronizes if and only if its blocks do.

This result reduces the scope of our analysis. In order to characterize synchronizing graphs families, we only need to focus on bi-connected graphs.

5. TWIN VERTICES

In this Section we introduce the idea of *twin vertices*, together with its main properties.

Definition 5.1. Consider two nodes u and v of a graph G. We say they are twins if the have the same set of neighbors: $\mathcal{N}_u = \mathcal{N}_v$.

Slightly modifying previous definition, we also say that two vertices are *adjacent twins* if they are adjacent and $\mathcal{N}_u \setminus \{v\} = \mathcal{N}_v \setminus \{u\}$. Concerning synchronization, twins vertices act as a *team* in order to get equilibrium in equation (3). Our first result concerns the necessary behavior of twins.

Lemma 5.1. Consider the system (3) with graph G. Let $\bar{\theta}$ be an equilibrium point of the system and v a vertex of G, with associated phasor V_v. Let T be the set of twins of v and $\mathcal{N}$ the set of common neighbors. If the real number $\alpha_v = \frac{1}{V_v}\sum_{w\in\mathcal{N}} V_w$ is nonzero, the twins of v are partially or fully coordinated with it, that is, the phasors V_h, with $h \in \mathcal{N}_v$ are all parallel to V_v. Moreover, if $\bar{\theta}$ is **stable**, the agents in T are fully coordinated.

Proof: Let $u \in T$ and consider the real numbers $\alpha_v = \frac{1}{V_v}\sum_{w\in\mathcal{N}} V_w$ and $\alpha_u = \frac{1}{V_u}\sum_{w\in\mathcal{N}} V_w$. Then, it follows that $\alpha_v.V_v = \alpha_u.V_u$, for all $u \in T$. If there are u_1, $u_2 \in T$ linearly independent, their respective α_{u_1} and α_{u_2} must be zero, and so are all numbers α in T. Then, if there is some $\alpha_u \neq 0$, $u \in T$, all phasors in T are parallel. So, all the nodes in T are partially or fully coordinated.

Now suppose that $\bar{\theta}$ is stable and that there are u_1, $u_2 \in T$ such that $u_1 = -u_2$. Then $\alpha_{u_1} = \frac{1}{V_{u_1}}\sum_{w\in\mathcal{N}} V_w = -\frac{1}{V_{u_2}}\sum_{w\in\mathcal{N}} V_w = -\alpha_{u_2}$ and we have at least one negative number α and $\bar{\theta}$ should be unstable, by Lemma 3.2, item iii). ■

We may define an equivalence relationship in the node set of a graph: two nodes are equivalent if they are adjacent twins. So, we can obtain a *quotient* graph by direct identification of equivalent nodes. The quotient graph can be seen as an induced subgraph of the original one. This leads us to the following results.

Theorem 5.1. The following two affirmations are true:

(1) any graph is the induced subgraph of a synchronizing one;
(2) any graph is *homeomorphic* to a non synchronizing. one.

We do not include the complete proof due to space reasons, but we hint it. The proof of the first assertion is based on the following observation. Given an arbitrary graph, we may start performing a *twins addition operation*. So, the relevance of a given vertex is enhanced because of the presence of its twins. If we do this carefully, we force that the only stable equilibria is the synchronization state. We say we can gain synchronizability by carefully adding enough numbers of twins vertices. The second statement is proved by taking an arbitrary edge and splitting it, transforming this single edge into a long path with the same terminal nodes. If we add enough number of intermediate nodes, we can find a non synchronized stable equilibrium point.

To conclude this Section, we mention that a relationship can be established between the equilibria set of a graph and its quotient. Moreover, since at an stable equilibrium point the twins must be synchronized, we may relate the stability properties of equilibrium points in the original graph and in the quotient. This will be used in the next Section.

6. FAMILIES OF SYNCHRONIZING GRAPHS

6.1 Trees and Complete Graphs

Next result gives us our first family of synchronizing graphs.

Theorem 6.1. A tree always synchronizes.

Proof: Let us consider K_2, the complete graph with two nodes. The only equilibrium point $\bar{\theta}$ of the two dimensional system given by (3) are the synchronization ($\bar{\theta}_1 = \bar{\theta}_2$) and the partial synchronization ($\bar{\theta}_1 = \bar{\theta}_2 + \pi$) which are, respectively, stable and unstable. So, K_2 synchronizes. Then, if we recursively apply Corollary 4.1, we conclude that every tree synchronizes. ■

This result has very important consequences and will let us to classify more families.

Theorem 6.2. Consider a graph G and its quotient $\tilde{G}$ with the twins equivalence relationship. then, if $\tilde{G}$ is a tree, then G synchronizes.

The proof of this result is not included here because of space reasons. We obtain now our second family of synchronizing graphs.

Theorem 6.3. A complete graph synchronizes.

Proof: At a complete graph, all the nodes are twins. Then, the quotient graph contains a single node. ■

Another way to prove synchronization of trees and complete graphs is by direct inspection of the equilibrium points. For the trees, the equilibria set contains only partially and fully synchronized points. For complete graphs, the only equilibrium points are partially and fully synchronized states and *balanced* states (where the agents' phases are symmetrically distributed on the circumference).

So far, we have proved that the most simple graphs, the trees, and the most complex, *all to all* graphs, synchronize. As we will see in the next Subsection, there are non synchronizing graphs *in the middle*. The synchronization property can not be directly related to the number of nodes, the number of neighbors or the number of links.

6.2 Cycles

Cycles are very simple graphs, where every agent has exactly two neighbors. They are the most simple biconnected graphs (for every two nodes, we have two distinct joining paths). The following result completely describes this family of graphs.

Theorem 6.4. A cycle synchronizes if and only if it has 5 or more nodes.

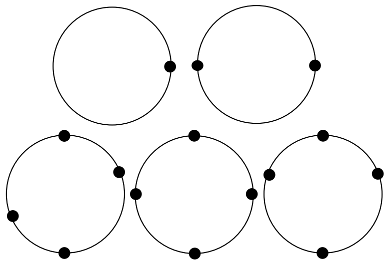

Fig. 2. A ring with four oscillators.

Proof: Let us denote by C_n the cycle with n elements. If $n = 3$, C_n is also complete and it synchronizes. If $n \geq 5$, we may obtain a non synchronized equilibrium point as follows. Define $\varphi = \frac{2\pi}{n}$ and consider the state

$$\bar{\theta} = [0, \varphi, 2\varphi, \ldots, (n-1)\varphi]^T$$

Then, involved the phase differences are all $\pm\varphi$. Since $n \geq 5$, we have $\cos(\pm\varphi) < 0$ and by Lemma 3.2-i), $\bar{\theta}$ is stable. The case $n = 4$ can not be analyzed by Jacobian linearization, since there are equilibria with all the phase differences equal to $\pm\frac{\pi}{2}$. The only equilibria of this system are shown in Fig. 2. The first case is the stable synchronized state. Except the *quadrature* case, the rest of the cases may be classified as unstable by Jacobian linearization. The quadrature case is also unstable, since all the numbers $\alpha_i(\bar{\theta})$, $i = 1, \ldots, 4$ are null and we apply Lemma 3.2-iii). ■

A direct application of this result is the finding of non synchronizing topologies, by detecting *attached* cycles at the interconnection graph.

6.3 Complete k-partite graphs

Consider a graph G, whose set of nodes can be split in k non intersecting subsets $G_1, G_2, \ldots, G_k$ such that for each

i, a particular element of G_i is connected to every other element of the rest of the subsets. G is called a *k-complete* graph. All the nodes of the same subset are non adjacent twins and then, at an stable equilibrium point, they must be synchronized, by Lemma 5.1. This result is the key property in order to prove that *complete k-partite* graphs synchronize.

We conclude this Section with a simple application, which is shown in Fig. 3. The picture shows the trajectories of a system which consists in two graphs (K_5 and K_3) acting separately, reaching consensus. At a given time, the two graphs are connected through a cut vertex, and they become fully synchronized.

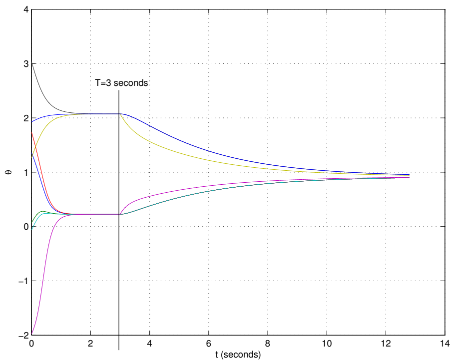

Fig. 3. Two graphs *glued* at a cut vertex at time $t = 3s$.

7. CONCLUSIONS

In this article we have presented the idea of synchronizing topologies for Kuramoto coupled oscillators. We think that the concept has many interesting application both in the analysis of existing interconnections and in the synthesis of networks. We have introduced some tools for dealing with algebraic graph theory and control theory elements. We have presented some synchronizing families, but we know we are far from a characterization of the large family of synchronizing topologies. Closing this work, we say some words on the class of regular graphs, in which all the nodes have the same number of neighbors, and its sub class or circulant graphs, which in addition have nice symmetry properties. We conjecture that for this family of graphs, synchronizability can be ensured through the presence of a minimum number of neighbors or working with the minimum size of induced cycles in the graph (the so called the *girth* of the graph). Robustness of these results will be addressed in further works.

REFERENCES

Acebrón, J., Bonilla, L., Pérez, C., Ritort, F., and Spigler, R. (2005). The Kuramoto model: A simple paradigm for synchronization. *Review of Modern Physics*, 77, pp. 137–185.

Aeyels, D. and Smet, F.D. (2008). A mathematical model for the dynamics of clustering. *Physica D*, 2517–2530.

Biggs, N. (1993). *Algebraic Graph theory*. Cambridge University Press.

Canale, E. and Monzón, P. (2007). Gluing Kuramoto coupled oscillators. In *Proceedings of the 46th IEEE Conference on Decision and Control*, 4596–4601. New Orleans.

Canale, E. and Monzón, P. (2008). Synchronizing graphs. Workshop on Spectral Graph Theory with applications on computer science, combinatorial optimization and chemistry.

Carareto, R., Orsatti, F., and Piqueira, J. (2009). Optimized network structure for full-synchronization. *Commun Nonlinear Sci Numer Simulat*, 14, 2536–2541.

Chopra, N. and Spong, M. (2009). On exponential synchronization of Kuramoto coupled oscillators. *IEEE Transactions on Automatic Control*, 54(2), 353–357.

Godsil, C. and Royle, G. (2001). *Algebraic Graph Theory*, volume 207 of *Springer Series Texts in Mathematics*. Springer, New York.

Jadbabaie, A., Motee, N., and Barahona, M. (2004). On the stability of the Kuramoto model of coupled nonlinear oscillators. In *Proc. of the American Control Conference*.

Khalil, H. (1996). *Nonlinear Systems*. Prentice-Hall.

Kuramoto, Y. (1975). in international symposium on mathematical problems in theoretical physics. *Lecture notes in Physics*, 39, 420.

Kuramoto, Y. (1984). Cooperative dynamics of oscillator community. *Progress of Theoretical Physics Suppl*, (79), 223–240.

Li, X. (2008). The role of degree-weigthed couplings in the synchronous onset of Kuramoto oscillator nertworks. *Physica A*, (6624-6630).

Lin, C. and Lin, M. (2009). The mathematical research for the Kuramoto model of the describing neuronal synchrony in the brain. *Commun Nonlinear Sci Numer Simulat*, (14), 3258–3260.

Marshall, J., Broucke, M., and Francis, B. (2004). Formations of vehicles in cyclic pursuit. *IEEE Transactions on Automatic Control*, 49(11), 1963–1974.

Monzón, P. and Paganini, F. (2005). Global considerations on the Kuramoto model of sinusoidally coupled oscillators. In *Proc. of the 44th IEEE CDC and the ECC*, 3923–3928. Sevilla, Spain.

Olfati-Saber, R., Fax, A., and Murray, R. (2007). Consensus and cooperation in networked multi-agent systems. *Proceedings og the IEEE*, 95(1), 215–233.

Rantzer, A. (2001). A dual to Lyapunov's stability theorem. *Systems and Control Letters*, 42(3), 161–168.

Rogge, J. and Aeyels, R. (2004). Stability of phase-locking solutions in a ring of unidirectionally coupled oscillators. *Journal of Physics A: Mathematical and General*, 37, 11135–11148.

Smet, F.D. and Aeyels, D. (2008). Clustering in a network of mutually attracting agents. In *47th IEEE Conference on Decision and Control*, 1827–1832. Mexico.

Strogatz, S. (1994). *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry and Engineering*. Perseus.

Verwoerd, M. and Mason, O. (2007). Conditions for the existence of fixed points in a finite system of Kuramoto oscillators. In *American Control Conference*, 4613–4618.

Wang, W. and Ghosh, B. (2007). Kuramoto models, coupled oscillators and laser networks. In *SICE Annual Conference*, 130–135.

Winfree, A. (1980). *The geometry of biological time*. Springer.