Stability conditions for a Stochastic Dynamic Optimizer for optimal dispatch policies in power systems with hydroelectrical generation

R. Chaer and P. Monzón

***Abstract*—This work analyzes the necessary and sufficient conditions for the stability of the stochastic dynamic optimization algorithm for the calculus of the water cost for one hydroelectrical generation plant. We present the theory and different simulations for parameters inside and outside the stability region. A particular relationship between the time integration step, the space discretization step and the maximal incoming and outcoming flows. In order to show some advantages and disadvantages of the method, we show a simulation of the Uruguayan generation system with four hydroelectrical generation plants. We show that the stability conditions impose a small time simulation step, i.e., large total simulation time. As a future research direction, we think that this time can be reduced using non linear integration methods**

Index terms: **Stochastic optimization, Power generation dispatch, Hydrotermal scheduling**.

I. INTRODUCTION

THE optimization of the operation of an hydrothermal system is a complex problem. The complexity comes from the fact that we are leading with system with reservoirs and then the problem of how to use the stocked resources is not only how much to use of each of the stocks but also when to use them.

The problem is formulated as an optimization with the objective of minimize the cost of fuel consumption at the thermal plants and the cost of fail in supply the demanded energy. The calculus is divided in a set of consecutive stages or time steps. In each stage the production costs of each thermal unit and of fail in supply the demanded energy, is supposed known. The water in the reservoirs hasn't an explicit cost value so the production cost of the hydroelectric plants is not defined. The use of stocked water today potentially increases the cost of stages in the future. The preservation of water today for a later use perhaps reduce the cost of some stages in the future but really increases the cost today due to the additional thermal generation that will be needed to substitute the avoided hydroelectric generation. That is, the problem is to found a policy of use of the stocked resources that result in a equilibrium between today and future costs.

Then, we face an optimization problem: minimize a cost function subject to several constrains. The are at least two well known strategies to face this problem. The more classical is called Stochastic Dynamic Programming (SDP) and the other is the called Stochastic Dual Dynamic Programming (SDDP).

The SDP computes the cost function the future back to the present. To proceed with the calculus, a discretization, both in time and space, is defined for each of the state variables of the system. This leads to the well known Bellman's "curse of dimensionality" that turns the SDP not applicable when the number of state variables increases [1], [2].

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The SDDP leads with the dimension of the state space using Benders cuts to approximate the cost function. A very good explanation of the method is given in [3]. The approximation is carried out in successive sweeps of the stages forward, computing the cost of a feasible solution and backward computing the cost of the relaxation. If the cost function and the constrains are convex, we obtain the exact solution. Without convexity, we have a gap. When the production costs of the thermal units are considered constant, the resulting cost functions are convex, linear, so the overall production cost is also convex and the method is applicable. When a more detailed production cost function is considered, a minimum operation point appears resulting in a non convex function. If the system is great enough the duality gap is irrelevant as shown in [4]. But in small systems, where the power of a unit is greater than 10% of the power of the demand the duality gap may be relevant and so the SDDP may be inappropriate.

The daily maximum of the power demand in Uruguay is about 1000MW. The greatest thermal unit in the system has a power of 125MW, so the system is very small and some care must be taken with dual optimization techniques. It is also true that classical SDP method are more suitable for distributed programming and with the permanent increasing of the power of computers at lower prices, it is foreseeable that SDP method can be implemented in spite of "the curse of dimensionality".

This work analyzes the necessary and sufficient conditions for the stability of the stochastic dynamic optimization algorithm for the calculus of the water cost for one hydroelectrical generation plant. We present the theory and different simulations for parameters inside and outside the stability region. A particular relationship between the time integration step, the space discretization step and the maximal incoming and outcoming flows.

In order to show some advantages and disadvantages of the method, we show a simulation of the Uruguayan generation system with four hydroelectrical generation plants. We show that the stability conditions impose a small time simulation step, i.e., large total simulation time. As a future research direction, we think that this time can be reduced using non linear integration methods.

The article is organized as follows. In Section II, we present the problem of dynamic programming, the derivation of the future cost function using a linear approximation and the distinct forms of performing the derivation. In Section III, the asymptotical analysis is performed and a convergence condition is stated. An Example shown the relevance of the number of water reservoirs is included in Section IV. Finally, some conclusions and comments end the work.

II. THE PROBLEM

As we have mentioned, we want to optimize the operation of a power system with one hydroelectrical generation plant. We perform a time discretization, introducing a time step. We denote by the integer k the actual time. The state of the system is given by the current volume V of the water reservoir. Its ranges from $V = 0$ (empty) to $V = V_{max}$ (full). Again, we discretize this level with step ΔV in order to have a discrete state j, ranging from 0 (empty) to N (full). It follows that

$$\Delta V = \frac{V_max}{N-1} \quad , \quad V_j = (j-1).\Delta V \quad , j = 1, \ldots, N$$

We will construct a future cost function $FC_k(V)$ that, for each time step k and every state j, tells us the optimal cost of the future operation of the system.

The Dynamic Programming Principle says that the operation cost from time k and volume V is equal to the operation cost from time $k+1$ and volume V' plus the cost of the dispatch decisions made at time k. We have the expression

$$FC_k(V) = \beta.FC_k\left(V'\right) + cs_k\left(V, u_k, r_k\right) \tag{1}$$

where β represents the effect of a discount rate that adapts to the actual time the future costs[1], V and V' are the state of the system at times k and $k+1$ respectively, cs_k is the running cost of the **optimal** dispatch decisions at time k. It computes the cost of fuel wasted in the thermal units and the cost associated with energy not supplied to the demand during the stage k. It depends on the actual state V_k, the control variables u_k and some random inputs r_k, like the water inflows to the reservoir. We also assume we known the relationship between V, the initial state at time k and the final state V':

$$V' = f\left(V, u_k, r_k\right)$$

We use a first order approximation of FC_k:

$$FC_{k+1}(V') \approx FC_{k+1}(V) + \frac{\partial FC_{k+1}}{\partial V}(V).(V'-V) \tag{2}$$

The total variation of the volume, $V'-V$, is

$$V' - V = Q'_k(V).\Delta T \tag{3}$$

with $Q'_k(V)$ the net water inflow to the reservoir and ΔT the duration of the stage. The net water inflow can be decomposed as follows:

$$Q'_k(V) = Q_E - Q_S = Q_E - (Q_T + Q_V + Q_P)$$

Q_E denotes the water flow that enters the reservoir. It includes rain and up streams water; Q_T is the turbinated flow, that is, water used to generate electric power; Q_V represents the water released and Q_P the flow lost by evaporation and filtration. Combining equations (1), (2) and (3), we obtain

$$FC_k(j) = \beta.\left[FC_{k+1}(j) + \frac{\partial FC_{k+1}}{\partial V}(j).Q'_k.\Delta T\right] + cs_k(j) \tag{4}$$

where we have simplified the notation of the volume.

We introduce a vector of future costs at the beginning of stage k:

$$c_k = \left[FC_k(1), FC_k(2), \ldots, FC_k(N)\right]^T$$

and the vector of dispatch costs (usually termal unit costs) at stage k:

$$y'_k = \left[cs_k(1), cs_k(2), \ldots, cs_k(N)\right]^T$$

Let q'_k be a diagonal matrix with

$$q'_k(i,i) = Q'_k(i) \quad , \quad i = 1, \ldots, N$$

Then, we can approximate the derivative $\frac{\partial FC_{k+1}}{\partial V}(j)$ as a linear combination of the elements of C_{k+1}, divided by the volume step ΔV:

$$\frac{\partial FC_{k+1}}{\partial V}(j) = \frac{1}{\Delta V}.D.c_{k+1}$$

[1]For example, in a daily time basis and with an annual discount rate of 12%, we have

$$\beta = \left(\frac{1}{1+0.12}\right)^{\frac{1}{365}}$$

Meaningful discount rates should not exceed the 12%.

Then, equation (4) can be re-written as

$$c_k = \beta.\left(1 + \frac{\Delta T}{\Delta V}.q'_k.D\right).c_{k+1} + y'_k \tag{5}$$

where D is a matrix that performs the derivative approximation. Several matrices D can be used. For example, if we use a *backward incremental quotient*:

$$\frac{\partial FC_{k+1}}{\partial V}(j) \approx \begin{cases} \frac{FC_{k+1}(j) - FC_{k+1}(j-1)}{\Delta V} & ; \quad j = 2, \ldots, N \\ \frac{FC_2(j) - FC_{k+1}(1)}{\Delta V} & ; \quad j = 1 \end{cases}$$

we have the matrix

$$D_{dec} = \begin{bmatrix} -1 & 1 & & & & & & \\ -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & -1 & 1 & & & \\ & & & & -1 & 1 & & \\ & & & & & -1 & 1 \end{bmatrix} \tag{6}$$

Observe that we have repeated the first row, in order to solve the border condition at $j=1$. In the same way, if we use a forward incremental quotient, we have

$$D_{inc} = \begin{bmatrix} -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & \ddots & \ddots & & & \\ & & & -1 & 1 & & \\ & & & & -1 & 1 & \\ & & & & & -1 & 1 \\ & & & & & -1 & 1 \end{bmatrix} \tag{7}$$

Matrix D_{dec} is suitable for handling qs, i.e., the outgoing vector component of the water flow vector q_k, since when the reservoir is empty, no water can be released, turbinated or evaporated. By a similar argument, we will use D_{inc} to deal with the vector qe_k of entering water. To get a good derivative approximation, we choose to use a linear combination of D_{dec} and D_{inc}. Our final first order approximation of the cost function is the following:

$$c_k = \beta.\left[1 + \frac{\Delta T}{\Delta V}.\left(qe_k.D_{inc} - qs_k.D_{dec}\right)\right].c_{k+1} + y'_k \tag{8}$$

with qe_k and qs_k diagonal matrices, constructed like q_k.

III. Asymptotical analysis

In this Section, we find conditions for the convergence of the algorithm showed in (8). It must be initialized at $k=+\infty$ (or big k). In order to formalize the analysis, we revert time, and consider the infinity at $k=0$. Let us put

$$A_k = \beta\left[I + \frac{\Delta T}{\Delta V}\left(qe_k D_{inc} - qs_k D_{dec}\right)\right] \quad , \quad b_k = y'_k$$

Then, we have the algorithm

$$c_{k+1} = A_k c_k + b_k \tag{9}$$

We emphasize that at each step, A_k and b_k stands for the optimal dispatch decisions, that involve the released or turbinated flow. We assume a known value QE_{max}, the maximum water flow that can enter to the reservoir. In the same way, QS_{max} is the maximum water flow that can leave the reservoir. Denote by $\mathcal{A}$ and $\mathcal{B}$ the sets where we choose matrices A_k and b_k from. These sets are convex and bounded.

We consider two different initial conditions c_0 and c_0 (these are the *final values* of the cost function). The algorithm (9) generates the sequences $\{c_k\}$ and $\{c'_k\}$. We prove that at each step, the sequences strictly approach each other. We use the infinity norm for a N-dimensional vector to measure the distance:

$$\|v\|_\infty = \max_{i=1,\ldots,N}\{|v_i|\}$$

At time $k+1$ we have

$$\begin{aligned} c_{k+1} &= A_k c_k + b_k \\ c'_{k+1} &= A'_k c'_k + b'_k \end{aligned} \tag{10}$$

Optimality of the couples (A_k, b_k) and (A'_k, b'_k) implies

$$\begin{aligned} c_{k+1} &\geq A'_k c_k + b'_k \\ c'_{k+1} &\geq A_k c'_k + b_k \end{aligned}$$

where the inequalities are at each component. Then,

$$\|c_{k+1} - c'_{k+1}\|_\infty = \left|\left(c_{k+1}\right)_i - \left(c'_{k+1}\right)_i\right|$$

and one of the two following inequalities must be true

$$\begin{aligned} \left|\left(c_{k+1}\right)_i - \left(c'_{k+1}\right)_i\right| &\leq \left|\left[A_k\left(c_k - c'_k\right)\right]_i\right| \\ \left|\left(c_{k+1}\right)_i - \left(c'_{k+1}\right)_i\right| &\leq \left|\left[A'_k\left(c_k - c'_k\right)\right]_i\right| \end{aligned} \tag{11}$$

We know that

$$\left|\left[A_k\left(c_k - c'_k\right)\right]_i\right| = \left|\sum_{j=1}^{N} A_{k_{ij}}\left(c_k - c'_k\right)_j\right|$$

Then

$$\left|\left[A_k\left(c_k - c'_k\right)\right]_i\right| \leq \|A\|_1 . \|c_k - c'_k\|_\infty$$

where $\|A\|_1$ denotes the matrix norm induced by the norm we have used for vectors [5]:

$$\|A\|_1 = \max_{\|v\|_\infty = 1}\{\|Av\|_\infty\} = \max_{i=1,\ldots,N}\left\{\sum_{j=1}^{N}|A_{ij}|\right\}$$

For every k, we the the inequality

$$\|c_{k+1} - c'_{k+1}\|_\infty \leq \max\left\{\|A_k\|_1, \|A'_k\|_1 .\right\}\|c_k - c'_k\|_\infty \tag{12}$$

and we conclude that, if every possible matrix $A_k \in \mathcal{A}$ has $\|A_k\|_1 < 1$, two different sequences generated by the algorithm (9) approach each other as time evolve. Let us compute $\|A_k\|_1$. Recall that

$$A_k = \beta . \left[I + \frac{\Delta T}{\Delta V}\left(qe_k D_{inc} - qs_k D_{dec}\right)\right]$$

Consider the non zero elements of the rows of A_k. For the first row, we have

$$A_k(1,1) = \beta . \left[1 - \frac{\Delta T}{\Delta V}\left[QE_k(1) - QS_k(1)\right]\right]$$

$$A_k(1,2) = \beta . \frac{\Delta T}{\Delta V}\left[QE_k(1) - QS_k(1)\right]$$

For the generic i-th row:

$$A_k(i,i-1) = \beta . \frac{\Delta T}{\Delta V} QS_k(i)$$

$$A_k(i,i) = \beta . \left[1 - \frac{\Delta T}{\Delta V}\left[QE_k(i) + QS_k(i)\right]\right]$$

$$A_k(i,i+1) = \beta . \frac{\Delta T}{\Delta V} QE_k(i)$$

The final row N verifies:

$$A_k(N,N-1) = \beta . \frac{\Delta T}{\Delta V}\left[QS_k(N) - QE_k(N)\right]$$

$$A_k(N,N) = \beta . \left[1 - \frac{\Delta T}{\Delta V}\left[QS_k(N) - QE_k(N)\right]\right]$$

First of all, observe that the sum of all the elements of the row is β. In particular, this implies that matrix A_k has an eigenvalue β with associated eigenvector the vector with all the components equal to 1. In second place, note that if $\frac{\Delta T}{\Delta V}$ is taken small enough, all the elements of A_k are non negative. It must be

$$\frac{\Delta T}{\Delta V} . \left[QE_k(i) + QS_k(i)\right] < 1 \quad , \ \forall \ i = 2, \ldots, N-1 \tag{13}$$

$$\frac{\Delta T}{\Delta V} . \left[QE_k(1) - QS_k(1)\right] < 1 \tag{14}$$

$$\frac{\Delta T}{\Delta V} . \left[QS_k(N) - QE_k(N)\right] < 1 \tag{15}$$

The only problematic terms are $A_k(1,2)$ and $A_k(N, N-1)$. Consider the first one. We are dealing with the reservoir at its lower level. Then, it must be $QS_k(1) \leq QE_k(1)$, since it is a restriction for an optimal dispatch. In the same way, if we consider the reservoir full of water, an optimal dispatch will ensure $QS_k(N) \geq QE_k(N)$. Then, a direct calculation gives

$$\|A_k\|_1 = \beta$$

and we obtain that every non zero discount rate satisfies the desire convergence condition. We can also conclude that every eigenvalue of A_k lies in the unit circle, since [5]

$$\max\{|\lambda_{A_k}|\} \leq \|A_k\|_1$$

So far, we have proved that the distance between any two distinct sequences generated by the algorithm (9) converges asymptotically to zero. Each of these sequences are monotone, since the cost function FC increases when we add a new stage. Let $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}_\mathcal{B}$ be bounds for the elements in $\mathcal{A}$ and $\mathcal{B}$ respectively. As we have seen, we can take $\mathcal{K}_\mathcal{A} < 1$, using the induced norm. Then, for a given initial condition c_0, at the first iteration we have

$$c_1 = A_0 c_0 + b_0$$

and

$$\|c_0\|_\infty \leq \|A_0\|_1 . \|c_0\|_\infty + \|b_0\|_\infty \leq K_\mathcal{A}\|c_0\|_\infty + K_\mathcal{B}$$

For the k-th iteration, we obtain the bound

$$\|c_k\|_\infty \leq \|c_0\|_\infty + K_\mathcal{B} . \left(\sum_{i=0}^{k-1} K_\mathcal{A}^i\right)$$

$$\|c_k\|_\infty \leq \|c_0\|_\infty + \frac{K_\mathcal{B}}{1 - K_\mathcal{A}}$$

Then, we always deal with bounded sequences. Then, the algorithm converges to the same result, for every initial condition.

IV. Example

The Uruguayan system has four hydro-plants: ”Bonete”,”Baygorria” and ”Palmar” over the ”Rio Negro” river and the bi-national plant ”Salto Grande” over the ”Río Uruguay” river. The three plants over the ”Rio Negro” are chained one after the other with Bonete at the upstream of Baygorria and this one at the upstream of Palmar. The more relevant parameters of these plants are shown in Table I.

In order to have a measure of the stability of the algorithm we have carried out a set of optimizations with different time-steps. In this optimization, mean values for the inflows and of the power of the different units where considered, so no stochastic data are in play. For the simulation we have chosen a volume discretization of five points for the lakes of Baygorria, Palmar and Salto Grande and a ten points discretization for Bonete.

TABLE I

DATA OF THE EXAMPLE OF SECTION IV.(*): ELEVATION IS MEASURED ABOVE SEA LEVEL; (**): THIS VALUES CORRESPOND TO THE 50% OF THE PLANT OWNED BY URUGUAY.

	Bonete	Baygorria	Palmar	Salto-UY	
Minimum elevation of the lake $[m]$	70	53	36	30	*
Maximum elevation of the lake $[m]$	81	56	44	35.5	*
Discharge elevation [m]	Baygorria	Palmar	7.5	5	*
Storage capacity of the lake $[Hm^3]$	8210	216	2575	3058	**
Mean inflow to the basin $[m^3/s]$	567	0	290	2358	
Maximum discharge flow $[m^3/s]$	680	828	1373	4200	**
Installed power $[MW]$	155	108	333	945	**

TABLE II

OPTIMIZATION PARAMETERS IN EXAMPLE OF SECTION IV.

	Bonete	Baygorria	Palmar	Salto-UY
Maximum inflow $[m^3/s]$	567	680	970	2358
Maximum outflow $[m^3/s]$	680	828	1373	4200
Time filling the lake (TFL) $[days]$	168	4	31	15
Time emptying the lake (TEL) $[days]$	139.7	3.0	21.7	8.4
Discretization steps $[n]$	10	5	5	5
Volume step $(V/(n-1))$ $[Hm^3]$	912.2	54.0	643.8	764.5
Time filling a volume step (TFVS) $[hours]$	446.9	22.1	184.3	90.1
Time emptying a volume step (TFVS) $[hours]$	372.64	18.12	130.26	50.56

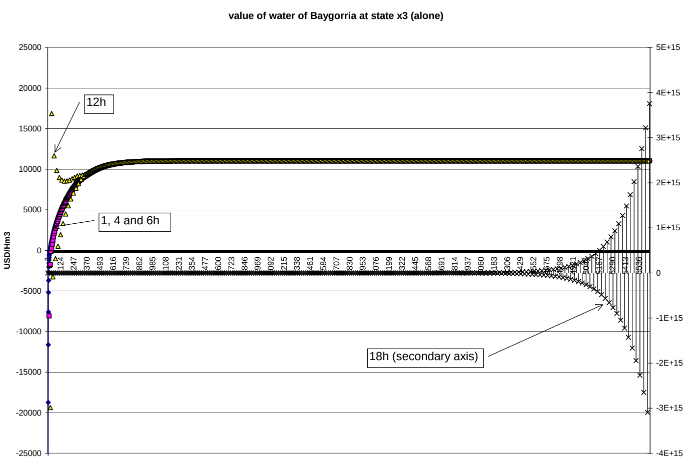

Fig. 1. Value of the water of Baygorria at state x_3 - only one hydroelectrical plant. (Reversed time axis in hours).

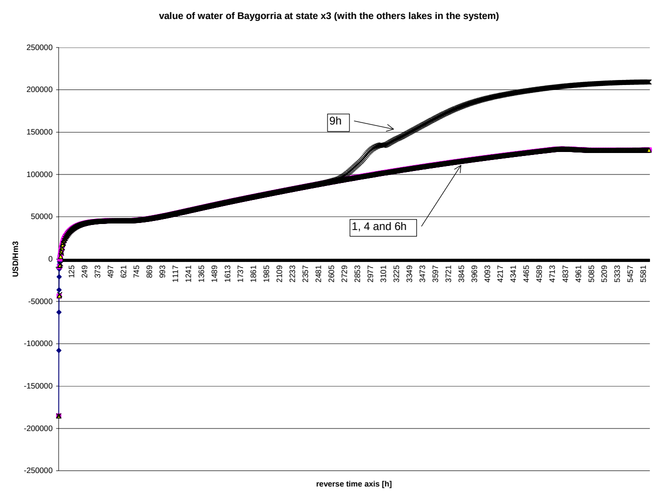

Fig. 2. Value of the water of Baygorria at state x_3 - with the other lakes in the system. (Reversed time axis in hours).

Knowing the maximum inflow and outflow of the lake, we can compute the *Time to Filling the Lake* (TFL), the *Time to Emptying the Lake* (TEL), the *Time to Fill a Volume Step* (TFVS) and the *Time to Emptying a Volume Step* (TEVS). These values are shown in Table II for the four lakes of Uruguay. The stability criterion (13)-)15) imposes an optimization time-step less than the minimum TFVS and less than the minimum TEVS. In our example the TEVS of Baygorria imposes a optimization time-step less than 18 hours.

The first example is considering only the lake of Baygorria with a discretization of 5 points of the state space. For each of the optimization stages the future cost function for the five state-points x_1, x_2, x_3, x_4 and x_5 is computed. Fig. 1 shows the result of the optimization for the point x_3 for stages of different time duration. The values plotted are derivative of the future cost function with respect to the volume of the lake, that is, the *value* of the stocked water. The trajectory corresponding to the time-step of 18 hours diverge and is plot against the secondary axis in the picture. This simple example shows the relationship between the discretization and the time steps.

Fig. 1 shows the value of the water of Baygorria for the five discretization points chosen for the representation of the state space at the last computed stage of the optimization process. Look that this stage is the first in time because the optimization is carried from the future to the present. This picture was build considering only the Baygorria lake. The values of the water of Baygorria for steps-times les than 6 hours are approximately the same. For a time step of 12 hours, the curve differs from the curve of 4 hours. The calculus is still stable but an error is present. As we increases the time step, the error increases. For a time step of 18 hours, the calculus is unstable and the algorithm diverges. When more than one lake is considered, the stability criterion differs from the case of only one lake in the system. Fig. 2 shows the same than fig. 1 but when the four lakes are considered for the optimization. In this case, the optimizations with time steps less than or equal to 6 hours are stable and the trajectories of the water value are visually the same. A time step of 9 hours is to large and the error are accumulated.

V. CONCLUSIONS

A criterion for the stability of a stochastic dynamic optimizer for optimal dispatch policies. This criterion imposes a maximum time step for the computation of the dispatch policy. A good selection of the time step is five times lesser than the theoretical maximum. This ensures a good error control, since we use a linear approximation for the future cost function. The stability condition was derived for a power system with only one reservoir and does not directly apply when more reservoirs are present. The example with four reservoir illustrates this fact. In the actual implementation in our simulator, the stability condition was improved using prediction-correction techniques, keeping the restrictions derived here for error control. The simulator is is freely available at http://iie.fing.edu.uy/simsee/.

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VI. BIOGRAPHIES

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