8

Almost Global Synchronization of Symmetric Kuramoto Coupled Oscillators

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1. Introduction

A few decades ago, Y. Kuramoto introduced a mathematical model of weakly coupled oscillators that gave a formal framework to some of the works of A.T. Winfree on biological clocks [Kuramoto (1975), Kuramoto (1984), Winfree (1980)]. The model proposes the idea that several oscillators can *interact* in a way such that the individual oscillation properties change in order to achieve a global behavior for the interconnected system. The Kuramoto model serves as a good representation of many systems in several contexts: biology, engineering, physics, mechanics, etc. [Ermentrout (1985), York (1993), Strogatz (1994), Dussopt et al. (1999), Strogatz (2000), Jadbabaie et a. (2003), Rogge et al. (2004), Marshall et al. (2004), Moshtagh et al. (2005)].

Recently, many works on the control community have focused on the analysis of the Kuramoto model, specially the one with sinusoidal coupling. The *consensus* or *collective synchronization* of the individuals is particularly important in many applications representing coordination, cooperation, emerging behavior, etc. Local stability properties of the consensus have been initially explored in [Jadbabaie et al. (2004)]. It must be noted that little attention has been devoted to the influence of the underlying interconnection graph on the stability properties of the system. The reason could be the fact that the local stability does not depend on the interconnection [van Hemmen et al. (1993)]. Global or *almost global* dynamical properties were studied in [Monzón et al. (2005), Monzón (2006), Monzón et al. (2006)]. In these works, the relevance of the interconnection graph of the system was hinted.

In the present chapter, we go deeper on the analysis of the relationships between the dynamical properties of the system and the algebraic properties of the interconnection graph, exploiting the strong algebraic structure that every graph has. We step forward into a classification of the interconnection graphs that ensure almost global attraction of the set of synchronized states.

In Section 2 we present the Kuramoto model for sinusoidally coupled oscillators, its general properties and the notion of almost global synchronization; in Section 3 we review some basic facts on algebraic graph theory; the symmetric Kuramoto model and the block analysis are presented in Sections 4 and 5; Section 6 gives some examples and applications of the main results; Section 7 presents the problem of classification of almost global synchronizing topologies.

2. The Kuramoto model

In the 1970s, Kuramoto proposed a model to describe a population of weakly coupled oscillators. In this model, each individual oscillator is described by its phase and the coupling between two individuals is a function of the phase difference. The general Kuramoto model takes the following form [Strogatz (2000)]:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}\left(\theta_i - \theta_j\right) \quad , \quad i = 1,\ldots,N$$

where Γ_{ij} are the *interaction* functions that represent the coupling and *N* is the total number of oscillators. Since each angle $\theta_i \in [0,2\pi)$, the corresponding state space is the *N*-dimensional torus T^N. We consider the particular case of sinusoidally coupled oscillators,

$$\dot{\theta}_i = \omega_i + \frac{K}{N}.\sum_{j\in N_i} \sin\left(\theta_i - \theta_j\right) \quad , \quad i = 1,\ldots,N \tag{1}$$

where N_i refers to the set of index of agents that affect the behavior of agent *i* -the *neighbors* of *i*- and *K* is the strength of the coupling. We will assume that all the agents have the same natural frequency. So, with a suitable shift, and simplifying the notation by eliminating the factor $\frac{K}{N}$ -this amount for to renormalizing time- we can write the previous model as

$$\dot{\theta}_i = \sum_{j\in N_i} \sin\left(\theta_i - \theta_j\right) \quad , \quad i = 1,\ldots,N \tag{2}$$

We want to emphasize the following aspects of system (2):

- The dynamic depends only on the phase difference of the oscillators. Then, there are several properties that are invariant under translations on the torus. For example, if $\overline{\theta}$ is an equilibrium point, so is[1] $\overline{\theta} + c.1_N$ for every $c \in [0,2\pi)$.
- As was done by Kuramoto [Kuramoto (1984)], we associate the individual oscillator phases to points running around the circle of radius 1 in the complex plane. Then, each oscillator can be described by the unitary phasor $V_i = e^{j\theta_i}$.

Equation (2) has always two kinds of *trivial* equilibria:

- We call **consensus** or **synchronization** the state where all the phase differences are zero, i.e. the diagonal of the state space. Every consensus state is of the form $\overline{\theta} = c.1_N$, with $c \in [0,2\pi)$. We have a closed curve of consensus points. Observe that at a consensus point, all the associated phasors coincide.

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[1] $\mathbf{1}_p$ denotes the column vector in R^p with all the elements equal to one. Analogously, $\mathbf{0}_p$ denotes the column vector in R^p with all the elements equal to zero.

- We say we have *partial synchronization* when all the phasors are parallel but they are not synchronized; i.e. most of the phases takes the value 0 (taking a suitable reference), but there are *m* agents with phase $\pm\pi$, for some $0 < 2m \le N$.
- The other equilibrium points have non-parallel phasors and we refer to them as *non synchronized*.

Example 2.1: Consider the graph *G* shown at the left of Figure 1. A non synchronized equilibrium point of (2) with interconnection graph *G* is given by

$$\overline{\theta} = [-160.95, 90, -19.09, 160.91, -90, 19.09, 0, 180]^T$$

and it is shown at the right of Figure 1 (the angles are measured in degrees). ♦

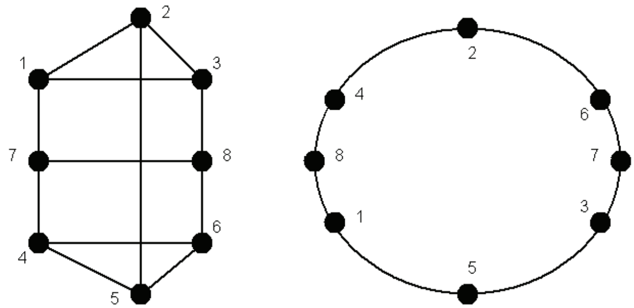


Figure 1. Phasor representation of the equilibrium point $\overline{\theta}$ of Example 2.1. The underlying graph is shown on the left

The key question we try to answer in this work is whether or not the system behavior of (2) reaches *consensus*, since this particular equilibrium may represent a desired behavior of the system. Recently, the Kuramoto model has received the attention of control theorists interested in the coordination and consensus of multi-agent systems (see [Jadbabaie et al (2004)] and references there in). We focus on the global properties of the consensus equilibrium. Since the system has many equilibria, we can not talk about global stability or global synchronization. But we may wonder if the system present the so called *almost global stability* property, that is, if the set of initial conditions that no lead to synchronization has zero Lebesgue measure. From an engineering point of view, this is a nice property [Rantzer (2001)], specially when it is combined with local stability. When the system has the almost global stability property, almost every initial condition leads to the synchronization of the system. So, we will use the expression *almost global synchronization* and the abbreviation a.g.s.

2.1 General properties

The following results are true for the general dynamic (2)

Proposition 2.1: At any equilibrium point $\overline{\theta}$ of (2), it must be true that the phasors $\sum_{h\in N_i} V_h$ and V_i are parallel in the complex plane, for every *i*.

Proof: For $i = 1,\ldots, N$, consider the number

$$\alpha_i = \sum_{h\in N_i} \frac{V_h}{V_i} = \sum_{h\in N_i} e^{j(\theta_h - \theta_i)} = \sum_{h\in N_i} \cos(\theta_h - \theta_i) + j.\sum_{h\in N_i} \sin(\theta_h - \theta_i)$$

Since $\overline{\theta}$ is an equilibrium point, α_i is a real number and $\sum_{h\in N_i} V_h = \alpha_i .V_i$. ♦

Important consequences of Proposition 2.1 will be presented in further sections. Nevertheless, we can write a direct corollary.

Corollary 2.1: Consider an agent *i* such that $N_i = \{k\}$, $i \neq k$. Then, at an equilibrium point $\overline{\theta}$, it must be true that $\overline{\theta}_i = \overline{\theta}_k$ or $\overline{\theta}_i = \overline{\theta}_k + \pi$. ♦

For example, if the underlying graph is a tree (see Section 3), an iterative application of Corollary 2.1 shows that the only equilibria are full or partial synchronized points.

To conclude this Section, we introduce the concept of *phase-locking solution*. We say that a solution $\theta(t)$ is *phase-locking* when the phase difference between any two agents remains constant in time. It follows that for $i = 1,..., N$, we have $\dot{\theta}_i = \Omega$ and $\theta_i(t) = \Omega t + \theta_{0i}$. For the particular case of $\Omega = 0$, we have the equilibrium points described above. Phase-locking solutions with $\Omega \neq 0$ correspond to closed periodic orbits in T^N and play important roles in many contexts, such pace generators or muscular contractions in biology [Ermentrout (1985)], cyclic pursuit problems [Marshall et al. (2004)] or circular polarization generation with antennas [Dussopt et al. (1999).

3. Brief review of algebraic graph theory

We will use a graph to naturally describe the interconnection topology between the agents in the Kuramoto model. In this Section we review the basic facts on algebraic graph theory that will be used along the article. A more detailed introduction to this theory can be found in [Biggs (1983); Cvetkovic et al. (1979)]. A graph *G* consists in a set of *n nodes* or *vertices* $VG = \{v_1,..., v_n\}$ and a set of *m links* or *edges* $EG = \{e_1,..., e_m\}$ that describes how the nodes are related to each other. If *n*=1 the graph is called *trivial*. We say that two nodes are neighbors or adjacent if there is a link in EG between them. If all the vertices are pairwise adjacent the graph is called *complete* or *all to all* and written K_n . A *walk* is a sequence $v_0, v_1,..., v_l$ of adjacent vertices. If the vertices are different except the first and the last which are equal ($v_i \neq v_j$, $0 < i < j$ and $v_0 = v_l$) the walk is called a *cycle*. A graph with no cycle is called *acyclic*. The graph is *connected* if there is a walk between any given pair of vertices. A *tree* is an acyclic connected graph and has *m*=*n*-1 edges. The graph is *oriented* if every link has a starting node and a final node. The topology of an oriented graph may be described by the *incidence* matrix *B* with *n* rows and *m* columns:

$$B_{ij} = \begin{cases} 1 \text{ if edge } j \text{ reaches node } i \\ -1 \text{ if edge } j \text{ leaves node } i \\ 0 \qquad \text{otherwise} \end{cases}$$

Observe that $B^T 1_n = 0$. The semidefinite matrix $L = B^T B$ is called the Laplacian of *G* and contains the *spectral* information of the graph. The *vertex space* and the *edge space* of *G* are the sets of real functions with domain *VG* and *EG* respectively, which we sometimes will identify, respectively, with the vectors spaces R^n and R^m. Thus, the incidence matrix *B* can be seen as a linear transformation from the edge space to the vertex space. The kernel of *B* is called the *cycle space* of the graph *G* and its elements are called *flows*. Every *flow* can be thought as a vector of weights assigned to every link in a way that the total algebraic sum at each node is zero. The cycle space is spanned by the flows determined by the cycles: given a cycle $v_0,...,v_l = v_0$, its associated flow $f_C(e)$ is ± 1 if *e* leaves some v_i and reaches $v_{i\pm 1}$ and 0 otherwise.

If the graph *G* is the union of two nontrivial graphs G_1 and G_2 with one and only one node v_i in common, then v_i is called a *cut-vertex* of *G*. A connected graph with more than two vertices and no cut-vertex is called *2-connected* and it follows that for every pair of nodes, there are at least two different walks between them. A *bridge* is a link with the following particular property: if it is removed, the resulting graph is not connected. Given a subset $V_1 \subset VG$, its *induced subgraph* is $\langle V_1 \rangle$, with vertex set V_1 and edge set $\{e \in EG : \ e \text{ joins vertices of } V_1\}$. The maximal induced subgraphs of *G* with no cut-vertex, are called the *blocks* of *G*. Every graph has the form of Figure 2: a collection of blocks joined by cut-vertices. For a complete graph, there is only one block, the graph itself. A tree can be seen as a collection of K_2.

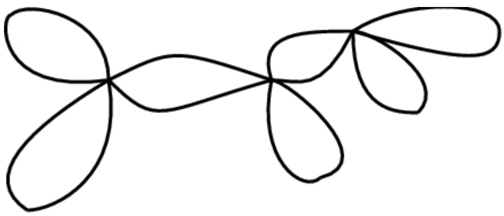


Figure 2. Representation of a graph as a union of blocks

The complement $\overline{G}$ of a graph *G* is another graph with the same nodes as *G* and such that two nodes are related in $\overline{G}$ if and only if they are not related in *G*. It follows that $G + \overline{G}$ is a complete graph, where the sum of two graphs with the same set of nodes is defined as a new graph which has all the edges of the original graphs.

We will use the following vector notation: given a *n*-dimensional vector $\overline{\theta} = [\theta_1,...,\theta_n]$, then $\overline{\theta}(i:j) = [\theta_i,...,\theta_j]$ and $\overline{\theta}(i) = \theta_i$.

4. Symmetric Kuramoto model

4.1. Dynamics

The dynamic of a given agent depends on the sine of its phase differences with its neighbors. Symmetry is characterized by $i \in N_k \Rightarrow k \in N_i$. As in [Jadbabaie et al. (2003)], we can build a directed graph G with the agents as nodes and the edges representing the relationships between agents. We only put one link between neighbors, with arbitrary orientation. Let M be the number of edges. We construct the incidence matrix $B_{N\times M}$ as in previous Section. In matrix notation, the dynamic (2) can be written as

$$\dot{\theta} = -B.\sin\left(B^T\theta\right) \tag{3}$$

We must emphasize that equation (3) does not depend on the particular orientation we have chosen for the links of the underlying graph. First of all, we show that the only phase-locking solutions of a symmetric system are the ones with $\Omega = 0$.

Lemma 4.1: The only phase-locking solutions of system (3) are equilibrium points.

Proof: Symmetry implies that the sum of all the phases is a constant magnitude of the system:

$$\frac{d}{dt}\sum_{i=1}^{N}\theta_i = \sum_{i=1}^{N}\dot{\theta}_i = 1^T.\dot{\theta} = -1^T.B.\sin\left(B^T\theta\right) = 0$$

since $B^T.1 = 0$. At a phase-locking solution, $\dot{\theta} = \Omega.1 = 0$. Then, $0 = 1^T.\dot{\theta} = \Omega.1^T.1 = N.\Omega$
So, $\Omega = 0$ and we have an equilibrium point. ♦

We remark that through this article, we deal with connected graph topologies.

4.1. Stability analysis

Local stability of the consensus point for system (3) was studied in [Jadbabaie et al. (2004)] using La Salle's invariance principle [Khalil (1996)]. The function

$$U(\theta) = M - 1_M^T.\cos\left(B^T\theta\right) \tag{4}$$

is non-negative, and such that the system can be written in the gradient form: $\dot{\theta} = -\nabla U(\theta)$. In particular this implies that the derivative of U along the trajectories is $\dot{U}(\theta) = -\left\|\dot{\theta}\right\|^2$. Hence, the function U is non-increasing along the trajectories. Since $U \equiv 0$ at the consensus set, it is a local Lyapunov function for the consensus set, meaning that if we start near enough to this set, we will converge to it. Since the state space is compact, every trajectory has a non-empty ω-limit set [Guckenheimer et al. (1983)]. La Salle's result ensures that every trajectory goes to the set

$$W = \left\{\theta : \quad \dot{U}(\theta) = -\left\|\dot{\theta}\right\|^2 = 0\right\}$$

which consists only of equilibrium points. In particular, this proves that the system admits no closed limit cycles and we recover the conclusion of Lemma 4.1. In order to establish almost global attraction of the consensus set (almost global synchronization, a.g.s.), it must be true that this set is the only attractor. Frequently, when we are dealing with an a.g.s. system, we will say that the underlying graph G is a.g.s.. The next Example shows a system without the a.g.s. property.

Example 4.1: Consider the case with N=6 in which the dynamics of the agents are as follows:

$$\dot{\theta}_i = \sin(\theta_{i-1} - \theta_i) + \sin(\theta_{i+1} - \theta_i) \qquad i = 1,\ldots,N$$

Here the configuration is circular; we identify θ_7 with θ_1 and θ_0 with θ_6. Consider the equilibrium point showed in Figure 3. Using an approach that will be presented later, it can be shown that this configuration is locally attractive. ♦

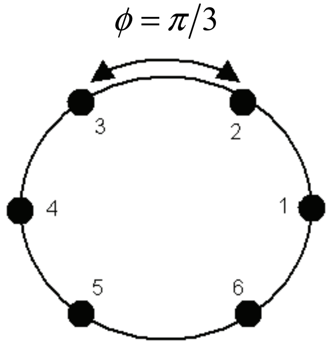


Figure 3. Stable non-consensus equilibrium for the Kuramoto model of Example 4.1

We thus see that guaranteeing almost global asymptotical consensus is more involved. We will analyze the stability of the equilibrium points using Jacobian linearization. A first order approximation of the system at an equilibrium point $\overline{\theta}$ takes the form $\dot{\delta\theta} = A.\delta\theta$, with $\delta\theta = \theta - \overline{\theta}$ and A the symmetric matrix $N \times N$ with entries

$$\begin{cases} a_{ii} = -\sum_{k \in N_i} \cos(\overline{\theta}_k - \overline{\theta}_i) = -\alpha_i \\ a_{hi} = \begin{cases} \cos(\overline{\theta}_k - \overline{\theta}_i) & , \quad h \in N_i \\ 0 & , \quad h \in N_i \end{cases} \end{cases}$$

with α_i defined as in Proposition 2.1. The matrix A can be written as

$$A = -B.diag\left(\cos\left(B^T\theta\right)\right)B^T \tag{5}$$

and can be seen as a *weighted* Laplacian, since $A = -L = -B.B^T$ at a consensus equilibrium. Two facts must be remarked. First of all, A is symmetric, reflecting the bidirectional influence of the agents. This implies that it is a diagonalizable matrix, with real eigenvalues. Note also that $A.1_N = 0$. Hence, A always has the zero eigenvalue, with associated eigenvector 1_N. We will analyze the transversal stability of the consensus set [Khalil (1996)], that is, the convergence to the consensus set.

The following results are true for general graph topologies. Their were originally introduced in [Monzón et al. (2005), Monzón (2006) and Monzón et al. (2006)].

Lemma 4.2: Let $\overline{\theta}$ be an equilibrium point of (3), such that at least one $\alpha_i < 0$. Then, $\overline{\theta}$ is unstable.

Proof: The numbers $-\alpha_i$ appear at the diagonal of the matrix symmetric *A*. So, a negative α_i implies that *A* has a positive eigenvalue. Then, $\overline{\theta}$ is unstable. ♦

Lemma 4.3: Let $\overline{\theta}$ be an equilibrium point of (3), such that $\cos\left(\overline{\theta}_k - \overline{\theta}_i\right) > 0$ for every $k \in N_i$, $i = 1,\ldots,N$. Then, $\overline{\theta}$ is stable.

Proof: Since the underlying graph *G* is connected, 0 is a simple eigenvalue of the Laplacian matrix $L = BB^T$ [Biggs, (1993)]. The linearization matrix *A* described in (5) is a *weighted* version of *L*. Since the weigths are all positive, i.e., the matrix $diag\left[\cos\left(B^T\theta\right)\right]$ is positive definite, $\overline{\theta}$ is stable. ♦

Example 4.2: Lemma 4.3 explains Example 4.1. In that case, the characteristic polynomial of the linear approximation has the roots 0 and -2 (simple), and $-\frac{1}{2}$ and $-\frac{3}{2}$ (double). Indeed, for large *N*, there can be equilibrium configurations with all neighboring angles lesser than $\frac{\pi}{2}$, and thus provide other attractors than the consensus set. ♦

Proposition 4.1: Let $\overline{\theta}$ be a partial consensus equilibrium point of (3). Then $\overline{\theta}$ is unstable.

Proof: At a partial equilibrium point, we have agents at phase 0 and agents at phase π. Define the vector $v = \cos\left(\overline{\theta}\right)$, which only contains 1 and -1. Then, an element of vector $B^T.v$ is null if the link related to the *l*-h row of B^T joins agents with distinct phases. Then, after some calculus, we have that $v.A^T.v = 4.c$, where *c* is the number of links that join agents of the two groups. Then, *A* has a positive eigenvalue and then, $\overline{\theta}$ is unstable. ♦

If for a given graph *G* we can prove that the only equilibrium points correspond to partial or total consensus, we can ensure the almost global stability of the synchronized state. This observation leads us to our first main result.

Lemma 4.4: Consider the system (3) with an associated graph *G* that is a tree. Then, the only equilibrium points are the trivial ones: partial or full consensus.

Proof: With an appropriate reference, a (partial or total) consensus state $\overline{\theta}$ is such that $\sin\left(B^T\theta\right) = 0$. In order to have only partial or total consensus equilibria, 0 must be the only solution of the equation: 0=*B.x*. That is, the cycle space must be trivial. Observe that for a connected graph, the matrix *B*, with *N* rows and *e* columns, has always rank *N*-1. Then, the previous equation has only the trivial solution when *e*=*N*-1, that is, it has full column rank. The only connected graphs with *N*-1 links are the trees. ♦

Theorem 4.1: Consider the system (3). If the associated graph *G* is a tree, it is almost globally stable.

Proof: The result is a direct consequence of Lemma 4.3 and Proposition 4.1. ♦

If we have several systems with underlying topology given by trees, we can interconnect them using single links, keeping the almost global synchronization property. The next Example illustrates that fact.

Example 4.3: A star graph is a connected tree graph that has a particular node, a *hub*, which is related with all of the rest of the nodes, while all the rest of the nodes are related to the hub only. The graph can be sketched as a star and it models several examples of centralized interactions between agents. It is a particular case of Theorem 4.1. The synchronized state is an almost global attractor. Moreover, if we have two star graphs and we couple them through their hubs, as in Figure 4, (or through any pair of agents), we obtain a new almost globally stable system (a kind of *synchronization preserving interconnection*). If we add one more link to a connected tree, we must have a cycle, and we may lose the almost global stability property, as in Example 4.1. ♦

To conclude this Section we present another important result. It states that complete graphs are always a.g.s. The result was originally hinted in several works [Jadbabaie et al. (2004); van Hemmen et al. (1993)]. The prove can be found in [Monzón et al. (2005)].

Theorem 4.2: Consider the system (3). If the underlying graph *G* is complete, the consensus set is almost globally stable. ♦

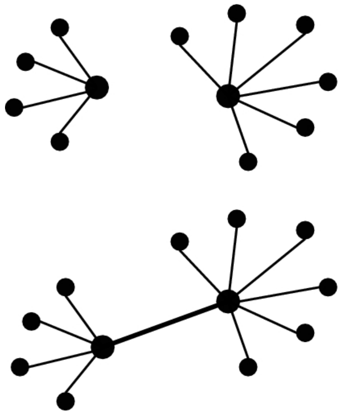


Figure 4. Two star graphs coupled through their hubs (Example 4.3)

5. Block analysis and synchronizing interconnection

In this Section we present some results that help to answer the question of whether or not a graph is a.g.s. They were originally presented in [Monzón et al. (2007); Canale et al. (2007)]. Here, we give a longer presentation.

From equation (3) we see that a phase angle vector θ is an equilibrium point if and only if $\sin\left(B^T\theta\right)$ is a flow on *G*. Thus, it should be possible that the equilibrium points of (3) could be obtained from the equilibrium points of the blocks of the graph *G*. In fact, this is exactly what happens. Furthermore, the stability of these equilibria depends only on the stability of the associated equilibrium points of the blocks. Firstly, we present some basic results. We include two different proofs for Lemma 5.1, in order to show two distinct interpretations of the same facts: one based on linear algebra, the other using graph theory elements. Then, we study the relationship between the equilibria of *G* and the equilibria of its blocks, which will follow directly from Lemma 5.1. After that we focus on the stability properties.

Lemma 5.1: Consider a graph *G*, with v a cut-vertex between G_1 and G_2. Then, an edge space element $f : EG \to R$ is a flow on *G*, if and only if $f\,|_{EG_1}$ and $f\,|_{EG_2}$ are a flows on G_1 and G_2 respectively.

Proof 1: Suppose that the *i* vertices of G_1 and its *k* edges come first in the chosen labelling. Suppose, also, that $v = v_i$, then *B* has the following form:

$$B = \left[\begin{array}{c|c} W_1 & 0_{(i-1)\times(m-k)} \\ \hline w_1^T & w_2^T \\ \hline 0_{(n-i)\times k} & W_2 \end{array}\right]$$

where w_1 and w_2 are column vectors with appropriate dimensions. With this notation, the incidence matrices of G_1 and G_2 are, respectively

$$B_1 = \left[\begin{array}{c} W_1 \\ \hline w_1^T \end{array}\right] \quad , \quad B_2 = \left[\begin{array}{c} w_2^T \\ \hline W_2 \end{array}\right]$$

Besides, B_1, as incidence matrix, verifies $1_i^T B_1 = 0$, thus $1_{(i-1)}^T W_1 + w_1^T = 0$, so

$$w_1^T = -1_{(i-1)}^T W_1 \tag{6}$$

Let *f* be a flow on *G*. In order to prove that $f_1 = f\,|_{EG_1}$ is a flow on G_1, we must show that $B_1.f_1 = 0$, i.e. $W_1.f_1 = 0_{(i-1)}$ and $w_1^T.f_1 = 0$. The former is true because since *f* is a flow on *G*, $B.f = 0$ and $W_1.f_1 = (B.f)(1:i-1)$. On the other hand, by (6), we have that, $w_1^T.f_1 = (-1_{(i-1)}^T.W_1)f_1 = -1_{(i-1)}^T.(W_1.f_1) = -1_{(i-1)}^T.0_{(i-1)} = 0$. With the same arguments, we obtain that $f_2 = f\,|_{EG_2}$ is a flow on G_2.

Conversely, if f_1 and f_2 are flows on G_1 and G_2 respectively, we have that $(Bf)(1:i-1) = B_1.f_1 = 0_i$ and $(Bf)(i+1:n) = B_2.f_2 = 0_{(n-i+1)}$. Finally, a direct calculation gives $(Bf)(i) = w_1^T.f_1 + w_2^T.f_2 = 0+0 = 0$. ♦

Proof 2: Following [Biggs (1993), Lemma 5.1, Theorem 5.2], given a spanning tree *T* of *G*, we obtain a basis of the cycle space in the following form: for each edge $e \in E' = EG \setminus ET$, we have a unique cycle $cyc(T,e)$ which determines a flow $f_{T,e}$. The set B of these flows is a basis of the cycle-space. However, since *v* is a cut-vertex, any cycle is included either in G_1 or in G_2, so its associated flow is null either in G_1 or in G_2. If we regard a flow on *G* which is null in EG_1 as a flow on G_2, we can split B into two sets B_1 and B_2, cycle-space basis of G_1 and G_2 respectively. Thus the cycle-space of *G* is the direct sum of the cycle-spaces of G_1 and G_2. ♦

Lemma 5.2: Let *G* be a graph, $V_1 \subset VG$ and $G_1 = \langle V_1 \rangle$ the subgraph of *G* induced by the vertices V_1 with incidence matrix B_1. Let $H: R \to R$ be any real function, $\overline{\theta}: VG \to R$ an element of the vertex-space of *G* and $f = H\left(B^T \overline{\theta}\right)$. Then, if

$$f_1 : f\,|_{EG_1} \qquad , \qquad \overline{\theta}_1 = \overline{\theta}\,|_{VG_1}$$

it is true that

$$f_1 = H\left(B_1^T \overline{\theta}_1\right)$$

Proof: Suppose that the *i* vertices and *k* edges of G_1 come first in the chosen labelling. Then, for some B', B'' and $\overline{\theta}_2$, we have that

$$B^T \overline{\theta} = \left[\begin{array}{c|c} B_1^T & 0_{i\times k} \\ \hline B' & B'' \end{array}\right]\begin{bmatrix} \overline{\theta}_1 \\ \overline{\theta}_2 \end{bmatrix} = \begin{bmatrix} B_1^T \overline{\theta}_1 \\ B'\overline{\theta}_1 + B''\overline{\theta}_2 \end{bmatrix}$$

Thus, $\left(B^T \overline{\theta}\right)(1:k) = B_1^T \overline{\theta}_1$, and $f_1 = f(1:k) = H\left(B^T \overline{\theta}\right)(1:k) = H\left[\left(B^T \overline{\theta}\right)(1:k)\right] = H\left(B_1^T \overline{\theta}_1\right)$. ♦

5.1 Equilibria

If $\theta_1 : VG_1 \to R$ is in the vector space of a subgraph G_1 of *G*, we will regard it also as its unique extension to the vector space of *G* which is null elsewhere of G_1. The same for an element of the edge space.

Proposition 5.1: Consider the graph *G* with a cut-vertex *v* between G_1 and G_2. If $\overline{\theta}$ is an equilibrium point of *G*, then $\overline{\theta}_1 = \overline{\theta}\,|_{VG_1}$ and $\overline{\theta}_2 = \overline{\theta}\,|_{VG_2}$ are equilibrium points of G_1 and G_2 respectively. Conversely, if $\overline{\theta}_1$ and $\overline{\theta}_2$ are equilibrium points of G_1 and G_2 respectively, there exists a real number α such that $\overline{\theta}'_2 = \overline{\theta}_2 + \alpha 1_{N-k}$ is an equilibrium point of G_2 and $\overline{\theta} = \overline{\theta}_1 + \overline{\theta}_2$ is an equilibrium point of *G*.

Proof: Let *B*, B_1, B_2, etc. like in Lemma 5.1. If $\overline{\theta}$ is an equilibrium point of *G*, then $f = \sin\left(B^T \overline{\theta}\right)$ is a flow on *G*, thus, by Lemma 5.1, $f_1 = f\,|_{EG_1}$ is a flow on G_1. Thus, it is enough to prove that $f_1 = \sin\left(B_1^T \overline{\theta}_1\right)$, which follows from Lemma 5.2, taking $H(x) = \sin(x)$ and noticing that G_1 is an induced subgraph of *G*. The case for G_2 follows by the same arguments.

Now, assume that $\overline{\theta}_1$ and $\overline{\theta}_2$ are equilibrium points of G_1 and G_2 respectively. Let $\alpha = \overline{\theta}_1(v) - \overline{\theta}_2(v)$, $\overline{\theta}'_2 = \overline{\theta}_2 + \alpha 1_{N-k}$, $\overline{\theta} = \overline{\theta}_1 + \overline{\theta}'_2$, and $f = \sin\left(B^T \overline{\theta}\right)$. Then, by Lemma 5.2, $f_1 = f\,|_{EG_1} = \sin\left(B_1^T \overline{\theta}_1\right)$ and $f_1 = f\,|_{EG_1} = \sin\left(B_1^T \overline{\theta}_1\right)$. On the other hand, due to the invariance of the system we have remarked on Section 2, the vector $\overline{\theta}'_2$ is also an equilibrium point of G_2, and then, f_1 and f_2 are flows in G_1 and G_2 respectively. Therefore, by Lemma 5.1, $f_1 + f_2$ is a flow on *G*. But $f = f_1 + f_2$, because $EG_1 \cap EG_2 = \Phi$. ♦

5.2 Stability analysis

We will relate the stability properties of the graph G with a cut-vertex with the stability properties of the subgraphs G_1 and G_2 joined by it. Since every equilibrium of G defines an equilibria for G_1 and G_2, we wonder whether or not the dynamical characteristics of these equilibria are or not the same. We will use Jacobian linearization. Recall that the zero eigenvalue is always present due to the invariance of the system by translations parallel to 1_n. If the multiplicity of the zero eigenvalue is more than one, Jacobian linearization may fail in classifying the equilibria. In this work, we assume that we always have a single null eigenvalue. We do not present here the study of this particular problem.

Theorem 5.1: Consider the graph G, with a cut-vertex v joining the subgraphs G_1 and G_2 of graph G. Let $\overline{\theta} \in R^n$ be an equilibrium point of G. Then, $\overline{\theta}$ is locally stable if and only if $\overline{\theta}1\,|_{VG_1}$ and $\overline{\theta}2\,|_{VG_2}$ are locally stable and coincide in $v = VG_1 \cap VG_2$.

Proof: Recall that the first order approximation of the system around an equilibrium point is given by

$$A_G = -B.diag\left(\cos\left(B^T\theta\right)\right)B^T$$

Suppose that G_1 has i vertices, that they come first in the chosen labelling and that v is the last of them $(v = v_i)$. Then, a direct calculation gives:

$$A_G = A_1 + A_2 \tag{7}$$

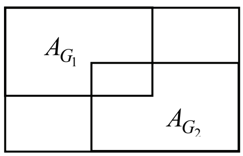
with

$$A_1 = \left[\begin{array}{c|c} A_{G_1} & 0_{i\times(n-i)} \\ \hline 0_{(n-i)\times i} & 0_{(n-i)\times(n-i)} \end{array}\right]$$

and

$$A_2 = \left[\begin{array}{c|c} 0_{(i-1)\times(i-1)} & 0_{(i-1)\times(n-i+1)} \\ \hline 0_{(n-i+1)\times(i-1)} & A_{G_2} \end{array}\right]$$

Observe that these matrices partially *overlap*, so the matrix A takes the form:

$A=$ 

First of all, we consider the case with $\overline{\theta}1$ and $\overline{\theta}2$ stable and $\overline{\theta}1(i) = \overline{\theta}2(1)$. Then, A_{G_1} and A_{G_2} are stable and equation (7) holds for $\overline{\theta} = \left(\overline{\theta}1, \overline{\theta}2(2:n-i)\right)$. So, A_G is the sum of two semidefinite negative matrices which gives rise a semidefinite negative one. Besides, the kernel of A_G has dimension 1, since if $A_G w = 0$, then $w^T A_G w = 0$. Thus, $w^T A_1 w + w^T A_2 w = 0$. But, $w^T A_1 w = w_1^T A_{G_1} w_1$ and $w^T A_2 w = w_2^T A_{G_2} w_2$ for $w_1 = w\,|_{VG_1}$ and

$w_2 = w|_{VG_2}$. Then $w_1^T A_{G_1} w_1 + w_2^T A_{G_2} w_2 = 0$. That can happen if only if $w_1^T A_{G_1} w_1 = 0$ and $w_2^T A_{G_2} w_2 = 0$. But the kernels of A_{G_1} and A_{G_2} are spanned by 1_i and 1_{n-i+1} respectively. Thus $w_1 = \alpha.1_i$ and $w_1 = \beta.1_{n-i}$. Since $w_1(i) = w_2(1) = w(i)$, we have $\alpha = \beta$ and $w = \alpha.1_n$. This proves the stability of A_G.

Now, we focus on the case with $\overline{\theta}_1$ or $\overline{\theta}_2$ unstable. We analyze the first case, since the other is similar. Suppose that A_{G_1} has a positive eigenvalue with associated eigenvector w_1, thus

$$w_1^T A_{G_1} w_1 > 0$$

Define the column vector

$$w = \begin{bmatrix} w_1 \\ w_1(i).1_{n-i} \end{bmatrix} = \begin{bmatrix} w_1(1:i-1) \\ w_1(i).1_{n-i+1} \end{bmatrix}$$

Then,

$$w^T A_G w = w_1^T A_{G_1} w_1 + w_1^2(i).1_{n-i+1}^T A_{G_2} 1_{n-i+1}$$

which actually is $w_1^T A_{G_1} w_1 > 0$ since $w_1^2(i).1_{n-i+1}^T A_{G_2} 1_{n-i+1} = 0$. Then, $\overline{\theta}$ is unstable. ♦

We are now ready to state and prove one of the main results of this Chapter.

Theorem 5.2: Consider the graph *G*, with a cut-vertex v_i joining the subgraphs G_1 and G_2. Then, G_1 and G_2 have the almost global synchronization property if and only if *G* does.

Proof: First of all, let $\overline{\theta}$ be an equilibrium point of *G*. According to Theorem 5.1., $\overline{\theta}$ is stable only if $\overline{\theta}_1 = \overline{\theta}|_{VG_1}$ and $\overline{\theta}_2 = \overline{\theta}|_{VG_2}$ are too. If G_1 and G_2 are a.g.s., the only locally stable set is the consensus, and since they have a vertex in common, the only locally stable equilibria of *G* is also the consensus and *G* is a.g.s.

In the other direction, if $\overline{\theta}_1$ is a locally stable equilibrium of G_1, we chose $\overline{\theta} = \left(\overline{\theta}_1, \overline{\theta}_1(i).1_{n-1}\right)$ and we construct a stable equilibrium for *G* (as we have mentioned before, a consensus equilibrium is always locally stable [Jadbabaie et al. (2004)]. Since *G* is a.g.s., $\overline{\theta}$, and so $\overline{\theta}_1$, must be consensus equilibrium points. ♦

Theorem 5.2 has many direct consequences. We point out some of them, with a brief hint of the respective proofs.

Proposition 5.2: Consider a graph *G* with a bridge e_k between the nodes v_i and v_j and let G_1 and G_2 be the connected components of $G \setminus \{e_k\}$. Then, *G* is a.g.s. if and only if G_1 and G_2 are.

Proof: If a graph has a *bridge*, the behavior of the system depends only on the parts connected by the bridge. Indeed, the bridge together with its ends vertices form a block, which is in fact a complete graph and its vertices are cut-vertices of the graph, as is shown in

Figure 5. Since any complete graph is a.g.s., the a.g.s. character of the original graph depends on the other blocks. ♦

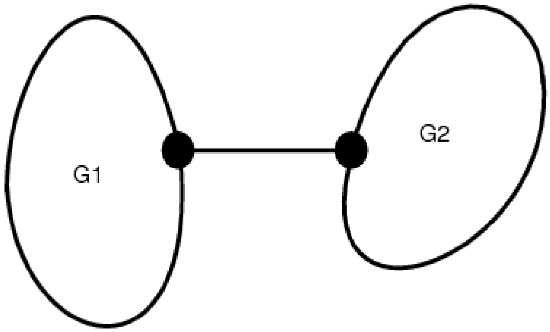


Figure 5. A graph with a bridge

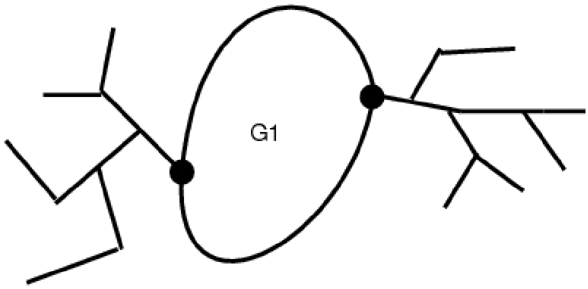


Figure 6. A graph with *arboricities*

We are now ready to present a different proof of Theorem 4.1:

Proof 2: We can iteratively apply Proposition 5.2, since in a tree, every link is a bridge.

If we have a graph with *arboricities*, like the one shown in Figure 6, we can neglect the trees in order to prove the a.g.s. property. ♦

*Corollary 5.1:*A graph with the structure shown in Figure 6 is a.g.s. if and only if the graph G_1 is.

Proof: The result is a straightforward application of Theorem 5.2. ♦

Now, we state an important result in order to classify a.g.s. graphs:

Theorem 5.3: A graph *G* is a.g.s. if and only if every block of *G* is a.g.s.

Proof: The graph *G* can be partitioned into its blocks. Then, *G* can be thought as a collection of subgraphs connected by cut-vertices. An iterative use of Theorem 5.2 leads us to the result. ♦

Theorem 5.3 reduces the characterization of the family of a.g.s. graphs to the analysis of 2-connected graphs. As an application, consider the case where we connect two a.g.s. graphs through another a.g.s. graph. In this way, we construct a new a.g.s. graph. Figures 7 and 8 illustrate the situation. Using the known fact that every complete graph is a.g.s., we derive the following result.

Theorem 5.4: If *G* is a graph such that all its blocks are complete graphs, then *G* is a.g.s.

Proof: As we have seen in Theorem 4.2, complete graphs are always a.g.s. So, the conclusion follows from Theorem 5.3. ♦

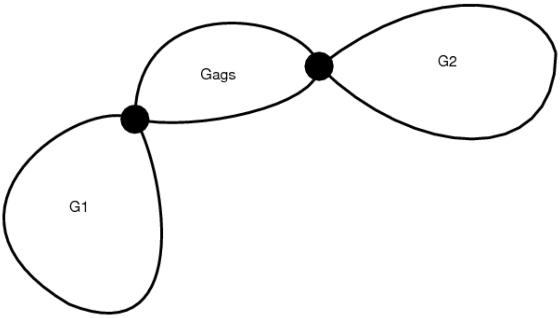


Figure 7. Two graphs connected by an a.g.s. graph

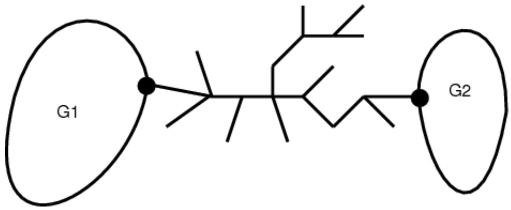


Figure 8. Two graphs connected by a tree

Finally, we present two direct consequences of Theorem 5.3. They are illustrated in Figure 9.

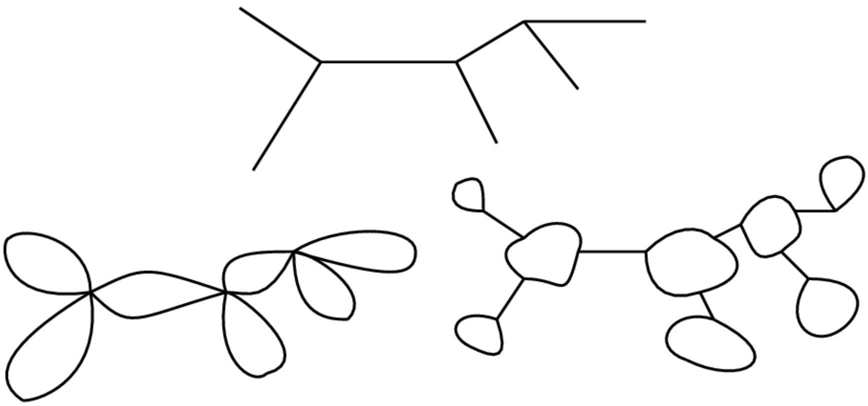


Figure 9. Situation of Propositions 5.3 and 5.4

Proposition 5.4: If G is a tree and we build a new graph K replacing some (or every) edges of G by an a.g.s. graph, then K is a.g.s. ♦

Proposition 5.5: If G is a tree and we build a new graph K replacing some (or every) nodes of G by an a.g.s. graph, then K is a.g.s. ♦

Previous results, specially Theorem 5.3, imply that in order to establish that a graph is a.g.s., we only need to deal with its blocks. So, we must focus in the general analysis of 2-connected graphs, as s*tructural pieces* of every connected graph. We know that complete graphs are a.g.s. 2-connected graphs. As long as we are able to find new a.g.s. 2-connected graphs, we are moving forward on the classification of all a.g.s. graphs.

6. Examples

In this Section we present some examples that illustrate applications of the theoretical results we have presented.

Example 6.1: Consider two Kuramoto systems with complete underlying interconnection graphs $G_1 = K_3$ and $G_1 = K_5$ (both a.g.s.). Starting from arbitrary initial conditions, each system quickly reaches a consensus state. At time T=3 seconds, we connect the two systems through a bridge between an arbitrary pair of agents. Then, the whole systems reaches a new consensus state. Observe that this convergency is slower than the previous (for the rate of local convergency, see [Jadbabaie et al. (2004)). Figure 10-left shows the results obtained from the simulation. They perfectly agree with Proposition 5.2. ♦

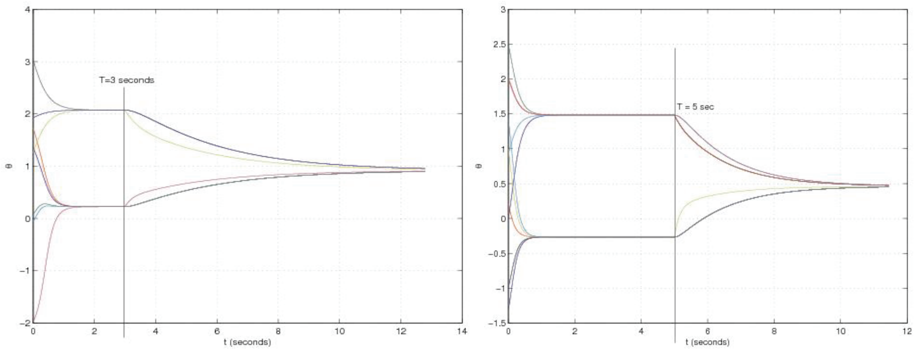


Figure 10. Left: two a.g.s. systems connected by a bridge; the connection takes place at time T=3 seconds. Right: two a.g.s. systems that become connected by a vertex; the connection takes place at time T=5 seconds

Example 6.2: Consider two a.g.s. systems, with underlying graphs $G_1 = K_5$ and $G_1 = K_7$. They run independently and at time T=5 seconds, an agent of the first system gets connected with some agents of the second one. Then, the new system has a new underlying graph G which has a vertex at this agent. Figure 11-right shows the evolution of the system. ♦

7. On the classification of A.G.S. graphs

In this Section, we introduce two operations on graphs. The first one transforms any connected graph into an a.g.s. graph. The second one destroy the a.g.s. property. Firstly, we introduce the idea of *twin vertices*.

Definition 7.1: We said that two vertices u and v are *twins* if their have the same common neighbors:

$$N_u \setminus \{v\} = N_v \setminus \{u\}$$

Previous definition does no assume that u and v are adjacent vertices. So, we will distinguish between two cases.

7.1 Adjacent twin vertices

The following Lemma generalizes previous results for complete graphs.

Lemma 7.1: Let $\overline{\theta}$ be a stable equilibrium point of (5), then any set of adjacent twin vertices should be synchronized.

Proof: Let $S = \{v_1,...,v_k\}$ a set of twin vertices with the set SN of adjacent twins and their common neighbors. Let α be the sum of all the phasors of SN . Then

$$\alpha = \sum_{j \in SN} V_j = V_i . \left(1 + \sum_{\substack{j \in SN \\ j \neq i}} \frac{V_j}{V_i} \right) = V_i .(1 + \alpha_i)$$

with α_i as in Proposition 2.1. First, notice that all the V_i should be parallel. Otherwise, let V_i and V_j be linearly independent. Since $V_i .(1+\alpha_i) = \alpha = V_j .(1+\alpha_j)$, we should have $(1+\alpha_i) = 0$, thus $\alpha_i = -1$ and, by Lemma 4.2, the equilibrium can not be stable. So, we have a group of say *a* vertices of SN in phase θ_0 and another group of *b*=*k*-*a* ones in phase $\theta_0 + \pi$. We claim that *b* should be zero. Indeed, let v_i and v_j vertices of *SN* in the first and second group respectively, then:

$$\alpha_i = (a-1) - b + \sum_{l \in SN \setminus S} \cos(\theta_0 - \theta_l)$$

and

$$\alpha_j = (b-1) - a + \sum_{l \in SN \setminus S} \cos(\theta_0 + \pi - \theta_l)$$

But, $\cos(\theta_0 + \pi - \theta_l) = -\cos(\theta_0 - \theta_l)$, thus $\alpha_i + \alpha_j = -2$ and at least one of them should be negative. This means that $\overline{\theta}$ is unstable. ♦

As a consequence of this Lemma, we have a new way to prove that any complete graph is a.g.s. since all its vertices are adjacent twins. But, as we will prove, we have even more, if the identification of adjacent twin vertices give rise a tree, then the graph is a.g.s. Since being adjacent (or itself) and twin is an equivalence relation we can make the *quotient graph* by this relation. In the quotient graph, the vertices are the classes of the equivalence and two vertices are adjacent in the quotient if the classes have adjacent vertices. We will say that a graph is a *twin cover* of its quotient graph.

Theorem 7.1: Consider a given graph *G* and its quotient graph G_Q by the adjacent-twin relation. If G_Q is a tree, *G* is a.g.s.

Proof: Let $\overline{\theta}$ be a stable equilibrium point of *G*. Then, $\sin(B^T \overline{\theta})$ is a flow on it. This flow gives rise the following flow in the quotient graph. Consider two adjacent vertices *u* and *v* in *G* which are not twins. Then, the classes $[u]$ and $[v]$ are adjacent in G_Q . Since $\overline{\theta}$ is stable, by Lemma 7.1, all the neighbors of u have the phase $\overline{\theta}_u$. In the same way, we define $\overline{\theta}_v$. Assign the number

$$\left|[u]\right|.\left|[v]\right|.\sin\left(\overline{\theta}_v - \overline{\theta}_u\right)$$

to the edge in G_Q joining the node classes $[u]$ and $[v]$ ($\left|[u]\right|$ denotes the number of elements of the class $[u]$). We affirm that this assignment is a flow in G_Q. Indeed,

$$\sum_{[v]\in N_{[u]}}\left|[u]\right|.\left|[v]\right|.\sin\left(\overline{\theta}_v - \overline{\theta}_u\right) = \left|[u]\right|.\sum_{[v]\in N_{[u]}}\left|[v]\right|.\sin\left(\overline{\theta}_v - \overline{\theta}_u\right)$$

Observe that if $v \in N_u \setminus [\mathbf{u}]$ in G, then, the term $\sin\left(\overline{\theta}_u - \overline{\theta}_v\right)$ appears $\left|[v]\right|$ times in the expression of $\dot{\theta}_u$. Then,

$$\sum_{[v]\in N_{[u]}}\left|[v]\right|.\sin\left(\overline{\theta}_v - \overline{\theta}_u\right) = \sum_{v\in N_u \setminus [\mathbf{u}]}\sin\left(\overline{\theta}_v - \overline{\theta}_u\right) = \dot{\theta}_u\Big|_{\overline{\theta}} = 0$$

So, the stable equilibrium point $\overline{\theta}$ of G induces another equilibrium point $\overline{\theta}_Q$ in G_Q. If G_Q is a tree, $\overline{\theta}_Q$ is a partial or full synchronized point. If it is a partial synchronization state, the phase value of each class in G_Q is 0 or π (taking a suitable reference) and $\overline{\theta}$ is also a partial synchronization state and so is unstable, which contradicts the hypothesis. Then, $\overline{\theta}_Q$ and $\overline{\theta}$ are consensus equilibrium and G is a.g.s. ♦

The opposite result is obviously not true. We present several corollaries that recover some known results and introduce tools for building a.g.s. graphs.

Corollary 7.1: Any complete graph is a.g.s.

Proof: Its quotient graph is the trivial one. ♦

Corollary 7.2: Any complete graph minus an edge is a.g.s.

Proof: Its quotient graph is a tree: the only one with three vertices. ♦

Corollary 7.3: Any complete graph minus any proper subset of the edges adjacent to a given vertex is a.g.s.

Proof: Its quotient graph is again the only tree with three vertices. The three groups of twins are: first the vertex that lost more edges, those who lost only one edge and those who did not lose any edge. ♦

The following Theorem shows that a connected graph G can be *enlarged*, adding twin vertices, in order to obtain a new a.g.s. graph.

Lemma 7.2: In a connected graph, no equilibrium but the synchronized one is possible with all phasors in a half of the unit circle.

Proof: Indeed, by absurd, suppose that there are unsynchronized vertices and without loss of generality that $\overline{\theta}_i \in [0,\pi]$ for all i, then $\overline{\theta}_{i_m} = \min\overline{\theta}_i < \max\overline{\theta}_i = \overline{\theta}_{i_M}$. We claim that there should exists an agent j achieving the minimum but unsynchronized with at least one of its neighbors. Indeed, it suffices to consider a walk from vertex i_m to vertex i_M and the first moment when the angle grows. Thus, for some j, for all $i \in N_j$ we have $\overline{\theta}_i - \overline{\theta}_j \geq 0$, and $\overline{\theta}_k - \overline{\theta}_j \geq 0$ for someb $k \in N_j$. But since the angles are in $[0,\pi]$, we hace such $\sin(\overline{\theta}_k - \overline{\theta}_j) > 0$. Therefore,

$$\sum_{i \in N_j} \sin(\overline{\theta}_i - \overline{\theta}_j) > 0$$

contradicting the equilibrium hypothesis. ♦

Theorem 7.2: Any connected graph G admits an a.g.s. twin cover.

Proof: Remember that by Lemma 7.1, twin vertices in a stable equilibrium should be synchronized. Thus, we can restrict our study to a set $V = \{v_1,\ldots,v_n\}$ of representants of the twins. We will identify V with the vertices of G. Furthermore, we will prove that given $\varepsilon > 0$, there is a twin cover such that for any stable equilibrium $\overline{\theta}$, the angle differences $\left|\overline{\theta}_i - \overline{\theta}_j\right|$ are less than ε for all pairs (v_i, v_j) of adjacent vertices. Thus, if the graph is connected with diameter D, the result will follow by taking $\varepsilon = \pi/D$ and applying Lemma 7.2. Notice that we can restrict our self to pairs (v_i, v_j) in a spanning tree.

Let us suppose that we have constructed the cover by splitting each vertex v_i of G in a number a_i of twins vertices. Then, the flow equation for (any twin of) vertex v_i in the new graph becomes:

$$\sum_{j \in N_i} a_j.\sin(\overline{\theta}_j - \overline{\theta}_i) = 0$$

Then, for any $k \in N_i$

$$a_k.\sin(\overline{\theta}_k - \overline{\theta}_i) = -\sum_{\substack{j \in N_i \\ j \neq k}} a_j.\sin(\overline{\theta}_j - \overline{\theta}_i)$$

and

$$\left|\sin(\overline{\theta}_k - \overline{\theta}_i)\right| \leq \frac{\sum_{j \in N_i, j \neq k} a_j}{a_k}$$

So, in order to find an upper bound for the difference $\left|\overline{\theta}_k - \overline{\theta}_i\right|$ it is enough to find an upper bound for the last term together with a lower one for $\cos(\overline{\theta}_k - \overline{\theta}_i)$. Now, we will construct the spanning tree *T*. Let S_i be the vertices at distance *i* from vertex v_1 (i.e. the sphere in the graph of center v_1 and ratio *i*). Then, sort each set S_i with an order $<_i$. We consider the following lexicographical order: given two vertices $v \in S_i$ and $w \in S_j$, we say that $v < w$ if $i < j$ or if $i = j$ and $v <_i w$. The order defined in this way is total, so we can relabel the vertices following this order, having $v_1 < v_2 < \ldots < v_n$. Next, set $a_i = (\Delta/\varepsilon)^{n-i}$ (rounded up) where Δ is the maximum degree of a vertex in *G*. Then *T* will be the spanning subgraph of *G* that joins vertices v_i and v_j if $a_i = \max_{l \in N_j}\{a_l\}$. We claim that *T* is a tree. Indeed, it is acyclic, because for each *i*>1, any vertex in S_i is adjacent to exactly one vertex in S_{i-1}. Besides any vertex reaches vertex v_1, thus *T* is connected as well.

Let us now find an upper bound for the sine of the difference between adjacent vertices of *T*. Let v_i and v_k be adjacent vertices of *T* with $i > k$. Then

$$\left|\sin\left(\overline{\theta}_k - \overline{\theta}_i\right)\right| \le \frac{\sum_{j \in N_i, j \ne k} a_j}{a_k} \le \frac{(\Delta - 1).\left[(\Delta/\varepsilon)^{n-k-1} + 1\right]}{(\Delta/\varepsilon)^{n-k}} < \varepsilon$$

for any $\varepsilon < \Delta$. On the other hand, since the equilibrium is stable we have that

$$a_i - 1 + \sum_{j \in N_i} a_j.\cos\left(\overline{\theta}_j - \overline{\theta}_i\right) \ge 0$$

Thus, by the same argument

$$\cos\left(\overline{\theta}_i - \overline{\theta}_k\right) \ge -\frac{a_i + \sum_{j \in N_i, j \ne k} a_j.\cos\left(\overline{\theta}_j - \overline{\theta}_i\right)}{a_k} > -\varepsilon.\left(1 + \Delta^{-1}\right)$$

Thus, choosing ε small enough we will have that the angles differ in less than any prescribed ε'. ♦

We can prove a *dual* version of this theorem which says that if we add an enough amount of vertices to an edge which is not a bridge we will obtain a non a.g.s. graph.

Theorem 7.3 Let *e* be an edge of a graph *G*. Then, if *e* is not a bridge, there is an integer n_0 such that the graph obtained from *G* by making $n > n_0$ subdivisions of *e* is not a.g.s.

Proof: The idea is the following. Consider the cycle C_n, with $n \ge 6$. As was mentioned in Example 4.1 and Lemma 4.3, C_n is not a.g.s. because $\overline{\theta}_i = \frac{2\pi}{n}$ is an *equally distributed* stable equilibrium point. Consider also the graph $G \setminus \{e\}$, obtained from *G* by removing the edge *e*. Take a edge of C_n, say *uv* and replace it by $G \setminus \{e\}$, joining the vertices of *e* with *u* and *v*.

The new graph we have obtained is the original *G* with the edge *e* split into several edges (see the sketch of Figure 11).

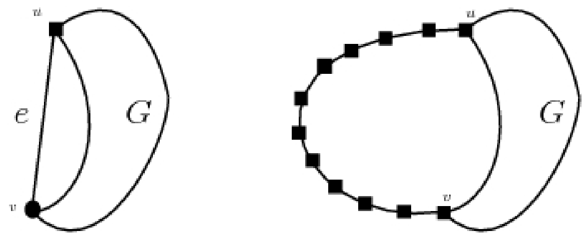


Figure 11. Situation of Theorem 7.3

The idea is the following: if *n* is large enough, the *force* induced by C_n will be weak enough to change the trivial equilibrium point of G to another still stable one.

Let $v_1,\ldots,v_m$ be the vertices of G and let $e = v_1 v_2$. Since *e* is not a bridge, $G' = G\setminus\{e\}$ is connected and 0_m is an stable equilibrium point of G'. Now, connect the vertices v_1 and v_2 of G' through a path $P_n : v_1 = w_1,\ldots,w_n = v_2$ to obtain a graph $\widetilde{G}$ with vertices $\widetilde{V} = \{w_1,\ldots,w_n,v_3,\ldots,v_m\}$. We want to prove that for *n* large enough, there exist an $\varepsilon > 0$ and angles θ_i^ε, $1 \le i \le m$, such that the point $\theta_i^\varepsilon : \widetilde{V} \to R$ defined by:

$$\theta^\varepsilon \,/\, \theta_x^\varepsilon = \begin{cases} i\varepsilon, \textbf{ if } x = w_i \\ \theta_i^\varepsilon, \textbf{ if } x = v_i \end{cases}$$

is a stable equilibrium point of $\widetilde{G}$. In order for θ^ε to be an equilibrium it must satisfies:

$$\sum_{y\in N_x} \sin\left(\theta_y^\varepsilon - \theta_x^\varepsilon\right) = 0 \qquad , \qquad x \in \widetilde{V}$$

where N_x is the set of neighbors of vertex *x* in graph $\widetilde{G}$. These equations are trivially fulfilled for $x = w_2,\ldots,w_{n-1}$. Thus, it remains the following set of equations:

$$\begin{cases} \sum\limits_{y\in N'_{v_1}} \sin\left(\theta_y^\varepsilon - \theta_{v_1}^\varepsilon\right) + \sin(\varepsilon) = 0 \\ \sum\limits_{y\in N'_{v_2}} \sin\left(\theta_y^\varepsilon - \theta_{v_2}^\varepsilon\right) - \sin(\varepsilon) = 0 \\ \sum\limits_{y\in N_x} \sin\left(\theta_y^\varepsilon - \theta_x^\varepsilon\right) = 0 \qquad x \in \widetilde{V}\setminus\{v_1, v_2\} \end{cases}$$

where *N* and *N′* denote neighbors in *G* and *G′* respectively. This system can be thought as an ε--perturbation of the system that defines the equilibrium of *G′*. Moreover, if we add an adequate equation, e.g. $\theta_{v_1} = 0$, the system verifies the hypothesis of the implicit function theorem for $\theta = 0_m$ and $\varepsilon = 0$. Thus, it implicitly defines the angles θ_x^ε as a function of ε, for each $x \in V_G$, in a neighborhood $(-\varepsilon_0, \varepsilon_0)$ of 0. Moreover, we will have that θ^ε is a C^∞ curve in R^n passing through 0_m for $\varepsilon = 0$.

Finally, in order to prove stability, we notice that when $\varepsilon = 0$, all the cosines $\cos(\theta_i^\varepsilon - \theta_j^\varepsilon)$ are positive, thus, the eigenvalues of the Jacobian linearization are negative, by Lemma 4.3. Thus, by the continuous dependence of the eigenvalues, ε_0 could be taken in such a way to assure the stability of equilibrium points θ^ε for each $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. Therefore it suffices to take $n_0 > 2\pi/\varepsilon_0$, and for each $n > n_0$, to set $\varepsilon = 2\pi/n$. ♦

7.2 Non adjacent twins

When the vertices are twins but not adjacent, previous arguments does not work, but something interesting can however be said. Indeed, let $S = \{v_1, ..., v_t\}$ a set of non adjacent twin vertices with the set SN of common neighbors. As in Proposition 2.1, let α be the sum of the phasors of *SN*. Then

$$\alpha = \sum_{j \in SN} V_j = V_i \cdot \sum_{j \in SN} \frac{V_j}{V_i} = \alpha_i . V_i \qquad , \quad i = 1, ..., t$$

So if two of them, say V_i and V_j are linearly independent, then, one of them is linearly independent to any of the others. So, α_k should be zero for any $k = 1, ..., t$.

Otherwise, if all of them are parallel, but non synchronized, we have a group of say *a* vertices of *SN* in a phase θ_0 and another group of $b = t - a$ ones in phase $\theta_0 + \pi$. Let v_i and v_j be in each group. Then:

$$\alpha_i = \sum_{l \in SN} \cos(\theta - \theta_l) \quad \text{and} \quad \alpha_j = \sum_{l \in SN} \cos(\theta + \pi - \theta_l)$$

But, $\cos(\theta + \pi - \theta_l) = -\cos(\theta - \theta_l)$, thus $\alpha_i + \alpha_j = 0$. As this argument could be repeated for any of the others pair of not synchronized vertices, if $a, b > 1$, we have a consistent homogeneous system of equations which has the null solution as the only one. Then, each α_i should be zero. If *a* or *b* is 1, either both α_i and α_j are null or some of them is negative. Summing all this up we have the following result.

Lemma 7.3: Let $\overline{\theta}$ be an equilibrium point of (3), then any set of *t* twin vertices should have their α_i equal to 0 if the equilibrium is stable or the synchronized twins are more than one. ♦

In that case, the matriz *A* of (5) will have a block of zeros (the one corresponding to the set of twins), thus either *A* has a kernel of dimension greater than one or it has positive eigenvalues and so is unstable. Thus we have the following result.

Proposition 7.1: Let $\overline{\theta}$ be a non degenerated stable equilibrium point of (3), then any set of non adjacent twin vertices should be synchronized. ♦

8. Conclusions

In this work we have introduced the idea of almost global synchronization (a.g.s.) of Kuramoto coupled oscillators. Local stability properties of the synchronization were recently stated and the are independent of the underlying interconnection graph. We have shown that the algebraic properties of this graph play a fundamental role when we look for global properties. Algebraic and dynamical properties are extremely related for these kind of systems. So, we presented the idea of a.g.s. graphs and started a characterization of this family of graphs. We have shown that the trees, the simplest graphs, are a.g.s. We have proved that complete graphs, the most complex, are also a.g.s. Several counterexamples illustrates that there are non a.g.s. graphs. We have proved that the characterization of a.g.s graphs can be reduced to the analysis of 2-connected graphs, since a graph is a.g.s. if and only if its block are. Typical techniques for graphs classification, like the use of homeomorphisms, can not be applied here, since we have shown that the a.g.s. property is not preserved by this way. Then, a different approach must be considered to go on with the classification.

9. References

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