UNIVERSIDAD DE LA REPÚBLICA URUGUAY

PEDECIBA

Tesis para el título de

Doctor en Matemática

Contributions to partially hyperbolic systems: coherence, transitivity and ergodicity.

Autor:
Luis Pedro PIÑEYRÚA RAMOS

Directores:
Dr. Rafael POTRIE
Dr. Martín SAMBARINO

Defendida el día 26 de agosto de 2022, delante del jurado compuesto por:

Sr. Sébastien ÁLVAREZ :	Universidad de la República	- Examinador
Sr. Lorenzo DÍAZ :	PUC - Rio de Janeiro	- Árbitro
Sr. Rafael POTRIE :	Universidad de la República	- Director
Sr. Enrique PUJALS :	CUNY - Graduate center	- Examinador
Sr. Martín REIRIS :	Universidad de la República	- Examinador

Acknowledgements

This is one of the hardest parts to write of the thesis, since it is very difficult to be fair with so many people who have been there one way or another.

First of all I want to thank my advisors Rafael Potrie and Martín Sambarino, they have been extremely generous with me during the whole project. I also want to express my admiration to both of them, it was a privilege and honor to have them as advisors. I want to thank Martín for his infinite patience with me, and for pushing me and encouraging me in every moment when I was down or lost. He has been my advisor since my undergraduate studies but he has also been my friend, and I am very grateful for that. Rafael has also been part of my mathematical education since the beginning of my studies. It is very hard to explain everything he has helped me all these years, and how generous he has been with his time. I can't think in a better mathematical role model to follow. Thank you for that.

I would like to thank the referees Álvaro Rovela and Lorenzo Díaz for the careful reading of this thesis and every valuable comment they gave me these past weeks. I also want to thank Sébastien Álvarez, Enrique Pujals and Martín Reiris for accepting being part of the jury and participating in a very important moment for me.

I want to deeply thank to Martin Leguil, for his friendship and commitment with the project we started a few years ago. Since we first met, he has been a good friend and he also has taught me beautiful mathematics.

During my stay in New York, I had the opportunity to work with Enrique Pujals, and it is difficult to explain how inspiring and stimulating it is to work with him. He is an amazing mathematician, but he is also a very humble and generous person, and I consider myself privileged to have shared some time with him. I also want to thank the hospitality of the Graduate Center of the City University of New York. Those three months there were very fruitful for me, personally and mathematically.

I would like to thank the entire uruguayan mathematical community. It is a pleasure for me to work in such a passionate community, and to share with others the love for mathematics. Special thanks to Marcos Barrios, León Carvajales, Verónica De Martino, Alejo García, Ernesto García, Victoria García, Gabriel Illanes, Joaquin Lema, Santiago Martinchich, Carolina Puppo, Emiliano Sequeira, Mario Shannon and Bruno Yemini. In addition to these people, I want to thank Juan Alonso, Diego Armentano, Joaquin Brum, Matías Carrasco, Nancy Guelman, Mariana Haim, Matilde Martínez, Pablo Lessa, Andrés Sambarino and Juliana Xavier. A special thanks to Alejandro Passeggi who showed me when I was a teenager, how beautiful mathematics can be. I wouldn't be here if it weren't for him. I also want to thank to Claudia Alfonso and Lydia Tappa for helping me with everything.

This thesis was possible thanks to the financial support of many institutions: Comisión Académica de Posgrado (CAP), Agencia Nacional de Investigación e Innovación (ANII), Programa de Desarrollo de las Ciencias Básicas (PEDECIBA) and Comisión Sectorial de Investigación Científica (CSIC Grupo Sistemas dinámicos 618). I want to thank also to the CMAT and IMERL for give me a good environment to develop this project.

I am deeply grateful to all my friends for every moment we shared, and for make my life happier, and also for making me a better person (and even a better mathematician).

I want to thank to my whole family for their love and support, and for always believing in me. I want to especially thank my two sisters, clari and cuca, for make me so happy, and for being with me at every step of life.

My father must be the most passionate person I know. It is difficult to express how much his passion for life has shaped me, thank you so much for your guidance and inspiration. I also want to thank my mother, wherever she is, I can still feel her love all around.

Finally I want to thank Florencia for everything. Her friendship, her love, her companion all these years, and for supporting me in my worst moments. This would not have been possible without you.

Abstract

This thesis is framed in the study of partially hyperbolic systems (PH). Within these systems, we cover three different topics: dynamical coherence (integrability of the center-stable and center-unstable bundles), robust transitivity and accessibility.

Concerning dynamical coherence, we prove that in certain isotopy classes, the existence of a PH diffeomorphism dynamically coherent implies that every diffeomorphism inside this isotopy class is dynamically coherent as well.

With respect to robust transitivity we present a new definition of SH (some hyperbolicity) property, which is an extension of the one introduced by Pujals and Sambarino. We prove that this new definition is C^1 open and then we give a condition that guarantees that certain PH diffeomorphisms with SH property are C^1 robustly transitive (we present a similar result in the flow case). We then build new examples of C^1 robustly transitive derived from Anosov diffeomorphisms.

Finally regarding the accessibility property, we worked on the Pugh-Shub accessibility conjecture, which says that the set of PH diffeomorphisms which are stably accessible is C^r open and dense among PH diffeomorphisms. In a joint work with M. Leguil, we prove that the conjecture is true, for the case of PH diffeomorphisms which are stably dynamically coherent, with two dimensional center bundle and a strong bunching condition.

Resumen

Esta tesis se enmarca dentro del estudio de los sistemas parcialmente hiperbólicos (PH). Dentro de estos sistemas, nos enfocamos en tres aspectos: la coherencia dinámica (integrabilidad de los fibrados centro-estable y centro-inestable), la transitividad robusta y la accesibilidad.

Respecto a la coherencia dinámica, se prueba que en ciertas clases de isotopía, la existencia de un difeomorfismo PH dinámicamente coherente implica que todo difeomorfismo dentro de esta misma clase de isotopía, también es dinámicamente coherente.

Sobre la transitividad robusta se presenta una nueva definición de SH (some hyperbolicity) que generaliza a la introducida por Pujals y Sambarino. Probamos que esta nueva SH es una propiedad C^1 abierta y luego se dan condiciones que garantizan que un difeomorfismo PH con propiedad SH sea C^1 robustamente transitivo (se presenta un resultado similar para el caso de flujos). Luego se construyen ejemplos nuevos de difeomorfismos derivados de Anosov C^1 robustamente transitivos.

Finalmente respecto a la accesibilidad, trabajamos en la conjetura de Pugh-Shub. Esta conjetura dice que el conjunto de los PH establemente accesibles es C^r abierto y denso dentro de los sistemas PH. En un trabajo en conjunto con Martín Leguil, probamos que la conjetura es cierta para el caso de PH establemente dinámicamente coherente, con fibrado central de dimensión 2 y una condición de center bunching fuerte.

Contents

Chapter 0

Introduction and presentation of the results

0.1 Introduction (English)

0.1.1 A brief introduction

This thesis is framed in the theory of dynamical systems, and more precisely in the study of partially hyperbolic systems. The main purpose of dynamical systems is to understand the asymptotic behaviour of orbits of a given law.

One may say that it all started with the law of universal gravitation, where Newton gave the differential equations that govern the motion of planets. The fundamental question was (and still is!) to determine if the solar system is stable in the long run. Now, one thing is to know the laws that govern the motion, and another completely different is to know the solutions or trajectories of this system. The problem is that, the vast majority of differential equations are not easily solvable, even when they do have solutions.

It was H. Poincaré when working on the three body problem, who realized that even the most simple equations lead to chaotic or unpredictable behaviour. He then proposed the qualitative study instead of the quanitative one, i.e. the study of the geometry or topology of the solutions, instead of the numerical or analytic approach, which were the usual methods by that time.

This unpredictability discovered by H. Poincaré in the three body problem was the cornerstone of dynamical system theory but it was not until the 60's with the appearance of the hyperbolic theory that it took form as a real subject. It's importance lies in the fact that uniform hyperbolicity turned out to be synonymous of chaos. This theory of hyperbolic dynamics was initiated with the works of Anosov, Sinai and Smale, and continued by Bowen, Franks, Manning, Mañé, Newhouse and Palis, just to name a few.

Since then, hyperbolic dynamics has been widely studied, and despite some important questions that remain open, the theory is practically closed. In part thanks to this success, dynamicists tried to push ideas form this theory to a more general setting, and partially hyperbolic systems arise as one natural generalization of uniform hyperbolicity (although there are other extensions like non-uniform hyperbolicity).

As the title says, the purpose of this thesis is to contribute to the study of partially hyperbolic systems and in particular we will cover three different topics which are at the core of the theory. The first one is the integrability of the center distribution, known as dynamical coherence, the second is robust transitivity, and the third is stable accessibility, and therefore stable ergodicity.

In what follows we are going to present these contributions.

0.1.2 Dynamically coherence

As we mentioned above, the theory of hyperbolic dynamical systems has been very fruitful since its appearance. Notably the results of D. A. Anosov [Ano67] about structural stability and stable ergodicity of globally hyperbolic diffeomorphisms, the works of J. Franks [Fra70], [Fra69] and A. Manning [Man74] about the classification of the (today called) Anosov diffeomorphisms on nilmanifolds, the codimension one case obtained by S. Newhouse [New70] and later the proof of the C^1 stability conjecture by R. Mañé [Mañ87a] are perhaps the most paradigmatic or illustrative results of the theory.

A fundamental tool in the proofs of these results is the stable manifold theorem, i.e. the integrability of the stable E^s and unstable E^u bundles of a uniformly hyperbolic diffeomorphism. Since these bundles are transversal, their corresponding integrated foliations fill the space at least locally.

For the partially hyperbolic case, given a diffeomorphism $f : M \to M$ with a splitting of the form $TM = E^s_f \oplus E^c_f \oplus E^u_f$, it is known that the strong bundles E^s_f, E^u_f integrate into unique f-invariant foliations $\mathcal{W}^s_f$ and $\mathcal{W}^u_f$ (see [HPS77]) and the same result holds for flows. However, the center bundle E^c_f can have many different behaviours and one hopes to be able to integrate the center bundle too, although this is not always the case. This represents the first important difference between global and partial hyperbolicity.

We say that a partially hyperbolic diffeomorphism f is *dynamically coherent* if the bundles $E^s_f \oplus E^c_f$ and $E^c_f \oplus E^u_f$ are integrable (and hence, the center bundle E^c_f is integrable too). Otherwise we say that f is *dynamically incoherent*. The first example of dynamically incoherent partially hyperbolic diffeomorphism was built in [Wil98] (see also [BW10]) on a six dimensional nilmanifold with 4-dim center bundle. Later in [HHU16], the authors built an example on the torus $\mathbb{T}^3$ (with 1-dim center bundle). In the later example on the 3-torus, the lack of differentiability on the bundles breaks the integrability of the center bundle, although there are curves tangent to E^c by Peano's theorem. In the 6-dimensional manifold example, despite having C^1 bundles, the Frobenius condition fails and thus no integrability is possible on the center bundle (we will see this example in detail in Subsection 1.3.2).

It still unknown weather dynamical coherence is a C^1 open condition (a related property is *plaque expansivity* and we will mention this on Section 1.4). For large classes of maps, in [FPS14] the authors obtained dynamical coherence for entire isotopy classes of linear Anosov diffeomorphisms on $\mathbb{T}^N$. This is the first result where the integrability of the center-stable and center-unstable bundles is obtained for a whole isotopy class of maps (the nilmanifold case of this result is proven in [Piñ]).

Recently in [Bar+] it is proven that in certain Seifert 3-manifolds, every partially hyperbolic diffeomorphism isotopic to the identity is dynamically coherent. On the other hand, in [Bon+20] the authors constructed new examples of partially hyperbolic diffeomorphisms which are robustly dynamically incoherent, and more recently in [Bar+21] the authors obtained entire isotopy classes of dynamically incoherent partially hyperbolic diffeomorphisms. All these results are somehow surprising, since on the one hand integrability is hard and quite technical to get, and on the other hand there is a lot of freedom to move inside isotopy classes (and there is no assumption on the behaviour on center bundles despite domination).

By the previous evidence, it seems that integrability (or not) of the center-stable and center-unstable bundles is a phenomenon that depends directly on the isotopy class of the diffeomorphism.

Our first contribution in this thesis is to go towards this ideas by generalizing the mechanisms obtained by T. Fisher, R. Potrie and M. Sambarino in [FPS14] to a more general setting. In that paper the authors proved that given a linear Anosov $A : \mathbb{T}^d \to \mathbb{T}^d$, then every partially hyperbolic diffeomorphism f isotopic to A (such that the isotopy is inside the set of partially hyperbolic diffeomorphisms) is dynamically coherent.

The first thing we do in this chapter is to capture the main ideas of that result and we provide a different approach to the problem in which similar techniques can be applied. In particular we are able to obtain dynamically coherence in entire isotopy classes for new kind of diffeomorphisms and manifolds (we will see these new examples in Subsection 2.1.2). We want to mention that there is an intrinsic technical difficulty in this passage from one case to another and moreover, there is a wrong proof in that paper, that we manage to solve it in our general context.

Let us give the following definition in order to state precisely the main theorems of this chapter. Given a dynamically coherent partially hyperbolic diffeomorphism $f : M \to M$ we will say that f is *fibered*, if it verifies the following two conditions:

- the foliations $\widetilde{\mathcal{W}}^{cs}_f$ and $\widetilde{\mathcal{W}}^{uu}_f$ have global product structure in the universal cover $\widetilde{M}$, and the same happens with $\widetilde{\mathcal{W}}^{cu}_f$ and $\widetilde{\mathcal{W}}^{ss}_f$.
- the induced map in the quotient by center leaves $\widetilde{f}_c : \widetilde{M}/\widetilde{\mathcal{W}^c_f} \to \widetilde{M}/\widetilde{\mathcal{W}^c_f}$ is a hyperbolic homeomorphism.

In fact we are going to call f fibered if it checks these two conditions, and in addition, it checks another two technical properties. We are going to give these technical conditions at the beginning of Chapter 2, but basically the above two properties capture the essence of what we mean with fibered partial hyperbolicity.

Notice that every linear Anosov diffeomorphism trivially checks the above definition (the center bundle can be any regrouping of intermediate bundles) and therefore linear Anosov are fibered partially hyperbolic diffeomorphisms (hence we obtain the result of [FPS14] as a particular case). We will see this, and different examples of fibered partially hyperbolic diffeomorphisms in Subsection 2.1.2.

Our main theorem here is the following:

Theorem A. *Let $f : M \to M$ be a fibered partially hyperbolic diffeomorphism. Let g be a partially hyperbolic diffeomorphism isotopic to f such that the isotopy is inside the set of partially hyperbolic diffeomorphisms (preserving the dimension of the bundles). Then g is dynamically coherent.*

From the proof of this theorem, we will be able to prove a classification result. Let us first say that two dynamically coherent partially hyperbolic diffeomorphisms f and g are *leaf conjugate* if there exists a homeomorphism $h\colon M \to M$, called a leaf conjugacy, such that h maps a f-center leaf to a g-center leaf, and $h \circ f(\mathcal{W}^c_f(\cdot)) = g \circ h(\mathcal{W}^c_f(\cdot))$. We then prove the following.

Theorem B. *Let $f : M \to M$ be a fibered partially hyperbolic diffeomorphism. Then, every partially hyperbolic g which is isotopic to f such that the isotopy is inside the set of partially hyperbolic diffeomorphism is leaf conjugate to f.*

0.1.3 Robust transitivity

In short, dynamical system theory is the study of motion and we want to understand the behaviour of most orbits. Tipically the structure of the orbits is very complicated, for example in some cases there are orbits that almost fill the whole space, making it indecomposable from the dynamical point of view. That is what is called *transitivity*: a dynamical system is said to be *transitive* if it has a dense forward orbit.

Even more interesting are the systems that present a dynamical feauture that is stable or robust (meaning that it persists under perturbations). We say that a dynamical system is *robustly transitive*, if there is a neighbourhood of the system (in some particular topology) such that every system in this neighbourhood is transitive.

The first example of a robustly transitive map was given by D. A. Anosov in [Ano67], where he proved that diffeomorphisms which posses a global hyperbolic structure are stable under C^1 perturbations. As a corollary, every transitive diffeomorphism of this type, like a hyperbolic matrix in the torus, is in fact C^1 robustly transitive.

Thanks to this result, globally hyperbolic diffeomorphisms are now called Anosov diffeomorphisms. In particular we call linear Anosov diffeomorphisms to the examples given by hyperbolic matrices on the torus (see Example 1.3.1) or hyperbolic automorphisms on nilmanifolds (Example 1.3.2).

Let us mention that every Anosov diffeomorphism on a nilmanifold is conjugated to a linear Anosov, and in consequence it must be transitive ([Fra70],[Man74]). However is still an open problem to determine which manifolds support Anosov diffeomorphisms, and if every Anosov diffeomorphism must be (robustly) transitive.

Years later M. Shub [Shu71] constructed the first non-Anosov C^1 robustly transitive diffeomorphism on the torus $\mathbb{T}^4$. This example is a skew product (see Example 1.3.4) of a torus $\mathbb{T}^2$ over an Anosov on $\mathbb{T}^2$ with two fixed points. Shub changed carefully the index of one fixed point in order to break the uniform hyperbolicity. A few years later R. Mañé improved this result and introduced an example on $\mathbb{T}^3$ [Mañ78]. Mañé's idea was to bifurcate the fixed point of a linear Anosov into three fixed points with different indexes and keeping the center manifold robustly dense. Both Shub's and Mañé's examples are isotopic to linear Anosov diffeomorphisms and by that reason they're called *derived from Anosov* examples (from now on DA maps, see Example 1.3.6).

Another way to construct robustly transitive diffeomorphisms was introduced by C. Bonatti and L. Díaz in [BD96]. Their technique is based on the existence of some particular hyperbolic subsets called *blenders*. With this technique, the authors were able to build examples C^1-close to time-t maps of Anosov flows (hence, isotopic to the identity) as well as examples C^1-close to the product of Anosov times the identity (therefore, with trivial action on the center).

All these non-hyperbolic examples share the feauture of being partially hyperbolic and this is not a coincidence. In [Mañ82] R. Mañé proved that every C^1 robustly transitive diffeomorphism on a surface must be conjugated to a linear Anosov (and therefore the manifold must be the torus $\mathbb{T}^2$ by [Fra70]). In the three-dimensional case, L. Díaz, E. Pujals and R. Ures [DPU99] proved that C^1 robust transitivity implies partial hyperbolicity (here the definition of partially hyperbolic is a little more general). Finally C. Bonatti, L. Díaz and E. Pujals generalized this result to higher dimensions by proving that C^1 robust transitivity imply dominated splitting [BDP03]. We want to remark that in [BV00] C. Bonatti and M. Viana built a C^1 robustly transitive diffeomorphism in the torus $\mathbb{T}^4$ that is not partially hyperbolic (although it necessarily has dominated splitting). In short, a dynamical assumption like robust transitivity implies strong geometric consequences.

A closely related property with transitivity is the minimality of the strong stable/unstable foliations. We say that a foliation $\mathcal{F}$ in a manifold M is *minimal* if every leaf is dense, i.e. $\overline{\mathcal{F}(x)} = M$ for every $x \in M$. It's easy to see that if the strong stable (or unstable) foliation of a partially hyperbolic diffeomorphism is minimal, then the diffeomorphism is transitive. Therefore, robust minimality of strong stable/unstable foliations implies robust transitivity.

In [PS06] the authors gave conditions to guarantee the robustness of these foliations in a C^1 neighbourhood. They were looking for a (robust) property that in addition to transitivity will imply robust transitivity. In that spirit they introduced what is called the *SH* (*Some hyperbolicity*) property. With this approach, the authors proved that SH property in addition to minimality of the strong unstable foliation implies C^1 robust minimality of the strong unstable foliation (and therefore C^1 robust transitivity). With this technique, they re obtained the examples of Shub's and Mañé's. We want to mention here, that one disadvantage of this approach is that it can not be applied to symplectic diffeomorphisms.

Recently in [HUY22], the authors gave a different condition to guarantee the C^1 robust minimality of the strong stable foliation for derived from Anosov diffeomorphisms in the three torus (and in consequence the C^1 robust transitivity). Moreover, with this approach they built an example with both stable and unstable foliations C^1 robustly minimal (the existence of such an example was unknown).

Despite these remarkable results, robustly transitive diffeomorphisms are not yet very well understood. In particular, in all examples mentioned above, the dominated splitting they have (which they must have according to [BDP03]), it is coherent with the dominated splitting of its Anosov part, i.e. the splitting has the same indexes as the linear Anosov. Recently R. Potrie in [Pot12] (page 152) constructed a robustly transitive example on $\mathbb{T}^3$ with dominated splitting, but in this case the example's dominated splitting is not coherent with its Anosov part[1].

Our contribution in this part of the thesis is the introduction of a more general concept of SH property, that we call SH-Saddle property. This new definition is a natural generalization of the previous SH definition and as a consequence, it can be applied to a larger number of cases. In particular, it has the advantage of being applicable in the symplectic context (something that the previous definition couldn't).

We want to mention that recently P. Carrasco and D. Obata showed in [CO21] that the example introduced in [BC14] is C^1 robustly transitive. This example although it is a skew product on $\mathbb{T}^4$, it has the particularity of having mixing behaviour on the center (which is two-dimensional) and thus makes it a new example. The authors mention in the paper that this example can't have the SH property (the original version). However, it follows directly from the proofs of their article, that the example has the SH-Saddle property. In consequence, it may be the case that every robustly transitive partially hyperbolic diffeomorphism has SH-Saddle property of some index.

Back to our constributions, by applying this new approach we give a sufficient condition for a derived from Anosov diffeomorphism to be C^1 robustly transitive. In fact we are able to produce new examples of C^1 robustly transitive diffeomorphisms. In particular, we can build examples for any dimension with as many different behaviours on center leaves as desire and moreover, the center bundle will not have a splitting into two subbundles. In consequence, the dominated splitting of this map is not going to be coherent with the hyperbolic splitting of the original linear Anosov.

Theorem C. *Let $A \in SL(d,\mathbb{Z})$ be a hyperbolic matrix such that it has a partially hyperbolic splitting of the form $\mathbb{R}^d = E^{ss} \oplus E^c \oplus E^{uu}$ and let $k = dimE^c$. Then there is $f : \mathbb{T}^d \to \mathbb{T}^d$ a C^1 robustly transitive partially hyperbolic diffeomorphism isotopic to A with $k+1$ fixed points: $p_0, p_1, \dots, p_k$ such that: $index(p_j) = j + dimE^{ss}$ for every $j = 0, \dots, k$.*

Moreover, the center bundle E^c_f does not admit a dominated splitting. In particular, the splitting of f is not coherent with the hyperbolic splitting of A.

[1]once again, the definition of partial hyperbolicity here is more general

We then move to the flow case, and translate the definition of SH-Saddle property from diffeomorphisms to flows. As we mention above, one advantage of our new definition of SH property is that it can be applied to the symplectic context. In particular we will be interested in geodesic flows, which always preserves a symplectic form.

There are many similarities between flows and diffeomorphisms concerning robust transitivity. For example, D. A. Anosov also showed in [Ano67] that, like in the diffeomorphism case, every hyperbolic flow is C^1 stable under perturbations. In consequence, hyperbolic flows which are transitive, are in fact C^1 robustly transitive. By this reason globally hyperbolic flows are now called Anosov flows.

The main example of an Anosov flow is the geodesic flow of a manifold of negative curvature (see Example 1.3.3). Moreover, for surfaces of -1 curvature the geodesic flow is in fact transitive, and since Anosov flows are C^1 stable, this example is C^1 robustly transitive. However, not every Anosov flow is transitive. Examples of non transitive Anosov flows were given by J. Franks and R. Williams in [FW80].

Another way to construct examples is given by suspensions (see Example 1.3.5). Since the suspension of a C^1 robustly transitive diffeomorphism gives a C^1 robustly transitive flow, we can construct many examples by taking the suspension of every robustly transitive diffeomorphism mentioned above.

Concerning classification results, the picture is a little different. When $\dim(M) = 3$, C. I. Doering proved that every C^1 robustly transitive vector field must be a transitive Anosov flow [Doe87]. This is not true in higher dimensions, for example the suspension of Mañé's derived from Anosov diffeomorphisms gives a non Anosov C^1 robustly transitive flow. Notice that the suspension of the Bonatti-Viana example gives a C^1 robustly transitive flow which has no dominated splitting at all. However C. Bonatti, N. Gourmelon and T. Vivier [BGV06], [Viv06] proved that the linear Poincaré flow of a C^1 robustly transitive flow must admit a dominated splitting. Once again, robust transitivity implies strong consequences.

In [Rug97] R. O. Ruggiero proved that if the geodesic flow $\varphi_t : T^1M \to T^1M$ of a compact, n-dimensional manifold without conjugate points is expansive, then it is topologically transitive. Our next theorem says that if in addition the flow is partially hyperbolic and with the SH-Saddle property, then it is robustly transitive. This theorem is motivated by the article of F. Carneiro and E. Pujals [CP14], where the authors built the first example of a transitive partially hyperbolic flow that is not an Anosov flow. This example verifies the SH-Saddle property although is not so clear that it is expansive and has no conjugate points.

Theorem D. *Let g_0 be a C^∞ Riemannian metric on a compact differentiable manifold M with no conjugate points and let $\varphi_t : T^1M \to T^1M$ be its geodesic flow. Suppose that φ_t is expansive with stable sets W^s and unstable sets W^u. Suppose that in addition φ_t is partially hyperbolic with a splitting $T(T^1M) = E^{ss} \oplus E^c \oplus \langle X \rangle \oplus E^{uu}$, and it has the SH-Saddle property of index (d_1, d_2) where $d_1 = dimW^s - dimE^{ss}$ and $d_2 = dimW^u - dimE^{uu}$. Then φ_t is C^1 robustly transitive.*

0.1.4 Accessibility and ergodicity

In 1871 L. Boltzmann stated his ergodic hypothesis when he was studying the motion of gases and thermodynamics. He wanted a property that could let him "characterize the probability of a state by the average time in which the system *is* in this state". Since then, ergodicity has played a key role in dynamical systems, physics and probability. Recall that a dynamical system $f \colon M \to M$ preserving a finite measure m is *ergodic* if every f-invariant set has zero or total measure.

After Birkhoff's ergodic theorem, E. Hopf proved in 1939 the ergodicity of the geodesic flow on a surface of constant negative curvature, introducing an argument to get ergodicity which is now called Hopf's argument [Hop39]. Twenty eight years later, D. A. Anosov [Ano67] improved Hopf's results by proving the ergodicity of the geodesic flow on surfaces of negative (non necessarily constant) curvature and compact manifolds of constant negative curvature. He also showed the ergodicity of uniformly hyperbolic diffeomorphisms, now called Anosov diffeomorphisms. Since hyperbolicity is a C^1-robust condition, Anosov diffeomorphisms became the first example of *stably ergodic* diffeomorphisms, that is, a C^r ergodic diffeomorphism (preserving a measure m) that remains ergodic after a C^1-small perturbation.

For almost thirty years Anosov diffeomorphisms were the only known examples of stably ergodic systems, until 1995 when M. Grayson, C. Pugh and M. Shub [GPS94] proved the C^2 stable ergodicity of the time-one map of the geodesic flow on surfaces of constant negative curvature, hence the first non-Anosov stably ergodic example. Despite being non globally hyperbolic, this example is partially hyperbolic. With the evidence of this work they formulated in a 1995 conference [PS96] the following conjecture:

Conjecture 0.1.1 (Pugh-Shub's stable ergodicity conjecture [PS96; PS97])**.** *On any compact connected Riemannian manifold, stable ergodicity is C^r-dense among the set of volume preserving partially hyperbolic diffeomorphisms, for any integer $r \geq 2$.*

They also proposed a program in order to prove this, and split the conjecture into two conjectures:

Conjecture 0.1.2 (Accessibility implies ergodicity)**.** *A C^2 partially hyperbolic volume preserving diffeomorphism with the essential accessibility property is ergodic.*

Here, *essential accessibility* is a measure-theoretic version of the accessibility property.

Conjecture 0.1.3 (Density of accessibility)**.** *For any integer $r \in [2, +\infty]$, stable accessibility is open and dense among the set of C^r partially hyperbolic diffeomorphisms, volume preserving or not.*

There has been a lot of progress on these conjectures, mostly depending on the topology and the dimension of the center bundle.

The main conjecture was proven in [HHU08] in the case where $\dim E^c = 1$ and for the C^r topology (in fact the authors showed C^∞-density). Recently in [ACW16] the conjecture was proved in its full generality (any center dimension) for the C^1 topology. Despite these remarkable results, in the C^r case for $r \geq 2$ the conjecture is far from being solved. Recently, M. Leguil and Z. Zhang [LZ22] obtained C^r-density of stable ergodicity for partially hyperbolic diffeomorphisms (for any center dimension) with a strong pinching condition, introducing a new technique based on random perturbations.

With respect to Conjecture 0.1.2, C. Pugh and M. Shub [PS00] proved that a C^2 volume preserving partially hyperbolic diffeomorphism that is dynamically coherent, center bunched and with the essential accessibility property is ergodic. The center bunching condition is required to compensate the lost of transversality between the strong stable and strong unstable bundles (due to the existence of a center bundle). The state-of-the-art on Conjecture 0.1.2 is the result of K. Burns and A. Wilkinson [BW10] where the authors improved Pugh-Shub's result by removing the dynamical coherence hypothesis, and weakening the center bunching condition. In other words, by these works, a possible strategy to show that stable ergodicity is typical in the C^r topology would be to go further towards Conjecture 0.1.3, i.e., that stable accessibility is C^r-dense.

Regarding Conjecture 0.1.3, in [DW03; ACW22] stable accessibility is obtained for a C^1-dense set of • all • volume preserving • symplectic partially hyperbolic diffeomorphisms. The authors strongly use C^1 techniques which seem hard to generalize to other topologies.

For the $\dim E^c = 2$ case, there has been many results lately. The first one is the remarkable result by F. Rodríguez-Hertz [Her05] where he classified the center accessibility classes and obtained stable ergodicity of certain automorphisms on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Elaborating on these ideas, in [HS17] V. Horita and M. Sambarino proved stable ergodicity for skew-products of surface diffeomorphisms over Anosov diffeomorphisms. Recently, A. Ávila and M. Viana [AV20] obtained C^1-openness of accessibility and C^r-density for certain *fibered* partially hyperbolic diffeomorphisms with 2-dimensional center bundle using different techniques.

The last part of this thesis is a joint work with M. Leguil [LP], where we made a contribution to the accessibility conjecture (Conjecture 0.1.3) by proving the C^r-density of accessibility for (stably) dynamically coherent partially hyperbolic diffeomorphism with 2-dimensional center bundle which satisfy some strong bunching condition, for any integer $r \geq 2$ (we will give this condition at the begining of Chapter 4). Given a Riemannian manifold M of dimension $d \geq 4$ and an integer $r \geq 2$, we denote by $\mathcal{PH}^r_*(M)$ to the set of these diffeomorphisms. We also denote by $\mathcal{PH}^r_*(M, \mathrm{Vol}) \subset \mathcal{PH}^r_*(M)$ to the subset of those that preserve volume.

Theorem E ([LP])**.** *For any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r_*(M)$, resp. $f \in \mathcal{PH}^r_*(M, \mathrm{Vol})$, with $\dim E^c_f = 2$, that is dynamically coherent and plaque expansive, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that g is stably accessible.*

In particular, by the work of Burns-Wilkinson [BW10], this implies that for any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r_*(M, \mathrm{Vol})$, with $\dim E^c_f = 2$, that is dynamically coherent and plaque expansive, and for any $\delta > 0$, there exists $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that g is stably ergodic.

One intermediate step is to show that trivial accessibility classes can be broken by C^r-small perturbations. This part of the proof also holds when the center is higher dimensional and only requires center bunching.

Theorem F ([LP])**.** *For any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$, resp. $f \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $\dim E^c_f \geq 2$, that is center bunched, dynamically coherent, and plaque expansive, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that $\mathrm{C}_g(x)$ is non-trivial, for all $x \in M$.*

We want to mention here that Theorem F was obtained in [HS17] (Theorem 2) for skew products over Anosov diffeomorphisms. The main difference between these two cases is that in their context, the center leaves are all compact and in our setting we don't make any assumption on the topology of the center leaves, although the ideas involved are quite similar.

0.2 Introducción (Español)

0.2.1 Una breve introducción

Esta tesis se enmarca en la teoría de los sistemas dinámicos, más precisamente en el estudio de los sistemas parcialmente hiperbólicos. El objetivo principal de los sistemas dinámicos es entender el comportamiento asintótico de las órbitas dadas por una ley de movimiento.

Podría situarse el comienzo de los sistemas dinámicos con el descubrimiento de la ley de la gravitación universal de Newton. La misma nos brinda las ecuaciones diferenciales que gobiernan el movimiento de los planetas. La pregunta era en ese momento (y todavía es hoy día!) determinar si el sistema solar es estable en el largo plazo. Ahora, una cosa es conocer las leyes que gobiernan el movimiento, y otra completamente diferente es conocer las soluciones o trajectorias del sistema. El problema es que la mayoría de las ecuaciones diferenciales no son fáciles de resolver, incluso cuando se sabe que éstas tienen solución.

Cuando H. Poincaré estaba trabajando en el problema de los 3 cuerpos, se dio cuenta de que hasta las más simples ecuaciones implicaban un comportamiento impredecible o caótico. A partir de ese descubrimiento, propuso el estudio cualitativo de las ecuaciones diferenciales en lugar del estudio cuantitativo, i.e. el estudio de la geometría o topología de las soluciones, en lugar del enfoque numérico o analítico, que era el usual en esa época.

Esta impredictibilidad descubierta por H. Poincaré en el problema de los tres cuerpos fue la piedra fundamental de la teoría de los sistemas dinámicos, pero no fue hasta la década del 60 con la aparición de la teoría hiperbólica que tomó forma propia. Su importancia radica en que la hiperbolicidad resultó ser sinónimo de caos. La teoría hiperbólica de los sistemas dinámicos fue iniciada con los trabajos de Anosov, Sinai y Smale, y luego continuada por Bowen, Franks, Manning, Mañé, Newhouse y Palis, por nombrar algunos.

Desde entonces la dinámica hiperbólica ha sido ampliamente estudiada, y a pesar de algunos problemas importantes que se mantienen abiertos, la teoría está practicamente cerrada. En parte gracias a este éxito, los dinamistas han intentado empujar las ideas de esta teoría a otros contextos más generales, y la hiperbolicidad parcial aparece como una generalización natural de la hiperbolicidad uniforme (aunque hay otras generalizaciones como por ejemplo la dinámica no uniformemente hiperbólica).

Como dice el título, el propósito de esta tesis es contribuir al estudio de los sistemas parcialmente hiperbólicos, y en particular nos vamos a enfocar en tres aspectos diferentes que están en el corazón de la teoría. El primero es la integrabilidad del fibrado central, conocido como coherencia dinámica, el segundo es la transitividad robusta y el tercero es la accesibilidad, y por ende la ergodicidad.

A continuación vamos a presentar estas contribuciones.

0.2.2 Coherencia dinámica

Como mencionamos recién, la teoría hiperbólica de los sistemas dinámicos ha sido fructífera desde su aparición. Los resultados de D. A. Anosov [Ano67] sobre la estabilidad estructural y la estabilidad ergódica de los difeomorfismos globalmente hiperbólicos, los trabajos de J. Franks [Fra70], [Fra69] y A. Manning [Man74] sobre la clasificación de los (hoy llamados) difeomorfismos de Anosov en nilvariedades, el caso de codimensión uno obtenido por S. Newhouse [New70] y la prueba de la conjetura de estabilidad C^1 de R. Mañé [Mañ87a] son quizás los resultados más paradigmáticos o ilustrativos de la teoría.

Una herramienta fundamental en las demostraciones de estos resultados es el teorema de la variedad estable, i.e. la integrabilidad de los fibrados estable E^s e inestable E^u de un difeomorfismo uniformemente hiperbólico. Como estos fibrados son transversales, sus correspondientes foliaciones llenan el espacio por lo menos localmente.

En el caso de la hiperbolicidad parcial, dado un difeomorfismo $f : M \to M$ con una descomposición de la forma $TM = E^s_f \oplus E^c_f \oplus E^u_f$, es sabido que los fibrados fuertes E^s_f y E^u_f integran a foliaciones únicas y f-invariantes $\mathcal{W}^s_f$ y $\mathcal{W}^u_f$ (ver [HPS77]) y el mismo resultado aplica para flujos. Sin embargo, el fibrado central E^c_f puede tener diferentes comportamientos y uno esperaría poder integrar el fibrado central también, aunque no siempre es posible. Esto representa la primera gran diferencia entre la hiperbolicidad parcial y la hiperbolicidad global.

Decimos que un difeomorfismo parcialmente hiperbólico f es *dinámicamente coherente* si los fibrados $E^s_f \oplus E^c_f$ y $E^c_f \oplus E^u_f$ son integrables (y en consecuencia, el fibrado central E^c_f también es integrable). En caso contrario decimos que f es *dinámicamente incoherente*. El primer ejemplo de un difeomorfismo parcialmente hiperbólico dinámicamente incoherente fue construido en [Wil98] (ver también [BW10]) en una nilvariedad de dimensión seis con un fibrado central de dimensión cuatro. Tiempo después en [HHU16], los autores construyen un ejemplo en el toro $\mathbb{T}^3$ (con fibrado central unidimensional). En este último ejemplo en el 3-toro, la falta de regularidad rompe con la integrabilidad del fibrado central, aunque siempre existen curvas tangentes a E^c debido al teorema de Peano. En el ejemplo en la variedad de dimensión 6, a pesar de tener fibrados C^1, la condición de Frobenius falla y ninguna integrabilidad es posible en el fibrado central (veremos este ejemplo en detalle en la Subsección 1.3.2).

Hasta la fecha no es posible determinar si la coherencia dinámica es una propiedad C^1 abierta (una propiedad relacionada es la *expansividad por placas*, mencionaremos esto en la Sección 1.4). Para grandes conjuntos de mapas, en [FPS14] los autores obtienen coherencia dinámica en clases enteras de isotopías de Anosov lineales en el toro $\mathbb{T}^N$. Este es el primer resultado en donde la integrabilidad de los fibrados centro-estable y centro-inestable es obtenida en toda una clase de isotopía de mapas (la generalización para el caso en nilvariedades está probada en [Piñ]).

Recientemente en [Bar+] se prueba que en algunas variedades de Seifert de dimensión 3, todo difeomorfismo parcialmente hiperbólico isotópico a la identidad es dinámicamente coherente. Por otro lado, en [Bon+20] los autores construyen nuevos ejemplos de difeomorfismos parcialmente hiperbólicos que son robustamente dinámicamente incoherentes, y un tiempo después en [Bar+21] los autores obienen clases de isotopías enteras de difeomorfismos parcialmente hiperbólicos dinámicamente incoherentes. Todos estos resultados son de alguna forma sorprendentes, ya que por un lado la integrabilidad suele ser difícil y técnica de obtener, y por otro lado hay mucha libertad para moverse dentro de una clase de isotopía (y no hay ninguna hipótesis extra en el fibrado central salvo la dominación).

Debido a estos resultados, parecería ser que la integrabilidad (o no) de los fibrados centro-estable y centro-inestable es un fenómeno que depende fuertemente de la clase de isotopía del difeomorfismo.

Nuestra primera contribución en esta tesis va en esta dirección, generalizando los mecanismos obtenidos por T. Fisher, R. Potrie y M. Sambarino en [FPS14] a un contexto más general. En dicho artículo los autores prueban que dado un Anosov lineal $A : \mathbb{T}^d \to \mathbb{T}^d$, todo difeomorfismo parcialmente hiperbólico f isotópico a A (cuya isotopía este dentro del espacio de los difeomorfismos parcialmente hiperbólicos) es dinámicamente coherente.

Lo primero que hacemos es captar la idea principal del resultado, y darle un enfoque diferente al problema, donde técnicas similares pueden ser aplicadas. Esto en particular nos permite obtener coherencia dinámica en clases de isotopías para nuevos tipos de difeomorfismos, y variedades (veremos estos nuevos ejemplos en la Subsección 2.1.2). Mencionamos aquí que existe una dificultad intrínseca en el pasaje de un caso al otro, y más aún, en el artículo mencionado arriba existe un error en una prueba que logramos solucionar en nuestro contexto.

Para enunciar los teoremas de este capítuo con precisión, presentamos la siguiente definición. Dado un difeomorfismo parcialmente hiperbólico dinámicamente coherente $f : M \to M$ decimos que f está *fibrado* si verifica las siguientes dos condiciones:

- las foliaciones $\widetilde{\mathcal{W}}^{cs}_f$ y $\widetilde{\mathcal{W}}^{uu}_f$ tienen estructura de producto global en el cubrimiento universal $\widetilde{M}$, y lo mismo ocurre con $\widetilde{\mathcal{W}}^{cu}_f$ y $\widetilde{\mathcal{W}}^{ss}_f$.
- el mapa inducido en el cociente por hojas centrales $\widetilde{f}_c : \widetilde{M}/\widetilde{\mathcal{W}^c_f} \to \widetilde{M}/\widetilde{\mathcal{W}^c_f}$ es un homeomorfismo hiperbólico.

En realidad, vamos a llamar f fibrado si verifica estos dos puntos y además verifica otras dos condiciones técnicas. Al comienzo del Capítulo 2 vamos a dar con precisión estas condiciones técnicas, pero los dos puntos que mencionamos arriba captan la esencia de lo que queremos decir con parcialmente hiperbólico fibrado.

Vale la pena mencionar que todo difeomorfismo de Anosov lineal verifica trivialmente las condiciones dadas arriba (el fibrado central puede ser cualquier reagrupamiento de subfibrados intermedios) y por eso, cualquier Anosov lineal es un difeomorfismo parcialmente hiperbólico fibrado (esto nos permite re obtener los resultados de [FPS14] como un caso particular). Veremos este y otros ejemplos de parcialmente hiperbólicos fibrados en la Subsección 2.1.2.

El teorema principal de este capítulo es el siguiente:

Theorem A. *Sea $f : M \to M$ un difeomorfismo parcialmente hiperbólico fibrado. Sea g un difeomorfismo parcialmente hiperbólico isotópico a f tal que la isotopía se encuentra dentro del conjunto de los difeomorfismos parcialmente hiperbólicos (preservando la dimensión de los fibrados). Entonces g es dinámicamente coherente.*

De la demostración de este teorema, surge como corolario un resultado de clasificación global. Decimos que dos difeomorfismos parcialmente hiperbólicos y dinámicamente coherentes f y g son *conjugados por hojas* si existe un homeomorfismo $h\colon M \to M$, llamado conjugación de hojas, tal que h manda hojas centrales de f en hojas centrales de g, y $h \circ f(\mathcal{W}^c_f(\cdot)) = g \circ h(\mathcal{W}^c_f(\cdot))$.

Theorem B. *Sea $f : M \to M$ un difeomorfismo parcialmente hiperbólico fibrado. Entonces, todo parcialmente hiperbólico g isotópico a f cuya isotopía se encuentre dentro del conjunto de difeomorfismos parcialmente hiperbólicos, es conjugado por hojas a f.*

0.2.3 Transitividad robusta

En resumen, la teoría de los sistemas dinámicos estudia el movimiento y queremos entender el comportamiento de la mayoría de las órbitas. Típicamente la estructura de las órbitas es muy complicada, por ejemplo en algunos casos existen órbitas que llenan el espacio por completo, haciendolo indescomponible desde el punto de vista dinámico. Esto es lo que se conoce como *transitividad*: un sistemas dinámico es *transitivo* si existe una órbita futura densa en el espacio.

Aún más interesantes son los sistemas que presentan una característica que sea estable o robusta (esto quiere decir que persiste por perturbaciones del sistema). Decimos que un sistema dinámico es *robustamente transitivo*, si existe un entorno del sistema (en alguna topología dada) tal que todo sistema en este entorno es transitivo.

El primer ejemplo de mapa robustamente transitivo fue dado por D. A. Anosov en [Ano67], donde el autor prueba que los difeomorfismos que poseen una estructura globalmente hiperbólica son estables bajo perturbaciones C^1. Como corolario directo de esto, se obtiene que todo difeomorfismo transitivo de este tipo, como por ejemplo una matriz hiperbólica en el toro, es de hecho C^1 robustamente transitivo.

Gracias a este resultado, los difeomorfismos globalmente hiperbólicos reciben el nombre de difeomorfismos de Anosov. En particular llamamos difeomorfismos de Anosov lineales a los ejemplos dados por matrices hiperbólicas en el toro (ver Ejemplo 1.3.1) o los automorfismos hiperbólicos en nilvariedades (Ejemplo 1.3.2).

Mencionamos aquí que, todo difeomorfismo de Anosov en una nilvariedad es conjugado a un Anosov lineal, y en consecuencia debe ser transitivo ([Fra70], [Man74]). Sin embargo, todavía es un problema abierto determinar qué variedades admiten difeomorfismos de Anosov, y si todo difeomorfismo de Anosov debe ser (robustamente) transitivo.

Algunos años después M. Shub [Shu71] construyó el primer ejemplo no Anosov de difeomorfismo C^1 robustamente transitivo en el toro $\mathbb{T}^4$. Este ejemplo es un skew-product (producto cruzado, ver Ejemplo 1.3.4) del toro $\mathbb{T}^2$ sobre un Anosov lineal en $\mathbb{T}^2$ con dos puntos fijos. La idea de Shub fue modificar con cuidado el índice de estos puntos fijos para romper con la hiperbolicidad uniforme. Algunos años después R. Mañé mejoró este resultado e introdujo un ejemplo en $\mathbb{T}^3$ [Mañ78]. La idea del ejemplo de Mañé fue bifurcar el punto fijo de un Anosov lineal en tres puntos fijos diferentes, manteniendo la hoja central robustamente densa. Los dos ejemplos de Shub y Mañé son isotópicos a Anosov lineales y por esta razón son llamados ejemplos *derivados de Anosov* (de aquí en adelante los denotaremos por DA, ver Ejemplo 1.3.6).

Otra forma de construir difeomorfismos robustamente transitivos fue introducida por C. Bonatti y L. Díaz en [BD96]. Su técnica se basa en la existencia de unos conjuntos hiperbólicos particulares llamados *blenders*. Con esta técnica, los autores fueron capaces de construir ejemplos C^1-cerca de tiempo t de flujos de Anosov (por ende, isotópicos a la identidad) así como ejemplos C^1-cerca del producto de Anosov por la identidad (por ende, con acción trivial en el fibrado central).

Todos estos ejemplos de difeomorfismos robustamente transitivos no Anosov, comparten la característica de ser parcialmente hiperbólicos, y esto no es una simple casualidad. En [Mañ82] R. Mañé probó que todo difeomorfismo C^1 robustamente transitivo en una superficie es conjugado a un Anosov lineal (y por ende, la variedad debe ser el toro $\mathbb{T}^2$ por [Fra70]). En el caso tridimensional, L. Díaz, E. Pujals y R. Ures [DPU99] probaron que la C^1 transitividad robusta implica hiperbolicidad parcial (la definición de hiperbolicidad parcial aquí es un poco más general). Finalmente C. Bonatti, L. Díaz y E. Pujals generalizan este último resultado a dimensiones mayores y prueban que la C^1 transitividad robusta implica descomposición dominada [BDP03]. Queremos remarcar aquí que en [BV00] C. Bonatti y M. Viana construyen un ejemplo de difeomorfismo C^1 robustamente transitivo en el toro $\mathbb{T}^4$ que no es parcialmente hiperbólico (aunque necesariamente tiene descomposición dominada). En resumen, una propiedad dinámica como la transitividad robusta implica fuertes restricciones geométricas.

Otra propiedad íntimamente relacionada con la transitividad es la minimalidad de las foliaciones estables/inestables fuertes. Decimos que una foliación $\mathcal{F}$ en una variedad M es *minimal* si toda hoja es densa, i.e. $\overline{\mathcal{F}(x)} = M$ para todo $x \in M$. Es fácil ver que si la foliación estable fuerte (o inestable fuerte) es minimal, entonces el difeomorfismo es transitivo. Como resultado de esto, la existencia de una foliación estable (inestable) fuerte robustamente minimal, implica la transitividad robusta.

En [PS06] se dan condiciones que garantizan la minimalidad de estas foliaciones en un entorno C^1. En ese trabajo, los autores buscaban una propiedad (robusta) que en presencia de transitividad implicara transitividad robusta. Con esta idea en mente, es que introducen la propiedad *SH* (*Some hyperbolicity*) y logran probar que la propiedad SH en presencia de una foliación estable minimal, implica la C^1 minimalidad robusta de la foliación estable (y en particular la C^1 transitividad robusta). Con esta técnica, además, re obtienen los ejemplos de Shub y Mañé. Queremos mencionar aquí que una desventaja de esta propiedad, es que no puede ser aplicada en el contexto simpléctico.

Recientemente en [HUY22], los autores dan diferentes condiciones que garantizan la minimalidad de la variedad estable fuerte en un entorno C^1, para difeomorfismos derivados de Anosov en el toro $\mathbb{T}^3$ (y en consecuencia la C^1 transitividad robusta). Más aún, con esta técnica construyen un ejemplo con ambas foliaciones estable e inestable fuertes C^1 robustamente minimales (la existencia de tal ejemplo era hasta ahora desconocida).

A pesar de estos importantes resultados, la transitividad robusta aún no es comprendida del todo. En particular, todos los ejemplos mencionados anteriormente tienen una descomposición dominada (una condición necesaria debido a [BDP03]) que es coherente con la descomposición hiperbólica de su parte lineal. Recientemente R. Potrie en [Pot12] (página 152) contruye un ejemplo robustamente transitivo en el toro $\mathbb{T}^3$ con descomposición dominada, pero en este caso la descomposición del ejemplo no es coherente con la descomposición de su parte lineal[2].

Nuestra contribución en esta parte de la tesis es la introducción de una definición más general de propiedad SH, que llamamos *SH-Silla*. Esta nueva definición aparece como una extensión natural de la definición original de SH y en consecuencia puede ser aplicada en contextos más generales. En particular, tiene la ventaja de ser aplicable al contexto simpléctico (algo que la definición original no podía).

Queremos mencionar aquí que recientemente P. Carrasco y D. Obata prueban en [CO21] que el ejemplo introducido en [BC14] es C^1 robustamente transitivo. Este ejemplo a pesar de ser un skew-product en $\mathbb{T}^4$, tiene la particularidad de tener comportamiento mezclado en el fibrado central (que es de dimensión 2) y eso lo convierte en un nuevo ejemplo. En dicho artículo, los autores mencionan que el ejemplo no puede tener la propiedad SH (la versión original de esta). Sin embargo, se desprende directamente de las pruebas del artículo, que el ejemplo si tiene la propiedad SH-Silla. En consecuencia, podría ocurrir que todo difeomorfismo parcialmente hiperboólico robustamente transitivo verifique la propiedad SH-Silla.

Volviendo a los resultados de esta tesis, aplicando este nuevo enfoque damos condiciones suficientes para que un difeomorfismo derivado de Anosov sea C^1 robustamente transitivo. Estas nuevas técnicas nos permiten construir nuevos ejemplos de difeomorfismos derivados de Anosov C^1 robustamente transitivos. En particular, podemos construir ejemplos en cualquier dimensión con todos los comportamientos posibles en las hojas centrales como se desee, y más aún, con un fibrado central que sea indescomponible en suma de subfibrados más pequeños. En consecuencia, la descomposición dominada de estos ejemplos no será coherente con la de su parte lineal.

[2] la definición de parcialmente hiperbólico aquí es más general

Theorem C. *Sea $A \in SL(d,\mathbb{Z})$ una matriz hiperbólica con una descomposición parcialmente hiperbólica de la forma $\mathbb{R}^d = E^{ss} \oplus E^c \oplus E^{uu}$ y sea $k = dimE^c$. Entonces, existe $f : \mathbb{T}^d \to \mathbb{T}^d$ un difeomorfismo parcialmente hiperbólico C^1 robustamente transitivo isotópico a A con $k+1$ puntos fijos: $p_0, p_1, \dots, p_k$ tal que: $index(p_j) = j + dimE^{ss}$ para todo $j = 0, \dots, k$.*

Más aún, el fibrado central E^c_f no admite una subdescomposición dominada. En particular, la descomposición de f no es coherente con la descomposición hiperbólica de A.

Luego de esto nos pasamos al caso de flujos, y trasladamos la definición de SH-Silla de difeomorfismos a flujos. Como ya mencionamos, una ventaja de nuestra nueva definición de SH es que puede ser aplicada al contexto simpléctico. En particular vamos a estar interesados en flujos geodésicos, que siempre preservan una forma simpléctica.

Existen muchas similitudes entre flujos y difeomorfismos respecto a la transitividad robusta. Por ejemplo D. A. Anosov también probó en [Ano67] que al igual que el caso de difeomorfismos, todo flujo hiperbólico es C^1 estable por perturbaciones. En particular, los flujos hiperbólicos transitivos son de hecho C^1 robustamente transitivos. Por esta razón los flujos globalmente hiperbólicos reciben el nombre de flujos de Anosov.

Los ejemplos más paradigmáticos de flujos de Anosov son los flujos geodésicos en variedades de curvatura negativa (ver Ejemplo 1.3.3). En el caso particular de superficies de curvatura constante -1 el flujo geodésico resulta transitivo, y por ende C^1 robustamente transitivo, debido a la estabilidad de los flujos de Anosov. Sin embargo no todo flujo de Anosov es transitivo, algunos ejemplos de este tipo de flujos fueron dados por J. Franks y R. Williams en [FW80].

Otra forma de construir ejemplos son las suspensiones (Ejemplo 1.3.5). Dado que la suspensión de un difeomorfismo C^1 robustamente transitivo, resulta ser un flujo C^1 robustamente transitivo, podemos construir varios ejemplos tomando la suspensión de todos los difeomorfismos C^1 robustamente transitivos mencionados anteriormente.

Respecto a los resultados de clasificación, el panorama es un poco diferente. Cuando $\dim(M) = 3$, C. I. Doering probó que todo campo de vectores C^1 robustamente transitivo es en realidad el campo de un flujo de Anosov transitivo [Doe87]. Esto no es cierto en dimensiones mayores, for ejemplo la suspensión del derivado de Anosov de Mañé es un flujo no Anosov C^1 robustamente transitivo. Mencionamos aquí que la suspensión del ejemplo de Bonatti-Viana nos da un ejemplo de un flujo C^1 robustamente transitivo que no tiene descomposición dominada. Sin embargo, C. Bonatti, N. Gourmelon y T. Vivier [BGV06], [Viv06] prueban que el flujo de Poincaré lineal de un flujo C^1 robustamente transitivo si admite una descomposición dominada. Nuevamente la transitividad robusta implica fuertes restricciones.

En [Rug97] R. O. Ruggiero prueba que si el flujo geodésico $\varphi_t : T^1M \to T^1M$ de una variedad compacta, n-dimensional sin puntos conjugados es expansivo, entonces es topológicamente transitivo. Nuestro siguiente teorema nos dice que si además el flujo es parcialmente hiperbólico y verifica la propiedad SH-Silla, entonces es robustamente transitivo. Este teorema está motivado por el artículo de F. Carneiro y E. Pujals [CP14], donde se construye el primer ejemplo de flujo geodésico transitivo parcialmente hiperbólico, no Anosov. Este ejemplo verifica la propiedad SH-Silla, aunque no es claro que sea expansivo o que no tenga puntos conjugados.

Theorem D. *Sea g_0 una métrica Riemanniana C^∞ en una variedad diferenciable y compacta M sin puntos conjugados, y sea $\varphi_t : T^1M \to T^1M$ su flujo geodésico. Supongamos que φ_t es expansivo con conjunto estable W^s y conjunto inestable W^u. Supongamos además que φ_t es parcialmente hiperbólico con una descomposición de la forma $T(T^1M) = E^{ss} \oplus E^c \oplus \langle X \rangle \oplus$*

E^{uu}, y que tiene la propiedad SH-Silla de índices (d_1, d_2) donde $d_1 = dimW^s - dimE^{ss}$ y $d_2 = dimW^u - dimE^{uu}$. Entonces φ_t es C^1 robustamente transitivo.

0.2.4 Accesibilidad y ergodicidad

En 1871 L. Boltzmann formuló su hipótesis ergódica cuando se encontraba estudiando el movimiento de los gases y la termodinámica. El quería una propiedad que le permitiera "caracterizar la probabilidad de un estado por el promedio temporal en que el sistema *está* en ese estado". Desde entonces, la ergodicidad ha jugado un papel clave en los sistemas dinámicos, la física y la probabilidad. Decimos que un sistema dinámico $f : M \to M$ que preserva una medida finita m es *ergódico* si cualquier conjunto f-invariante tiene medida total o nula.

Luego del celebrado teorema ergódico de Birkhoff, E. Hopf provó en 1939 que el flujo geodésico de una superficie con curvatura constante negativa es ergódico, introduciendo un método para obtener la ergodicidad que hoy se le conoce como el argumento de Hopf. Veintiocho años después, D. A. Anosov [Ano67] mejoró los resultados de Hopf probando la ergodicidad de los flujos geodésico en superficies de curvatura negativa (no necesariamente constante) y para variedades compactas de curvatura negativa constante. También probó la ergodicidad de los difeomorfismos uniformemente hiperbólicos, hoy llamados difeomorfismos de Anosov. Debido a que la hiperbolicidad es una propiedad C^1-robusta, los difeomorfismos de Anosov se convirtieron en el primer ejemplo de difeomorfismos *establemente ergódicos*, es decir, difeomorfismos ergódicos C^r (que preservan una medida m) que permanecen ergódicos después de perturbaciones C^1 pequeñas.

Por casi treinta años los difeomorfismos de Anosov fueron los únicos ejemplos de sistemas establemente ergódicos, hasta 1995 cuando M. Grayson, C. Pugh y M. Shub [GPS94] probaron la C^2 estabilidad ergódica del tiempo-1 del flujo geodésico en superficies de curvatura constante negativa, convirtiendose en el primer ejemplo de difeomorfismo establemente ergódico no Anosov. A pesar de no ser globalmente hiperbólico, el ejemplo es parcialmente hiperbólico. Con estos resultados en mente, los autores formulan en una conferencia en 1995 [PS96] la siguiente conjetura:

Conjecture 0.2.1 (Conjetura de ergodicidad de Pugh y Shub [PS96; PS97])**.** *En una variedad Riemanniana compacta y conexa, la estabilidad ergódica es C^r densa en el conjunto de los difeomorfismos parcialmente hiperbólicos que preservan volumen, para todo entero $r \geq 2$.*

También propusieron un programa con el fin de probar la conjetura, dividiendola en dos subconjeturas:

Conjecture 0.2.2 (Accesibilidad implica ergodicidad)**.** *Un difeomorfismo parcialmente hiperbólico C^2 que preserva volumen con la propiedad de accesibilidad esencial, es ergódico.*

La *accesibilidad esencial* es una versión un poco diferente (más débil) a la propiedad de accesibilidad.

Conjecture 0.2.3 (Densidad de accesibilidad)**.** *Para cualquier entero $r \in [2, +\infty]$, la accesibilidad estable es abierta y densa dentro del conjunto de difeomorfismos C^r parcialmente hiperbólicos, preserven volumen o no.*

Han habido grandes avances respecto a estas conjeturas, la mayoría dependiendo de la topología y de la dimensión del fibrado central.

La conjetura principal fue probada en [HHU08] para el caso $\dim E^c = 1$ y para la topología C^r (en realidad, los autores prueban densidad C^∞). Recientemente en

[ACW16] la conjetura fue probada en toda su generalidad (cualquier dimensión del fibrado central) para la topología C^1. A pesar de estos notables resultados, en la topología C^r, con $r \geq 2$ la conjetura está lejos de ser probada. Recientemente M. Leguil y Z. Zhang [LZ22] obtienen la C^r-densidad de estabilidad ergódica para difeomorfismos parcialmente hiperbólicos (con cualquier dimensión del fibrado central) con una condición de pinching fuerte, introduciendo una nueva técnica basada en perturbaciones aleatorias (random perturbations).

Con respecto a la Conjetura 0.2.2 C. Pugh y M. Shub en [PS00] probaron que un difeomorfismo C^2 parcialmente hiperbólico que preserva volumen, dinámicamente coherente, center bunched y con la propiedad de accesibilidad esencial, es ergódico. La condición de center bunching es necesaria para compensar la falta de transversalidad entre los fibrados estable e inestable fuertes (debido a la existencia del fibrado central). El estado-del-arte de la Conjetura 0.2.2 es el resultado de K. Burns y A. Wilkinson [BW10] donde los autores mejoran los resultados de Pugh y Shub, quitando la hipótesis de coherencia dinámica, y mejorando la condición de center bunching. En otras palabras, gracias a estos trabajos, una posible estrategia para mostrar que la estabilidad ergódica es típica en la topología C^r es ir hacia la prueba de la Conjetura 0.2.3, i.e., que la accesibilidad estable es C^r-densa.

Con respecto a la Conjetura 0.2.3, en [DW03; ACW22] la accesibilidd estable es obtenida para un conjunto C^1-denso de difeomorfismos parcialmente hiperbólicos (que preservan volumen, simplécticos, etc). En esos trabajos, los autores utilizan fuertemente técnicas C^1 que no parecen fáciles de aplicar a otras topologías.

Para el caso $\dim E^c = 2$, han habido muchos resultados en los últimos años. El primero es el notable trabajo de F. Rodríguez-Hertz [Her05] donde se clasifican las clases de accesibilidad centrales y se obtiene la estabilidad ergódica de ciertos automorfismos en el toro $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$. Profundizando sobre estas ideas, en [HS17] V. Horita y M. Sambarino prueban la estabilidad ergódica para skew-products de superficies sobre difeomorfismos de Anosov. Recientemente A. Ávila y M. Viana [AV20] obtienen la C^1 estabilidad de las clases abiertas y la C^r-densidad para algunos parcialmente hiperbólicos fibrados con central de dimensión 2, utilizando técnicas diferentes.

La última parte de esta tesis es un trabajo en conjunto con M. Leguil [LP], donde hacemos una contribución a la conjetura de accesibilidad (Conjetura 0.2.3) probando la C^r densidad de accesibilidad ($r \geq 2$) para difeomorfismos parcialmente hiperbólicos con central de dimensión 2 que son (robustamente) dinámicamente coherentes y que satisfacen una condición de bunching fuerte (daremos esta condición de bunching fuerte al comienzo del Capítulo 4). Dada una variedad Riemanniana M de dimensión $d \geq 4$ y un entero $r \geq 2$, notamos por $\mathcal{PH}^r_*(M)$ al conjunto de estos difeomorfismos. También notamos por $\mathcal{PH}^r_*(M, \mathrm{Vol}) \subset \mathcal{PH}^r_*(M)$ al subconjunto de estos que preservan volumen.

Theorem E ([LP]). *Para todo difeomorfismo parcialmente hiperbólico $f \in \mathcal{PH}^r_*(M)$, resp. $f \in \mathcal{PH}^r_*(M, \mathrm{Vol})$, con $\dim E^c_f = 2$, dinámicamente coherente y plaque-expansive, y para todo $\delta > 0$, existe un difeomorfismo parcialmente hiperbólico $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, con $d_{C^r}(f, g) < \delta$, tal que g es establemente accesible.*

En particular, por los trabajos de Burns-Wilkinson [BW10], esto implica que para todo difeomorfismo parcialmente hiperbólico $f \in \mathcal{PH}^r_*(M, \mathrm{Vol})$, con $\dim E^c_f = 2$, dinámicamente coherente y plaque-expansive, y para todo $\delta > 0$, existe un difeomorfismo parcialmente hiperbólico $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, con $d_{C^r}(f, g) < \delta$, tal que g es establemente ergódico.

Un paso intermedio en la prueba, es mostrar que las cases de accesibilidad triviales se pueden romper por perturbaciones C^r pequeñas. Esta parte de la prueba también funciona cuando el fibrado central tiene dimensión mayor que 2 y solo requiere center bunching.

Theorem F ([LP]). *Para todo difeomorfismo parcialmente hiperbólico $f \in \mathcal{PH}^r(M)$, resp. $f \in \mathcal{PH}^r(M, \mathrm{Vol})$, con $\dim E^c_f \geq 2$, dinámicamente coherente, plaque-expansive y center bunched, y para todo $\delta > 0$, existe un difeomorfismo parcialmente hiperbólico $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, con $d_{C^r}(f, g) < \delta$, tal que $\mathrm{C}_g(x)$ es no trivial, para todo $x \in M$.*

Queremos mencionar aquí que el Teorema F fue obtenido en [HS17] (Theorem 2) para el caso de skew-products sobre difeomorfismos de Anosov. La principal diferencia entre estos dos resultados es que en el contexto de skew-product, las hojas centrales son todas compactas y en nuestro contexto no hacemos ninguna suposición sobre la topología de las hojas centrales, aunque las ideas involucradas en las pruebas son bastante similares.

0.3 Organization of the thesis

This thesis is organized as follows:

- In Chapter 1 we introduce some definitions and well known results that we are going to use along this work.
- In Chapter 2 we study dynamically coherence in isotopy classes of fibered partially hyperbolic diffeomorphisms and prove Theorem A and Theorem B.
- Chapter 3 is devoted to robust transitivity. We introduce the SH-Saddle property and we prove that it is a C^1 open condition among partially hyperbolic diffeomorphisms. We then prove Theorem C and Theorem D.
- Finally in Chapter 4 we deal with the accessibility property and prove Theorem E and Theorem F.

Chapter 1 is only for backround material, therefore readers with knowledge on the field can skip this part and pass directly to the following chapters. The next three chapters 2, 3 and 4, can be read independently since they don't use any result in common. Besides, since we haven't been very rigorous with the statements of the results, a brief introduction has been placed at the beginning of each chapter.

Chapter 1

Preliminaries

1.1 Basic concepts and dynamical systems

1.1.1 Differentiable manifolds

Let X be a topological space. We say that X is a *topological manifold of dimension d* if every point $x \in X$ has a neighborhood U which is homeomorphic to an open set of $\mathbb{R}^d$. In general since topology can be a little lax, we will need some additional structure on our space that allow us to do geometry. We say that a *differentiable manifold of dimension d* is a subset M and a family of bijective maps $\varphi_\alpha : U_\alpha \subset \mathbb{R}^d \to M$, of open sets U_α of $\mathbb{R}^d$ on M such that:

1. $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$.

2. For every pair α, β such that $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W \neq \emptyset$, we have that $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open sets in $\mathbb{R}^d$ and the functions $\varphi_\beta^{-1} \circ \varphi_\alpha$ are differentiable.

3. The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal within all the ones satisfying 1 and 2.

The pair $(U_\alpha, \varphi_\alpha)$ (and the function φ_α) with $p \in \varphi_\alpha(U_\alpha)$ is called a *parametrization* or *coordinate system* of M at the point p. We call $\varphi_\alpha(U_\alpha)$ a coordinate neighborhood of p. A family $\{(U_\alpha, \varphi_\alpha)\}$ satisfying conditions 1 and 2 is called a *differentiable structure* on M. By a slightly abuse of notation, we will assume that a differentiable structure satisfies condition 3 too, since we can always complete such a family.

A differentiable structure on a set M induces a natural topology on the manifold M, such that the functions φ_α are continuous. Just define $A \subset M$ as an open set if and only if $\varphi_\alpha^{-1}(A \cap \varphi_\alpha(U_\alpha))$ is open in $\mathbb{R}^d$ for every α. It is easy to see that this is a well defined topology.

Moreover a differentiable structure allows us to define differentiable functions via local coordinates. We say that a function $f : M_1 \to M_2$ between differentiable manifolds is *differentiable* at $p \in M_1$ if there are parametrizations $\varphi_\beta : U_\alpha \subset \mathbb{R}^{d_2} \to M_2$ with $f(p) \in \varphi_\beta(U_\beta)$, and $\varphi_\alpha : U_\alpha \subset \mathbb{R}^{d_1} \to M_1$ such that $\varphi_\beta^{-1} \circ f \circ \varphi_\alpha : U_\alpha \to \mathbb{R}^{d_2}$ is differentiable on $\varphi_\alpha^{-1}(p)$. We say that f is differentiable if it's differentiable on every point of the manifold M_1. This definition does not depend on the choice of parametrizations due to condition 2. In the same way, we say that a function $f : M_1 \to M_2$ is of class C^r if in local coordinates it is of class C^r, i.e. their first r derivatives exists and are continuous. We are going to call a *curve* on M to a differentiable function $\alpha : (-\epsilon, \epsilon) \subset \mathbb{R} \to M$.

Now given a point $p \in M$, we say that a vector v is *tangent* to p if there is a curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(0) = p$ and $\alpha'(0) = v$ (tangent vectors are just velocity vectors as in Euclidean spaces). We denote by T_pM to the set of all tangent vectors to M at the point p and we call T_pM the *tangent space* of M on p. It's easy to see that T_pM

is a vector space of dimension $d = \dim(M)$. More generally, we are going to call the set $TM = \{(x, v) : x \in M,\ v \in T_xM\}$ the *tangent bundle* of M, that is the union of all tangent spaces of M. It's easy to see that a differentiable structure on M induces a differentiable structure on TM, that makes it a differentiable manifold of dimension $\dim(TM) = 2d$.

Now that we have tangent spaces, we can define the derivative of a function. Let M_1 and M_2 be two differentiable manifolds and let $f : M_1 \to M_2$ be a differentiable funcion. Take $p \in M_1$ and $v \in T_pM$, and take $\alpha : (-\epsilon, \epsilon) \to M$ a corresponding curve associated to v, that is $\alpha(0) = p$ and $\alpha'(0) = v$. Call $\beta(t) = (f \circ \alpha)(t)$. Then the map $Df_p : T_pM_1 \to T_{f(p)}M_2$ given by $Df_p(v) = \beta'(0)$ is called the *derivative of f at p*. This map is a linear transformation and does not depend on the choice of the curve α.

A *vector field* on a differentiable manifold M is a function $X : M \to TM$, such that $X(p) \in T_pM$ for every $p \in M$. We say that the vector field X is differentiable if the map $X : M \to TM$ is differentiable (with their respective differentiable structures). In the same way, we say that the vector field is C^r if $X : M \to TM$ is C^r. In local coordinates, given a parametrization $\varphi : U \subset \mathbb{R}^d \to M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

where $a_i : U \to \mathbb{R}$ are functions on U and $\{\frac{\partial}{\partial x_i}\}$ is a basis of T_pM associated to φ. This way we have that X is a differentiable vector field iff the functions a_i are differentiable for every i. We are going to note by $\mathfrak{X}^r(M)$ to the set of C^r vector fields on M.

In the same way, we say that a *distribution* or *subbundle* E of dimension k on M is a continuous family of k-dimensional subspaces $E_x \subset T_xM$. By continuity we mean that for every $x \in M$ there is U a neighbourhood of x and $X_1, \ldots, X_k$ linearly independent continuous vector fields defined on U such that for every $y \in U$ we have $E(y) = \langle X_1(y), \ldots, X_k(y) \rangle$. We say that the distribution is of class C^r if the vector fields can be chosen of class C^r.

1.1.2 Riemannian geometry

A differentiable manifold (Hausdorff and with numerable basis) allows us to introduce a special type of metric on the manifold in order to do geometry.

A *Riemannian metric* on a differentiable manifold M is a correspondance which associates an inner product g_x to every T_xM which varies differentiably with respect to $x \in M$. By differentiability we mean that if $\varphi : U \subset \mathbb{R}^d \to M$ is a parametrization, $q \in \varphi(U)$, and $\frac{\partial}{\partial x_i}$ is a basis of T_qM then the funcions $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$ are differentiable. We call the functions g_{ij} the local representations of the metric and we are going to call the pair (M, g) a *Riemannian manifold*.

Recall that the tangent bundle is the set $TM = \{(x, v) : x \in M,\ v \in T_xM\}$ and it is a differentiable manifold of dimension $2d$. Therefore, we can see the metric g as a smooth section $g : M \to Symm_2^+(TM)$, where $Symm_2^+(TM)$ is the set of positive definite symmetric and bilinear forms.

Notice that we can equip a differentiable manifold M with many different Riemannian metrics. However, if the manifold is compact all metrics are equivalent in the following sense: given two Riemannian metrics g_1 and g_2 there are constants $\alpha, \beta > 0$ such that for every $x \in M$ and $v \in T_xM$ we have:

$$\alpha \|v\|_1 \leq \|v\|_2 \leq \beta \|v\|_1$$

where $\|\cdot\|_k$ denotes the norm associated to the metric g_k, for $k = 1, 2$.

The *unitary tangent bundle* will be the restriction of TM to unitary vectors:

$$T^1M = \{(x, v) \in TM : g_x(v, v) = \|v\|^2 = 1\}$$

Notice that the tangent bundle TM does not depend on choice of the Riemannian metric, but the unitary tangent bundle does. However, given two different Riemannian metrics g_1 and g_2, their corresponding unitary tangent bundles $T^1_{g_1}M$ and $T^1_{g_2}M$ are diffeomorphic.

Volume form

Another important tool form Riemannian geometry is the concept of volume form on the manifold M. Let $\varphi : U \subset \mathbb{R}^d \to M$ be a parametrization and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM and let $X_i(p) = \frac{\partial}{\partial x_i}(q)$ be written by $X_i(p) = \sum_{i,j} a_{ij}e_j$ then we have

$$g_{ij}(p) = \langle X_i(p), X_j(p)\rangle = \sum_{jl} a_{ij}a_{kl}\langle e_j, e_l\rangle = \sum_j a_{ij}a_{kj}$$

Then we define the volume by the equation

$$\mathrm{Vol}(X_1(p), \ldots, X_n(p)) = \det(a_{ij}) = \sqrt{\det(g_{ij})}(p)$$

This definition does not depend on the choice of the parametrization. We call Vol the *volume form* of M associated to the Riemannian metric g.

Geodesic flow and exponential map

Given a Riemannian metric g, we are goint to note by ∇ to the Levi-Civita conexion associated to this metric, and by $\frac{D}{dt}$ to the covariant derivative associated to this conexion. We say that a parametrized curve $\gamma : I \to M$ is a *geodesic* if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ for every $t \in I$. In particular, for this kind of curves we have that the lenght of $\gamma'(t)$ is constant. We say that a geodesic is *normalized* when this constant is equal to 1. Now given a curve $\gamma : I \to M$ which in local coordinates $\varphi : U \subset \mathbb{R}^d \to M$ has the form $\gamma(t) = (x_1(t), \ldots, x_d(t))$ we have that γ is a geodesic if and only if:

$$0 = \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k \left\{\frac{dv^2x_k}{dt^2} + \sum_{i,j}\Gamma^k_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}\right\} X_k$$

Or equivalently, for every $k = 1, \ldots, d$ we have:

$$\frac{d^2x_k}{dt} + \sum_{i,j}\Gamma^k_{i,j}\frac{dx_i}{dt}\frac{dx_j}{dt} = 0 \tag{1.1}$$

where Γ^k_{ij} are defined by $\nabla_{X_i}X_j = \sum_k \Gamma^k_{ij}X_k$ and are called the *Christoffel symbols* of the conexion. By a change of variables we can transform this system of differentiable equations of second order into a first order system. Notice that any differentiable curve $t \mapsto \gamma(t)$ on M determines a unique curve $t \mapsto \left(\gamma(t), \frac{d\gamma}{dt}(t)\right)$ on TM. Moreover the curve γ is a geodesic if and only if the curve

$$t \mapsto \left(x_1(t), \ldots, x_d(t), \frac{dx_1}{dt}(t), \ldots, \frac{dx_d}{dt}(t)\right)$$

verifies the system:

$$\begin{cases} \frac{dx_k}{dt} & = y_k \\ \frac{dy_k}{dt} & = -\sum_{i,j} \Gamma_{ij}^k y_i y_j \end{cases} \tag{1.2}$$

where $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ is a local system on TU. We thus obtain a vector field $G : TM \to T(TM)$ defined in local coordinates by Equation (1.2). We call this vector field the *geodesic field*. Notice that the trajectories of G are of the form $t \mapsto (\gamma(t), \gamma'(t))$. Integrating this vector field by classical theorems of differential equations, we obtain a flow which is called the *geodesic flow*.

Moreover, for every $p \in M$ there is an open set $\mathcal{U}$ in TU where $\varphi : U \to M$ is a coordinate system on p and $(p, 0) \in \mathcal{U}$, there is $\delta > 0$ and a C^∞ function $\phi : (-\delta, \delta) \times \mathcal{U} \to TU$ such that the curve $t \mapsto \phi(t, q, v)$ is the only trajectory of G with initial conditions $\phi(0, q, v) = (q, v)$ for every $(q, v) \in \mathcal{U}$. This allow us to define the following function (at least locally). Let $\mathcal{U}$ be a sufficiently small open set in TU, then the function $\exp : \mathcal{U} \to M$ given by

$$\exp(q, v) = \gamma(1, q, v) = \gamma\left(\|v\|, q, \frac{V}{\|v\|}\right), \quad (q, v) \in \mathcal{U}$$

is called the *exponential map*. Notice that the map exp is differentiable. In general we are going to fix a point $q \in M$ and consider the function

$$\exp_q : B(0, \epsilon) \subset TqM \to M \text{ given by } \exp_q(v) = \exp(q, v)$$

Now by definition of the exponential map we have that $D(\exp_q)_0(v) = v$, then $D(\exp_q)_0$ is the identity on T_qM and by the inverse function theorem, we have that there is $\epsilon > 0$ such that $\exp_q : B(0, \epsilon) \to M$ is a diffeomorphism onto its image (an open set of M). If $\exp_p$ is a diffeomorphism on a neighbourhood V of $0 \in T_pM$, then $\exp_p(V) = U$ is called a *normal* neighbourhood of p. If $\overline{B(0, \epsilon)} \subset V$ then we call $B(p, \epsilon) = \exp_p(B(0, \epsilon))$ a *normal ball* centered at p of radius ϵ.

We end this subsection with one last definition concerning the critical points of the exponential map. We say that a Riemannian metric g on M has *no conjugate points*, if the exponential map $\exp_p : T_pM \to M$ is non singular, for every $p \in M$.

Curvature

Given a Riemannian manifold, a curvature R is a correspondance which associates for every two vector fields X, Y a function $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

where ∇ is the Riemannian conexion. Notice that in the euclidean case, we have that $\Gamma_{ij}^k = 0$ for every k, i, j and therefore $R(X, Y)Z = 0$ for every $X, Y, Z \in \mathfrak{X}(M)$. This tell us that in a sense, the curvature measures how far we are from being Euclidean.

The curvature is bilinear in $\mathfrak{X}(M) \times \mathfrak{X}(M)$ and moreover it verifies the Bianchi identity:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

We can see the curvature operator in local coordinates: let (U, x) be a coordinate system, and let $\frac{\partial}{\partial x_i} = X_i$ and let $R(X_i, X_j)X_k = \sum_l R_{ijk}^l X_l$. Then the numbers R_{ijk}^l are the components of the curvature R in (U, x). Given three vector fields

$$X = \sum_i u^i X_i,\ Y = \sum_j v^j X_j,\ Z = \sum_k w^k X_k$$

we have by linearity that

$$R(X,Y)Z = \sum_{i,j,k,l} R^l_{ijk} u^i v^j w^k X_l$$

The curvature operator allows us to define a more geometric concept of curvature. Given a point $p \in M$ and a plane $\sigma \subset T_pM$, take two vectors $X, Y \in T_pM$ that form a basis of σ. Then the *sectional curvature of σ at p* is:

$$K_p(\sigma) = \frac{\langle R(X,Y)X, Y\rangle}{|X \wedge Y|^2}$$

where $|X \wedge Y| = \sqrt{|X|^2|Y|^2 - \langle X, Y\rangle^2}$ is the area of the bidimensional parallelogram determined by X and Y. It can be seen that this number does not depend on the choice of the basis and therefore the sectional curvature is well defined.

1.1.3 Diffeomorphisms and flows

Given a topological space X, we say that $f : X \to X$ is a homeomorphism if it is continuous, invertible and its inverse f^{-1} is continuous. In the differentiable setting, given a differentiable manifold M, we say that $f : M \to M$ is a *diffeomorphism* if f is bijective, differentiable and its inverse function f^{-1} is differentiable too. Given a differentiable manifold M we are going to note by

$$\text{Diff}(M) = \{f : M \to M \text{ diffeomorphism}\}$$

to the set of all diffeomorphisms on M. In the same way, for $r \geq 1$ we are going to note by $\text{Diff}^r(M)$ to the set of all C^r diffeomorphisms on M. In the space $\text{Diff}^r(M)$ we can introduce a natural topology: we say that two diffeomorphisms $f, g \in \text{Diff}^r(M)$ are ϵ close in the C^r topology, if the first r derivatives of f and g are ϵ close.

In general we will be interested in maps which have the same behaviour under a change of coordinates: we say that two diffeomorphisms $f : M \to M$ and $g : N \to N$ are topologically equivalent or *conjugated* if there exist a homeomorphism $h : M \to N$ such that $h \circ f = g \circ h$. This relation is represented in the diagram below:

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ N & \xrightarrow{g} & N \end{array}$$

We will be interested in $\mathbb{R}$ actions besides $\mathbb{Z}$ actions. Let M be a Riemannian manifold and $r \geq 0$. A C^r-flow on M is a C^r-map $\varphi : \mathbb{R} \times M \to M$ such that $\varphi(0, x) = x$ and $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for every $t, s \in \mathbb{R}$ and $x \in M$. In general we are going to note by $\varphi_t(x) = \varphi(t, x)$, and just called φ_t a flow.

One particularly important example of a flow, is given by the geodesic flow of a Riemannian metric as defined in the previous section. Recall that the geodesic flow of a Riemannian metric g is the flow given by:

$$\phi_t : TM \to TM, \quad \phi_t(x,v) = (\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t))$$

where $\gamma_{(x,v)}$ is the geodesic for the metric g with initial conditions $\gamma_{(x,v)}(0) = x$ and $\gamma'_{(x,v)}(0) = v$, for $x \in M$ and $v \in T_xM$. Since the speed of the geodesics is constant, we can restrict the flow to the unit tangent bundle T^1M.

1.1.4 Dynamically defined sets

As we mentioned in the introduction, we are interested in studying the orbit structure of dynamical systems. The following subsets play a major role in this study. Let us fix a topological space X and a homeomorphism $f : X \to X$. Given point $x \in X$ we are going to call the *orbit* of x to the set

$$\mathcal{O}(f,x) = \{f^n(x) : n \in \mathbb{Z}\}$$

The points with finite orbit has special interest, in particular the ones with only one point. We denote by

$$Fix(f) = \{x \in X : f(x) = x\}$$

to the set of fixed points and moreover we denote by

$$Per(f) = \{x \in X : f^p(x) = x, \text{ for some } p \in \mathbb{Z}^+\}$$

to the set of periodic points. When the orbit of a point $x \in M$ is an infinite set, we can look at the accumulation points of the orbit (both for the future and for the past). We then define the *omega limit set* and the *alpha limit set* of x as the sets

$$\omega(f,x) = \{y \in M : f^{n_k}(x) \to y \text{ for some } \{n_k\} \subset \mathbb{Z}^+,\ n_k \to +\infty\}$$
$$\alpha(f,x) = \{y \in M : f^{n_k}(x) \to y \text{ for some } \{n_k\} \subset \mathbb{Z}^-,\ n_k \to -\infty\}$$

The union of all this sets is called the *limit set*, that is

$$L(f) = \overline{\bigcup_{x \in X} \alpha(f,x) \cup (f,x)}$$

We say that a point $x \in X$ is *recurrent for the future* if $x \in \omega(f,x)$ and analogously with the past, i.e. a point $x \in X$ is *recurrent for the past* if $x \in \alpha(f,x)$. A little weaker notion of recurrent points is the following. We say that a point $x \in X$ is *non-wandering* if for every neighborhood U of x there is $n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all non-wandering points is called the *non-wandering set* and we note it by $\Omega(f)$, i.e.

$$\Omega(f) = \{x \in X : \forall U \text{ neighborhood of } x, \exists n \in \mathbb{Z} : f^n(U) \cap U \neq \emptyset\}$$

It is easy to see that we have the following inclusions:

$$Fix(f) \subseteq Per(f) \subseteq L(f) \subseteq \Omega(f)$$

These inclusions are not equalities in general, there are many counterexamples for any of the previous inclusions.

Additionally to the previous sets, there are more dynamically defined susbets of a dynamical system, but the previous ones are enough for our purposes.

Notice that we have defined these sets for the discrete case (when f is a homeomorphism), but for the continuous case (for a flow) we have exactly the same subsets, we only have to change the time variable from $\mathbb{Z}$ to $\mathbb{R}$.

1.2 Invariant structures.

Along the thesis, we will be interested in systems which preserve some geometric structure invariant by the dynamics. We begin with the most paradigmatic case, the uniform hyperbolicity.

1.2.1 Uniform hyperbolicity

Definition 1.2.1 (Anosov diffeomorphism)**.** *We say that a diffeomorphism $f : M \to M$ is Anosov or globally hyperbolic, if there exists a Df-invariant splitting $TM = E^s_f \oplus E^u_f$ of the tangent bundle, a Riemannian metric $\|\cdot\|$ and constants $\lambda_s, \lambda_u, C > 0$ with:*

$$0 < \lambda_s < 1 < \lambda_u$$

such that for any $(x,t) \in TM$ and $n > 0$, it holds

$$\begin{aligned} \|D_x f^n(v)\| &< C\lambda_s^n \|v\|, && \text{if } v \in E^s_f(x) \setminus \{0\}, \\ \|D_x f^{-n}(v)\| &< C\lambda_u^{-n} \|v\|, && \text{if } v \in E^u_f(x) \setminus \{0\}. \end{aligned}$$

We call E^s_f and E^u_f the stable and unstable subbundles respectively.

It is usual to ask for the subbundles to be continuous, but this is a direct consequence of the inequalities in Definition 1.2.1. To see this, suppose that $x_n \to x$. By taking a subsequence, we can suppose that $\dim E^s_f(x_n) = k$, for every $n \in \mathbb{N}$. Now take $\{v^n_1, v^n_2, \dots, v^n_k\}$ an orthonormal basis of $E^s_f(x_n)$ and $\{v^n_{k+1}, \dots, v^n_d\}$ an orthonormal basis of $E^u_f(x_n)$. We can assume too that $v^n_j \to v_j$ when $n \to +\infty$. Notice that $\{v_1, \dots, v_k\}$ and $\{v_{k+1}, \dots, v_d\}$ are orthonormal subsets too. Call $E(x) = \langle v_1, \dots, v_k \rangle$ and $F(x) = \langle v_{k+1}, \dots, v_d \rangle$. Now given $v \in E(x)$ with $\|v\| = 1$ we can take a sequence $v_n \in E^s_f(x_n)$, with $\|v_n\| = 1$ converging to v. Then for a fixed $m \in \mathbb{N}$ we have

$$\|Df^m_x(v)\| = \lim_{n\to+\infty} \|Df^m_{x_n}(v_n)\| \leq C\lambda_s^m$$

This implies that $E(x) \subseteq E^s_f(x)$. In the same way we get $F(x) \subseteq E^u_f(x)$ and in particular we have $E \cap F = \{0\}$. This implies that $E(x) = E^s_f(x)$ and $F(x) = E^u_f(x)$ proving the continuity of the bundles.

Now suppose that $\|\cdot\|$ is the Riemannian metric given in the Anosov definition, and let $\|\cdot\|_*$ be another Riemannian metric. Since any two Riemannian metrics on M are equivalent, we know there are constants $\alpha, \beta > 0$ such that $\alpha\|\cdot\| \leq \|\cdot\|_* \leq \beta\|\cdot\|$. Now given $v \in E^s_f(x)$ we have that

$$\|Df^m_x(v)\|_* \leq \beta\|Df^m_x(v)\| \leq C\beta\lambda_s^m\|v\| \leq \frac{C\beta}{\alpha}\lambda_s^m\|v\|_*$$

Hence the bundle E^s_f is uniformly contracting with one metric if and only if it is uniformly contracting with the other metric. We thus have obtained the following remark.

Remark 1.2.2. *Definition 1.2.1 does not depend on the choice of the Riemannian metric.*

The previous remark tell us that we can choose among all Riemannian metrics, to the one that is easier to work with. In particular we will be interested in a metric such that the contraction and expansion is seen at the first step. We can define this special metric in the following way. For vectors $v \in E^s_f(x)$ let's define

$$\|v\|_s := \sum_{n\geq 0} \|Df^n_{f^n(x)}(v)\| > \|v\|$$

Notice that it is well defined since

$$\sum_{n\geq 0} \|Df^n_{f^n(x)}(v)\| \leq \sum_{n\geq 0} C\lambda_s^n \|v\| = \frac{C}{1-\lambda_s}\|v\| < \infty$$

Then we have that $\|v\| < \|v\|_s \leq \frac{C}{1-\lambda_s}\|v\|$ and in particular $\frac{C}{1-\lambda_s} > 1$. Then we have:

$$\|Df_x(v)\|_s = \sum_{n=0}^{\infty} \|Df^n_{f(x)}(v)\| = \|v\|_s - \|v\| \leq \|v\|_s - \frac{1-\lambda_s}{C}\|v\|_s = \left(1 - \frac{1-\lambda_s}{C}\right)\|v\|_s$$

and we see the contraction of the bundle in one step. We can do the same with the unstable bundle E^u_f and obtain a norm $\|\cdot\|_u$. Finally for a vector $v = (v_s, v_u) \in E^s_f \oplus E^u_f$ we define the norm $\|v\|_* = \max\{\|v_s\|_s, \|v_u\|_u\}$. Then we have obtained what is called an *adapted metric*, i.e. a metric where $C = 1$ in Definition 1.2.1.

If we look at flows instead of diffeomorphisms we get the following definition.

Definition 1.2.3 (Anosov flow). *Given a flow $\phi_t : M \to M$ in the manifold M generated by a vector field $X : M \to TM$ we say that it is an Anosov flow if there is a $D\phi$-invariant splitting $TM = E^s \oplus \langle X \rangle \oplus E^u$ of the tangent bundle TM, and constants $\lambda_s, \lambda_u > 0$ with:*

$$0 < \lambda_s < 1 < \lambda_u$$

such that for any $(x,t) \in TM$ and $t \geq 0$, it holds

$$\begin{aligned} \|D_x\phi_t(v)\| &\leq \lambda_s^t \|v\| && \text{if } v \in E^s(x) \setminus \{0\}, \\ \|D_x\phi_{-t}(v)\| &\leq \lambda_u^{-t} \|v\| && \text{if } v \in E^u(x) \setminus \{0\}. \end{aligned}$$

We call E^s and E^u the stable and unstable subbundles respectively.

Recall that every Riemannian metric has a natural dynamical system associated to it; the geodesic flow. Then we can translate the Anosov definition to metrics.

Definition 1.2.4. *We say that a C^∞ Riemannian metric is Anosov, if its corresponding geodesic flow, is an Anosov flow.*

1.2.2 Partial hyperbolicity

In this subsection we introduce the dynamical systems we are going to work with along the thesis. Let us fix a compact Riemannian manifold M of dimension $m \geq 3$. Recall that we denote by Vol the volume form, and we denote by $\|\cdot\|$ the norm on TM associated to the Riemannian metric. There are many defintions in the literature, we are going to use the following.

Definition 1.2.5 (Partial hyperbolicity). *We say that a diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a nontrivial Df-invariant splitting $TM = E^s_f \oplus E^c_f \oplus E^u_f$ of* *the tangent bundle and continuous functions* $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u : M \to \mathbb{R}^+$ *with*

$$\lambda_s < 1 < \lambda_u, \qquad \lambda_s < \lambda_c^- \leq \lambda_c^+ < \lambda_u, \tag{1.3}$$

such that for any $(x, v) \in TM$*, it holds*

$$\begin{aligned} \|D_x f(v)\| &< \lambda_s(x)\|v\|, && \text{if } v \in E^s_f(x) \setminus \{0\}, \\ \lambda_c^-(x)\|v\| < \|D_x f(v)\| &< \lambda_c^+(x)\|v\|, && \text{if } v \in E^c_f(x) \setminus \{0\}, \\ \lambda_u(x)\|v\| &< \|D_x f(v)\|, && \text{if } v \in E^u_f(x) \setminus \{0\}. \end{aligned}$$

As in the Anosov case, partial hyperbolicity does not depend on the choice of the Riemannian metric. Notice that in the definition above, there is no constant $C > 0$ like in the Anosov case. This is because for partially hyperbolic diffeomorphisms there is an adapted metric too. In this case the proof is not that simple, but N. Gourmelon did it on [Gou07] by pushing the same idea above.

For any integer $r \geq 1$, we will denote by $\mathcal{PH}^r(M)$ to the set of all partially hyperbolic diffeomorphisms of M of class C^r; we also denote by $\mathcal{PH}^r(M, \mathrm{Vol}) \subset \mathcal{PH}^r(M)$ to the subset of volume preserving partially hyperbolic diffeomorphisms.

The analogous definition for flows is the following.

Definition 1.2.6 (Partially hyperbolic flow)**.** *We say that a flow* $\varphi_t : M \to M$ *generated by a vector field* $X : M \to TM$ *is partially hyperbolic if there exists a nontrivial* $D\varphi$*-invariant splitting* $TM = E^{ss} \oplus E^c \oplus \langle X \rangle \oplus E^{uu}$ *of the tangent bundle* TM*, a Riemannian metric* $\|\cdot\|$ *and continuous functions* $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u : M \to \mathbb{R}^+$ *with*

$$\lambda_s < 1 < \lambda_u, \qquad \lambda_s < \lambda_c^- \leq \lambda_c^+ < \lambda_u, \tag{1.4}$$

such that for any $(x, v) \in TM$ *and* $t \geq 0$*, it holds*

$$\begin{aligned} \|D_x\varphi_t(v)\| &< \lambda_s(x)^t\|v\|, && \text{if } v \in E^s(x) \setminus \{0\}, \\ \lambda_c^-(x)^t\|v\| < \|D_x\varphi_t(v)\| &< \lambda_c^+(x)^t\|v\|, && \text{if } v \in E^c(x) \setminus \{0\}, \\ \lambda_u(x)^t\|v\| &< \|D_x\varphi_t(v)\|, && \text{if } v \in E^u(x) \setminus \{0\}. \end{aligned}$$

Remark 1.2.7. *If* $\varphi_t : M \to M$ *is a partially hyperbolic flow with* $\dim E^c_\varphi = c$*, then for every* $T \in \mathbb{R}$*, the diffeomorphism* $f := \varphi_T : M \to M$ *is partially hyperbolic with* $\dim E^c_f = c + 1$*.*

Like in Definition 1.2.4 we can translate the partially hyperbolic definition to Riemannian metrics.

Definition 1.2.8. *We say that a* C^∞ *Riemannian metric is partially hyperbolic if its corresponding geodesic flow is partially hyperbolic.*

1.2.3 Dominated splitting

A more general concept than partial hyperbolicity is what is called dominated splitting. It was introduced by Liao and Mañé when working on the stability conjecture.

Definition 1.2.9. *Let* $f : M \to M$ *be a diffeomorphism on a differentiable manifold* M*. We say that* f *has dominated splitting if there exists a* Df*-invariant splitting* $TM = E_1 \oplus \cdots \oplus E_k$ *of the tangent bundle, and constants* $C > 0$ *and* $\lambda \in (0, 1)$ *such that for* $x \in M$ *and every pair of vectors* $v_j \in E_j(x) \setminus \{0\}$ *and* $v_{j+1} \in E_{j+1} \setminus \{0\}$ *and* $n \geq 0$ *it holds*

$$\frac{\|Df^n_x(v_j)\|}{\|v_j\|} \leq C\lambda^n \frac{\|Df^n_x(v_{j+1})\|}{\|v_{j+1}\|} \tag{1.5}$$

Like in the Anosov case, the distributions E_j varies continuously with the point $x \in M$. Moreover, the dominated splitting does not depend on the choice of the Riemannian metric, and it also has an adapted metric such that $C = 1$ [Gou07].

If we call $m(A)$ to the minimum norm or co-norm of a matrix A, then the domination Equation (1.5) above can be expressed by:

$$\|Df^n|_{E_j(x)}\| \leq C\lambda^n m(Df^n|_{E_{j+1}(x)}), \text{ for evey } x \in M,\ n \geq 0$$

Notice that dominated splitting does not necessarily implies there is contraction or expansion in some of the bundles, it just tell us that there are directions which are *dominant* with respect to others. With this new definition, we can say that a partially hyperbolic diffeomorphism is a map with a dominated splitting of the form $TM = E_1 \oplus \cdots \oplus E_k$ and such that the bundle E_1 uniformly contracts, and the bundle E_k uniformly expands.

1.3 Examples

The most common examples of partially hyperbolic systems are automorphisms on torus (or nilmanifolds), time-one maps of geodesic flows (on non-positive curvature) and skew-products over Anosov diffeomorphisms. Recentley new examples on three manifolds were built by the works of Bonatti, Gogolev, Hammerlindl, Parwani and Potrie [BPP16], [BGP16], [Bon+20]. In this section we are going to briefly present some of these examples.

1.3.1 Automorphisms on the torus $\mathbb{T}^d$

Let $A \in \mathrm{SL}(d, \mathbb{Z})$ be a matrix with integer coeficients and determinant one. Since the matrix A is $\mathbb{Z}^d$-invariant, it induces a diffeomorphism f_A in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ by the equation $\Pi \circ A = f_A \circ \Pi$, where $\Pi : \mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection.

Suppose the matrix has a dominated splitting of the form $\mathbb{R}^d = E_A^{ss} \oplus E_A^{ws} \oplus E_A^{wu} \oplus E_A^{uu}$. If we call $E_A^s = E_A^{ss} \oplus E_A^{ws}$ and $E_A^u = E_A^{wu} \oplus E_A^{uu}$, then with the splitting $\mathbb{R}^d = E_A^s \oplus E_A^u$ the induced example f_A is an Anosov diffeomorphism on the torus $\mathbb{T}^d$.

On the other hand we can think the example as a partially hyperbolic diffeomorphisms by taking the center bundle as $E_A^c = E_A^{ws} \oplus E_A^{wu}$. Then with the splitting $R^d = E_A^{ss} \oplus E_A^c \oplus E_A^{uu}$ the map f_A is a partially hyperbolic diffeomorphism.

In the same way, if $A \in \mathrm{SL}(d, \mathbb{Z})$ is a matrix with a splitting of the form $\mathbb{R}^d = E_A^s \oplus E_A^c \oplus E_A^u$, where E_A^c is the generalized eigenspace associated to the eigenvalues of modulus equal to one. Then the example is a partially hyperbolic diffeomorphism.

1.3.2 Automorphisms on nilmanifolds

The example we just saw in the torus can be generalized to nilmanifolds. For our purposes on this thesis we are going to see a specific construction. The example we are going to present appeared for the first time in [Sma67] and it is attributed by S. Smale to A. Borel. The example orginally was presented as an Anosov diffeomorphism in a compact orientable manifold that is not a torus. Years later A. Wilkinson [Wil98] observed that putting together weak sub bundles, one creates a partially hyperbolic diffeomorphism whose center distribution is not integrable. For a more detailed presentation of these examples see [Sma67], [BW08] or [Ham13]. We now give a brief description of this example.

Take $\mathcal{H}$ the Heisenberg group, that is the subgroup of matrices in $\mathrm{SL}(3,\mathbb{R})$ of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

with $x, y, z \in \mathbb{R}$. Identifying (x, y, z) with the upper triangular matrix, the product in $\mathcal{H}$ has the form:

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$$

Then we have that $\mathcal{H}$ is a connected, simply connected, nilpotent Lie group, diffeomorphic to $\mathbb{R}^3$, and it is clearly non abelian. Its corresponding Lie subalgebra $\mathfrak{h}$ is generated by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These matrices satisfy the following relations: $[X, Z] = [Y, Z] = 0$ and $[X, Y] = Z$. If we identify (a, b, c) with $aX + bY + cZ \in \mathfrak{h}$ the exponential map $\exp : \mathfrak{h} \to \mathcal{H}$ is a diffeomorphism and its formula is given by

$$\exp(a, b, c) = \begin{pmatrix} 1 & a & c + \frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Now consider the group $\mathcal{G} = \mathcal{H} \times \mathcal{H}$ with the direct product group structure. We get that $\mathcal{G}$ is a connected, simply connected nilpotent Lie group diffeomorphic to $\mathbb{R}^6$. Its Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ is generated by $\{X_1, Y_1, Z_1, X_2, Y_2, Z_2\}$. Note that the only non-trivial relations are

$$[X_1, Y_1] = Z_1 \text{ and } [X_2, Y_2] = Z_2$$

Now identify $(c, b, a, a', b', c') \in \mathbb{R}^3 \times \mathbb{R}^3$ with $aX_1 + bY_1 + cZ_1 + a'X_2 + b'Y_2 + c'Z_2 \in \mathfrak{g}$. Take a matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ and suppose that $\lambda > 1$ and $\lambda^{-1} < 1$ are their eigenvalues. Now λ and λ^{-1} are units in the ring of integers. The field $\mathbb{Q}(\lambda)$ is a quadratic extension of $\mathbb{Q}$; it's Galois involution σ interchanges λ and λ^{-1}. Now if we take $\tilde{\Gamma} \subset \mathfrak{g}$ as the set of vectors of the form:

$$\left(\frac{1}{2}w, v, u, \sigma(u), \sigma(v), \sigma\left(\frac{1}{2}w\right)\right)$$

with $u, v, w \in \mathbb{Z}[\lambda]$ the ring of algebraic integers in $\mathbb{Q}(\lambda)$. It can be proved that $\tilde{\Gamma}$ is an irreducible and cocompact lattice of $\mathfrak{g}$. Then it's easy to see that $\Gamma = \exp(\tilde{\Gamma})$ is a discrete and cocompact subgroup of $\mathcal{G}$. Now for any pair of real numbers α and β, the linear map B

$$B : (c, b, a, a', b', c') \mapsto (c\lambda^{\alpha+\beta}, b\lambda^{\beta}, a\lambda^{\alpha}, a'\lambda^{-\alpha}, b'\lambda^{-\beta}, c'\lambda^{-\alpha-\beta})$$

is an automorphism of $\mathfrak{g}$ and induces an homomorphism $F_B : \mathcal{G} \to \mathcal{G}$ whose derivative at the identity is B. If $\alpha, \beta \in \mathbb{Z}$ the automorphism B preserves $\tilde{\Gamma}$ and we obtain a diffeomorphism $f_B : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$. If one of $\alpha, \beta, \alpha + \beta$ is non zero, then f_B is partially hyperbolic and if all three are non zero, f_B is Anosov. Assume that $\alpha + \beta > \beta \geq \alpha > 0$. In this case f_B is Anosov: the center bundle is trivial, the stable bundle E^s is generated by X_2, Y_2, Z_2 and the unstable bundle E^u by X_1, Y_1, Z_1. This way we get an Anosov diffeomorphism $f_B : \mathcal{G}/\Gamma \to \mathcal{G}/\Gamma$ on a six dimensional nilmanifold that is not a torus (because its Lie algebra/group is non abelian).

This is the form in which this example originally appeared in [Sma67], but as we mentioned above there are several ways in which one can think about this example. These are the following:

- In [Wil98] A. Wilkinson made the following observation: take the stable bundle E^s generated by Z_2, the unstable bundle E^u is generated by Z_1 and the center bundle E^c generated by the remaining fields X_1, Y_1, X_2 and Y_2. With this splitting f_B is a partially hyperbolic diffeomorphism. The interesting thing about this example is that the center bundle E^c is not integrable because is not closed under the Lie bracket operation: $[X_1, Y_1] = Z_1 \in E^u$.

- A third way of seing this is due to A. Hammerlindl. One chooses the bundle E^u to be generated by Z_1, Y_1 and X_1, the center bundle E^c generated by X_2 and Y_2 and the stable bundle E^s generated by Z_2.

- We can see the example in a fourth way, a much simpler one: the unstable bundle E^u is generated by Z_1, Y_1, the center bundle E^c generated by X_1, X_2 and the stable bundle E^s generated by Y_2 and Z_2.

1.3.3 Geodesic flows

Recall that given a C^∞ Riemannian metric g, we have a flow associated to this metric called the geodesic flow, and it is given by

$$\phi_t : TM \to TM, \quad \phi_t(x, v) = (\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t))$$

where $\gamma_{(x,v)}$ is the geodesic for the metric g with initial conditions $\gamma_{(x,v)}(0) = x$ and $\gamma'_{(x,v)}(0) = v$, for $x \in M$ and $v \in T_xM$. Since the speed of the geodesics is constant, we can restrict the flow to the unit tangent bundle T^1M.

D. A. Anosov showed in [Ano67] that if the seccional curvature of the metric is negative at every point, then the geodesic flow is an Anosov flow (see Definition 1.2.3). This is the most paradigmatic example of an Anosov flow. Recall that the time 1 map of an Anosov flow gives a partially hyperbolic diffeomorphism, and thus we obtain another example with discrete time.

In [CP14] F. Carneiro and E. Pujals built the first examples of C^∞ Riemannian metrics such that their geodesic flows are partially hyperbolic but non Anosov. Moreover, some of these geodesic flows are transitive. Again taking the time 1 map of these flows, we get partially hyperbolic diffeomorphisms.

1.3.4 Direct products and skew-products

Take $A : M \to M$ any of the examples above, and take N another manifold of any dimension. Then the diffeomorphism $f : M \times N \to M \times N$ given by $f = A \times Id$, that is $f(x, y) = (Ax, y)$ is a partially hyperbolic diffeomorphism. Clearly the center bundle is $E^c_f = E^c_A \times N$. This example is called a direct product example.

We can take another kind of product examples. Let $A : M \to M$ be an Anosov diffeomorphism and consider N another compact manifold. Consider an open set $\mathcal{U} \subset \text{Diff}^r(M)$ such that if $h \in \mathcal{U}$ then $f \times h : M \times N \to M \times N$ is partially hyperbolic with fibers $\{x\} \times N$. Let $g : M \to \mathcal{U}$ be a continuous map. For a fixed $x \in M$, denote by g_x to the map $g(x) : N \to N$. Then the diffeomorphism $F : M \times N \to M \times N$ defined by $F(x, y) = (Ax, g_x(y))$ is called a skew-product. By definition F is partially hyperbolic.

1.3.5 Suspension constructions

Take any diffeomorphism $f : M \to M$ on a closed manifold M. Take the product space $M \times \mathbb{R}$ and consider the equivalence relation

$$(x, s_1) \sim (y, s_2) \iff s_1 - s_2 \in \mathbb{Z} \text{ and } f^{s_1 - s_2}(x) = y$$

Denote by $\widehat{M}$ to the quotient space (which is a closed manifold) and $p : M \times \mathbb{R} \to \widehat{M}$ to the canonical projection. Then the flow $\varphi : \mathbb{R} \times M \times \mathbb{R} \to M \times \mathbb{R}$ given by $\varphi(t, (x, s)) := (x, t + s)$ induces a flow $\widehat{\varphi} : \mathbb{R} \times \widehat{M} \to \widehat{M}$ by the equation $\widehat{\varphi}(t, p(x, s)) = p(\varphi(t, (x, s)))$. The flow $\widehat{\varphi}$ is called the *suspension flow*.

Now if the diffeomorphism f is Anosov, then the suspension flow $\widehat{\varphi}$ is an Anosov flow. In the same way, if the map f is partially hyperbolic, then the suspension flow is a partially hyperbolic flow.

1.3.6 Derived from Anosov

The last kind of examples of maps we are going to mention are the *derived from Anosov* diffeomorphisms. These examples are built by deforming a linear Anosov by a specific isotopy, in order to change the index of a given fixed point, but keeping the partially hyperbolic structure. These maps were introduced by R. Mañé in [Mañ78]. In Section 3.4 we are going to see this kind of examples in detail.

1.4 Integrability of distributions

By a k-dimensional C^0 foliation $\mathcal{F}$ with C^1 leaves we mean a partition of the manifold M into k-dimensional, complete, connected C^1 submanifolds $\mathcal{F}(x)$ that depends continuously with the point $x \in M$. Another way of saying this, is that for every point $x \in M$ there is a neighbourhood U and a homeomorphism $\varphi : \mathbb{D}^k \times \mathbb{D}^{d-k} \to U$ such that for each $y \in \mathbb{D}^{d-k}$ the set $\mathcal{F}_U(\varphi(-, y)) := \varphi(\mathbb{D}^k, y)$ (called the *local leaf*) is contained in $\mathcal{F}(\varphi(0, y))$ and $\varphi(\cdot, y) : \mathbb{D}^k \to \mathcal{F}_U(\varphi(0, y))$ is a C^1 diffeomorphism which depends continuously on $y \in \mathbb{D}^{d-k}$ in the C^1 topology.

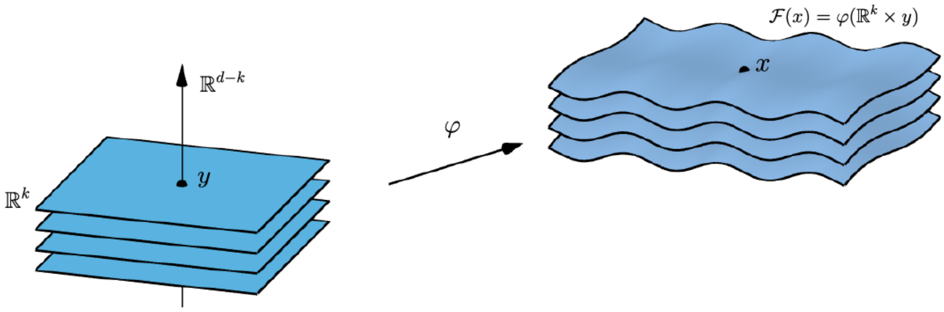


FIGURE 1.1: Foliation $\mathcal{F}$

When the local chart (U, φ) at x can be chosen C^r with C^l leaves we say $\mathcal{F}$ is a C^r foliation with C^l leaves. We remark that the regularity of the local chart is always smaller or equal to the regularity of the leaves ($l \geq r$).

Given a k-dimensional distribution $E \subset TM$, we say that E is *integrable* if there exists a C^0 foliation $\mathcal{F}$ with C^1 leaves which are everywhere tangent to E, i.e. $T_x\mathcal{F}(x) = E(x)$ for every $x \in M$. We call such a foliation an *integral foliation* of E. We say that E is *uniquely integrable* if it's integrable with integral foliation $\mathcal{F}$ and in addition any C^1 curve everywhere tangent to E lies on a single leaf of $\mathcal{F}$, i.e. every $\alpha : I \to M$ satisfying $\alpha'(t) \in E(\alpha(t))$ for every $t \in I$, is contained in $\mathcal{F}(\alpha(0))$. Notice that unique integrability implies that E has a unique integral foliation, although the reciprocal is not true. The typical example is given by the distribution E tangent to the foliation $\mathcal{F}$ on the real plane $\mathbb{R}^2$ given by leaves of form $\{(t, (t+c)^3) : t \in \mathbb{R}\}$. Despite being $\mathcal{F}$ the only integral foliation tangent to E, it is not uniquely integrable: the curve $\{(t, 0) : t \in \mathbb{R}\}$ is tangent to E but it doesn't belong to any leaf of $\mathcal{F}$. Thus unique integrability is slightly stronger than having a unique integral foliation.

When the distribution E is C^1, that is when the local chart can be chosen C^1, the problem of integrability was solved by G. Frobenius, who proved that a C^1 distribution E is uniquely integrable if the distribution E is closed by the Lie bracket operation, i.e. for every pair of vector fields X, Y on M such that $X, Y \in E$, we have that $[X, Y] \in E$. Therefore in the differentiable case it's enough to see how the Lie bracket behaves in order to get integrability. A proof of Frobenius Theorem can be found in [War71].

However in our context the distributions are only continuous and therefore another techniques are needed to get integrability (the Lie bracket doesn't make any sense). To be more precise it is well known that if $f \in \mathcal{PH}(M)$, then the stable and unstable bundles E^s_f and E^u_f are only Hölder continuous. Nevertheless the celebrated stable manifold theorem says that the strong bundles E^u_f and E^s_f are uniquely integrable ([HPS77]). Their corrresponding unique integral foliations are called the *strong unstable* and *strong stable* foliations respectively, and we note them by $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$. We want to remark that in general the stable/unstable foliations are not C^1 even if the diffeomorphism is highly regular: in [Ano67] there is an example of a C^∞ Anosov diffeomorphism whose distributions are only Hölder continuous.

Notice that since E^u_f and E^s_f are Df-invariant, then unique integrability (or having a unique integral foliation) implies that their corresponding integral foliations $\mathcal{W}^*_f$ are invariant under the dynamics, i.e., $f(\mathcal{W}^*_f(\cdot)) = \mathcal{W}^*_f(f(\cdot))$ for $* = u, s$.

Despite the stable manifold theorem, we don't have *a priori* integrability of the rest of the bundles E^{cs}_f, E^{cu}_f and E^c_f. This fact leads to the following definition.

Definition 1.4.1 (Dynamical coherence)**.** *A partially hyperbolic diffeomorphism f is dynamically coherent if the center-unstable bundle $E^{cu}_f := E^c_f \oplus E^u_f$ and the center-stable bundle $E^{cs}_f := E^c_f \oplus E^s_f$ are integrable. Their corresponding integral foliations are called the center-unstable foliation, resp. the center-stable foliation and are noted by $\mathcal{W}^{cu}_f$, $\mathcal{W}^{cs}_f$.*

Notice that dynamical coherence implies that the center distribution E^c_f is integrable too: if $f \in \mathcal{PH}(M)$ is dynamically coherent, then for any $x \in M$ the set $\mathcal{W}^c_f(x) := \mathcal{W}^{cs}_f(x) \cap \mathcal{W}^{cu}_f(x)$ integrates E^c_f and we call $\mathcal{W}^c_f$ the center foliation. On the other hand, the integrability of E^c_f does not imply dynamically coherence: if E^c_f integrates into $\mathcal{W}^c_f$ and if we take $\mathcal{W}^{cs}_f(x) = \cup_{y \in \mathcal{W}^c_f(x)} \mathcal{W}^s_f(y)$ we obtain a plaque tangent to $E^{cs}_f(x)$ but the union of this plaques is not going to be a foliation necessary.

We do have the following proposition.

Proposition 1.4.2 (Proposition 2.4 in [BW08]). *Let $f \in \mathcal{PH}(M)$ be dynamically coherent. Then, the foliations $\mathcal{W}^u_f$ and $\mathcal{W}^c_f$ subfoliate $\mathcal{W}^{cu}_f$, while $\mathcal{W}^s_f$ and $\mathcal{W}^c_f$ subfoliate $\mathcal{W}^{cs}_f$.*

The previous proposition was included in the original definition of dynamically coherent in [PS97] but it was soon realized to be a consequence of integrability. In [BW08] there is a long discussion about all the possible definitions of dynamical coherence that have been used since its introduction and every implications between them. We want to remark also that we don't require unique integrability of the bundles E^{cs}_f and E^{cu}_f in the definition above, although every known example of dynamical coherence is uniquely integrable.

It is an open question whether dynamical coherence is a C^1-open condition among $\mathcal{PH}(M)$. A closely related property is *plaque expansiveness*. Before introducing it, we need another definition.

Definition 1.4.3. *Given $\epsilon > 0$ we say that a sequence $\{x_n\}_{n\in\mathbb{Z}} \subset M$ is a ϵ-pseudo orbit with respect to f if $d(f(x_n), x_{n+1}) < \epsilon$ for every $n \in \mathbb{Z}$. In addition, we say that the pseudo orbit respects $\mathcal{W}^c_f$ if $f(x_n) \in \mathcal{W}^c_f(x_{n+1})$ for every $n \in \mathbb{Z}$.*

Definition 1.4.4 (Plaque expansiveness). *We say that $f \in \mathcal{PH}(M)$ is plaque expansive (see* [HPS77, Section 7]*) if f is dynamically coherent and there exists $\varepsilon > 0$ with the following property: if $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are ε-pseudo orbits which respect $\mathcal{W}^c_f$ and such that $d(p_n, q_n) \leq \varepsilon$ for all $n \geq 0$, then $q_n \in \mathcal{W}^c_f(p_n)$.*

It is known that plaque expansiveness is a C^1-open condition (see Theorem 7.4 in [HPS77]). The importance of plaque expansivity lies on the following theorem.

Theorem 1.4.5 (Theorem 7.1 [HPS77], see also Theorem 1 in [PSW12]). *Let us assume that f is dynamically coherent and plaque expansive. Then any $g \in \mathcal{PH}^1(M)$ which is sufficiently C^1-close to f is also dynamically coherent and plaque expansive. Moreover, there exists a homeomorphism $\mathfrak{h} = \mathfrak{h}_g : M \to M$, called a leaf conjugacy, such that $\mathfrak{h}$ maps a f-center leaf to a g-center leaf, and $\mathfrak{h} \circ f(\mathcal{W}^c_f(\cdot)) = g \circ \mathfrak{h}(\mathcal{W}^c_f(\cdot))$.*

As a result, every $f \in \mathcal{PH}(M)$ dynamically coherent and plaque-expansive is C^1 stably dynamically coherent. The problem then, is to decide when a partially hyperbolic diffeomorphism is plaque exapansive. This problem is open in its full generality although plaque expansivity has been obtained in several cases:

- when the center foliation $\mathcal{W}^c_f$ is C^1 (or E^c_f is C^1 or both E^{cs}_f and E^{cu}_f are C^1) this was proved in [HPS77].
- when $Df|_{E^c_f}$ is an isometry this was proved in [HHU07], originally mentioned in [HPS77] without proof.
- when the center foliation $\mathcal{W}^c_f$ is uniformly compact, i.e. every center leaf is compact and there is a uniform bound on the volumes, this was proved in [Car11]

Notice that Examples 1.3.1, 1.3.3, 1.3.4 and 1.3.5 mentioned in Section 1.3 are dynamically coherent and fall into one of the previous cases, hence each one of these examples is stably dynamically coherent. In Chapter 2 we are going to treat the three possible cases of Example 1.3.2.

We finish this section by adding another important definition (which we already mentioned in the introduction) that arises from Theorem 1.4.5 and it is related with the topological stability of a partially hyperbolic diffeomorphism.

Definition 1.4.6 (Leaf conjugacy). *We say that two dynamically coherent partially hyperbolic diffeomorphisms $f, g : M \to M$ are leaf conjugate if there exists a homeomorphism $\mathfrak{h} \colon M \to M$, called a leaf conjugacy, such that $\mathfrak{h}$ maps a f-center leaf to a g-center leaf, and $\mathfrak{h} \circ f(\mathcal{W}^c_f(\cdot)) = g \circ \mathfrak{h}(\mathcal{W}^c_f(\cdot))$.*

Leaf conjugacy is the analogous to topological conjugacy for Anosov diffeomorphisms in the partially hyperbolic case (notice we need dynamically coherence for this definition to make sense). Then by Theorem 1.4.5 every $f \in \mathcal{PH}(M)$ dynamically coherent and plaque expansive is topologically stable in the sense mentioned above.

1.5 Holonomies

Let us assume that f is a partially hyperbolic dynamically coherent diffeomorphism. In the following, for any $* \in \{s, c, u, cs, cu\}$ we denote by $d_{\mathcal{W}^*_f}$ the leafwise distance, and for any $x \in M$ and for any $\epsilon > 0$, we denote by

$$\mathcal{W}^*_f(x, \varepsilon) := \{y \in \mathcal{W}^*_f(x) : d_{\mathcal{W}^*_f}(x, y) < \varepsilon\}$$

the ε-ball in $\mathcal{W}^*_f$ of center x and radius ε.

Take $x_1 \in M$ and let $x_2 \in \mathcal{W}^s_f(x_1)$. By transversality, there are neighbourhoods $\mathcal{U}^{cu}_1$ of x_1 in $\mathcal{W}^{cu}_f(x_1)$ and $\mathcal{U}^{cu}_2$ of x_2 in $\mathcal{W}^{cu}_f(x_2)$ such that for any $z \in \mathcal{U}^{cu}_1$, the local stable leaf through z intersects $\mathcal{U}^{cu}_2$ at a unique point, denoted by $H^s_{f,x_1,x_2}(z) \in \mathcal{U}^{cu}_2$. We thus get a well defined local homeomorphism

$$H^s_{f,x_1,x_2} \colon \mathcal{U}^{cu}_1 \to \mathcal{U}^{cu}_2$$

called the *stable holonomy map*. Since f is dynamically coherent the image of the restriction $H^s_{f,x_1,x_2}|_{\mathcal{U}^{cu}_1 \cap \mathcal{W}^c_f(x_1)}$ to the center leaf $\mathcal{W}^c_f(x_1)$ is contained in the center leaf $\mathcal{W}^c_f(x_2)$. We define the unstable holonomy in the same way.

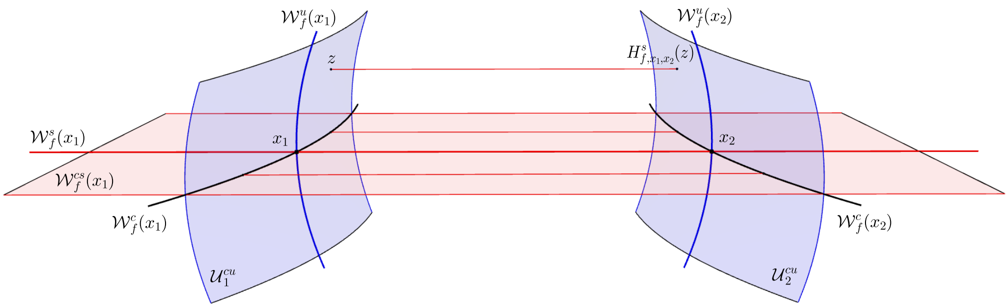


FIGURE 1.2: Stable holonomy

Notice that *a priori* we have no additional information about the regularity of these holonomies beyond continuity. In next subsections we'll see that under some extra hypothesys, we can provide more regularity to the holonomies.

1.5.1 θ-pinching

Definition 1.5.1 (θ-pinching). *Let $f \in \mathcal{PH}(M)$ with functions $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u$ as in Definition 1.2.5. We say that f is θ-pinching for some $\theta \in (0, 1)$ if there are constants $\overline{\lambda_u}$ and $\overline{\lambda_s}$*

which verifies $\overline{\lambda_s} < \|Df^{-1}\|^{-1} \leq \|Df\| < \overline{\lambda_u}$ *and such that*

$$\overline{\lambda_s}^{\theta} < \frac{\lambda_u}{\lambda_c^+} \text{ and } \overline{\lambda_u}^{\theta} < \frac{\lambda_c^-}{\lambda_s} \tag{1.6}$$

Notice that given any $f \in \mathcal{PH}(M)$ with functions $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u$, we can take $\overline{\lambda_u} = \max_{x \in M}\{\|Df_x\|\}$ and $\overline{\lambda_s} = \min_{x\in M}\{\|Df_x^{-1}\|\}$. Then for θ sufficiently close to 0 we always get the θ-pinching condition in Equation (1.6). Therefore, every partially hyperbolic diffeomorphism is θ-pinching for some $\theta \in (0,1)$, possibly close to 0.

The importance of the pinching condition comes from the following theorem which relates the pinching condition with the regularity of the u, s holonomies.

Theorem 1.5.2 (Theorem A in [PSW97]). *If $f \in \mathcal{PH}^1(M)$ satisfies the pinching condition for some $\theta \in (0,1)$, then local stable/unstable holonomy maps between center leaves are uniformly θ-Hölder.*

1.5.2 Center bunching

We can ask for a little stronger condition on the derivatives than pinching (which is always satisfied for every $f \in \mathcal{PH}(M)$).

Definition 1.5.3 (Center bunching). *We say that $f \in \mathcal{PH}(M)$ is center bunched if the functions $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u$ in (1.3) can be chosen such that*

$$\max(\lambda_s, (\lambda_u)^{-1}) < \frac{\lambda_c^-}{\lambda_c^+} \tag{1.7}$$

Unlike the pinching condition, not every partially hyperbolic diffeomorphism is center-bunched, but every known example is arbitrarily C^1 close to a center bunched one. The analogous result relating the bunching condition and the regularity of the u, s holonomies is again due to C. Pugh, M. Shub and A. Wilkinson.

Theorem 1.5.4 (Theorem B in [PSW97]). *If $f \in \mathcal{PH}^2(M)$ is dynamically coherent and center bunched, then local stable/unstable holonomy maps between center leaves are C^1 when restricted to some center-stable/center-unstable leaf.*

1.6 Accessibility

Given $f \in \mathcal{PH}(M)$, a *f-accessibility sequence* is a sequence $[x_1, \ldots, x_k]$ of $k \geq 1$ points in M such that for any $i \in \{1, \ldots, k-1\}$, the points x_i and x_{i+1} belong to the same stable or unstable leaf of f.

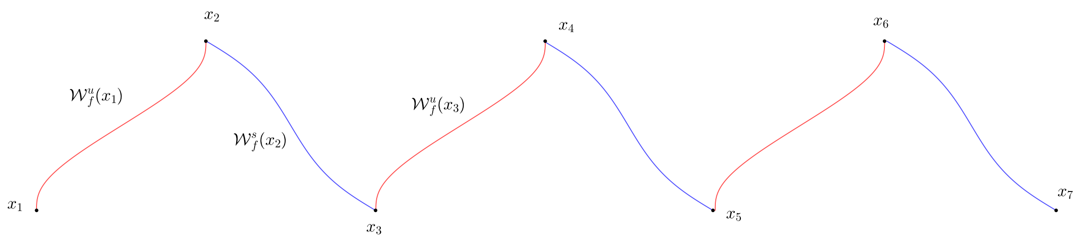


FIGURE 1.3: A f-accessibility sequence $[x_1, x_2, \ldots, x_7]$

In particular, the points x_1 and x_k can be connected by some f-*path*, i.e., a continuous path in M obtained by concatenating finitely many arcs in $\mathcal{W}^s_f$ or $\mathcal{W}^u_f$. We will refer to the points $x_1, \dots, x_k$ as the *corners* of the accessibility sequence $[x_1, \dots, x_k]$. We say that two points belonging to the same f-path are su-related. This is an equivalence relationship and their equivalence classes are called *accessibility classes*. That is

$$\mathrm{Acc}_f(x) := \{y \in M : \text{there is a } f\text{-path from } x \text{ to } y\}$$

Notice that you can have infinitely many equivalence classes, for example if we take $f = A \times Id : \mathbb{T}^n \times N \to \mathbb{T}^n \times N$ like Example 1.3.4, then $\mathrm{Acc}_f(x) = p_1(x) \times N$ for every $x \in \mathbb{T}^n \times N$ (where $p_1 : \mathbb{T}^n \times N \to \mathbb{T}^n$ is the projection on the first coordinate) and we have as many classes as points in the torus $\mathbb{T}^n$. On the other hand, the time-one map of the geodesic flow of a surface of constant negative curvature has only one accessibility class. We will be interested in maps having this last property.

Definition 1.6.1 (Accessibility). *We say that $f \in \mathcal{PH}(M)$ is accessible if there is only one accessibility class, i.e. there is $x \in M$ such that $\mathrm{Acc}_f(x) = M$.*

Moreover we will be interested in maps which are accessible and such that every map in a sufficiently small neighbourhood is accessible too.

Definition 1.6.2 (Stable accessibility). *We say that $f \in \mathcal{PH}(M)$ is stably accessible if there exists $\mathcal{U}$ a C^1 neighbourhood of f such that every $g \in \mathcal{U}$ is accessible.*

1.6.1 Center accessibility classes

When f is dynamically coherent, we define the *center accessibility class of x* as the set

$$C_f(x) := \mathrm{cc}(\mathrm{Acc}_f(x) \cap \mathcal{W}^c_f(x, 1), x)$$

i.e. the connected component containing x of the intersection of the accessibility class of x and the local center leaf through x. Similarly, for any $\varepsilon > 0$, we let $C_f(x, \varepsilon) := \mathrm{cc}(\mathrm{Acc}_f(x) \cap \mathcal{W}^c_f(x, \varepsilon), x)$. Therefore, instead of looking at the accessibility class on M, we can look at the accessibility class inside the center leaf. This naturally decreases the difficulty of classifying the accessibility classes. In particular for open accessibility classes, this idea is reflected in the following lemma.

Lemma 1.6.3. *The following are equivalent:*

1. $\mathrm{Acc}_f(x)$ *is an open subset.*
2. $\mathrm{Acc}_f(x)$ *has non-empty interior.*
3. $C_f(x)$ *is an open subset of* $\mathcal{W}^c_f(x)$.
4. $C_f(x)$ *has non-empty interior (in* $\mathcal{W}^c_f(x)$*).*

Proof. If we have 1, then we trivially have 2, 3 and 4. On the other hand if $C_f(x)$ is an open subset in $\mathcal{W}^c_f(x)$ then we can saturate this set by stable and unstable leaves, and by local product structure we obtain an open set on M. The proofs of the other equivalences are basically the same. □

As a result, if we want to determine the "shape" or the topology of a given accessibility class, it is a better idea to see what is the structure on the center leaf. Notice that center accessibility classes are connected subsets of the center leaves, but classify connected subsets is an immeasurable problem in general. However in lower dimensions there are a lot of important results.

Case dim$E^c = 1$

Suppose that we have dim$E^c_f = 1$. We will see that the simple topology of $\mathbb{R}$ has strong consequences in the structure of accessibility classes. Fix a sufficiently small $\sigma > 0$ in order to have local product structure. Given a point $x \in M$ we take the following points: $x_1 \in \mathcal{W}^u_f(x, \sigma/10)$, $x_2 \in \mathcal{W}^s_f(x_1, \sigma/10)$, $x_3 = \mathcal{W}^u_f(x_2, \sigma) \cap \mathcal{W}^{cs}_f(x, \sigma)$, and $x_4 = \mathcal{W}^s_f(x_3, \sigma) \cap \mathcal{W}^c_f(x, \sigma)$. This gives a f-accessibility sequence $[x, x_1, x_2, x_3, x_4]$. By continuously decreasing the size of the first two legs (i.e. making σ goes to 0), we obtain that $x_4 \in C_f(x)$. Since $\mathcal{W}^c_f(x)$ is 1-dimensional and the connected subsets of the real line are only points or segments we have only two possibilities:

- $x_4 = x$ for every f-accessibility sequence $[x, x_1, x_2, x_3, x_4]$,
- $x_4 \neq x$ for some f-accessibility sequence $[x, x_1, x_2, x_3, x_4]$

The first case implies that the bundle $E^{su}_f = E^s_f \oplus E^u_f$ is integrable, this is known as *trivial* accessibility class (see the left image on Figure 1.4 below). The second case implies that $C_f(x)$ contains a non-trivial interval and hence a non-empty interior. Then by Lemma 1.6.3 we get that $\mathrm{Acc}_f(x)$ is open (see the right image on Figure 1.4).

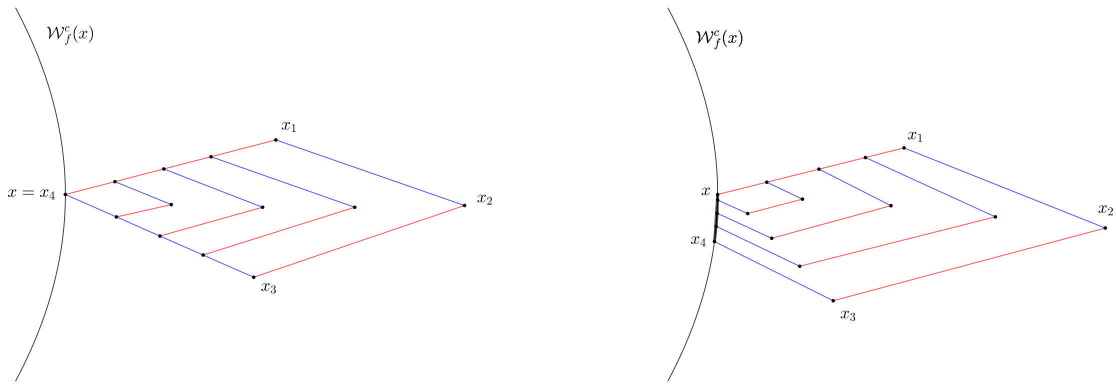


FIGURE 1.4: Dichotomy between center accessibility classes: Integrability of E^{su} bundle vs open classes

To sum up, when the center bundle is 1-dimensional we have a dichotomy for accessibility classes: either we have trivial accessibility classes (E^{su}_f is integrable), or we have open accessibility classes. In particular, in order to open a trivial accessibility class, it is enough to break the integrability of E^{su}_f bundle. This idea was exploited by Ph. Didier to obtain stability of accessibility in [Did03], and by J. Rodriguez-Hertz, F. Rodriguez-Hertz and R. Ures in [HHU08] where they proved the Pugh-Shub conjecture in the 1 dimensional center case.

Case dim$E^c = 2$

We now investigate the structure of accessibility classes when the center bundle has dimension two, that is $f \in \mathcal{PH}(M)$ and dim$E^c_f = 2$. We can make the same construction of a f-accessibility sequence $[x, x_1, x_2, x_3, x_4]$ as above, but in this case we don't have a dichotomy between center accessibility classes because the connected subsets of a 2-dimensional plane can be very complicated. However in [Her05] F. Rodriguez-Hertz made the following remarkable result.

Theorem 1.6.4 (Proposition 5.2 in [Her05]). *Let $f \in \mathcal{PH}(M)$ and assume $\dim E^c_f = 2$. Let $x \in M$. Then one and only one of the following holds:*

- $C_f(x)$ *is open.*
- $C_f(x) = x$.
- $C_f(x)$ *is a topological one dimensional manifold.*

This classification result of center accessibility classes was used by F. Rodriguez-Hertz to prove that certain linear automorphisms on the torus $\mathbb{T}^N$ with 2-dimensional center are C^5 stably ergodic. Later in [HS17] V. Horita and M. Sambarino used this classification to prove the Pugh-Shub conjecture for skew-product surface diffeomorphisms over Anosov.

Case dim$E^c \geq 3$

When $\dim E^c_f \geq 3$ there is no information about the center accessibility classes. It has been conjectured in [Wil13] (Conjecture 1.3) that the same phenomenon on the 1 and 2 dimensional center case, occurs in any case, for any dimension of the center bundle. For example in the $\dim E^c_f = 3$ case, it should be reasonable to expect that given a point $x \in M$ the center accessibility class $C_f(x)$ should be point, a continuous curve, a topological plane or an open set. This however, is an open problem until now, although it seems reasonable to be true.

C^1 homogeneity

The classification of center accessibility classes is also motivated by the following fact. Take two points $x, y \in M$ that are su-related by a f-accessibility sequence $[x_1, \ldots, x_k]$ such that $x = x_1$ and $y = x_k$. For $j \in \{1, \ldots, k-1\}$ we let

$$H^{*_{j-1}}_{f,x_{j-1},x_j} \colon \mathcal{W}^c_{f,\mathrm{loc}}(x_{j-1}) \to \mathcal{W}^c_{f,\mathrm{loc}}(x_j)$$

be the holonomy map where $*_j \in \{s, u\}$ is such that $x_{j+1} \in \mathcal{W}^{*_j}_f(x_j)$.

By concatenating these local holonomy maps along the arcs of γ we get a well defined map $H_{f,\gamma} \colon \mathcal{W}^c_{f,\mathrm{loc}}(x_1) \to \mathcal{W}^c_{f,\mathrm{loc}}(x_k)$, i.e.,

$$H_{f,\gamma} := H^{*_{k-1}}_{f,x_{k-1},x_k} \circ \cdots \circ H^{*_1}_{f,x_1,x_2}$$

Notice that by definition, for every $z \in \mathcal{W}^c_{f,\mathrm{loc}}(x)$ we have that $H_{f,\gamma}(z) \in \mathrm{Acc}_f(z)$. In particular $H_{f,\gamma}(C_f(x) \cap \mathcal{W}^c_{f,\mathrm{loc}}(x)) = C_f(y) \cap \mathcal{W}^c_{f,\mathrm{loc}}(y)$. This motivates the following definition.

Definition 1.6.5. *Let M be a Riemannian manifold. A subset $N \subset M$ is said to be C^r-homogeneous, if for every pair of points $x, y \in N$ there are neighborhoods U_x, U_y and a C^r-diffeomorphism $\varphi : U_x \to U_y$ such that $\varphi(U_x \cap N) = U_y \cap N$ and $\varphi(x) = y$.*

By the previous observation we have that center accessibility classes are C^0-homogeneous subsets because the holonomy maps between center leaves are continuous (in fact, local homeomorphisms). Now if we ask for $f \in \mathcal{PH}(M)$ to be center bunched (see Definition 1.5.3) then the holonomy maps $H^{*_{j-1}}_{f,x_{j-1},x_j} : \mathcal{W}^c_{f,\text{loc}}(x_{j-1}) \to \mathcal{W}^c_{f,\text{loc}}(x_j)$ are C^1 according to Theorem 1.5.4. Therefore, the map $H_{f,\gamma}$ above is C^1 and the center accessibility classes are C^1-homogeneous sets.

Given a manifold M, the most common example of a C^1-homogeneous set is a C^1-submanifold $N \subset M$. Then the question on the opposite direction becomes natural: is a C^1-homogeneous set necessarily a C^1 submanifold? In [RSS96] the authors partially answered the question by proving that every C^1 homogeneous set that is locally compact, must be a C^1-submanifold.

To sum up, if $f \in \mathcal{PH}(M)$ is center bunched, then the holonomies are C^1, and in consequence center accessibility classes are C^1 homogeneous subsets. In particular if the center accessibility class is 1-dimensional (a topological curve like in the classification of [Her05]), then it is in fact a C^1 curve. This is what Horita and Sambarino used in [HS17] to prove that 1 dimensional center accessibility classes form a C^1 lamination.

Chapter 2

Dynamical coherence of partially hyperbolic isotopic to fibered PH

In this chapter we are going to prove Theorems A and B. In Section 2.1 we introduce a few definitons and some classical well known results, and then we restate the main theorems in a more specific setting. In Section 2.2 we prove an integrability criterion for partially hyperbolic diffeomorphisms isotopic to fibered partially hyperbolic diffeomorphisms. In Section 2.3 we obtain dynamically coherence in the whole isotopy class of a fibered partially hyperbolic diffeomorphism and prove Theorem A. Finally in Section 2.4 we deal with leaf conjugacy and prove Theorem B.

2.1 Preliminaries

2.1.1 Definitions and notations

Let $f : X \to X$ be a homeomorphism on a metric space $(X, dist)$. We define the *stable set* and the *stable set of size* ϵ of a point $x \in X$ as the sets:

$$\mathcal{W}^s_f(x) = \{y \in X : dist(f^n(x), f^n(y)) \to_{n\to+\infty} 0\}$$
$$\mathcal{W}^s_f(x,\epsilon) = \{y \in X : dist(f^n(x), f^n(y)) < \epsilon \text{ for all } n \in \mathbb{N}\}$$

In the same way but looking at the past, we define the *unstable set* and the *unstable set of size* ϵ of a point $x \in X$ as the sets:

$$\mathcal{W}^u_f(x) = \{y \in X : dist(f^{-n}(x), f^{-n}(y)) \to_{n\to+\infty} 0\}$$
$$\mathcal{W}^u_f(x,\epsilon) = \{y \in X : dist(f^{-n}(x), f^{-n}(y)) < \epsilon \text{ for all } n \in \mathbb{N}\}$$

Definition 2.1.1 (Hyperbolic homeomorphisms)**.** *We say that a homeomorphism $f : X \to X$ on a metric space $(X, dist)$ is uniformly hyperbolic if there are constants $C > 0$, $\lambda > 1$, $\epsilon > 0$ and $\delta > 0$ such that:*

1. *$dist(f^n(x_1), f^n(x_2)) \leq C\lambda^{-n} dist(x_1, x_2)$, for all $x_1, x_2 \in \mathcal{W}^s_f(x,\epsilon)$, $n \geq 0$.*
2. *$dist(f^{-n}(x_1), f^{-n}(x_2)) \leq C\lambda^{-n} dist(x_1, x_2)$, for all $x_1, x_2 \in \mathcal{W}^u_f(x,\epsilon)$, $n \geq 0$.*
3. *if $dist(x_1, x_2) < \delta$ then $\mathcal{W}^s_f(x_1,\epsilon)$ and $\mathcal{W}^u_f(x_2,\epsilon)$ intersect at exactly one point denoted by $[x_1, x_2]$ and this point depends continuoulsy with $(x_1, x_2) \in X \times X$.*

Originally it was said that a homeomorphism f satisfying the previous definition had *hyperbolic coordinates*. In [Mañ87b] (Chapter IV, Section 9) R. Mañé presents a slightly different definition of a hyperbolic homeomorphism that the one we stated, but later J. Ombach observed in [Omb96] that they are both equivalent. We state it this way because we think is the most natural one. Hyperbolic coordinates or hyperbolic homeomorphisms, appeared on the seventies in the attempt to give a topological description of the concept of hyperbolicity (see [Wal78], [Omb86], [Omb87], [Omb96]). Notice that Definition 2.1.1 is purely topological.

Now recall that according to Definition 1.4.1 from Chapter 1, we say that a partially hyperbolic diffeomorphism $f : M \to M$ with a splitting $TM = E^{ss}_f \oplus E^c_f \oplus E^{uu}_f$ is dynamically coherent if the center-unstable bundle $E^{cu}_f := E^c_f \oplus E^{uu}_f$ and the center-stable bundle $E^{cs}_f := E^{ss}_f \oplus E^c_f$ integrate respectively to invariant foliations $\mathcal{W}^{cu}_f$, $\mathcal{W}^{cs}_f$ called the center-unstable and the center-stable foliation respectively. This implies in addition, that we have a center foliation $\mathcal{W}^c_f(x) := \mathcal{W}^{cs}_f(x) \cap \mathcal{W}^{cu}_f(x)$ which is also f-invariant and tangent to E^c_f.

This center foliation $\mathcal{W}^c_f$ gives a partition of the manifold M and thus we have a well defined quotient space $M/\mathcal{W}^c_f$. We are going to note by $p : M \to M/\mathcal{W}^c_f$ to the projection into equivalence classes. Moreover, we have an induced map in the quotient space:

$$f_c : M/\mathcal{W}^c_f \to M/\mathcal{W}^c_f \quad \text{given by} \quad p \circ f = f_c \circ p$$

The idea of this map is to "cancel" the non-hyperbolic behaviour of the partially hyperbolic diffeomorphism f in order to get some hyperbolicity in the quotient space.

We have the same behaviour on the universal cover. Let $\pi : \widetilde{M} \to M$ be the universal cover of M and recall that $M = \widetilde{M}/\Gamma$ where $\Gamma = \pi_1(M)$ acts on $\widetilde{M}$ by isometries. In this case we have that $\widetilde{\mathcal{W}^c_f}$ gives a partition of the manifold $\widetilde{M}$ and we have a well defined quotient space $\widetilde{M}/\widetilde{\mathcal{W}^c_f}$. We are going to note by $\widetilde{p} : \widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{W}^c_f}$ to the projection into equivalence classes, and its corresponding induced map will be:

$$\widetilde{f}_c : \widetilde{M}/\widetilde{\mathcal{W}^c_f} \to \widetilde{M}/\widetilde{\mathcal{W}^c_f} \quad \text{given by} \quad \widetilde{p} \circ \widetilde{f} = \widetilde{f}_c \circ \widetilde{p}$$

Additionally since the center leaves are invariant by the dynamics, we can define a function $\pi_c : \widetilde{M}/\widetilde{\mathcal{W}^c_f} \to M/\mathcal{W}^c_f$ by the equation $\pi_c \circ \widetilde{p} = p \circ \pi$. It is clear that this map is well defined because if $\widetilde{x} \in \widetilde{M}$ then $\widetilde{p}(\widetilde{x}) = \widetilde{\mathcal{W}^c_f}(\widetilde{x})$ and therefore

$$\pi_c \circ \widetilde{p}(\widetilde{x}) = \pi_c(\widetilde{\mathcal{W}^c_f}(\widetilde{x})) = \mathcal{W}^c_f(\pi(\widetilde{x})) = \mathcal{W}^c_f(x) = p(x) = p \circ \pi(\widetilde{x})$$

Notice that *a priori*, nothing tell us that the previous quotient spaces will have nice properties. The following is the main object of this chapter.

Definition 2.1.2 (Fibered partially hyperbolic)**.** *Let $f : M \to M$ be a dynamically coherent partially hyperbolic diffeomorphism of class C^r. We say that f is fibered if:*

1. *the foliations $\widetilde{\mathcal{W}^{cs}_f}$ and $\widetilde{\mathcal{W}^{uu}_f}$ have global product structure;*
   the foliations $\widetilde{\mathcal{W}^{cu}_f}$ and $\widetilde{\mathcal{W}^{ss}_f}$ have global product structure.
2. *For every $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ we have that $d_H(\widetilde{\mathcal{W}^c_f}(\widetilde{x}), \widetilde{\mathcal{W}^c_f}(\widetilde{y})) < \infty$. As a consequence, the Hausdorff distance in $\widetilde{M}$ induces a distance (noted by dist) in the quotient space $\widetilde{M}/\widetilde{\mathcal{W}^c_f}$.*
3. *the map $\widetilde{f}_c : \widetilde{M}/\widetilde{\mathcal{W}^c_f} \to \widetilde{M}/\widetilde{\mathcal{W}^c_f}$ is a hyperbolic homeomorphism.*
4. *there exists a linear Anosov $A : \mathbb{R}^{d-c} \to \mathbb{R}^{d-c}$ with a splitting $\mathbb{R}^{d-c} = E^{ss}_A \oplus E^{uu}_A$, and a bi-Lipschitz homeomorphism $h : \widetilde{M}/\widetilde{\mathcal{W}^c_f} \to \mathbb{R}^{d-c}$ such that $A \circ h = h \circ \widetilde{f}_c$.*

The following diagram illustrates all maps involved in the previous definition:

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\ \widetilde{f}\ } & \widetilde{M} \\
\widetilde{p}\downarrow & & \downarrow\widetilde{p} \\
\widetilde{M}/\widetilde{\mathcal{W}^c_f} & \xrightarrow{\ \widetilde{f}_c\ } & \widetilde{M}/\widetilde{\mathcal{W}^c_f} \\
h\downarrow & & \downarrow h \\
\mathbb{R}^{d-c} & \xrightarrow{\ A\ } & \mathbb{R}^{d-c}
\end{array}$$

Lemma 2.1.3. *Condition 2 in the definiton above implies the following: given $K > 0$ there is $C > 0$ such that, if $d(\widetilde{x}, \widetilde{y}) \leq K$ then $dist(\widetilde{p}(\widetilde{x}), \widetilde{p}(\widetilde{y})) \leq C$.*

Proof. Let D be a compact fundamental domain and let $\gamma_1, \ldots, \gamma_k \in \Gamma$ be such that if $\widetilde{x} \in D$ then $B(\widetilde{x}, K) \subset \bigcup_{i=1}^{k} \gamma_i \cdot D =: \widehat{D}$. By compactness and since $\widetilde{p} : \widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{W}^c_f}$ is continuous, there exists $C > 0$ such that if $x, y \in \widehat{D}$ then $dist(\widetilde{p}(\widetilde{x}), \widetilde{p}(\widetilde{y})) \leq C$.

Now if $\widetilde{z}, \widetilde{w} \in \widetilde{M}$ and $d(\widetilde{z}, \widetilde{w}) \leq K$, there is γ such that $\gamma \cdot \widetilde{z} \in D$ and this implies that $\gamma \cdot \widetilde{w} \in D$ too. We conclude that: $dist(\widetilde{p}(\widetilde{z}), \widetilde{p}(\widetilde{w})) = dist(\widetilde{p}(\gamma \cdot \widetilde{x}), \widetilde{p}(\gamma \cdot \widetilde{y})) \leq C$ □

Let us give a simple notation that will be useful in the whole chapter. If A is a hyperbolic matrix with a splitting $\mathbb{R}^{d-c} = E^{ss}_A \oplus E^{uu}_A$ and $v \in \mathbb{R}^{d-c}$ we are going to note by

$$\Pi^{\sigma}_v : \mathbb{R}^{d-c} \longrightarrow v + E^{\sigma}_A$$

for $\sigma = ss, uu$ to the corresponding orthogonal projections.

Now let $f : M \to M$ be a fibered partially hyperbolic diffeomorphism. From now on for simplicity, we are going to note by

$$[\widetilde{x}] := h \circ \widetilde{p}(\widetilde{x}) \in \mathbb{R}^{d-c} \text{ for every } \widetilde{x} \in \widetilde{M}.$$

Recall that $M = \widetilde{M}/\Gamma$ where $\gamma = \pi_1(M)$ acts on $\widetilde{M}$ by isometries. Then we can define an action of Γ in $\widetilde{M}/\widetilde{\mathcal{W}^c_f}$ given by the equation

$$\gamma \cdot \widetilde{p}(\widetilde{x}) := \widetilde{p}(\gamma \cdot \widetilde{x})$$

Since for every $\gamma \in \Gamma$ we have that $\gamma \cdot \widetilde{\mathcal{W}^c_f}(\widetilde{x}) = \widetilde{\mathcal{W}^c_f}(\gamma \cdot \widetilde{x})$ the action is well defined and moreover for every $\gamma \in \Gamma$ and every $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ we have:

$$\begin{aligned}
dist(\gamma \cdot \widetilde{p}(\widetilde{x}), \gamma \cdot \widetilde{p}(\widetilde{y})) &= dist(\widetilde{p}(\widetilde{x}), \widetilde{p}(\widetilde{y})) = d_H(\widetilde{\mathcal{W}^c_f}(\gamma \cdot \widetilde{x}), \widetilde{\mathcal{W}^c_f}(\gamma \cdot \widetilde{y})) \\
&= d_H(\widetilde{\mathcal{W}^c_f}(\widetilde{x}), \widetilde{\mathcal{W}^c_f}(\widetilde{y})) = dist(\widetilde{p}(\widetilde{x}), \widetilde{p}(\widetilde{y}))
\end{aligned}$$

and the action preserves the distance. In the same way we can define an action of Γ in $\mathbb{R}^{d-c} = Im(h)$ by the equation:

$$\gamma \cdot [\widetilde{x}] := [\gamma \cdot \widetilde{x}]$$

Notice that since $[\widetilde{x}] := h \circ \widetilde{p}(\widetilde{x})$ this is equivalent to $\gamma \cdot h \circ \widetilde{p}(\widetilde{x}) := h \circ \widetilde{p}(\gamma \cdot \widetilde{x})$. Then the action is well defined, but it doesn't necessarily preserves the distance. However we have the following estimate.

Lemma 2.1.4. *There exists a constant $K > 0$ s.t. for every $\gamma \in \Gamma$ and every $[\tilde{x}], [\tilde{y}] \in \mathbb{R}^{d-c}$ we have:*

$$\|\gamma \cdot [\tilde{x}] - \gamma \cdot [\tilde{y}]\| \leq K \|[\tilde{x}] - [\tilde{y}]\|$$

Proof. Let $C_h > 0$ and $C_{h^{-1}} > 0$ be the Lipschitz constants of h and h^{-1} respectively. Then given $[\tilde{x}], [\tilde{y}] \in \mathbb{R}^{d-c}$ we have:

$$dist(h^{-1}[\tilde{x}], h^{-1}[\tilde{y}]) \leq C_{h^{-1}} \|[\tilde{x}] - [\tilde{y}]\|$$

Since Γ acts on $\widetilde{M}/\widetilde{\mathcal{W}}^c_f$ preserving the distance, given $\gamma \in \Gamma$ we have:

$$dist(\gamma \cdot h^{-1}[\tilde{x}], \gamma \cdot h^{-1}[\tilde{y}]) = dist(h^{-1}[\tilde{x}], h^{-1}[\tilde{y}]) \leq C_{h^{-1}} \|[\tilde{x}] - [\tilde{y}]\|$$

This implies that:

$$\|\gamma \cdot [\tilde{x}] - \gamma \cdot [\tilde{y}]\| = \|h(\gamma \cdot h^{-1}[\tilde{x}]) - h(\gamma \cdot h^{-1}[\tilde{y}])\| \leq C_h . C_{h^{-1}} \|[\tilde{x}] - [\tilde{y}]\|$$

Taking $K = C_h . C_{h^{-1}}$ we obtain the lemma. □

Remark 2.1.5. *In the previous lemma we used that the map h^{-1} in Definition 2.1.2 is Lipschitz. This is the only part of the chapter when we use this property, but it will have strong consequences.*

2.1.2 Examples of fibered partially hyperbolic diffeomorphisms

The following are some examples of fibered partially hyperbolic diffeomorphisms.

Anosov automorphisms.

Let $A \in SL(d, \mathbb{Z})$ be a hyperbolic matrix with a splitting of the form $\mathbb{R}^d = E^{ss}_A \oplus E^{ws}_A \oplus E^{wu}_A \oplus E^{uu}_A$. This matrix induces an Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ as we saw in Example 1.3.1. Then we can see f as a fibered partially hyperbolic with trivial fibers. In this case $\mathbb{R}^d/\widetilde{\mathcal{W}}^c_f = \mathbb{R}^d$ and $\widetilde{f}_c = A$ and the four conditions above are trivially satisfied.

On the other hand we can see f as a partially hyperbolic diffeomorphism by taking the center bundle as $E^c_f = E^{ws}_A \oplus E^{wu}_A$. Since E^c_A is a linear subspace, we get that f is dinamically coherent and moreover f has global product structure (as in 1). The quotient space is $\mathbb{R}^d/\widetilde{\mathcal{W}}^c_f = E^{ss}_A \oplus E^{uu}_A = \mathbb{R}^{d-c}$ and the map $\tilde{p}$ can be seen as the orthogonal projection $\Pi^{su} : \mathbb{R}^d \to E^{ss}_A \oplus E^{uu}_A = \mathbb{R}^{d-c}$ proving point 2. The quotient map is $\widetilde{f}_c = A|_{E^{ss}_A \oplus E^{uu}_A}$ and we get 3. Point 4 is not needed since $\widetilde{f}_c$ is already linear (or in this case $h = Id$). Therefore f is a fibered partially hyperbolic diffeomorphism.

Partially hyperbolic automorphisms.

Let $A \in SL(d, \mathbb{Z})$ be a matrix with a splitting of the form $\mathbb{R}^d = E^{ss}_A \oplus E^c_A \oplus E^{uu}_A$, where E^c_A is the generalized eigenspace associated to the eigenvalues of modulus equal to one. Like in the Anosov case (see Example 1.3.1), the matrix A induces a map $f : \mathbb{T}^d \to \mathbb{T}^d$ which is a dynamically coherent partially hyperbolic diffeomorphism. In the same way as above since f is linear, it's clear that f has global product structure as in 1, the quotient space is $\mathbb{R}^d/\widetilde{\mathcal{W}}^c_f = E^{ss}_A \oplus E^{uu}_A$ and the map $\tilde{p}$ is the orthogonal projection $\Pi^{su} : \mathbb{R}^d \to E^{ss}_A \oplus E^{uu}_A$ proving 2. Finally observe that $\widetilde{f}_c = A|_{E^{ss}_A \oplus E^{uu}_A}$ and thus we get point 3. Once again we don't need point 4 since $\widetilde{f}_c$ is already linear. We conclude that f is fibered partially hyperbolic.

Anosov $\times$ Identity.

Let $A \in SL(d, \mathbb{Z})$ be a hyperbolic matrix with a splitting of the form $\mathbb{R}^d = E^s_A \oplus E^u_A$. This matrix induces an Anosov diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ as we already saw. Let N be any other manifold of any dimension and let $g : \mathbb{T}^d \times N \to T^d \times N$ be the map $g = f \times Id$. Then g is a dynamically coherent partially hyperbolic diffeomorphism with global product structure (see Example 1.3.4). The center leaves are of the form $\mathcal{W}^c_g(x, y) = \{x\} \times N$ and then its quotient space is $(\mathbb{R}^d \times \widetilde{N})/\widetilde{\mathcal{W}}^c_g = \mathbb{R}^d = E^{ss}_A \oplus E^{uu}_A$, the projection $\widetilde{p}_g : \mathbb{R}^d \times \widetilde{N} \to (\mathbb{R}^d \times \widetilde{N})/\widetilde{\mathcal{W}}^c_g = \mathbb{R}^d$ is just the projection on the first coordinate and the induced map is $\widetilde{g}_c = A|_{E^s_A \oplus E^u_A}$. This shows points 1, 2 and 3. Again $h = Id$ in this case and we have 4. Therefore g is fibered partially hyperbolic.

Dominated splitting examples.

Generalizing the previous example take $f : M \to M$ any of the previous fibered partially hyperbolic diffeomorphisms and let N be a manifold of any dimension. Take a map $g : N \to N$ such that its behaviour is dominated by f: there exist $\lambda \in (0, 1)$ such that $\|Dg_y\| \leq \lambda m(Df_x|_{E^u_f})$ and $\|Df_x|_{E^s_f}\| \leq \lambda m(Dg_y)$ for every $x \in M$, $y \in N$. Then the map $F : M \times N \to M \times N$ defined by $F = f \times g$ is a dynamically coherent partially hyperbolic diffeomorphism. Since the center leaves are $\mathcal{W}^c_F(x, y) = \mathcal{W}^c_f(x) \times N$ the quotient space is $(\widetilde{M} \times \widetilde{N})/\widetilde{\mathcal{W}}^c_F = \widetilde{M}/\widetilde{\mathcal{W}}^c_f$. Moreover since f is fibered, we have that F has global product structure as in point 1. The projection $\widetilde{p}_F$ is the function $\widetilde{p}_F(\widetilde{x}, \widetilde{y}) = \widetilde{p}_f(\widetilde{x})$ where $\widetilde{p}_f : \widetilde{M} \to \widetilde{M}/\widetilde{\mathcal{W}}^c_f$ proving 2 and the induced map is just $\widetilde{F}_c = \widetilde{f}_c$ getting point 3. If we take $h = h_f$ we have 4 and therefore F is a fibered partially hyperbolic diffeomorphism.

Skew-products

Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be an Anosov diffeomorphism induced by some hyperbolic matrix as above and let G be a compact Lie group. Take a smooth function $\theta : N \to G$ and consider the map $F : N \times G \to N \times G$ given by $F(x, g) = (f(x), \theta(x)g)$. Then it is easy to see that F is a dynamically coherent partially hyperbolic diffeomorphism with global product structure in the universal cover proving 1. The center leaves are given by $\mathcal{W}^c_F(x, g) = \{x\} \times G$, and therefore we have point 2. By the same reason the projection into equivalence classes $\widetilde{p}_F$ is just the projection into the first coordinate and the induced map is just $\widetilde{F}_c = \widetilde{f}$ getting point 3. Since the Anosov in the base f is linear, we have 4 and therefore F is a fibered partially hyperbolic diffeomorphism.

Fiberings

More general than the previous examples, we have the systems that fiber over partially hyperbolic diffeomorphisms. Take $f : \mathbb{T}^d \to \mathbb{T}^d$ be a fibered partially hyperbolic diffeomorphism with a splitting of the form $T\mathbb{T}^d = E^s_f \oplus E^c_f \oplus E^u_f$. Take a fibration $N \hookrightarrow M \xrightarrow{\pi} \mathbb{T}^d$, i.e. $\pi^{-1}(\{x\}) \simeq N$ for every $x \in \mathbb{T}^d$, and denote by $N(x) = \pi^{-1}(\{x\})$ to the fiber through x. Consider a lift $F : M \to M$, that is a map such that $\pi \circ F = f \circ \pi$.

Then if we ask for the lift F to verify:

$$\|Df_{\pi(x)}|_{E^s_f}\| < m(DF_x|_{TN(x)}) \leq \|DF_x|_{TN(x)}\| < m(Df_{\pi(x)}|_{E^u_f})$$

then F is partially hyperbolic and dynamically coherent. Moreover since the map in the base f is a fibered p.h. we have that F has global product structure 1 and center leaves in the universal cover are $\widetilde{\mathcal{W}^c_F}(\tilde{x}) = \widetilde{\mathcal{W}^c_f}(\pi(\tilde{x})) \times \tilde{N}$ proving 2. It is direct to check that the projection map $\tilde{p}_F$ is just the composition $\tilde{p}_f \circ \pi$, showing point 3. Finally taking $h_F = h_f$ we get point 4 and F is a fibered partially hyperbolic diffeomorphism.

2.1.3 Shadowing and stability

The main object of study in this chapter are fibered partially hyperbolic diffeomorphisms and their induced hyperbolic homeomorphisms on their respective quotient spaces. As we mentioned in the introduction, uniform hyperbolicity is a robust property. In this subsection we are going to see that this fundamental relation between hyperbolicity and stability is given by a property which is in the core of hyperbolicity: the *shadowing property*.

The pseudo-orbit tracing property or shadowing property originally appeared in the works of R. Bowen where he studied the ergodic properties of Anosov or Axiom A diffeomorphisms. This shadowing property was quickly generalized to many contexts (for example hyperbolic homeomorphisms) but we are just going to state the simplest version since it is enough for our purposes.

Let $f : X \to X$ be a homeomorphism on a metric space $(X, dist)$ and take a C^0-perturbation of size $K > 0$, i.e. a map $g : X \to X$ such that $dist(f(x), g(x)) < K$ for every $x \in X$. Then given a point $x \in X$ the g-orbit of x, i.e. the sequence $x_n := g^n(x)$ satisfies the following: $dist(f(x_n), x_{n+1}) = dist(f(x_n), g(x_n)) < K$. Hence the g-orbit of x is *almost* an orbit of f in the sense that it is allowed to make *jumps* of length smaller than K. This simple observation leads to the following classical definition.

Definition 2.1.6. *Given a homeomorphism $f : X \to X$ on a metric space $(X, dist)$ and $\delta > 0$ we say that a sequence of points $\{x_n\}_{n\in\mathbb{Z}} \subset X$ is a K-pseudo orbit (with respect to f) if $dist(f(x_n), x_{n+1}) < K$ for every $n \in \mathbb{Z}$.*

Now the real problem is to determine which condition must satisfy the homeomorphism f in order to get a precise relationship between the set of K-pseudo orbits (with respect to f) and the truly orbits of f. The key ingredient turned out to be uniform hyperbolicity as the following classical lemma shows. We are going to see a specific statement of the lemma, the one that best suits for our purposes, although there are more general versions.

Lemma 2.1.7 (Shadowing lemma for hyperbolic automorphisms). *Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a hyperbolic matrix. Then given $K > 0$ there is $\alpha > 0$ such that for every K-pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$ there is a unique $y \in \mathbb{R}^d$ such that $\|A^n(y) - x_n\| < \alpha$ for every $n \in \mathbb{Z}$.*

We say that the A-orbit of y is the one that α-shadows the pseudo orbit $\{x_n\}_{n\in\mathbb{Z}}$.

We are not going to see a proof of this lemma, the interested reader can found a proof in [Sam09]. The only thing we are going to mention is that the constant α depends on λ_s, λ_u (eigenvalues of A) and K.

The following theorem establishes the relation between the shadowing property and the C^0-stability.

Theorem 2.1.8 (Stability of fibered partially hyperbolic diffeomorphisms)**.** *Let $f : M \to M$ be a fibered partially hyperbolic diffeomorphism. Then for every $g : M \to M$ such that $\sup\{dist(\widetilde{f}(\widetilde{x}), \widetilde{g}(\widetilde{x}))\} < K < \infty$ for some lift $\widetilde{g} : \widetilde{M} \to \widetilde{M}$, there exist a continuous and surjective map $H_g : \widetilde{M} \to \mathbb{R}^{d-c}$ and a number $\alpha = \alpha(f, K) > 0$ such that:*

1. $A \circ H_g = H_g \circ \widetilde{g}$
2. $d_{C^0}(H_g, h \circ \widetilde{p}) < \alpha$
3. *the map H_g varies continuously with g in the C^0 topology.*
4. *H_g is Γ invariant.*

Proof. Let f be a fibered partially hyperbolic diffeomorphism. By hypothesys, there exists a linear Anosov $A : \mathbb{R}^{d-c} \to \mathbb{R}^{d-c}$ and a bi Lipschitz homeomorphism $h : \widetilde{M}/\widetilde{\mathcal{W}}^c_f \to \mathbb{R}^{d-c}$ such that $A \circ h = h \circ \widetilde{f}_c$.

Let $g \in \mathcal{PH}(M)$ be such that $\sup\{dist(\widetilde{f}(\widetilde{x}), \widetilde{g}(\widetilde{x})) : \widetilde{x} \in \widetilde{M}\} = K < \infty$ for some lift $\widetilde{g} : \widetilde{M} \to \widetilde{M}$ on the universal cover. Now for this $K > 0$ we know by Lemma 2.1.3 that there is $C > 0$ such that if $d(\widetilde{x}, \widetilde{y}) \leq K$ then $dist(\widetilde{p}(\widetilde{x}), \widetilde{p}(\widetilde{y})) \leq C$.

Given a point $\widetilde{x} \in \widetilde{M}$ we define the following sequence:

$$G_n(\widetilde{x}) := h \circ \widetilde{p}(\widetilde{g}^n(\widetilde{x})) = [\widetilde{g}^n(\widetilde{x})]$$

We claim that $\{G_n(\widetilde{x})\}_{n\in\mathbb{Z}}$ is a C_hC-pseudo orbit with respect to A where C_h is the Lipschitz constant of h. First observe that:

$$A(G_n(\widetilde{x})) = A \circ h \circ \widetilde{p}(\widetilde{g}^n(\widetilde{x})) = h \circ \widetilde{p} \circ \widetilde{f}(\widetilde{g}^n(\widetilde{x}))$$

Then we have that:

$$\begin{aligned}
\|A(G_n(\widetilde{x})) - G_{n+1}(\widetilde{x})\| &= \|A \circ h \circ \widetilde{p}(\widetilde{g}^n(\widetilde{x})) - h \circ \widetilde{p}(\widetilde{g}^{n+1}(\widetilde{x})\| \\
&= \|h \circ \widetilde{p} \circ \widetilde{f}(\widetilde{g}^n(\widetilde{x})) - h \circ \widetilde{p}(\widetilde{g}^{n+1}(\widetilde{x})\| \\
&\leq C_h\, dist(\widetilde{p} \circ \widetilde{f}(\widetilde{g}^n(\widetilde{x})), \widetilde{p}(\widetilde{g}^{n+1}(\widetilde{x}))) \\
&= C_h\, dist(\widetilde{p}(\widetilde{f}(\widetilde{g}^n(\widetilde{x}))), \widetilde{p}(\widetilde{g}(g^n(\widetilde{x}))))
\end{aligned}$$

Since $d(\widetilde{f}(\widetilde{x}), \widetilde{g}(\widetilde{x})) \leq K$ for every $\widetilde{x} \in \widetilde{M}$, we have that $d(\widetilde{f}(\widetilde{g}^n(\widetilde{x})), \widetilde{g}(g^n(\widetilde{x}))) \leq K$ and therefore $dist(\widetilde{p}(\widetilde{f}(\widetilde{g}^n(\widetilde{x}))), \widetilde{p}(\widetilde{g}(g^n(\widetilde{x})))) \leq K$. We conclude that $\|A(G_n(\widetilde{x})) - G_{n+1}(\widetilde{x})\| \leq C_hK$ proving that $\{G_n(\widetilde{x})\}$ is a C_hC-pseudo orbit with respect to A. Since $A : \mathbb{R}^{d-c} \to \mathbb{R}^{d-c}$ is a hyperbolic automorphism, we can apply the Shadowing Lemma 2.1.7 and obtain a unique vector $v \in \mathbb{R}^{d-c}$ such that $\|A^n v - G_n(\widetilde{x})\| < \alpha$ for every $n \in \mathbb{Z}$. Notice that α depends only on f, C_h and K (and C_h depends on f). Therefore the map $H_g : \widetilde{M} \to \mathbb{R}^{d-c}$ given by $H_g(\widetilde{x}) = v$ is well defined. Now by definition we have:

$$G_{n+1}(\widetilde{x}) = h \circ \widetilde{p}(\widetilde{g}^{n+1}(\widetilde{x})) = h \circ \widetilde{p}(\widetilde{g}^n(\widetilde{g}(\widetilde{x}))) = G_n(\widetilde{g}(\widetilde{x}))$$

Then

$$\|A^n(A \circ H_g)(\widetilde{x})) - G_n(\widetilde{g}(\widetilde{x}))\| = \|A^{n+1}(H_g(\widetilde{x})) - G_{n+1}(\widetilde{x})\| < \alpha$$

and the uniqueness in the Shadowing Lemma implies that

$$H_g(\widetilde{g}(x)) = A(H_g(\widetilde{x})) \text{ or equivalently } A \circ H_g = H_g \circ \widetilde{g}$$

proving point 1. By definition we have that $\|A^n \circ H_g(\widetilde{x}) - h \circ \widetilde{p}(\widetilde{g}^n(\widetilde{x}))\| < \alpha$, thus taking $n = 0$ gives $\|H_g(\widetilde{x}) - h \circ \widetilde{p}(\widetilde{x}))\| < \alpha$ for every $\widetilde{x} \in \widetilde{M}$, proving point 2.

To see the continuity of H_g suppose that the sequence $\{\tilde{x}_k\}_{k\in\mathbb{N}} \subset \tilde{M}$ is such that $\tilde{x}_k \to \tilde{x}$ as $k \to \infty$, and fix some integer $l \in \mathbb{Z}$. Then,

$$\begin{aligned}\|A^l(\lim_{k\to\infty} H_g(\tilde{x}_k)), h \circ \tilde{p}(\tilde{g}^l(\tilde{x}))\| &= \|A^l(\lim_{k\to\infty} H_g(\tilde{x}_k)) - h \circ \tilde{p} \circ \tilde{g}^l(\lim_{k\to\infty} \tilde{x}_k)\| \\ &= \lim_{k\to\infty} \|A^l(H_g(\tilde{x}_k)) - h \circ \tilde{p}(\tilde{g}^l(\tilde{x}_k))\| < \alpha\end{aligned}$$

Since $l \in \mathbb{Z}$ is arbitrary, by the uniqueness of the shadowing we get $\lim_{k\to\infty} H_g(\tilde{x}_k) = H_g(\tilde{x})$ and H_g is continuous. Since $d_{C^0}(H_g, h \circ \tilde{p}) < \alpha$ by a degree argument we get that H_g is surjective.

To prove the continuous variation with respect to g, take some $\epsilon > 0$ and fix some large $N_0 \in \mathbb{N}$ such that every vector $v \in \mathbb{R}^{d-c}$ with $\|v\| \geq \epsilon$ verifies: $\|A^{N_0}(v)\| > 2\alpha + C_h C$ or $\|A^{-N_0}(v)\| > 2\alpha + C_h C$. We always have this N_0 since A is hyperbolic. Let $\mathcal{U}(g)$ be the C^0 neighbourhood of g s.t. for every $g' \in \mathcal{U}(g)$, $\tilde{x} \in \tilde{M}$ and $|j| \leq N_0$ we have $d(\tilde{g}^j(\tilde{x}), \tilde{g'}^j(\tilde{x})) < K$. Now take $g' \in \mathcal{U}(g)$, $\tilde{x} \in \tilde{M}$ and $|j| \leq N_0$:

$$\begin{aligned}\|A^j(H_g(\tilde{x})) - A^j(H_{g'}(\tilde{x}))\| &\leq \|A^j(H_g(\tilde{x})) - h \circ \tilde{p}(\tilde{g}^j(\tilde{x}))\| \\ &+ \|h \circ \tilde{p}(\tilde{g}^j(\tilde{x})) - h \circ \tilde{p}(\tilde{g'}^j(\tilde{x}))\| \\ &+ \|h \circ \tilde{p}(\tilde{g'}^j(\tilde{x})) - A^j(H_{g'}(\tilde{x}))\| \\ &\leq \alpha + C_h C + \alpha = 2\alpha + C_h C\end{aligned}$$

where the first and third inequalities come from the shadowing property, and the second one because $\|h \circ \tilde{p}(\tilde{g}^j(\tilde{x})) - h \circ \tilde{p}(\tilde{g'}^j(\tilde{x}))\| \leq C_h dist(\tilde{p}(\tilde{g}^j(\tilde{x})), \tilde{p}(\tilde{g'}^j(\tilde{x}))) \leq C_h C$ since $d(\tilde{g}^j(\tilde{x}), \tilde{g'}^j(\tilde{x})) < K$. This implies $\|H_g(\tilde{x}) - H_{g'}(\tilde{x})\| < \epsilon$ by the above condition and therefore we get point 3.

To finish the proof we have to prove that H_g is Γ-invariant. Recall that by definition we have $[\tilde{x}] = h \circ \tilde{p}(\tilde{x})$ and $\gamma \cdot [\tilde{x}] = [\gamma \cdot \tilde{x}]$. First notice that if we call $\varphi : \Gamma \to \Gamma$ the induced map of $\tilde{f}$ in the fundamental group, we get that for every $\gamma \in \Gamma$ and every $\tilde{x} \in \tilde{M}$:

$$\tilde{f}(\gamma \cdot \tilde{x}) = \varphi(\gamma) \cdot \tilde{f}(\tilde{x})$$

and the same happens for every g as in the hypothesys: $\tilde{g}(\gamma \cdot \tilde{x}) = \varphi(\gamma) \cdot \tilde{g}(\tilde{x})$. By induction we get that $\tilde{f}^n(\gamma \cdot \tilde{x}) = \varphi^n(\gamma) \cdot \tilde{f}^n(\tilde{x})$. In a similar way we have:

$$\begin{aligned}A(\gamma \cdot [\tilde{x}]) &= A([\gamma \cdot \tilde{x}]) = A(h \circ \tilde{p}(\gamma \cdot \tilde{x})) = h \circ \tilde{p} \circ \tilde{f}(\gamma \cdot \tilde{x}) \\ &= h \circ \tilde{p}(\varphi(\gamma) \cdot \tilde{f}(\tilde{x})) = \varphi(\gamma) \cdot h \circ \tilde{p}(\tilde{f}(\tilde{x})) \\ &= \varphi(\gamma) \cdot A \circ h \circ \tilde{p}(\tilde{x}) = \varphi(\gamma) \cdot A([\tilde{x}])\end{aligned}$$

By induction we get that $A^n(\gamma \cdot [\tilde{x}]) = \varphi^n(\gamma) \cdot A^n([\tilde{x}])$. Finally just observe that:

$$\begin{aligned}G_n(\gamma \cdot \tilde{x}) &= h \circ \tilde{p}(\tilde{g}^n(\gamma \cdot \tilde{x})) = h \circ \tilde{p}(\varphi^n(\gamma) \cdot \tilde{g}^n(\tilde{x})) \\ &= \varphi^n(\gamma) \cdot h \circ \tilde{p}(\tilde{g}^n(\tilde{x})) = \varphi^n(\gamma) \cdot G_n(\tilde{x})\end{aligned}$$

To sum up, for every $\gamma \in \Gamma$, $\tilde{x} \in \tilde{M}$ and $n \in \mathbb{Z}$ we have

$$\begin{aligned}\|A^n(\gamma \cdot H_g(\tilde{x})) - G_n(\gamma \cdot \tilde{x})\| &= \|\varphi^n(\gamma) \cdot A^n(H_g(\tilde{x})) - \varphi^n(\gamma) \cdot G_n(\tilde{x})\| \\ &\leq C_h C_{h^{-1}} \|A^n(H_g(\tilde{x})) - G_n(\tilde{x})\| < C_h C_{h^{-1}} \alpha\end{aligned}$$

where the first inequality comes from Lemma 2.1.4. By uniqueness of the Shadowing Lemma, we get that $H_g(\gamma \cdot \tilde{x}) = \gamma \cdot H_g(\tilde{x})$, proving point 4. $\square$

Remark 2.1.9. *As we mentioned in Remark 2.1.5, we only use the fact that h^{-1} is Lipschitz in order to prove Lemma 2.1.4, and we have just used this lemma to prove Point 4 in the theorem above. In short, we need h^{-1} to be Lipschitz in order to get the Γ invariance of H_g.*

Remark 2.1.10. *In case $g = f$ we get $H_f = h \circ \widetilde{p}$. In many parts of this chapter we will note H_f instead of $h \circ \widetilde{p}$.*

Remark 2.1.11. *If $g \in \mathcal{PH}(M)$ is isotopic to f and we take a lift $\widetilde{g}$, then we always have that*

$$\sup\{d(\widetilde{f}(\widetilde{x}), \widetilde{g}(\widetilde{x})) : \widetilde{x} \in \widetilde{M}\} < K < \infty$$

Therefore Theorem 2.1.8 applies and we get the map H_g.

2.1.4 Main results

From now on $f \in \mathcal{PH}(M)$ will be a fibered partially hyperbolic diffeomorphism and we are going to consider the subset $\mathcal{PH}_f(M) \subseteq \mathcal{PH}(M)$ of partially hyperbolic diffeomorphisms such that:

$$\mathcal{PH}_f(M) = \left\{ \begin{array}{c} g \in \mathcal{PH}(M) \text{ which are isotopic to } f \text{ and such that} \\ \dim E^\sigma_g = \dim E^\sigma_f, \text{ for } \sigma = ss, c, uu \end{array} \right\}$$

By Theorem 2.1.8 (and Remark 2.1.11) we have that for every $g \in \mathcal{PH}_f(M)$ there is a continuous and surjective map $H_g : \widetilde{M} \to \mathbb{R}^{d-c}$ such that $A \circ H_g = H_g \circ \widetilde{g}$, i.e. $\widetilde{g}$ is semiconjugated to the linear Anosov A. The first direct consequence of this semiconjugacy is the following:

$$\text{if } \widetilde{y} \in \widetilde{\mathcal{W}}^{ss}_g(\widetilde{x}) \text{ then } H_g(\widetilde{y}) \in E^{ss}_A + H_g(\widetilde{x})$$

and the same happens with the unstable manifold:

$$\text{if } \widetilde{y} \in \widetilde{\mathcal{W}}^{uu}_g(\widetilde{x}) \text{ then } H_g(\widetilde{y}) \in E^{uu}_A + H_g(\widetilde{x})$$

This is easy to see since $\widetilde{y} \in \widetilde{\mathcal{W}}^{ss}_g(\widetilde{x})$ if and only if $d(\widetilde{g}^n(\widetilde{y}), \widetilde{g}^n(\widetilde{x})) \to 0$ for $n \to +\infty$. This implies that $\|H_g \circ \widetilde{g}^n(\widetilde{y}) - H_g \circ \widetilde{g}^n(\widetilde{x})\| \to 0$ and by the semiconjugacy relation this is the same as $\|A^n(H_g(\widetilde{y}) - H_g(\widetilde{x}))\| \to 0$. By hyperbolicity this can only happen if $H_g(\widetilde{y}) \in E^{ss}_A + H_g(\widetilde{x})$. The same calculation works for the past.

On the other hand suppose there are points $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ such that their orbits are at finite distance at any time (this is the "ideal" picture of the behaviour on center leaves), then since A is uniformly hyperbolic we have $H_g(\widetilde{x}) = H_g(\widetilde{y})$. This motivates the following definition, which is the analogous to the one introduced in [FPS14].

Definition 2.1.12 (Center fibered)**.** *We say that a dynamically coherent $g \in \mathcal{PH}_f(M)$ is center-fibered (CF) if $H_g^{-1}(H_g(\widetilde{x})) = \widetilde{\mathcal{W}}^c_g(\widetilde{x})$ for every $\widetilde{x} \in \widetilde{M}$.*

In particular this means that two different center leaves of $\widetilde{g}$ are sent by H_g to two different points in $\mathbb{R}^{d-c}$.

Now given a fibered partially hyperbolic diffeomorphism f, we are going to note:

$$\mathcal{PH}^0_f(M) = \left\{ \begin{array}{c} \text{connected componentes of } \mathcal{PH}_f(M) \text{ which contains a} \\ \text{DC and CF p.h.d. with global product structure} \end{array} \right\}$$

We remark that the partially hyperbolic diffeomorphism f is itself center fibered by definition because $H_f = h \circ \widetilde{p}$, then for every $\widetilde{x} \in \widetilde{M}$ we have that:

$$(H_f)^{-1}(H_f)(\widetilde{x}) = (h \circ \widetilde{p})^{-1}(h \circ \widetilde{p})(\widetilde{x}) = \widetilde{p}^{-1}(\widetilde{p}(\widetilde{x})) = \widetilde{\mathcal{W}}^c_f(\widetilde{x})$$

Then the set $\mathcal{PH}^0_f(M)$ is a non-empty open set with at least one connected component. Let us mention here that in [FG14] it is proved that given a linear Anosov $A : \mathbb{T}^d \to \mathbb{T}^d$ (with $d \geq 10$), the space of Anosov diffeomorphisms homotopic to A has infinitely many connected components. In particular, this implies that $\mathcal{PH}^0_f(M)$ may have more than one connected component (besides the one containing f).

With this new notations we can restate the main result of this chapter.

Theorem 2.1.13. *Every $g \in \mathcal{PH}^0_f(M)$ is dynamically coherent and center-fibered.*

A direct consequence from the proof of this theorem, is that it implies to have plaque expansiveness in the whole connected component. Applying Theorem 1.4.5 and a connectedness argument, we can obtain the following classification result.

Theorem 2.1.14. *Any two diffeomorphisms in the same connected component of $\mathcal{PH}^0_f(M)$ are leaf conjugate. In particular every $g \in \mathcal{PH}^0_f(M)$ in the same connected component of f is leaf conjugate to f.*

Let us summarize the main steps of the proofs of these theorems.

1. We first state an integrability criterion for partially hyperbolic diffeomorphisms isotopic to a fixed fibered partially hyperbolic diffeomorphism. This criterion is a generalization of the one introduced in [FPS14] and it is based on the concepts of σ-properness, global product structure (GPS) and strong almost dynamical coherence (SADC). This is done in Section 2.2.

2. We then study the C^1 openess and C^1 closedness of all these properties: σ properness, GPS and SADC. In particular, once this is achieved, we get that if there is a partially hyperbolic diffeomorphism g which satisfies all these properties, we have that every partially hyperbolic diffeomorphism in the same connected component of g satisfies these properties as well. We then can apply the integrability criterion mentioned in Point 1 to every partially hyperbolic diffeomorphism in that connected component. This is shown in Section 2.3.

3. We then pass to the proof of Theorem 2.1.13. The only thing we must check, is the other way around; let $g \in \mathcal{PH}_f(M)$ be a partially hyperbolic diffeomorphism isotopic to a fibered p.h. f, such that g is dynamically coherent and center-fibered (and has global product structure), then g is σ-proper and SADC. This is done in Subsection 2.3.5.

4. We then pass to the proof of Theorem 2.1.14. To do this, we just prove that every g in the above conditions must be plaque expansive, and by classical arguments we prove the leaf conjugacy between g and the fibered p.h. f. The proof of this theorem is made in Section 2.4.

2.2 Integrability for fibered partially hyperbolic diffeomorphisms

In this section we are going to see an integrability criterion for partially hyperbolic diffeomorphisms isotopic to fibered partially hyperbolic diffeomorphisms. This criterion is a generalization of the one given in [FPS14] and therefore it can be applied in a larger number of cases.

2.2.1 σ-Properness

Recall that given $g \in \mathcal{PH}_f(M)$, for any $* \in \{ss, uu\}$, for any $\tilde{x} \in \widetilde{M}$, and for any $\epsilon > 0$, we denote by

$$\widetilde{\mathcal{W}}^*_g(\tilde{x}, \epsilon) := \{\tilde{y} \in \widetilde{\mathcal{W}}^*_g(\tilde{x}) : d_{\widetilde{\mathcal{W}}^*_g}(\tilde{x}, \tilde{y}) < \epsilon\}$$

to the ϵ-ball in $\widetilde{\mathcal{W}}^*_g$ of center $\tilde{x}$ and radius ϵ, where $d_{\widetilde{\mathcal{W}}^*_g}$ denotes the leafwise distance, that is the distance induced by the Riemannian metric in $\widetilde{M}$ restricted to the leaves.

In order to avoid any confusion (since we are working in different spaces), for any $* \in \{ss, uu\}$, for any $v \in \mathbb{R}^{d-c}$, and for any $\epsilon > 0$ we denote by

$$D^*_A(v, \epsilon) := \{w \in E^*_A + v : \|v - w\| < \epsilon\}$$

The following definition is the analogous of the one introduced in [FPS14].

Definition 2.2.1 (σ-proper)**.** *For $\sigma = ss, uu$ we say that $g \in \mathcal{PH}_f(M)$ is σ-**proper** if for every $\tilde{x} \in \widetilde{M}$ the map H_g restricted to $\widetilde{\mathcal{W}}^\sigma_g(\tilde{x})$ is uniformly proper. More precisely, for every $R > 0$ there exists $R' > 0$ such that*

$$(H_g)^{-1}(D^\sigma_A(H_g(\tilde{x}), R)) \cap \widetilde{\mathcal{W}}^\sigma_g(\tilde{x}) \subset \widetilde{\mathcal{W}}^\sigma_g(\tilde{x}, R') \ \textit{for every} \ \ \tilde{x} \in \widetilde{M}$$

Remark 2.2.2. *In the previous definition, we can take $R = 1$ by uniform hyperbolicity of the strong bundles, and the cocompactness of $\widetilde{M}$.*

The definition of σ-properness can be expressed in a different and more geometric way. The next lemma gives the desire equivalence. We omit its proof since it is the same as Lemmas 3.2 and 3.4 in [FPS14]. Given $g \in \mathcal{PH}_f(M)$ we say that g has condition:

(I^σ) If the function H_g is injective restricted to $\widetilde{\mathcal{W}}^\sigma_g$-leaves.

(S^σ) If the function H_g is surjective restricted to $\widetilde{\mathcal{W}}^\sigma_g$-leaves.

Then if g verifies both conditions, the map $H_g|_{\widetilde{\mathcal{W}}^\sigma_g(\tilde{x})} : \widetilde{\mathcal{W}}^\sigma_g(\tilde{x}) \to E^\sigma_A + H_g(\tilde{x})$ is a homeomorphism.

Lemma 2.2.3. *If $g \in \mathcal{PH}_f(M)$ then, g is σ-proper if and only if g satisfies properties (I^σ) and (S^σ). Moreover (I^σ) implies (S^σ).*

2.2.2 Strong almost dynamically coherence

Given a subset $K \subset \widetilde{M}$ and $R > 0$ we call $B(K, R)$ the R-neighbourhood of K, that is, the set of points in $\widetilde{M}$ that are less than R from some point in K:

$$B(K, R) = \{\tilde{x} \in \widetilde{M} : \text{there is } \tilde{y} \in K \text{ s.t. } d(\tilde{x}, \tilde{y}) < R\}$$

This includes the case $K = \tilde{x}$ and:

$$B(\tilde{x}, R) = \{\tilde{y} \in \tilde{M} : d(\tilde{x}, \tilde{y}) < R\}$$

Definition 2.2.4 (Almost parallel foliations)**.** *Given $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ two foliations in $\tilde{M}$, we say they are almost parallel if there exists $R > 0$ such that for every $\tilde{x} \in \tilde{M}$, there are points $\tilde{x_1}, \tilde{x_2} \in \tilde{M}$ such that:*

- $\tilde{\mathcal{F}}_1(\tilde{x}) \subset B(\tilde{\mathcal{F}}_2(\tilde{x_1}), R)$ *and* $\tilde{\mathcal{F}}_2(\tilde{x_1}) \subset B(\tilde{\mathcal{F}}_1(\tilde{x}), R)$
- $\tilde{\mathcal{F}}_2(\tilde{x}) \subset B(\tilde{\mathcal{F}}_1(\tilde{x_2}), R)$ *and* $\tilde{\mathcal{F}}_1(\tilde{x_2}) \subset B(\tilde{\mathcal{F}}_2(\tilde{x}), R)$

It's easy to see that this is an equivalence relation. Moreover the condition can be expressed in terms of the Hausdorff distance: for every $\tilde{x} \in \tilde{M}$, there exist $\tilde{x_1}, \tilde{x_2} \in \tilde{M}$ such that $d_H(\tilde{\mathcal{F}}_1(\tilde{x}), \tilde{\mathcal{F}}_2(\tilde{x_1})) < R$ and $d_H(\tilde{\mathcal{F}}_2(\tilde{x}), \tilde{\mathcal{F}}_1(\tilde{x_2})) < R$.

Definition 2.2.5 (SADC)**.** *We say that $g \in \mathcal{PH}_f(M)$ is **strongly almost dynamically coherent** (SADC) if there exists foliations $\mathcal{F}^{cs}_g$, $\mathcal{F}^{cu}_g$ (not necessarily invariant) such that:*

- $\mathcal{F}^{cs}_g$, $\mathcal{F}^{cu}_g$ *are transverse to* E^{uu}_g, E^{ss}_g *respectively,*
- $\tilde{\mathcal{F}}^{cs}_g$, $\tilde{\mathcal{F}}^{cu}_g$ *are almost parallel to the foliations* $\tilde{\mathcal{W}}^{cs}_f$, $\tilde{\mathcal{W}}^{cu}_f$ *respectively.*

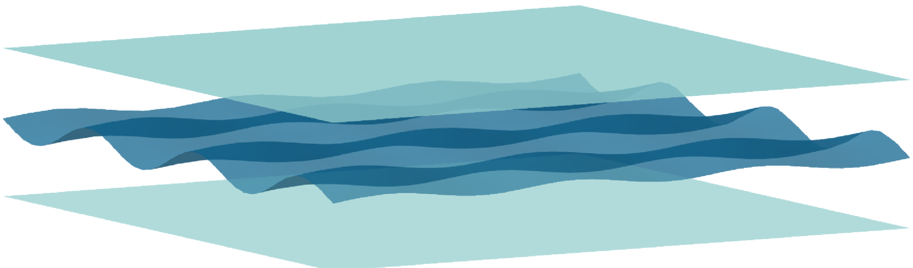


FIGURE 2.1: Almost parallel foliations $\mathcal{F}^{cs}_g$ and $\mathcal{W}^{cs}_f$

The previous name (SADC) comes from [Pot12] where Potrie defines the concept of *almost dynamically coherent* as a partially hyperbolic diffeomorphism with foliations $\mathcal{F}^{cs}_f$, $\mathcal{F}^{cu}_f$ transverse to E^{uu}_f, E^{ss}_f. In fact in that paper the author proved for dimension 3 that these foliations are almost parallel to E^{cs}_A, E^{cu}_A. In higher dimension this is not clear, that's why in [FPS14] they added the *stronger* hypothesis.

Definition 2.2.6 (SADC with GPS)**.** *Given $g \in \mathcal{PH}_f(M)$ which is SADC with their corresponding foliations $\mathcal{F}^{cs}_g$ and $\mathcal{F}^{cu}_g$, we say that g has global product structure if $\mathcal{F}^{cs}_g$ and $\tilde{\mathcal{W}}^{uu}_g$ have global product structure (GPS) and, $\mathcal{F}^{cu}_g$ and $\tilde{\mathcal{W}}^{ss}_g$ have global product structure.*

2.2.3 Integrability criterion

The following is an integrability criterion for partially hyperbolic diffeomorphisms isotopic to fibered partially hyperbolic diffeomorphisms.

Theorem 2.2.7 (Integrability criterion)**.** *Assume that $g \in \mathcal{PH}_f(M)$ verifies the following conditions:*

- *g is uu-proper.*
- *g is* SADC *with global product structure.*

Then the bundle E^{cs}_g is integrable into a g-invariant foliation $\mathcal{W}^{cs}_g$ that verifies $H_g^{-1}(E^{ss}_A + H_g(\tilde{x})) = \widetilde{W}^{cs}_g(\tilde{x})$. Moreover, $\widetilde{\mathcal{W}}^{cs}_g$ and $\widetilde{\mathcal{W}}^{uu}_g$ have global product structure.

Proof. The idea of the proof is pretty clear: take the foliation $\widetilde{\mathcal{F}}^{cs}_g$ given by the SADC property and iterate it backwards by $\tilde{g}$ hoping that in the limit it will converge to the desired foliation. Specifically the goal is to show that $\{(H_g)^{-1}(H_g(\tilde{y}) + E^{ss}_A) : \tilde{y} \in \widetilde{M}\}$ is the center-stable foliation of $\tilde{g}$.

First observe that this partition of $\widetilde{M}$ is $\tilde{g}$-invariant:

$$\begin{aligned}
\tilde{g}^{-1}((H_g)^{-1}(H_g(\tilde{y}) + E^{ss}_A)) &= (H_g \circ \tilde{g})^{-1}(H_g(\tilde{y}) + E^{ss}_A) = (A \circ H_g)^{-1}(H_g(\tilde{y}) + E^{ss}_A) \\
&= H_g^{-1}(A^{-1}(H_g(\tilde{y}) + E^{ss}_A)) = H_g^{-1}(A^{-1}(H_g(\tilde{y})) + E^{ss}_A) \\
&= H_g^{-1}(H_g(\tilde{g}^{-1}\tilde{y}) + E^{ss}_A)
\end{aligned}$$

Moreover the partition is invariant by deck translations since H_g is Γ-invariant. Now take the foliation $\widetilde{\mathcal{F}}^{cs}_g$ given by the SADC property. Since it is almost parallel to $\widetilde{\mathcal{W}}^{cs}_f$ and H_g is at bounded Hausdorff distance from $H_f = h \circ \tilde{p}$ we have that $H_g(\widetilde{\mathcal{F}}^{cs}_g(\tilde{x}))$ is also at bounded Hausdorff distance from some translate of E^{ss}_A for every $\tilde{x} \in \widetilde{M}$.

Since $\widetilde{\mathcal{W}}^{uu}_g$ and $\widetilde{\mathcal{F}}^{cs}_g$ have global product structure, we can see the leaves of $\widetilde{\mathcal{F}}^{cs}_g$ (and then of $\tilde{g}^{-n}(\widetilde{\mathcal{F}}^{cs}_g)$) as graphs of functions from $\mathbb{R}^{cs}$ to $\mathbb{R}^{uu}$. Since the foliation $\widetilde{\mathcal{F}}^{cs}_g$ is uniformly transverse to E^{uu}_g we know there are local product structure boxes of uniform size in $\widetilde{M}$, i.e. there is $\epsilon > 0$ s.t. $\forall \tilde{x} \in \widetilde{M}$ there is a neighbourhood $V_{\tilde{x}} \supseteq B(\tilde{x}, \epsilon)$ and C^1-local coordinates $\psi_{\tilde{x}} : \mathbb{D}^{cs} \times \mathbb{D}^{uu} \to V_{\tilde{x}}$ such that:

- $\psi_{\tilde{x}}(\mathbb{D}^{cs} \times \mathbb{D}^{uu}) = V_{\tilde{x}}$
- For every $\tilde{y} \in B(\tilde{x}, \epsilon) \subseteq V_{\tilde{x}}$ we have that if we call $W^{\tilde{x}}_n(\tilde{y})$ to the connected component of $V_{\tilde{x}} \cap \tilde{g}^{-n}(\widetilde{\mathcal{F}}^{cs}_g(\tilde{g}^n(\tilde{y})))$ that contains $\tilde{y}$ then

  $$\psi_{\tilde{x}}^{-1}(W^{\tilde{x}}_n(\tilde{y})) = \text{graph}(h^{\tilde{x},\tilde{y}}_n)$$

  where $h^{\tilde{x},\tilde{y}}_n : \mathbb{D}^{cs} \to \mathbb{D}^{uu}$ is a C^1 function with bounded first derivatives.

This way we get that the set $\{h^{\tilde{x},\tilde{y}}_n\}_{n\in\mathbb{N}}$ is precompact in the space of Lipschitz functions $\mathbb{D}^{cs} \to \mathbb{D}^{uu}$ ([HPS77]). Therefore the leaves of $\tilde{g}^{-n}(\widetilde{\mathcal{F}}^{cs}_g)$ have convergent subsequences. From this point we have to deal with two problems: the first one is that *a priori* there could be a leaf with more than one limit, and second, that in the limit, different leaves might merge. We will handle these two problems in the same way.

For every $\tilde{y} \in B(\tilde{x}, \epsilon)$, we call $\mathcal{J}^{\tilde{x}}_{\tilde{y}}$ to the set of indices such that for every $\alpha \in \mathcal{J}^{\tilde{x}}_{\tilde{y}}$ there is a Lipschitz function $h^{\tilde{x},\tilde{y}}_{\infty,\alpha} : \mathbb{D}^{cs} \to \mathbb{D}^{uu}$ and a subsequence $n_j \to +\infty$ such that:

$$h^{\tilde{x},\tilde{y}}_{\infty,\alpha} = \lim_{j\to+\infty} h^{\tilde{x},\tilde{y}}_{n_j}$$

Every $h^{\tilde{x},\tilde{y}}_{\infty,\alpha}$ has its corresponding graph, and we note $W^{\tilde{x}}_{\infty,\alpha}(\tilde{y})$ to the image by $\psi_{\tilde{x}}$ of this graph. The following claim is crucial for the theorem.

Claim 2.2.8. *For every $\tilde{z} \in B(\tilde{x}, \epsilon)$ and every $\alpha \in \mathcal{J}_{\tilde{z}}^{\tilde{x}}$, we have that $H_g(W_{\infty,\alpha}^{\tilde{x}}(\tilde{z})) \subseteq H_g(\tilde{z}) + E_A^{ss}$.*

Proof. Take $\tilde{z} \in B(\tilde{x}, \epsilon)$ and $\alpha \in \mathcal{J}_{\tilde{z}}^{\tilde{x}}$. Then by hypothesis there is subsequence $n_j \to +\infty$ such that $W_{n_j}^{\tilde{x}}(\tilde{z}) \to W_{\infty,\alpha}^{\tilde{x}}(\tilde{z})$. Given $\tilde{y} \in W_{\infty,\alpha}^{\tilde{x}}(\tilde{z})$ we want to prove that $H_g(\tilde{y}) \in H_g(\tilde{z}) + E_A^{ss}$. Call $\tilde{z}_{n_j} = \widetilde{W}_g^{uu}(\tilde{y}) \cap W_{n_j}^{\tilde{x}}(\tilde{z})$ (see Figure 2.2 below). Then $\tilde{z}_{n_j} \to \tilde{y}$ when $j \to +\infty$ and $\tilde{g}^{n_j}(\tilde{z}_{n_j}) \in \widetilde{\mathcal{F}}_g^{cs}(\tilde{g}^{n_j}(\tilde{z}))$. If $H_g(\tilde{y}) = H_g(\tilde{z})$ we're done. Suppose by the contrary that $H_g(\tilde{z}) \neq H_g(\tilde{y})$. Then $H_g(\tilde{z}_{n_j}) \to H_g(\tilde{y}) \neq H_g(\tilde{z})$. Note that $\tilde{z}$ and $\tilde{z}_{n_j}$ belong to the same leaf $\widetilde{\mathcal{F}}_g^{cs}$, and the same for $\tilde{g}^{n_j}(\tilde{z})$ and $\tilde{g}^{n_j}(\tilde{z}_{n_j})$.

Since $\mathcal{F}_g^{cs}$ is almost parallel to $\widetilde{\mathcal{W}}_f^{cs}$ and H_g is C^0-close to $H_f = h \circ \tilde{p}$, we have that there is constant $C_1 > 0$ such that for any $j \in \mathbb{N}$:

$$\|\Pi_0^{uu}\left(H_g(\tilde{g}^{n_j}(\tilde{z})) - H_g(\tilde{g}^{n_j}(\tilde{z}_{n_j}))\right)\| < C_1$$

Therefore by semiconjugacy we deduce that

$$\|\Pi_0^{uu}\left(A^{n_j}(H_g(\tilde{z})) - A^{n_j}(H_g(\tilde{z}_{n_j}))\right)\| < C_1$$

Since A is hyperbolic this implies $H_g(\tilde{z}) - H_g(\tilde{z}_{n_j}) \in E_A^{ss}$ for every $j \in \mathbb{N}$. Finally taking the limit when $j \to +\infty$ we get $H_g(\tilde{z}) - H_g(\tilde{y}) \in E_A^{ss}$. □

We are going to solve the two problems mentioned above in the same way. Suppose first that $\tilde{z} \in B(\tilde{x}, \epsilon)$ has two different limits $W_{\infty,\alpha}^{\tilde{x}}(\tilde{z})$ and $W_{\infty,\beta}^{\tilde{x}}(\tilde{z})$. Then there are points $\tilde{z}_1 \in W_{\infty,\alpha}^{\tilde{x}}(\tilde{z})$ and $\tilde{z}_2 \in W_{\infty,\beta}^{\tilde{x}}(\tilde{z})$ that belong to the same $\widetilde{W}_g^{uu}$-leaf. The previous claim implies that $H_g(\tilde{z}_1)$ and $H_g(\tilde{z}_2)$ belong to $H_g(\tilde{z}) + E_A^{ss}$ and this can happen if and only if $H_g(\tilde{z}_1) = H_g(\tilde{z}_2)$ which contradicts the injectivity of $H_g|_{\widetilde{W}_g^{uu}}$.

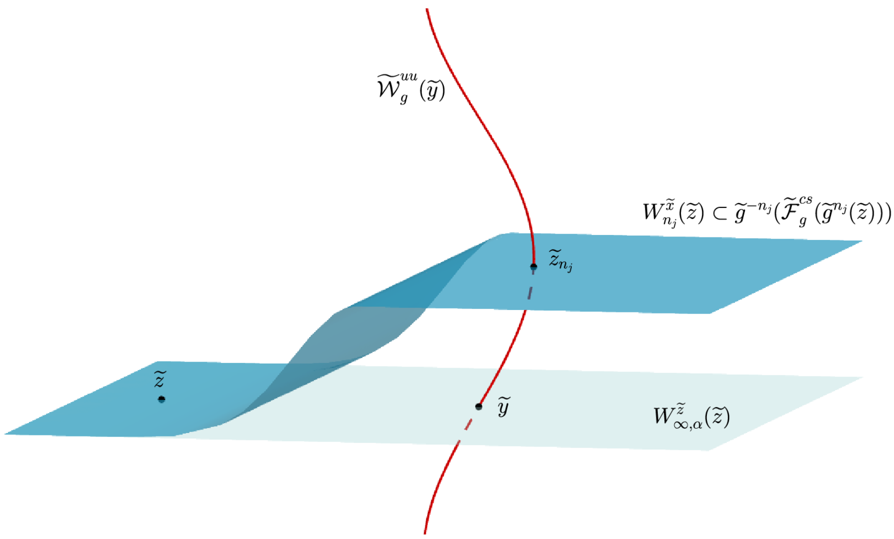


FIGURE 2.2: Plaques does not merge.

For the second problem we manage the same way. Let's suppose there are points $\tilde{z}_1 \neq \tilde{z}_2$ in $B(\tilde{x}, \epsilon)$ such that their limits $W_{\infty,\alpha}^{\tilde{x}}(\tilde{z}_1)$ and $W_{\infty,\beta}^{\tilde{x}}(\tilde{z}_2)$ have non empty intersection. Then we get two points $\tilde{y}_1 \in W_{\infty,\alpha}^{\tilde{x}}(\tilde{z}_1)$ and $\tilde{y}_2 \in W_{\infty,\beta}^{\tilde{x}}(\tilde{z}_2)$ inside the same

$\widetilde{\mathcal{W}}^{uu}_g$-leaf. Again the previous claim implies $H_g(\widetilde{z}_1) = H_g(\widetilde{z}_2)$ and this contradicts the injectivity of $H_g|_{\widetilde{\mathcal{W}}^{uu}_g}$.

To sum up, we obtained that for every $\widetilde{x} \in \widetilde{M}$ and every $\widetilde{y} \in B(\widetilde{x}, \epsilon)$, the limit $W^{\widetilde{x}}_\infty(\widetilde{y})$ of the $W^{\widetilde{x}}_n(\widetilde{y})$ leaves is unique, and for every pair of points $\widetilde{y}, \widetilde{z} \in B(\widetilde{x}, \epsilon)$, their limits are disjoint or coincide. These limits are also $\widetilde{g}$-invariant. To get that it is truly a foliation, it's enough to observe the following: given two points $\widetilde{z}, \widetilde{w} \in B(\widetilde{x}, \epsilon)$, we have that $W^{\widetilde{x}}_\infty(\widetilde{z})$ and $\widetilde{\mathcal{W}}^{uu}_g(\widetilde{w})$ intersect in a unique point. Since the leaves of $\widetilde{\mathcal{W}}^{uu}_g$ varies continuously and the plaques of $W^{\widetilde{x}}_\infty$ either coincide or are disjoint, we get a continuous function from $\mathbb{D}^{cs} \times \mathbb{D}^{uu}$ to a neighbourhood of $\widetilde{x}$ which sends horizontal disks to $W^{\widetilde{x}}_\infty$-plaques. This proves that these plaques form a foliation. Since the leaves of the foliations are tangent to small cones around the E^{cs}_g direction and these leaves are $\widetilde{g}$-invariant, we get that the foliation is tangent to E^{cs}_g. Finally observe that the foliation $\widetilde{\mathcal{W}}^{cs}_g$ has the same properties that $\widetilde{\mathcal{F}}^{cs}_g$. Thus we have global product structure between $\widetilde{\mathcal{W}}^{cs}_g$ and $\widetilde{\mathcal{W}}^{uu}_g$. □

Corollary 2.2.9. *If $g \in \mathcal{PH}_f(M)$ verifies the following conditions:*

- *g is uu and ss proper.*
- *g is* SADC *with global product structure.*

Then g is dynamically coherent, center fibered and has global product structure.

We end this section with a proposition which finishes the proof of the equivalence between dynamically coherence and center fibered, with σ properness and SADC (in presence of global product structure).

Proposition 2.2.10. *If $g \in \mathcal{PH}_f(M)$ is dynamically coherent, center fibered and has global product structure, then it is σ-proper ($\sigma = ss, uu$) and* SADC *with global product structure.*

Proof. Take a dynamically coherent and center fibered $g \in \mathcal{PH}_f(M)$, such that $\widetilde{\mathcal{W}}^{cs}_g$ and $\widetilde{\mathcal{W}}^{uu}_g$ have global product structure, and $\widetilde{\mathcal{W}}^{cu}_g$ and $\widetilde{\mathcal{W}}^{ss}_g$ have global product structure. Suppose that there is $\widetilde{y} \in \widetilde{\mathcal{W}}^{uu}_g(\widetilde{x})$ such that $H_g(\widetilde{y}) = H_g(\widetilde{x})$. Then by center fibered this implies that $\widetilde{y} \in \widetilde{\mathcal{W}}^{c}_g(\widetilde{x}) \subset \widetilde{\mathcal{W}}^{cs}_g(\widetilde{x})$. But then $\{\widetilde{x}, \widetilde{y}\} \in \widetilde{\mathcal{W}}^{uu}_g(\widetilde{x}) \cap \widetilde{\mathcal{W}}^{cs}_g(\widetilde{x})$ which violates the global product structure. This implies that $H_g|_{\widetilde{\mathcal{W}}^{uu}_g}$ is a homeomorphism, and therefore g is uu-proper by Lemma 2.2.3. The case ss-proper is exactly the same. Now recall that $\mathcal{W}^{cs}_g$ and $\mathcal{W}^{cu}_g$ are uniformly transverse to E^{uu}_g and E^{ss}_g respectively, and so in order to prove that g is SADC, it remains to show that $\widetilde{\mathcal{W}}^{cs}_g$ and $\widetilde{\mathcal{W}}^{cu}_g$ are almost parallel to the center-stable and center-unstable foliations of f. This is quite direct, since σ-properness, global product structure and center fibered implies that $H_g(\widetilde{\mathcal{W}}^{cs}_g(\widetilde{x})) = E^{ss}_A + H_g(\widetilde{x})$ and $H_g(\widetilde{\mathcal{W}}^{cu}_g(\widetilde{x})) = E^{uu}_A + H_g(\widetilde{x})$ for every $\widetilde{x} \in \widetilde{M}$. This implies SADC because H_g is at bounded distance from $h \circ \widetilde{p} = H_f$. □

2.3 Dynamical coherence is open and closed

To obtain the main theorem of this chapter, we have to prove that SADC, σ-properness ($\sigma = ss, uu$) and global product structure (between the strong stable/unstable manifolds and the ones given by SADC) are C^1 open and closed properties among $\mathcal{PH}_f(M)$. Then we can apply Corollary 2.2.9 to a whole connected component as long as it contains a diffeomorphism with such properties.

2.3.1 SADC is C^1 open and closed

Proposition 2.3.1. SADC *is a* C^1 *open property among* $\mathcal{PH}_f(M)$.

Proof. This is pretty direct since the same foliation works by the continuity of the E^{ss} and E^{uu} bundles. Take $g \in \mathcal{PH}_f(M)$ with SADC property and let $\mathcal{F}^{cs}_g$, $\mathcal{F}^{cu}_g$ be the foliations given by the SADC property. These foliations are transverse to E^{uu}_g, E^{ss}_g and their lifts are almost parallel to $\widetilde{\mathcal{W}}^{cs}_f$ and $\widetilde{\mathcal{W}}^{cu}_f$ respectively. Then $\angle(\widetilde{\mathcal{F}}^{cs}_g(\tilde{x}), E^{uu}_g)(\tilde{x}) > \epsilon$ for every $\tilde{x} \in \widetilde{M}$ and there is $\mathcal{U}(g)$ a neighbourhood of g in the C^1 topology s.t. for every $g' \in \mathcal{U}(f)$ we have $\angle(E^{uu}_{g'}(\tilde{x}), E^{uu}_g(\tilde{x})) < \frac{\epsilon}{2}, \forall \tilde{x} \in \widetilde{M}$. Take $\mathcal{F}^{cs}_{g'} = \mathcal{F}^{cs}_g$, then

$$
\begin{aligned}
\angle(\widetilde{\mathcal{F}}^{cs}_{g'}(\tilde{x}), E^{uu}_{g'}(\tilde{x})) + \frac{\epsilon}{2} &> \angle(\widetilde{\mathcal{F}}^{cs}_{g'}(\tilde{x}), E^{uu}_{g'}(\tilde{x})) + \angle(E^{uu}_{g'}(\tilde{x}), E^{uu}_g(\tilde{x})) \\
&\geq \angle(\widetilde{\mathcal{F}}^{cs}_{g'}(\tilde{x}), E^{uu}_g(\tilde{x})) \\
&= \angle(\widetilde{\mathcal{F}}^{cs}_g(\tilde{x}), E^{uu}_g(\tilde{x})) > \epsilon > 0
\end{aligned}
$$

This implies that $\angle(\widetilde{\mathcal{F}}^{cs}_{g'}(\tilde{x}), E^{uu}_{g'}(\tilde{x})) > \frac{\epsilon}{2}$ for every $\tilde{x} \in \widetilde{M}$. Then every $g' \in \mathcal{U}(g)$ has foliations $\mathcal{F}^{cs}_{g'}$, $\mathcal{F}^{cu}_{g'}$ transverse to $E^{uu}_{g'}$, $E^{ss}_{g'}$ and thus each $g' \in \mathcal{U}(g)$ verifies SADC. □

Proposition 2.3.2. SADC *is a* C^1 *closed property among* $\mathcal{PH}_f(M)$.

Proof. Take $g_n \in \mathcal{PH}_f(M)$ such that $g_n \to^{C^1} g$ and every $g_n \to g$ is SADC. Call $E^{cs}_n = E^{ss}_{g_n} \oplus E^c_{g_n}$ and let $\mathcal{F}^{cs}_n$, $\mathcal{F}^{cu}_n$ be the foliations given by the SADC property for every $n \in \mathbb{N}$. By the C^1 convergence we have $E^{cs}_n \to E^{cs}_g$ and $E^{uu}_n \to E^{uu}_g$. Let $\eta = \angle(E^{cs}_g, E^{uu}_g)$ (minimum bound of the angle). Now since $E^{cs}_n \to E^{cs}_g$ there is $n_1 > 0$ such that $\angle(E^{uu}_g, E^{cs}_{n_1}) > \frac{\eta}{2}$. Take $\mathcal{F}^{cs}_{n_1}$ foliation uniformly transverse to $E^{uu}_{n_1}$. Then there is $n_2 > 0$ such that $g^{-n_2}_{n_1}(\mathcal{F}^{cs}_{n_1})$ is contained in a cone centered at $E^{cs}_{n_1}$ of radius $\frac{\eta}{2}$. Thus $g^{-n_2}_{n_1}(\mathcal{F}^{cs}_{n_1})$ is uniformly transverse to E^{uu}_g. To finish the proof, just notice that since g_n is isotopic to f, it fixes the class of foliations almost parallel to any $\tilde{f}$-invariant foliation. Then $\tilde{g}^{-n_2}_{n_1}(\widetilde{\mathcal{F}}^{cs}_{n_1})$ is almost parallel to $\widetilde{\mathcal{W}}^{cs}_f$ and g is SADC. □

2.3.2 σ-proper is C^1 open

The following remark refers to a classical fact about hyperbolicity that we'll be useful.

Remark 2.3.3. *Given* $f \in \mathcal{PH}(M)$, *there exist constants* $1 < \lambda_f < \Delta_f$ *and there exists* $\mathcal{U}$ *a* C^1*-neighbourhood of* f *s.t. for every* $g \in \mathcal{U}$, $\tilde{x} \in \widetilde{M}$ *and* $R > 0$ *we have:*

$$\widetilde{\mathcal{W}}^{uu}_g(\tilde{g}(\tilde{x}), \lambda_f R) \subset \tilde{g}(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R)) \subset \widetilde{\mathcal{W}}^{uu}_g(\tilde{g}(\tilde{x}), \Delta_f R)$$

Analogously for $\widetilde{\mathcal{W}}^{ss}_g$ *by applying* $\tilde{g}^{-1}$.

Proposition 2.3.4. *For* $\sigma = ss, uu$, *being* σ*-proper is a* C^1 *open property among* $\mathcal{PH}_f(M)$.

Proof. Given $g \in \mathcal{PH}_f(M)$ that is σ-proper, we must find a neighbourhood $\mathcal{U}(g)$ in the C^1 topology such that every $g' \in \mathcal{U}(g)$ is σ-proper. Remark 2.2.2 says that it's enough to find a neighbourhood $\mathcal{U}(g)$ and $R_1 > 0$ such that for every $g' \in \mathcal{U}(g)$ and $\tilde{x} \in \widetilde{M}$:

$$(H_{g'})^{-1}(D^{\sigma}_A(H_{g'}(\tilde{x}), 1)) \cap \widetilde{\mathcal{W}}^{\sigma}_{g'}(\tilde{x}) \subseteq \widetilde{\mathcal{W}}^{\sigma}_{g'}(\tilde{x}, R_1)$$

Since g is σ-proper, we know $H_g|_{\widetilde{\mathcal{W}}_g^\sigma(\tilde{x})} : \widetilde{\mathcal{W}}_g^\sigma(\tilde{x}) \to H_g(\tilde{x}) + E_A^\sigma$ is a homeomorphism. Then there is $R_1 > 0$ s.t.

$$H_g(\widetilde{\mathcal{W}}_g^\sigma(\tilde{x}, R_1)^c) \cap D_A^\sigma(H_g(\tilde{x}), 2) = \emptyset$$

Call $A_{r,R,g'}^\sigma(\tilde{x})$ the annulus $\widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x}, R) \setminus \widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x}, r)$ for $R > r > 0$. Then for $R_2 > \Delta_g R_1$ we have that

$$H_g(A_{R_1,R_2,g}^\sigma(\tilde{x})) \cap D_A^\sigma(H_g(\tilde{x}), 2) = \emptyset$$

where we take $\Delta_g > 1$ like in Remark 2.3.3. Now since H_g is continuous and Γ-invariant, it is uniformly continuous. Then there is $\epsilon_1 > 0$ s.t. if $d(\tilde{x}, \tilde{y}) < \epsilon_1$ then $d(H_g(\tilde{x}), H_g(\tilde{y})) < 1/4$. Take the following C^1-neighbourhoods:

- From uniform hyperbolicity there is $\mathcal{U}_1(g)$ such that the constants Δ_g and λ_g are uniform in $\mathcal{U}_1(g)$ (see Remark 2.3.3).
- The continuous variation of the leaves in the C^1 topology says that for every $\epsilon_1 > 0$ and $R_2 > 0$, there is $\mathcal{U}_2(g)$ and $\delta > 0$ s.t. for every $g' \in \mathcal{U}_2(g)$ and every pair of points $\tilde{x}, \tilde{y}$ with $d(\tilde{x}, \tilde{y}) < \delta$ we have $d_{C^1}(\widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x}, R_2), \widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{y}, R_2)) < \epsilon_1$.
- Take $\mathcal{U}_3(g) = \{g' \in \mathcal{PH}_f(M) : d_{C^0}(H_{g'}, H_g) < 1/4\}$.

Finally take $\mathcal{U}_g := \mathcal{U}_1(g) \cap \mathcal{U}_2(g) \cap \mathcal{U}_3(g)$. Now, let $g' \in \mathcal{U}(g)$ and $\tilde{x}, \tilde{y}$ such that $\tilde{y} \in A_{R_1,R_2,g'}^\sigma(\tilde{x})$. Then there is $\tilde{z} \in A_{R_1,R_2,g}^\sigma(\tilde{x})$ such that $d(\tilde{z}, \tilde{y}) < \epsilon_1$ and from uniform continuity we get $d(H_g(\tilde{z}), H_g(\tilde{y})) < 1/4$. Since $\tilde{z} \in A_{R_1,R_2,g}^\sigma(\tilde{x})$ and $d(H_g(\tilde{z}), H_g(\tilde{y})) < 1/4$, applying the triangular inequality we obtain:

$$\begin{aligned} 2 &< \|H_g(\tilde{z}) - H_g(\tilde{x})\| \leq \|H_g(\tilde{z}) - H_g(\tilde{y})\| + \|H_g(\tilde{y}) - H_g(\tilde{x})\| \\ &\leq 1/4 + \|H_g(\tilde{y}) - H_g(\tilde{x})\| \end{aligned}$$

Therefore $\|H_g(\tilde{y}) - H_g(\tilde{x})\| > 2 - 1/4$. Once again the triangular inequality gives:

$$\begin{aligned} 2 - 1/4 &< \|H_g(\tilde{y}) - H_g(\tilde{x})\| \\ &\leq \|H_g(\tilde{y}) - H_{g'}(\tilde{y})\| + \|H_{g'}(\tilde{y}) - H_{g'}(\tilde{x})\| + \|H_{g'}(\tilde{x}) - H_g(\tilde{x})\| \\ &\leq 1/4 + \|H_{g'}(\tilde{y}) - H_{g'}(\tilde{x})\| + 1/4 \end{aligned}$$

and we conclude that $\|H_{g'}(\tilde{y}) - H_{g'}(\tilde{x})\| > 2 - 3/4 > 1$, which means

$$H_{g'}(A_{R_1,R_2,g'}^\sigma(\tilde{x})) \cap D_A^\sigma(H_{g'}(\tilde{x}), 1) = \emptyset \text{ for every } \tilde{x} \in \widetilde{M} \tag{2.1}$$

Finally this implies

$$(H_{g'})^{-1}(D_A^\sigma(H_{g'}(\tilde{x}), 1)) \cap \widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x}) \subseteq \widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x}, R_1) \text{ for every } \tilde{x} \in \widetilde{M}$$

If it weren't the case, there will be $\tilde{y} \in \widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x})$ such that $H_{g'}(\tilde{y}) \in D_A^\sigma(H_{g'}(\tilde{x}), 1)$ but $\tilde{y} \notin \widetilde{\mathcal{W}}_{g'}^\sigma(\tilde{x}, R_2)$. By the choice of Δ_g we know that there is $n \in \mathbb{Z}$ s.t. $\widetilde{g'}^n(\tilde{y}) \in A_{R_1,R_2,g'}^\sigma(\widetilde{g'}^n(\tilde{x}))$ and $H_{g'}(\widetilde{g'}^n(\tilde{x})) \in \widetilde{\mathcal{W}}_{g'}^\sigma(\widetilde{g'}^n(\tilde{x}), 1)$. This contradicts (2.1) above. □

2.3.3 SADC + σ-proper + GPS is C^1 open

In this subsection we are going to prove that given $g \in \mathcal{PH}_f(M)$ which is σ-proper and SADC with global product structure, then every g' sufficiently C^1 close to g is σ-proper and SADC with global product structure (maybe with a different foliation than the original one).

Proposition 2.3.5. *Let $g \in \mathcal{PH}_f(M)$ be such that g is σ-proper, for $\sigma = ss, uu$, and SADC with global product structure. Then there is a C^1 neighbourhood $\mathcal{U}$ of g such that every $g' \in \mathcal{U}$ is σ proper and SADC with global product structure.*

Proof. Take $g \in \mathcal{PH}_f(M)$ such that g is σ-proper for $\sigma = ss, uu$, and SADC with their corresponding foliations $\widetilde{\mathcal{F}}^{cs}_g$ and $\widetilde{\mathcal{F}}^{cu}_g$, and suppose that $\widetilde{\mathcal{F}}^{cs}_g$ and $\widetilde{\mathcal{W}}^{uu}_g$ have global product structure (the other case is symmetric). By Theorem 2.2.7 we now that g is dynamically coherent, center-fibered and $\widetilde{\mathcal{W}}^{cs}_g$ and $\widetilde{\mathcal{W}}^{uu}_g$ have global product structure.

Now we can replace $\widetilde{\mathcal{F}}^{cs}_g$ by $\widetilde{\mathcal{W}}^{cs}_g$ in the SADC definition of g (i.e. with these new foliations g is still SADC by Proposition 2.2.10). We have to do this interchange because we need Γ-invariance of the foliations (this will be clear in a moment).

By Proposition 2.3.1 we know there is a C^1 neighbourhood $\mathcal{U}_1$ of g such that every $g' \in \mathcal{U}_1$ is SADC (applying the proposition to $\widetilde{\mathcal{W}}^{cs}_g$).

On the other hand by Proposition 2.3.4 we know there is a C^1 neighbourhood $\mathcal{U}_2$ of g such that every $g' \in \mathcal{U}_2$ is σ-proper. Moreover we know there is $R_1 > 0$ such that:

$$(H_{g'})^{-1}(D^\sigma_A(H_{g'}(\widetilde{x}), 1)) \cap \widetilde{\mathcal{W}}^\sigma_{g'}(\widetilde{x}) \subseteq \widetilde{\mathcal{W}}^\sigma_{g'}(\widetilde{x}, R_1) \tag{2.2}$$

for every $\widetilde{x} \in \widetilde{M}$ and $g' \in \mathcal{U}_2$.

Claim 2.3.6. *There is a C^1 neighbourhood $\mathcal{U}_3$ of g such that for every $g' \in \mathcal{U}_3$ and every $\widetilde{x} \in \widetilde{M}$ we have that:*

$$\widetilde{\mathcal{W}}^{uu}_{g'}(\widetilde{x}, R_1) \cap \widetilde{\mathcal{W}}^{cs}_g(\widetilde{x}) = \{\widetilde{x}\} \tag{2.3}$$

Proof. Just notice that for every $\widetilde{x} \in \widetilde{M}$ there is $\epsilon(\widetilde{x}) > 0$ and a C^1 neighbourhood $\mathcal{U}(\widetilde{x})$ of g such that for every $g' \in \mathcal{U}(\widetilde{x})$ and every $\widetilde{y} \in B(\widetilde{x}, \epsilon(\widetilde{x}))$ Equation (2.3) holds. Since $\widetilde{\mathcal{W}}^{cs}_g$ is Γ invariant, we can restrict ourselves to a compact fundamental domain. Then, we can cover this fundamental domain by finite balls $B(\widetilde{x}_1, \epsilon(\widetilde{x}_1)), \dots, B(\widetilde{x}_N, \epsilon(\widetilde{x}_N))$ and take $\mathcal{U}_3 = \cap_{j=1}^N \mathcal{U}(\widetilde{x}_j)$. This proves the claim. □

To end the proof of the proposition, take $g' \in \mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3$ and take two points $\widetilde{x}, \widetilde{y} \in \widetilde{M}$. Now it is easy to see that $\widetilde{\mathcal{W}}^{uu}_{g'}(\widetilde{x}) \cap \widetilde{\mathcal{W}}^{cs}_g(\widetilde{y})$ is non empty. By Equation (2.2) and Equation (2.3) of the claim, we have that $\widetilde{\mathcal{W}}^{uu}_{g'}(\widetilde{x}) \cap \widetilde{\mathcal{W}}^{cs}_g(\widetilde{y})$ is exactly one point. This proves the global product structure between $\widetilde{\mathcal{W}}^{uu}_{g'}$ and $\widetilde{\mathcal{W}}^{cs}_g$. □

Remark 2.3.7. *In the proof of the previous claim, we need the foliation to be Γ-invariant, in order to restrict ourselves to points in a fundamental domain, and then later to be able to take a finite cover. That's why we interchange $\widetilde{\mathcal{F}}^{cs}_g$ with $\widetilde{\mathcal{W}}^{cs}_g$ in the proof.*

2.3.4 SADC + σ-proper + GPS is C^1 closed

The previous proposition shows that σ-properness and SADC with global product structure are C^1 open among $\mathcal{PH}_f(M)$. To finish the proof of the main theorem we have to prove that they are also C^1-closed properties. This is the most difficult part of the theorem. For the proof we are going to use once again Theorem 2.2.7. Recall that we already know that SADC is C^1 closed by Proposition 2.3.2.

Before getting into the proof, recall that if A is a hyperbolic matrix with a splitting $\mathbb{R}^{d-c} = E^{ss}_A \oplus E^{uu}_A$, for $\sigma = ss, uu$ and $\widetilde{x} \in \widetilde{M}$ we denote by

$$\Pi^\sigma_{\widetilde{x}} : \mathbb{R}^{d-c} \to \widetilde{x} + E^\sigma_A$$

to the corresponding orthogonal projection.

Theorem 2.3.8. *For $\sigma = ss, uu$, being σ-proper and SADC with global product structure is a C^1-closed property in $\mathcal{PH}_f(M)$.*

Proof. Take a sequence $\{g_k\} \subset \mathcal{PH}_f(M)$ with $g_k \to g$ in the C^1 topology, such that for every $k \in \mathbb{N}$, g_k is σ-proper and SADC with global product structure. By Proposition 2.3.2 we know that g is SADC. We have to prove that g is σ-proper and that we have global product structure. We are going to prove case $\sigma = uu$, but the case $\sigma = ss$ is completely symmetric.

Note that every g_k is in the hypothesis of Theorem 2.2.7, then for every $k \in \mathbb{N}$ there is a g_k-invariant foliation $\mathcal{W}^{cs}_{g_k}$ tangent to $E^{ss}_{g_k} \oplus E^c_{g_k}$ such that:

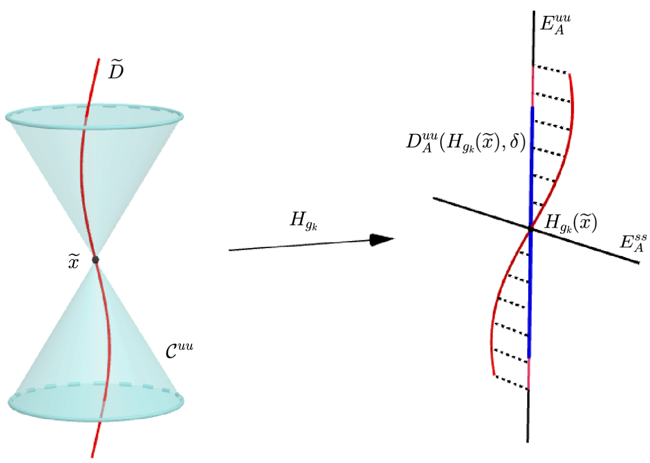
$$\widetilde{\mathcal{W}}^{cs}_{g_k}(\tilde{x}) = (H_{g_k})^{-1}(H_{g_k}(\tilde{x}) + E^{ss}_A) \tag{2.4}$$

Then by center-fibered we have that:

$$H_{g_k}(\tilde{x}) = H_{g_k}(\tilde{y}) \text{ if and only if } \tilde{y} \in \widetilde{\mathcal{W}}^c_{g_k}(\tilde{x}) \tag{2.5}$$

Claim 2.3.9. *Given $\epsilon > 0$, there exists $\delta > 0$, a cone field $\mathcal{C}^{uu}$ around E^{uu}_g and k_0 such that if $k \geq k_0$ and $\widetilde{D}$ is a disk tangent to $\mathcal{C}^{uu}$ of internal radius larger than ϵ and centered at $\tilde{x}$, then*

$$D^{uu}_A(H_{g_k}(\tilde{x}), \delta) \subset \Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\widetilde{D})$$



Proof. This is because $g_k \to g$ in the C^1 topology, and so $E^\sigma_k \to E^\sigma_g$ for every σ. Then M has a finite cover of local product structure boxes B of size smaller than $\epsilon > 0$ such that for $k \geq k_0$ large enough, these are local product structure boxes for g_k too. Moreover, we can take these boxes B small enough in order to have the following:

- The boxes $2B$ and $3B$ are also local product structure boxes for g_k.
- For every B of the covering and every disk $D \subset M$ tangent to $\mathcal{C}^{uu}$ of internal radius larger than ϵ and centered at a point $x \in B$ we have that D intersects in a unique point in $3B$ every center-stable plaque of $\mathcal{W}^{cs}_{g_k}$ which intersects $2B$.

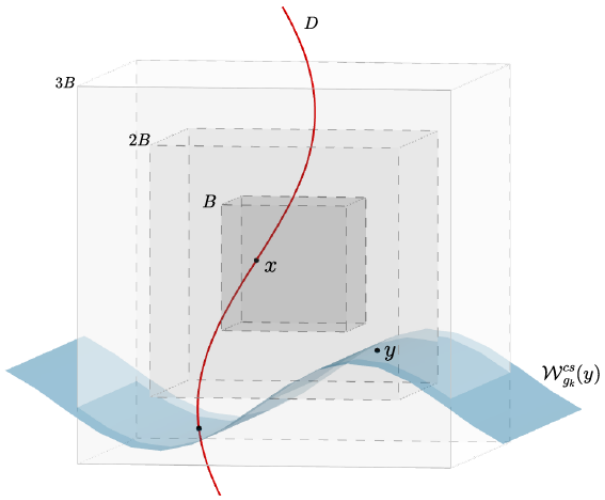


FIGURE 2.3: Boxes with local product structure.

We can lift this cover by boxes and obtain a cover of $\widetilde{M}$ with the same properties as above. The previous condition plus Equation (2.5) implies that:

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(2\widetilde{B}) \subset \Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\widetilde{D})$$

where $\widetilde{D} \subset \widetilde{M}$ is a lift of a disk $D \subset M$ as above. Using the injectivity of H_{g_k} restricted to $\widetilde{\mathcal{W}}^{uu}_{g_k}$ leaves, we have that given a connected component $2\widetilde{B}$ of a lift we have $int(\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(2\widetilde{B})) \neq \emptyset$ and every point $\tilde{y} \in \widetilde{B}$ verifies that $\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\tilde{y}))$ lies in the interior of $\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(2\widetilde{B})$. Since there are finite boxes (in M), there is a uniform $\delta > 0$ such that $\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\widetilde{B}))$ is at bounded δ distance from the boundary of $\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(2\widetilde{B})$ independently of the box $\widetilde{B}$. We deduce that every disk $\widetilde{D}$ of internal radius ϵ and centered at $\tilde{x}$ and tangent to a small cone around E^{uu}_g verifies that $\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\widetilde{D})$ contains $D^{uu}_A(H_{g_k}(\tilde{x}), \delta)$ as desired. □

Claim 2.3.10. *For k sufficiently large enough and for every pair of points $\tilde{x}, \tilde{y} \in \widetilde{M}$, we have that $\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})$ and $\widetilde{\mathcal{W}}^{cs}_{g_k}(\tilde{y})$ have non-trivial intersection.*

Proof. Given two points $\tilde{x}, \tilde{y} \in \widetilde{M}$, take $\mathcal{S}$ the segment in $E^{uu}_A + H_{g_k}(\tilde{x})$ that connects $H_k(\tilde{x})$ and $\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\tilde{y}))$. Fix $\epsilon > 0$ and take the corresponding $\delta > 0$, the cones $\mathcal{C}^{uu}$ and $k_0 > 0$ from the previous claim. Then we have that

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, \epsilon)) \supset D^{uu}_A(H_{g_k}(\tilde{x}), \delta)$$

In the same way we get that

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, 2\epsilon)) \supset D^{uu}_A(H_{g_k}(\tilde{x}), 2\delta)$$

We can apply inductively the same argument, and since the segment $\mathcal{S}$ is compact, we get $m \in \mathbb{N}$ such that

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, m\epsilon)) \supset D^{uu}_A(H_{g_k}(\tilde{x}), m\delta) \supset \mathcal{S}$$

Then there is a point $\tilde{z} \in \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, m\epsilon) \subset \widetilde{\mathcal{W}}^{uu}_g(\tilde{x})$ such that

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\tilde{z}) = \Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}(\tilde{y})$$

Then $H_{g_k}(\tilde{z}) - H_{g_k}(\tilde{y}) \in E^{ss}_A$ and this implies by Equation (2.4) that $\tilde{z} \in \widetilde{\mathcal{W}}^{cs}_{g_k}(\tilde{y})$. We conclude that $\tilde{z} \in \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}) \cap \widetilde{\mathcal{W}}^{cs}_{g_k}(\tilde{y})$ as desire.

□

Claim 2.3.11. *For k sufficiently large, the foliations $\widetilde{\mathcal{W}}^{uu}_g$ and $\widetilde{\mathcal{W}}^{cs}_{g_k}$ have global product structure. Equivalently, the map $\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}|_{\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})} : \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}) \to H_{g_k}(\tilde{x}) + E^{uu}_A$ is a homeomorphism.*

Proof. By the previous claim, we only have to prove that the intersection between $\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})$ and $\widetilde{\mathcal{W}}^{cs}_k(\tilde{y})$ is unique for every pair of points $\tilde{x}, \tilde{y} \in \widetilde{M}$. Since the leaf $\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})$ intersects transversely $\widetilde{\mathcal{W}}^{cs}_k(\tilde{y})$ for every $\tilde{x}, \tilde{y}$ and $H_{g_k}(\widetilde{\mathcal{W}}^{cs}_{g_k}(\tilde{y})) = H_{g_k}(\tilde{y}) + E^{ss}_A$ we have that $H_{g_k}(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}))$ is topologically transverse to $H_{g_k}(\tilde{y}) + E^{ss}_A$. This implies that

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} : H_{g_k}(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})) \to H_{g_k}(\tilde{x}) + E^{uu}_A$$

is a covering and since E^{uu}_A is contractible, it must be injective. This proves that $\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}$ restricted to $\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})$ is a homeomorphism onto $H_{g_k}(\tilde{x}) + E^{uu}_A$. □

This claim proves that g is SADC with global product structure. To finish the proof of the theorem we must prove there is $R > 0$ such that:

$$(H_g)^{-1}(D^{uu}_A(H_g(\tilde{x}), 1)) \cap \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}) \subset \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R)\ , \ \ \forall \tilde{x} \in \widetilde{M}$$

Fix $\tilde{x} \in \widetilde{M}$. We know that $d_{C^0}(H_{g_k}, H_g) < K_*$ for some constant $K_* > 0$. The previous claim says that that the restriction of $\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}$ to $\widetilde{\mathcal{W}}^{uu}_g(\tilde{x})$ is a homeomorphism onto $H_{g_k}(\tilde{x}) + E^{uu}_A$. Then there is $R_1 = R_1(x) > 0$ such that

$$\Pi^{uu}_{H_{g_k}(\tilde{x})} \circ H_{g_k}((\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R_1))^c) \cap D^{uu}_A(H_{g_k}(\tilde{x}), 1 + 2K_*) = \emptyset$$

Take $\tilde{y} \in \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R_1)^c$. Then applying the triangular inequality we obtain

$$\begin{aligned}
1 + 2K_* &< \|\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\tilde{x})) - \Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\tilde{y}))\| \\
&\leq \|\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\tilde{x})) - \Pi^{uu}_{H_{g_k}(\tilde{x})}(H_g(\tilde{x}))\| \\
&+ \|\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_g(\tilde{x})) - \Pi^{uu}_{H_{g_k}(\tilde{x})}(H_g(\tilde{y}))\| \\
&+ \|\Pi^{uu}_{H_{g_k}(\tilde{x})}(H_g(\tilde{y})) - \Pi^{uu}_{H_{g_k}(\tilde{x})}(H_{g_k}(\tilde{y}))\| \\
&< K_* + \|H_g(\tilde{x}) - H_g(\tilde{y})\| + K_*
\end{aligned}$$

where we are using the fact that orthogonal projections does not increase the norm. Thus $\|H_g(\tilde{x}) - H_g(\tilde{y})\| > 1$ and therefore we get

$$H_g(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R_1)^c) \cap D^{uu}_A(H_{g_k}(\tilde{x}), 1) = \emptyset$$

which is the same as

$$(H_g)^{-1}(D^{uu}_A(H_g(\tilde{x}), 1)) \cap \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}) \subset \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R_1)$$

Then we have proved that the function φ is well defined where

$$\varphi(x) = \inf\{R > 0 : (H_g)^{-1}(D^{uu}_A(H_g(\tilde{x}), 1)) \cap \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}) \subset \widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, R)\}$$

By Remark 2.2.2 we have to prove that φ is uniformly bounded in $\widetilde{M}$ for getting uu-proper. Since φ is Γ-periodic (because H_g is Γ periodic), it's enough to restrict ourselves to points in a fundamental domain which is compact. Thus it is enough to show that if $\tilde{x}_n \to \tilde{x}$ then $\varphi(\tilde{x}_n) \leq \varphi(\tilde{x})$. To prove this, note that $H_g(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, \varphi(\tilde{x})))$ contains $D^{uu}_A(H_g(\tilde{x}), 1)$. Now for every $\epsilon > 0$ we can find $\delta > 0$ such that

$$D^{uu}_A(H_g(\tilde{x}), 1 + \delta) \subset H_g(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}, \varphi(\tilde{x}) + \epsilon))$$

By continuous variation of the $\widetilde{\mathcal{W}}^{uu}_g$-leaves and since H_g is continuous, we deduce that for n large enough $H_g(\widetilde{\mathcal{W}}^{uu}_g(\tilde{x}_n, \varphi(\tilde{x}) + \epsilon))$ contains $D^{uu}_A(H_g(\tilde{x}_n), 1)$. This shows that $\limsup \varphi(\tilde{x}_n) \leq \varphi(\tilde{x}) + \epsilon$. Since the choice of $\epsilon > 0$ was arbitrary, we get the desire result. □

2.3.5 Proof of the Theorem 2.1.13 (Theorem A)

In this subsection we are going to finish the proof of Theorem 2.1.13 (Theorem A in the introduction). Let $g \in \mathcal{PH}_f(M)$ be a diffeomorphism in the same connected component of a partially hyperbolic diffeomorphism g' such that:

- g' is dynamically coherent.
- g' is center fibered.
- $\widetilde{\mathcal{W}}^{cs}_{g'}$ and $\widetilde{\mathcal{W}}^{uu}_{g'}$ have GPS and, $\widetilde{\mathcal{W}}^{cu}_{g'}$ and $\widetilde{\mathcal{W}}^{ss}_{g'}$ have GPS.

Then by Proposition 2.2.10 we have that g' is σ proper and SADC (and has global product structure).

Propositions 2.3.1, 2.3.2, 2.3.4, 2.3.5 and Theorem 2.3.8 tell us that σ-proper, SADC and global product structure are open and closed properties in the C^1 topology among $\mathcal{PH}_f(M)$. In particular this implies that g is σ-proper, SADC and has global product structure. By Theorem 2.2.7 (and Corollary 2.2.9) we get that g is dynamically coherent, center fibered and has global product structure. This ends the proof.

2.4 Leaf conjugacy and proof of Theorem 2.1.14 (Theorem B)

In this section we are going to prove Theorem 2.1.14 (Theorem B in the introduction). For the proof we're going to show that center-fibered implies plaque expansiveness. Then we can conclude by Theorem 1.4.5 and a connectedness argument.

Proposition 2.4.1. *Every $g \in \mathcal{PH}^0_f(M)$ is plaque expansive.*

Proof. Take $g \in \mathcal{PH}^0_f(M)$. We know from Theorem 2.1.13 that g is dynamically coherent and center fibered. Now take $\epsilon > 0$ and two ϵ-pseudo orbits $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ such that:

(i) $g(x_n) \in \mathcal{W}^c_g(x_{n+1})$, for every $n \in \mathbb{Z}$.

(ii) $g(y_n) \in \mathcal{W}^c_g(y_{n+1})$, for every $n \in \mathbb{Z}$.

(iii) $d(x_n, y_n) < \epsilon$, for every $n \in \mathbb{Z}$.

Then, we have to prove that x_0 and y_0 belong to the same center leaf. To do so, first take two lifts $\tilde{x}_0$ and $\tilde{y}_0$ of x_0 and y_0 respectively such that $d(\tilde{x}_0, \tilde{y}_0) < \epsilon$. Since ϵ is small enough, we have a unique pair of sequences $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$ and $\{\tilde{y}_n\}_{n\in\mathbb{Z}}$ that check points (i),(ii) and (iii).

Notice that center fibered imply that $H_g(\tilde{g}(\tilde{x}_n)) = H_g(\widetilde{x_{n+1}})$ and $H_g(\tilde{g}(\tilde{y}_n)) = H_g(\widetilde{y_{n+1}})$. By semiconjugacy we get

$$A(H_g(\tilde{x}_n)) = H_g(\tilde{g}(\tilde{x}_n)) = H_g(\widetilde{x_{n+1}})$$

$$A(H_g(\tilde{y}_n)) = H_g(\tilde{g}(\tilde{y}_n)) = H_g(\widetilde{y_{n+1}})$$

Then $\{H_g(\tilde{x}_n)\}_{n\in\mathbb{Z}}$ and $\{H_g(\tilde{y}_n)\}_{n\in\mathbb{Z}}$ are orbits of the linear map $A : \mathbb{R}^{d-c} \to \mathbb{R}^{d-c}$ and

$$\begin{aligned}
\|A^n(H_g(\tilde{x}_0) - H_g(\tilde{y}_0))\| &= \|A^n(H_g(\tilde{x}_0)) - A^n(H_g)(\tilde{y}_0))\| = \|H_g(\tilde{x}_n) - H_g(\tilde{y}_n)\| \\
&\leq \|H_g(\tilde{x}_0) - H_f(\tilde{x}_0)\| + \|H_f(\tilde{x}_0) - H_f(\tilde{y}_0)\| \\
&+ \|H_f(\tilde{y}_0) - H_g(\tilde{y}_0)\| \leq 2K^* + \epsilon
\end{aligned}$$

for every $n \in \mathbb{Z}$ and some constant $K^* > 0$. Since A is hyperbolic, this can happen if and only if $H_g(\tilde{x}_0) = H_g(\tilde{y}_0)$. By center-fibered we conclude that $\tilde{y}_0 \in \widetilde{\mathcal{W}}^c_g(\tilde{x}_0)$ and therefore $y_0 \in \mathcal{W}^c_g(x_0)$ proving that g is plaque-expansive. □

Proof of Theorem 2.1.14. Take g_0 and g_1 diffeomorphisms in the same connected component of $\mathcal{PH}^0_f(M)$, and a continuous path $\{g_t\}_{t\in[0,1]} \subset \mathcal{PH}^0_f(M)$ connecting g_0 and g_1.

By Theorem 2.1.13 every g_t is dynamically coherent and center fibered. Then by Proposition 2.4.1 every g_t is plaque expansive. We can apply Theorem 1.4.5 (Theorem 7.1 in [HPS77]) to every g_t and obtain a neighbourhood $\mathcal{U}(t)$ such that every partially hyperbolic in $\mathcal{U}(t)$ is leaf conjugate to g_t. Since $[0,1]$ is compact and connected, we can cover $\{g_t\}_{t\in[0,1]}$ by a finite union $\cup_{i=1}^{i=l}\mathcal{U}(t_i)$. Since leaf-conjugacy is an equivalence relation we conclude that g_0 is leaf conjugate to g_1. □

Chapter 3

Some hyperbolicity and robust transitivity

In this chapter we are going to treat robust transitivity. In the first section we present a few simple facts about transitivity. In Section 3.2 we introduce the SH-Saddle property, and we prove that it is a C^1-open condition among partially hyperbolic diffeomorphisms. This new definition is a generalization of the SH condition introduced by E. Pujals and M. Sambarino in [PS06]. This new approach allows us to treat the symplectic case (in particular geodesic flows), something that the previous definition couldn't. In Section 3.3 we state a result that we are going to apply several times in the whole chapter. In Section 3.4 we are going to apply the previous results to build new derived from Anosov (DA) examples and prove Theorem C. The novelty of these examples is that they have mixed behaviour on center leaves, in particular they present a dominated splitting non coherent with its Anosov part, a difference with its predecessors DA examples. Finally in Section 3.6 we extend the SH-Saddle property for flows (with emphasis in geodesic flows), and we get a criterion for Riemannian metrics that guarantees robust transitivity for their corresponding geodesic flows (Theorem D).

3.1 Transitivity

Recall that a diffeomorphism $f : M \to M$ is said to be *transitive* if there is $x \in M$ such that $\overline{\mathcal{O}^+(f,x)} = M$. This definition is simple and easy to understand, however when it comes to work, it can be a little difficult to deal with. The following proposition gives an equivalent definition that is more manageable.

Proposition 3.1.1. *Given a diffeomorphism $f : M \to M$ the following are equivalent:*

- *f is topologically transitive,*
- *for every pair of open sets U and V there is $N \in \mathbb{Z}$ such that $f^N(U) \cap V \neq \emptyset$.*

Proof. If f is transitive, then we have a point $x \in M$ such that $\overline{\mathcal{O}^+(f,x)} = M$. This implies that for every pair of open sets U and V there are positive integers $n > m \in \mathbb{N}$ such that $f^m(x) \in U$ and $f^n(x) \in V$. Then $f^{n-m}(U) \cap V \neq \emptyset$.

The other equivalence is a little more subtle. Since M is a differentiable manifold, it has a numerable basis of the topology $\{B_n : n \in \mathbb{N}\}$. Now for every $n \in \mathbb{N}$ take the subset $A_n = \{y \in M : f^k(y) \in B_n \text{ for some } k \geq 0\}$. By hypothesys A_n is open and dense, then the set $R = \cap_{n\geq 0} A_n$ is a residual set. Finally notice that for every $x \in R$ we have $\omega(f,x) = M$, which implies that $\overline{\mathcal{O}^+(f,x)} = M$. □

In the same way, we say that a flow $\varphi_t : M \to M$ is *transitive* if there is a point $x \in M$ such that $\overline{\mathcal{O}^+(\varphi,x)} = M$. Notice that if for a given $T \in \mathbb{R}^+$ the diffeomorphism

$f := \varphi_T : M \to M$ is transitive, then the flow φ is transitive. This is clear since:

$$M = \overline{\bigcup_{n\in N} \varphi_T^n(x)} = \overline{\bigcup_{n\in N} \varphi_{T+n}(x)} \subseteq \overline{\bigcup_{t\in\mathbb{R}^+} \varphi_t(x)} \subseteq M$$

The opposite direction is not true, a simple counterexample is the linear irrational flow in the torus where every orbit is dense, but the orbits of the time 1 map leaves invariant some transversal sections. Nevertheless, we obtain the following remark.

Remark 3.1.2. *Given a flow $\varphi_t : M \to M$, if there is $T \in \mathbb{R}^+$ such that φ_T is transitive as a diffeomorphism, then the flow φ_t is transitive.*

3.2 SH-Saddle property

In this section we introduce the main definition of this chapter. We begin with a few simple definitions. Let V be a $\mathbb{R}$-vector space with an inner product. A *cone* in V is a subset $\mathcal{C}$ such that there is a non-degenerate quadratic form $B : V \to \mathbb{R}$ such that

$$\mathcal{C} = \{v \in V : B(v) \le 0\}$$

Analogously we can express the cone $\mathcal{C}$ according to a decomposition $V = E \oplus F$:

$$\mathcal{C} = \{v = (v_E, v_F) : \|v_E\| \le a\|v_F\|\} \tag{3.1}$$

for some $a > 0$. In this case we observe that $B(v) = -a^2\|v_F\|^2 + \|v_E\|^2$. We are going to say that the number a in Equation (3.1) is the *size* of the cone. In some cases we will note by $\mathcal{C}_a$ instead of $\mathcal{C}$ to make emphasis on the size of $\mathcal{C}$. The *dimension of a cone* is the maximal dimension of any subspace contained in the cone.

Recall that for $f \in \mathcal{PH}(M)$ we have a splitting of the form $TM = E_f^{ss} \oplus E_f^c \oplus E_f^{uu}$. We are going to note by $c = \dim E_f^c$. Then given $f \in \mathcal{PH}(M)$ a *d-center cone* in $x \in M$ is simply a cone $\mathcal{C}(x)$ in $E_f^c(x)$ of dimension $d \le c$. We now introduce the main definitions of this chapter. Recall that

$$\mathcal{W}_f^*(x,\varepsilon) := \{y \in \mathcal{W}_f^*(x) : d_{\mathcal{W}_f^*}(x,y) < \varepsilon\}$$

is the ε-ball in $\mathcal{W}_f^*$ of center x and radius ε for $* \in \{ss, uu\}$.

Definition 3.2.1 (SH-Saddle property for unstable foliations)**.** *Given $f \in \mathcal{PH}(M)$ we say that the strong unstable foliation $\mathcal{W}_f^{uu}$ has the SH-Saddle property of index $d \le c$ if there are constants $L > 0$, $a > 0$, $\lambda_0 > 1$ and $C > 0$ such that the following hold.*

For every point $x \in M$, there is a point $x^u \in \mathcal{W}_f^{uu}(x, L)$ such that:

1. *There is a d-center cone field of size a along the forward orbit of x^u which is Df-invariant, i.e. there exist $\mathcal{C}_a^u(f^l(x^u)) \subset E_f^c(f^l(x^u))$ such that $Df(\mathcal{C}_a^u(f^l(x^u))) \subset \mathcal{C}_a^u(f^{l+1}(x^u))$ for every $l \ge 0$.*
2. *$\|Df^n_{f^l(x^u)}(v)\| \ge C\lambda_0^n\|v\|$ for every $v \in \mathcal{C}_a^u(f^l(x^u))$ and every $l, n \ge 0$.*

Notice that condition 1 is equivalent to the following:

1'. For every $l \ge 0$ there is a splitting $T_{f^l(x^u)} = E^{c_1}(f^l(x^u)) \oplus E^{c_2}(f^l(x^u))$ which is Df-invariant for the future, and dominated: there is $\lambda > 1$ s.t. for every $l, n \ge 0$

$$\|D^n|_{E^{c_1}(f^l(x^u))}\|C\lambda^n \le \|Df^n|_{E^{c_2}(f^l(x^u))}\|$$

Remark 3.2.2. *In case the strong unstable foliation has SH-Saddle property of index $d = c$ where $c = dimE^c_f$, we get the original definition of SH property introduced in [PS06].*

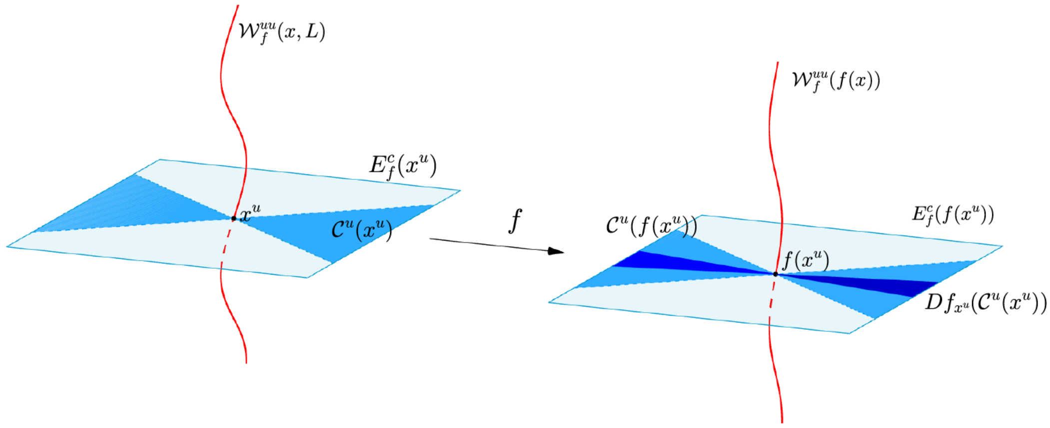


FIGURE 3.1: Strong unstable manifold with SH-Saddle property

We can make an analogous definition for the strong stable foliation. In this case, we ask for the invariance of the cones for the past.

Definition 3.2.3 (SH-Saddle property for stable foliations)**.** *Given $f \in \mathcal{PH}(M)$ we say that the strong stable foliation $\mathcal{W}^{ss}_f$ has the SH-Saddle property of index $d \leq c$ if there are constants $L > 0$, $a > 0$, $\lambda_0 > 1$ and $C > 0$ such that the following hold.*

For every point $x \in M$, there is a point $x^s \in \mathcal{W}^{ss}_f(x, L)$ such that:

1. *There is a d-center cone field of size a along the backward orbit of x^s which is Df^{-1}-invariant, i.e. there exist $\mathcal{C}^s_a(f^l(x^s))$ such that $Df^{-1}(\mathcal{C}^s_a(f^l(x^s))) \subset \mathcal{C}^s_a(f^{l-1}(x^s))$ for every $l \leq 0$.*
2. *$\|Df^n_{f^l(x^s)}(v)\| \geq C\lambda_0^{-n}\|v\|$ for every $v \in \mathcal{C}^s_a(f^l(x^s)))$ and every $l, n \leq 0$.*

Definition 3.2.4 (SH-Saddle property)**.** *We say that $f \in \mathcal{PH}(M)$ has (d_1, d_2) SH-Saddle property if the following conditions hold:*

1. *$\mathcal{W}^{ss}_f$ has the SH-Saddle property of index d_1.*
2. *$\mathcal{W}^{uu}_f$ has the SH-Saddle property of index d_2.*

Remark 3.2.5. *Notice that not necessarily we have $d_1 + d_2 = c$, in fact in many cases we are going to have $d_1 + d_2 < c$. In some parts of this chapter, for simplicity and when is not needed we are going to omit the indexes (d_1, d_2) and we're just going to say that a partially hyperbolic diffeomorphism has the SH-Saddle property.*

Like in the uniformly or partially hyperbolic setting, the SH-Saddle property is independent of the Riemannian metric. This is the aim of the next proposition.

Proposition 3.2.6. *The SH-Saddle property does not depend on the choice of the Riemannian metric.*

Proof. First notice that given two Riemannian metrics $\|\cdot\|_1$ and $\|\cdot\|_2$ on a compact manifold, there are positive constants α and β such that $\alpha\|\cdot\|_1 \leq \|\cdot\|_2 \leq \beta\|\cdot\|_1$.

Now let $S \subset TM$ be a Df-invariant subbundle with a splitting of the form $S = E \oplus F$ and suppose there is a cone $\mathcal{C}_1 \subset S$ that can be written as

$$\mathcal{C}_1 = \{v = (v_E, v_F) : \|v_E\|_1 \leq a\|v_F\|_1\}$$

for some $a > 0$. Then if we take the cone $\mathcal{C}_2 \subset S$ defined by

$$\mathcal{C}_2 = \{v = (v_E, v_F) : \|v_E\|_2 \leq \frac{\beta}{\alpha} a\|v_F\|_2\}$$

a direct calculation shows that $\mathcal{C}_2$ is Df-invariant if and only if $\mathcal{C}_1$ is Df-invariant. Moreover $\mathcal{C}_1$ is uniformly expanding: $\|Df^l(v)\|_1 \geq C\lambda^l\|v\|_1$ for every $v \in \mathcal{C}_1$ and $l \geq 0$, if and only if $\mathcal{C}_2$ is uniformly expanding: $\|Df^l(v)\|_2 \geq (C/\alpha)\lambda^l\|v\|_2$ for every $v \in \mathcal{C}_2$ and $l \geq 0$. This proves that the SH-Saddle property does not depend on the Riemannian metric. □

As a corollary of the previous proposition we get the following fact.

Proposition 3.2.7. *A partially hyperbolic diffeomorphism f has the SH-Saddle property if and only if f^k has the SH-Saddle property for some $k \in \mathbb{N}$.*

3.2.1 SH-Saddle property and hyperbolic subsets

Let's see a different approach of the SH-Saddle property in order to get a better understanding of what it means. Let $f \in \mathcal{PH}(M)$ be such that its unstable foliation has the SH-Saddle property of index $d \leq c$ and let $L > 0$, $a > 0$, $\lambda_0 > 1$ and $C > 0$ be the constants given by Definition 3.2.1. We can define the following subset:

$$H^+_{\lambda_0,d}(f) = \{x \in M : \text{conditions 1 and 2 of Definition 3.2.1 are satisfied}\} \quad (3.2)$$

Then the unstable foliation has the SH-Saddle property of index d if and only if

$$H^+_{\lambda_0,d}(f) \cap \mathcal{W}^{uu}_f(x, L) \neq \emptyset \text{ for every } x \in M.$$

In the same way let $f \in \mathcal{PH}(M)$ be such that its stable foliation has the SH-Saddle property of index d and let $L > 0$, $a > 0$, $\lambda_0 > 1$ and $C > 0$ be the constants given by Definition 3.2.3, then we can define the following subset:

$$H^-_{\lambda_0,d}(f) = \{x \in M : \text{conditions 1 and 2 of Definition 3.2.3 are satisfied}\} \quad (3.3)$$

and the stable foliations has the SH-Saddle property of index d if and only if

$$H^-_{\lambda_0,d}(f) \cap \mathcal{W}^{ss}_f(x, L) \neq \emptyset \text{ for every } x \in M.$$

Remark 3.2.8. *The sets $H^\sigma_{\lambda_0,d}(f)$ are closed subsets of M, for $\sigma = +, -$.*

Now suppose that the unstable foliation of $f \in \mathcal{PH}(M)$ has SH-Saddle property of index $d \leq c$ where $c = \dim E^c_f$. Then we can take the following subset:

$$\Lambda^+_f = \overline{\bigcup\{\omega(x) : x \in H^+_{\lambda_0,d}(f)\}}$$

Now observe that the set Λ_f^+ is a hyperbolic subset if $c = d$ but in case $c < d$ is not necessarily hyperbolic. However, it does have a dominated splitting of the form

$$T_{\Lambda_f^+}M = E \oplus F$$

where $\dim E = \dim E_f^{ss} + (c - d)$, $\dim F = \dim E_f^{uu} + d$ and the bundle F is uniformly expanding.

Remark 3.2.9. *Notice that despite Λ_f^+ is not hyperbolic, we can associate a stable set $\mathcal{W}^s(\Lambda_f^+)$ to it. Lets assume for a moment that this stable set is a manifold (this is the "ideal" picture). In this case the dimension of the stable manifold is smaller or equal to $\dim E$, and we have proved that $\mathcal{W}^s(\Lambda_f^+) \cap \mathcal{W}_f^{uu}(x) \neq \emptyset$ for every $x \in M$ where $\dim E + \dim E_f^{uu} < d$. In the next subsection we will see that this non-transverse intersection is robust, and hence it resembles in a sense to the existence of a blender.*

3.2.2 SH-Saddle property is C^1-open

In this subsection we are going to prove that the SH-Saddle property is C^1 open among $\mathcal{PH}(M)$. According to Definition 3.2.4 we only have to prove that having an unstable manifold with SH-Saddle property 3.2.1, and having a stable manifold with SH-Saddle property 3.2.3 are C^1 open properties. We are going to focus on the unstable case, since the stable case is completely symmetric. We begin with a few simple lemmas that only uses the properties of the C^1 topology.

Lemma 3.2.10. *Suppose that the unstable foliation of $f \in \mathcal{PH}(M)$ has SH-Saddle property of index d. Then there is $\epsilon_0 > 0$ such that every $y \in M$ satisfying $d(y, H^+_{\lambda_0,d}(f)) < \epsilon_0$ has a d-center cone $\mathcal{C}^u(y)$. Moreover there is $\delta_0 > 0$ such that if $d(y, H^+_{\lambda_0,d}(f)) < \delta_0$ then $Df(\mathcal{C}^u(y)) \subseteq \mathcal{C}^u(f(y))$.*

Proof. We know that for every $x \in H^+_{\lambda_0,d}(f)$ there is a cone $\mathcal{C}^u(x)$ which is Df-invariant. Now for the first part of the lemma just notice that since the family of center cones comes from a non-degenerate quadratic form, we can extend this quadratic form to neighbours by continuity. For the second part just observe that f is uniformly continuous. □

Since the family of cones varies continuously, the same family of cones in the lemma above is still invariant for every g sufficiently close to f. Then we obtain the following.

Lemma 3.2.11. *Suppose that the unstable foliation of $f \in \mathcal{PH}(M)$ has the SH-Saddle property of index d, and let $\epsilon_0 > 0$ and $\delta_0 > 0$ be as in Lemma 3.2.10. Then there is a C^1-neighbourhood $\mathcal{U}_0(f)$ of f such that if $g \in \mathcal{U}_0(f)$ and $d(y, H^+_{\lambda_0,d}(f)) < \delta_0$ then $Dg(\mathcal{C}^u(y)) \subseteq \mathcal{C}^u(g(y))$.*

Now we are ready to prove the main theorem of this section.

Theorem 3.2.12. *Suppose that the unstable foliation of $f \in \mathcal{PH}(M)$ has SH-Saddle property of index d. Then there are constants $\lambda > 1$, $L > 0$ and a C^1-neighbourhood $\mathcal{V}$ of f such that, if $g \in \mathcal{V}$ then $H^+_{\lambda,d}(g) \cap \mathcal{W}_g^{uu}(x, L) \neq \emptyset$ for every $x \in M$ (i.e.: the unstable foliation $\mathcal{W}_g^{uu}$ has the SH-Saddle property of index d with constants $\lambda > 1$ and $L > 0$).*

Proof. Take $f \in \mathcal{PH}(M)$ such that its strong unstable foliation has the SH-Saddle property of index d. That means there are constants $\lambda_0 > 1$ and $L_0 > 0$ such that

Definition 3.2.1 holds. Then we have:

$$H^+_{\lambda_0,d}(f) \cap \mathcal{W}^{uu}_f(x, L_0) \neq \emptyset \text{ for every } x \in M.$$

Let $\epsilon_0 > 0$, $\delta_0 > 0$ and $\mathcal{U}_0(f)$ be as in Lemma 3.2.10 and Lemma 3.2.11. Take $c > 0$ such that $\frac{\lambda_0}{1+c} = \lambda_1 > 1$. Take $\epsilon > 0$, $\delta_1 \in (0, \delta_0)$ and $\mathcal{U}_1(f) \subseteq \mathcal{U}_0(f)$ such that if $g \in \mathcal{U}_1(f)$, $d(x, y) < \delta_1$ and $v \in T_xM$ has $\|v\| = 1$ then:

$$\|Df_x(v) - Dg_y(w)\| < \epsilon$$

where $w = P_{x,y}(v) \in T_yM$ is the parallel transport of v from x to y. We can take $\epsilon > 0$ small enough such that if $d(x, y) < \delta_1$ and $g \in \mathcal{U}_1(f)$ then:

$$\frac{1}{1+c} \leq \frac{\|Df_x\|}{\|Dg_y\|} \leq 1+c \;\text{ and }\; \frac{1}{1+c} \leq \frac{m\{Df_x\}}{m\{Dg_y\}} \leq 1+c \tag{3.4}$$

Finally let $K^+ = \sup\{\|Df|_{E^c(x)}\| : x \in M\}$ and $K^- = \inf\{m\{Df|_{E^c(x)}\} : x \in M\}$. We can assume that K^+ and K^- are C^1-uniform on a neighbourhood $\mathcal{U}_2(f) \subseteq \mathcal{U}_1(f)$.

Let $m_1 \in \mathbb{Z}^+$ be large enough such that $(\lambda_u)^{m_1} > 2$ and for any $g \in \mathcal{U}_2(f)$ and any $x \in M$ we have

$$\mathcal{W}^{uu}_g(g^{m_1}(x), L_0) \subset g^{m_1}(\mathcal{W}^{uu}_g(x, \delta_1/4)) \tag{3.5}$$

Now take $m_2 \in \mathbb{Z}^+$ sufficiently large, and take λ_2 such that

$$C\lambda_1^{m_2}.(K^-)^{m_1} \geq \lambda_2 > 1 \tag{3.6}$$

Let $\mathcal{U}_3(f)$ and $\delta_2 \in (0, \delta_1/2)$ be such that if $d(x, y) < \delta_2$ and $g \in \mathcal{U}_3(f)$, then $d(f^j(x), g^j(y)) < \delta_1$, for $0 \leq j \leq m_2$.

Finally take $\mathcal{U}_4(f)$ such that for every $g \in \mathcal{U}_4(f)$ we have

$$d_H(\mathcal{W}^{uu}_g(x, L_0), H^+_{\lambda_0,d}(f)) < \delta_2 \tag{3.7}$$

We claim that every $g \in \mathcal{V} = \mathcal{U}_4(f)$ has unstable manifold with SH-Saddle property of index d. In fact, we are going to see that g^{k_0} has this property for $k_0 = m_1 + m_2$, with constans $2L$ and $\lambda_2 > 1$ (where λ_2 comes from Equation (3.6)). Then we conclude by Proposition 3.2.7.

To see this, take $g \in \mathcal{V}$ and $x \in M$. We know there are points $x_0 \in H^+_{\lambda_0,d}(f)$ and $z_0^u \in \mathcal{W}^{uu}_g(x, L)$ such that $d(x_0^u, z_0^u) < \delta_2$. Notice that since $\delta_2 < \delta_0$ we know there is a center cone $\mathcal{C}^u(z_0^u)$.

Now let $v \in \mathcal{C}^u(z_0^u)$. Since $d(x_0, z_0^u) < \delta_2$ we have that $d(f^j(x_0), g^j(z_0^u)) < \delta_1$ for $0 \leq j \leq m_2$. Then we have:

$$\|Dg^{m_2}_{z_0^u}(v)\| \geq \frac{\|Df^{m_2}_{x_0^u}(w)\|}{(1+c)^{m_2}} \geq C\left(\frac{\lambda_0}{1+c}\right)^{m_2} = \lambda_1^{m_2}\|w\| \tag{3.8}$$

where $w = P_{z_0^u, x_0^u}(v)$ is the parallel transport of v from z_0^u to x_0^u. Now,

$$\|Dg^{k_0}_{z_0^u}(v)\| = \|Dg^{m_1}_{g^{m_2}(z_0^u)}(Dg^{m_2}_{z_0^u}(v))\| \geq (K^-)^{m_1}\lambda_1^{m_2}\|v\| \geq \lambda_2\|v\| \tag{3.9}$$

Now by (3.5), we can apply the same argument to $\mathcal{W}^{uu}_g(g^{k_0}(z_0^u), L)$, and we can find points $x_1 \in H^+_{\lambda_0,d}(f)$ and $z_1^u \in \mathcal{W}^{uu}_g(g^{k_0}(z_0^u), L)$ such that $d(x_1, z_1^u) < \delta_2 < \epsilon_0$. Then, there is a center cone $\mathcal{C}^u(z_1^u)$ and for every vector $v \in \mathcal{C}^u(z_1^u)$ we have $\|Dg^{k_0}_{z_1^u}(v)\| \geq$

$\lambda_2\|v\|$. Call $y_1^u = g^{-k_0}(z_0^u)$. Now, by (3.5) we have that $g^{-m_1}(z_1^u) \in D_g^{uu}(g^{m_2}(y_1^u), \delta_1/4)$ and this implies that

$$d(y_1^u, x_0) \leq d(y_1^u, y_0^u) + d(y_0^u, x_0) < \frac{\delta_1}{4} + \delta_2 \leq \frac{\delta_1}{4} + \frac{\delta_1}{2} < \delta_1$$

and there is a d-center cone $\mathcal{C}^u(y_1^u)$. Moreover we have that

$$d(g^j(y_1^u), g^j(z_0^u)) < \delta_1 \text{ for every } 0 \leq j \leq m_2$$

and by applying the same calculations as in (3.8) and (3.9) we have

$$\|Dg^{2k_0}_{y_1^u}(v)\| \geq (\lambda_2)^2\|v\| \tag{3.10}$$

Inductively, we can find sequences $\{z_n^u\}_{n\in\mathbb{N}}$, $\{y_n^u\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$, which verify the following:

- $z_{n+1}^u \in \mathcal{W}_g^{uu}(g^{k_0}(z_n^u), L)$.
- $x_n \in H^+_{\lambda_0,d}(f)$.
- $d(z_n^u, x_n) < \delta_2$.
- $y_n^u = g^{-k_0 n}(z_n^u)$.

By the same arguments as above, the distance between y_n^u and x_0 is

$$d(y_n^u, x_0) \leq \frac{\delta_1}{2} + \frac{\delta_1}{4} + \dots \frac{\delta_1}{2^{n-1}} < \delta_1$$

and there is a d-center cone $\mathcal{C}^u(g^j(y_n^u)))$ such that $Dg(\mathcal{C}^u(g^j(y_n^u)))) \subset \mathcal{C}^u(g^{j+1}(y_n^u)))$ for every $j \in \{0, \dots, nk_0\}$. Moreover $y_n^u \in \mathcal{W}_g^{uu}(x, 2L)$.

By the same reasons than above, if $v \in \mathcal{C}^u(g^{ik_0}(y_n^u))$ we have

$$\|Dg^{jk_0}_{g^{ik_0}(y_n^u)}(v)\| \geq (\lambda_2)^j\|v\| \text{ for every } 0 \leq i+j \leq n$$

Finally, if we take $y \in \mathcal{W}_g^{uu}(x, 2L)$ as an accumulation point of $\{y_n^u\}_{n\in\mathbb{N}}$ we obtain that there is a d-center cone $\mathcal{C}^u(g^l(y)) \subset E_g^c(g^l(y))$ such that $Dg(\mathcal{C}^u(g^l(y))) \subset \mathcal{C}^u(g^{l+1}(y))$ for every $l \geq 0$ and $\|Dg^{jk_0}(v)\| \geq \lambda_2^j\|v\|$, for every $v \in \mathcal{C}^u(g^{lk_0}(y))$ and $j, l > 0$. □

Since the C^1-openess of the SH-Saddle property for stable manifolds is completely analogous we get the following corollary.

Corollary 3.2.13. *The SH-Saddle property is C^1-open among $\mathcal{PH}(M)$.*

We end this section with a key corollary from Theorem 3.2.12 that we're going to use for the proof of the main theorem of this chapter.

Corollary 3.2.14. *Let $f \in \mathcal{PH}(M)$ be such that its unstable foliation has the SH-Saddle property of index d and let $\lambda > 1$, $\delta_1 > 0$ and $\mathcal{V}$ as in the Theorem 3.2.12. Take $g \in \mathcal{V}$, $x^u \in H^+_{\lambda,d}(g)$ and D^u a center disk of dimension d tangent to $\mathcal{C}^u_{x^u}$. Then there is $N > 0$ such that $g^n(D^u)$ contains a disk W^{cu} of diameter bigger than $2\delta_1$ for every $n \geq N$.*

Analogously with the stable foliation.

Proof. Just notice that if $g \in \mathcal{V} \subseteq \mathcal{U}_1(f)$ and $d(x, y) < \delta_1$ (here $\mathcal{U}_1(f)$ and δ_1 come from Equation (3.4)), then every point in D^u is expanded by $\lambda > 1$ for the future in the $\mathcal{C}_g^u$ direction. Basically $\text{diam}(g(D^u)) \geq \lambda \text{diam}(D^u)$ and so eventually by induction we obtain a center disk with diameter bigger than $2\delta_1$. □

3.3 A criterion for openess

In this section we are going to state a result that we are going to apply in many cases. Roughly speaking it says that given a continuous function between topological spaces of the same dimension, and such that the fibers (preimages of points) of the funcion are small enough, then the image contains an open set.

The ideas comes from dimension theory (see for example [HW41]) and basically says in which conditions a map can not decrease its topological dimension. The version we are going to use comes from [LZ22] which is an improvement from a result of [BK02] (Proposition 3.2). We begin with a few definitions.

Definition 3.3.1. *Suppose $f : X \to Y$ is a continuous function between metric spaces. We say that $y \in Y$ is a stable value if there is $\epsilon > 0$ such that if $d_{C^0}(f, g) < \epsilon$ then $y \in Im(g)$.*

Remark 3.3.2. *Let $Y = \mathbb{R}^n$ and suppose that $f : X \to \mathbb{R}^n$ has a stable value y, then $Im(f)$ contains an open set. This is easy to see: take $\epsilon > 0$ from the definition of stable value, and take a vector $v \in \mathbb{R}^d$ with $\|v\| < \epsilon$. Then the map $g : X \to \mathbb{R}^d$ defined by $g(x) = f(x) - v$ satisfies $d_{C^0}(f, g) \leq \|v\| < \epsilon$. Since y is a stable value, there is a point $x \in X$ such that $g(x) = y$ and this is equivalent to $f(x) = y + v$. Since v was arbitrary we get $B_{\mathbb{R}^n}(y, \epsilon) \subset Im(f)$.*

Definition 3.3.3. *Given a continuous function $f : X \to Y$ and $\delta > 0$ we say that f is $\delta-$light if for every $y \in Y$ the connected components of $f^{-1}(y)$ have diameter smaller than δ.*

Proposition 3.3.4 (Theorem F in [LZ22])**.** *Given $d \in \mathbb{N}$ and $r > 0$ there is $\rho = \rho(d, r) > 0$ such that every ρ-light map $f : [-r, r]^d \to \mathbb{R}^d$ has a stable value.*

The version stated in [LZ22] is for maps $f : [0, 1]^d \to \mathbb{R}^d$ but the proof can be adapted to maps $f : [-r, r]^d \to \mathbb{R}^d$ for a fixed $r > 0$. Now combining this proposition and Remark 3.3.2 we have the following corollary.

Corollary 3.3.5. *Fix $d \in \mathbb{N}$ and $r > 0$, and take the corresponding $\rho = \rho(d, r) > 0$ from Proposition 3.3.4. Then the image of every ρ-light map $f : [-r, r]^d \to \mathbb{R}^d$ contains an open set.*

3.4 Derived from Anosov revisited

In this section we are going to prove Theorem C. In particular, we are going to build examples of robustly transitive derived from Anosov diffeomorphisms with any center dimension and with as many different behaviours on center leaves as desire. In particular, we are going to build examples with dominated splitting (a necessary condition according to [BDP03]) that is not coherent with the hyperbolic dominated splitting of its Anosov part, as in every previous example constructed this way ([Mañ78],[Shu71], [BD96] & [BV00], see also [Pot12]).

3.4.1 Robust transitivity for DA diffeomorphisms

Take $\mathbb{R}^d$ and let $p : \mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$ be the canonical projection. Take $A \in SL(d, \mathbb{Z})$ a hyperbolic matrix and call f_A to the diffeomorphism induced in the torus $\mathbb{T}^d$, i.e. $f_A \circ p = p \circ A$. By a slightly abuse of notation we are going to note $f_A = A$. Suppose that A admits a dominated spiltting of the form $\mathbb{R}^d = E^{ss}_A \oplus E^{ws}_A \oplus E^{wu}_A \oplus E^{uu}_A$ and call $E^s_A = E^{ss}_A \oplus E^{ws}_A$ and $E^u_A = E^{uu}_A \oplus E^{wu}_A$. We are going to note by $\Pi^\sigma : \mathbb{R}^d \to E^\sigma_A$ to the canonical projections, for $\sigma = ws, ss, s, wu, uu, u$.

Now given A as above, let $\mathcal{PH}_A(\mathbb{T}^d)$ be the set

$$\mathcal{PH}_A(\mathbb{T}^d) = \left\{ f \in \mathcal{PH}(\mathbb{T}^d) : f \simeq A,\ \dim E_f^{ss} = \dim E_A^{ss},\ \dim E_f^{uu} = \dim E_A^{uu} \right\}$$

where $f \simeq A$ means the maps are isotopic. Given $f \in \mathcal{PH}_A(\mathbb{T}^d)$ and a lift $\widetilde{f}$ to $\mathbb{R}^d$, we know from Theorem 2.1.8 (see also Remark 2.1.11) that there exist a continuous and surjective map $H_f : \mathbb{R}^d \to \mathbb{R}^d$ such that $A \circ H_f = H_f \circ \widetilde{f}$. The map H_f is $\mathbb{Z}^d$-invariant and therefore it induces a continuous and surjective map $h_f : \mathbb{T}^d \to \mathbb{T}^d$ such that $h_f \circ f = A \circ h_f$. Moreover, the map H_f varies continuously with the diffeomorphism f in the C^0-topology and the distance $d_{C^0}(H_f, Id_{\widetilde{M}}) = d_{C^0}(h_f, Id_M) < \infty$. In particular we have that $d_{C^0}(H_f, Id) \to 0$ when $f \to A$ in the C^0 topology (see the proof of Theorem 2.1.8 for more details).

Notice that we are making an abuse of notation since the map H_f is determined by $\widetilde{f}$ instead of f. But this is not a problem since given two lifts $\widetilde{f}_1$ and $\widetilde{f}_2$ there is an integer vector $v \in \mathbb{Z}^d$ such that $\widetilde{f}_1 - \widetilde{f}_2 = v$ and this implies that $H_{f_2} = H_{f_1} + w$, where $w = -(A - Id)^{-1}(v)$:

$$\begin{aligned} H_{f_2} \circ \widetilde{f}_2(\widetilde{x}) &= H_{f_1}(\widetilde{f}_2(\widetilde{x})) + w = H_{f_1}(\widetilde{f}_1(\widetilde{x}) - v) + w \\ &= H_{f_1} \circ \widetilde{f}_1(\widetilde{x}) - v + w = A \circ H_{f_1}(\widetilde{x}) - v + w \\ &= A(H_{f_1}(\widetilde{x}) + w) - Aw + w - v = A \circ H_{f_2}(\widetilde{x}) - (A - Id)^{-1}(w) - v \\ &= A \circ H_{f_2}(\widetilde{x}) \end{aligned}$$

Observe that the matrix $A - Id$ is invertible since A is hyperbolic.

Now given $f \in \mathcal{PH}_A(\mathbb{T}^d)$ and $\widetilde{x} \in \mathbb{R}^d$ we are going to call the *fiber* of $\widetilde{x} \in \mathbb{R}^d$ to the set $H_f^{-1}(H_f(\widetilde{x}))$. By the previous observation given two lifts $\widetilde{f}_1$ and $\widetilde{f}_2$ there is a vector $w \in \mathbb{R}^d$ such that $H_{f_2} = H_{f_1} + w$ and this implies that

$$H_{f_2}^{-1}(H_{f_2}(\widetilde{x})) = H_{f_1}^{-1}(H_{f_1}(\widetilde{x}))$$

and the fiber does not depend on the choice of the lift. As a result we can define the function *size of the fiber*

$$\Lambda : \mathcal{PH}_A(\mathbb{T}^d) \times \mathbb{R}^d \to \mathbb{R}_{\geq 0} \quad \text{by} \quad \Lambda(f, \widetilde{x}) = \operatorname{diam}(H_f^{-1}(H_f(\widetilde{x}))).$$

We also note by

$$\Lambda(f) = \sup\{\Lambda(f, \widetilde{x}) : \widetilde{x} \in \mathbb{R}^d\} \tag{3.11}$$

to the supremum of sizes within all fibers. Since $d_{C^0}(H_f, Id_{\mathbb{R}^d}) < \infty$ this supremum is always finite and we get a well defined function $\Lambda : \mathcal{PH}_A(\mathbb{T}^d) \to \mathbb{R}_{\geq 0}$. Notice that f is conjugated to A if and only if $\Lambda(f) = 0$.

It's easy to see that the function Λ does not depend continuously on f, however we have an upper semicontinuity property as the following lemma shows.

Lemma 3.4.1. *Let $f \in \mathcal{PH}_A(\mathbb{T}^d)$. Then for every $\epsilon > 0$ there exist $\delta > 0$ such that: if $d_{C^0}(f, g) < \delta$ then $\Lambda(g) < \Lambda(f) + \epsilon$.*

Proof. Take $f \in \mathcal{PH}_A(\mathbb{T}^d)$ and $\epsilon > 0$. Suppose by contradiction that the lemma is false. Then for every $n > 0$ there is $g_n \in \mathcal{PH}_A(\mathbb{T}^d)$ with $d_{C^0}(g_n, f) \leq 1/n$, and points $\widetilde{x}_n, \widetilde{y}_n \in \mathbb{R}^d$ such that $d(\widetilde{x}_n, \widetilde{y}_n) \geq \Lambda(f) + \epsilon$ and $H_{g_n}(\widetilde{x}_n) = H_{g_n}(\widetilde{y}_n)$. We can assume that $\widetilde{x}_n \to \widetilde{x}$ and $\widetilde{y}_n \to \widetilde{y}$, and in consequence $d(\widetilde{x}, \widetilde{y}) \geq \Lambda(f) + \epsilon$. Then by the triangular inequality we have

$$\begin{aligned} d(H_f(\tilde{x}), H_f(\tilde{y})) &\leq d(H_f(\tilde{x}), H_f(\tilde{x_n})) + d(H_f(\tilde{x_n}), H_{g_n}(\tilde{x_n})) + d(H_{g_n}(\tilde{x_n}), H_{g_n}(\tilde{y_n})) \\ &+ d(H_{g_n}(\tilde{y_n}), H_f(\tilde{y_n})) + d(H_f(\tilde{y_n}), H_f(\tilde{y})) \\ &\leq d(H_f(\tilde{x}), H_f(\tilde{x_n})) + 2/n + d(H_f(\tilde{y}), H_f(\tilde{y_n})) \to 0 \end{aligned}$$

and this implies $H_f(\tilde{x}) = H_f(\tilde{y})$. As a result, the points $\tilde{x}$ and $\tilde{y}$ belong to the same fiber which implies $d(\tilde{x}, \tilde{y}) \leq \Lambda(f)$. But then we have $\Lambda(f) + \epsilon \leq d(\tilde{x}, \tilde{y}) \leq \Lambda(f)$ which is a contradiction. □

Now we are ready to prove the main theorem of this section.

Theorem 3.4.2 (Robust transitivity criterion)**.** *Let $A \in SL_d(\mathbb{Z})$ be a hyperbolic matrix with spliting $\mathbb{R}^d = E^{ss}_A \oplus E^{ws}_A \oplus E^{wu}_A \oplus E^{uu}_A$. Take $f \in \mathcal{PH}_A(\mathbb{T}^d)$ with (d_1, d_2) SH-Saddle property where $d_1 = dimE^{ws}_A$ and $d_2 = dimE^{wu}_A$. Then there is $\tau = \tau(f) > 0$ s.t. if $\Lambda(f) < \tau$ then f is C^1 robustly transitive. In fact C^1 robustly topologically mixing.*

Proof. Take $f \in \mathcal{PH}_A(\mathbb{T}^d)$ with (d_1, d_2) SH-Saddle property such that $d_1 = \dim E^{ws}_A$ and $d_2 = \dim E^{wu}_A$. Let $\mathcal{V}$, $\lambda > 1$ and $\delta_1 > 0$ be as in Theorem 3.2.12.

Let us define the following constants:

$$\begin{aligned} \rho_s &= \rho(\dim E^s_A, \delta_1) \\ \rho_u &= \rho(\dim E^u_A, \delta_1) \\ \tau &= \min\{\rho_s, \rho_u\} \end{aligned}$$

where $\rho(d, r)$ are given by Proposition 3.3.4. We claim that the theorem holds for this $\tau > 0$ and for proving this we are going to find a C^1-neighbourhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ is transitive.

First observe that since $\Lambda(f) < \tau(f)$, then by Lemma 3.4.1 applied to $\epsilon = \tau(f) - \Lambda(f) > 0$, we know there is $\delta_0 > 0$ such that if $d_{C^0}(f, g) < \delta_0$ then $\Lambda(g) < \Lambda(f) + \epsilon = \tau(f)$.

Now take $\mathcal{U}(f) = \mathcal{V} \cap \{g \in \mathcal{PH}_A(\mathbb{T}^d) : d_{C^0}(f, g) < \delta_0\}$. We claim that every $g \in \mathcal{U}(f)$ is transitive (in fact topologically mixing). By Proposition 3.1.1 in order to get transitivity for $g \in \mathcal{U}(f)$, we have to prove that for any two open sets $U_1, U_2 \subset \mathbb{T}^d$ there is $n \in \mathbb{Z}$ such that $g^n(U_1) \cap U_2 \neq \emptyset$.

Take two points $x_1 \in U_1$ and $x_2 \in U_2$, and let $n_1 \in \mathbb{N}$ be such that $g^{-n_1}(U_1) \supset \mathcal{W}^{ss}_g(g^{-n_1}(x_1), L)$ and $g^{n_1}(U_2) \supset \mathcal{W}^{uu}_g(g^{n_1}(x_2), L)$. Take $x^s \in H^-_{\lambda, d_1}(g) \cap \mathcal{W}^{ss}_g(g^{-n_1}(x_1), L)$ and $x^u \in H^+_{\lambda, d_2}(g) \cap \mathcal{W}^{uu}_g(g^{n_1}(x_2), L)$ given by (d_1, d_2) SH-Saddle property.

Now take $D^s \subset \mathcal{W}^c_g(x^s)$ a center disk of dimension d_1 tangent to $\mathcal{C}^s_{x^s}$ and $D^u \subset \mathcal{W}^c_g(x^u)$ a center disk of dimension d_2 tangent to $\mathcal{C}^u_{x^u}$. We can take D^s, D^u small enough such that $D^s \subset g^{-n_1}(U_1)$ and $D^u \subset g^{n_1}(U_2)$. Recall that $\mathcal{C}^s$ and $\mathcal{C}^u$ are the cones invariant for the past and the future respectively given by SH-Saddle property. Moreover, $\mathcal{C}^s$ and $\mathcal{C}^u$ uniformly expand vectors for the past and the future respectively.

Now take $D_1 = \cup_{x \in D^s} \mathcal{W}^{ss}_g(x, \theta)$ and $D_2 = \cup_{x \in D^u} \mathcal{W}^{uu}_g(x, \theta)$. We can choose $\theta > 0$ small enough such that $D_1 \subset g^{-n_1}(U_1)$ and $D_2 \subset g^{n_1}(U_2)$. Notice that D_1 is a disk of dimension equal to $\dim E^s_A$ and D_2 is a disk of dimension equal to $\dim E^u_A$. Now by Corollary 3.2.14 there is $n_2 \in \mathbb{N}$ such that $g^{-n}(D^s)$ contains a disk of diameter bigger than $2\delta_1$ and $g^n(D^u)$ contains a disk of diameter bigger than $2\delta_1$ for every $n \geq n_2$.

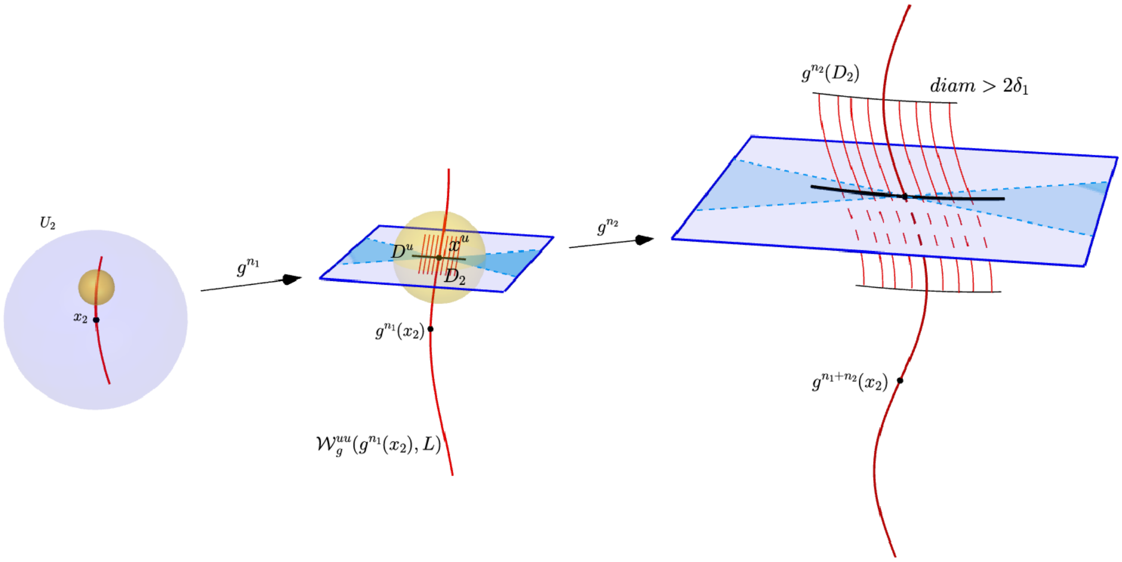


FIGURE 3.2: Obtaining a disk of diameter bigger than $2\delta_1$

Now the idea is to use Corollary 3.3.5 applied to the functions $\Pi^s \circ H_g$ and $\Pi^u \circ H_g$ to conclude that the images of the sets $g^{-n_2}(D_1)$ and $g^{n_2}(D_2)$ by h_g contain topological disks of complementary dimensions and that they have the appropriate inclination. Then the hyperbolicity of the matrix A will do the mixing, and we can translate this mixing of A to the diffeomorphism g.

Observe that $g \in \mathcal{U}(f)$ which implies that $\Lambda(g) < \tau$ and in particular we have that H_g is τ-light (see Definition 3.3.3). Moreover we claim the following.

Claim 3.4.3. *The function $\Pi^s \circ H_g$ is τ-light when restricted to $\tilde{g}^{-n_2}(\widetilde{D_1})$ and the function $\Pi^u \circ H_g$ is τ-light when restricted to $\tilde{g}^{n_2}(\widetilde{D_2})$.*

Proof. We are going to see the case $\Pi^s \circ H_g$ since the other one is symmetric. Now notice that $\tilde{g}^{-n}(\widetilde{D_1})$ contains a disk of size bigger than $2\delta_1$ for every $n \geq n_2$ and the disk $\tilde{g}^{-n}(\widetilde{D_1})$ is tangent to a cone $\mathcal{C}^s$ which is uniformly expanding for the past. Thus by the semiconjugacy relation $H_g \circ \tilde{g} = A \circ H_g$ we know that $H_g(\widetilde{D_1})$ can not intersect E^u_A more than once, otherwise there would be different points in $\widetilde{D_1}$ such that their distance by past iterates of $\tilde{g}$ goes to zero, and this is impossible since the cones $\mathcal{C}^s$ are expanding for the past. In consequence the fibers of $\Pi^s \circ H_g$ have the same size of the fibers of H_g, and so $\Pi^s \circ H_g$ is τ-light restricted to $\tilde{g}^{-n_2}(\widetilde{D_1})$. ☐

To sum up, we have a continuous map $\Pi^s \circ H_g : \tilde{g}^{-n_2}(\widetilde{D_1}) \to E^s_A \simeq \mathbb{R}^{\dim E^s_A}$ such that its domain $\tilde{g}^{-n_2}(\widetilde{D_1})$ contains a disk $[-\delta_1, \delta_1]^{\dim E^s_A}$ and by our choice of τ we have that $\tau \leq \rho(\dim E^s_A, \delta_1)$. Then just notice that we are in hypothesys of Corollary 3.3.5 and therefore $\Pi^s \circ H_g(\tilde{g}^{-n_2}(\widetilde{D_1})) \subset E^s_A$ contains an open set. The same argument shows that $\Pi^u \circ H_g(\tilde{g}^{n_2}(\widetilde{D_2})) \subset E^u_A$ contains an open set.

Since A is a hyperbolic matrix and the topological disks have complementary dimensions and with the right inclination, we know there is $n_3 \in \mathbb{N}$ such that for every $n \geq n_3$ we have that $A^n(H_g(\tilde{g}^{n_2}(\widetilde{D_2}))) \cap (H_g(\tilde{g}^{-n_2}(\widetilde{D_1})) + V_n) \neq \emptyset$ for some $V_n \in \mathbb{Z}^d$. This implies that $H_g \circ g^n(\tilde{g}^{n_2}(\widetilde{D_2}))) \cap (H_g(\tilde{g}^{-n_2}(\widetilde{D_1})) + V_n) \neq \emptyset$. Since H_g is at bounded distance to the identity, we know that there is $n_4 \in \mathbb{N}$ such that for every $n \geq n_4$, we have $\tilde{g}^n(\tilde{g}^{n_2}(\widetilde{D_2}))) \cap (\tilde{g}^{-n_2}(\widetilde{D_1}) + V_n) \neq \emptyset$. Then since $p : \mathbb{R}^d \to \mathbb{T}^d$

satisfies $p \circ \tilde{g} = g \circ p$ we have that:

$$\emptyset \neq g^n(g^{n_2}(D_2))) \cap g^{-n_2}(D_1) \subset g^{n+n_1+n_2}(U_2) \cap g^{-n_1-n_2}(U_1)$$

for every $n \geq n_4$ and this is equivalent to

$$\emptyset \neq g^{n+2n_1+2n_2}(U_2) \cap U_1, \quad \text{for every } n \geq n_4.$$

Finally if we take $N = n_4 + 2(n_1 + n_2)$ we have that $g^n(U_2) \cap U_1 \neq \emptyset$ for every $n \geq N$ proving that g is topologically mixing. This ends the proof. □

Corollary 3.4.4. *Let $A \in SL_d(\mathbb{Z})$ be a hyperbolic matrix with spliting $\mathbb{R}^d = E_A^{ss} \oplus E_A^{ws} \oplus E_A^{wu} \oplus E_A^{uu}$ and let $f \in \mathcal{PH}_A(\mathbb{T}^d)$ with (d_1, d_2) SH-Saddle property where $d_1 = dim E_A^{ws}$ and $d_2 = dim E_A^{wu}$. If $\Lambda(f) = 0$ then f is C^1 robustly transitive.*

Proof. Since f has SH-Saddle property, we know that $\tau(f) > 0$. Then we trivially have $\Lambda(f) = 0 < \tau(f)$ and we conclude by Theorem 3.4.2. □

3.4.2 Derived from Anosov is always SH-Saddle

In this subsection we are going to show that every derived from Anosov diffeomorphism has always the SH-Saddle property for a given index (actually the same index as its linear part). We begin by explaining what we mean with derived from Anosov diffeomorphisms.

Take $A \in \mathrm{SL}(d, \mathbb{Z})$ a hyperbolic matrix and suppose that admits a dominated spilting of the form $\mathbb{R}^d = E_A^{ss} \oplus E_A^{ws} \oplus E_A^{wu} \oplus E_A^{uu}$ as in the last subsection. Denote by $d_1 = \mathrm{dim} E_A^{ws}$ and $d_2 = \mathrm{dim} E_A^{wu}$.

From now on we are going to consider a partially hyperbolic $f : \mathbb{T}^d \to \mathbb{T}^d$ defined the following way. Take $\epsilon > 0$ and call $U = B(0, \epsilon) \subset \mathbb{R}^d$. Take $f_t : \mathbb{R}^d \to \mathbb{R}^d$ an isotopy such that:

1. $f_0 = A$ and $f_1 = f$
2. $f_t|_{U^c} = A|_{U^c}$, for every $t \in [0, 1]$
3. $\mathrm{dim} E_{f_t}^{ss} = \mathrm{dim} E_A^{ss}$ and $\mathrm{dim} E_{f_t}^{uu} = \mathrm{dim} E_A^{uu}$, for every $t \in [0, 1]$.

We can assume that ϵ is small enough in order to send f_t to the quotient $\mathbb{T}^d$. It is clear that a diffeomorphism f built this way belongs to $\mathcal{PH}_A(\mathbb{T}^d)$. From now on, we are going to say that a diffeomorphism satisfying points 1, 2 and 3 is a *derived from Anosov* (DA) diffeomorphism.

Lemma 3.4.5. *Let $f : \mathbb{T}^d \to \mathbb{T}^d$ be a derived from Anosov diffeomorphism (i.e. satisfying points 1, 2 and 3). Then if $\epsilon > 0$ is sufficiently small, f has the (d_1, d_2) SH-Saddle property.*

Proof. First by taking an iterate we can suppose that $\|Df_x|_{E^{uu}(x)}\| > 4$ for every $x \in \mathbb{T}^d$. Now take $0 < \epsilon < 1/4$. Then for every $x \in \mathbb{T}^d$, there is a point $z_0^x \in \mathcal{W}_f^{uu}(x, 1)$ such that $\mathcal{W}_f^{uu}(z_0^x, 1/4) \cap U = \emptyset$. Call $D_0 = \overline{\mathcal{W}_f^{uu}(z^x, 1/4)}$. In the same way since $f(D_0) = \mathcal{W}_f^{uu}(f(z^x), 1)$, we can find a disk $D_1 = \overline{\mathcal{W}_f^{uu}(z_1^x, 1/4)} \subset f(D_0)$ such that $D_1 \cap U = \emptyset$. Inductively we get a sequence of unstable disks $\{D_n\}_{n \geq 0}$ such that $D_n \cap U = \emptyset$ for every $n \geq 0$ and $f^{-1}(D_n) \subset D_{n-1}$ (see Figure 3.3 below). Finally the point $x^u = \bigcap_{n \geq 0} f^{-n}(D_n)$ never meets U in the future. Since f is equal to A outside U we get that the point x^u is hyperbolic for the future, and so the unstable manifold $\mathcal{W}_f^{uu}$ has SH-Saddle property of index d_2.

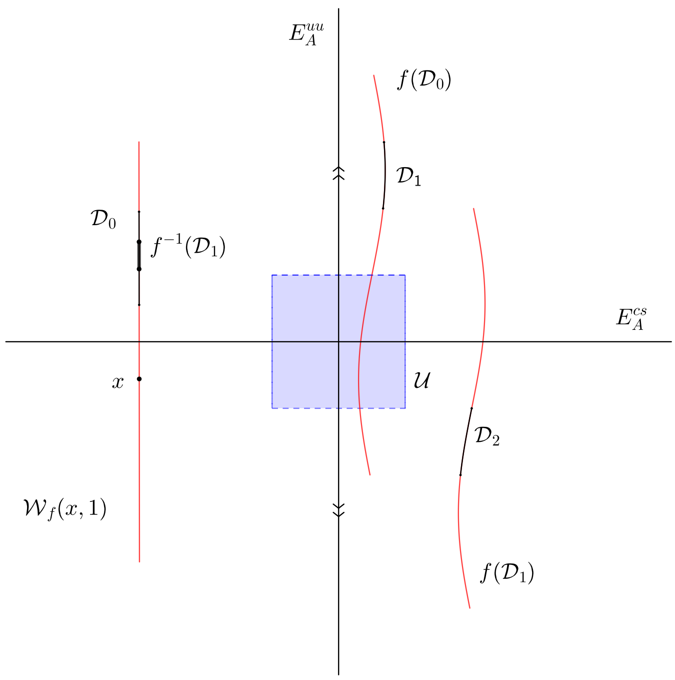


FIGURE 3.3: Finding a point whose forward orbit never meets $\mathcal{U}$

In the same way we can find a point x^s in every strong stable leaf of large 1, such that the past orbit of x^s never meets U. Once again since $f = A$ outside U, the same argument as above shows that $\mathcal{W}^{ss}_f$ has SH-Saddle property of index d_1. ☐

3.4.3 Expansive DA diffeomorphisms

In this subsection we are going to build examples of expansive DA diffeomorphisms which are partially hyperbolic, but not Anosov. The idea is to introduce an isotopy in a small neighbourhood of a fixed point p, in order to make the derivative of p (restricted to a center subbundle) equal to the identity, and keeping the rest of the manifold hyperbolic. As a result, these examples will be partially hyperbolic, expansive and not Anosov. These examples will be used in the next section, as the first step in the construction of the examples of Theorem C.

For the construction of the local isotopies, we are going to use an auxiliary function that will be used many times.

Lemma 3.4.6. *Let $b > 0$. Then for every $\epsilon > 0$ (arbitrarily small) there exist a function $\beta : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ such that:*

1. *β is C^∞, decreasing and $-\epsilon \leq \beta'(t)t \leq 0$.*
2. *β is supported in $[0, \epsilon]$.*
3. *$\beta(0) = b$.*

Proof. Take $r_0 < \epsilon$. Since the integral $\int_0^{r_0} \frac{\epsilon}{t} dt$ is divergent, we can find a function ψ supported in $[0, r_0]$ such that $\int_0^{r_0} \psi(t)dt = b$ and $\psi(t) \leq \frac{\epsilon}{t}$ (see Figure 3.4 below). Now just take β as:

$$\beta(t) = b - \int_0^t \psi(s)ds$$

This function clearly satisfies the lemma. ☐

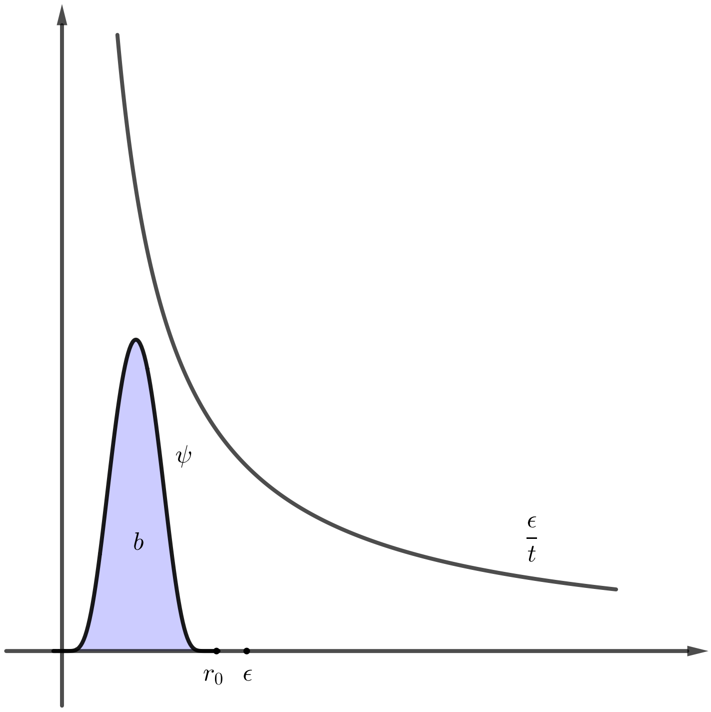


FIGURE 3.4: Bump function ψ

Along this section we are going to perform different isotopies depending on the type of local behaviour we are looking for, i.e. increase or decrease the index of a fixed point, mixing two subbundles, etc. Recall that for our purposes we need the examples to be expansive, and so the construction has to be made with some care.

Two dimensional center bundle

We begin with the case of a fixed point of saddle type in dimension two. This will be useful in order to mix two different subbundles. The case when the center bundle is entirely contracting or expanding will be treated later.

Lemma 3.4.7. *Let $A \in SL(2, \mathbb{Z})$ and take $\epsilon > 0$ sufficiently small. Then, there exists a diffeomorphism $g : \mathbb{T}^2 \to \mathbb{T}^2$ such that:*

- *$g(x) = Ax$, for every $x \in B(0, \epsilon)^c$.*
- *g is expansive.*
- *$Dg_0 = Id$.*

Proof. Take a matrix $A \in SL(2, \mathbb{Z})$ and suppose the eigenvalues of A are λ and μ with $0 < \lambda < 1 < \mu$. Let E^s be the eigenspace associated to λ, and let E^u be the eigenspace associated to μ. Then we have that $\mathbb{R}^2 = E^s \oplus E^u$. Fix a small $\epsilon > 0$ (sufficiently small in order to send the map to the quotient) and $r_0 \in (0, \epsilon)$. Let β_1 be the function given by Lemma 3.4.6 for $b = 1 - \lambda$ and its corresponding function ψ_1, and let β_2 be the function given by Lemma 3.4.6 for $b = \mu - 1$ and its corresponding function ψ_2. According to the decomposition $\mathbb{R}^2 = E^s \oplus E^u$ we define the function $g : \mathbb{R}^2 \to \mathbb{R}^2$ by the equation:

$$g(x, y) = (\lambda x, \mu y) + (\beta_1(r)x, -\beta_2(r)y)$$

where $r = x^2 + y^2$. Notice that if $r \geq r_0$ then $g(x, y) = A(x, y)$.

In particular $Dg_{(x,y)} = A$ for every (x, y) such that $x^2 + y^2 \geq r_0$. In case $r < r_0$ the differential is:

$$Dg_{(x,y)} = \begin{bmatrix} \lambda + \beta_1(r) + 2x^2\beta_1'(r) & 2xy\beta_1'(r) \\ -2xy\beta_2'(r) & \mu - \beta_2(r) - 2y^2\beta_2'(r) \end{bmatrix}$$

In particular we have that $Dg_0 = Id$ and therefore g is not hyperbolic. In case $r > 0$ we have that

$$\lambda + \beta_1(r) + 2x^2\beta_1'(r) < \lambda + \beta_1(0) = 1$$

and

$$\mu - \beta_2(r) - 2y^2\beta_2'(r) > \mu - \beta_2(0) = 1$$

Now take the family of cones in $\mathbb{R}^2$

$$\mathcal{C}^u(x,y) = \{(a,b) \in \mathbb{R}^2 : |a| \leq |b|\}$$

We claim that this familiy of cones is Dg-invariant. This is clear if $r > r_0$ since $g = A$, but for points close to zero (r small) this is not so clear. To prove this, we have to take a little more care with the funcions β_1 and β_2 (in particular with the functions ψ_1, ψ_2).

Therefore in order to get invariance of the cones, we have to prove that if $(a, b) \in \mathcal{C}^u$ then $Dg(a, b) = (a_1, b_1) \in \mathcal{C}^u$, and this occurs if and only if $|a_1| \leq |b_1|$. By the equations above we have that:

$$\begin{aligned} a_1 &= a\left(\lambda + \beta_1(r) + 2x^2\beta_1'(r)\right) + b\left(2xy\beta_1'(r)\right) \\ b_1 &= a\left(-2xy\beta_2'(r)\right) + b\left(\mu - \beta_2(r) - 2y^2\beta_2'(r)\right) \end{aligned}$$

Now let us first take $0 < r_1 < r_0$ to be determined and take $\rho > 0$ and $l > 0$ such that $2\rho + l < r_1$. We define the function ψ_1 in the following way, like in Figure 3.5 below:

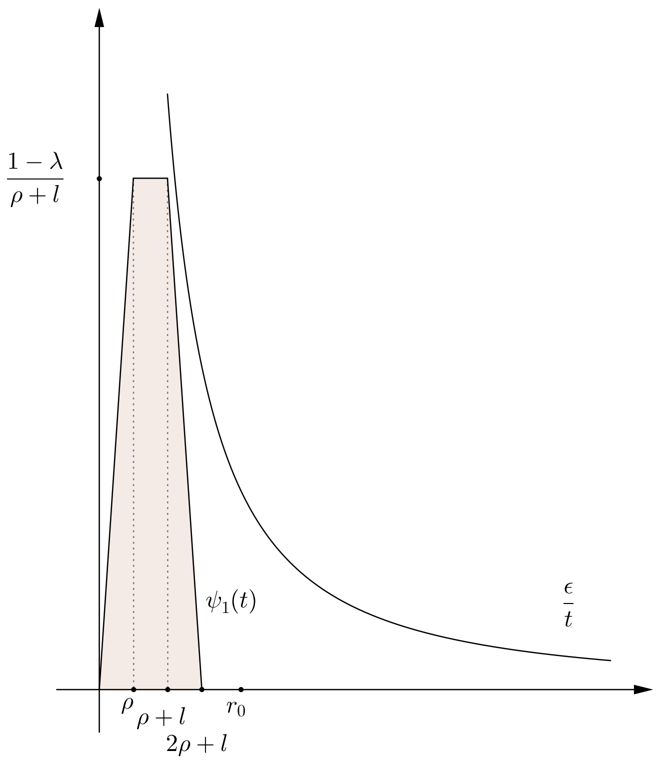


FIGURE 3.5: Bump function ψ_1

$$\psi_1(r)=\begin{cases}\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r & if \quad r\in[0,\rho]\\ \frac{1-\lambda}{\rho+l} & if \quad r\in(\rho,\rho+l)\\ -\left(\frac{1-\lambda}{\rho(\rho+l)}\right)t+(1-\lambda)\left(\frac{1}{\rho+l}+\frac{1}{\rho}\right) & if \quad r\in[\rho+l,2\rho+l]\end{cases}$$

Notice that $\int_0^{r_1}\psi_1(t)dt=1-\lambda$. Taking r_1 sufficiently small, we have that $\psi_1(r)<\epsilon/r$. In the same way we can take the function ψ_2 by:

$$\psi_2(r)=\begin{cases}\left(\frac{\mu-1}{\rho(\rho+l)}\right)r & if \quad r\in[0,\rho]\\ \frac{\mu-1}{\rho+l} & if \quad r\in(\rho,\rho+l)\\ -\left(\frac{\mu-1}{\rho(\rho+l)}\right)t+(\mu-1)\left(\frac{1}{\rho+l}+\frac{1}{\rho}\right) & if \quad r\in[\rho+l,2\rho+l]\end{cases}$$

In this case $\int_0^{r_1}\psi_2(t)dt=\mu-1$, and taking r_1 sufficiently small we have $\psi_2(r)<\epsilon/r$. Recall that we want to prove that if $(a,b)\in\mathcal{C}^u$ then $Dg(a,b)=(a_1,b_1)\in\mathcal{C}^u$, and this happens if and only if $|a_1|\leq|b_1|$. We will obtain this inequality by studying different cases depending on r (the square of the distance of the point (x,y) to the origin).

Case 1: $r\in[0,\rho]$.
By our definitions above we have:

$$\beta_1(r)=1-\lambda-\left(\frac{1-\lambda}{\rho(\rho+l)}\right)\frac{r^2}{2} \quad\text{and}\quad \beta_1'(r)=-\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r$$

Then we have that:

$$\lambda+\beta_1(r)+2x^2\beta_1'(r)=1-\left(\frac{1-\lambda}{\rho(\rho+l)}\right)\frac{r^2}{2}-2x^2\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r$$

and in consequence we obtain that a_1 is equal to:

$$a_1=a\left(1-\left(\frac{1-\lambda}{\rho(\rho+l)}\right)\frac{r^2}{2}-2x^2\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r\right)+b\left(-2xy\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r\right)$$

Since $|a|\leq|b|$, by taking absolute value and applying triangular inequality we get:

$$\begin{aligned}|a_1| &\leq |b|\left(1-\left(\frac{1-\lambda}{\rho(\rho+l)}\right)\frac{r^2}{2}-2x^2\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r\right)+|b|\left(2|xy|\left(\frac{1-\lambda}{\rho(\rho+l)}\right)r\right)\\ &= |b|\left(1-r\left(\frac{1-\lambda}{\rho(\rho+l)}\right)\left(\frac{r}{2}+2x^2-2|xy|\right)\right)\\ &= |b|\left(1-\frac{r}{2}\left(\frac{1-\lambda}{\rho(\rho+l)}\right)(r+4x^2-4|xy|)\right)\leq|b|\end{aligned}$$

where the last inequality holds as long as $r+4x^2-4|xy|=5x^2+y^2-4|xy|>0$. We claim that this is always the case: if $|xy|=xy$ we have to show that $5x^2+y^2-4xy>0$.

By solving the second degree equation in y we obtain that

$$y = \frac{4x \pm \sqrt{16x^2 - 20x^2}}{2}$$

and this has no real roots. Since for $y = 0$ we have $5x^2 \geq 0$, we obtain the desire inequality. The case where $|xy| = -xy$ is completely the same since the discriminant in the equation above is the same. We conclude that $r + 4x^2 - 4|xy| \geq 0$, and moreover $|a_1| < |b|$ if $r > 0$.

In the same way we have that:

$$\beta_2(r) = \mu - 1 - \left(\frac{\mu - 1}{\rho(\rho + l)}\right)\frac{r^2}{2} \text{ and } \beta_2'(r) = -\left(\frac{\mu - 1}{\rho(\rho + l)}\right) r$$

and then

$$\mu - \beta_2(r) - 2y^2\beta_2'(r) = 1 + \left(\frac{\mu - 1}{\rho(\rho + l)}\right)\frac{r^2}{2} + 2y^2\left(\frac{\mu - 1}{\rho(\rho + l)}\right) r$$

We then have that b_1 is equal to

$$b_1 = a\left(2xy\left(\frac{\mu - 1}{\rho(\rho + l)}\right) r\right) + b\left(1 + \left(\frac{\mu - 1}{\rho(\rho + l)}\right)\frac{r^2}{2} + 2y^2\left(\frac{\mu - 1}{\rho(\rho + l)}\right) r\right)$$

Since $|a| \leq |b|$, taking absolute value and by the triangular inequality we have:

$$\begin{aligned}
|b_1| &\geq |b|\left(1 + \left(\frac{\mu - 1}{\rho(\rho + l)}\right)\frac{r^2}{2} + 2y^2\left(\frac{\mu - 1}{\rho(\rho + l)}\right) r\right) - |b|\left(2|xy|\left(\frac{\mu - 1}{\rho(\rho + l)}\right) r\right) \\
&= |b|\left(1 + r\left(\frac{\mu - 1}{\rho(\rho + l)}\right)\left(\frac{r}{2} + 2y^2 - 2|xy|\right)\right) \\
&= |b|\left(1 + \frac{r}{2}\left(\frac{\mu - 1}{\rho(\rho + l)}\right)\left(r + 4y^2 - 4|xy|\right)\right) \geq |b|
\end{aligned}$$

where the last inequality holds as long as: $r + 4y^2 - 4|xy| = x^2 + 5y^2 - 4|xy| \geq 0$. This is exactly the same equation we solve above, and thus we conclude that $|b_1| \geq |b|$ and moreover, $|b_1| > |b|$ if $r > 0$. Then we conclude that $|a_1| \leq |b| \leq |b_1|$.

Case 2: $r \in [\rho, \rho + l]$**.**
In this case we obtain that β_1 verifies:

$$\beta_1(r) = 1 - \lambda - \frac{\rho}{2}\left(\frac{1 - \lambda}{\rho + l}\right) - (r - \rho)\left(\frac{1 - \lambda}{\rho + l}\right) \text{ and } \beta_1'(r) = -\left(\frac{1 - \lambda}{\rho + l}\right)$$

Then we have that:

$$\lambda + \beta_1(r) + 2x^2\beta_1'(r) = 1 - \frac{\rho}{2}\left(\frac{1 - \lambda}{\rho + l}\right) - (r - \rho)\left(\frac{1 - \lambda}{\rho + l}\right) - 2x^2\left(\frac{1 - \lambda}{\rho + l}\right)$$

therefore a_1 is equal to:

$$a_1 = a\left(1 - \frac{\rho}{2}\left(\frac{1 - \lambda}{\rho + l}\right) - (r - \rho)\left(\frac{1 - \lambda}{\rho + l}\right) - 2x^2\left(\frac{1 - \lambda}{\rho + l}\right)\right) + b\left(-2xy\left(\frac{1 - \lambda}{\rho + l}\right)\right)$$

Since $|a| \leq |b|$, taking absolute value and applying triangular inequality we get:

$$\begin{aligned}
|a_1| &\leq |b|\left(1 - \frac{\rho}{2}\left(\frac{1-\lambda}{\rho+l}\right) - (r-\rho)\left(\frac{1-\lambda}{\rho+l}\right) - 2x^2\left(\frac{1-\lambda}{\rho+l}\right)\right) \\
&+ |b|\left(-2xy\left(\frac{1-\lambda}{\rho+l}\right)\right) \\
&= |b|\left(1 - \left(\frac{1-\lambda}{\rho+l}\right)\left(\frac{\rho}{2} + (r-\rho) + 2x^2 - 2|xy|\right)\right) \\
&= |b|\left(1 - \left(\frac{1-\lambda}{\rho+l}\right)\left(-\frac{\rho}{2} + 3x^2 + y^2 - 2|xy|\right)\right) \leq |b|
\end{aligned}$$

where the last inequality holds as long as $-\frac{\rho}{2} + 3x^2 + y^2 - 2|xy| \geq 0$. Notice that

$$-\frac{\rho}{2} + 3x^2 + y^2 - 2|xy| = \left(-\frac{\rho}{2} + \frac{x^2+y^2}{2}\right) + \left(\frac{5x^2+y^2}{2} - 2|xy|\right)$$

The first term in the right expression is greater or equal to zero since $\rho \leq r$. The second term is exactly $\frac{5x^2+y^2-4|xy|}{2}$, which is exactly the same equation we solve in case 1.

In the same way we have that:

$$\beta_2(r) = \mu - 1 - \frac{\rho}{2}\left(\frac{\mu-1}{\rho+l}\right) - (r-\rho)\left(\frac{\mu-1}{\rho+l}\right) \quad \text{and} \quad \beta_2'(r) = -\left(\frac{\mu-1}{\rho+l}\right)$$

and then we obtain:

$$\mu - \beta_2(r) - 2y^2\beta_2'(r) = 1 + \frac{\rho}{2}\left(\frac{\mu-1}{\rho+l}\right) + (r-\rho)\left(\frac{\mu-1}{\rho+l}\right) + 2y^2\left(\frac{\mu-1}{\rho+l}\right)$$

We then have that b_1 is equal to

$$b_1 = a\left(2xy\left(\frac{\mu-1}{\rho+l}\right)\right) + b\left(1 + \frac{\rho}{2}\left(\frac{\mu-1}{\rho+l}\right) + (r-\rho)\left(\frac{\mu-1}{\rho+l}\right) + 2y^2\left(\frac{\mu-1}{\rho+l}\right)\right)$$

Since $|a| \leq |b|$, taking absolute value and by the triangular inequality we have:

$$\begin{aligned}
|b_1| &\geq |b|\left(1 + \frac{\rho}{2}\left(\frac{\mu-1}{\rho+l}\right) + (r-\rho)\left(\frac{\mu-1}{\rho+l}\right) + 2y^2\left(\frac{\mu-1}{\rho+l}\right)\right) \\
&- |b|\left(2|xy|\left(\frac{\mu-1}{\rho+l}\right)\right) \\
&= |b|\left(1 + \left(\frac{\mu-1}{\rho+l}\right)\left(\frac{\rho}{2} + (r-\rho) + 2y^2 - 2|xy|\right)\right) \\
&= |b|\left(1 + \left(\frac{\mu-1}{\rho+l}\right)\left(-\frac{\rho}{2} + x^2 + 3y^2 - 2|xy|\right)\right) \geq |b|
\end{aligned}$$

by the same estimates than above. Moreover we have that $|b_1| > |b|$ if $r > 0$. Then we conclude that $|a_1| \leq |b| \leq |b_1|$.

Case 3: $r \in [\rho + l, 2\rho + l]$
This is the simplest case, since we are far enough to zero and so we omit the calculations.

To sum up, we have proved that the cone $\mathcal{C}^u$ is Dg-invariant. To finish the proof, just take the norm $\|(a,b)\|_1 := \max\{|a|,|b|\}$ in $\mathbb{R}^2$, then we have that vectors in $\mathcal{C}^u$ are expanded for the future: if $v=(a,b)\in\mathcal{C}^u$ then $|a|\le|b|$ and thus $\|v\|_1=|b|$. Since $Dg(v)=(a_1,b_1)\in\mathcal{C}^u$, this implies that $\|Dg(v)\|_1=|b_1|$ and we have just proved that $|b|\le|b_1|$. In short, we have that $\|Dg(v)\|_1\ge\|v\|_1$ (moreover $\|Dg_{(x,y)}(v)\|_1>\|v\|_1$ if $(x,y)\neq(0,0)$). This proves that g is expansive as we wanted to show. □

Now that we have the map $g:\mathbb{R}^2\to\mathbb{R}^2$ constructed in Lemma 3.4.7, we can build a partially hyperbolic example in $\mathbb{R}^4$ whose center leaves behaves like the map g.

Lemma 3.4.8. *Let $A\in SL(4,\mathbb{Z})$ be a matrix with four eigenvalues $\lambda^{ss},\lambda,\mu,\mu^{uu}$ such that $0<\lambda^{ss}<\lambda<1<\mu<\mu^{uu}$ and take $\epsilon>0$. Then there is a partially hyperbolic diffeomorphism $f:\mathbb{T}^4\to\mathbb{T}^4$ with a splitting of the form $\mathbb{R}^4=E^{ss}_f\oplus E^c_f\oplus E^{uu}_f$, such that $dimE^c_f=2$ and verifies the following:*

- *$f(x)=Ax$ for every $B(0,\epsilon)^c$.*
- *f is expansive.*
- *$Df_0|_{E^c_f}=Id$.*

Proof. Let $A\in SL(4,\mathbb{Z})$ be a matrix with four eigenvalues $\lambda^{ss},\lambda,\mu,\mu^{uu}$ such that:

$$0<\lambda^{ss}<\lambda<1<\mu<\mu^{uu}$$

We can assume that in the basis given by the eigenspaces associated to the eigenvalues we have that: $A(x,y,z,t)=(\lambda x,\mu y,\lambda^{ss}z,\mu^{uu}t)$. Take the same functions β_1 and β_2 as in Lemma 3.4.7 and define the map $f:\mathbb{R}^4\to\mathbb{R}^4$ by:

$$f(x,y,z,t)=(\lambda x,\mu y,\lambda^{ss}z,\mu^{uu}t)+\rho(w)(\beta_1(r)x,-\beta_2(r)y,0,0)$$

where ρ is a bump function supported in $[0,\epsilon]$ and $w=z^2+t^2$. If $\|(x,y,z,t)\|\ge\epsilon$ we have that $f=A$. For points with $\|(x,y,z,t)\|<\epsilon$ the differential of f at a point (x,y,z,t) is:

$$\begin{bmatrix} \lambda+\rho(w)\left(\beta_1(r)+2x^2\beta_1'(r)\right) & \rho(w)\left(2xy\beta_1'(r)\right) & 2xz\rho'(w)\beta_1(r) & 2tz\rho'(w)\beta_1(r) \\ -\rho(w)\left(2xy\beta_2'(r)\right) & \mu-\rho(w)\left(\beta_2(r)-2y^2\beta_2'(r)\right) & -2yz\rho'(w)\beta_2(r) & 2yt\rho'(w)\beta_2(r) \\ 0 & 0 & \lambda^{ss} & 0 \\ 0 & 0 & 0 & \mu^{uu} \end{bmatrix}$$

In this case the subspace $E^c_f=\{(x,y,0,0)\}$ is Df invariant, and it is quite direct to see that $Df|_{E^c}$ is basically Dg like above (we have to deal with the funcion ρ but is not a problem). In particular we have that:

$$Df_0=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda^{ss} & 0 \\ 0 & 0 & 0 & \mu^{uu} \end{bmatrix}$$

The strong bundles are not going to be the canonical ones, but if we ask to the strong eigenvalues λ^{ss} and μ^{uu} to be sufficiently far from 1 (and we can do this by iterating the matrix), the same strong cones for the matrix A are going to be Df invariant. In consequence Df is hyperbolic outside 0, and therefore $f:\mathbb{T}^4\to\mathbb{T}^4$ is an expansive derived from Anosov diffeomorphism. □

Remark 3.4.9. *The construction in Lemma 3.4.8 can be made with no restriction on the dimensions of the strong subbundles (which were one dimensional in the example above). Indeed, the contruction only uses the isotopy in dimension 2 we made in Lemma 3.4.7, and the domination of the external strong subbundles.*

Higher dimensional center bundle

In the example of Lemma 3.4.8, the center bundle is two dimensional (and it behaves hyperbolic on the center). Here we are going to treat the case when the center bundle has dimension bigger than 2. We begin with the case when the center bundle is contractive (every center eiganvalue has modulus smaller than 1), and then we insert this example as the center leaf of a higher dimensional example like we did before.

Lemma 3.4.10. *Let $A \in M_{k\times k}(\mathbb{R})$ be a diagonal matrix with k eigenvalues such that*

$$0 < \lambda_1 \leq \cdots \leq \lambda_k < 1$$

Then, for every $\epsilon > 0$ there is a diffeomorphism $g : \mathbb{R}^k \to \mathbb{R}^k$ such that:

- $g(x) = Ax$ *for every* $x \in B(0, \epsilon)^c$.
- g *is expansive.*
- $Dg_0 = Id_{k\times k}$.

Proof. Take a matrix $A \in M_{k\times k}(\mathbb{R})$ as above. Then the eigenvalues of A verify that:

$$0 < \lambda_1 \leq \cdots \leq \lambda_k < 1$$

Fix $\epsilon > 0$ small, and take λ such that $\lambda_k < \lambda < 1$. In particular $\lambda > \lambda_j$ for every $j = 1, \ldots, k$. Now take a number $c \in (0, \epsilon)$ such that: $c < \frac{1-\lambda}{k(\lambda-\lambda_1)}$. Now, for this $c > 0$ take the function β given by Lemma 3.4.6 for $b = 1$. In particular, the function β verifies:

- β is C^∞, decreasing and $-c \leq \beta'(t)t \leq 0$.
- β is supported in $[0, \epsilon]$.
- $\beta(0) = 1$.

Moreover, we can ask for β to be equal to 1 in a small interval $[0, \delta]$. We can always have this small $\delta > 0$, since the integral $\int_0^r \frac{c}{t} dt$ is divergent (see the details in the proof of Lemma 3.4.6). Now we can define the map $g_1 : \mathbb{R}^k \to \mathbb{R}^k$ by

$$g_1(x_1, \ldots, x_k) = A(x_1, \ldots, x_k) + \beta(r)((\lambda - \lambda_1)x_1, \ldots, (\lambda - \lambda_k)x_k)$$

where $r = x_1^2 + \cdots + x_k^2$. Since $supp(\beta) \subseteq [0, \epsilon]$ we have that if $\|x\| > \epsilon$ then $g_1 = A$. The differential of g_1 in a point x is:

$$D(g_1)_x = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_k \end{bmatrix} + \beta(r) \begin{bmatrix} \lambda - \lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda - \lambda_k \end{bmatrix} + M(x)$$

where $M(x)$ is the matrix given by

$$M(x) = 2\beta'(r)\begin{bmatrix} (\lambda-\lambda_1)x_1^2 & (\lambda-\lambda_1)x_1x_2 & \dots & (\lambda-\lambda_1)x_1x_k \\ (\lambda-\lambda_2)x_1x_2 & (\lambda-\lambda_2)x_2^2 & \dots & (\lambda-\lambda_2)x_2x_k \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda-\lambda_k)x_1x_k & (\lambda-\lambda_k)x_2x_k & \dots & (\lambda-\lambda_k)x_k^2 \end{bmatrix}$$

In particular, since $\beta(0) = 1$ we have that $D(g_1)_0 = A + \beta(0)(\lambda Id - A) + M(0) = \lambda Id$. Now take a point $x \in \mathbb{R}^k$ and a vector $v \in \mathbb{R}^k$, then we have that:

$$D(g_1)_x(v) = Av + \beta(r)(\lambda Id - A)v + M(x)v$$

Assume the vector v is equal to $v = (a, \dots, a) \in \mathbb{R}^k$ for a given $a \in \mathbb{R}$, and denote by $D(g_1)_x(v) = (a_1, \dots, a_k)$. If we prove that $|a_j| < |a|$ for every $j = 1, \dots, k$, we obtain that $D(g_1)_x$ is a contraction (by taking the norm of the maximum). Let's take a look at the first coordinate a_1:

$$a_1 = a\left(\lambda_1 + \beta(r)(\lambda-\lambda_1) + 2\beta'(r)(\lambda-\lambda_1)\sum_{j=1}^{k} x_1x_j\right)$$

By taking absolute value, and applying the triangular inequality we obtain:

$$|a_1| \le |a|\left(|\lambda_1 + \beta(r)(\lambda-\lambda_1)| + 2|\beta'(r)(\lambda-\lambda_1)|\sum_{j=1}^{k} |x_1x_j|\right)$$

Notice that $0 \le (x_i + x_j)^2 = x_i^2 + x_j^2 + 2x_ix_j$ and this implies that $2|x_ix_j| \le x_i^2 + x_j^2 \le r$. Now recall that: $|\beta'(r)| \le \frac{c}{r} < \frac{1-\lambda}{k(\lambda-\lambda_1)r}$ and in particular we have that

$$2|\beta'(r)(\lambda-\lambda_1)|\sum_{j=1}^{k} |x_1x_j| < 1-\lambda$$

This implies that:

$$|a_1| < |a|\left(|\lambda_1 + \beta(r)(\lambda-\lambda_1)| + 1 - \lambda\right)$$

Since β is a decreasing function, we have that $1 = \beta(0) \ge \beta(r)$ and then:

$$|a_1| < |a|\left(|\lambda_1 + \beta(r)(\lambda-\lambda_1)| + 1 - \lambda\right) \le |a|\left(|\lambda_1 + (\lambda-\lambda_1)| + 1 - \lambda\right) = |a|$$

The exact same calculation shows that $|a_j| < |a|$ for every $j = 1, \dots, k$. This shows that $D(g_1)_x$ is a contraction (for the norm of the maximum) for every $x \in \mathbb{R}^k$ and in particular, g_1 is expansive.

Notice that since $\beta(r) = 1$ for every $r \in [0, \delta]$, we have that $g_1(x) = \lambda x$ for every $x \in B(0, \delta)$. Now take the function $h : \mathbb{R}^k \to \mathbb{R}^k$ given by $h(x) = (1-r)x$ where $r = \|x\|^2$ and consider a bump function $\rho : [0, +\infty) \to \mathbb{R}$ such that:

- $\rho(t) = 1$ for every $t \in [0, \delta/2]$.
- $\rho(t) = 0$ for every $t \ge \delta$.

Now we define $g : \mathbb{R}^k \to \mathbb{R}^k$ given by the equation:

$$g(x) = \rho(r)h(x) + (1-\rho(r))g_1(x)$$

where $r = \|x\|^2$. The first direct observation is that if $\|x\| \geq \delta$ then $g(x) = g_1(x) = Ax$. On the other hand, if $r = \|x\|^2 \leq \delta$ then we have that $g_1(x) = \lambda x$ and therefore

$$g(x) = \rho(r)h(x) + (1 - \rho(r))\lambda x = [\rho(r)(1 - r) + (1 - \rho(r))\lambda]x$$

and the function g is radial. Denote by $\alpha(r) := \rho(r)(1 - r) + (1 - \rho(r))\lambda$, then it is direct to see that

$$\alpha(r) = \rho(r)(1 - r) + (1 - \rho(r))\lambda = \rho(r)(1 - r - \lambda) + \lambda \leq 1 - r$$

As a result g sends every sphere of radius R to a sphere of radius $\alpha(R)R$ which is strictly smaller than R. This implies that Dg_x is a contraction for every $x \in B(0, \delta)$. To see this, just notice that given $x \in \mathbb{R}^k$ we have that $T_x\mathbb{R}^k = T_xS_{\|x\|} + \langle x \rangle$ where $S_{\|x\|}$ is the sphere centered at 0 of radius $\|x\|$. The same happens with $g(x)$, i.e. $T_{g(x)}\mathbb{R}^k = T_{g(x)}S_{\alpha(\|x\|)\|x\|} + \langle x \rangle$ and the differential of g at x restricted to this subspace is exactly

$$Dg_x|_{T_xS_{\|x\|}} = \alpha(\|x\|)Id$$

which is a contraction. The other direction $\langle x \rangle$ is exactly the same, and therefore Dg_x is a contraction, hence g is expansive. To finish the proof just observe that if $r = \|x\|^2 < \delta/2$ we have that $g(x) = (1 - r)x$ and in particular $Dg_0 = Id$. □

As a corollary of the previous lemma, by applying the same trick as in Lemma 3.4.8 (with a suitable bump function) we can embed the example above as the center leaf of a higher dimensional manifold. We thus obtain the following lemma.

Lemma 3.4.11. *Let $A \in SL(d, \mathbb{Z})$ with a splitting of the form $\mathbb{R}^d = E^{ss}_A \oplus E^c_A \oplus E^{uu}_A$ s.t. $dimE^c_A = k$ and E^c_A is the eigenspace associated to the eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_k < 1$. Then for every $\epsilon > 0$ small, there is a partially hyperbolic diffeomorphism $f : \mathbb{T}^d \to \mathbb{T}^d$ with a splitting of the form $T\mathbb{T}^d = E^{ss}_f \oplus E^c_f \oplus E^{uu}_f$ such that $dimE^\sigma_f = dimE^\sigma_A$ for $\sigma = ss, c, uu$, and moreover:*

- *$f(x) = Ax$ for every $x \in B(0, \epsilon)^c$.*
- *f is expansive.*
- *$Df_0|_{E^c_f} = Id$*

3.4.4 Proof of Theorem C

The idea of the proof is to construct the examples by steps. The first step is to apply the results of the last subsection, i.e. given a matrix A, we introduce an isotopy in a small neighbourhood of fixed point p in order to make the derivative of p equal to the identity (when restricted to the center bundle), and keeping the rest of the manifold hyperbolic. As a result, these examples will be expansive and not Anosov. Then, since the isotopy is made in a sufficiently small neighbourhood of the fixed point, by Lemma 3.4.5 we have that this map has the SH-Saddle property. Notice that we can make these perturbations in as many different fixed points as desire since Lemma 3.4.5 still works for finite fixed points. Then by C^1 small perturbations we can change the index of the fixed points as desire by classical Franks Lemma. Finally we obtain robust transitivity by applying Theorem 3.4.2.

We begin with the case when $d = 4$ since it is quite direct for our previous results and illustrates the general ideas. We then prove the general case.

Proof for case $d = 4$

Take a matrix $A \in SL(4, \mathbb{Z})$ with four eigenvalues $\lambda^{ss}, \lambda, \mu, \mu^{uu}$ such that:

$$0 < \lambda^{ss} < \lambda < 1 < \mu < \mu^{uu}$$

This induces a splitting of the form $\mathbb{R}^4 = E^{ss} \oplus E^{ws} \oplus E^{wu} \oplus E^{uu}$ and we take the center bundle as $E^c = E^{ws} \oplus E^{wu}$. We can assume that in the basis given by the eigenspaces associated to the eigenvalues we have that:

$$A(x, y, z, t) = (\lambda x, \mu y, \lambda^{ss} z, \mu^{uu} t)$$

Moreover, we can assume that the linear Anosov A has four different fixed points: $Fix(A) = \{p_0, p_1, p_2, p_3\}$ (we are making an abuse of notation here, by calling A instead of f_A, the induced map in the torus). We just have to iterate the matrix a few times in order to have four different fixed points.

Now notice that the procedure made in Subsection 3.4.3 works as well. First, for every fixed point p_j (with $j = 0, 1, 2$) take a small neighbourhood U_j (notice that we are not going to perturb p_3 since it already has index 2). We can take them small enough to be disjoint. Second, just notice that the isotopy procedure we made in Lemma 3.4.7 is only local, and therefore it can be applied in different disjoint neighbourhoods. Hence the same proof as in Lemma 3.4.8 shows that we can make an isotopy whose support is contained in $U_0 \cup U_1 \cup U_2$ in order to get a partially hyperbolic diffeomorphism $f_1 : \mathbb{T}^4 \to \mathbb{T}^4$ such that:

- $f_1(x) = Ax$, for every $x \in (U_0 \cup U_1 \cup U_2)^c$.
- f_1 is hyperbolic outside $Fix(f_1) = \{p_0, p_1, p_2\}$ (thus f_1 is expansive).
- $D(f_1)_{p_j}|_{E^c(p_j)} = Id$ for $j = 0, 1, 2$.

The second item above, shows that f_1 is expansive and then we have that $\Lambda(f_1) = 0$. The first point shows that f_1 is SH-Saddle of index (1,1). To see this, we just have to observe that the same proof of Lemma 3.4.5 shows that f_1 have the SH-Saddle property as well. In that proof, the only property we use is the fact that for a point p outside U and given a small $\delta > 0$, there is always a point p_1 such that

$$\mathcal{W}^{uu}_f(p_1, \delta) \subset f(\mathcal{W}^{uu}_f(p, \delta)) \cap (U^c)$$

and by an induction argument we find a point whose forward orbit never meets U, and the same happens for the past. By the same arguments, by taking the strong bundles E^{ss} and E^{uu} sufficiently contractive and expanding, and taking the neighbourhoods U_j sufficiently small, we also have this property, i.e. for every point p outside $U_0 \cup U_1 \cup U_2$, and given a small $\delta > 0$, there is always a point p_1 such that

$$\mathcal{W}^{uu}_{f_1}(p_1, \delta) \subset f_1(\mathcal{W}^{uu}_{f_1}(p, \delta)) \cap (U_0 \cup U_1 \cup U_2)^c$$

Then we can find a point that never meets $U_0 \cup U_1 \cup U_2$ for the future, and the same for the past, proving that f_1 is SH-Saddle of index (1,1).

Now since f_1 is SH-Saddle we have that $\tau(f_1) > 0$, and by expansiveness we also have $\Lambda(f_1) = 0$. Then a direct application of Theorem 3.4.2 shows that f_1 is C^1 robustly transitive. Let's call $\mathcal{U}_1$ to the C^1 neighbourhood of f_1 such that every $h \in \mathcal{U}_1$ is transitive.

To end the proof of the theorem, we are going to change the indexes of the fixed points p_0 and p_1, and to put a complex eigenvalue in p_2. First take the two matrixes B_0 and B_1 given by:

$$B_0 = \begin{bmatrix} 1-\eta & 0 & 0 & 0 \\ 0 & 1-\eta & 0 & 0 \\ 0 & 0 & \lambda^{ss} & 0 \\ 0 & 0 & 0 & \mu^{uu} \end{bmatrix} \quad B_1 = \begin{bmatrix} 1+\eta & 0 & 0 & 0 \\ 0 & 1+\eta & 0 & 0 \\ 0 & 0 & \lambda^{ss} & 0 \\ 0 & 0 & 0 & \mu^{uu} \end{bmatrix}$$

Then for η sufficiently small we have that the matrixes B_0 and B_1 are ϵ close to $D(f_1)_{p_0}$ and $D(f_1)_{p_1}$ respectively. Now in order to mix the two center subbundles, take the matrix B_2 with the form:

$$B_2 = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & \lambda^{ss} & 0 \\ 0 & 0 & 0 & \mu^{uu} \end{bmatrix}$$

where $a \pm ib$ are the complex eigenvalues of B_2. It is possible to take a and b such that a is close to 1, b is close to 0 (the modulus of $a \pm ib$ can be smaller, bigger or equal to 1 for our purposes). For suitable values of a and b we can assure that B_2 is ϵ close to $D(f_1)_{p_2}$. Then by Franks Lemma [Fra71], there is a diffeomorphism $f \in \mathcal{U}_1$ such that:

- $f(x) = f_1(x)$ for every $x \in (U_0 \cup U_1 \cup U_2)^c$.
- $f(p_j) = f_1(p_j) = p_j$ for $j = 0, 1, 2$.
- $Df_{p_j} = B_j$ for $j = 0, 1, 2$.

In particular $index(p_0) = 3$ and $index(p_1) = 1$ (recall that $index(p_3) = 2$). Since Df_{p_2} has a center complex eigenvalue, the center bundle of f can not be decomposed into two 1-dimensional subbundles. To sum up, the map $f : \mathbb{T}^4 \to \mathbb{T}^4$ is a C^1 robustly transitive derived from Anosov diffeomorphism, and verifies all the properties of Theorem C.

Proof of the general case

For the proof of the general case we proceed like we did above. Let $A \in SL(d, \mathbb{Z})$ be a hyperbolic matrix with a splitting of the form:

$$\mathbb{R}^d = E_A^{ss} \oplus E_A^{ws} \oplus E_A^{wu} \oplus E_A^{uu}$$

where we take as the center bundle to $E_A^c = E_A^{ws} \oplus E_A^{wu}$. Suppose in addition that the center subbundles E_A^{ws} and E_A^{wu} can be decomposed into 1-dimensional subbundles, i.e.:

$$\mathbb{R}^d = E_A^{ss} \oplus E_1^{ws} \oplus \cdots \oplus E_m^{ws} \oplus E_1^{wu} \oplus \cdots \oplus E_l^{wu} \oplus E_A^{uu}$$

where $E_j^{w\sigma}$ is the eigenspace associated to the eigenvalue λ_j^σ for $\sigma = s, u$. In particular the eigenvalues verify:

$$\lambda_1^s \leq \cdots \leq \lambda_m^s < 1 < \lambda_1^u \leq \cdots \leq \lambda_l^u$$

In short $m = dimE_A^{ws}$, $l = dimE_A^{wu}$ and $k = dimE_A^c = m + l$.

Notice that 0 is a fixed point of A and $index(0) = dimE_A^{ss} + m$. Now by iterating the matrix if necessary we can take $k = m + l$ different fixed points of A (here we are making an abuse of notation once again), $Fix(A) = \{p_1, \ldots, p_m, q_1, \ldots, q_l\}$. For every $j = 1, \dots, m$ take a neighbourhood U_j of p_j, and for every $j = 1, \dots, l$ take a neighbourhood V_j of q_j. We can assume that they are small enough to be disjoint.

Like before, we proceed like in Subsection 3.4.3. Notice that the isotopies made in that subsection were only local. Therefore a direct application of Lemma 3.4.11 implies that there is a partially hyperbolic diffeomorphism $g : \mathbb{T}^d \to \mathbb{T}^d$ with a splitting of the form

$$T\mathbb{T}^d = E^{ss}_g \oplus E^{ws}_g \oplus E^{wu}_g \oplus E^{uu}_g$$

where $dimE^{\sigma}_A = dimE^{\sigma}_g$ for $\sigma = ss, ws, wu, uu$, and moreover:

- $g(x) = Ax$ for every $x \in (U_1 \cup \cdots \cup U_m \cup V_1 \cup \cdots \cup V_l)^c$.
- g is expansive.
- $Dg_{p_j}|_{E^{ws}} = Id$ for every $j = 1, \dots, m$.
- $Dg_{q_j}|_{E^{wu}} = Id$ for every $j = 1, \dots, l$.

Once again, by taking the neighbourhoods U_i and V_j sufficiently small, Lemma 3.4.5 implies that g has the SH-Saddle property of index (m, l). The second point above says that g is expansive, and hence $\Lambda(g) = 0$. Then by Theorem 3.4.2 we have that g is C^1 robustly transitive. Let $\mathcal{U}$ be the C^1 neighbourhood of g such that every $h \in \mathcal{U}$ is transitive, and let $\epsilon > 0$ be such that $B_{C^1}(g, \epsilon) \subset \mathcal{U}$.

Now for this ϵ, take m hyperbolic matrixes $B_1, \dots, B_m$ which are ϵ close to $Dg_{p_1}, \dots, Dg_{p_m}$, and such that $index(B_j) = dimE^{ss}_A + j$. Notice that we can always have these matrixes since $Dg_{p_j}|_{E^{ws}} = Id$ for every $j = 1, \dots, m$. In the same way, take l hyperbolic matrixes $C_1, \dots, C_l$ which are ϵ close to $Dg_{q_1}, \dots, Dg_{q_l}$, and such that $index(C_j) = dimE^{ss}_A + m + j$. Notice that we can always have these matrixes since $Dg_{q_j}|_{E^{wu}} = Id$ for every $j = 1, \dots, l$.

Finally by applying Franks Lemma [Fra71] once again, we know there is a partially hyperbolic diffeomorphism $f \in \mathcal{U}$ with a splitting of the form:

$$T\mathbb{T}^d = E^{ss}_f \oplus E^{c}_f \oplus E^{uu}_f$$

where $dimE^{\sigma}_f = dimE^{\sigma}_A$ for $\sigma = ss, c, uu$, and moreover:

- $f(x) = g(x) = Ax$ for every $x \in (U_1 \cup \cdots \cup U_m \cup V_1 \cup \cdots \cup V_l)^c$.
- $f(p_j) = p_j$ for every $j = 1, \dots, m$.
- $f(q_j) = q_j$ for every $j = 1, \dots, l$.
- $Df_{p_j} = B_j$ for every $j = 1, \dots, m$.
- $Df_{q_j} = C_j$ for every $j = 1, \dots, l$.

In particular we have $k + 1$ fixed points (we are including 0 here) with indexes going from $dimE^{ss}_A$ to $dimE^{ss}_A + k$. To end the proof we have to mix the center subbundles E^{ws}_m and E^{wu}_1. To do this, we just have to take another different fixed point p and apply the same isotopy as in Lemma 3.4.7. We thus obtain our example and this finish the proof of Theorem C.

3.5 The Berger-Carrasco-Obata example

As we mentioned in the introduction, in this section we are going to treat an example introduced by P. Berger and P. Carrasco in [BC14]. The example was introduced originally as a C^2 robustly non-uniformly hyperbolic diffeomorphism. Almost every point has both negative and positive Lyapunov exponents and the center direction (which is two dimensional) does not admit any dominated splitting, since it has mixing behaviour.

Later in [Oba18] it was proved that the example is indeed C^2 stably ergodic. We mention here that to obtain ergodicity the author doesn't use the accessibility property. Finally in [CO21] the authors proved that the example is C^1 robustly topologically mixing (hence C^1 robustly transitive). The mixing behaviour along the center disables the example of having the SH property according to Pujals-Sambarino definition. However, in this section we are going to see that the example has the SH-Saddle property of index (1,1).

We mention that we are not going to obtain robust transitivity of the example by a SH-Saddle argument as above, we are just going to see that the example has SH-Saddle property, and we hope that this will contribute to the study of this property.

We begin by presenting the example. Let $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ be the two torus, and let $N > 0$ be a positive integer. We consider the *standard map* given by $s_N(x,y) = (2N\sin x + 2x - y, x)$. Take $A \in SL(2,\mathbb{Z})$ a hyperbolic matrix and define for each N the skew-product $f_N : \mathbb{T}^4 \to \mathbb{T}^4$ given by

$$f_N(x,y,z,w) = (s_N(x,y) + P_x(A^N(z,w)), A^{2N}(z,w)) \text{ where } P_x(x,y) = (x,0)$$

We observe that $Df_N|_{\mathbb{R}^2\times\{0\}} = Ds_N$. The main theorem in [CO21] says that there is $N_0 > 0$ such that for any $N \geq N_0$ the map f_N is C^1 robustly topologically mixing. We are going to deduce from the calculations in their article that it has the SH-Saddle property of index (1,1).

A simple computation of the characteristic polynomial of Ds_N tell us that the eigenvalues have the form $N\cos x + 1 \pm \sqrt{N\cos x(N\cos x + 2)}$. From here we deduce that when x is different from $\pi/2$ or $3\pi/2$ then Ds_N is hyperbolic. Now take $I_N = (-2N^{-3/10}, 2N^{-3/10})$ and write $C = \{\pi/2 + I_N\} \cup \{3\pi/2 + I_N\}$. We take the *bad regions* $C^u = C \times S^1 \times \mathbb{T}^2$ and $C^s = S^1 \times C \times \mathbb{T}^2$ and the *good regions* $G^u = \mathbb{T}^4 \setminus C^u$ and $G^s = \mathbb{T}^4 \setminus C^s$. In the good regions Ds_N is hyperbolic, and since G^u and G^s are compact, we have uniform constants, named σ_1 and σ_2. Then, if a point belongs to G^u then Ds_N is hyperbolic, and if a point belongs to G^s then, Ds_N^{-1} is hyperbolic too.

Now for each $\theta > 0$ we define the horizontal cone of size θ along the center as $\mathcal{C}_\theta^{hor} = \{v = (v_x, v_y) \in E^c : \|v_y\| \leq \theta\|v_x\|\}$ and the vertical cone of size θ along the center as $\mathcal{C}_\theta^{ver} = \{v = (v_x, v_y) \in E^c : \|v_x\| \leq \theta\|v_y\|\}$. Fix $\theta = N^{-3/5}$ then Lemma 2.8 of [CO21] says that if a point $m \in G^u$ then the family of center cones are Ds_N-invariant for the future. The same happens for the past in the region G^s.

Proposition 3.5.1. *The map f_N has (1,1) SH-Saddle property.*

Proof. First, Lemma 2.8 of [CO21] gives us the desire family of cones Df_N-invariant to the future and the past. For $N > 0$ large enough Lemma 3.1 of [CO21] says that every strong unstable leaf of large greater than $L > \lambda^N(\|P_x(e^u)\| + 3\lambda^N)^1 > 0$ has a point $m \in \mathcal{W}_{f_N}^{uu}(x, L)$ such that the forward orbit of m doesn't meet the bad region, i.e. $f_N^k(m) \in G^u$ for every $k \geq 0$. This implies that the center behavies hyperbolic to the future for m, and this implies that the strong unstable foliation $\mathcal{W}_{f_N}^{uu}$ has SH-Saddle property of index 1. In the same way, we have that the strong stable foliation $\mathcal{W}_{f_N}^{ss}$

has SH-Saddle property of index 1. Therefore, f_N has SH-Saddle property of index (1,1). □

3.6 Partially hyperbolic geodesic flows

In this section we are going to adapt the previous techniques from diffeomorphisms to flows, with particular emphasis on geodesic flows. More precisely, we are going to give sufficient conditions for a C^∞ partially hyperbolic Riemannian metric to be C^2 robustly transitive. Recall that we say that a Riemannian metric is transitive, if its corresponding geodesic flow is transitive.

In Subsection 3.6.1 we are going to translate the definition of SH-Saddle property from diffeomorphisms to flows. In Subsection 3.6.2 we state some general results concerning the topological stability of geodesic flows. Finally in Subsection 3.6.3 we give a criterion to get robust transitivity for partially hyperbolic geodesic flows with SH-Saddle property, analogous to the one given in Section 3.4 for diffeomorphisms.

3.6.1 SH-Saddle property for flows

In this subsection we are going to translate the definitions from diffeomorphisms to flows. In particular we are going to be interested in geodesic flows.

Recall that a flow $\varphi_t : M \to M$ generated by a vector field $X : M \to TM$ is partially hyperbolic if the tangent bundle TM splits into $D\varphi$ invariant continuous subbundles $TM = E^{ss} \oplus E^c \oplus \langle X \rangle \oplus E^{uu}$ such that

$$\|D\varphi_t(v^{ss})\| < \lambda_{ss}^t < \|D\varphi_t(v^c)\| < \lambda_{uu}^t < \|D\varphi_t(v^{uu})\| \text{ for } t > 0$$

for some Riemannian metric $\|\cdot\|$ and some $\lambda_{ss} < 1 < \lambda_{uu}$ and all unit vectors $v^{ss} \in E^{ss}$, $v^c \in E^c$ and $v^{uu} \in E^{uu}$.

Definition 3.6.1. *Given a partially hyperbolic flow $\varphi_t : M \to M$, we say that the strong unstable foliation $\mathcal{W}^{uu}_\varphi$ has SH-Saddle property of index $d \leq c$ if there exist $T \in \mathbb{R}$ such that the induced partially hyperbolic diffeomorphism $f = \varphi_T$ has strong unstable foliation $\mathcal{W}^{uu}_f$ with SH-Saddle property of index d. Analogously for the strong stable foliation.*

In addition, we have the SH-Saddle property for flows.

Definition 3.6.2 (SH-Saddle for flows)**.** *We say that a partially hyperbolic flow $\varphi_t : M \to M$ has the SH-Saddle property of index (d_1, d_2) if there is $T \in \mathbb{R}$ such that $f = \varphi_T$ has (d_1, d_2) SH-Saddle property as a partially hyperbolic diffeomorphism.*

Remark 3.6.3. *Recall that if $\varphi_t : M \to M$ is a partially hyperbolic flow with $\dim E^c_\varphi = c$ then $f := \varphi_T : M \to M$ is a partially hyperbolic diffeomorphism with $\dim E^c_f = c + 1$.*

Notice that if two flows $\varphi, \psi : \mathbb{R} \times M \to M$ are C^1-close, then their time-T maps $f = \varphi(T, \cdot)$ and $g = \psi(T, \cdot)$ are C^1-close diffeomorphisms. In particular Theorem 3.2.12 implies that SH-Saddle property is C^1 open among partially hyperbolic flows. We summarize this observation in the following proposition.

Proposition 3.6.4. *The SH-Saddle property is C^1-open among partially hyperbolic flows.*

Proof. Take a partially hyperbolic flow $\varphi : \mathbb{R} \times M \to M$ with SH-Saddle property. By definition, we have that there is $T \in \mathbb{R}$ such that the map $f = \varphi(T, \cdot)$ has SH-Saddle property as a partially hyperbolic diffeomorphism. By Theorem 3.2.12 there is a C^1

neighbourhood $\mathcal{B}$ of f such that every partially hyperbolic diffeomorphism $g \in \mathcal{B}$ has SH-Saddle property. Now just take a sufficiently C^1 small neighbourhood $\mathcal{V}$ of φ such that for every flow $\psi \in \mathcal{V}$, we have $\psi(T, \cdot) \in \mathcal{B}$. As a result, every flow $\psi \in \mathcal{V}$ has SH-Saddle property. □

Now since every Riemannian metric has its corresponding geodesic flow, we can translate the definition of SH-Saddle property to partially hyperbolic Riemannian metrics. Recall that a Riemannian metric is said to be partially hyperbolic if its corresponding geodesic flow is partially hyperbolic (see Definition 1.2.6 and Definition 1.2.8).

Definition 3.6.5 (SH-Saddle property for Riemannian metrics)**.** *A C^∞ partially hyperbolic Riemannian metric has the SH-Saddle property if its induced geodesic flow has SH-Saddle property.*

Notice that a C^2-small perturbation on the metric imply a small perturbation on the geodesic field, and therefore a C^1-small perturbation on the flow. Then by a similar argument as above, we get the following proposition.

Proposition 3.6.6. *The SH-Saddle property is a C^2 open property among C^∞ partially hyperbolic Riemannian metrics.*

Proof. Suppose that g_0 is a C^∞ Riemannian metric such that its geodesic flow $\varphi_t^{g_0} : T^1M \to T^1M$ is partially hyperbolic and has SH-Saddle property. By Proposition 3.6.4 we know there is a C^1-neighbourhood $\mathcal{V}$ such that every flow $\phi \in \mathcal{V}$ has SH-Saddle property. Then we just have to take $\mathcal{U}$ a C^2 neighbourhood of g_0 in the space of C^∞ Riemannian metrics such that for every metric $g \in \mathcal{U}$, its geodesic flow $\varphi_t^g \in \mathcal{V}$. □

3.6.2 Expansiveness and topological stability

We begin this subsection introducing some definitions and well known results concerning expansive geodesic flows. Let $\varphi_t : X \to X$ be a continuous flow on a metric space (X, d). The flow φ_t is said to be *expansive* if there exists a constant $\epsilon > 0$ such that for every $x \in X$ we have the following property: if for a given $y \in X$ there exists a continuous and surjective map $r_y : \mathbb{R} \to \mathbb{R}$ with $r_y(0) = 0$ such that

$$d(\varphi_t(x), \varphi_{r_y(t)}(y)) \leq \epsilon \text{ for every } t \in \mathbb{R}$$

then there exists $t_0 \in \mathbb{R}$ such that $\varphi_{t_0}(x) = y$. We call ϵ the *expansivity constant*. In other words, every two different orbits of an ϵ-expansive flow are ϵ-separeted eventually in time.

We say that a continuous flow $\varphi_t : X \to X$ is *topologically stable* if there exists a C^0-neighbourhood $\mathcal{V}$ of φ_t such that, for every flow $\psi_t \in \mathcal{V}$ there are continuous and surjective functions $h : X \to X$ and $r : X \times \mathbb{R} \to \mathbb{R}$ with $r(x, 0) = 0$ such that

$$h \circ \varphi_t(p) = \psi_{r(p,t)} \circ h(p) \text{ for every } t \in \mathbb{R}, p \in X$$

From now on and in the rest of this section, every flow would be a geodesic flow, i.e. φ_t will be the geodesic flow associated to a Riemannian manifold (M, g) and X will be the unitary tangent bundle T^1M. We remark that the unitary tangent bundle depends on the choice of the metric, but given two Riemannian metrics g_1 and g_2 their unitary tangent bundles $T^1_{g_1}M$ and $T^1_{g_2}M$ are diffeomorphic (see Subsection 1.1.2). Recall that a Riemannian manifold has no conjugate points if the exponential map is non singular at every point (see also Subsection 1.1.2).

A fundamental property in hyperbolic dynamics is the existence of stable and unstable manifolds. These manifolds have local product structure, and are invariant by the dynamics. For expansive geodesic flows of Riemannian manifolds without conjugate points, we have an analogous result due to R. O. Ruggiero.

Theorem 3.6.7 (Theorem 1 in [Rug97]). *Let (M, g_0) be a C^∞ compact Riemannian manifold of dimension n with no conjugate points. Let $\varphi_t : T^1M \to T^1M$ be the geodesic flow on the unitary tangent bundle and assume that φ_t is ϵ-expansive. Then:*

1. *for every point $\theta \in T^1M$ the sets:*

$$W^s(\theta) = \{\eta \in T^1M : \lim_{t\to+\infty} d(\varphi_t(\theta), \varphi_t(\eta)) = 0\}$$
$$W^u(\theta) = \{\eta \in T^1M : \lim_{t\to-\infty} d(\varphi_t(\theta), \varphi_t(\eta)) = 0\}$$

   are C^0 submanifolds of dimension $n-1$

2. *The sets W^s and W^u give continuous foliations of T^1M which induce a local product structure.*

The sets $W^s(\theta)$ and $W^u(\theta)$ are called the *stable* and *unstable sets* of θ respectively. By local product structure in Point 2 above we mean the following: for every $\theta \in T^1M$ there is a local transverse section Σ_θ of θ and a homeomorphism $F : (-1,1)^{2(n-1)} \to \Sigma_\theta$ such that:

1. $F((-1,1)^{n-1} \times \{y_0\})$ is a subset of the connected component of

$$\bigcup_{t\in\mathbb{R}} \varphi_t(W^s(F(0, y_0))) \cap \Sigma_\theta$$

   containing $F(0, y_0)$ for every $y_0 \in (-1,1)^{n-1}$

2. $F(\{x_0\} \times (-1,1)^{n-1})$ is a subset of the connected component of

$$\bigcup_{t\in\mathbb{R}} \varphi_t(W^u(F(x_0, 0))) \cap \Sigma_\theta$$

   containing $F(x_0, 0)$ for every $x_0 \in (-1,1)^{n-1}$

The sets

$$W^{cs}(\theta) = \bigcup_{t\in\mathbb{R}} \varphi_t(W^s(\theta)) \text{ and } W^{cu}(\theta) = \bigcup_{t\in\mathbb{R}} \varphi_t(W^u(\theta))$$

are called the *center stable* and *center unstable* sets of θ respectively.

As a consequence of the above result, R. O. Ruggiero proved the following theorem.

Theorem 3.6.8 (Theorem 2 in [Rug97]). *Let (M, g_0) be a C^∞ compact Riemannian manifold with no conjugate points such that the geodesic flow $\varphi_t : T^1M \to T^1M$ is expansive. Then the set of closed orbits is dense and φ_t is topologically transitive.*

The stable and unstable manifolds in the uniformly hyperbolic case allowed us to have shadowing properties as we saw in Chapter 2. Since we have stable/unstable sets for expansive flows with no conjugate points, the same shadowing lemma holds for this kind of geodesic flows. The proof once again is due to R. O. Ruggiero.

Lemma 3.6.9 ([Rug96]). *Let $\varphi_t : T^1M \to T^1M$ be a geodesic flow of a compact, n-dimensional manifold (M, g) without conjugate points, such that φ_t is ϵ-expansive. Then, there exists a C^0 neighbourhood $\mathcal{V}_\epsilon$ of φ_t such that every C^1 flow $\psi_t : T^1M \to T^1M$ in $\mathcal{V}_\epsilon$ has the property that given $\theta \in T^1M$, there exists $\theta_0 \in T^1M$ and and increasing surjective function $r : \mathbb{R} \to \mathbb{R}$ with $r(0) = 0$ such that $d(\varphi_t(\theta_0), \psi_{r(t)}(\theta)) \leq \epsilon/2$.*

The proof of this lemma has basically two ingredients. First the fact that expansive homeomorphisms have stable and unstable sets, and this implies the shadowing property, similar to the case we saw in Chapter 2. Second the fact that geodesic flows have no singularities, and thus the manifold can be covered by finite boxes with local product structure. This allows the author to bring the shadowing ideas from the discrete case to the continuous case. For more details the interested reader can see [Rug96] or references therein (for example [Lew83] or [Pat90]).

As in the discrete case the Shadowing Lemma has important consequences concerning the stability of the systems.

Theorem 3.6.10 ([Rug96]). *Let $\varphi_t : T^1M \to T^1M$ be a geodesic flow of a compact, n-dimensional manifold (M, g) without conjugate points. If φ_t is expansive, then it is topologically stable.*

The previous theorem says the following: given $\varphi_t : T^1M \to T^1M$ a geodesic flow of a compact manifold (M, g) without conjugate points such that φ_t is ϵ-expansive, there is a C^0-neighbourhood $\mathcal{V}_\epsilon$ of φ_t such that, for every flow $\psi_t \in \mathcal{V}_\epsilon$ there are continuous and surjective functions $h : T^1M \to T^1M$ and $r : T^1M \times \mathbb{R} \to \mathbb{R}$ with $r(x, 0) = 0$ such that:

$$h \circ \psi_t(\theta) = \varphi_{r(\theta,t)} \circ h(\theta) \ \text{ for every } \ t \in \mathbb{R},\ \theta \in T^1M$$

Remark 3.6.11. *Like in the discrete case, we have control of the size of the fibers of the semiconjugacy map h above: for every $\delta > 0$ there is a C^0 neighbourhood $\mathcal{V}_\delta$ of φ_t such that, for every flow $\psi_t \in \mathcal{V}_\delta$ we have that $diam(h^{-1}(h(\theta))) < \delta$ for every $\theta \in T^1M$.*

Notice that topological stability is a weaker notion than structural stability. The problem is that the function f mentioned above is continuous and surjective, but in most cases will not be injective. We had the same problem in the discrete case for diffeomorphisms isotopic to Anosov: we have the semiconjugacy to the linear part but not a real conjugacy. We are going to solve this problem in the same way: despite having no trivial fibers, if we are sufficiently close to a fixed Riemannian metric which is expansive, the fibers of the map h will have small size compared with the size of the center disks given by the SH-Saddle property. We then conclude by the same arguments of the previous section for diffeomorphisms.

3.6.3 Proof of Theorem D

Let us recall the statement of the theorem.

Theorem 3.6.12. *Let g_0 be a C^∞ Riemannian metric on a compact differentiable manifold M with no conjugate points and let $\varphi_t : T^1M \to T^1M$ be its geodesic flow. Suppose that φ_t is expansive with stable sets W^s and unstable sets W^u. Suppose that in addition φ_t is partially hyperbolic with a splitting $T(T^1M) = E^{ss} \oplus E^c \oplus \langle X \rangle \oplus E^{uu}$, and it has the SH-Saddle property of index (d_1, d_2) where $d_1 = dimW^s - dimE^{ss}$ and $d_2 = dimW^u - dimE^{uu}$. Then φ_t is C^1 robustly transitive (or C^2 among metrics).*

Proof. The idea of the proof is pretty similar to the one of Theorem 3.4.2 for diffeomorphisms. The main difference is that here we have an extra dimension given by the flow direction, and instead of semiconjugating to the linear part, we semiconjugate to the original geodesic flow.

Let $\varphi_t : T^1M \to T^1M$ be the geodesic flow of the metric g_0. By hypothesys, we know that there is $T \in \mathbb{R}$ such that $f = \varphi_T : T^1M \to T^1M$ has the SH-Saddle property of index (d_1, d_2). Notice that f is a partially hyerbolic diffeomorphism with a splitting of the form $T(T^1M) = E^{ss}_f \oplus E^c_f \oplus E^{uu}_f$ where $E^{ss}_f = E^{ss}$, $E^{uu}_f = E^{uu}$ and $E^c_f = E^c \oplus \langle X \rangle$.

Despite having an extra dimension in the center bundle $E^c_f = E^c \oplus \langle X \rangle$, since f is the time T map of a geodesic flow, we have that $\|Df|_{\langle X \rangle}\| = 1$. Therefore, the cones given by the SH-Saddle property are contained in E^c (it is imposible to have expansion or contraction in the X direction).

Since SH-Saddle property is C^1 open among partially hyperbolic diffeomorphisms by Theorem 3.2.12, we know there are constants $\lambda > 1$, $L > 0$, $\delta_1 > 0$ and a C^1-neighbourhood $\mathcal{V}$ of f such that, if $g \in \mathcal{V}$ then:

$$H^-_{\lambda,d_1}(g) \cap \mathcal{W}^{ss}_g(\theta, L) \neq \emptyset \ \text{ for every } \theta \in T^1M$$
$$H^+_{\lambda,d_2}(g) \cap \mathcal{W}^{uu}_g(\theta, L) \neq \emptyset \ \text{ for every } \theta \in T^1M$$

Moreover, by Corollary 3.2.14, for every $g \in \mathcal{V}$, $\theta^u \in H^+_{\lambda,d_2}(g)$ and D^u a center disk of dimension d_2 tangent to $\mathcal{C}^u_{\theta^u}$, there is $N > 0$ such that $g^n(D^u)$ contains a disk W^{cu} of diameter bigger than $2\delta_1$ for every $n \geq N$. The same happens with the stable manifold: for every $g \in \mathcal{V}$, $\theta^s \in H^-_{\lambda,d_1}(g)$ and D^s a center disk of dimension d_1 tangent to $\mathcal{C}^s_{\theta^s}$, there is $N > 0$ such that $g^{-n}(D^s)$ contains a disk W^{cs} of diameter bigger than $2\delta_1$ for every $n \geq N$.

By Proposition 3.6.4 and Proposition 3.6.6 we can take a C^2 neighbourhood $\mathcal{V}_2$ of g_0, and a C^1 neighbourhood $\mathcal{V}_1$ of φ_t such that for every $g \in \mathcal{V}_2$, we have that its corresponding geodesic flow ψ_t belongs to $\mathcal{V}_1$, and hence, its time T map belongs to $\mathcal{V}$. Now let us define the following constants:

$$\begin{aligned} \rho_s &= \rho(\dim W^s, \delta_1) \\ \rho_u &= \rho(\dim W^u, \delta_1) \\ \tau &= \min\{\rho_s, \rho_u\} \end{aligned}$$

where $\rho(d,r)$ are given by Proposition 3.3.4.

According to Theorem 3.6.10 since φ_t is ϵ expansive, there is a C^0-neighbourhood $\mathcal{V}_\epsilon$ of φ_t such that, for every flow $\psi_t \in \mathcal{V}_\epsilon$ there are continuous and surjective functions $h : T^1M \to T^1M$ and $r : T^1M \times \mathbb{R} \to \mathbb{R}$ with $r(x,0) = 0$ such that:

$$h \circ \psi_t(\theta) = \varphi_{r(\theta,t)} \circ h(\theta) \ \text{ for every } \ t \in \mathbb{R},\ \theta \in T^1M \tag{3.12}$$

Moreover, we can take $\mathcal{V}_\epsilon$ sufficiently small such that for every flow $\psi_t \in \mathcal{V}_\epsilon$ we have that $\operatorname{diam}(h^{-1}(h(\theta))) < \tau$ for every $\theta \in T^1M$ (see Remark 3.6.11).

Now take $\mathcal{U} = \mathcal{V}_1 \cap \mathcal{V}_\epsilon$. We claim that every flow $\psi_t \in \mathcal{U}$ is transitive. By Proposition 3.1.1 and Remark 3.1.2 it is enough to prove that there is $T > 0$ such that for any two open sets $U_1, U_2 \subset T^1M$ there is $n \in \mathbb{Z}$ such that $g^n(U_1) \cap U_2 \neq \emptyset$, where $g = \psi_T$.

Now we repeat the argument we did for diffeomorphisms in Theorem 3.4.2. Take two points $\theta_1 \in U_1$ and $\theta_2 \in U_2$, and let $n_1 \in \mathbb{N}$ be such that $g^{-n_1}(U_1) \supset \mathcal{W}^{ss}_g(g^{-n_1}(\theta_1), L)$ and $g^{n_1}(U_2) \supset \mathcal{W}^{uu}_g(g^{n_1}(\theta_2), L)$. Take $\theta^s \in H^-_{\lambda,d_1}(g) \cap \mathcal{W}^{ss}_g(g^{-n_1}(\theta_1), L)$ and $\theta^u \in H^+_{\lambda,d_2}(g) \cap \mathcal{W}^{uu}_g(g^{n_1}(\theta_2), L)$ given by (d_1, d_2) SH-Saddle property. Now take $D^s \subset \mathcal{W}^c_g(\theta^s)$ a center disk of dimension d_1 tangent to $\mathcal{C}^s_{\theta^s}$ and $D^u \subset \mathcal{W}^c_g(\theta^u)$ a center disk of dimension d_2 tangent to $\mathcal{C}^u_{\theta^u}$. We can take D^s, D^u small enough such that $D^s \subset g^{-n_1}(U_1)$ and $D^u \subset g^{n_1}(U_2)$. Recall that $\mathcal{C}^s$ and $\mathcal{C}^u$ are the cones invariant for the past and the future respectively given by SH-Saddle property. Moreover, $\mathcal{C}^s$ and $\mathcal{C}^u$ uniformly expand vectors for the past and the future respectively.

Now take $D_1 = \cup_{\theta \in D^s} \mathcal{W}_g^{ss}(\theta, \delta)$ and $D_2 = \cup_{\theta \in D^u} \mathcal{W}_g^{uu}(\theta, \delta)$. We can choose $\delta > 0$ small enough such that $D_1 \subset g^{-n_1}(U_1)$ and $D_2 \subset g^{n_1}(U_2)$. Notice that D_1 is a disk of dimension equal to dimW^s and D_2 is a disk of dimension equal to dimW^u. Now by what we mentioned above, there is $n_2 \in \mathbb{N}$ such that $g^{-n}(D^s)$ contains a disk of size bigger than $2\delta_1$ and $g^n(D^u)$ contains a disk of size bigger than $2\delta_1$ for every $n \geq n_2$.

Now we use again Corollary 3.3.5 applied to the functions $\pi^s \circ p \circ h$ and $\pi^u \circ p \circ h$ where the funcions π^s and π^u are the local projections restricted to the local section Σ_θ given by the local product structure of the flow, and the funcion p is the local projection from T^1M to Σ_θ (projection given by the flow lines).

Observe that $\psi_t \in \mathcal{U}$ which implies that the semiconjugacy h is τ-light (see Definition 3.3.3). Moreover we have:

Claim 3.6.13. *The function $\pi^s \circ p \circ h$ is τ-light when restricted to $g^{-n_2}(D_1)$ and the function $\pi^u \circ p \circ h$ is τ-light when restricted to $g^{n_2}(D_2)$.*

Proof. Let's see the case $\pi^s \circ p \circ h$ since the other one is symmetric. Now notice that $g^{-n}(D_1)$ contains a disk of size bigger than $2\delta_1$ for every $n \geq n_2$ and the disk $g^{-n}(D_1)$ is tangent to a cone $\mathcal{C}^s$ which is uniformly expanding for the past. Thus by the semiconjugacy relation in Equation (3.12) we know that $p \circ h(D_1)$ can not intersect W^u more than once, otherwise there would be different points in D_1 such that their distance by past iterates of g goes to zero, and this is impossible since the cones $\mathcal{C}^s$ are expanding for the past. In consequence the fibers of $\pi^s \circ p \circ h$ have the same size of the fibers of $p \circ h$, and so $\pi^s \circ p \circ h$ is τ-light restricted to $g^{-n_2}(D_1)$. □

Now we are in hypothesys of Corollary 3.3.5 and therefore $\pi^s \circ p \circ h(g^{-n_2}(D_1)) \subset W^s$ contains an open set. The same argument shows that $\pi^u \circ p \circ h(g^{n_2}(D_2)) \subset W^u$ contains an open set. Since the flow φ_t is expansive (and transitive by Theorem 3.6.8) and the topological disks have complementary dimensions and with the right inclination, we know there is $t_1 > 0$ such that $\varphi_{t_1}(h(g^{n_2}(D_2))) \cap h(g^{-n_2}(D_1)) \neq \emptyset$. By the semiconjugacy relation and since h is close to the identity, there is $t_2 \in \mathbb{R}$ such that

$$\emptyset \neq \psi_{t_2}(g^{n_2}(D_2))) \cap g^{-n_2}(D_1)$$

and this implies that

$$\emptyset \neq \psi_{t_2+Tn_2}(D_2) \cap \psi_{-Tn_2}(D_1) \subset \psi_{t_2+T(n_1+n_2)}(U_2) \cap \psi_{-T(n_1+n_2)}(U_1)$$

which is equivalent to

$$\emptyset \neq \psi_{t_2+2T(n_1+n_2)}(U_2) \cap U_1$$

Since the choice of U_1 and U_2 was arbitrary, this proves that ψ_t is transitive. □

Chapter 4

Stable accessibility

In this chapter we prove Theorem E and Theorem F. We begin by introducing the statements of the results and giving an outline of the proofs of these theorems and the general structure of the chapter.

4.1 Main results

We fix a compact Riemannian manifold M of dimension $d \geq 4$ and an integer $r \geq 2$. Our main result is about the C^r-density of the accessibility property for partially hyperbolic diffeomorphisms with two-dimensional center which are stably dynamically coherent and satisfy some strong bunching condition (this bunching condition will be presented in Section 4.2, as Definition 4.2.5). We will denote by $\mathcal{PH}^r_*(M)$ to the set of partially hyperbolic diffeomorphisms satisfying this strong bunching condition. We also denote by $\mathcal{PH}^r_*(M, \mathrm{Vol}) \subset \mathcal{PH}^r_*(M)$ the subset of volume preserving partially hyperbolic diffeomorphisms.

Theorem E ([LP]). *For any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r_*(M)$, resp. $f \in \mathcal{PH}^r_*(M, \mathrm{Vol})$, with $\dim E^c_f = 2$, that is dynamically coherent and plaque expansive, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that g is stably accessible.*

In particular, by the work of Burns-Wilkinson [BW10], this implies that for any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r_*(M, \mathrm{Vol})$, with $\dim E^c_f = 2$, that is dynamically coherent and plaque expansive, and for any $\delta > 0$, there exists $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that g is stably ergodic.

One intermediate step is to show that trivial accessibility classes can be broken by C^r-small perturbations. This part of the proof also holds when the center is higher dimensional and only requires center bunching.

Theorem F ([LP]). *For any partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$, resp. $f \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $\dim E^c_f \geq 2$, that is center bunched, dynamically coherent, and plaque expansive, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathcal{PH}^r(M)$, resp. $g \in \mathcal{PH}^r(M, \mathrm{Vol})$, with $d_{C^r}(f, g) < \delta$, such that $\mathrm{C}_g(x)$ is non-trivial, for all $x \in M$.*

Let us briefly summarize the main steps of the proof:

1. we study the structure of local center accessibility classes, i.e., the set of points which can be attained within some small center disk around a given point, following accessibility sequences with a given number of legs of prescribed size; in particular, we identify which are the configurations to break in order to make each accessibility class open;

2. given a small center disk $\mathcal{D}$, we construct continuous families of local accessibility sequences at points in $\mathcal{D}$; these families depend on the nature of the center accessibility class of the base point (which can be zero, one or two-dimensonal), and allow us to have sufficiently many "degrees of freedom" to create local accessibility after perturbation;

3. once these families are constructed, we design families of perturbations, localized near one of the corners of the accessibility sequences, and which depend in a differentiable way on the perturbation parameter;

4. we study the variation of the endpoint of these accessibility sequences once the perturbation parameter is turned on, and show that for suitable perturbations, we obtain a submersion from the space of perturbations to the phase space; in particular, bad configurations in phase space (non-open accessibility classes) correspond to special configurations in the space of perturbations, which can be broken to create local accessibility;

5. we globalize the argument using spanning families.

Let us say a few more words about the previous points. The details about point (1) are given in Section 4.3. For partially hyperbolic diffeomorphisms with two-dimensional center that are center bunched, it is known (by the works of Rodriguez-Hertz [Her05], Rodriguez-Hertz and Vásquez [HV20] etc.) that center accessibility classes are zero, one or two-dimensional submanifolds (see Theorem 1.6.4 in Chapter 1). Moreover, Horita-Sambarino [HS17] have studied the organization of center accessibility classes within a small center disk all of whose points have non-trivial center accessibility classes; in particular, they have shown that the set of one-dimensional center accessibility classes of points in the disk forms a C^1 lamination. In Section 4.3, we go further in this direction, and investigate the variation of center accessibility classes for perturbations of a given partially hyperbolic diffeomorphism. In particular, we show that if the center accessibility class of a point x remains one-dimensional after perturbation, it stays in a certain "cone" around x. This is the part of the chapter where we need the strong bunching condition (see Definition 4.2.5 below), in order to get some regularity on the holonomies.

The construction of loops mentioned in point (2) is outlined in Section 4.4. Indeed, in the subsequent argument, given a point x whose accessibility class is not open, we need to construct (non-trivial) *closed* accessibility sequences at x; moreover, we show that it is possible to construct these loops in such a way that they depend nicely on x.

The details about point (3) are in Section 4.5, and follow the arguments of [LZ22]. Given a point $x \in M$ that is non-periodic, we construct a family $\{\gamma(t)\}_{t\in[0,1]}$ of contractible *us-loops* at (f, x), and we define a family of perturbations such that the support of the perturbations is contained in some small neighbourhood of the first corner of $\gamma(1)$. By taking the loop sufficiently small, the first return time to the support of the perturbation can be made arbitrarily large, and we show that it induces a change of the holonomy along the continuation of $\gamma(1)$ for the perturbed diffeomorphisms. More precisely, by the results of [LZ22], we get a submersion from the space of perturbations to the phase space – here, the local center leaf of x.

The submersion property is sufficient to show that after perturbation, the center accessibility class of x can be made non-trivial. This part of the proof is explained in Subsection 4.6.1 and holds in a more general setting, as it does not require the center to be two-dimensional. When the center accessibility class of x is one-dimensional, by point (1), it varies continuously with respect to the diffeomorphism in the C^1 topology. In particular, if the center accessibility class of x were one-dimensional for every diffeomorphism in a C^r-neighbourhood of f, then all those classes would stay in some cone around the point x; but this is in contradiction with the submersion property for the family of perturbations we construct. The details of this part are given in Subsection 4.6.2.

The details about point (5) are given in Section 4.7, where we explain how to globalize the arguments in order to verify the accessibility property, through the notion of spanning family of center disks, as in [DW03] (see Subsection 4.7.1). In Subsection 4.7.2, given some small center disk in the family, we explain how by a C^r-small perturbation, it is possible to make the center accessibility class of each point in the disk non-trivial. One difficulty is that the perturbation used to break trivial center accessibility classes may create new trivial classes in other places (at points with non-trivial, but very small center accessibility classes). The idea to bypass this difficulty is to take two families of us-loops which we can perturb "independently", in order to increase the codimension of "bad" situations for which the center accessibility class of some point in the disk would be trivial. Once all classes in the disk are non-trivial, we have to make a further perturbation to make all these classes simultaneously open. One important step in the argument is the aforementioned result (inspired by the work of Horita-Sambarino [HS17]) that within the center disk, one-dimensional center accessibility classes vary C^1-continuously both in perturbation space and phase space. In particular, if the center disk is chosen sufficiently small, then the set of tangent directions associated to one-dimensional classes (even for small perturbations of the diffeomorphism f) stay in a small cone that is uniform in the points of the disk. Thanks to the submersion property, we can then choose a perturbation for which each point x in the center disk will have a point y in its center accessibility class lying outside this cone, which forces the accessibility class of x to be open. There again, one difficulty is to check that the perturbations we make preserve the accessibility classes which were already open. Repeating the same argument for each center disk in the spanning family, we thus construct a C^r-small perturbation of f that is accessible.

4.2 Preliminaries

In this section we introduce some definitions, preliminaries and well known results that we will use along the chapter. Some of these preliminares were already introduced in Chapter 1, but we also introduce them in this section in order to make the chapter self-contained.

Recall that given a compact Riemannian manifold M of dimension $m \geq 3$, we say that a diffeomorphism $f : M \to M$ is partially hyperbolic if there exists a nontrivial Df-invariant splitting $TM = E^s_f \oplus E^c_f \oplus E^u_f$ of the tangent bundle and continuous functions $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u : M \to \mathbb{R}^+$ with

$$\lambda_s < 1 < \lambda_u, \qquad \lambda_s < \lambda_c^- \leq \lambda_c^+ < \lambda_u, \tag{4.1}$$

such that for any $(x, v) \in TM$, it holds

$$\begin{aligned} \|D_x f(v)\| &< \lambda_s(x)\|v\|, && \text{if } v \in E^s_f(x) \setminus \{0\}, \\ \lambda_c^-(x)\|v\| < \|D_x f(v)\| &< \lambda_c^+(x)\|v\|, && \text{if } v \in E^c_f(x) \setminus \{0\}, \\ \lambda_u(x)\|v\| &< \|D_x f(v)\|, && \text{if } v \in E^u_f(x) \setminus \{0\}. \end{aligned}$$

For any integer $r \geq 1$, we denote by $\mathcal{PH}^r(M)$ the set of all partially hyperbolic diffeomorphisms of M of class C^r; we also denote by $\mathcal{PH}^r(M, \mathrm{Vol}) \subset \mathcal{PH}^r(M)$ the subset of volume preserving partially hyperbolic diffeomorphisms.

In the rest of this chapter, we fix an integer $r \geq 1$ and we consider a partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$. We will denote $d_s := \dim E^s_f$ and $d_u := \dim E^u_f$. Recall that the strong bundles E^u_f and E^s_f are uniquely integrable to continuous foliations $\mathcal{W}^u_f$ and $\mathcal{W}^s_f$ respectively, called the *strong unstable* and *strong stable* foliations.

4.2.1 Dynamical coherence, plaque expansiveness

Recall from Section 1.4 that a partially hyperbolic diffeomorphism f is *dynamically coherent* if the *center-unstable* bundle $E^{cu}_f := E^c_f \oplus E^u_f$ and the *center-stable* bundle $E^{cs}_f := E^c_f \oplus E^s_f$ integrate respectively to foliations $\mathcal{W}^{cu}_f$, $\mathcal{W}^{cs}_f$, called the *center-unstable foliation*, resp. the *center-stable foliation*, where $\mathcal{W}^u_f$ subfoliates $\mathcal{W}^{cu}_f$, while $\mathcal{W}^s_f$ subfoliates $\mathcal{W}^{cs}_f$. In this case, the collection $\mathcal{W}^c_f$ obtained by intersecting the leaves of $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$ is a foliation which integrates E^c_f, and subfoliates both $\mathcal{W}^{cs}_f$ and $\mathcal{W}^{cu}_f$; it is called the *center foliation*.

In the following, for any $* \in \{s, c, u, cs, cu\}$, we denote by $d_{\mathcal{W}^*_f}$ the leafwise distance, and for any $x \in M$, for any $\epsilon > 0$, we denote by $\mathcal{W}^*_f(x, \varepsilon) := \{y \in \mathcal{W}^*_f(x) : d_{\mathcal{W}^*_f}(x, y) < \varepsilon\}$ the ε-ball in $\mathcal{W}^*_f$ of center x and radius ε.

It is an open question whether dynamical coherence is a C^1-open condition. A closely related property is *plaque expansiveness*.

Definition 4.2.1 (Plaque expansiveness)**.** *We say that f is* plaque expansive *(see* [HPS77, Section 7]*) if f is dynamically coherent and there exists $\varepsilon > 0$ with the following property: if $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ are ε-pseudo orbits which respect $\mathcal{W}^c_f$ such that $d(p_n, q_n) \leq \varepsilon$ for all $n \geq 0$, then $q_n \in \mathcal{W}^c_f(p_n)$. It is known that plaque expansiveness is a C^1-open condition (see* Theorem 7.4 *in* [HPS77]*).*

The following result is due to Hirsch-Pugh-Shub.

Theorem 4.2.2 (Theorem 7.1, [HPS77], see also Theorem 1 in [PSW12])**.** *Let us assume that f is dynamically coherent and plaque expansive. Then any $g \in \mathcal{PH}^1(M)$ which is sufficiently C^1-close to f is also dynamically coherent and plaque expansive. Moreover, there exists a homeomorphism $\mathfrak{h} = \mathfrak{h}_g : M \to M$, called a* leaf conjugacy*, such that $\mathfrak{h}$ maps a f-center leaf to a g-center leaf, and $\mathfrak{h} \circ f(\mathcal{W}^c_f(\cdot)) = g \circ \mathfrak{h}(\mathcal{W}^c_f(\cdot))$.*

4.2.2 Holonomies

Let us assume that the diffeomorphism f is dynamically coherent. Let $x_1 \in M$ and let $x_2 \in M$ be sufficiently close to x_1.[1] By transversality, there exist a neighbourhood $\mathcal{U}_1$ of x_1 within $\mathcal{W}^{cu}_f(x_1)$ and a neighbourhood $\mathcal{U}_2$ of x_2 within $\mathcal{W}^{cu}_f(x_2)$ such that for any $z \in \mathcal{U}_1$, the local stable leaf through z intersects $\mathcal{U}_2$ at a unique point, denoted by $H^s_{f,x_1,x_2}(z) \in \mathcal{U}^{cu}_{x_2}$. We thus get a well defined local homeomorphism

$$H^s_{f,x_1,x_2} : \mathcal{U}_1 \to \mathcal{U}_2 \subset \mathcal{W}^{cu}_f(x_2),$$

called the *stable holonomy map*. Note that as a consequence of dynamical coherence, if $x_2 \in \mathcal{W}^s_{f,\mathrm{loc}}(x_1)$, then the image of the restriction $H^s_{f,x_1,x_2}|_{\mathcal{U}_1 \cap \mathcal{W}^c_f(x_1)}$ to the center leaf

[1] In the rest of the chapter, all the constructions will be local.

$\mathcal{W}^c_f(x_1)$ is contained in the center leaf $\mathcal{W}^c_f(x_2)$. Unstable holonomies are defined in a similar way, following local unstable leaves.

Definition 4.2.3 (Center bunching)**.** *We say that $f \in \mathcal{PH}(M)$ is center bunched if the functions $\lambda_s, \lambda_c^-, \lambda_c^+, \lambda_u$ in (4.1) can be chosen such that*

$$\max(\lambda_s, (\lambda_u)^{-1}) < \frac{\lambda_c^-}{\lambda_c^+} \tag{4.2}$$

Theorem 4.2.4 (see [HPS77] and Theorem B in [PSW12])**.** *If $f \in \mathcal{PH}^2(M)$ is dynamically coherent and center bunched, then local stable/unstable holonomy maps between center leaves are C^1 when restricted to some center-stable/center-unstable leaf and have uniformly continuous derivatives.*

Indeed, the authors prove that the strong stable/unstable foliation is C^1 when restricted to a center-stable/unstable leaf. However, from their proof, it is not clear how the holonomies $H^s_{f,x_1,x_2}|_{\mathcal{W}^c_{f,\mathrm{loc}}(x_1)}$, resp. $H^u_{f,x_1,x_2}|_{\mathcal{W}^c_{f,\mathrm{loc}}(x_1)}$ vary in the C^1-topology with the choices of the points x_1 and $x_2 \in \mathcal{W}^s_{f,\mathrm{loc}}(x_1)$, resp. $x_2 \in \mathcal{W}^u_{f,\mathrm{loc}}(x_1)$. This question is investigated in Obata's work [Oba], where it is shown that under some stronger bunching condition, these holonomy maps vary continuously with the choices of the base points x_1, x_2.

Definition 4.2.5 (see [Oba])**.** *For any integer $r \geq 1$, we denote by $\mathcal{PH}^r_*(M)$ the set of all partially hyperbolic diffeomorphisms $f \in \mathcal{PH}^r(M)$ such that, for some $\theta \in (0,1)$,*

$$\|D_x f|_{E^s_f}\|^\theta < \frac{m(D_x f|_{E^c_f})}{\|D_x f|_{E^c_f}\|}, \quad \frac{\|D_x f|_{E^c_f}\|}{m(D_x f|_{E^c_f})} < m(D_x f|_{E^u_f})^\theta,$$
$$\|D_x f|_{E^s_f}\| < m(D_x f|_{E^c_f}) m(D_x f|_{E^s_f})^\theta,$$
$$\|D_x f|_{E^c_f}\| \cdot \|D_x f|_{E^u_f}\|^\theta < m(D_x f|_{E^u_f}).$$

Note that any diffeomorphism $f \in \mathcal{PH}^r_*(M)$ is automatically center bunched.

Theorem 4.2.6 (Theorem 0.3 in [Oba])**.** *Assume that $f \in \mathcal{PH}^2_*(M)$. Then, for $* = s, u$, the family $\{H^*_{f,x_1,x_2}|_{\mathcal{W}^c_{f,\mathrm{loc}}(x_1)}\}_{x_1 \in M, x_2 \in \mathcal{W}^*_{f,\mathrm{loc}}(x_1)}$ is a family of C^1 maps depending continuously in the C^1-topology with the choices of the points x_1 and $x_2 \in \mathcal{W}^*_{f,\mathrm{loc}}(x_1)$.*

4.2.3 Accessibility classes

A *f-accessibility sequence* is a sequence $[x_1, \ldots, x_k]$ of $k \geq 1$ points in M such that for any $i \in \{1, \ldots, k-1\}$, the points x_i and x_{i+1} belong to the same stable or unstable leaf of f. In particular, the points x_1 and x_k can be connected by some *f-path*, i.e., a continuous path in M obtained by concatenating finitely many arcs in $\mathcal{W}^s_f$ or $\mathcal{W}^u_f$. We will refer to the points $x_1, \ldots, x_k$ as the *corners* of the accessibility sequence $[x_1, \ldots, x_k]$.

For any point $x \in M$, we denote by $\mathrm{Acc}_f(x)$ the *accessibility class* of x. By definition, it is the set of all points $y \in M$ which can be connected to x by some f-path. We also let

$$\mathrm{C}_f(x) := \mathrm{cc}(\mathrm{Acc}_f(x) \cap \mathcal{W}^c_f(x,1), x)$$

be the connected component containing x of the intersection of the accessibility class of x and the local center leaf through x. Similarly, for any $\varepsilon > 0$, we let $\mathrm{C}_f(x,\varepsilon) := \mathrm{cc}(\mathrm{Acc}_f(x) \cap \mathcal{W}^c_f(x,\varepsilon), x)$. By definition, accessibility classes form a partition of M.

We say that the diffeomorphism f is *accessible* if this partition is trivial, i.e., the whole manifold M is a single accessibility class; we say that f is *stably accessible* if the diffeomorphisms which are sufficiently C^1-close to f are accessible.

Moreover, given any f-accessibility sequence $\gamma = [x_1, \dots, x_k]$, we let $H_{f,\gamma} \colon \mathcal{W}^c_{f,\mathrm{loc}}(x_1) \to \mathcal{W}^c_{f,\mathrm{loc}}(x_k)$ be the holonomy map obtained by concatenating the local holonomy maps along the arcs of γ, i.e.,

$$H_{f,\gamma} := H^{*_{k-1}}_{f,x_{k-1},x_k} \circ \cdots \circ H^{*_1}_{f,x_1,x_2}, \tag{4.3}$$

where for $j \in \{1, \dots, k-1\}$, $*_j \in \{s,u\}$ is such that $x_{j+1} \in \mathcal{W}^{*_j}_f(x_j)$.

The next lemma is elementary; it follows from the local product structure and the continuous dependence of the invariant foliations with respect to the diffeomorphism.

Lemma 4.2.7 (Continuation of accessibility sequences). *Let $\gamma = [x_0, x_1, \dots, x_k]$ be a f-accessibility sequence, for some integer $k \geq 0$. Then there exist a neigbourhood $\mathcal{O}$ of x_0 and a C^1-neighbourhood $\mathcal{U}$ of f such that for any point $x \in \mathcal{O}$, and for any diffeomorphism $g \in \mathcal{U}$, there exists a natural continuation $\gamma^{x,g} = [x, x_1^{x,g}, \dots, x_k^{x,g}]$ of γ for x and g. Indeed, the g-accessibility sequence $\gamma^{x,g}$ is defined as*

$$\begin{aligned} x_1^{x,g} &:= H^{*_0}_{g,x,x_1}(x); \\ x_2^{x,g} &:= H^{*_1}_{g,x_1^{x,g},x_2}(x_1^{x,g}); \\ &\dots \\ x_k^{x,g} &:= H^{*_{k-1}}_{g,x_{k-1}^{x,g},x_k}(x_{k-1}^{x,g}); \end{aligned}$$

here, for each $j \in \{0, \dots, k-1\}$, we let $*_j \in \{s,u\}$ be such that $x_{j+1} \in \mathcal{W}^{*_j}_f(x_j)$. Moreover, $\gamma^{x,g}$ depends continuously on the pair (x,g).

Definition 4.2.8. *Given a point $x \in M$ and an integer $n \geq 2$, a $2n$ us-loop at (f, x_0) is a f-accessibility sequence $\gamma = [x_0, x_1, x_2, \dots, x_{2n}] \in M^{2n+1}$ with $2n$ legs such that*

$$\begin{aligned} x_1 &\in \mathcal{W}^u_{f,\mathrm{loc}}(x_0), \\ x_2 &\in \mathcal{W}^s_{f,\mathrm{loc}}(x_1), \dots \\ \dots\ x_{2n-1} &\in \mathcal{W}^u_{f,\mathrm{loc}}(x_{2n-2}) \cap \mathcal{W}^{cs}_{f,\mathrm{loc}}(x_0), \\ x_{2n} &:= H^s_{f,x_{2n-1},x}(x_{2n-1}) \in \mathcal{W}^c_{f,\mathrm{loc}}(x_0). \end{aligned}$$

We define $2n$ su-loops accordingly (with $x_1 \in \mathcal{W}^s_{f,\mathrm{loc}}(x_0)$ etc.).

The length *of a $2n$ us-loop $\gamma = [x_0, x_1, x_2, \dots, x_{2n}] \in M^{2n+1}$ is defined as*

$$\ell(\gamma) := d_{\mathcal{W}^u_f}(x_0, x_1) + \sum_{i=1}^{n-1} \left[d_{\mathcal{W}^s_f}(x_{2i-1}, x_{2i}) + d_{\mathcal{W}^u_f}(x_{2i}, x_{2i+1}) \right] + d_{\mathcal{W}^{cs}_f}(x_{2n-1}, x_0).$$

Moreover, we say that the us-loop γ is

- closed, *if $x_{2n} = x_0$;*
- trivial, *if $x_0 = x_1 = x_2 = \cdots = x_{2n}$;*
- non-degenerate, *if x_1 is distinct from the other corners $x_0, x_2, \dots, x_{2n}$.*

We also denote by $\overline{\gamma}$ the $2n$ su-loop $\overline{\gamma} := [x_{2n}, x_{2n-1}, \dots, x_2, x_1, x_0] \in M^{2n+1}$ at (f, x_{2n}). Finally, given an integer $m \geq 2$ and a $2m$ us-loop $\gamma' = [x_{2n}, x'_1, \dots, x'_{2m}]$ at (f, x_{2n}), the *concatenation $\gamma\gamma'$ of γ and γ' is the $2(m+n)$ us-loop $\gamma\gamma' := [x_0, x_1, \dots, x_{2n}, x'_1, \dots, x'_{2m}]$ at (f, x_0).*

Definition 4.2.9. *Given $x \in M$ and $n \geq 2$, a one-parameter family $\gamma = \{\gamma(t) = [x, x_1(t), \dots, x_{2n}(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (f, x) is said to be* continuous *if for any $i = 1, \dots, 2n$, the map $t \mapsto x_i(t)$ is continuous. We define $\ell(\gamma) := \sup_{t\in[0,1]} \ell(\gamma(t))$.*

4.2.4 Structure of center accessibility classes

Let M be a compact Riemannian manifold of dimension $d \geq 4$. Let $r \geq 2$ be some integer, and let $f \in \mathcal{PH}^r(M)$ be a partially hyperbolic diffeomorphism with $\dim E^c_f = 2$ that is center bunched and dynamically coherent.

By Theorem 4.2.4, for $* = s, u$, the $*$-holonomy maps are C^1 when restricted to a $\mathcal{W}^{c*}_f$ leaf; by C^1-homogeneity arguments, this allowed [Her05; HV20] to obtain a classification of center accessibility classes.

Theorem 4.2.10 ([Her05; HV20])**.** *For any point $x \in M$, and for any sufficiently small $\varepsilon > 0$, the local center accessibility class $\mathrm{C}_f(x, \varepsilon)$ can be either*

- *trivial, i.e., reduced to a point;*
- *a one-dimensional submanifold of $\mathcal{W}^c_f(x)$;*
- *open; in this case, $\mathrm{Acc}_f(x)$ is also open.*

In the following, for any subset $\mathscr{S} \subset M$, we let

- $\Gamma^0_f(\mathscr{S}) := \{x \in \mathscr{S} : \mathrm{C}_f(x) \text{ is trivial}\}$;
- $\Gamma^1_f(\mathscr{S}) := \{x \in \mathscr{S} : \mathrm{C}_f(x) \text{ is one-dimensional}\}$;
- $\Gamma_f(\mathscr{S}) := \Gamma^0_f(\mathscr{S}) \cup \Gamma^1_f(\mathscr{S})$.

In particular, $\mathscr{S} \setminus \Gamma_f(\mathscr{S})$ is the set of points $x \in \mathscr{S}$ whose accessibility class $\mathrm{Acc}_f(x)$ is open. When $\mathscr{S} = M$, we abbreviate $\Gamma^0_f(\mathscr{S}), \Gamma^1_f(\mathscr{S}), \Gamma_f(\mathscr{S})$ respectively as $\Gamma^0_f, \Gamma^1_f, \Gamma_f$.

4.3 Variation of one-dimensional center accessibility classes

In this section, given an integer $r \geq 2$, we prove that the set of one-dimensional center accessibility classes varies continuously in the C^1 topology with respect to $f \in \mathcal{PH}^r_*(M)$. The idea of the proof is similar to Proposition 2.19 from [HS17] where it is proved that for a fixed partially hyperbolic diffeomorphism, and for a given center disk, the one-dimensional accessibility classes form a C^1-lamination. To prove this, we have to see that for a given $x \in M$, the direction $T_x\mathrm{C}_f(x)$ varies continuously with respect to f in the C^1 topology.

Let us fix an integer $r \geq 2$. We denote by $\mathscr{F} \subset \mathcal{PH}^r_*(M)$ the set of C^r dynamically coherent, plaque expansive, partially hyperbolic diffeomorphisms with two-dimensional center which satisfy the bunching condition in Definition 4.2.5. Let $f \in \mathscr{F}$. By center bunching, for $* = s, u$, for any $x \in M$, $y \in \mathcal{W}^*_{f,\mathrm{loc}}(x)$, the holonomy map $H^*_{f,x,y}$ is C^1 when restricted to the leaf $\mathcal{W}^{c*}_{f,\mathrm{loc}}(x)$. For any C^1 neighbourhood $\mathcal{U}$ of f, we will denote by $\mathcal{U}^{\mathscr{F}}$ the set $\mathcal{U}^{\mathscr{F}} := \mathcal{U} \cap \mathscr{F}$.

In the following, we will need to have uniform control of the differential of the holonomies $H^*_{f,x,y}$ in two ways:

- with respect to the points x, y (in the same stable/unstable manifold);
- when the diffeomorphism f is replaced with another C^r partially hyperbolic diffeomorphism in a C^1-neighbourhood of f.

This is the content of the next lemma.

Lemma 4.3.1 (See [Oba], and also [Bro; BW]). *Let $f \in \mathscr{F}$. Then there exists a C^1 neighbourhood $\mathcal{U}$ of f such that for $* = s, u$ and $\mathcal{U}^{\mathscr{F}} = \mathcal{U} \cap \mathscr{F}$, the family of C^1 maps $\{H^*_{g,x,y}|_{\mathcal{W}^c_g(x)}\}_{g\in\mathcal{U}^{\mathscr{F}}, x\in M, y\in\mathcal{W}^*_f(x)}$ depends continuously in the C^1 topology with the choices of the points x, y and of the map $g \in \mathcal{U}^{\mathscr{F}}$.*

Remark 4.3.2. *In fact, Obata [Oba] shows that for $* = s, u$, the family of holonomy maps $\{H^*_{f,x,y}|_{\mathcal{W}^c_f(x)}\}_{x\in M, y\in\mathcal{W}^*_f(x)}$ depends continuously in the C^1 topology with the choices of the points x, y, when f is dynamically coherent and satisfies a strong bunching condition. For our purpose, we also need to have a uniform control with respect to the diffeomorphism g in a C^1-small neighbourhood of f. It is indeed possible as the estimates in [Oba] are written in terms of the functions as in (4.1) controlling the growth rates along the different invariant bundles, which depend continuously on the map g in the C^1 topology.*

The holonomy map associated to some accessibility sequence is obtained by composing the holonomy maps between two consecutive corners (recall (4.3)). By the previous lemma, we thus have:

Corollary 4.3.3. *Let $f \in \mathscr{F}$, and let $\gamma = [x_1, x_2, \ldots, x_k] \in M^k$ be a f-accessibility sequence for some integer $k \geq 1$. We take a small neighbourhood $\mathcal{O} \subset M$ of x_1 and a C^1 neighbourhood $\mathcal{U}$ of f such that for any $x \in \mathcal{O}$ and for any $g \in \mathcal{U}$, the continuation $\gamma^{x,g} = [x, x_2^{x,g}, \ldots, x_k^{x,g}]$ of γ starting at x given by Lemma 4.2.7 is well defined.*

Then, the family of C^1 maps $\{H_{g,\gamma^{x,g}}|_{\mathcal{W}^c_g(x)}\}_{x\in\mathcal{O}, g\in\mathcal{U}^{\mathscr{F}}}$ depends continuously in the C^1 topology with the choices of the point $x \in \mathcal{O}$ and the map $g \in \mathcal{U}^{\mathscr{F}}$.

For any point $x \in M$ and any subset $\mathscr{G} \subset \mathscr{F}$, we let $\mathscr{G}_1(x) \subset \mathscr{G}$ be the subset of maps f for which the center accessibility class $\mathrm{C}_f(x)$ is one-dimensional. For any $f \in \mathscr{G}_1(x)$, and for any sufficiently small $\theta, \varepsilon > 0$, we let $\mathscr{C}_f(x, \theta, \varepsilon) \subset \mathcal{W}^c_f(x) \cap B(x, \varepsilon)$ be the set of points in the ε-ball $B(x, \varepsilon)$ centered at x which belong to (the image by the exponential map of) the cone of angle θ around $T_x\mathrm{C}_f(x)$, i.e.,

$$\mathscr{C}_f(x, \theta, \varepsilon) := \exp_x\{y \in T_xM : \angle(y, T_x\mathrm{C}_f(x)) < \theta\} \cap B(x, \varepsilon). \tag{4.4}$$

The main result of this section is the following:

Proposition 4.3.4. *Take $f \in \mathscr{F}$ and $x \in M$ such that $\mathrm{C}_f(x)$ is one-dimensional, i.e., $f \in \mathscr{F}_1(x)$. Then, for every $\theta > 0$ there exists a C^1 neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}_1^{\mathscr{F}}(x)$, the angle at x between $\mathrm{C}_f(x)$ and $\mathrm{C}_g(x)$ satisfies*

$$\angle(T_x\mathrm{C}_f(x), T_x\mathrm{C}_g(x)) < \theta.$$

Moreover, there exists $\varepsilon_0 > 0$ such that for any $g \in \mathcal{U}_1^{\mathscr{F}}(x)$ and $\varepsilon \in (0, \varepsilon_0)$, it holds

$$\mathrm{C}_g(x, \varepsilon) \subset \mathscr{C}_f(x, \theta, \varepsilon).$$

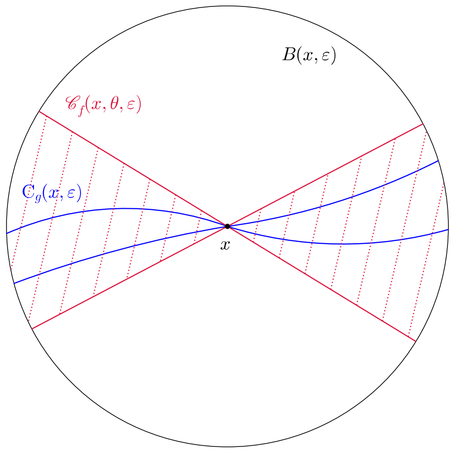


FIGURE 4.1: Variation of 1-dimensional center accessibility classes.

The idea of the proof consists in showing some "uniform" homogeneity of one-dimensional center accessibility classes $\mathrm{C}_g(x)$ of all maps $g \in \mathcal{U}^{\mathscr{F}}$, for a sufficiently small C^1 neighbourhood $\mathcal{U}$ of a fixed $f \in \mathscr{F}$. Indeed, the tangent spaces at two points x, y in the same center accessibility class are naturally related through the differential of the holonomy map along an accessibility sequence connecting x to y. Since everything we are doing here is local, we are able to compare angles and norms of vectors in different tangent spaces, using trivialization charts as follows. Recall that $d := \dim M$, and that we denote $d_s := \dim E^s_f$, $d_u := \dim E^u_f$.

Lemma 4.3.5 (see Construction 9.1, [LZ22]). *There exist C^2-uniform constants $\overline{h} = \overline{h}(f) > 0$ and $\overline{C} = \overline{C}(f) > 1$ such that for any $x \in M$, there exists a C^r volume preserving map $\phi = \phi_x \colon (-\overline{h}, \overline{h})^d \to M$ such that $\phi(0_{\mathbb{R}^d}) = x$ and*

1. *$\mathcal{W}^c_f(x, \frac{\overline{h}}{5}) \subset \phi\big((-\frac{\overline{h}}{4}, \frac{\overline{h}}{4})^2 \times \{0\}^{d_u+d_s}\big) \subset \phi\big((-\frac{2\overline{h}}{3}, \frac{2\overline{h}}{3})^2 \times \{0\}^{d_u+d_s}\big) \subset \mathcal{W}^c_f(x, \overline{h})$;*
2. *$\|\phi\|_{C^2} < \overline{C}$;*
3. *$D\phi(0, \mathbb{R}^2 \times \{0_{\mathbb{R}^{d_u+d_s}}\})$, $D\phi(0, \{0_{\mathbb{R}^2}\} \times \mathbb{R}^{d_u} \times \{0_{\mathbb{R}^{d_s}}\})$, $D\phi(0, \{0_{\mathbb{R}^{2+d_u}}\} \times \mathbb{R}^{d_s})$ are respectively equal to $E^c_f(x), E^u_f(x), E^s_f(x)$;*
4. *for any $y \in \phi((-\overline{h}, \overline{h})^d)$, $\Pi^c D(\phi^{-1})_y \colon E^c_f(y) \to \mathbb{R}^2$ has determinant in $\big(\overline{C}^{-1}, \overline{C}\big)$, where $\Pi^c \colon \mathbb{R}^d \simeq \mathbb{R}^2 \times \mathbb{R}^{d_u+d_s} \to \mathbb{R}^2$ is the canonical projection;*
5. *for any $\zeta > 0$, there exists a C^1-uniform constant $\overline{h}_\zeta = \overline{h}_\zeta(f) \in (0, \overline{h})$ so that if $h \in (0, \overline{h}_\zeta)$, then for any $y \in \phi((-h, h)^d)$, the norm of $\Pi^c(D\phi^{-1})_y \colon E^{su}_f(y) \to \mathbb{R}^2$ is smaller than ζ.*

In the following, we will denote by Π^c_x the map $\Pi^c_x := \Pi^c \circ \phi_x^{-1} \colon M \to \mathbb{R}^2$.

Before giving the proof of Proposition 4.3.4, let us state an elementary lemma and introduce a notation. Let $\alpha \colon [0,1] \to M$ be a C^1 arc of M and given $\epsilon > 0$, consider an ϵ tubular neighbourhood $\mathscr{N}_{\alpha,\epsilon}$ of α. This tubular neighbourhood is diffeomorphic to $[0,1] \times [-\epsilon, \epsilon]^{d-1}$. We identify points in $\mathscr{N}_{\alpha,\epsilon}$ with pairs (t,s), where $t \in [0,1]$ and $s \in [-\epsilon, \epsilon]^{d-1}$. We call the boundary $\{0\} \times [-\epsilon, \epsilon]^{d-1}$ the *left side* of $\mathscr{N}_{\alpha,\epsilon}$, and we call the boundary $\{1\} \times [-\epsilon, \epsilon]^{d-1}$ its *right side*. We denote by $\xi \colon \mathscr{N}_{\alpha,\epsilon} \to \alpha$ the projection $\xi \colon (t,s) \mapsto \alpha(t)$.

Lemma 4.3.6. *With the notation above, given $\delta > 0$, there exists $\epsilon > 0$ such that if $\beta\colon [0,1] \to \mathscr{N}_{\alpha,\epsilon}$ is a C^1 curve in $\mathscr{N}_{\alpha,\epsilon}$ from the left to the right side, then there exists some $(t,s) = \beta(\tilde{t})$ with $\tilde{t} \in [0,1]$ such that the angle between α and β satisfies*

$$\angle(\dot{\alpha}(t), \dot{\beta}(\tilde{t})) < \delta.$$

Proof of Proposition 4.3.4. Let us show the first part. Suppose by contradiction that for some $\eta > 0$, there exists a sequence $(g_n)_{n\geq 0} \in \mathscr{F}^{\mathbb{N}}$ of maps such that $g_n \to f$ in the C^1 topology, with $g_n \in \mathscr{F}_1(x)$ and

$$\angle(T_x C_f(x), T_x C_{g_n}(x)) > \eta, \quad \text{for all } n \geq 0. \tag{4.5}$$

Since $C_f(x)$ is one-dimensional, for some integer $n \geq 2$, there exists a $2n$ us-loop $\gamma = [x, x_1, \ldots, x_{2n}]$ at (f,x) such that $x_{2n} \neq x$. By shrinking the size of the legs, we get a one-parameter family $\{\gamma(t) = [x, x_1(t), \ldots, x_{2n}(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (f,x), where $\gamma(1) = \gamma$ and $\gamma(0)$ is trivial. By Lemma 4.2.7, there exists a C^1 neighbourhood $\widetilde{\mathcal{U}}$ of f such that for any $g \in \widetilde{\mathcal{U}}$ and for any $t \in [0,1]$, there exists a one-parameter family $\{\gamma^{x,g}(t) = [x, x_1^{x,g}(t), \ldots, x_{2n}^{x,g}(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (g,x) such that $\gamma^{x,g}(0)$ is the trivial loop. We also denote $\alpha_{g,x}\colon t \mapsto x_{2n}^{x,g}(t) \in C_g(x)$.

For each pair $(g,t) \in \widetilde{\mathcal{U}} \times [0,1]$ we have the corresponding holonomy map $H_g^t := H_{g,\gamma^{x,g}(t)}|_{\mathcal{W}^c_{g,\mathrm{loc}}(x)}\colon \mathcal{W}^c_{g,\mathrm{loc}}(x) \to \mathcal{W}^c_{g,\mathrm{loc}}(x)$. Given some small $h > 0$, and assuming that $\widetilde{\mathcal{U}}$ is sufficiently small, then for every map $g \in \widetilde{\mathcal{U}}^{\mathscr{F}}$, we take a C^1 chart $\phi_{x,g}\colon (-h,h)^2 \to \mathcal{W}^c_{g,\mathrm{loc}}(x)$ as in Lemma 4.3.5; as center leaves vary continuously with respect to g in the C^1 topology, the map $g \mapsto \phi_{x,g}$ depends continuously on g in the C^1 topology. After replacing H_g^t with $\phi_{x,g}^{-1} \circ H_g^t \circ \phi_{x,g}$, we can compare angles and norms of vectors for diffeomorphisms in a neighbourhood of f; by a slight abuse of notation, we will still denote this map by H_g^t for simplicity. By Corollary 4.3.3, and by compactness of $[0,1]$, we deduce that the family of holonomy maps $\{H_g^t\}_{(t,g)\in[0,1]\times\widetilde{\mathcal{U}}^{\mathscr{F}}}$ is uniformly C^1. In particular, for any $\delta > 0$, there exists a C^1 neighbourhood $\mathcal{U}_\delta$ of f such that for $\mathcal{U}_\delta^{\mathscr{F}} := \mathcal{U}_\delta \cap \mathscr{F}$, it holds

$$\sup_{(t,g)\in[0,1]\times\mathcal{U}_\delta^{\mathscr{F}}} \|DH_g^t - DH_f^t\| < \delta. \tag{4.6}$$

Therefore, for every $\theta > 0$, there exist $\delta_0 > 0$, $\rho_0 > 0$ such that for $g \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$ and for any $t \in [0,1]$, if $y \in \mathcal{W}^c_{g,\mathrm{loc}}(x)$ is such that $d(x,y) < \rho_0$ and if the vectors $v, w \in \mathbb{R}^2$ satisfy $\angle(v,w) > \theta$, then we have

$$\angle(D_x H_f^t(v), D_y H_g^t(w)) > \delta_0. \tag{4.7}$$

As invariant manifolds depend continuously on the diffeomorphism $g \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$, for any $\epsilon_0 > 0$, there exists $\rho(\epsilon_0) > 0$ such that for any $y \in B(x, \rho(\epsilon_0))$, for any $t \in [0,1]$ and for any $g \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$ (taking a smaller δ_0 if necessary), it holds

$$d(H_f^t(x), \xi(H_g^t(y))) < \epsilon_0. \tag{4.8}$$

Since the center accessibility class $C_f(x)$ is C^1, the map $C_f(x) \ni z \mapsto T_z C_f(x)$ is continuous, hence, if $\epsilon_0 > 0$ is chosen sufficiently small, then for any $t \in [0,1]$, $g \in \mathcal{U}_{\delta_0}^{\mathscr{F}}$, and $y \in B(x, \rho(\epsilon_0))$, we have

$$\angle(T_{H^t_f(x)} \mathrm{C}_f(x), T_{\xi(H^t_g)(y)} \mathrm{C}_f(x)) < \frac{\delta_0}{2}. \tag{4.9}$$

Now we argue as in Proposition 2.19 of [HS17]. For $\theta = \eta$ (recall (4.5)) we take $\delta_0 = \delta_0(\theta) > 0$ as in (4.7) and we set $\delta := \frac{\delta_0}{2} > 0$.

Since $g_n \to f$, we can take n large enough so that $g_n \in \mathcal{U}^{\mathscr{F}}_{\delta_0}$ and such that the arc $\{\alpha_{g_n,x}(t)\}_{t\in[0,1]}$ is a curve that crosses $\mathscr{N}_{\alpha_{f,x},\epsilon_0}$ from the left to the right side. Set $\beta := \alpha_{g_n,x}$. Note that if $t \in [0,1]$ is such that $\beta(t) \in \mathscr{N}_{\alpha_{f,x},\epsilon_0}$, then

$$\mathrm{Span}(\dot\beta(t)) = D_x H^t_{g_n}(T_x \mathrm{C}_{g_n}(x)). \tag{4.10}$$

Then, by (4.5), (4.7), (4.9), (4.10), we deduce that for any $t \in [0,1]$,

$$\begin{aligned}
&\angle(\dot\beta(t), T_{\xi(\beta(t))} \mathrm{C}_f(x)) = \angle(D_x H^t_{g_n}(T_x \mathrm{C}_{g_n}(x)), T_{\xi(H^t_{g_n}(x))} \mathrm{C}_f(x)) \\
&\geq \angle(D_x H^t_{g_n}(T_x \mathrm{C}_{g_n}(x)), T_{H^t_f(x)} \mathrm{C}_f(x)) - \angle(T_{H^t_f(x)} \mathrm{C}_f(x), T_{\xi(H^t_{g_n}(x))} \mathrm{C}_f(x)) \\
&= \angle(D_x H^t_{g_n}(T_x \mathrm{C}_{g_n}(x)), D_x H^t_f(T_x \mathrm{C}_f(x))) - \angle(T_{H^t_f(x)} \mathrm{C}_f(x), T_{\xi(H^t_{g_n}(x))} \mathrm{C}_f(x)) \\
&> \delta_0 - \frac{\delta_0}{2} = \delta,
\end{aligned}$$

which contradicts Lemma 4.3.6.

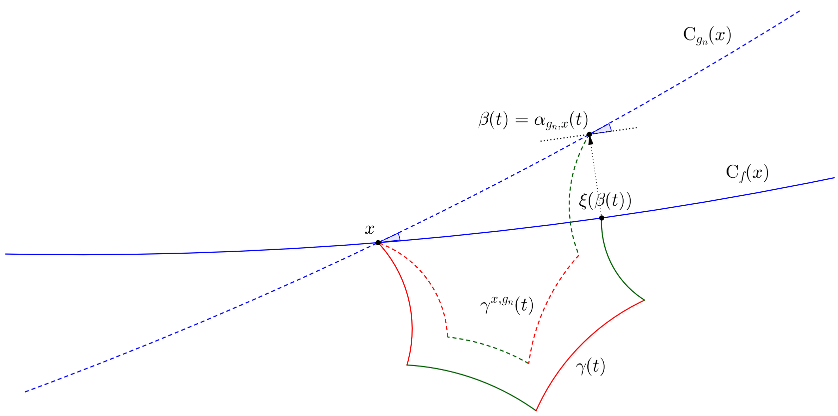


FIGURE 4.2: Tangent spaces to $\mathrm{C}_f(x)$, resp. $\mathrm{C}_{g_n}(x)$ at $\xi(\beta(t))$, resp. $\beta(t)$.

The proof of the second part of the proposition is similar. We know from the previous part that given $\theta > 0$ there is a C^1 neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}^{\mathscr{F}}_1(x)$, it holds $\angle(T_x \mathrm{C}_f(x), T_x \mathrm{C}_g(x)) < \theta$. Now we want to see the variation of the leaves at uniform (small) scale. Let us then suppose by contradiction that there are sequences $g_n \to f$ and $x_n \to x$ such that $x_n \in \mathrm{C}_{g_n}(x, \frac{1}{n}) \setminus \mathscr{C}_f(x, \theta, \frac{1}{n})$. By Lagrange Mean Value Theorem, this implies that there is $y_n \in \mathrm{C}_{g_n}(x, \frac{1}{n})$ such that $\angle(T_{y_n} \mathrm{C}_{g_n}(x), T_x \mathrm{C}_f(x)) > \theta$. Take $\delta_0 = \delta_0(\theta) > 0$, $\epsilon_0 > 0$ sufficiently small, and $n > 0$ sufficiently large such that $g_n \in \mathcal{U}^{\mathscr{F}}_{\delta_0}$, $y_n \in B(x, \rho(\epsilon_0))$, and such that the curve $\beta_1 \colon t \mapsto H^t_{g_n}(y_n)$ crosses $\mathscr{N}_{\alpha_{f,x},\epsilon_0}$ from the left to the right side. Now we argue as above, the only difference being that the role of the “big angle” is played by

$\angle(T_{y_n}\mathrm{C}_{g_n}(x), T_x\mathrm{C}_f(x))$ instead of $\angle(T_x\mathrm{C}_{g_n}(x), T_x\mathrm{C}_f(x))$: for any $t \in [0,1]$, it holds

$$\begin{aligned}
&\angle(\dot{\beta}_1(t), T_{\xi(\beta_1(t))}\mathrm{C}_f(x)) = \angle(D_{y_n}H^t_{g_n}(T_{y_n}\mathrm{C}_{g_n}(x)), T_{\xi(H^t_{g_n}(y_n))}\mathrm{C}_f(x)) \\
&> \angle(D_{y_n}H^t_{g_n}(T_{y_n}\mathrm{C}_{g_n}(x)), T_{H^t_f(x)}\mathrm{C}_f(x)) - \angle(T_{H^t_f(x)}\mathrm{C}_f(x), T_{\xi(H^t_{g_n}(y_n))}\mathrm{C}_f(x)) \\
&= \angle(D_{y_n}H^t_{g_n}(T_{y_n}\mathrm{C}_{g_n}(x)), D_xH^t_f(T_x\mathrm{C}_f(x))) - \angle(T_{H^t_f(x)}\mathrm{C}_f(x), T_{\xi(H^t_{g_n}(x))}\mathrm{C}_f(x)) \\
&> \delta_0 - \frac{\delta_0}{2} = \delta,
\end{aligned}$$

which again contradicts Lemma 4.3.6. This concludes the proof. □

As it will be used in the proof, let us recall the following result of [HS17]:

Proposition 4.3.7 (Corollary 2.21, [HS17]). *Let $\mathcal{C}$ be a center disk of f such that $\mathcal{C} \cap \Gamma^0_f = \emptyset$. Then the set $\Gamma^1_f(\mathcal{C})$ of points with 1-dimensional center accessibility classes in $\mathcal{C}$ admits a C^1 lamination whose leaves are the manifolds $\mathrm{C}_f(y) \cap \mathcal{C}$, $y \in \Gamma^1_f(\mathcal{C})$.*

4.4 Construction of adapted accessibility loops

Let $r \geq 2$, and let us consider a partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$ with $\dim E^c_f \geq 2$ that is center bunched, dynamically coherent, and plaque expansive. In this section, given a point $x \in M$, we build suitable loops starting at x which will later be used to construct perturbations to break non-open accessibility classes. The loops which we construct will depend on whether the accessibility class of the point x is already open or not. In fact, although the accessibility class of x is a homogeneous set, when working with specific families of loops with a prescribed number of legs of a certain size, the set of points which we can reach from x moving along these loops may not exhibit the global structure of the accessibility class (for example, if the class of x is open, to be able to reach any point in a neighborhood of x, we may need to consider very long accessibility paths instead of local ones), which leads us to the following definitions.

Fix a subset $\mathscr{S} \subset M$. For any $\sigma > 0$, we let $\tilde{\Gamma}^0_f(\mathscr{S}, \sigma)$ be the set of all points $x \in \mathscr{S}$ whose center accessibility class is *locally trivial* in the following sense: for any 4 us-loop $\gamma = [x, x_1, x_2, x_3, x_4]$ at (f, x) such that $\ell(\gamma) < 10^{-2}\sigma$, we have $x_4 = x$. We also set $\tilde{\Gamma}^0_f(\mathscr{S}) := \cup_{\sigma \in (0,1)} \tilde{\Gamma}^0_f(\mathscr{S}, \sigma)$. When $\mathscr{S} = M$, we abbreviate $\tilde{\Gamma}^0_f(\mathscr{S}, \sigma), \tilde{\Gamma}^0_f(\mathscr{S})$ respectively as $\tilde{\Gamma}^0_f(\sigma), \tilde{\Gamma}^0_f$.

The next lemma explains how to construct *closed* us-loops at points x whose center accessibility class is (locally) one-dimensional; it will be useful later to show that after a C^r-small perturbation, the accessibility class of x can be made open.

Lemma 4.4.1. *There exist C^2-uniform constants $\sigma_0 = \sigma_0(f) > 0$, $K_0 = K_0(f) > 0$ such that for any $\sigma \in (0, \sigma_0)$, for any point $x_0 \in \Gamma^1_f \setminus \tilde{\Gamma}^0_f(\sigma)$, if $\phi = \phi_{x_0}$ is the chart given by Lemma 4.3.5, then for any point $x \in \Gamma^1_f \cap \phi(B(0_{\mathbb{R}^d}, \frac{K_0}{10}\sigma))$, there exists a non-degenerate closed us-loop $\gamma_x = [x, x_1, \ldots, x_9, x]$ at (f, x) such that*

1. *$\ell(\gamma_x) < \sigma$;*
2. *$B(z_1, K_0\sigma) \cap \{z, z_2, \ldots, z_9\} = \emptyset$, where $z = \phi^{-1}(x)$, and $z_i = \phi^{-1}(x_i)$, for each integer $i = 2, \ldots, 9$;*
3. *the map $\Gamma_f \cap \phi(B(0_{\mathbb{R}^d}, \frac{K_0}{10}\sigma)) \ni x \mapsto \gamma_x$ is continuous.*

Proof. Fix some small $\sigma > 0$, let $x_0 \in \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$, and let $\sigma' \in (\frac{\sigma}{30}, \frac{\sigma}{20})$. By definition, there exists a non-degenerate 4 us-loop $\gamma = [x_0, x_1, x_2, x_3, x_4]$ such that

- $x_1 \in \mathcal{W}_f^u(x_0)$, with $\frac{\sigma'}{2} < d_{\mathcal{W}_f^u}(x_0, x_1) < \sigma'$;
- $x_2 \in \mathcal{W}_f^s(x_1)$, with $\frac{\sigma'}{2} < d_{\mathcal{W}_f^s}(x_1, x_2) < \sigma'$;
- $x_3 := H^u_{f,x_2,x_0}(x_2) \in \mathcal{W}_f^u(x_2, \sigma') \cap \mathcal{W}_f^{cs}(x_0, \sigma')$;
- $x_4 := H^s_{f,x_3,x_0}(x_3) \in \mathcal{W}_f^s(x_3, \sigma') \cap \mathcal{W}_f^c(x_0, \sigma')$, with $x_4 \in \mathrm{C}_f(x) \setminus \{x_0\}$.

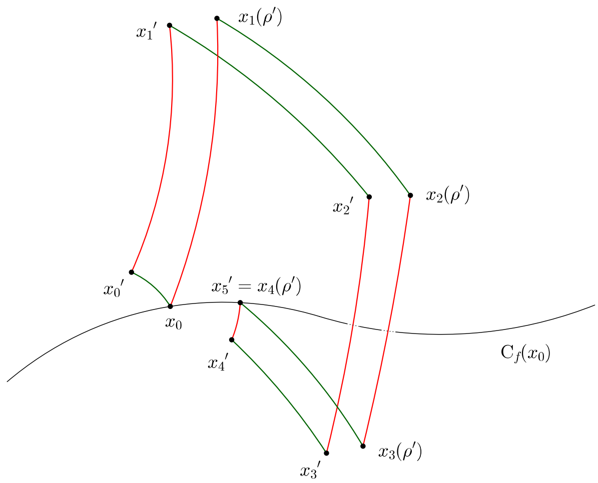


FIGURE 4.3: Construction of a non-degenerate closed us-loop.

As $\mathrm{C}_f(x_0)$ is one-dimensional, for the chart $\phi = \phi_{x_0}$ given by Lemma 4.3.5, we can assume that $\phi^{-1}(\mathrm{C}_f(x_0, \sigma)) = (-\rho_1, \rho_2) \times \{0_{\mathbb{R}^{d-1}}\} \simeq (-\rho_1, \rho_2)$, with $\rho_1, \rho_2 > 0$, $x_0 \simeq 0$, and $x_4 \simeq \rho \in (0, \rho_2)$. By varying the size of the legs, we can construct a continuous family $\{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_4(t)]\}_{t \in [\frac{\rho}{2}, \rho]}$ of non-degenerate 4 us-loops at (f, x_0) such that $x_4(t) \simeq t \in [\frac{\rho}{2}, \rho]$.

Let us take $x_0' \in \mathcal{W}_f^s(x_0, \frac{\sigma'}{10}) \setminus \mathcal{W}_f^s(x_0, \frac{\sigma'}{20})$ and $t_0 \in [\frac{\rho}{2}, \rho)$ close to ρ. As in Lemma 4.2.7, we let $\gamma^{x_0', f}(t_0) = [x_0', x_1', x_2', x_3', x_4']$ be the natural continuation of $\gamma(t_0)$ starting at x_0' in place of x_0. Since $\mathcal{W}_f^{cu}(x_4') = \mathcal{W}_f^{cu}(x_4) = \mathcal{W}_f^{cu}(x_0)$, we can also define $\{x_5'\} := H^u_{f,x_4',x_0}(x_4') \in \mathcal{W}^u_{f,\mathrm{loc}}(x_4') \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$, and we set $\gamma' := [x_0, x_0', x_1', \dots, x_5']$. In particular, $x_5' \in \mathrm{C}_f(x, \sigma)$, and $x_5' \simeq \rho'$ for some $\rho' \in (0, \rho)$. As $x_5' = x_4(\rho')$, we can concatenate the 4 us-loop $\gamma(\rho')$ at (f, x) with the 6 us-loop $\overline{\gamma'}$ at (f, x_5') to produce a closed 10 us-loop $\gamma_{x_0} := \overline{\gamma'}\gamma(\rho')$ at (f, x_0). By construction, γ_{x_0} is non-degenerate, and we have $\ell(\gamma_{x_0}) < \sigma$.

Let us check that $d(x_1', x_1(\rho')) > \frac{\sigma}{800}$ provided that σ is taken sufficiently small. By definition, we have $d_{\mathcal{W}_f^s}(x_0, x_0') \in [\frac{\sigma}{600}, \frac{\sigma}{200}]$. Since we work in a σ-neighbourhood of x_0, and as the map $z \mapsto E_f^u(z)$ is Hölder continuous (see [PSW12]), we deduce that the distance between the unstable bundles at any two points $z_1 \in \mathcal{W}_f^u(x_0', \sigma)$, $z_2 \in \mathcal{W}_f^u(x_0, \sigma)$ is at most $\tilde{c}_1 \sigma^\theta$, for two C^2-uniform constants $\theta = \theta(f) > 0$, $\tilde{c}_1 = \tilde{c}_1(f) > 0$.

Integrating the discrepancy along the unstable arcs from x_0' to x_1' and from x_0 to $x_1(\rho')$ yields

$$d(x_1', x_1(\rho')) \geq d(x_0', x_0) - \tilde{c}_2\sigma^\theta \times \sigma \geq \frac{\sigma}{600} - \tilde{c}_2\sigma^{1+\theta},$$

for some constant $\tilde{c}_2 > 0$. We conclude that $d(x_1', x_1(\rho')) \geq \frac{\sigma}{800}$ provided that σ is chosen sufficiently small, i.e., $\sigma \in (0, \sigma_0)$, for some C^2-uniform constant $\sigma_0 = \sigma_0(f) > 0$. Moreover, by construction, $B(x_1, \frac{\sigma}{100}) \cap \{x_0, x_2, x_3, x_4\} = \emptyset$. Similarly, we have $B(x_1', \frac{\sigma}{200}) \cap \{x_0, x_0', x_2', x_3', x_4', x_5'\} = \emptyset$ and $B(x_1(\rho'), \frac{\sigma}{200}) \cap \{x_0, x_2(\rho'), x_3(\rho'), x_4(\rho')\} = \emptyset$.

Let us now explain how this construction can be performed for points x near x_0 whose center accessibility class is also one-dimensional. By Lemma 4.2.7, for any point $x \in M$ which is sufficiently close to x_0, and for any $t \in [\frac{\rho}{2}, \rho]$, the us-loop $\gamma(t)$ admits a natural continuation $(\gamma(t))^{x,f} =: \check{\gamma}^x(t)$ that is a 4 us-loop at (f, x). Moreover, the map $t \mapsto \check{\gamma}^x(t)$ is continuous. Similarly, the 6 su-loop γ' has a natural continuation $(\gamma')^{x,f} = [x, (x_0')^{x,f}, \ldots, (x_5')^{x,f}]$. The point $(x_5')^{x,f}$ depends continuously on x, hence we can choose a continuous map $\rho'(\cdot)$ such that $\rho'(x_0) = \rho'$ and such that the endpoint of $\check{\gamma}^x(\rho'(x))$ coincides with the endpoint $(x_5')^{x,f}$ of $(\gamma')^{x,f}$. In particular, the continuations $(\gamma')^{x,f}$, $\check{\gamma}^x(\rho'(x))$ depend continuously on x. We conclude that the closed 10 us-loop $\gamma_x := \overline{(\gamma')^{x,f}}\check{\gamma}^x(\rho'(x))$ at (f, x) depends continuously on the point x in a small neighbourhood of x_0. In particular, for x sufficiently close to x_0, we have $\ell(\gamma_x) < \sigma$. □

Actually, given a small center disk $\mathcal{D}$, we will need to construct *closed* us-loops at points $x \in \mathcal{D}$ whose center accessibility class is not open, i.e., either zero or one-dimensional. Let us introduce some notation. Fix some small $\sigma > 0$. For any $x \in \Gamma_f = \Gamma_f^0 \cup \Gamma_f^1$, we let

- let $\overline{\Gamma}_f(x) := \tilde{\Gamma}_f^0(\sigma)$, and $n(x) := 2$, if $x \in \tilde{\Gamma}_f^0(\sigma)$;
- otherwise, let $\overline{\Gamma}_f(x) := \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$, and $n(x) := 5$, if $x \in \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$.

Lemma 4.4.2. *There exist C^2-uniform constant $\tilde{K} = \tilde{K}(f) \in (0,1)$, $\tilde{\sigma} = \tilde{\sigma}(f) > 0$ such that for any integer $k_0 \geq 1$,[2] for any $\sigma \in (0, \tilde{\sigma})$, and for any point $x_0 \in \Gamma_f$ there exists a continuous map $\overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}\sigma) \ni x \mapsto \gamma^x$ such that $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \ldots, x_{2n}^x(t)]\}_{t \in [0,1]}$ is a continuous family of $2n$ us-loops at (f, x), with $n := n(x_0) \in \{2, 5\}$, $\ell(\gamma^x) < \sigma$, such that $\gamma^x(0)$ is trivial, and for any integer $k \in \{1, \ldots, k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop.*

Proof. Let $k_0 \geq 1$ be some integer. Let $\sigma_0 = \sigma_0(f) > 0$, $K_0 = K_0(f) > 0$ be as in Lemma 4.4.1, let $\overline{h} = \overline{h}(f) > 0$ and $\phi = \phi_{x_0} \colon (-\overline{h}, \overline{h})^d \to M$ be given by Lemma 4.3.5, set $\tilde{\sigma} = \tilde{\sigma}(f) := \min(\overline{h}, \sigma_0) > 0$, and take some small $\sigma \in (0, \tilde{\sigma})$.

We consider a point $x_0 \in \Gamma_f$ and set $n := n(x_0) \in \{2, 5\}$. We distinguish between two cases.

(1) If $x_0 \in \tilde{\Gamma}_f^0(\sigma)$, then there exists a non-degenerate closed $2n$ us-loop $\tilde{\gamma} = [x_0, x_1, x_2, x_3, x_0]$ at (f, x_0) with $n = 2$, $\ell(\tilde{\gamma}) < \frac{\sigma}{2}$ and $B(z_1, K_0\sigma) \cap \{0_{\mathbb{R}^d}, z_2, z_3\} = \emptyset$, where $z_i := \phi^{-1}(x_i)$, for $i = 1, 2, 3$. By decreasing continuously the size of the legs of $\tilde{\gamma}$, we obtain a family of $2n$ us-loops $\{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_0]\}_{t \in [0,1]}$ at (f, x_0) such that $\gamma(0)$ is trivial and $\gamma(1) = \tilde{\gamma}$. Moreover, by choosing the map $t \mapsto \gamma(t)$ carefully, we can ensure that for any $k \in \{1, \ldots, k_0\}$, it holds

[2] We will apply this lemma with $k_0 = 1$ or 2 in the following.

$B(z_1(\frac{k}{k_0}), \frac{K_0}{2}\sigma) \cap \{0_{\mathbb{R}^d}, z_2(\frac{k}{k_0}), z_3(\frac{k}{k_0})\} = \emptyset$, where $z_i(\frac{k}{k_0}) := \phi^{-1}\big((x_i)(\frac{k}{k_0})\big)$, for $i = 1,2,3$, and $d\big(z_1(\frac{k}{k_0}), z_1(\frac{k'}{k_0})\big) \geq \frac{K_0}{2k_0}\sigma$,[3] for all $k' \in \{1,\dots,k_0\} \setminus \{k\}$.

For any point $x \in \tilde{\Gamma}_f^0(\sigma) \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$ with $d(0_{\mathbb{R}^d}, \phi^{-1}(x)) \leq \frac{K_0}{10}\sigma$, and for $t \in [0,1]$, let $\gamma^x(t) = [x, x_1^x(t), x_2^x(t), x_3^x(t), x]$ be the closed $2n$ us-loop whose corners are:

- $x_1^x(t) := H^u_{f,x,x_1(t)}(x) \in \mathcal{W}^u_{f,\mathrm{loc}}(x) \cap \mathcal{W}^{cs}_{f,\mathrm{loc}}(x_1(t))$;
- $x_2^x(t) := H^s_{f,x_1^x(t),x_2(t)}(x_1^x(t)) \in \mathcal{W}^s_{f,\mathrm{loc}}(x_1^x(t)) \cap \mathcal{W}^{cu}_{f,\mathrm{loc}}(x_2(t))$;
- $x_3^x(t) := H^u_{f,x_2^x(t),x}(x_2^x(t)) \in \mathcal{W}^u_{f,\mathrm{loc}}(x_2^x(t)) \cap \mathcal{W}^{cs}_{f,\mathrm{loc}}(x)$.

We let γ^x be the continuous family $\gamma^x := \{\gamma^x(t)\}_{t\in[0,1]}$. If σ is sufficiently small, then $\ell(\gamma^x) < \sigma$, and for any $k \in \{1,\dots,k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop at (f,x). Let $z_0^x := \phi^{-1}(x)$, and $z_i^x(\frac{k}{k_0}) := \phi^{-1}\big((x_i^x)(\frac{k}{k_0})\big)$, for $i = 1,2,3$. Arguing as in the proof of Lemma 4.4.1, we have $B(z_1^x(\frac{k}{k_0}), \frac{K_0}{5}\sigma) \cap \{z_0^x, z_2^x(\frac{k}{k_0}), z_3^x(\frac{k}{k_0})\} = \emptyset$, and $d\big(z_1^x(\frac{k}{k_0}), z_1^x(\frac{k'}{k_0})\big) \geq \frac{K_0}{5k_0}\sigma$, for all $k' \in \{1,\dots,k_0\} \setminus \{k\}$, provided that σ is sufficiently small.

(2) Otherwise, we have $x_0 \in \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$. By Lemma 4.4.1, after possibly taking K_0 smaller, then for any point $x \in \Gamma_f \cap \mathcal{W}^c_{f,\mathrm{loc}}(x_0)$ such that $d(0_{\mathbb{R}^d}, \phi^{-1}(x)) \leq \frac{K_0}{10}\sigma$, there exists a non-degenerate closed $2n$ us-loop $\gamma_x = [x, x_1, \dots, x_{2n-1}, x]$ at (f,x) with $n = 5$, $\ell(\gamma_x) < \frac{\sigma}{2}$, such that the map $\Gamma_f \cap \phi(B(0_{\mathbb{R}^d}, \frac{K_0}{10}\sigma)) \ni x \mapsto \gamma_x$ is continuous, and such that $B(z_1^x, K_0\sigma) \cap \{z_0^x, z_2^x, \dots, z_{2n-1}^x\} = \emptyset$, where $z_0^x := \phi^{-1}(x)$, and $z_i^x := \phi^{-1}\big(x_i^x\big)$, for each integer $i = 1,\dots,2n-1$.

By decreasing continuously the size of the legs of γ_x, keeping $x_{2n-1}(t) \in \mathcal{W}^{cs}_f(x)$ and letting $x_{2n}(t) := H^s_{f,x_{2n-1}(t),x}(x_{2n-1}(t))$, we obtain a continuous family $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \dots, x_{2n}^x(t)]\}_{t\in[0,1]}$ of $2n$ us-loops at (f,x) such that $\gamma^x(0)$ is trivial, $\gamma^x(1) = \gamma_x$, and $\ell(\gamma^x) < \sigma$.

Moreover, by choosing carefully the map $t \mapsto \gamma^x(t)$, we can ensure that for any integer $k \in \{1,\dots,k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop at (f,x). Indeed, as in the proof of Lemma 4.4.1, we consider a one-parameter family $(\check{\gamma}^x(t))_{t\in[0,1]}$ of 4 us-loops at (f,x) such that $\check{\gamma}^x(0)$ is the trivial loop and such that the first corners of $\check{\gamma}^x(t)$ and $\check{\gamma}^x(t')$ are distinct for $t \neq t' \in [0,1]$. We can also perform the same construction as in Lemma 4.4.1 in order to obtain a closed 10-us loop $\gamma^x(t)$ at the times $t = 1, \frac{k_0-1}{k_0}, \frac{k_0-2}{k_0}, \dots, \frac{1}{k_0}$, and such that $B(z_1^x(\frac{k}{k_0}), \frac{K_0}{5}\sigma) \cap \{z_0^x, z_2^x(\frac{k}{k_0}), \dots, z_{2n-1}^x(\frac{k}{k_0})\} = \emptyset$, where we let $z_i^x(\frac{k}{k_0}) := \phi^{-1}\big((x_i^x)(\frac{k}{k_0})\big)$, for $i = 1,\dots,2n-1$, and such that $d\big(z_1^x(\frac{k}{k_0}), z_1^x(\frac{k'}{k_0})\big) \geq \frac{K_0}{5k_0}\sigma$, for all $k' \in \{1,\dots,k_0\} \setminus \{k\}$. □

We will also need to construct certain us/su-paths for all points in a small center disk. Take $f \in \mathscr{F}$ and let $\sigma > 0$ be small. We assume that for some point $x_0 \in M$, and some constant $K > 0$, it holds $x \notin \tilde{\Gamma}_f^0(\sigma)$, for all $x \in \mathcal{W}^c_f(x_0, K\sigma)$. Fix $\theta > 0$ small. By Proposition 4.3.7 and Proposition 4.3.4, there exists a C^1 neighbourhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}^{\mathscr{F}}$ and for any $x \in \Gamma_g^1 \cap \mathcal{W}^c_f(x_0, K\sigma)$, it holds

$$\Pi^c_x C_g(x, 10\sigma) \subset \mathscr{C}_1 \tag{4.11}$$

[3] For instance, we choose the map $t \mapsto x_1(t) \in \mathcal{W}^u_{f,\mathrm{loc}}(x_0)$ in such a way that $d(z_1(t), z_1(t')) = d(0_{\mathbb{R}^d}, z_1) \cdot |t - t'|$, for all $t, t' \in [0,1]$.

where $\mathscr{C}_1 \subset \mathbb{R}^2$ is the cone of angle θ centered at $0_{\mathbb{R}^2}$, and $\Pi_x^c \colon M \to \mathbb{R}^2$ is the map in Lemma 4.3.5 for f. In the following, we let $\mathscr{C} := (\mathbb{R}^2 \setminus \mathscr{C}_1) \cup \{0_{\mathbb{R}^2}\}$, we denote by $\mathscr{C}_*^+, \mathscr{C}_*^-$ the two components of the set $\mathscr{C} \setminus \{0_{\mathbb{R}^2}\}$, and let $\mathscr{C}^+ := \mathscr{C}_*^+ \cup \{0_{\mathbb{R}^2}\}$, $\mathscr{C}^- := \mathscr{C}_*^- \cup \{0_{\mathbb{R}^2}\}$. Assume that $\mathscr{C}^+$, resp. $\mathscr{C}^-$ is the top, resp. bottom component in Figure 4.4.

Lemma 4.4.3. *Take f, x_0, σ, θ, $\mathcal{U}$ as above, and let $\mathscr{C}$, $\mathscr{C}^+$, and $\mathscr{C}^-$ as defined above. After possibly taking K smaller, there exist continuous maps $\mathcal{W}_f^c(x_0, K\sigma) \ni x \mapsto \gamma_1^x, \gamma_2^x$ such that for any $x \in \mathcal{W}_f^c(x_0, K\sigma)$, $\gamma_1^x = [x, \alpha_1^x, \dots, \omega_1^x]$, resp. $\gamma_2^x = [x, \alpha_2^x, \dots, \omega_2^x]$, is a non-degenerate closed 10 us-loop, resp. 10 su-loop at (f, x) such that $\ell(\gamma_1^x), \ell(\gamma_2^x) < \sigma$, such that the endpoints $\omega_1^x = H_{\gamma_1^x}(x)$, $\omega_2^x = H_{\gamma_2^x}(x)$ satisfy*

$$\left(\Pi_x^c \omega_1^x, \Pi_x^c \omega_2^x\right) \in \left(\mathscr{C}^+ \times \mathscr{C}^-\right) \cup \left(\mathscr{C}^- \times \mathscr{C}^+\right),$$

and such that for $\star = 1, 2$, for some C^2-uniform constant $\hat{K}_\star > 0$, we have

$$B(\alpha_\star^x, \hat{K}_\star \sigma) \cap \{z\} = \emptyset, \text{ for any corner } z \neq \alpha_\star^x \text{ of } \gamma_\star^x. \tag{4.12}$$

Proof. As $x_0 \in \Gamma_f^1 \setminus \tilde{\Gamma}_f^0(\sigma)$, there exists a non-degenerate 4 us-loop $\gamma = [x_0, x_1, x_2, x_3, x_4]$ with $\ell(\gamma) < \sigma$ and $x_4 \in C_f(x_0) \setminus \{x_0\}$. By shrinking the size of the legs, we construct a continuous family $\{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_4(t)]\}_{t \in [0,1]}$ of non-degenerate 4 us-loops at (f, x_0) such that $\gamma(0)$ is trivial and $\gamma(1) = \gamma$.

Assuming that $K > 0$ is sufficiently small, the family $\{\gamma(t)\}_{t \in [0,1]}$ extends to a continuous map $\mathcal{W}_f^c(x_0, K\sigma) \ni x \mapsto \gamma^x = \{\gamma^x(t)\}_{t \in [0,1]}$ such that for each $x \in \mathcal{W}_f^c(x_0, K\sigma)$, and for each $t \in [0,1]$, $\gamma^x(t) = [x, x_1^x(t), x_2^x(t), x_3^x(t), x_4^x(t)]$ is a 4 us-loop at (f, x), and $\gamma^x(0)$ is trivial. Moreover, up to reparametrization, there exists $\vartheta > 0$ such that for each $x \in \mathcal{W}_f^c(x_0, K\sigma)$, it holds

$$\begin{cases} \left\{\Pi_x^c\left(x_4^x(t)\right)\right\}_{t \in [\frac{1}{4}, \frac{1}{3}]} \subset B\left(\Pi_x^c\left(x_4\left(\frac{1}{3}\right)\right), \frac{1}{2}\vartheta\right) \subset B\left(0_{\mathbb{R}^2}, 3\vartheta\right), \\ \left\{\Pi_x^c\left(x_4^x(t)\right)\right\}_{t \in [\frac{1}{2}, \frac{2}{3}]} \subset B\left(\Pi_x^c\left(x_4\left(\frac{2}{3}\right)\right), \frac{1}{2}\vartheta\right) \subset \mathscr{C}_1^r \cap \left(B\left(0_{\mathbb{R}^2}, 7\vartheta\right) \setminus B\left(0_{\mathbb{R}^2}, 4\vartheta\right)\right), \\ \left\{\Pi_x^c\left(x_4^x(t)\right)\right\}_{t \in [\frac{3}{4}, 1]} \subset B\left(\Pi_x^c\left(x_4(1)\right), \frac{1}{2}\vartheta\right) \subset \mathscr{C}_1^r \cap \left(B\left(0_{\mathbb{R}^2}, 10\vartheta\right) \setminus B\left(0_{\mathbb{R}^2}, 8\vartheta\right)\right), \end{cases} \tag{4.13}$$

denoting by $\mathscr{C}_1^r$ the connected component of $\mathscr{C}_1 \setminus \{0_{\mathbb{R}^2}\}$ containing $\Pi_{x_0}^c(x_4)$. Moreover, after possibly changing the parametrization by t, we can also assume that for all $x \in \mathcal{W}_f^c(x_0, K\sigma)$, we have

$$d_{\mathcal{W}_f^u}(x_1^x(t), x) \geq \frac{1}{200}\sigma, \quad \text{for all } t \in \left[\frac{1}{4}, 1\right]. \tag{4.14}$$

Now, as in Lemma 4.4.1, we take $x_0' \in \mathcal{W}_f^s(x_0)$ such that $\frac{1}{200}\sigma \leq d_{\mathcal{W}_f^s}(x_0, x_0') \leq \frac{1}{100}\sigma$. For any $t \in [0,1]$, let $\tilde{\gamma}(t) = [y_0(t), \dots, y_4(t)]$ be the natural continuation of $\gamma(t)$ starting at $y_0(t) = x_0'$ in place of x_0. As $\mathcal{W}_f^{cu}(y_4(t)) = \mathcal{W}_f^{cu}(x_4) = \mathcal{W}_f^{cu}(x_0)$, we may also define $\{y_5(t)\} := \mathcal{W}_{f,\mathrm{loc}}^u(y_4(t)) \cap \mathcal{W}_{f,\mathrm{loc}}^c(x_0)$, and set $\gamma_*(t) := [x_0, y_0(t), \dots, y_4(t), y_5(t)]$. In the same way, for each point $x \in \mathcal{W}_f^c(x_0, K\sigma)$, we let $\gamma_*^x(t) = [x, y_0^x(t), \dots, y_5^x(t)]$ be the continuation of $\gamma_*(t)$ starting at x given by Lemma 4.2.7.

For each $(x, t) \in \mathcal{W}_f^c(x_0, K\sigma) \times [0,1]$, we denote by $\check{\gamma}^x(t)$ the continuation of $\overline{\gamma_*^x(\frac{2}{3})}$ starting at $x_4^x(t)$ as in Lemma 4.2.7, and by concatenation, we obtain the 10 us-loop $\gamma_1^x(t) := \gamma^x(t)\check{\gamma}^x(t) = [x, \alpha_1^x(t), \dots, \omega_1^x(t)]$. If σ is sufficiently small, x_0' is very close to

x_0, and by (4.13), for any $x \in \mathcal{W}_f^c(x_0, K\sigma)$, it holds

$$\Pi_x^c\left(\omega_1^x\left(\frac{1}{4}\right)\right) \notin \mathscr{C}_1^r, \qquad \Pi_x^c(\omega_1^x(1)) \in \mathscr{C}_1^r.$$

Since the set $\{\omega_1^x(t)\}_{t\in[0,1]}$ of endpoints is connected, its image under Π_x^c has to cross the cone $\mathscr{C} = \mathscr{C}^+ \cup \mathscr{C}^-$. We then let $t^x \in [0,1]$ be the smallest $t \in [0,1]$ such that $\Pi_x^c(\omega_1^x(t)) \in \mathscr{C}_1^r$; we also denote by $\gamma_1^x = [x, \alpha_1^x, \ldots, \omega_1^x]$ the 10 us-loop $\gamma_1^x(t^x) = \gamma^x(t^x)\check{\gamma}^x(t^x)$, with $\alpha_1^x := \alpha_1^x(t^x)$ and $\omega_1^x := \omega_1^x(t^x)$. In particular, we have $\Pi_x^c(\omega_1^x) \in \mathscr{C}$; without loss of generality, we assume that $\Pi_x^c(\omega_1^x) \in \mathscr{C}^+$.

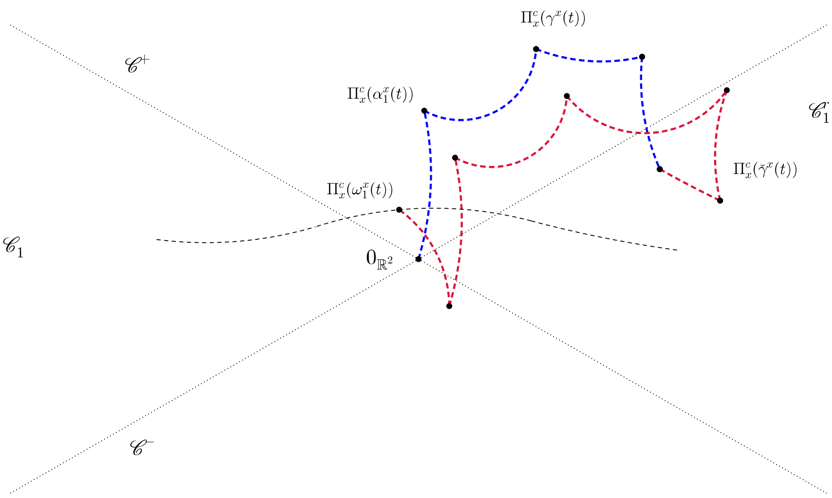


FIGURE 4.4: Construction of the loop γ_1^x.

For each $s \in [0,1]$, we also denote by $\gamma_2^x(s) = [x, \alpha_2^x(s), \ldots, \omega_2^x(s)]$ the 10 su-loop obtained by taking the continuation of $\overline{\gamma_1^x(s)}$ starting at x in place of $\omega_1^x(s)$. In this case, arguing as above, we see that for certain values $s \in [0,1]$, it holds $\Pi_x^c(\omega_2^x(s)) \in \mathscr{C}^-$; we then let $s^x \in [0,1]$ be the largest $s \in [0,1]$ with that property, and we define the 10 su-loop $\gamma_2^x := \gamma_2^x(s^x)$, with $\gamma_2^x = [x, \alpha_2^x, \ldots, \omega_2^x]$, and $\Pi_x^c(\omega_2^x) \in \mathscr{C}^-$.

Besides, (4.12) follows from arguments similar to those in Lemma 4.4.1, using (4.14), and since x_0' was chosen such that $d_{\mathcal{W}_f^s}(x_0, x_0') \geq \frac{1}{200}\sigma$. □

4.5 A submersion from the space of perturbations to the phase space

As above, we consider a partially hyperbolic diffeomorphism $f \in \mathcal{PH}^r(M)$, $r \geq 2$, with $\dim E_f^c \geq 2$ that is center bunched, dynamically coherent, and plaque expansive. In Subsection 4.5.1, we recall some general results from [LZ22] about random perturbations and the changes those perturbations induce on certain holonomy maps. In Subsection 4.5.2, we construct a family of perturbations and show how the results of the previous part can be applied to the particular setting we are interested in.

4.5.1 Random perturbations

As in [LZ22], we will use the following suspension construction to show that certain holonomy maps are differentiable with respect to the perturbation parameter. The idea is to incorporate the perturbation parameter into a higher dimensional partially hyperbolic diffeomorphism, which, under some assumptions, is still dynamically coherent and center bunched.

Definition 4.5.1 (C^r deformation). *Let $I \geq 1$ be some integer, and let $\mathcal{U}$ be an open neighbourhood of $\{0\}$ in $\mathbb{R}^I$. A C^r map $\hat{f} \colon \mathcal{U} \times M \to M$ satisfying $\hat{f}(0,\cdot) = f$ and $\hat{f}(b,\cdot) \in \mathcal{PH}^r(M)$ for all $b \in \mathcal{U}$ is called a C^r* deformation at f with I-parameters. *We associate with $\hat{f}$ the suspension map $T(\hat{f})$ defined by*

$$T = T(\hat{f}) \colon \mathcal{U} \times M \to \mathcal{U} \times M, \quad (b,x) \mapsto (b, \hat{f}(b,x)), \tag{4.15}$$

and we denote $f_b := \hat{f}(b,\cdot)$. If in addition $f_b \in \mathcal{PH}^r(M, \mathrm{Vol})$ for all $b \in \mathcal{U}$, then $\hat{f}$ is said to be volume preserving.

Definition 4.5.2 (Infinitesimal C^r deformation). *Let $I \geq 1$ be an integer. A C^r map $V \colon \mathbb{R}^I \times M \to TM$ is called an* infinitesimal C^r deformation with I-parameters *if*

1. *for each $B \in \mathbb{R}^I$, $V(B,\cdot)$ is a C^r vector field on M;*
2. *for each $x \in M$, $B \mapsto V(B,x)$ is a linear map from $\mathbb{R}^I$ to T_xM.*

Remark 4.5.3. *Given $I \geq 1$, an infinitesimal C^r deformation V with I-parameters, and some small $\epsilon > 0$, we associate with V a C^r deformation at f with I-parameters, denoted by $\hat{f}$, which is defined by*

$$\hat{f}(b,x) := \mathcal{F}_{V(b,\cdot)}(1, f(x)), \quad \forall\, (b,x) \in \mathcal{U} \times M,$$

where $\mathcal{U} = B(0,\epsilon) \subset \mathbb{R}^I$ and for any $B \in \mathbb{R}^I$, $\mathcal{F}_{V(B,\cdot)} \colon \mathbb{R} \times M \to M$ denotes the C^r flow generated by the vector field $V(B,\cdot)$. In this case, we say that $\hat{f}$ is generated by V. *If in addition $V(B,\cdot)$ is divergence-free for each $B \in \mathbb{R}^I$, then $\hat{f}$ is volume preserving as in Definition 4.5.1, and we say that V is* volume preserving.

Lemma 4.5.4 (Lemma 4.11 in [LZ22]). *Let $I \geq 1$ be some integer, let $\mathcal{U} \subset \mathbb{R}^I$ be an open neighbourhood of $\{0\}$, and let $\hat{f} \colon \mathcal{U} \times M \to M$ be a C^r deformation at f with I-parameters. If $\mathcal{U}$ is chosen sufficiently small, then the map $T = T(\hat{f})$ is a C^r dynamically coherent partially hyperbolic system for some T-invariant splitting*

$$T_{(b,x)}(\mathcal{U} \times M) \simeq T_b\mathcal{U} \oplus T_xM = E_T^s(b,x) \oplus E_T^c(b,x) \oplus E_T^u(b,x),$$

for all $(b,x) \in \mathcal{U} \times M$. Moreover, for any $(b,x) \in \mathcal{U} \times M$, we have

$$E_T^*(b,x) = \{0\} \oplus E_{f_b}^*(x), \qquad \mathcal{W}_T^*(b,x) = \{b\} \times \mathcal{W}_{f_b}^*(x), \qquad \text{for } * = u,s,$$

and

$$E_T^c(b,x) = Graph(v_b(x,\cdot)) \oplus E_{f_b}^c(x), \tag{4.16}$$

for a unique linear map $v_b(x,\cdot) \colon T_b\mathcal{U} \to E_{f_b}^{su}(x) := E_{f_b}^s(x) \oplus E_{f_b}^u(x)$.

If in addition f is center bunched, then, after reducing the size of $\mathcal{U}$, u/s-holonomy maps between local center leaves of T (within distance 1) are C^1 when restricted to some cu/cs-leaf, with uniformly continuous, uniformly bounded derivatives.

Let $I \geq 1$ be some integer, let $\mathcal{U} \subset \mathbb{R}^I$ be some small neighbourhood of $\{0\}$ in $\mathbb{R}^I$, let $\hat{f} \colon \mathcal{U} \times M \to M$ be a C^1 deformation at f with I-parameters, and let $T = T(\hat{f})$.

Definition 4.5.5 (Lift of a us/su-loop). *For any point $x \in M$, for any integer $n \geq 2$, and for any $2n$ us/su-loop $\gamma = [x, x_1, \ldots, x_{2n}]$ at (f,x), we define the* lift *of γ as*

$$\hat{\gamma} := [(0,x), (0,x_1), \ldots, (0,x_{2n})].$$

In particular, by Lemma 4.5.4, $\hat{\gamma}$ is a $2n$ us/su-loop at $(T,(0,x))$.

Remark 4.5.6. *In the following, we will mostly consider us-loops; for that reason, we will state the technical lemmas needed for the proof only for us-loops, but similar results hold for su-loops as well.*

Similarly to Lemma 4.2.7, given a point $x \in M$ and a us-loop at (f,x), we can define a natural continuation for the C^1 deformation $\hat{f}$ with I-parameters we consider:

Definition 4.5.7. *Let $x \in M$, let $n \geq 2$, we say that $\gamma = \{\gamma(t) = [x, x_1(t), \dots, x_{2n}(t)]\}_{t\in[0,1]}$ is a continuous family of $2n$ us-loops at (f,x) if for each $t \in [0,1]$, $\gamma(t)$ is a $2n$ us-loop, and for each $i = 1, \dots, 2n$, the map $t \mapsto x_i(t)$ is continuous. Given such a family, for any $t \in [0,1]$, we let $\hat{\gamma}(t)$ be the lift of $\gamma(t)$ as above. Then by continuity, there exists a C^2-uniform constant $\hat{\delta} = \hat{\delta}(T,\gamma) > 0$ such that $B(0,\hat{\delta}) \subset \mathcal{U}$, and for any $(b,y,t) \in \mathcal{W}^c_T((0,x),\hat{\delta}) \times [0,1]$, for some constant $\hat{h} = \hat{h}(T,\gamma) > 0$, the following intersections exist and are unique:*

- $\{(b, \hat{x}_1^{b,y}(t))\} := \mathcal{W}^u_{T,\mathrm{loc}}((b,y),\hat{h}) \cap \mathcal{W}^{cs}_{T,\mathrm{loc}}((0,x_1(t)),\hat{h})$;
- $\{(b, \hat{x}_2^{b,y}(t))\} := \mathcal{W}^s_{T,\mathrm{loc}}((b,\hat{x}_1^{b,y}(t)),\hat{h}) \cap \mathcal{W}^{cu}_{T,\mathrm{loc}}((0,x_2(t)),\hat{h}) \dots$
- $\dots \{(b, \hat{x}_{2n-1}^{b,y}(t))\} := \mathcal{W}^u_{T,\mathrm{loc}}((b,\hat{x}_{2n-2}^{b,y}(t)),\hat{h}) \cap \mathcal{W}^{cs}_{T,\mathrm{loc}}((0,x),\hat{h})$;
- $\{(b, \hat{x}_{2n}^{b,y}(t))\} := \mathcal{W}^s_{T,\mathrm{loc}}((b,\hat{x}_{2n-1}^{b,y}(t)),\hat{h}) \cap \mathcal{W}^{c}_{T,\mathrm{loc}}((0,x),\hat{h})$.

We thus have a continuous family of $2n$ us-loops at (f_b,y), denoted by $\{\hat{\gamma}^{b,y}(t)\}_{t\in[0,1]}$:

$$\hat{\gamma}^{b,y}(t) := [y, \hat{x}_1^{b,y}(t), \dots, \hat{x}_{2n}^{b,y}(t)], \quad \forall t \in [0,1].$$

We define the map

$$\widehat{\psi} = \widehat{\psi}(T,x,\gamma) \colon \begin{cases} \mathcal{W}^c_T((0,x),\hat{\delta}) \times [0,1] & \to \quad \mathcal{W}^c_T(0,x), \\ (b,y,t) & \mapsto \quad H_{T,\hat{\gamma}(t)}(b,y) = (b, \hat{x}_{2n}^{b,y}(t)). \end{cases} \tag{4.17}$$

For any $(b,y) \in \mathcal{W}^c_T((0,x),\hat{\delta})$, we thus get a map $\psi = \psi(T,x,\gamma)$:

$$\psi(b,y,\cdot) := \pi_M \widehat{\psi}(b,y,\cdot) \colon [0,1] \to \mathcal{W}^c_{f_b}(y), \tag{4.18}$$

where $\pi_M \colon \mathcal{U} \times M \to M$ denotes the canonical projection.

Definition 4.5.8. *Let $I \geq 1$ be some integer. For any infinitesimal C^r deformation with I-parameters $V \colon \mathbb{R}^I \times M \to TM$, we define*

$$\mathrm{supp}(V) := \{x \in M \mid \exists B \in \mathbb{R}^I \text{ such that } V(B,x) \neq 0\}.$$

Given an open neighbourhood $\mathcal{U}$ of $\{0\}$ in $\mathbb{R}^I$, and a C^r deformation at f with I-parameters $\hat{f} \colon \mathcal{U} \times M \to M$, we define

$$\mathrm{supp}(\hat{f}) := \{x \in M \mid \exists b \in \mathcal{U} \text{ such that } \hat{f}(b,x) \neq f(x)\}.$$

We introduce the following definitions in order to control return times of a map to the support of a deformation; they are motivated by the fact that for very large return times, it is possible to achieve a good control on how certain holonomies change after perturbation.

Definition 4.5.9. *For any subsets $A, B \subset M$, and for $* \in \{+,-\}$, we define*

$$R(f,A,B) := \inf\{n \geq 0 \mid f^n(A) \cap B \neq \emptyset \text{ or } f^{-n}(A) \cap B \neq \emptyset\};$$
$$R_*(f,A,B) := \inf\{n \geq 1 \mid f^{*n}(A) \cap B \neq \emptyset\}.$$

We abbreviate $R(f,A,A)$, $R_*(f,A,A)$ respectively as $R(f,A)$, $R_*(f,A)$. Similarly, for a C^1 deformation $\hat{f}\colon \mathcal{U} \times M \to M$ of f, and for $* \in \{+,-\}$, we set

$$R(\hat{f},A,B) := \inf\{n \geq 0 \mid \exists b \in \mathcal{U} \text{ s.t. } \hat{f}(b,\cdot)^n(A) \cap B \neq \emptyset \text{ or } \hat{f}(b,\cdot)^{-n}(A) \cap B \neq \emptyset\},$$
$$R_*(\hat{f},A,B) := \inf\{n \geq 1 \mid \exists b \in \mathcal{U} \text{ s.t. } \hat{f}(b,\cdot)^{*n}(A) \cap B \neq \emptyset\},$$

and we abbreviate $R(\hat{f},A,A)$, $R_*(\hat{f},A,A)$ respectively as $R(\hat{f},A)$, $R_*(\hat{f},A)$.

In the following, most of the time[4], we restrict ourselves to the case of deformations with 2-parameters, i.e., we take a small neighbourhood $\mathcal{U} \subset \mathbb{R}^2$ of $\{0_{\mathbb{R}^2}\}$, we let $\hat{f}\colon \mathcal{U} \times M \to M$ be a C^1 deformation at f with 2-parameters generated by an infinitesimal C^1 deformation with 2-parameters $V\colon \mathbb{R}^2 \times M \to TM$, and we set $T = T(\hat{f})$.

Definition 4.5.10 (Adapted deformation)**.** *Let $x \in M$, let $n \geq 2$ be some integer, and let $\gamma = [x, x_1, \ldots, x_{2n}]$ be a $2n$ us-loop or su-loop at (f,x) with $\ell(\gamma) < \sigma$ for some small $\sigma > 0$. Given two constants $C, R_0 > 0$, we say that an infinitesimal C^r deformation V is* adapted to *(γ, σ, C, R_0) if*

1. *$\sigma\|\partial_b\partial_x V\|_M + \|\partial_b V\|_M < C$;*
2. *$R(f,\{z\},\text{supp}(V)) > R_0$ for $z = x, x_2, \ldots, x_{2n}$;*
3. *$R_\pm(f,\{x_1\},\text{supp}(V)) > R_0$.*

Proposition 4.5.11 (see Proposition 5.6, [LZ22])**.** *For any $C, \kappa > 0$, there exist C^2-uniform constants $R_0 = R_0(f,C,\kappa) > 0$ and $\kappa_0 = \kappa_0(f,C,\kappa) > 0$ such that the following is true.*

Let $x \in M$, let $n \geq 2$ be some integer, and let $\gamma = [x, x_1, \ldots, x_{2n}]$ be a $2n$ us-loop at (f,x) of length $\sigma > 0$ such that there exists an infinitesimal C^r deformation V that is adapted to (γ, σ, C, R_0). In the following, we denote by $B = (B_1, B_2)$ an element of $T_0\mathcal{U} \simeq \mathbb{R}^2$. Assume that for all $z \in \{x, x_2, \ldots, x_{2n}\}$, we have

$$D_B(\pi_c V(B,z)) = 0, \tag{4.19}$$

while

$$\left|\det\left(B \mapsto D_B(\pi_c V(B,x_1))\right)\right| > \kappa, \tag{4.20}$$

where $\pi_c\colon TM \to E^c_f$ denotes the canonical projection.

Then, the map

$$\Xi\colon \begin{cases} T_0\mathcal{U} & \to & E^c_f(x_{2n}), \\ B & \mapsto & \hat{\pi}_c DH_{T,\hat{\gamma}}(B + \nu_0(x,B)), \end{cases}$$

satisfies

$$\det \Xi \geq \kappa_0,$$

where $\hat{\gamma}$ is the lift of γ for T, and $\hat{\pi}_c\colon E^c_T(0,x_{2n}) = Graph(\nu_0(x_{2n},\cdot)) \oplus E^c_f(x_{2n}) \to E^c_f(x_{2n})$ denotes the canonical projection.

[4]Except in Subsection 4.7.2 where deformations with 4-parameters are needed.

4.5.2 Construction of C^r deformations at f

In the following, we assume that $\dim E^c_f = 2$. Recall that $\Pi^c\colon \mathbb{R}^d \simeq \mathbb{R}^2 \times \mathbb{R}^{d_u+d_s} \to \mathbb{R}^2$ is the canonical projection, and that Π^c_x is the map $\Pi^c_x := \Pi^c \circ \phi_x^{-1}\colon M \to \mathbb{R}^2$.

Lemma 4.5.12. *Let $\tilde{K} = \tilde{K}(f) \in (0,1)$, $\tilde{\sigma} = \tilde{\sigma}(f) > 0$ be as in Lemma 4.4.2. Then, for any $R_0 > 0$, for any integer $k_0 \geq 1$, for any $\sigma \in (0,\tilde{\sigma})$, and for any point $x_0 \in \Gamma_f$ satisfying $R_\pm(f, B(x_0, 10\sigma)) > R_0$,[5] there exists an infinitesimal C^r deformation at f with $2k_0$-parameters $V\colon \mathbb{R}^{2k_0} \times M \to TM$ such that $\operatorname{supp}(V) \subset B(x_0, 10\sigma)$,[5] and there exists a continuous map $\overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}\sigma) \ni x \mapsto \gamma^x$ such that $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \dots, x_{2n}^x(t)]\}_{t\in[0,1]}$ is a continuous family of $2n$ us-loops at (f,x), with $n := n(x_0) \in \{2,5\}$, $\ell(\gamma^x) < \sigma$, such that $\gamma^x(0)$ is trivial, and for any integer $k \in \{1,\dots,k_0\}$, we have:*

1. *$\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop;*
2. *V is adapted to $(\gamma^x(\frac{k}{k_0}), \sigma, \widetilde{C}, R_0)$, for some C^2-uniform constant $\widetilde{C} = \widetilde{C}(f, k_0) > 0$;*
3. *for any $z \in \{x, x_2^x(\frac{k}{k_0}), \dots, x_{2n-1}^x(\frac{k}{k_0})\}$, it holds*

$$D_B(\pi_c V(B,z)) = 0,$$

and there exists a 2-dimensional vector space $E_k \subset \mathbb{R}^{2k_0}$ such that

$$\left|\det\left(E_k \ni B \mapsto D_B\left(\pi_c V\left(B, x_1^x\left(\frac{k}{k_0}\right)\right)\right)\right)\right| > \tilde{\kappa},$$

for some C^2-uniform constant $\tilde{\kappa} = \tilde{\kappa}(f) > 0$, where $\pi_c\colon TM \to E^c_f$ denotes the canonical projection.

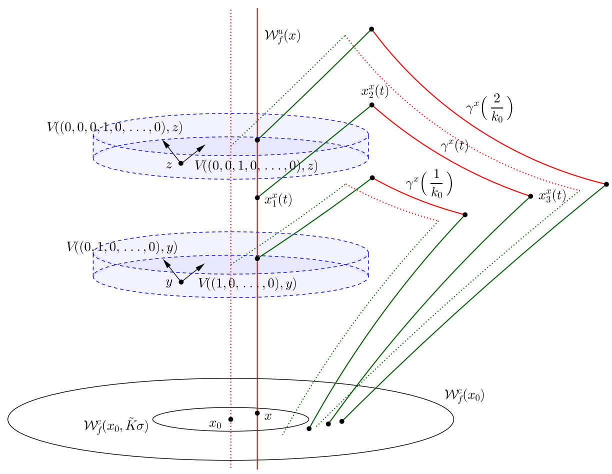


FIGURE 4.5: Localization of the perturbations.

Actually, we will apply Lemma 4.5.12 with $k_0 = 1$ or 2 in the following. The construction in the proof of Lemma 4.5.12 is adapted from [LZ22].

[5] Recall Definition 4.5.8 and Definition 4.5.9.

Proof of Lemma 4.5.12. Let $R_0 > 0$, and let $k_0 \geq 1$ be some integer. Let $\tilde{K} = \tilde{K}(f) \in (0,1)$, $\tilde{\sigma} = \tilde{\sigma}(f) > 0$ be as in Lemma 4.4.2, and take some small $\sigma \in (0, \tilde{\sigma})$.

We consider a point $x_0 \in \Gamma_f$ such that $R_\pm(f, B(x_0, 10\sigma)) > R_0$, and set $n := n(x_0) \in \{2,5\}$. We let $\bar{h} = \bar{h}(f) > 0$ and $\phi = \phi_{x_0} \colon (-\bar{h}, \bar{h})^d \to M$ be given by Lemma 4.3.5.

For each $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}^c_f(x_0, \tilde{K}\sigma)$ we let $\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \ldots, x_{2n}^x(t)]\}_{t\in[0,1]}$ be the continuous family of $2n$ us-loops at (f,x) constructed in Lemma 4.4.2.

For each integer $k \in \{1, \ldots, k_0\}$, we let $z_k = (z_k^c, z_k^u, z_k^s) := z_1^{x_0}(\frac{k}{k_0}) \in (-\bar{h}, \bar{h})^d = (-\bar{h}, \bar{h})^2 \times (-\bar{h}, \bar{h})^{d_u} \times (-\bar{h}, \bar{h})^{d_s}$. We define a collection of functions and vector fields as follows.

For each $1 \leq j \leq 2$, let $U_j \colon (-\frac{1}{5}, \frac{1}{5})^2 \times (-\frac{1}{5}, \frac{1}{5})^{d_u} \times (-\frac{1}{3}, \frac{1}{3})^{d_s} \to \mathbb{R}^d$ be a compactly supported C^∞ divergence-free vector field such that U_j restricted to $(-\frac{1}{10}, \frac{1}{10})^2 \times (-\frac{1}{10}, \frac{1}{10})^{d_u} \times (-\frac{1}{5}, \frac{1}{5})^{d_s}$ is equal to the constant vector e_j, where $e_1 := (1, 0, 0, \ldots, 0) \in \mathbb{R}^d$ and $e_2 := (0, 1, 0, \ldots, 0) \in \mathbb{R}^d$. Moreover, we can assume that U_j satisfies $\|U_j\|_{C^1} < C_*$ for some constant $C_* = C_*(d) > 0$.

For any $x_c \in \mathbb{R}^2$, $x_u \in \mathbb{R}^{d_u}$, $x_s \in \mathbb{R}^{d_s}$, for any $\lambda_c, \lambda_u, \lambda_s > 0$, and for any $z_c \in \mathbb{R}^2, z_u \in \mathbb{R}^{d_u}, z_s \in \mathbb{R}^{d_s}$, we set:

$$\Lambda^{\lambda_c, \lambda_u, \lambda_s}_{x_c, x_u, x_s}(z_c, z_u, z_s) = (x_c + \lambda_c z_c, x_u + \lambda_u z_u, x_s + \lambda_s z_s).$$

For any $j \in \{1,2\}$, any $x_u \in \mathbb{R}^{d_u}$, we let $U^\sigma_{j,x_u} \colon (-\bar{h}, \bar{h})^d \to \mathbb{R}^d$ be the vector field

$$U^\sigma_{j,x_u} = U_j\big(\Lambda^{\tilde{K}\sigma, \frac{\tilde{K}}{k_0}\sigma, \tilde{K}\sigma}_{0_{\mathbb{R}^2}, x_u, 0_{\mathbb{R}^{d_s}}}\big)^{-1}.$$

The support of U^σ_{j,x_u} is contained in

$$\Big(-\frac{\tilde{K}}{5}\sigma, \frac{\tilde{K}}{5}\sigma\Big)^2 \times \Big(x_u + \big(-\frac{\tilde{K}}{5k_0}\sigma, \frac{\tilde{K}}{5k_0}\sigma\big)^{d_u}\Big) \times \Big(-\frac{\tilde{K}}{3}\sigma, \frac{\tilde{K}}{3}\sigma\Big)^{d_s}.$$

Moreover, for any $z_c \in (-\frac{\tilde{K}}{10}\sigma, \frac{\tilde{K}}{10}\sigma)^2$, for any $z_u \in x_u + (-\frac{\tilde{K}}{10k_0}\sigma, \frac{\tilde{K}}{10k_0}\sigma)^{d_u}$ and for any $z_s \in \big(-\frac{\tilde{K}}{5}\sigma, \frac{\tilde{K}}{5}\sigma\big)^{d_s}$, it holds

$$U^\sigma_{j,x_u}(z_c, z_u, z_s) = e_j.$$

We set

$$V^\sigma_{j,x_u} := D\phi(U^\sigma_{j,x_u}).$$

The vector field V^σ_{j,x_u} is divergence-free and satisfies:

$$\sigma\|\partial_x V^\sigma_{j,x_u}\|_M + \|V^\sigma_{j,x_u}\|_M < \widetilde{C}_0, \tag{4.21}$$

for some C^2-uniform constant $\widetilde{C}_0 = \widetilde{C}_0(f, k_0) > 0$.

Let $V \colon \mathbb{R}^{2k_0} \times M \to TM$ be the infinitesimal C^r deformation defined as

$$\begin{aligned} V(B, \cdot) := \big(B_{1,1}V^\sigma_{1,z_1^u} + B_{2,1}V^\sigma_{2,z_1^u}\big) + \big(B_{1,2}V^\sigma_{1,z_2^u} + B_{2,2}V^\sigma_{2,z_2^u}\big) + \cdots + \\ + \big(B_{1,k_0-1}V^\sigma_{1,z^u_{k_0-1}} + B_{2,k_0-1}V^\sigma_{2,z^u_{k_0-1}}\big) + \big(B_{1,k_0}V^\sigma_{1,z^u_{k_0}} + B_{2,k_0}V^\sigma_{2,z^u_{k_0}}\big), \end{aligned} \tag{4.22}$$

for all $B = \sum_{k=1}^{k_0} B_{1,k}u_{2k-1} + B_{2,k}u_{2k} \in \mathbb{R}^{2k_0}$, where $(u_i)_{i=1}^{2k_0}$ denotes the canonical basis of $\mathbb{R}^{2k_0}$.

By definition, the map V is linear in B. Moreover, by (4.21), (4.22), it holds

$$\sigma\|\partial_b\partial_x V\|_M + \|\partial_b V\|_M < \widetilde{C}, \tag{4.23}$$

with $\widetilde{C} := 2k_0\widetilde{C}_0 > 0$.

As $\ell(\gamma^x) < \sigma$, we have $\mathrm{supp}(V) \subset B(x_0, 10\sigma)$, and for any integer $k \in \{1, \dots, k_0\}$, it holds $x_2^x(\frac{k}{k_0}), x_3^x(\frac{k}{k_0}), \dots, x_{2n-1}^x(\frac{k}{k_0}) \in B(x_0, 10\sigma)$, for all $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}\sigma)$. Recall that by assumption, we have

$$R_\pm(f, B(x_0, 10\sigma)) > R_0. \tag{4.24}$$

By (4.23) and (4.24), we conclude that V is adapted to $(\gamma^x(\frac{k}{k_0}), \sigma, \widetilde{C}, R_0)$.

By construction, for any $z \in \{x, x_2^x(\frac{k}{k_0}), \dots, x_{2n-1}^x(\frac{k}{k_0})\}$, it holds

$$D_B(\pi_c V(B, z)) = 0,$$

and

$$\left|\det\left(E_k \ni B \mapsto D_B\big(\pi_c V\big(B, x_1^x(\frac{k}{k_0})\big)\big)\right)\right| > \tilde{\kappa},$$

for some C^2-uniform constant $\tilde{\kappa} = \tilde{\kappa}(f) > 0$, where $E_k := \mathrm{Span}(u_{2k-1}, u_{2k}) \subset \mathbb{R}^{2k_0}$, and $\pi_c\colon TM \to E_f^c$ denotes the canonical projection. □

Corollary 4.5.13. *For any integers $k_0 \geq 1$, $r \geq 2$, for any $\delta > 0$, there exist C^2-uniform constants $\tilde{K}_0 = \tilde{K}_0(f) \in (0,1)$, $\tilde{\sigma}_0 = \tilde{\sigma}_0(f, k_0) > 0$, $\tilde{R}_0 = \tilde{R}_0(f, k_0) > 0$ and $\tilde{\delta}_0 = \tilde{\delta}_0(f, r, \delta) > 0$ such that for any $\sigma \in (0, \tilde{\sigma}_0)$, for any point $x_0 \in \Gamma_f$ satisfying $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_0$, there exists a C^r deformation $\hat{f}\colon B(0_{\mathbb{R}^{2k_0}}, \tilde{\delta}_0) \times M \to M$ at f with $2k_0$-parameters generated by an infinitesimal C^r deformation $V\colon \mathbb{R}^{2k_0} \times M \to TM$, such that $\mathrm{supp}(\hat{f}) \subset B(x_0, 10\sigma)$,[6] and there exists a continuous map $\overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}_0\sigma) \ni x \mapsto \gamma^x$, such that*

1. *$\gamma^x = \{\gamma^x(t) = [x, x_1^x(t), \dots, x_{2n}^x(t)]\}_{t \in [0,1]}$ is a continuous family of $2n$ us-loops at (f, x) as in Lemma 4.5.12, with $n := n(x_0) \in \{2, 5\}$, $\ell(\gamma^x) < \sigma$, such that $\gamma^x(0)$ is trivial, and for any integer $k \in \{1, \dots, k_0\}$, $\gamma^x(\frac{k}{k_0})$ is a non-degenerate closed us-loop;*

2. *let $T = T(\hat{f})$, let $\psi_x := \psi(T, x, \gamma^x)$ be the map defined in (4.18), let $\Pi_x^c\colon M \to \mathbb{R}^2$ be the map given by Lemma 4.3.5, and for $k \in \{1, \dots, k_0\}$, let $\Phi^{(k)}\colon (b, x) \mapsto \Pi_x^c \psi_x(b, x, \frac{k}{k_0})$; then, the map*

$$\Phi\colon \begin{cases} B(0_{\mathbb{R}^{2k_0}}, \tilde{\delta}_0) \times \left(\overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}_0\sigma)\right) & \to \mathbb{R}^{2k_0} \\ (b, x) & \mapsto \left(\Phi^{(k)}(b, x)\right)_{k=1,\dots,k_0} \end{cases}$$

is continuous; besides, for any $x \in \overline{\Gamma}_f(x_0) \cap \mathcal{W}_f^c(x_0, \tilde{K}_0\sigma)$, $\Phi(\cdot, x)$ is C^1, and

$$|\det D_b|_{b=0}(\Phi(\cdot, x))| > \tilde{\kappa}_0,$$

for some C^2-uniform constant $\tilde{\kappa}_0 = \tilde{\kappa}_0(f, k_0) > 0$;

3. *$d_{C^r}(f, f_b) < \delta$, for all $b \in B(0_{\mathbb{R}^{2k_0}}, \tilde{\delta}_0)$, where $f_b := \hat{f}(b, \cdot) \in \mathcal{PH}^r(M)$.*

Proof. Fix two integers $k_0 \geq 1$ and $r \geq 2$. Let $\tilde{K}_0 := \tilde{K}(f) > 0$, $\tilde{\sigma} := \tilde{\sigma}(f) > 0$, $\widetilde{C} = \widetilde{C}(f, k_0) > 0$ and $\tilde{\kappa} = \tilde{\kappa}(f) > 0$ be the constants given by Lemma 4.5.12, and let $\tilde{R}_0 := R_0(f, \widetilde{C}, \tilde{\kappa}) > 0$, $\kappa_0 := \kappa_0(f, \widetilde{C}, \tilde{\kappa}) > 0$ be the constants given by Proposition 4.5.11. Given $\sigma \in (0, \tilde{\sigma})$, we consider a point $x_0 \in \Gamma_f$ such that $R_\pm(f, B(x_0, 10\sigma), B(x_0, 10\sigma)) > \tilde{R}_0$, and set $n := n(x_0) \in \{2, 5\}$.

[6] Recall Definition 4.5.8.

We let $V\colon \mathbb{R}^{2k_0}\times M\to TM$ be the infinitesimal C^r deformation at f with $2k_0$-parameters given by Lemma 4.5.12. Take a point $x\in\overline{\Gamma}_f(x_0)\cap\mathcal{W}_f^c(x_0,\tilde{K}_0\sigma)$ and let $\gamma^x=\{\gamma^x(t)=[x,x_1^x(t),\dots,x_{2n}^x(t)]\}_{t\in[0,1]}$ be the continuous family of $2n$ us-loops at (f,x) given by Lemma 4.5.12. Recall that the map $x\mapsto\gamma^x$ is continuous. Let $\hat{f}$ be the C^r deformation at f with 2-parameters generated by V, and let $T=T(\hat{f})$. By the properties of V in Lemma 4.5.12, we have $\mathrm{supp}(\hat{f})\subset B(x_0,10\sigma)$.

For any $t\in[0,1]$, let $\hat{\gamma}^x(t)$ be the lift of $\gamma^x(t)$ for T, and let us denote by $\hat{\pi}_c\colon E_T^c(0,x)=Graph(v_0(x,\cdot))\oplus E_f^c(x)\to E_f^c(x)$ the canonical projection. Fix an integer $k\in\{1,\dots,k_0\}$. We let $\Xi_x^{(k)}$ be the map defined as

$$\Xi_x^{(k)}\colon\left\{\begin{array}{rcl}\mathbb{R}^2\simeq E_k & \to & E_f^c(x),\\ B & \mapsto & \hat{\pi}_c DH_{T,\hat{\gamma}^x(\frac{k}{k_0})}(B+v_0(x,B)).\end{array}\right.$$

By points (2)-(3) of Lemma 4.5.12, and by Proposition 4.5.11, it holds

$$|\det\Xi_x^{(k)}|\geq\kappa_0. \tag{4.25}$$

Let $\widehat{\psi}_x:=\widehat{\psi}(T,x,\gamma^x)$ and $\psi_x:=\pi_M\widehat{\psi}_x$ be the maps defined in (4.17)-(4.18), and let $\Pi_x^c\colon M\to\mathbb{R}^2$ be the map given by Lemma 4.3.5. Let $\hat{\delta}>0$ be such that $\hat{\delta}<\hat{\delta}(T,\gamma^x)$ for all $x\in\overline{\Gamma}_f(x_0)\cap\mathcal{W}_f^c(x_0,\tilde{K}_0\sigma)$, with $\hat{\delta}(T,\gamma^x)>0$ as in Definition 4.5.7. Let

$$\Phi^{(k)}\colon\left\{\begin{array}{rcl}\mathcal{W}_T^c((0,x_0),\hat{\delta}) & \to & \mathbb{R}^2,\\ (b,x) & \mapsto & \Pi_x^c\psi_x(b,x,\frac{k}{k_0}).\end{array}\right.$$

As $x\mapsto\gamma^x$ is continuous, the maps $x\mapsto\psi_x$ and $\Phi^{(k)}$ are continuous as well.

For each $x\in\overline{\Gamma}_f(x_0)\cap\mathcal{W}_f^c(x_0,\tilde{K}_0\sigma)$, and for each $B\in\mathbb{R}^2\simeq E_k$, we have

$$\begin{aligned}D\Phi_x^{(k)}(0,B+v_0(x,B))&=D\Pi_x^c\pi_M DH_{T,\hat{\gamma}^x(\frac{k}{k_0})}(B+v_0(x,B))\\&=D\Pi_x^c\Big[\hat{\pi}_c DH_{T,\hat{\gamma}^x(\frac{k}{k_0})}(B+v_0(x,B))+v_0(x,B)\Big],\end{aligned}\tag{4.26}$$

where $\Phi_x^{(k)}:=\Phi^{(k)}(\cdot,x)$, and $\pi_M\colon\mathbb{R}^2\times M\to M$ denotes the projection onto the second coordinate.

By Lemma 4.3.5 there exists a constant $D>0$ such that for $\zeta>0$ small, if $\sigma\in(0,\overline{h}_\zeta(f))$, then for any $x\in\overline{\Gamma}_f(x_0)\cap\mathcal{W}_f^c(x_0,\tilde{K}_0\sigma)$ and for any $B\in\mathbb{R}^2$, it holds

$$\|D\Pi_x^c v_0(x,B)\|\leq D\zeta\|B\|. \tag{4.27}$$

If $\zeta>0$ is sufficiently small (depending only on κ_0), then for any $\sigma\in(0,\tilde{\sigma}_0)$, with $\tilde{\sigma}_0:=\min(\tilde{\sigma},\overline{h}_\zeta(f))>0$, and for any $x\in\overline{\Gamma}_f(x_0)\cap\mathcal{W}_f^c(x_0,\tilde{K}_0\sigma)$, by (4.25)-(4.26)-(4.27), we deduce that

$$\left|\det D_b|_{b=0}(\Phi_x^{(k)}|_{E_k})\right|>\frac{1}{2}\kappa_0,$$

which concludes the proof of point (2), for $\tilde{\kappa}_0:=\left(\frac{1}{2}\kappa_0\right)^{k_0}>0$.

Finally, point (3) is a direct observation. □

4.6 Local accessibility

Let us fix an integer $r \geq 2$, and let us consider $f \in \mathscr{F}$, where as before, $\mathscr{F} \subset \mathcal{PH}^r_*(M)$ is the set of C^r dynamically coherent, plaque expansive, partially hyperbolic diffeomorphisms with two-dimensional center, which satisfy some strong bunching condition as in Definition 4.2.5.

In this section, we show that it is possible to make the accessibility class of any non-periodic point open by a C^r-small perturbation. First, we explain how to break trivial accessibility classes, and then, we show how to open one-dimensional accessibility classes, based on some transversality arguments.

4.6.1 Breaking trivial accessibility classes

Proposition 4.6.1. *For any non-periodic point $x_0 \in M$, for any $\delta > 0$, and for any $\sigma > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that $x_0 \notin \tilde{\Gamma}^0_g(\sigma)$; in particular, the center accessibility class $\mathrm{C}_g(x_0)$ is at least one-dimensional.*

Proof. Take a non-periodic point $x_0 \in M$. Fix some small $\delta > 0$, let $k_0 := 1$, and let $\tilde{\sigma}_0 = \tilde{\sigma}_0(f,1) > 0$, $\tilde{R}_0 = \tilde{R}_0(f,1) > 0$ and $\tilde{\delta}_0 = \tilde{\delta}_0(f,r,\delta) > 0$ be the constants given by Corollary 4.5.13. As x_0 is non-periodic, then for $\sigma \in (0,\tilde{\sigma}_0)$ sufficiently small, it holds $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_0$. Assume that $x_0 \in \tilde{\Gamma}^0_f(\sigma)$ (otherwise there is nothing to prove).

By Corollary 4.5.13, for $n := n(x_0) = 2$, there exist a continuous family

$$\gamma = \gamma^{x_0} = \{\gamma(t) = [x_0, x_1(t), x_2(t), x_3(t), x_0]\}_{t \in [0,1]} \tag{4.28}$$

of 4 us-loops at (f, x_0) such that $\ell(\gamma) < \sigma$, $\gamma(0)$ is trivial, $\gamma(1)$ is a non-degenerate closed 4 us-loop, a C^r deformation $\hat{f} \colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \times M \to M$ at f with 2-parameters, so that $\mathrm{supp}(\hat{f}) \subset B(x_0, 10\sigma)$, and such that the map

$$\Phi_{x_0} \colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \ni b \mapsto \Pi^c_x \psi(b, x_0, 1) \tag{4.29}$$

is C^1 and satisfies

$$|\det D_b|_{b=0} \Phi_{x_0}| > \tilde{\kappa}_0, \tag{4.30}$$

for some C^2-uniform constant $\tilde{\kappa}_0 = \tilde{\kappa}_0(f,1) > 0$. Recall that in (4.29), $\Pi^c_x \colon M \to \mathbb{R}^2$ is the map defined in Lemma 4.3.5, $T = T(\hat{f})$, and $\psi = \psi(T, x_0, \gamma^{x_0})$.

Moreover, by Definition 4.5.7, for all $b \in B(0_{\mathbb{R}^2}, \tilde{\delta}_0)$ and all $t \in [0,1]$, we have $\psi(b, x_0, t) \in \mathcal{W}^c_{f_b}(x_0) \cap \mathrm{Acc}_{f_b}(x_0) = \mathrm{C}_{f_b}(x_0)$, where $f_b := \hat{f}(b, \cdot)$. Besides, (4.30) implies that the map $\psi(\cdot, x_0, 1)$ is a submersion in a neighbourhood of $0_{\mathbb{R}^2}$, hence

$$\psi(\cdot, x_0, 1)^{-1}\{x_0\} \cap B(0_{\mathbb{R}^2}, \delta_1) = \{0_{\mathbb{R}_2}\},$$

for some sufficiently small $\delta_1 \in (0, \tilde{\delta}_0)$. Fix $b \in B(0_{\mathbb{R}^2}, \delta_1) \setminus \{0_{\mathbb{R}^2}\}$ such that $g := f_b \in \mathscr{F}$; we have $\psi(b, x_0, 1) \in \mathrm{C}_g(x_0) \setminus \{x_0\}$, and $[0,1] \ni t \mapsto \psi(b, x_0, t) \in \mathrm{C}_g(x_0)$ is a non-trivial g-path connecting x_0 to $\psi(b, x_0, 1) \neq x_0$ within $\mathrm{C}_g(x_0)$. In particular, $x_0 \notin \tilde{\Gamma}^0_g(\sigma)$; in fact, by Theorem 4.2.10, $\mathrm{C}_g(x_0)$ is at least one-dimensional. Moreover, by point (3) of Corollary 4.5.13, we have $d_{C^r}(f,g) < \delta$, which concludes the proof. □

4.6.2 Opening one-dimensional accessibility classes

The following result shows that local accessibility can be achieved near non-periodic points after a C^r-small perturbation.

Proposition 4.6.2. *For any non-periodic point $x_0 \in M$, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that the accessibility class $\mathrm{Acc}_g(x_0)$ is open.*

Proof. Let us consider a non-periodic point $x_0 \in M$. Let $k_0 := 1$, and let $\tilde{\sigma}_0 = \tilde{\sigma}_0(f,1) > 0$, $\tilde{R}_0 = \tilde{R}_0(f,1) > 0$ be the constants given by Corollary 4.5.13. As x_0 is non-periodic, we can fix $\sigma \in (0, \tilde{\sigma}_0)$ such that $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_0$.

Assume by contradiction that $\mathrm{Acc}_f(x_0)$ is stably not open in the C^r topology. In other words, by Theorem 4.2.10, for some $\delta_1 > 0$, and for every diffeomorphism $g \in \mathcal{PH}_*^r(M)$ such that $d_{C^r}(f,g) < \delta_1$, we have $x_0 \in \Gamma_g$. Fix some small $\delta \in (0, \delta_1)$. By Proposition 4.6.1, there exists a C^r diffeomorphism $f_0 \in \mathscr{F}$ with $d_{C^r}(f, f_0) < \frac{\delta}{2}$ such that $x_0 \notin \tilde{\Gamma}^0_{f_0}(\frac{\sigma}{2})$. In particular, by our choice of δ, we have $x_0 \in \Gamma^1_{f_0} \setminus \tilde{\Gamma}^0_{f_0}(\frac{\sigma}{2})$. Besides, there exists $\delta_2 \in (0, \frac{\delta}{2})$ such that for any diffeomorphism g satisfying $d_{C^r}(f_0, g) < \delta_2$, we have $g \in \mathscr{F}$, and $x_0 \in \Gamma^1_g \setminus \tilde{\Gamma}^0_g(\sigma)$. In particular, with the notations in Section 4.3, $\mathcal{U}^{\mathscr{F}} = \mathcal{U}_1^{\mathscr{F}}(x_0)$, where $\mathcal{U}$ is a δ_2-neighbourhood of f_0 in the C^r topology.

By Corollary 4.5.13, for $n := n(x_0) = 5$, there exist a continuous family

$$\gamma = \gamma^{x_0} = \{\gamma(t) = [x_0, x_1(t), x_2(t), \dots, x_{10}(t)]\}_{t \in [0,1]} \tag{4.31}$$

of 10 us-loops at (f_0, x_0) such that $\ell(\gamma) < \sigma$, $\gamma(0)$ is trivial, $\gamma(1)$ is a non-degenerate closed 10 us-loop, a C^r deformation $\hat{f}\colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \times M \to M$ at f_0 with 2-parameters, with $\tilde{\delta}_0 = \tilde{\delta}_0(f_0, r, \delta) > 0$, so that for the map $\Pi^c_{x_0}\colon M \to \mathbb{R}^2$ defined in Lemma 4.3.5, $T = T(\hat{f})$, and $\psi = \psi(T, x_0, \gamma^{x_0})$, the map $\Phi_{x_0}\colon B(0_{\mathbb{R}^2}, \tilde{\delta}_0) \ni b \mapsto \Pi^c_{x_0}\psi(b, x_0, 1)$ is C^1, and for some constant $\tilde{\kappa}_0 = \tilde{\kappa}_0(f,1) > 0$, it holds

$$|\det D_b|_{b=0}\Phi_{x_0}| > \tilde{\kappa}_0. \tag{4.32}$$

Fix some small $\theta > 0$. It follows from the previous discussion and Proposition 4.3.4 that for $\delta_0 \in (0, \tilde{\delta}_0)$, $\varepsilon_0 > 0$ sufficiently small, then for all $b \in B(0_{\mathbb{R}^2}, \delta_0)$, the diffeomorphism $f_b := \hat{f}(b, \cdot)$ satisfies $f_b \in \mathcal{U}_1^{\mathscr{F}}(x_0)$, and

$$\mathrm{C}_{f_b}(x_0, \varepsilon_0) \subset \mathscr{C}_{f_0}(x_0, \theta, \varepsilon_0), \tag{4.33}$$

where $\mathscr{C}_{f_0}(x_0, \theta, \varepsilon_0)$ is as in (4.4). Let us set

$$\mathscr{C}(x_0, \theta) := \Pi^c_{x_0}\big(\mathscr{C}_{f_0}(x_0, \theta, \varepsilon_0)\big).$$

By Definition 4.5.7, and since $\psi(0, x_0, 1) = x_0$,[7] for $\tilde{\delta}_1 \in (0, \delta_0)$ sufficiently small, we have $\psi(b, x_0, 1) \in \mathrm{C}_{f_b}(x_0, \varepsilon_0)$, for all $b \in B(0_{\mathbb{R}^2}, \tilde{\delta}_1)$, and by (4.33), we deduce that

$$\Phi_{x_0}\big(B(0_{\mathbb{R}^2}, \tilde{\delta}_1)\big) \subset \Pi^c_x\big(\mathrm{C}_{f_b}(x_0, \varepsilon_0)\big) \subset \mathscr{C}(x_0, \theta).$$

On the one hand, by the definition of the cone $\mathscr{C}(x_0, \theta)$, we have $\mathbb{R}^2 \setminus \Phi_{x_0}\big(B(0_{\mathbb{R}^2}, \tilde{\delta}_1)\big) \supset \Delta_0$ for some straight line Δ_0 through the origin $0_{\mathbb{R}^2}$. But on the other hand, it follows from (4.32) that $\Phi_{x_0}\big(B(0_{\mathbb{R}^2}, \tilde{\delta}_1)\big)$ contains an open neighbourhood of $0_{\mathbb{R}^2}$, a contradiction. By Theorem 4.2.10, we conclude that for some $b \in B(0_{\mathbb{R}^2}, \tilde{\delta}_0)$, $\mathrm{Acc}_{f_b}(x_0)$ is open; moreover, by construction, $g := f_b \in \mathscr{F}$ satisfies

$$d_{C^r}(f,g) \leq d_{C^r}(f, f_0) + d_{C^r}(f_0, f_b) < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

[7] Recall that $\gamma(1)$ is a closed 10 us-loop at (f_0, x_0).

which concludes the proof. □

4.7 C^r-density of accessibility

In this section, we conclude the proof of our main results stated in Section 4.1. As above, we fix an integer $r \geq 2$, and let $f \in \mathscr{F}$, where $\mathscr{F} \subset \mathcal{PH}^r_*(M)$ is the set of C^r dynamically coherent, plaque expansive, partially hyperbolic diffeomorphisms with two-dimensional center, which satisfy some strong bunching condition as in Definition 4.2.5. Our goal is to conclude the proof of our main result (Theorem E):

Proposition 4.7.1. *For any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ with $d_{C^r}(f, g) < \delta$ such that g is stably accessible.*

4.7.1 Spanning c-families

For the proof of Proposition 4.7.1, we combine ideas from the last section with some global argument; this is done by means of spanning families of center-disks; this notion was already present in the work of Dolgopyat-Wilkinson [DW03] and is also used in [LZ22].

Definition 4.7.2 (c-disk)**.** *For each $x \in M$ and $\sigma > 0$, $\mathcal{C} = \mathcal{W}^c_f(x, \sigma)$ is called the* center disk *of f (or c-disk of f for short) centered at x with radius σ. We set $\varrho(\mathcal{C}) := \sigma$, and for any $\theta \in (0, 1]$, we also define $\theta\mathcal{C} := \mathcal{W}^c_f(x, \theta\sigma)$.*

Definition 4.7.3. *A collection of disjoint center disks $\mathcal{D} = \{\mathcal{C}_1, \ldots, \mathcal{C}_J\}$ is called a* family of center disks for *f (or c-*family for *f for short). In addition, we set*

$$\underline{r}(\mathcal{D}) := \inf_{\mathcal{C} \in \mathcal{D}} \{\varrho(\mathcal{C})\}, \quad \overline{r}(\mathcal{D}) := \sup_{\mathcal{C} \in \mathcal{D}} \{\varrho(\mathcal{C})\}.$$

*Given $\theta \in (0, 1)$ and $k \geq 1$, we say that $\mathcal{D}$ is a (θ, k)-*spanning *c-family for f if*

$$M = \bigcup_{\mathcal{C} \in \mathcal{D}} \bigcup_{x \in \theta\mathcal{C}} Acc_f(x, k),$$

where $Acc_f(x, k)$ denotes the set of all points $y \in M$ which can be connected to x by a f-accessibility sequence with at most k legs of length less than one.

Given any subset $\mathcal{C} \subset M$, and $\sigma \geq 0$, we set $(\mathcal{C}, \sigma) := \{x \in M \,|\, d(x, \mathcal{C}) \leq \sigma\}$. Given $\sigma \geq 0$ and a c-family $\mathcal{D} = \{\mathcal{C}_1, \ldots, \mathcal{C}_J\}$ for f, we set

$$(\mathcal{D}, \sigma) := \bigcup_{j=1}^{J} (\mathcal{C}_j, \sigma).$$

*We say that $\mathcal{D}$ is σ-*sparse *if for any two distinct $\mathcal{C}, \mathcal{C}' \in \mathcal{D}$, $(\mathcal{C}, \sigma), (\mathcal{C}', \sigma)$ are disjoint. Any c-family for f is σ-sparse for some $\sigma > 0$.*

Proposition 4.7.4 (Corollary 6.2, [LZ22])**.** *Assume that $f \in \mathcal{PH}^1(M)$ is dynamically coherent, plaque expansive, and that the fixed points of f^k are isolated for all $k \geq 1$. Then for every $\overline{R} > 1$, there exist C^1-uniform constants $\overline{N} = \overline{N}(f, \overline{R}) > 0$, $\overline{\rho} = \overline{\rho}(f, \overline{R}) \in (0, \overline{R}^{-1})$ and $\overline{\sigma} = \overline{\sigma}(f, \overline{R}) > 0$ such that the following is true. For all diffeomorphism g sufficiently C^1-close to f, there exists a $(\frac{1}{40}, 4)$-spanning c-family $\mathcal{D}_g$ for g with at most $\overline{N}$ elements such that*

1. $\overline{\rho} < \underline{r}(\mathcal{D}_g) \leq \overline{r}(\mathcal{D}_g) < \overline{R}^{-1}$;
2. $\mathcal{D}_g$ *is* $\overline{\sigma}$*-sparse;*
3. $R_\pm(g,(\mathcal{D}_g,\overline{\sigma})) > \overline{R}$.

Moreover, the map $g \mapsto \mathcal{D}_g$ *can be chosen to be continuous.*

4.7.2 Density of diffeomorphisms with no trivial accessibility class (proof of Theorem F)

The following result strengthens Proposition 4.6.1.

Proposition 4.7.5. *There exist* C^2*-uniform constants* $\tilde{\sigma}_1 = \tilde{\sigma}_1(f) > 0$, $\tilde{K}_1 = \tilde{K}_1(f) \in (0,1)$ *and* $\tilde{R}_1 = \tilde{R}_1(f) > 0$ *such that for any* $\delta > 0$, *for any* $\sigma \in (0,\tilde{\sigma}_1)$, *for any point* $x_0 \in M$ *satisfying* $R_\pm(f,B(x_0,10\sigma)) > \tilde{R}_1$, *there exists a partially hyperbolic diffeomorphism* $g \in \mathscr{F}$ *such that* $d_{C^r}(f,g) < \delta$ *and such that for some* $\delta' = \delta'(x_0,g) > 0$, *we have* $x \notin \tilde{\Gamma}_h^0(\sigma)$, *for all* $x \in \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$ *and for all* $h \in \mathscr{F}$ *with* $d_{C^1}(g,h) < \delta'$. *In particular, the center accessibility class* $\mathrm{C}_h(x)$ *of each point* $x \in \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$ *is at least one-dimensional.*

Remark 4.7.6. *In order to deal with all the points in a given center disk, the idea is to increase the codimension of "bad" configurations; this is done by considering two 4 us-loops at each point in the center disk, and show that we can construct a perturbation in such a way that for each of those points, at least one of the endpoints of the 4 us-loops is not the original point.*

Proof. Fix some small $\delta > 0$, let $k_0 := 2$, and let $\tilde{K}_1 := \tilde{K}_0(f) \in (0,1)$, $\tilde{\sigma}_1 := \tilde{\sigma}_0(f,2) > 0$, $\tilde{R}_1 := \tilde{R}_0(f,2) > 0$ and $\tilde{\delta}_1 := \tilde{\delta}_0(f,r,\delta) > 0$ be the constants given by Corollary 4.5.13. Let us take $\sigma \in (0,\tilde{\sigma}_1)$, and let us consider a point $x_0 \in \tilde{\Gamma}_f^0(\sigma)$ satisfying $R_\pm(f,B(x_0,10\sigma)) > \tilde{R}_1$.

By Corollary 4.5.13, for $n := n(x_0) = 2$, there exists a continuous map $\tilde{\Gamma}_f^0(\sigma) \cap \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma) \ni x \mapsto \gamma^x$ such that $\gamma^x = \{\gamma^x(t) = [x,x_1^x(t),x_2^x(t),x_3^x(t),x]\}_{t\in[0,1]}$ is a continuous family of 4 us-loops at (f,x), with $\ell(\gamma^x) < \sigma$, such that $\gamma^x(0)$ is trivial, for $k = 1,2$, $\gamma^x(\frac{k}{2})$ is a non-degenerate closed us-loop, and there exists a C^r deformation $\hat{f}\colon B(0_{\mathbb{R}^4},\tilde{\delta}_1) \times M \to M$ at f with 4-parameters, so that $\mathrm{supp}(\hat{f}) \subset B(x_0,10\sigma)$, and such that the map

$$\Phi\colon \left\{ \begin{array}{rcl} B(0_{\mathbb{R}^4},\tilde{\delta}_1) \times \left(\tilde{\Gamma}_f^0(\sigma) \cap \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)\right) & \to & \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \\ (b,x) & \mapsto & (\Phi^{(1)}(b,x),\Phi^{(2)}(b,x)) \end{array} \right.$$

is continuous and satisfies

$$|\det D_b|_{b=0}(\Phi(\cdot,x))| > \tilde{\kappa}_0,$$

for some C^2-uniform constant $\tilde{\kappa}_0 = \tilde{\kappa}_0(f,2) > 0$. Recall that $\Pi_x^c\colon M \to \mathbb{R}^2$ is the map given by Lemma 4.3.5, $T = T(\hat{f})$, $\psi_x = \psi(T,x,\gamma^x)$, and $\Phi^{(k)}(\cdot,x) := \Pi_x^c\psi_x(\cdot,x,\frac{k}{2})$, for $k = 1,2$.

By Lemma 4.2.7, we can extend the map $x \mapsto \gamma^x = \{\gamma^x(t)\}_{t\in[0,1]}$ to all the points $x \in \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma)$ (note that for $x \in \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma) \setminus \tilde{\Gamma}_f^0(\sigma)$, the us-loops $\gamma^x(\frac{1}{2}),\gamma^x(1)$ may not be closed). Considering the associated maps $\psi_x = \psi(T,x,\gamma^x)$ and $\Phi^{(k)}(\cdot,x) = \Pi_x^c\psi_x(\cdot,x,\frac{k}{2})$, for $k = 1,2$, we can thus extend Φ to a map

$$\Phi\colon \left\{ \begin{array}{rcl} B(0_{\mathbb{R}^4},\tilde{\delta}_1) \times \mathcal{W}_f^c(x_0,\tilde{K}_1\sigma) & \to & \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \\ (b,x) & \mapsto & (\Phi^{(1)}(b,x),\Phi^{(2)}(b,x)) \end{array} \right.$$

such that

$$|\det D_b|_{b=0}(\Phi(\cdot,x))| > \frac{1}{2}\tilde{\kappa}_0.$$

Take $\hat{\delta} > 0$ suitably small, and let

$$\Psi\colon \left\{ \begin{array}{rcl} \mathcal{W}_T^c((0,x_0),\hat{\delta}) & \to & \mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \\ (b,x) & \mapsto & (\Pi_x^c(x), \Phi^{(1)}(b,x), \Phi^{(2)}(b,x)). \end{array} \right.$$

For any point $x \in \mathcal{W}_f^c(x_0, \tilde{K}_1\sigma)$, the map $\Phi(\cdot,x)$ is a submersion, and thus, the map Ψ is uniformly transverse to the diagonal

$$\Sigma_0 := \{(z,z,z) : z \in \mathbb{R}^2\} \subset \mathbb{R}^6.$$

Therefore, $\Psi^{-1}(\Sigma_0)$ is a submanifold of codimension 4. Let $\pi_B\colon \mathcal{W}_T^c((0,x_0),\hat{\delta}) \to \mathbb{R}^4$, $(b,x) \mapsto b$. Let $b \in B(0_{\mathbb{R}^4}, \tilde{\delta}_1) \setminus \pi_B(\Psi^{-1}(\Sigma_0))$, and let $g := f_b := \hat{f}(b,\cdot) \in \mathscr{F}$. Then, for any $x \in \mathcal{W}_f^c(x_0, \tilde{K}_1\sigma)$, we have $\Psi((b,x)) \notin \Sigma_0$, i.e., $\psi_x(b,x,\frac{1}{2}) \in \mathrm{C}_g(x) \setminus \{x\}$ or $\psi_x(b,x,1) \in \mathrm{C}_g(x) \setminus \{x\}$. We conclude that $x \notin \tilde{\Gamma}_g^0(\sigma)$.

Actually, the same holds for any diffeomorphism h that is sufficiently C^1-close to g. Indeed, for any $x \in \mathcal{W}_f^c(x_0, \tilde{K}_1\sigma)$, let γ_1^x, γ_2^x be the 4 us-loops at (g,x) coming from $\gamma^x(\frac{1}{2})$, $\gamma^x(1)$, with respective endpoints $\psi_x(b,x,\frac{1}{2})$, $\psi_x(b,x,1)$. For any diffeomorphism h which is C^1-close to g, we let $\gamma_1^{x,h}$, $\gamma_2^{x,h}$ be the respective continuations of γ_1^x, γ_2^x given by Lemma 4.2.7, and we set

$$\widetilde{\Psi}(h,x) := \left(\Pi_x^c(x), \Pi_x^c H_{h,\gamma_1^{x,h}}(x), \Pi_x^c H_{h,\gamma_2^{x,h}}(x)\right).$$

By our choice of b, and by compactness, there exists $\varepsilon_0 > 0$ such that

$$d(\widetilde{\Psi}(g,x), \Sigma_0) = d(\Psi(b,x), \Sigma_0) > \varepsilon_0,$$

for all $x \in \mathcal{W}_f^c(x_0, \tilde{K}_1\sigma)$. Thus, there exists $\delta' > 0$ such that for any diffeomorphism h with $d_{C^1}(g,h) < \delta'$, and for any $x \in \mathcal{W}_f^c(x_0, \tilde{K}_1\sigma)$, it holds

$$d(\widetilde{\Psi}(h,x), \Sigma_0) > \frac{\varepsilon_0}{2} > 0.$$

Therefore, $H_{h,\gamma_1^{x,h}}(x) \in \mathrm{C}_h(x) \setminus \{x\}$ or $H_{h,\gamma_2^{x,h}}(x) \in \mathrm{C}_h(x) \setminus \{x\}$, so that $x \notin \tilde{\Gamma}_h^0(\sigma)$, which concludes the proof. □

We can now give the proof of Theorem F.

Corollary 4.7.7. *There exists a C^2-uniform constant $\hat{\sigma}_1 = \hat{\sigma}_1(f) > 0$ such that for any $\sigma \in (0, \hat{\sigma}_1)$, and for any $\delta > 0$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that for some $(\frac{1}{40}, 4)$-spanning c-family $\mathcal{D}_g$ for g, it holds $x \notin \tilde{\Gamma}_g^0(\sigma)$, for all $\mathcal{C} \in \mathcal{D}_g$, and for all $x \in \frac{1}{20}\mathcal{C}$. In particular, the center accessibility class $\mathrm{C}_g(x)$ of each point $x \in M$ is non-trivial.*

Proof. Fix some small $\delta > 0$. By Kupka-Smale's Theorem (see for instance [Kal97]), C^r-generically, periodic points are hyperbolic. Therefore, without loss of generality, we can assume that the fixed points of f^k are isolated, for all $k \geq 1$.

Let $\tilde{\sigma}_1 = \tilde{\sigma}_1(f) > 0$, $\tilde{K}_1 = \tilde{K}_1(f) \in (0,1)$ and $\tilde{R}_1 = \tilde{R}_1(f) > 0$ be the constants given by Proposition 4.7.5. For $\overline{R} > \max(\tilde{R}_1, 1)$, let $\overline{N} = \overline{N}(f, \overline{R}) > 0$, $\overline{\rho} = \overline{\rho}(f, \overline{R}) \in (0, \overline{R}^{-1})$ and $\overline{\sigma} = \overline{\sigma}(f, \overline{R}) > 0$ be the constants given by Proposition 4.7.4. Then, there exists a constant $\delta_0' \in (0,\delta)$ such that for any diffeomorphism g with $d_{C^1}(f,g) < \delta_0'$, there exists a $(\frac{1}{40},4)$-spanning c-family $\mathcal{D}_g$ for g with at most $\overline{N}$ elements such that the map $g \mapsto \mathcal{D}_g$ is continuous, and

$$\overline{\rho} < \underline{r}(\mathcal{D}_g) \leq \overline{r}(\mathcal{D}_g) < \overline{R}^{-1}; \qquad \mathcal{D}_g \text{ is } \overline{\sigma}\text{-sparse}; \qquad R_\pm(g,(\mathcal{D}_g,\overline{\sigma})) > \overline{R}.$$

Take $\sigma \in \left(0, \min\left(\tilde{\sigma}_1, \frac{\overline{\sigma}}{10}\right)\right)$, and let $z_1, z_2, \dots, z_\ell$, $\ell \geq 1$, be a finite collection of points such that for any diffeomorphism g with $d_{C^1}(f,g) < \delta_0'$, we have $g \in \mathscr{F}$, and

$$\bigcup_{\mathcal{C}\in\mathcal{D}_g} \frac{1}{20}\mathcal{C} \subset \bigcup_{i=1}^{\ell} \mathcal{W}_f^c(z_i, \tilde{K}_1\sigma) \subset (\mathcal{D}_g, 10\sigma). \tag{4.34}$$

As $\sigma \in (0,\tilde{\sigma}_1)$ and $R_\pm(f, B(z_1, 10\sigma)) > \tilde{R}_1$, we can apply Proposition 4.7.5 to get a diffeomorphism $f_1 \in \mathscr{F}$ such that for some $\delta_1' \in (0, \delta'(z_1, f_1))$, it holds $B_{C^r}(f_1,\delta_1') \subset B_{C^r}(f,\delta_0')$, and $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_1, \tilde{K}_1\sigma)$ and for all $h \in B_{C^r}(f_1,\delta_1')$.

Similarly, as $R_\pm(f_1, B(z_2, 10\sigma)) > \tilde{R}_1$, we can apply Proposition 4.7.5 to get a diffeomorphism $f_2 \in \mathscr{F}$ such that for some $\delta_2' > 0$, it holds $B_{C^r}(f_2,\delta_2') \subset B_{C^r}(f_1,\delta_1') \subset B_{C^r}(f,\delta_0')$, and $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_2, \tilde{K}_1\sigma)$ and for all $h \in B_{C^r}(f_2,\delta_2')$; in fact, as $B_{C^r}(f_2,\delta_2') \subset B_{C^r}(f_1,\delta_1')$, we have $x \notin \tilde{\Gamma}_h^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_1, \tilde{K}_1\sigma) \cup \mathcal{W}_f^c(z_2, \tilde{K}_1\sigma)$.

Recursively, we thus obtain a diffeomorphism $g = f_\ell \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta_0' < \delta$ and such that $x \notin \tilde{\Gamma}_g^0(\sigma)$, for all $x \in \mathcal{W}_f^c(z_1, \tilde{K}_1\sigma) \cup \cdots \cup \mathcal{W}_f^c(z_\ell, \tilde{K}_1\sigma)$. By (4.34), we conclude that for each $\mathcal{C} \in \mathcal{D}_g$, and for each $x \in \frac{1}{20}\mathcal{C}$, we have $x \notin \tilde{\Gamma}_g^0(\sigma)$. In particular, as $\mathcal{D}_g$ is a $(\frac{1}{40},4)$-spanning c-family for g, the center accessibility class $\mathrm{C}_g(x)$ of each point $x \in M$ is non-trivial. □

Remark 4.7.8. *In fact, Corollary 4.7.7 also holds when the center dimension* $\dim E_f^c$ *is larger than* 2*. Indeed, the proof relies on the submersion from the space of perturbations to the phase space – here, some center leaf – constructed in Lemma 4.5.12 and Corollary 4.5.13, which can be carried out also when* $\dim E_f^c > 2$.

4.7.3 Density of accessibility (proof of Theorem E)

In this part, we conclude the proof of Proposition 4.7.1 (Theorem E). Let us start with the following result, which strengthens Proposition 4.6.2.

Proposition 4.7.9. *There exist C^2-uniform constants $\tilde{\sigma}_2 = \tilde{\sigma}_2(f) > 0$, $\tilde{K}_2 = \tilde{K}_2(f) \in (0,1)$ and $\tilde{R}_2 = \tilde{R}_2(f) > 0$ such that for any $\delta > 0$, for any $\sigma \in (0,\tilde{\sigma}_2)$, for any point $x_0 \in M$ satisfying $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}_2$, there exists a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f,g) < \delta$ and such that for some $\delta'' = \delta''(x_0,g) > 0$, it holds $\mathrm{Acc}_h(x_0) \supset B(x_0, \tilde{K}_2\sigma)$, for all $h \in \mathscr{F}$ with $d_{C^1}(g,h) < \delta''$.*

Proof. Fix some small $\delta > 0$. Let $\tilde{\sigma}_1 = \tilde{\sigma}_1(f) > 0$, $\tilde{K}_1 = \tilde{K}_1(f) \in (0,1)$ and $\tilde{R}_1 = \tilde{R}_1(f) > 0$ be the constants in Proposition 4.7.5. Let $x_0 \in M$ be a point satisfying $R_\pm(f, B(x_0, 10\sigma)) > \tilde{R}$, for some $\tilde{R} > \tilde{R}_1$ and $\sigma \in (0, \tilde{\sigma}_1)$, and take $\tilde{K} \in (0, \tilde{K}_1)$. Then, by Proposition 4.7.5, there exists a partially hyperbolic diffeomorphism $f_1 \in \mathscr{F}$ such that $d_{C^r}(f,f_1) < \frac{\delta}{2}$ and such that for some $\delta' \in (0, \frac{\delta}{2})$, we have $x \notin \tilde{\Gamma}_g^0(\sigma)$, for all $x \in B(x_0, \tilde{K}\sigma)$ and for all $g \in \mathscr{F}$ with $d_{C^1}(f_1, g) < \delta'$.

In the following, for any $x \in \Gamma_g^1 \cap \mathcal{W}_{f_1}^c(x_0, \tilde{K}\sigma)$, we denote by $\Pi_x^c \colon M \to \mathbb{R}^2$ the map in Lemma 4.3.5 for the diffeomorphism f_1. By Proposition 4.3.7 and Proposition 4.3.4, if δ' is sufficiently small, then for any $g \in \mathscr{F}$ with $d_{C^1}(f_1,g) < \delta'$ and for any

$x \in \Gamma_g^1 \cap \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, it holds

$$\Pi_x^c \mathrm{C}_g(x, 10\sigma) \subset \mathscr{C}_1, \tag{4.35}$$

for some cone $\mathscr{C}_1$ centered at $0_{\mathbb{R}^2}$; as in Section 4.4, we let $\mathscr{C} := (\mathbb{R}^2 \setminus \mathscr{C}_1) \cup \{0_{\mathbb{R}^2}\}$, and let $\mathscr{C}^+$, $\mathscr{C}^-$ be the closures of the two connected components of $\mathscr{C} \setminus \{0_{\mathbb{R}^2}\}$. For any $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, we let $\gamma_1^x = [x, \alpha_1^x, \dots, \omega_1^x]$, resp. $\gamma_2^x = [x, \alpha_2^x, \dots, \omega_2^x]$ be the non-degenerate closed 10 us-loop, resp. non-degenerate closed 10 su-loop at (f_1, x) given by Lemma 4.4.3 for f_1 in place of f, with

$$\left(\Pi_x^c \omega_1^x, \Pi_x^c \omega_2^x\right) \in \left(\mathscr{C}^+ \times \mathscr{C}^-\right) \cup \left(\mathscr{C}^- \times \mathscr{C}^+\right). \tag{4.36}$$

In the following, we will define a new deformation $\hat{f}$ obtained by considering infinitesimal deformations localized near the points α_1^x and α_2^x for $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$. Arguing as in Lemma 4.5.12, for $\star = 1, 2$, we can construct an infinitesimal C^r deformation at f_1 with 2-parameters $V_\star \colon \mathbb{R}^2 \times M \to TM$ such that $\mathrm{supp}(V_\star) \subset B(x_0, 10\sigma)$, and such that for some constants $\widetilde{C} > 0$, $\tilde{\kappa} > 0$, we have: for any $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$,

1. $V_\star$ is adapted to $(\gamma_\star^x, \sigma, \widetilde{C}, \widetilde{R})$;
2. for any corner $z \neq \alpha_\star^x$ of $\gamma_\star^x$, it holds

   $$D_B(\pi_c V_\star(B, z)) = 0,$$

   where $\pi_c \colon TM \to E_f^c$ denotes the canonical projection, and

   $$\left|\det D_B\left(\pi_c V_\star\left(B, x_1^x\left(\frac{k}{k_0}\right)\right)\right)\right| > \tilde{\kappa}.$$

Indeed, for $\star = 1, 2$, as the map $\mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma) \ni x \mapsto \gamma_\star^x$ is continuous, and by (4.12), we can construct the infinitesimal deformation $V_\star$ such that the $\mathrm{supp}(V)$ is localized around the set $\{\alpha_\star^x\}_x$ of the first corners of the loops $\gamma_\star^x$.

Let then $V \colon \mathbb{R}^4 \times M \to TM$ be the infinitesimal C^r deformation defined as

$$V(B, \cdot) := B^1 V_1(\cdot) + B^2 V_2(\cdot), \quad \forall\, B = (B^1, B^2) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

In particular, V satisfies $\mathrm{supp}(V) \subset B(x_0, 10\sigma)$, for any $x \in \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, V is adapted to $(\gamma_1^x, \sigma, \widetilde{C}, \widetilde{R})$ and $(\gamma_2^x, \sigma, \widetilde{C}, \widetilde{R})$, and for any corner $z \neq \alpha_1^x, \alpha_2^x$ of γ_1^x, γ_2^x,

$$D_B(\pi_c V(B, z)) = 0,$$

while for $E_1 := \mathbb{R}^2 \times \{0_{\mathbb{R}^2}\}$, $E_2 := \{0_{\mathbb{R}^2}\} \times \mathbb{R}^2$, we have

$$\left|\det\left(E_\star \ni B \mapsto D_B\left(\pi_c V\left(B, \alpha_\star^x\right)\right)\right)\right| > \tilde{\kappa}, \quad \star = 1, 2. \tag{4.37}$$

For some small $\delta_1 > 0$, let us consider the C^r deformation $\hat{f} \colon B(0_{\mathbb{R}^4}, \delta_1) \times M \to M$ at f_1 with 4-parameters generated by the infinitesimal C^r deformation V. As before, for any $b \in B(0_{\mathbb{R}^4}, \delta_1)$, we set $f_b := \hat{f}(b, \cdot)$. By (4.35), if δ_1 and σ are sufficiently small, then for all $b \in B(0_{\mathbb{R}^4}, \delta_1)$, and for all $x \in \Sigma_b(\sigma) := \Gamma_{f_b}^1 \cap \mathcal{W}_{f_1}^c(x_0, \widetilde{K}\sigma)$, it holds

$$\Pi_x^c \mathrm{C}_{f_b}(x, 10\sigma) \subset \mathscr{C}_1. \tag{4.38}$$

Let $T = T(\hat{f})$ be as in (4.15). We denote by $\hat{\gamma}_1^x$, $\hat{\gamma}_2^x$ the respective lifts of γ_1^x and γ_2^x for T according to Definition 4.5.5. By (4.37), thanks to Proposition 4.5.11, and arguing as in Corollary 4.5.13, we obtain:

Lemma 4.7.10. *The map*

$$\Phi\colon \begin{cases} B(0_{\mathbb{R}^4}, \delta_1) \times \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma) & \to \quad \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2 \\ (b, x) & \mapsto \quad \left(\Pi^c_x H_{T,\hat{\gamma}_1^x}(b,x), \Pi^c_x H_{T,\hat{\gamma}_2^x}(b,x)\right) \end{cases}$$

satisfies

$$\left|D_b|_{b=0}\Phi(\cdot, x) - D_b|_{b=0}\Phi(\cdot, y)\right| \leq \rho(\sigma), \qquad \forall\, x, y \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma), \tag{4.39}$$

for some function $\rho\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\lim_{\sigma \to 0} \rho(\sigma) = 0$*, and there exists* $\kappa > 0$ *such that for any* $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$*, it holds*

$$\left|\det D_b|_{b=0}\big(\Phi(\cdot, x)\big)\right| > \kappa. \tag{4.40}$$

Indeed, for $\star = 1, 2$, since the map $\mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma) \ni x \mapsto \gamma_\star^x$ is continuous, it follows from Lemma 4.3.1 and Corollary 4.3.3 that the partial derivatives of the holonomies $H_{T,\hat{\gamma}_\star^x}$ with respect to b are uniformly close, for all $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$. Hence, by the definition of Φ, and by Proposition 4.5.11, the maps $\{\Phi(\cdot, x)\}_{x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)}$ are uniform submersions, which gives (4.39) and (4.40).

By (4.36), for each $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, we have

$$\Phi(0, x) \in \left(\mathscr{C}^+ \times \mathscr{C}^-\right) \cup \left(\mathscr{C}^- \times \mathscr{C}^+\right).$$

Let us denote by S^+, resp. S^- the set of all points $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$ such that $\Phi(0, x) \in \mathscr{C}^+ \times \mathscr{C}^-$, resp. $\Phi(0, x) \in \mathscr{C}^- \times \mathscr{C}^+$, so that $S^+ \cup S^- = \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$. By (4.39)-(4.40), there exists a perturbation parameter $b \in B(0_{\mathbb{R}^4}, \delta_1)$ such that

$$\begin{aligned} \Pi^c_x H_{T,\hat{\gamma}_1^x}(b, x) &\in \mathscr{C}^+_* = \mathscr{C}^+ \setminus \{0_{\mathbb{R}^2}\}, \qquad \text{for all } x \in S^+, \\ \Pi^c_x H_{T,\hat{\gamma}_2^x}(b, x) &\in \mathscr{C}^-_* = \mathscr{C}^- \setminus \{0_{\mathbb{R}^2}\}, \qquad \text{for all } x \in S^-. \end{aligned}$$

As $S^+ \cup S^- = \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$, we deduce that for each $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$,

$$\text{either} \quad \Pi^c_x H_{T,\hat{\gamma}_1^x}(b, x) \notin \mathscr{C}_1, \quad \text{or} \quad \Pi^c_x H_{T,\hat{\gamma}_2^x}(b, x) \notin \mathscr{C}_1.$$

By (4.38), we deduce that $\Sigma_b(\sigma) = \emptyset$, i.e., $\Gamma^1_{f_b} = \emptyset$. Therefore, by Theorem 4.2.10, the accessibility class $\mathrm{Acc}_{f_b}(x)$ of each point $x \in \mathcal{W}^c_{f_1}(x_0, \widetilde{K}\sigma)$ is open. Moreover, if δ_1 is sufficiently small, then by construction, the diffeomorphism $g := f_b$ satisfies

$$d_{C^r}(f, g) \leq d_{C^r}(f, f_1) + d_{C^r}(f_1, f_b) < \delta,$$

which concludes the proof of Proposition 4.7.9. □

Proof of Proposition 4.7.1. Fix $\delta > 0$ arbitrarily small. Let $\tilde{\sigma}_2 = \tilde{\sigma}_2(f) > 0$, $\tilde{K}_2 = \tilde{K}_2(f) \in (0, 1)$ and $\tilde{R}_2 = \tilde{R}_2(f) > 0$ be the C^2-uniform constants given by Proposition 4.7.9. By Proposition 4.7.4, there exist C^1-uniform constants $\overline{N} = \overline{N}(f, \tilde{R}_2) > 0$, $\overline{\rho} = \overline{\rho}(f, \tilde{R}_2) \in$

$(0, \tilde{R}_2^{-1})$ and $\overline{\sigma} = \overline{\sigma}(f, \tilde{R}_2) > 0$ such that for all diffeomorphism g sufficiently C^1-close to f, there exists a $(\frac{1}{40}, 4)$-spanning c-family $\mathcal{D}_g$ for g with at most $\overline{N}$ elements such that

1. $\overline{\rho} < \underline{r}(\mathcal{D}_g) \leq \overline{r}(\mathcal{D}_g) < \tilde{R}_2^{-1}$;
2. $\mathcal{D}_g$ is $\overline{\sigma}$-sparse;
3. $R_\pm(g, (\mathcal{D}_g, \overline{\sigma})) > \tilde{R}_2$.

and such that the map $g \mapsto \mathcal{D}_g$ is continuous. Let $\sigma \in (0, \frac{1}{10}\min(\tilde{\sigma}_2, \overline{\sigma}))$. By compactness, we can take a finite collection of points $x_1, \ldots, x_m \in M$ such that

$$\frac{1}{20}\mathcal{D}_f \subset U := \bigcup_{i=1}^{m} B(x_i, \tilde{K}_2\sigma) \subset (\mathcal{D}_f, \overline{\sigma}).$$

Note that $x_i \in M$ satisfies $R_\pm(f, B(x_i, 10\sigma)) > \tilde{R}_2$, for each $i \in \{1, \ldots, m\}$. Therefore, we can apply Proposition 4.7.9 inductively to get a partially hyperbolic diffeomorphism $g \in \mathscr{F}$ such that $d_{C^r}(f, g) < \delta$ and such that $\mathrm{Acc}_g(x_i) \supset B(x_i, \tilde{K}_2\sigma)$, for all $i \in \{1, \ldots, m\}$. By connectedness of the disks in $\mathcal{D}_f$, each center disk in the family $\frac{1}{20}\mathcal{D}_f$ is contained in a single accessibility class for g. Moreover, if δ is sufficiently small, and by continuity of the map $h \mapsto \mathcal{D}_h$, each center disk in the family $\frac{1}{40}\mathcal{D}_g$ is contained in a single accessibility class for g. As $\mathcal{D}_g$ is a $(\frac{1}{40}, 4)$-spanning c-family for g, we deduce that g is accessible, as wanted. □

Bibliography

[ACW16] A. Avila, S. Crovisier, and A. Wilkinson. "Diffeomorphisms with positive metric entropy". In: *Publications Mathematiques de I'IHES* 124 (2016), pp. 589–602.

[ACW22] A. Avila, S. Crovisier, and A. Wilkinson. "Symplectomorphisms with positive metric entropy". In: *Proceedings of the London Mathematical Society* 124.5 (2022), pp. 691–712.

[Ano67] D. V. Anosov. "Geodesic flows on closed Riemann manifolds with negative curvature". In: *Proc. Steklov Inst. Math.* 90 (1967). English translation. Providence, R.I.:Amer. Math. Soc. 1969.

[AV20] A. Avila and M. Viana. "Stable accessibility with 2-dimensional center". In: *Asterisque* 416 (2020). Quelques aspects de la théorie des systèmes dynamiques: un hommage à Jean-Christophe Yoccoz. II, pp. 301–320.

[Bar+] T. Barthelmé et al. "Partially hyperbolic diffeomorphisms homotipic to the identity in dimension 3 part 2: branching foliations". To appear in *Geometry and Topology*. URL: `https://arxiv.org/abs/2008.04871`.

[Bar+21] T. Barthelmé et al. "Dynamical incoherence for a large class of partially hyperbolic diffeomorphisms". In: *Ergodic Theory and Dynamical Systems* 41.11 (2021), pp. 3227–3243.

[BC14] P. Berger and P. Carrasco. "Non-uniformly hyperbolic diffeomorphisms derived from the standard map". In: *Comm. Math. Phys.* 329 (2014), pp. 239–262.

[BD96] C. Bonatti and L. J. Díaz. "Persistent nonhyperbolic transitive diffeomorphisms". In: *Annals of Mathematics* 143.2 (1996), pp. 357–396.

[BDP03] C. Bonatti, L. J. Díaz, and E. Pujals. "A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources". In: *Annals of Mathematics* 158.2 (2003), pp. 355–418.

[BGP16] C. Bonatti, A. Gogolev, and R. Potrie. "Anomalous partially hyperbolic II: stably ergodic examples". In: *Invent. Math.* 206.3 (2016), pp. 801–836.

[BGV06] C. Bonatti, N. Gourmelon, and T. Vivier. "Perturbations of the derivative along periodic orbits". In: *Ergodic Theory and Dynamical Systems* 26.5 (2006), pp. 1307–1337.

[BK02] M. Bonk and B. Kleiner. "Rigidity for quasi-Möbius group actions". In: *J. Diff. Geom.* 61.1 (2002), pp. 81–106.

[Bon+20] C. Bonatti et al. "Anomalous partially hyperbolic diffeomorphisms III: abundance and incoherence". In: *Geometry and Topology* 24.4 (2020), pp. 1751–1790.

[BPP16] C. Bonatti, K. Parwani, and R. Potrie. "Anomalous partially hyperbolic I: dynamically coherent examples". In: *Ann. Sci. Éc. Norm. Supér.* 49.16 (2016), pp. 1387–1402.

[Bro] A. Brown. "Smoothness of stable holonomies inside center-stable manifolds and the C^2 hypothesis in Pugh-Shub and Ledrappier-Young theory". URL: https://arxiv.org/abs/1608.05886.

[BV00] C. Bonatti and M. Viana. "SRB measures for partially hyperbolic systems whose central direction is mostly contracting". In: *Israel Journal of Mathematics* 115 (2000), pp. 157–193.

[BW] K. Burns and A. Wilkinson. "A note on stable holonomy between centers".

[BW08] K. Burns and A. Wilkinson. "Dynamical coherence and center bunching". In: *Discrete and Continuous Dynamical System A* 22 (2008). (Pesin birthday issue), pp. 89–100.

[BW10] K. Burns and A. Wilkinson. "On the ergodicity of partially hyperbolic systems". In: *Annals of Mathematics* 171.1 (2010), pp. 451–489.

[Car11] P. Carrasco. "Compact dynamical foliations". PhD thesis. University of Toronto, Mar. 2011.

[CO21] P. Carrasco and D. Obata. "A new example of robustly transitive diffeomorphism". In: *Math. Research Letters* 28.3 (2021), pp. 665–679.

[CP14] F. Carneiro and E. Pujals. "Partially hyperbolic geodesic flows". In: *Ann. Ins. H. Poincaré* 31.5 (2014), pp. 985–1014.

[Did03] Ph. Didier. "Stability of accessibility". In: *Ergod. Th. & Dynam. Sys.* 23.6 (2003), pp. 1717–1731.

[Doe87] C.I. Doering. "Persistenly transitive vector fields on three-dimensional manifolds". In: *Proc. on Dynamical Systems and Bifurcation Theory* 160 (1987), pp. 59–89.

[DPU99] L.J. Díaz, E. Pujals, and R. Ures. "Partial hyperbolicity and robust transitivity". In: *Acta Math.* 183.1 (1999), pp. 1–43.

[DW03] D. Dolgopyat and A. Wilkinson. "Stable accessibility is C^1-dense". In: *Astérisque* 287 (2003). Geometric methods in dynamics, pp. 33–60.

[FG14] F. T. Farrel and A. Gogolev. "The space of Anosov diffeomorphisms". In: *Journal of the London Mathematical Society* 89.2 (2014), pp. 383–396.

[FPS14] T. Fisher, R. Potrie, and M. Sambarino. "Dynamical coherence of partially hyperbolic diffeomorphisms of tori isotopic to Anosov". In: *Mathematische Zeitschrift* 278 (2014), pp. 149–168.

[Fra69] J. Franks. "Anosov diffeomorphisms on tori". In: *Transactions of the American Mathematical Society* 145 (1969), pp. 117–124.

[Fra70] J. Franks. "Anosov diffeomorphisms". In: *Proc. Sympos. Pure Math.* 14 (1970), pp. 61–93.

[Fra71] J. Franks. "Necessary conditions for stability". In: *Transactions of the American Mathematical Society* 158.2 (1971), pp. 301–308.

[FW80] J. Franks and R. Williams. "Anomalous Anosov flows". In: *Global theory of dynamical systems* 819 (1980), pp. 158–174.

[Gou07] N. Gourmelon. "Adapted metrics for dominated splittings". In: *Ergodic Theory and Dynamical Systems* 27.6 (2007), pp. 1839–1849.

[GPS94] M. Grayson, C. Pugh, and M. Shub. "Stably ergodic diffeomorphisms". In: *Ann. of Math.* 140.2 (1994), pp. 295–329.

[Ham13] A. Hammerlindl. “Partial hyperbolicity on 3-dimensional nilmanifolds”. In: *Discrete and Continuous Dynamical Systems* 33.8 (2013), pp. 3641–3669.

[Her05] F. Rodríguez Hertz. “Stable ergodicity of certain linear automorphisms of the torus”. In: *Annals of Mathematics* 162 (2005), pp. 65–107.

[HHU07] M. A. Rodríguez Hertz, F. Rodríguez Hertz, and R. Ures. “A survey on partially hyperbolic systems”. In: *Fields institute communications* 51 (2007), pp. 35–88.

[HHU08] F. Rodríguez Hertz, M.A. Rodríguez Hertz, and R. Ures. “Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1D-center bundle”. In: *Invent. Math.* 172.2 (2008), pp. 353–381.

[HHU16] M. A. Rodríguez Hertz, F. Rodríguez Hertz, and R. Ures. “A non dynamically coherent example in $\mathbf{T}^3$”. In: *Annals IHP (C): Analyse nonlineaire* 33.4 (2016), pp. 1023–1032.

[Hop39] E. Hopf. “Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung”. In: *Ber. Verh. Sächs. Akad. Wiss. Leipzig* 91 (1939), pp. 261–304.

[HPS77] M. Hirsch, C. Pugh, and M. Shub. “Invariant manifolds”. In: *Springer Lecture Notes in Math.* 583 (1977).

[HS17] V. Horita and M. Sambarino. “Stable Ergodicity and Accessibility for certain Partially Hyperbolic Diffeomorphisms with Bidimensional Center Leaves”. In: *Comment. Math. Helv.* 92.3 (2017), pp. 467–512.

[HUY22] M. A. Rodríguez Hertz, R. Ures, and J. Yang. “Robust minimality of strong foliations for DA diffeomorphisms: *cu*-volume expansion and new examples”. In: *Trans. Amer. Math. Soc.* 375.6 (2022), pp. 4333–4367.

[HV20] M. A. Rodríguez Hertz and C. Vásquez. “Structure of accessibility classes”. In: *Disc. and Cont. Dyn. Sys. Series A* 40.8 (2020), pp. 4653–4664.

[HW41] W. Hurewicz and H. Wallman. *Dimension theory*. Vol. 4. Princeton Mathematical Series, 2. Princeton University Press, 1941.

[Kal97] V. Kaloshin. “Some prevalent properties of smooth dynamical systems”. In: *Proceedings of the Steklov Institute of Mathematics* 213 (1997), pp. 123–151.

[Lew83] J. Lewowicz. “Persistence in expansive systems”. In: *Ergod. Theor. & Dynam. Sys.* 3.4 (1983), pp. 567–578.

[LP] M. Leguil and L. P. Piñeyrúa. “Accessibility for dynamically coherent partially hyperbolic diffeomorphisms with 2d center”. URL: https://arxiv.org/abs/2112.12762.

[LZ22] M. Leguil and Z. Zhang. “C^r-prevalence of stable ergodicity for a class of partially hyperbolic systems”. In: *Journal of Eur. Math. Soc. (JEMS)* 24.9 (2022), pp. 3379–3438.

[Man74] A. Manning. “There are no new Anosov on tori”. In: *Amer. Jour. of Math.* 96.3 (1974), pp. 422–429.

[Mañ78] R. Mañé. “Contributions to the stability conjecture”. In: *Topology* 17.4 (1978), pp. 383–396.

[Mañ82] R. Mañé. “An ergodic closing lemma”. In: *Annals of Mathematics* 116.3 (1982), pp. 503–540.

[Mañ87a] R. Mañé. “A proof of the C^1 stability conjecture”. In: *Publ. Math. IHES* 66 (1987), pp. 161–210.

[Mañ87b] R. Mañé. *Ergodic theory and differentiable dynamics*. Springer, 1987.

[New70] S. Newhouse. "On codimension one Anosov diffeomorphisms". In: *Amer. Jour. of Math.* 92.3 (1970), pp. 761–770.

[Oba] D. Obata. "On the holonomies of strong stable foliations". URL: https://www.imo.universite-paris-saclay.fr/~obata/holonomies.pdf..

[Oba18] D. Obata. "On the stable ergodicity of Berger-Carrasco's example". In: *Ergodic Theory and Dynamical Systems* 40.4 (2018), pp. 1008–1056.

[Omb86] J. Ombach. "Equivalent conditions for hyperbolic coordinates". In: *Topology and applications* 23 (1986), pp. 87–90.

[Omb87] J. Ombach. "Consequences of the pseudo orbit tracing property and expansiveness". In: *Journal of the Australian Mathematical Society (series A)* 43 (1987), pp. 301–313.

[Omb96] J. Ombach. "Shadowing, expansiveness and hyperbolic homeomorphisms". In: *Jorunal of the Australian Mathematical society (Series A)* 61 (1996), pp. 57–72.

[Pat90] M. Paternain. "Expansive flows and the fundamental group". PhD thesis. 1990.

[Piñ] L.P. Piñeyrúa. "Dynamical coherence of partially hyperbolic diffeomorphisms on nilmanifolds isotopic to Anosov". URL: https://arxiv.org/abs/1910.05279.

[Pot12] R. Potrie. "Partially hyperbolicity and attracting regions in 3-dimensional manifolds". PhD thesis. 2012. URL: https://arxiv.org/abs/1207.1822.

[PS00] C. Pugh and M. Shub. "Stable ergodicity and julienne quasi-conformality". In: *J. Eur. Math. Soc. (JEMS)* 2.1 (2000), pp. 1–52.

[PS06] E. Pujals and M. Sambarino. "A sufficient condition for robustly minimal foliations". In: *Ergodic Theory & Dynamical Systems* 26.1 (2006), pp. 281–289.

[PS96] C. Pugh and M. Shub. "Stable ergodicity and partial hyperbolicity". In: *International Conference on Dynamical Systems, Montevideo, 1995*. Vol. 362. Longman, Harlow, 1996, pp. 182–187.

[PS97] C. Pugh and M. Shub. "Stably ergodic dyamical systems and partial hyperbolicity". In: *J. Complexity* 13.1 (1997), pp. 125–179.

[PSW12] C. Pugh, M. Shub, and A. Wilkinson. "Hölder foliations, revisited". In: *Journal of Modern Dynamics* 6.1 (2012), pp. 79–120.

[PSW97] C. Pugh, M. Shub, and A. Wilkinson. "Hölder foliations". In: *Duke Math. J.* 86.3 (1997), pp. 517–546.

[RSS96] D. Repovs, A. B. Skopenkov, and E. V. Scepin. "C^1-homogeneous compacta in $\mathbb{R}^n$ are C^1-submanifolds of $\mathbb{R}^n$". In: *Proc. Amer. Math. Soc.* 124.4 (1996), pp. 1219–1226.

[Rug96] R. O. Ruggiero. "On a conjecture about expansive geodesic flows". In: *Ergod. Theor. & Dynam. Sys.* 16.3 (1996), pp. 545–553.

[Rug97] R. O. Ruggiero. "Expansive geodesic flows in manifolds with no conjugate points". In: *Ergod. Theor. & Dynam. Sys.* 17.1 (1997), pp. 211–225.

[Sam09] M. Sambarino. "Hiperbolicidad y estabilidad". Ediciones IVIC - Caracas. 2009.

[Shu71] M. Shub. "Topological transitive diffeomorphism on $\mathbb{T}^4$". In: *Lecture notes in Mathematics* (1971).

[Sma67] S. Smale. "Differentiable dynamical systems". In: *Bulletin of the AMS* 73 (1967), pp. 747–817.

[Viv06] T. Vivier. "Projective hyperbolicity and fixed points". In: *Ergodic Theory and Dynamical Systems* 26.3 (2006), pp. 923–936.

[Wal78] P. Walters. "On the pseudo orbit tracing property and its relationship to stability". In: *The structure of attractors in dynamical systems* 668 (1978), pp. 231–244.

[War71] F. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer, 1971.

[Wil13] A. Wilkinson. "The cohomological equation for partially hyperbolic diffeomorphisms". In: *Astérisque* 358 (2013), pp. 75–165.

[Wil98] A. Wilkinson. "Stable ergodicity of the time-one map of a geodesic flow". In: *Ergodic Theory and Dynamical Systems* 18.6 (1998), pp. 1545–1587.