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Asymptotic Symmetries and Phase Space Extensions

In Gravity and Gauge Theories

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Oh, let the sun beat down upon my face
And stars to fill my dream
I'm a traveller of both time and space
To be where I have been

—Led Zeppelin, *Kashmir*

Abstract

The aim of this thesis is to provide new insights in the structure of the asymptotic symmetries for gravity and gauge theories by studying phase space extensions.

In the first part of this work, dedicated to asymptotic symmetries in General Relativity, we present a correction term for the super-angular momentum and an extension of the phase space of gravity at null infinity. In this extension the superrotation group $\text{Diff}(S^2)$ acts canonically, thus generalizing the Bondi-Metzner-Sachs group as a symmetry of the system. We discussed this results in the context of covariant phase space formalism, making connections with the extended corner symmetry results. By considering Eintein-Yang-Mills theory, we show an extension of the phase space of gravity coupled to a non-abelian gauge theory where the generalized Bondi-Metzner-Sachs group acts canonically.

The second part deals with phase space extensions in Yang-Mills and Maxwell Theories. First, we construct the linearized extension for the asymptotic symmetries in Yang-Mills in order to accommodate a large gauge transformation with associated subleading charges. Then, the inclusion of higher order large gauge transformations in the abelian case is done, where it is shown that we can obtain an infinite hierarchy of asymptotic symmetries. Each asymptotic symmetry has a corresponding sub^n-leading charge, compatible with the sub^n-leading soft photon theorems. Finally, in the non-abelian case, working in the self dual sector of the theory we propose an extended phase space along with a perturbative-like method to compute the asymptotic symmetry algebra.

Resumen

El objetivo de esta tesis es aportar nuevos resultados sobre la estructura de las simetrías asintóticas para gravedad y teorías de gauge, mediante extensiones de espacios de fase.

En la pimer parte de este trabajo, dedicada a las simetrías asintóticas en Relatividad General, presentamos un término correctivo en el super-momento angular y un espacio de fase extendido para gravedad en infinito nulo. En esta extensión, el grupo de super-rotaciones $\text{Diff}(S^2)$ actúa canónicamente, generalizando el grupo de simetrías asintótico de Bondi-Metzner-Sachs. En el contexto del formalismo de espacios de fase covariantes, conectamos nuestros resultados con los trabajos en el grupo de simetrías en esquinas extendido. Tomando Einstein-Yang-Mills, mostramos que la extensión del espacio de fase gravitacional acoplado a una teoría de gauge mantiene la acción canónica del grupo Bondi-Metzner-Sachs generalizado.

La segunda parte de la tesis trata las extensiones de espacios de fase en las teorías de Yang-Mills y Maxwell. Primero, construímos la extensión linearizada para el grupo de simetrías asintóticas en Yang-Mills, de forma que contenga las transformaciones de gauge de orden r asociadas a las cargas subdominantes. Luego, hacemos la inclusión de transformaciones de gauge de orden más grande en el caso abeliano, mostrando que se puede obtener una jerarquía infinita de simetrías asintóticas. Cada simetrías está asociada a una carga sub$^n-$dominante, compatible con los teoremas sub$^n-$dominantes para fotones suaves. Finalmente, en el caso no abeliano y trabajando en el sector autodual de la teoría, proponemos una extensión del espacio de fase junto con un método perturbativo para calcular el álgebra de simetrías asintóticas.

Contents

Introduction

0.1 Introduction

General Relativity and Gauge Theories [1] are clear examples of the deep roots that Geometry has in the current physical theories that model nature. One can argue that the principle of invariance, coming from any type of gauge symmetry, is the philosophical descendant of Euclid's Elements: there are classes of objects which are invariant with respect to their particular position in space. In a modern paraphrasing of this idea, physical laws are independent of any particular coordinate system or local trivialization. As in any geometric theory, the study of symmetries is central. They contain the information to understand, organize and classify the objects within the confines of the model.

The aim of this thesis is to provide new insights in the structure of the symmetries in gravity and gauge theories, by showing extensions of the structures defined on their solution spaces in order to accommodate more physical information, in the form of conserved quantities.

In the following paragraphs we give a (shallow) review of the topic.

Symmetries and gauge theories

In a physical theory, the connection between symmetries and physics is provided by the seminal works by Noether [1], where it is shown that conserved quantities and the existence of a variational principle that is invariant under symmetry transformations are equivalent.

The theories on which Noether's theorems are valid are called *Lagrangian Theories*, where a variational principle can be stated in terms of the action (integral of the lagrangian). The equations of motion are given by the Euler-Lagrange equations. Roughly speaking, Noether's theorems establish a one to one correspondence between symmetries of the Lagrangian (i.e.,

[1]The word gauge has different meanings in different contexts. In some texts Gauge Theories refer to any field theory invariant under local transformations, such as Maxwell, Yang-Mills or General Relativity. In others, the definition is restricted to those that are invariant under local transformations based only on actions of Lie groups, leaving General Relativity in a special category. Throughout this thesis we are using both connotations, the meaning in each case will be clear from the context. The general term "gauge symmetry" denotes *any* type of local symmetry, whether it comes from diffeomorphisms or Lie group actions.

field transformations that leave the action invariant) and conserved quantities associated to *currents*.

Within the family of symmetries that leaves the Lagrangian invariant we distinguish between *global symmetries* and *local symmetries*. Local symmetries, or gauge symmetries, are the set of symmetries that can be parametrized using arbitrary functions on the spacetime, while global symmetries are transformations independent of the points in spacetime.

Gauge theories, by definitions, are those lagrangian theories that admit non-trivial local symmetries. These symmetries generate *gauge transformations*, which are linear maps from local function to the set of global symmetries of the Lagrangian. In the absence of boundaries, pure gauge-invariant theories (such as gravity) have no non-trivial global symmetries, and therefore via the Noether theorem the only conserved current is the trivial one. Therefore, boundaries are essential to the study of the conserved quantities. This leads us to the boundary conditions analysis.

Boundary conditions

The Euler-Lagrange equations are, usually, an intricate system of highly non-linear partial differential equations (PDE). Such is the case of general relativity, where the equations to solve are

$$\mathrm{Ric} - \frac{1}{2} R g = 8\pi T, \tag{1}$$

or Yang-Mills case,

$$d(*\mathcal{F}) + [\mathcal{A}, *\mathcal{F}] = 4\pi J. \tag{2}$$

As any PDE, we have to provide a domain on which this equations are solved and a set of suitable boundary conditions to have a well-defined structure in the solution space. The first step is to set the topology of the spacetime under consideration. In most of the radiative cases, one assume that the topology is that of asymptotically flat manifolds: each Cauchy slice is diffeomorphic to a spacial slice of Minkowski, outside a connected compact region. Within this compact region are the sources (black holes, neutron stars, charged objects) of the fields. In a neighbourhood of null infinity we have vacuum solutions.

The second ingredient are the boundary condition. The boundaries in a spacetime are located at infinity and in the source region, e.g. the near horizon region of black holes. In this thesis we will be interested in the former. The boundary at infinity can be regarded as part of the space after a compactification procedure. Thus, it acquires a physical meaning and posses a geometric structure. Since we are in a Lorentzian theory, the structure of infinity is not as simple as in the Euclidean case. We have time (future and past), null (future and past) and spatial infinity, denoted by $i^{\pm}$, $\mathscr{I}^{\pm}$ and i^0 respectively. Upon fixing certain coordinates, which fixes partially the gauge, one can define the solution spaces by prescribing the decay rate of the fields near infinity. The space of null generator of $\mathscr{I}^{\pm}$ is called *celestial sphere*.

The choice of $\mathscr{I}^{\pm}$ as the boundaries has many purposes: it gives a geometrical definition of asymptotic flatness, definitions for incoming and outgoing radiation (whether it is gravitational or other) and provides a framework with a natural kinematical structure to define the $\mathcal{S}-$matrix for gravity, among others (see the seminal paper [2] by Penrose). Regarding the $\mathcal{S}-$matrix program, extensive research has been conducted since the first works by Ashtekar and collaborators, [3–6].

Radiative solutions and their symmetries

In order to study relevant families of solutions, certain frameworks have to be adapted. As we mention above, $\mathscr{I}$ is naturally suited to study radiative solutions.

In a gauge theory such as electromagnetism or Yang-Mills, radiative fields have very precise decays in a neighbourhood of $\mathscr{I}$, due to the finiteness of the energy integrals. Therefore, once a coordinate system has been selected, the fall offs can be immediately computed by imposing the vector potential to decay as $1/r$, the Euclidean distance to the origin, towards null infinity.

In gravity, the notion of energy is not locally but globally defined. This lead to the concept of *asymptotically flat spaces*, which tries to capture the essence of a localized system emitting gravitational radiation. Bondi, Metzner, van der Burg [7] and Sachs [8, 9], introduce a set of coordinates well suited to study such spaces, called *Bondi gauge*. In such coordinates, some of the metric coefficients are taken to vanish, and this imply that some of the gauge freedom is resolved. The surviving local symmetries (called *residual gauge symmetries*) are set to satisfy the decay prescription for the metric components that do not vanish. They are responsible for one of the main discoveries in those works: when decaying to a flat metric, far away from sources, the gravitational field not only exhibits Minkowski's isometries (the Poincaré group) but an infinite dimensional group, called the *BMS group*. This group can be seen as the Lorentz subgroup times (semi-directly) with the infinite abelian subgroup of *supertranslations*. This symmetries act on the radiative modes of the metric, which are located on null infinity, $\mathscr{I}^+ \cup \mathscr{I}^-$, leaving fixed the celestial sphere metric.

The key feature of this discovery is that any strengthening of the boundary conditions that restricts the supertranslations imply a cancelling of the radiative modes. In other words, the infinite dimensional BMS group and radiative degrees of freedom on the boundary are two faces of the same coin.

In terms of the symmetries in the solutions, the general covariance on general relativity imply that *any* diffeomorphic solution remains a solution of the equations of motion. But in any given generic solution, the group of exact symmetries contains only the identity: there are no Killing fields in an arbitrary solution to the Einstein equations. Nevertheless, the residual gauge symmetries can be shown to satisfy *asymptotically* the Killing equation,

$$\nabla_{(a}\xi_{b)} \overset{r\to+\infty}{\longrightarrow} 0, \tag{3}$$

which is the expected behaviour if one want to recover Poincaré. The BMS leaves fix the celestial metric, defined in the space of null generator of $\mathscr{I}^+$.

Noether theorems: what is conserved?

In terms of the Noether's theorems, one expect conserved quantities if in the presence of symmetries. This implies that the LGT or the BMS elements have charges associated to them. Nevertheless, the concept of charge is not a trivial one when dealing with gauge invariant theories. In the case of gravity, general covariance implies an ambiguity in the definitions of the Noether charges, since *every* symmetry is the trivial global symmetry. The generalized Noether theorem, proved by Barnich, Brandt and Henneaux in [10], allows to deal with codimension 2 forms carrying the conserved quantities.

The case of electromagnetism in Minkowski background is more clear: we have a gauge symmetry, parametrized by a function Λ, which leaves invariant the action. The Euler-Lagrange equations are a linear PDE system,

$$d * \mathcal{F} = 4\pi J, \tag{4}$$

which immediately implies that the conserved quantity (J) is a corner term. By imposing the minimal boundary condition that allow radiative degrees of freedom at $\mathscr{I}^+$, the possible Λ's reduce to $\Lambda(x) = \lambda(x) + o(r^0)$, where the leading function is independent of u. By taking $\lambda \equiv 1$, we have the total electric charge of the system,

$$Q = \int_{S^2} *\mathcal{F}. \tag{5}$$

In the general case of an arbitrary function on the sphere, the gauge freedom is what is called *Large Gauge transformations* (LGT). The charges are a generalization of the usual electric charge, the latter computed taking $\lambda(x) \equiv 1$. These charges can be shown to be conserved by Campiglia and Eyheralde [11] when going from $\mathscr{I}^-$ to $\mathscr{I}^+$.

In gravity, the charge definition is more subtle. With the techniques from covariant phase space formalism, in [12] Iyer and Wald define the charges associated to a Cauchy slice, in terms of the symplectic structure.

Link with quantum theory

The underlying structure of the symmetries in both general relativity and gauge theories is fundamental to the problem of quantization. The charges become observables whose canonical commutation with the fields generate the quantum symmetries.

In a quantum theory, one seeks an $\mathcal{S}$-matrix that contains the scattering information between in states and out states in the form of *scattering amplitudes*. The more symmetries we have for the $\mathcal{S}-$matrix, the more information we have about the quantum theory.

It is natural to ask whether the BMS group provides a symmetry of the $\mathcal{S}$-matrix. In the seminal work [13], Strominger showed how to map BMS^+ with BMS^-, and defined a diagonal group, $BMS^0 \subset BMS^+ \times BMS^-$, which satisfies that its infinitesimal generators commute with the $\mathcal{S}$-matrix. The explicit conservation law,

$$b^+\mathcal{S} - \mathcal{S}b^- = 0, \quad \forall (b^+, b^-) \in BMS^0 \tag{6}$$

can be written as a Ward identity between any given scattering amplitude with the same amplitude with a soft graviton insertion. It is in this point when the soft theorems for gauge bosons appeared as the link between asymptotic symmetries and the $\mathcal{S}-$matrix.

Roughly stated, the soft gauge boson theorem says the following: given an amplitude, the amplitude with one additional gauge boson exhibits a universal behaviour as the momentum ω of the added gauge boson tends to zero (is *soft*), in the sense that it can be written as the same amplitude times a factor of $1/\omega$. The factorization can be carried on to the next order in the soft factor, schematically,

$$\lim_{\omega\to 0}\mathcal{A}^{\pm}_{n+1}(q)=\left[\frac{1}{\omega}S_n^{(0)\pm}+S_n^{(1)\pm}+O(\omega)\right]\mathcal{A}_n, \tag{7}$$

where $\mathcal{A}^{\pm}_{n+1}$ is a scattering amplitude of n particles of $\pm$ helicity and one gauge boson of four momentum $(\omega,\vec{q})$, and $\mathcal{A}^{\pm}_{n+1}$ is the same scattering amplitude of n particles. $S_n^{(0)\pm}$ and $S_n^{(1)\pm}$ are the leading and subleading soft factors, respectively.

The leading soft theorems were discovered by Weinberg, [14]. The subleading soft photon by Low, [15] and the subleading soft graviton by Cachazo and Strominger, [16].

In [17], He, Lysov, Mitra and Strominger showed that if we assume BMS^0 is a symmetry group of the quantum $\mathcal{S}-$matrix, the Ward identities obtained in [13] are precisely Weinberg's leading soft graviton theorem. The same result was proven for massless QED, by He, Mitra, Porfyriadis and Strominger in [18], showing that the soft photon theorem implies a Ward identity for the $\mathcal{S}-$matrix and the charges associated to the LGT's. This opened the door for a sequence of results in both gravity and gauge theories regarding the connection between soft theorems and Ward identities for asymptotic symmetries acting in the $\mathcal{S}-$matrix, see e.g. [16,18–28] and references therein.

An asymptotic symmetry for the subleading soft graviton

One of the main results of the literature regarding soft theorems and asymptotic symmetries can be state as follows: in a gauge theory enhancing the symmetries of the $\mathcal{S}-$matrix leads to include more and more LGT, enlarging the asymptotic symmetries group. The price to pay is a relaxation of the boundary conditions.

With respect to gravity, there have been some efforts to enhance the Lorentz subgroup of BMS into an infinite dimensional group. Barnich and Troessaert [29] proposed an extension into the conformal transformations group on the two dimensional sphere. This extension carries the introduction of fields with singularities, focusing only on local properties of symmetries. The new group includes all Virasoro transformations, which applied to the celestial sphere were called *superrotations*, in analogy with the supertranslations. In [30] it was argued that the subleading soft graviton theorem ([16]) can provide Ward identities associated to an extended version of the BMS group. However, due to the singular behaviour of the local conformal Killing vectors, it was not clear how to obtain the subleading soft theorem form Ward identities.

Campiglia and Laddha [21, 22] introduced and extension based on the soft theorems for gravitons (regarding the infrared behaviour of the fields near null infinity), allowing the group of diffeomorphisms Diff(S^2) to be Lorentz's group extension. As we have stated before, the more symmetries we add, the more relaxed the boundary conditions have to be. A natural question to ask is what kind of solutions are the ones invariant under the generalized BMS action. In other words, how many degrees of freedom can we add on $\mathscr{I}$, before exhausting the generalized BMS symmetries. The answer, my friend, is *not* blowin' in the wind, but it is actually in the celestial sphere metric.

As it was showed in [21], the residual gauge transformations satisfy

$$\nabla_a \xi^a \overset{r \to +\infty}{\longrightarrow} 0, \tag{8}$$

and the leading order in the metric is now a dynamical variable. This implies that the metric on the celestial sphere is no longer fixed. The usual condition in the radiative space is that the metric on the celestial sphere is the standard round metric. With the above condition, the possible metrics are such that the area element is the same as the round one. Therefore, the metric is not fixed and can be taken as a "boundary field".

The superrotation charges associated to the entire null infinity were computed in [22], restricting on the valuation of variations to the case of the round metric on the celestial sphere. Nevertheless, the charge algebra does not Poisson-closes, and therefore a more detailed analysis is needed. In [31] the surface charges associated to finite cuts of $\mathscr{I}^+$ were computed using covariant phase space methods. Their algebra was computed, showing extension terms. In the case of charges on the entire null infinity, i.e., the fluxes of the surface charges, transformation properties of the supermomentum imply that there *should not* be any central extensions.

In [32], we show that an extension of the phase space of gravity at null infinity can be defined such that the superrotation group (Diff(S^2)) acts canonically. This result is the content of chapter 1.

By considering Eintein-Yang-Mills theory, we test the compatibility of the previous extension in other contexts. In chapter 2 we show an extension of the phase space of gravity coupled to a non-abelian gauge theory where the generalized BMS group acts canonically.

Following the same logic as with the gauge theories, in [25, 33] it is argued that the sub^2-leading soft graviton is equivalent to the conservation of asymptotic charges associated to a new class of vector fields not contained in the generalized BMS. Further improvement in this direction has been made by Freidel, Pranzetti and Raclariu, see [28] and references therein.

The phase space extension problem

We arrive at the main subject of this thesis: the study of the phase space extensions in the classical theories due to the enlargement of the asymptotic symmetries groups.

As we have seen, the extra symmetries are not free, they have a price that the relaxation of the boundary conditions. In general, this implies that the variational principle cannot be translated with the new boundary condition to the old fields. Two main problems arise when relaxing the boundary conditions. First, as it was shown in [24], divergences in the asymptotic symplectic structure arise, and therefore some kind of projection or renormalization procedure is needed. Second, even after the asymptotic structure can be defined, the charges can be divergent.

Regarding the first problem, several techniques can be found in the literature to handle divergences in the symplectic structure. At spatial infinity, in [34] it was shown that under parity condition for the fields, a suitable boundary term can be added to the gravitational Hamiltonian, reproducing Poincaré algebra. At null infinity, in [31] it was shown that a renormalization of the symplectic potential can be done, such that the dynamics on $\mathscr{I}^+$ can be well defined as a variational problem.

The second problem can be reformulated as to what is the meaning of the relaxation of the boundary conditions at $\mathscr{I}$. The case of radiative data is clear, since the energy flux reaching $\mathscr{I}$ is non-vanishing, and there is non-trivial brackets between the charges. In most of the cases, when dealing with LGT in gravity and the fixing of the boundary conditions, the most artistic part of the research begins. Given the symmetries we want to include, the recursive application of infinitesimal variations on the metric eventually stabilizes or produce increasingly larger solution spaces, see e.g. [35,36]. A natural question is then whether the subsequent enlargement of the phase space can be understood in terms of a clear method.

Covariant phase space formalism

Covariant phase space methods are useful in this particular set ups, [12,37,38]. In [39], Freidel and Donnelly proposed a general procedure to associate a gauge-invariant classical phase space to a spatial slice with boundary by introducing new degrees of freedom on the boundary. Generally speaking, a boundary in a Cauchy slice breaks the gauge invariance of the theory, since there are now residual gauge transformations that do not leave fixed the boundary conditions. Such is the case of the GBMS, where an arbitrary diffeomorphism changes the metric in the celestial sphere, or the case of $O(r)$ LGT in Yang-Mills. The boundary degrees of freedom transform under a group of surface symmetries, and a counter term is added to the symplectic potential in order to restore gauge-invariance. The residual gauge symmetries are then view as generators of the surface symmetries. Recently, the existence of an universal symmetry group for boundaries ("corners") at both finite distance and null infinite, called the *extended corner symmetry group*, has been studied (e.g. [40–47]. The canonical representation of the different types of large gauge symmetries acting on the boundary, in particular supertranslations ([46]), has provided insights in the symplectic structure of the phase space of local subsystems.

A missing piece in the literature is the connection between the extension proposed in [32], where the supertranslations and superrotations act canonically, and the results obtained regarding the extended corner symmetry. In chapter 3 we will provide this missing piece.

Sub^n-leading soft photon/gluon/graviton theorems and large$\underline{r}$ gauge transformations

There are still much more soft theorems from which try to construct asymptotic symmetries and charges, both in gravity and in gauge theories.

In [20], it was shown that the subleading soft photon theorem can be interpreted as an infinitesimal symmetry of the $\mathcal{S}-$matrix, and explicit formulas for the charges were computed. In [24], it was shown that this charges can be obtained via the symplectic structure of the radiative phase space electromagnetism, and that they are associated to large $U(1)$ gauge transformations decaying as $O(r)$. In chapter 4 we will provide an answer regarding the sub-leading charges in Yang-Mills.

Beyond subleading charges, in the abelian case, sub^n-leading soft theorems (due to Hamada and Shiu in [48] and Li, Lin and Zhang in [49]) imply the conservation laws of new charges, [50]. Under this new perspective, large gauge transformations are the ones that generate this conserved charges (as it was shown for a particular set of conserved quantities by Seraj in [51] for electromagnetism and by Compère, Oliveri and Seraj in [52] for gravity), and therefore it is natural to ask more about the phase space structure that enables them to act canonically. As we will see in chapter 5, this can be done.

In the non-abelian case, the situation is more complicated. The non linearities imply that, if the charges are associated to LGT via a symplectic structure, then one has to be able to derive the complete hierarchy of charges, up to arbitrary order. In other words, if we want that the charge algebra resembles the variation algebra, then the mere presence of commutators imply an additive gradation in the charge algebra, schematically

$$[O(r^n), O(r^m)] = O(r^{m+n}). \tag{9}$$

The understanding of this structure remains an open problem. , and a partial answer with respect to the general case in the self-dual sector will be presented in chapter 6.

0.2 Original contributions

The content of this thesis is based on the following papers,

1. M. Campiglia and J. Peraza, *Generalized BMS charge algebra*, Phys. Rev. D **101** (2020) no.10, 104039, doi:10.1103/PhysRevD.101.104039, [arXiv:2002.06691 [gr-qc]].

2. M. Campiglia and J. Peraza, *Charge algebra for non-abelian large gauge symmetries at $O(r)$*, JHEP **12** (2021), 058, doi:10.1007/JHEP12(2021)058, [arXiv:2111.00973 [hep-th]].

3. J. Peraza, *Renormalized electric and magnetic charges for $O(r^n)$ large gauge symmetries* , to appear. Preprint: [arXiv:2301.05671 [hep-th]].

4. S. Nagy and J. Peraza, *Radiative phase space extensions at all orders in r for self-dual Yang-Mills and Gravity*, to appear. Preprint: [arXiv:2211.12991 [hep-th]].

We provide new results within the body of this thesis, which will appear in future works:

- Chapter 2: Einstein-Yang-Mills charge algebra containing the GBMS group.

- Chapter 3: The connection between the boundary terms in [32] with the ones proposed in the literature [43,46,53,54].

0.3 Conventions

Throughout this thesis, we assume that the base manifold $\mathcal{M}$ is a Lorentzian manifold, with signature $(-+++)$.

We denote spacetime indices by greek letters from middle alphabet, $\mu, \nu, \rho, \sigma, \ldots$. When working with coordinates on a Riemannian manifold (a sphere, for example), we will denote the indices with the first letters of the roman alphabet, $a, b, c, \ldots$.

Scalar fields will be denoted as $\phi, \varphi, \psi, \ldots$. Vector fields on $\mathcal{M}$ are denoted by $\chi, \xi, \ldots$, while vector fields on a 2-surface will be denoted by $V, W, Y, \ldots$

Covariant derivatives associated to metrics are denoted by ∇ for $\mathcal{M}$, D for a metric on the sphere, and ∂ for the round metric on the sphere.

A nomenclature difference with respect to the literature ([55], [31],etc.) is the definition of charge: what we call charge is usually the integral of the charge flux, whereas charge is a codimension 2 form ([12]). This change in denomination is due to the equivalence between Ward identities and the soft theorem statements regarding the scattering processes. We will discussed this item in chapter 3.

We take units where $G = c = 1$. The commutator convention we use in Yang-Mills theory is the following for the covariant derivative,

$$\mathcal{D} := \partial + [\mathcal{A}, \cdot] \tag{10}$$

Regarding notation used when integrating quantities on null infinity $\mathscr{I}^+$, we denote

$$\int_{\mathscr{I}} \omega, \quad \int_{\partial \mathscr{I}} \omega, \quad \int_{S^2} \omega, \tag{11}$$

the integrals on $\mathscr{I}$, on $\partial \mathscr{I} := \mathscr{I}^+_+ - \mathscr{I}^+_-$ and the celestial sphere S^2.

Part I

General Relativity

Asymptotic symmetries in gravity

In this chapter we present the results of [32], where we proved that an extended symplectic structure can be defined at null infinity on which Diff(S^2) acts canonically.

After reviewing Bondi coordinates and Bondi gauge, we introduce the main objects of our study: the residual gauge transformations that act on the solution space, and whose action lead to non-vanishing charges.

We review the structure of the BMS group, its action on solution space and its charges. Two different possible extensions have been proposed to the original BMS group, depending on which extension for the superrotations is chosen: one can consider the conformal Killing vector (CKV) on the sphere, or the whole group Diff(S^2). The action of both groups on the celestial metric is different, the later leading to non-trivial asymptotic behaviour and a new extension of the phase space [24].

1.1 Bondi coordinates

Throughout the thesis we will be using Bondi retarded coordinates, (u, r, x^a), where future null infinity is given by $\mathscr{I}^+ = \{r = +\infty\}$. These coordinates are useful in several applications, since they are well adapted to gravitational wave physics ([2,7–9]), from radiation going out a binary system to scattering processes from past to future null infinity. Advance coordinates (v, r, x^a) are taken when considering $\mathscr{I}^-$.

To construct this coordinates, let us consider a foliation of the spacetime M by a family of null hypersurfaces labeled by u. Their normal vector, $n^\mu = g^{\mu\nu}\partial_\nu u$, satisfies $n^\mu n_\mu = 0$ by definition, so this gives us the first fixing condition, $g^{uu} = 0$. Next, we define angular coordinates on the transverse two-dimensional Riemannian surfaces to the family of null hypersurfaces, which we will denote by x^a. The condition we impose is $n^\mu \partial_\mu x^a = 0$, which implies $g^{ua} = 0$.

Finally, we impose some condition on r: it satisfies the equation

$$\partial_r \left(\frac{\det g_{ab}}{r^4} \right) = 0. \tag{1.1}$$

The condition given in [7–9] was $\det g_{ab} = r^2 \det \mathring{q}_{ab}$, with $\mathring{q}_{ab}$ the standard metric on the round sphere S^2. Here, we use (1.1) since it allows Weyl rescalings.

A general expression of the metric satisfying the above gauge conditions can be written as follows,

$$g = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{ab}\left(dx^a - U^a du\right)\left(dx^b - U^b du\right), \tag{1.2}$$

where β, V, g_{ab} and U_a are functions of (u, r, x^a). The decay rates of each function near $\mathscr{I}^+$ will be computed by consistency with Einstein equations, once we define the boundary conditions.

To fix ideas, let see how Minkowski metric is represented in Bondi gauge. Take the coordinate change

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}, \quad u = r - x^0, \quad \hat{x}^i = \frac{x^i}{x^3 + r}, \tag{1.3}$$

where $i = 1, 2$. Then, the metric reads

$$\eta = -du^2 - 2dudr + r^2 \overset{\circ}{q} d\hat{x}^a d\hat{x}^b, \tag{1.4}$$

with $\overset{\circ}{q}$ the round metric in S^2. Observe that a hypersurface at constant r, by taking the limit $r \to +\infty$ tends to $\mathscr{I}^+ \cup \mathscr{I}^-$, while a hypersurface at constant u is 45° with respect to the $t = 0$ slice, and parametrizes the spheres at $\mathscr{I}^+$.

If we want to work in a neighbourhood of $\mathscr{I}^-$, the coordinate we have to take is the retarded time

$$v = r + x^0, \tag{1.5}$$

instead of u. The correspondent Bondi gauge functions for the metric (1.4) are,

$$\beta = 0, \quad V = -2r, \quad g_{ab} = r^2 \overset{\circ}{q}_{ab}, \tag{1.6}$$

with $\overset{\circ}{q}_{ab}$ the round metric on the sphere. This gives us already some boundary conditions candidates to start identifying the asymptotically flat spacetimes. As it is showed in Figure 1.1, the decays we will take are such that on the exterior region of certain compact set of a Cauchy slice, the fields are close to Minkowski solution.

Bondi gauge provides well adapted coordinates to represent $\mathscr{I}^+$, since it is parametrized by $\{(u, x^a)\}$ and hast the topology of $\mathbb{R} \times S^2$. Its two boundaries, denoted $\mathscr{I}^+_\pm$, correspond to the spheres at $u = \pm\infty$. Then, $\partial\mathscr{I}^+ = \mathscr{I}^+_- \cup \mathscr{I}^+_+$, with orientation provided by the vector ∂_u.

The *celestial sphere* is defined as the space of null generators to $\mathscr{I}^+$, denoted as S_∞. It can be showed ([5]) that S_∞ has the topology of S^2, and can be parametrized by $\{x^a\}$.

1.2 Residual Gauge Transformations

Fixing the gauge reduces the possible diffeomorphisms acting on the solution to give another solution, in the sense that they have to preserve the form of (1.2). The residual gauge transformations are the remaining diffeomorphisms that have not been fixed by the gauge choice. In other words, diffeomorphisms that preserve the gauge fixing conditions.

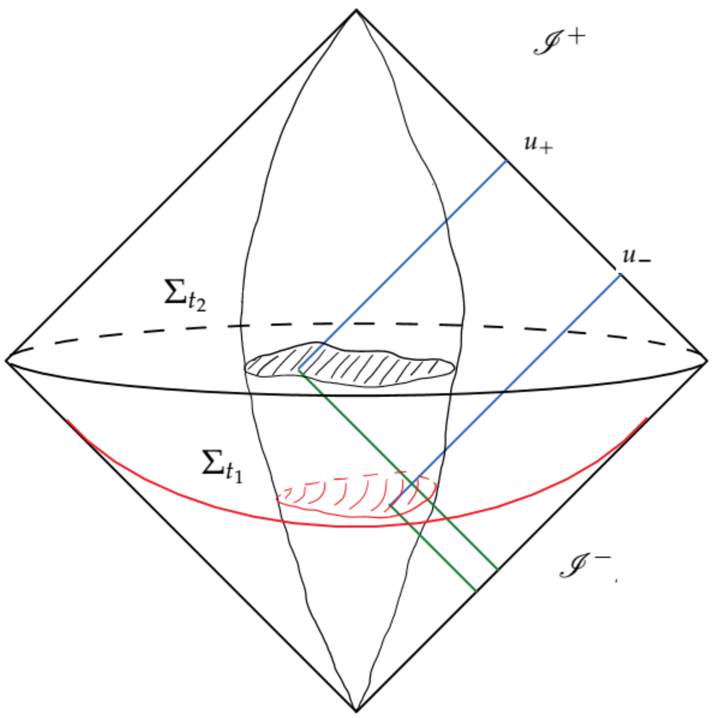

Figure 1.1: Asymptotically flat spacetimes. $\mathscr{I}^{\pm}$ indicate the conformal infinity, and scattering processes (indicated by green and blue lines) occur in the confines of a compact region on each Σ_t Cauchy slice.

In the particular case of Bondi gauge, we are looking for diffeomorphisms generated by a vector field ξ^μ such that

$$\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{ra} = 0, \quad g^{ab}\mathcal{L}_\xi g_{ab} = 4c(u, x^a), \tag{1.7}$$

where c is a function. This equations can be solved [55] as follows,

$$\xi^u = F(u, x^a), \tag{1.8}$$

$$\xi^a = V^a(u, x^a) + I^a, \quad I^a = -\partial_b F \int_r^\infty e^{2\beta} g^{ab} dr, \tag{1.9}$$

$$\xi^r = -\frac{r}{2}\left(\nabla_a^g V^a - 2c(u, x^a) + \nabla_a^g I^a - \partial_b F U^b + \frac{1}{2} F g^{-1} \partial_u g\right), \tag{1.10}$$

where $g = \det(g_{ab})$ and ∇^g is the Levi-Civita connection associated to g. F and c are two free function on $\mathscr{I}$, and V^a is a vector field on the sphere for each u. [1] The boundary conditions will provide us new constraints to the possible residual gauge transformations, since they will imply certain fall off for the field near $\mathscr{I}^+$.

[1]Not to be confused with V appearing in (1.2), which is a function of (r, u, x^a) in the component uu of the metric.

1.3 Solution space

We want to define a phase space structure on solutions spaces. The first step on this direction is to define a precise set of solutions from which one can construct jet bundles, giving certain differential structure.

In this work, we will be dealing with *asymptotically flat* spacetimes. They are central in the understanding of isolated systems and their gravitational radiation, e.e. in the case of binaries systems [56]. Outside regions with sources we have a vacuum spacetime and therefore we can assume that the further away we are from them, the more Minkowskian the metric is. This limit to vacuum metric could be at various rates, which will contain information regarding the sources.

In the case of scattering of gravitational or any other type or radiation, we can use the notion of asymptotically flatness as it is shown in figure 1.1: incoming radiation (green) interacts with sources inside certain region of spacetime. At $\mathscr{I}^+$ the "observers" receive the outgoing radiation (blue).

1.3.1 Boundary conditions

The formal definition of asymptotic flatness is the following: outside a compact region of some Cauchy slice in a spacetime, $\mathcal{K} \subset \Sigma$, we can map $\Sigma \setminus \mathcal{K}$ to the set $R^3 \setminus B$ with the euclidean metric, for some ball B, via a diffeomorphism. Within $\mathcal{K}$ there could be in principle any matter source. Outside $\mathcal{K}$, the matter content vanishes, and therefore we are in vacuum space, so the equation of motion is simply

$$R_{\mu\nu} = 0. \tag{1.11}$$

The previous definition implies certain fall-offs for the metric coefficients in Bondi gauge, compatible with (1.6), which resembles the fall offs for the functions of the Kerr family [57],

$$\beta = o(r^0), \quad \frac{V}{r} = o(r^2), \quad U^a = o(r^0), \quad g_{ab} = r^2 q_{ab} + r C_{ab} + o(r), \tag{1.12}$$

where q_{ab} and C_{ab} are functions of (u, x^a). By a conformal rescaling by the function $e^\psi = \frac{1}{r}$, we can define a metric q_{ab} on each $S^2 \subset \mathscr{I}^+$. The pull back of the Bondi metric to this compactified space gives

$$q_{ab} dx^a dx^b \tag{1.13}$$

on $\mathscr{I}^+$, and therefore a degenerate metric (no du terms). The 2-dimensional metric q_{ab} can be thought as the metric on the celestial sphere S_∞: by fixing a null normal $\mathbf{n}$ on $\mathscr{I}^+$, we can define a tangential $\mathbf{t}$, whose flow is parametrized by the coordinate u.

Regarding the nature of the metric q_{ab}, we can distinguish two different types of prescriptions [58]:

- Asymptotically flat case (AF): (1.12) approaches Minkowski metric as $r \to +\infty$. This implies

$$q_{ab} = \overset{\circ}{q}_{ab}. \tag{1.14}$$

- Asymptotically locally flat (ALF)[2]: the local volume form is asymptotically Minkowski,

$$\sqrt{q} = \sqrt{\overset{\circ}{q}}. \tag{1.15}$$

Both definitions have a characterization in terms of the gauge symmetries that we are imposing. Indeed, by taking the fall off and the residual gauge transformations computed in section 1.2, we see that the residual gauge symmetries for the AF solutions in Bondi gauge satisfy [55]

$$\nabla_\mu \xi_\nu \to 0 \quad (r \to +\infty), \tag{1.16}$$

which geometrically states that the residual gauge diffeomorphisms are asymptotically Killing fields, having a clear conceptual meaning. On the other hand, the residual gauge symmetries in the ALF case are the ones that satisfy [23],

$$\nabla_\nu \xi^\nu \to 0 \quad (r \to +\infty), \tag{1.17}$$

which are the asymptotically divergence-free vector fields, preserving the volume form in the (compactified) transversal spheres. As we will show in this chapter, this volume preserving condition has deep consequences in the symplectic structure of the phase space.

1.3.2 Solution space

To obtain a parametrization of the solution space, we are solving Einstein equations, while imposing the gauge fixing conditions. Since we want the *asymptotic behaviour* of the solutions near $\mathscr{I}^+$, or in other words, the decay rates far enough of any localized source, we need to compute recursively the coefficients of the functions in (1.12).

Consider the following expansion in r of the transversal metric,

$$g_{ab} = r^2 q_{ab} + r C_{ab} + C^{(0)}_{ab} + \frac{1}{r} C^{(-1)}_{ab}, \tag{1.18}$$

with q_{ab} satisfying the ALF condition. The gauge fixing condition on the luminosity coordinate r (1.1) impose several identities for the traces of the successive terms in (1.18), [55,58],

$$\begin{aligned} q^{ab} C_{ab} &= 0 & (1.19)\\ C^{(0)}_{ab} &= \frac{1}{4} q_{ab} C^{cd} C_{cd} + \mathcal{C}^{(0)}_{ab}(u, x^a), & (1.20)\\ C^{(-1)}_{ab} &= \frac{1}{2} q_{ab} \mathcal{C}^{(0)}_{ab} C_{cd} + \mathcal{C}^{(-1)}_{ab}(u, x^a), & (1.21)\\ \dots & \quad , \end{aligned}$$

[2] The definition of ALF involves more generally the topology of S_∞ and is used in other contexts, e.g. [59]. Here we use it as a minimal extension of the AF case.

where $\mathcal{C}^{(n)}_{ab}$ can be thought of as functions on $\mathscr{I}^+$, with $q^{ab}\mathcal{C}^{(0)}_{ab} = q^{ab}\mathcal{C}^{(-1)}_{ab} = 0$. We will denote as D_a the Levi-Civita connection associated to q_{ab}, and indices $a, b, c, ...$ will be assumed to be lowered and raised with q_{ab}.

Next, we have to solve the Einstein vacuum equations, (1.11). Here we present the results, the reader can find a more detail exposition in [55], [58]. Each component of the Ricci tensor gives certain relations between the functions,

- $R_{rr} = 0$: the radial equation establishes a relation between β and the coefficients $C_{ab}, \mathcal{C}^{(0)}_{ab}, \mathcal{C}^{(-1)}_{ab}, ...,$ with an integration constant fixed to zero so that the leading order of $e^{2\beta}$ is 1,

$$\beta = -\frac{1}{32r^2}C^{ab}C_{ab} + O(1/r^3) \tag{1.22}$$

- $R_{ra} = 0$: determines (completely) the $r-$expansion of U^a in terms of $C_{ab}, \mathcal{C}^{(0)}_{ab}, \mathcal{C}^{(-1)}_{ab}, ...$ and an integration constant $N^a(u, x^a)$, known as the *Bondi angular momentum aspect*. The first terms of the expansion are,

$$U^a = -\frac{1}{2r^2}D_bC^{ab} - \frac{2}{3r^3}\left(N^a - \frac{1}{2}C^{ab}D^cC_{bc} - \frac{1}{3}D_b\mathcal{C}^{(0)ab}\right) - \frac{2\ln r}{3r^3}D_b\mathcal{C}^{(0)ab} + O(1/r^4) \tag{1.23}$$

To avoid logarithmic terms, it must be imposed $D_b\mathcal{C}^{(0)ab} = 0$, [29]. The definition of N^a as integration constant varies in the literature, [55,60,61],

$$N_a^{FN} = N_a + \frac{1}{4}C_{ab}D_cC^{bc} + \frac{3}{32}D_a(C_{bc}C^{bc}) \tag{1.24}$$

$$N_a^{HPS} = N_a^{FN} - uD_aM, \tag{1.25}$$

where FN stands for Flanagan-Nichols and HPS for Hawking-Perry-Strominger, and M is the Bondi mass aspect, defined next.

- $R_{ab} = 0$: the trace, $g^{ab}R_{ab}$ is equivalent (due to the gauge condition $g^{uu} = g^{ua} = 0$ and the previous two equations) to $R_{ur} = 0$, which gives V in terms of g_{ab} and an integration constant, M, called *Bondi mass aspect*,

$$\frac{V}{r} = -r\partial_u \ln\sqrt{q} - \frac{1}{2}R[q] + \frac{2M}{r} + O(1/r^2) \tag{1.26}$$

The other couple of equations coming form $R_{ab} = 0$ imply a dynamical condition on the metric g_{ab} in terms of the conformal factor,

$$\partial_u q_{ab} = (\partial_u \ln\sqrt{q})q_{ab} \tag{1.27}$$

By the ALF condition, $\sqrt{q} = \sqrt{\mathring{q}}$, and therefore $\partial_u q_{ab} = 0$, which implies that the metric on the celestial sphere is $u-$independent.

- $R_{uu}, R_{ua} = 0$: They imply evolution equations for the Bondi mass and angular momentum aspects,

$$\partial_u M = -\frac{1}{8}N_{ab}N^{ab} + \frac{1}{4}D_a D_b N^{ab} + \frac{1}{8}D_a D^a R[q] \tag{1.28}$$

$$\begin{aligned}\partial_u N^a = & D_a M + \frac{1}{16}D_a(N_{bc}C^{bc}) - \frac{1}{4}N^{bc}D_a C_{bc} - \frac{1}{4}D_b(C^{bc}N_{ac} - N^{bc}C_{ac}) \\ & -\frac{1}{4}D_b D^b D^c C_{ac} + \frac{1}{4}D_b D_a D_c C^{bc} + \frac{1}{4}C_{ab}D^b R[q]\end{aligned} \tag{1.29}$$

Similar evolution equations can be computed for N_a^{FN} and N_a^{HPS} using (1.24) and (1.25).

Observe that C_{ab} is a completely arbitrary input for the equations. We can give C_{ab} a geometrical interpretation by considering the congruence of null geodesics with constant u, which asymptotically (as $r \to 0$) reaches $\mathscr{I}^+$, [62]. Consider the derivative of the generator of the null congruence, ∂_r, in the compactified spacetime,

$$\tilde{\nabla}_\mu(\partial_r)_\nu = \partial_\mu(\delta^u_\nu g_{ur}) - \tilde{\Gamma}^u_{\mu\nu}g_{ur}, \tag{1.30}$$

and take the transversal part with respect to the null congruence,

$$\tilde{\nabla}_a(\partial_r)_b = -\tilde{\Gamma}^u_{ab}g_{ur} = \frac{1}{2}\partial_r g_{ab} - \frac{1}{r}g_{ab} = -\frac{1}{2}C_{ab} + o(1), \tag{1.31}$$

thus, C_{ab} is proportional to the shear of the congruence of geodesics, and thus indicates the focusing due to curvature in the path of the geodesics. As such, it encodes the two polarizations present in gravitational waves. Its time derivative,

$$N_{ab} = \partial_u C_{ab}, \tag{1.32}$$

is known as the *Bondi news tensor*, which measure the energy flux. It is worth to mention that Raychadhuri's equation for the null congruence is equivalent to the radial equation $R_{rr} = 0$ for β, (1.22)

We are in conditions to define the solution spaces. In the case of AF conditions, we consider the following set of initial conditions, viewed as fields on $\mathscr{I}$,

$$\mathcal{S}_{BMS} = \{g_{\mu\nu}[\mathring{q}_{ab}, C_{ab}, M, N^a, ..., C^{(n)}_{ab}, ...] | R_{\mu\nu}[g] = 0\}. \tag{1.33}$$

In the case of ALF, we consider,

$$\mathcal{S}_{GBMS} = \{g_{\mu\nu}[q_{ab}, C_{ab}, M, N^a, ..., C^{(n)}_{ab}, ...] | R_{\mu\nu}[g] = 0, \sqrt{q} = \sqrt{\mathring{q}}\}. \tag{1.34}$$

Observe that $\mathcal{S}_{BMS}$ is a proper subset of $\mathcal{S}_{GBMS}$. The names *BMS* and *GBMS* will be explained in the next section.

1.4 Asymptotic symmetries

The residual gauge symmetries act on the solution space we defined in the last subsection, by taking one solution to another one. This group is what is called the *asymptotic symmetries group*, initially discovered by Bondi , Metzner [7] and Sachs [8], and called the BMS group. This action changes the functions M and N^a, which are the mass and the angular momentum aspects respectively. This implies that this group of diffeomorphisms generates non-trivial charges on the phase space of solutions, [4].

A renew interested on BMS appear when Barnich and Troessaert [55], inspired by 2-d CFT, defined an extended version of the BMS, which includes an infinite dimensional "superrotation" group, generated by infinitesimal local conformal transformations. On the ground of the equivalence between asymptotic symmetries and subleading soft graviton theorems [16], a different extension of the BMS group was proposed by Campiglia and Laddha [21] such that the solution space where the group acts is $\mathcal{S}_{GBMS}$.

In subsection 1.4.1 we review the construction of the standard BMS group, which correspond to the AF boundary conditions. In section 3.5 we present the construction of the generalized BMS group (GBMS).

1.4.1 BMS action review

Due to the preservation of the asymptotic decays for β, V and U^a, we can deduce some conditions on the functions F and V^a from section 1.2, by imposing

$$\mathcal{L}_\xi g_{ur} = \mathcal{L}_\xi g_{ua} = O(1/r^2). \tag{1.35}$$

These equations give,

$$\partial_u F = \frac{1}{2} D_a V^a - c, \quad \partial_u V^a = 0. \tag{1.36}$$

The last equation establishes that V^a is indeed a vector on the sphere, while the first equations is a linear ODE,

$$F = \sqrt[4]{q}\left(f(x^a) + \frac{1}{2}\int_{-\infty}^{u} \frac{1}{\sqrt[4]{q}}(D_a V^a - 2c)du \right). \tag{1.37}$$

These are the most general diffeomorphisms compatible with teh decays (1.12), parametrized by two sphere fields, f and V^a, and one $\mathscr{I}$ function, c. Next, we impose the AF condition. By computing the equation $\mathcal{L}_\xi g_{ab} = r^2 \delta_\xi g_{ab} + O(r)$,

$$\delta_\xi q_{ab} = \mathcal{L}_V q_{ab} - (D_c V^c - 2c) q_{ab}, \quad \delta_\xi \sqrt{q} = 4c\sqrt{q}. \tag{1.38}$$

Since $q_{ab} = \overset{\circ}{q}_{ab}$ is fixed, the Weyl rescalings $q_{ab} \mapsto e^{2c} q_{ab}$ are excluded from the group. Also, $\mathcal{L}_V q_{ab} - D_c V^c q_{ab} = 0$ imply that V^a are the conformal Killing vectors (CKV) of the sphere. This group is isomorphic to the (proper orthochronous) *Lorentz group*, SO(3,1), which generates the Lorentz algebra $\mathfrak{so}(3,1)$.

Thus, F simplifies, and we can integrate the first equations in (1.36),

$$F = f + u\frac{1}{2}D_a V^a =: f + u\alpha, \tag{1.39}$$

absorbing $\sqrt[4]{q}$ in f, with the explicit u-linear behaviour and the definition $\alpha := \frac{1}{2}D_a V^a$. The asymptotic vectors generating the diffeomorphims are given by a pair (f, V^a) of celestial sphere fields. We will denote ξ_f and ξ_V the diffeomorphisms generated by f and V respectively, and schematically write

$$\begin{aligned} \xi_f &= f\partial_u + ..., & (1.40)\\ \xi_V &= V^a\partial_a + u\alpha\partial_u - r\alpha\partial_r + ..., & (1.41) \end{aligned}$$

where ... indicates $O(r^{-1})\partial_a + O(r^{-1})\partial_u + O(1)\partial_r$. We can compute the algebra straightforward from the Lie bracket between two diffeomorphism generators, giving,

$$[\xi_f, \xi_{f'}] = 0, \quad [\xi_V, \xi_f] = \xi_{V(f)}, \quad [\xi_V, \xi_{V'}] = \xi_{[V,V']}, \tag{1.42}$$

where $[\cdot,\cdot]$ denotes the Lie bracket and $V(f) = V^a\partial_a f - \alpha f$.

The diffeomorphisms generated by f are known as *supertranslations*, since they correspond to the angle dependent shift $u \mapsto u + f(x^a)$ in the time direction on $\mathscr{I}^+$. As it was showed in [8], the group of asymptotic symmetries BMS contains the Poincaré group of Minkowski,

$$ISO(3,1) = SO(3,1) \ltimes \mathfrak{t} < SO(3,1) \ltimes \mathfrak{s} = BMS_4, \tag{1.43}$$

where $\mathfrak{t}$ is the four-dimensional abelian group of translations, and $\mathfrak{s}$ is the infinite dimensional abelian group of supertranslations [3].

1.4.2 Generalizations of BMS

The first attempts in extending the BMS group to a larger group was done by Barnich and Troessaert ([29,55,63]), where it was proposed to maintain the CKV equation, $\delta_V q_{ab} = 0$, except at a finite number of points on the celestial sphere. This led to the extension of BMS_4 by meromorphic superrotations,

$$EBMS_4 = (\text{Diff}(S^1) \times \text{Diff}(S^1)) \ltimes \mathfrak{s}^*, \tag{1.44}$$

where $\mathfrak{s}^*$ is the abelian ideal generated by the new supertranslations, which by consistency must also admit poles.

Based on the equivalence between the subleading soft graviton theorem [16, 64, 65] and Ward identities for the $\mathcal{S}-$matrix, a second proposal for the extension was given by Campiglia and Laddha, [21], where they consider the group of diffeomorphisms on the sphere, Diff(S^2), instead of the finite dimensional Lorentz group $SO(3,1)$. This lead to the generalized BMS

[3] To prove that ISO(3,1) is indeed a subgroup, and not only a subset, one has to show that $\mathfrak{t}$ maps to a finite dimensional ideal in $\mathfrak{s}$, [8]

group (GBMS),

$$GBMS_4 = \text{Diff}(S^2) \ltimes \mathfrak{s}, \tag{1.45}$$

where now we have an arbitrary diffeomorphism on the superrotation sector. In other words, $\text{Diff}(S^2)$ are the new symmetries that the subleading soft graviton theorem provides to the $\mathcal{S}-$matrix via the Ward identity. This implies that the action on the leading part of the boundary metric is not trivial,

$$\delta_V q_{ab} = \mathcal{L}_V q_{ab} - 2\alpha q_{ab} \neq 0, \tag{1.46}$$

and therefore the celestial sphere metric is no longer fixed. However, it can be showed that

$$\delta_V \sqrt{q} = 0, \tag{1.47}$$

which together with $\delta_f q_{ab} = 0$ shows that this group is exactly the *asymptotic symmetries group* of $\mathcal{S}_{GBMS}$! Observe that the Weyl rescalings are still excluded. The algebra of vector remains the same as in (1.42)

This enlargement of the symmetries imply a non-trivial structure for the symplectic form. In particular, since we are varying the metric up to leading order, one expects (as it is the case) that the symplectic potential contains radial divergences, and therefore the computation of the charges will be ill-define due to divergences in the integrals [23].

The representation of the algebra on the solution space is the following,

$$\delta_f q_{ab} = 0, \qquad \delta_V q_{ab} = \mathcal{L}_V q_{ab} - 2\alpha q_{ab}, \tag{1.48}$$
$$\delta_f C_{ab} = f N_{ab} - 2D_a D_b f^{TF}, \quad \delta_V C_{ab} = \mathcal{L}_V C_{ab} - \alpha C_{ab} + u\alpha N_{ab} - 2u D_a D_b \alpha^{TF}, \tag{1.49}$$
$$\delta_f N_{ab} = f\partial_u N_{ab}, \qquad \delta_V N_{ab} = \mathcal{L}_V N_{ab} + u\alpha\partial_u N_{ab} - 2D_a D_b \alpha^{TF} \tag{1.50}$$

where X_{ab}^{TF} for a symmetric tensor X_{ab} denotes the *trace free* tensor component,

$$X_{ab}^{TF} := X_{ab} - \frac{1}{2} q_{ab} X_c^{\ c}. \tag{1.51}$$

The actions on the Bondi mass aspect M and momentum aspect N^a are the following,

$$\begin{aligned}
\delta_{(f,V)} M &= [F\partial_u + \mathcal{L}_Y + 3\alpha] M + \frac{1}{4} D_a D_b \alpha C^{ab} \\
&\quad + \frac{1}{4} D_a F D^a R[q] + \frac{1}{4} N^{ab} D_a D_b F + \frac{1}{2} D_a F D_b N^{ab},
\end{aligned} \tag{1.52}$$

$$\begin{aligned}
\delta_{(f,V)} N_a &= [F\partial_u + \mathcal{L}_Y + 2\alpha] N_a + 3M D_a F - \frac{3}{16} D_a F N_{bc} C^{bc} + \frac{1}{2} D_b F N^{bc} C_{ac} \\
&\quad - \frac{1}{32} D_a \alpha C_{cd} C^{cd} + \frac{1}{4}(D^b F R[q] + D^b D_c D^c F) C_{ab} - \frac{3}{4} D_b F (D^b D^c C_{ac} - D_a D_c C^{bc}) \\
&\quad + \frac{3}{8} D_a (D_c D_b F C^{bc}) + \frac{1}{2} D_a D_b F^{TF} D_c C^{bc},
\end{aligned} \tag{1.53}$$

with $F = f + u\alpha$.

1.5 Phase space structure on $\mathcal{S}_{GBMS}$

In this section we review the definition of the radiative phase space, and later we provide several results concerning the action of Diff(S^2) on the solution space, and will define the corresponding phase space for GBMS.

1.5.1 Radiative phase space

Radiative phase space [4,5] consists in taking the action of the BMS group in Bondi frame, and relax a little bit the decay for C_{ab},

$$\Gamma_{rad} := \{\hat{C}(u,x)|\mathring{q}^{ab}\hat{C}_{ab} = 0, \quad \partial_u\hat{C}_{ab} \overset{u\to\pm\infty}{=} O(1/|u|^{1+\epsilon}), \quad D_{[a}D^b\hat{C}_{b]c} \overset{u\to\pm\infty}{=} 0\} \supset \Gamma_{\mathring{q}_{ab}}. \tag{1.54}$$

On this space, one can actually compute a (finite) symplectic form from the Einstein-Hilbert lagrangian,

$$\Omega(\delta,\delta') = \int_{\mathscr{I}} \delta N^{ab} \wedge \delta' C_{ab}\sqrt{q}dud^2x \tag{1.55}$$

One can compute the charges associated to δ_f and δ_V straightforwardly, by using the definition,

$$\Omega(\delta,\delta_V) = \delta Q_{\xi} \tag{1.56}$$

By imposing the decays in (1.54), the non-integrable terms vanish (see chapter 3), and the expressions for the charges are

$$P^0_f = \int_{\mathscr{I}} \partial_u C^{ab}\delta_f C_{ab}\sqrt{q}dud^2x, \tag{1.57}$$

for the supermomentum, and

$$J^0_V = \int_{\mathscr{I}} \partial_u C^{ab}\delta_V C_{ab}\sqrt{q}dud^2x \tag{1.58}$$

for the super angular momentum. These charges can be shown to form a closed Poisson algebra, resembling (1.42),

$$\{P^0_{f_1}, P^0_{f_2}\} = 0, \quad \{J^0_V, P^0_f\} = P^0_{V(f)}, \quad \{J^0_{V_1}, J^0_{V_2}\} = J^0_{[V_1,V_2]}. \tag{1.59}$$

1.5.2 Decays in $u \to \pm\infty$ for $\mathcal{S}_{GBMS}$

The first step in understanding the solution space (1.34), is to see the effect of the extension from $SO(3,1)$ to Diff(S^2). We define a *Bondi frame* as the solutions in which the celestial metric is that of the round metric, $q_{ab} = \mathring{q}_{ab}$, [66].

In [31] it is shown that one needs an $u-$independent tensor, the Geroch tensor, to appropriately parametrize the gravitational field at $\mathscr{I}^+$. The proof consists in taking a finite diffeomorphism and studying its action on the complex plane coordinates. Here, we follow the proof given in [32].

First, we show the presence of a $u-$linear part in C_{ab}, by the successive action of variations due to a vector field V on the shear C_{ab}. Let us start with Minkowski spacetime. We have the following metric coefficients,

$$\beta = 0, \quad U^a = 0, \quad \frac{V}{r} = -1, \quad g_{ab} = \mathring{q}_{ab}, \tag{1.60}$$

and consider the infinitesimal action of an arbitrary diffeomorphism generated by the functions (f, V),

$$\begin{aligned} \delta_f q_{ab} &= 0, & \delta_V q_{ab} &= \mathcal{L}_V \mathring{q}_{ab} - 2\alpha \mathring{q}_{ab}, & (1.61)\\ \delta_f C_{ab} &= -2\mathring{D}_a \mathring{D}_b f^{TF}, & \delta_V C_{ab} &= -2u \mathring{D}_a \mathring{D}_b \alpha^{TF} & (1.62)\end{aligned}$$

where $\mathring{D}$ is the Levi-Civita connection of $\mathring{q}$. Now, we have a shear $C_{ab} + \delta_V C_{ab}$ linear in u. If we act again with another δ_W, we will also have a linear term in u. Then, the general form of the shear will be

$$C_{ab} = \hat{C}_{ab}(u, x^a) + u T_{ab}(x^a), \tag{1.63}$$

where $\hat{C}_{ab}(u, x^a)$ is at most $O(1)$ and T is independent of u. The fall off in u that we take for $\hat{C}_{ab}$ are the following,

$$\partial_u \hat{C}_{ab} = O(1/|u|^{2+\epsilon}). \tag{1.64}$$

This fall offs for the shear $\hat{C}_{ab}$ is such that the expressions (1.57) and (1.58) remain to be finite, since they are the ones coming from the leading soft graviton theorem, and are also compatible with a $O(1)$ subleading soft theorem, [21]. However, they are too restrictive for a generic gravitational scattering process, where the fall offs are given taking $\epsilon = 0$ (quadratic decay), which corresponds to the logarithmic subleading soft theorem [67, 68]. As we will see below, the tensor T_{ab} can be constructed entirely form q_{ab} and vanishes in Bondi frame.

By inspecting the variations δ_f, δ_V on C_{ab}, we arrive at the transformations for $\hat{C}_{ab}$ and T_{ab},

$$\begin{aligned} \delta_V \hat{C}_{ab} &= \mathcal{L}_V \hat{C}_{ab} - \alpha \hat{C}_{ab} + u\alpha \hat{N}, & (1.65)\\ \delta_V T_{ab} &= \mathcal{L}_V T_{ab} - 2D_a D_b \alpha^{TF} + \cancelto{0}{u^2 \alpha \partial_u T_{ab}}, & (1.66)\\ \delta_f \hat{C}_{ab} &= f\partial_u \hat{C}_{ab} - 2D_a D_b f^{TF} + f T_{ab} & (1.67)\\ \delta_f T_{ab} &= 0, & (1.68)\end{aligned}$$

which agrees with the fact that T_{ab} depends only on q_{ab}, since for the variations to stabilize (preserve the fall off (1.64)) it is necessary to impose $\partial_u T_{ab}$ (we will show this explicitly in the next section).

We will also require that $\hat{C}_{ab}$ is asymptotically flat as $u \to \pm\infty$, [13], in the sense that the

Weyl tensor vanishes approaching $\mathscr{I}^+_\pm$. In a Bondi frame, this condition implies

$$\lim_{u\to\pm\infty} D_{[a}D^b\hat{C}_{b]c} = 0, \tag{1.69}$$

which we will call vanishing of the *magnetic part* of $\hat{C}_{ab}$. Observe that since the magnetic part vanishes, the functions $\hat{C}^\pm_{ab}$ are Hessians of a certain scalar function $\hat{C}^\pm$ in each end $\mathscr{I}^+_\pm$, i.e.,

$$\hat{C}^\pm_{ab} = -2(D_aD_b\hat{C}^\pm)^{TF} = \overset{0}{S}{}^{\hat{C}^\pm}_{ab} \tag{1.70}$$

This condition have to be extended to non-Bondi frames, which is our task for the next subsections. Given a metric q_{ab} on the sphere, we can now define the *radiative phase space* that is compatible with the GBMS group,

$$\Gamma_q := \{\hat{C}(u,x^a)|q^{ab}\hat{C}_{ab} = 0, \quad \partial_u\hat{C}_{ab} \overset{u\to\pm\infty}{=} O(1/|u|^{2+\epsilon}), \quad \text{mag}(\hat{C}_{bc}) \overset{u\to\pm\infty}{=} 0\}, \tag{1.71}$$

where mag() denotes the magnetic part of the tensor, which remains to be defined. The full GBMS pahse space,

$$\Gamma_{GBMS} := \{\hat{C}(u,x^a), q_{ab}(x^a)|q^{ab}\hat{C}_{ab} = 0, \ \partial_u\hat{C}_{ab} \overset{u\to\pm\infty}{=} O(1/|u|^{2+\epsilon}), \ \text{mag}(\hat{C}_{bc}) \overset{u\to\pm\infty}{=} 0, \ \sqrt{q} = \sqrt{\mathring{q}}\}. \tag{1.72}$$

1.5.3 Finite action of Diff(S^2)

In this section we will show that T_{ab} only depends on q_{ab}, by explicitly constructing the orbit of Bondi frames under the action of Diff(S^2). We also provide some results that will be useful for the next sections and chapter 3.

Consider a diffeomorphism $\phi : (M, g) \to (M, g_{bf})$ such that preserves the Bondi gauge, not infinitesimally, but finitely,

$$\phi(r,u,x^a) = (R, U, X^A), \tag{1.73}$$

where R, U, X^A are the initial coordinates, in a Bondi frame with metric g_{bf}, and the metric g in coordinates (r, u, x^a) being the pullback of g_{bf},

$$\langle\xi,\chi\rangle_g := \langle\nabla_\xi\phi, \nabla_\chi\phi\rangle_{g_{bf}}. \tag{1.74}$$

By the decays we are assuming on the metric, $g_{ab} = q_{ab}r^2 + ...$, and equation (1.74), we can assume the following $1/r$-expansions,

$$\begin{aligned} R &= R^{(1)}(x)r + R^{(0)}(u,x) + O(1/r), && (1.75)\\ U &= U^{(1)}(x)u + O(1/r), && (1.76)\\ X^A &= \phi^A(x) + \frac{1}{r}X^{(-1)A}(u,x), && (1.77) \end{aligned}$$

where the leading orders do not depend on u due to $\partial_u q_{ab} = 0$, and $g_{rr} = g_{ra} = 0$. Also, it can be seen that $U^{(1)} = \frac{1}{R^{(1)}}$.

The angular part of g_{ab} is given by

$$\begin{aligned} g_{ab}(r,u,x) &= r^2(R^{(1)})^2 \frac{\partial\phi^C}{\partial x^a}\frac{\partial\phi^D}{\partial x^b} q_{CD} + \\ & r\left(R^{(1)}\partial_a\phi^A\partial_b\phi^B(C_{ab}(R^{(1)-1}u,\phi) + 2R^{(1)}q_{AB}(\phi) + R^{(1)}X^{(-1)C}\partial_C q_{AB}(\phi))\right. \\ & \left. + 2R^{(1)2}\partial_a\phi^A\partial_b X^{(-1)B}q_{AB}(\phi) + 2R^{(1)-2}\partial_a R^{(1)}\partial_b R^{(1)}\right) + ..., \end{aligned} \tag{1.78}$$

where we will denote as q^{ϕ}_{ab} and C^{ϕ}_{ab} as the leading and subleading term, respectively. We will obtain expressions for both quantities in terms of ϕ.

The determinant condition (1.15) implies $\det(q^{\phi}\mid_x) = \det(q\mid_x)$ (but, of course, not equal to $\det(q\mid_{\phi(x)})$!), and therefore

$$R^{(1)} = \frac{\det^{1/4}(q^{\phi})}{\det^{1/2}(J\phi)\det^{1/4}(q\circ\phi)}, \tag{1.79}$$

where $J\phi$ is the Jacobian matrix of ϕ. This fixes $R^{(1)}$. The subleading determinant condition, $q^{ab}_{\phi}C^{\phi}_{ab} = 0$, fixes $R^{(0)}$ in terms of the previous functions. Next, the pullback of g_{ra} gives

$$g_{ra}(r,u,x) = -u\partial_a \ln R^{(1)} + X^{(-1)}\frac{\partial\phi^B}{\partial x^a}(R^{(1)})^2 q_{AB} + O(1/r), \tag{1.80}$$

which should vanish. If we write

$$X^{(-1)A} = \partial_a\phi^A Y^a, \tag{1.81}$$

i.e., the pushforward of the sphere vector Y^a, then $g_{ra} = 0$ implies

$$Y^a = u q^{ab}_{\phi}\partial_a \ln R^{(1)}. \tag{1.82}$$

This implies, in particular, that we can construct the vector field generating the diffeomorphism on the sphere, Y^a, entirely in terms of q^{ϕ}_{ab}.

We can now compute C^{ϕ}_{ab} in terms of ϕ. By taking the covariant derivative of q^{ϕ}_{ab}, D^{ϕ}_{ab}, the expression simplifies,

$$C^{\phi}_{ab} = R^{(1)}\frac{\partial\phi^A}{\partial x^a}\frac{\partial\phi^B}{\partial x^b}C_{AB}(\phi) + 2u\left(D^{\phi}_a \ln R^{(1)} D^{\phi}_b \ln R^{(1)} + D^{\phi}_a D^{\phi}_b \ln R^{(1)}\right)^{TF}. \tag{1.83}$$

First, observe the extra term that is linear in u, as we saw it has to be the case in the last section. Let us define

$$T_{ab} := 2\left(D^{\phi}_a \ln R^{(1)} D^{\phi}_b \ln R^{(1)} + D^{\phi}_a D^{\phi}_b \ln R^{(1)}\right)^{TF}, \quad \psi := \ln R^{(1)}, \tag{1.84}$$

and observe that ψ is the conformal factor that restore the area identity, for any diffeomorphism, rescaling the metric,

$$q^{\phi}_{ab}(x) = e^{2\psi}\partial_a\phi^A\partial_b\phi^B q_{AB}(\phi(x)). \tag{1.85}$$

In fact, this appearance of a Weyl transformation is not just a coincidence: first, we are in two dimensions, so every riemannian metric is in the Yamabe class of the round metric, or, in other terms, every riemannian metric is conformal to $\mathring{q}$. Second, right from the beginning, we are taking a slice in Diff$(S^2) \ltimes \mathcal{W}$, with $\mathcal{W}$ the Weyl subgroup, such that the Weyl parameter is completely determined by the diffeomorphism (given by α), and therefore not entering in the dynamics of the phase space.

The curvature for the metric (1.85),

$$R(q^{\phi}) = 2(e^{-2\psi} - \Delta^{\phi}\psi), \tag{1.86}$$

where $\Delta^{\phi} := e^{-2\phi}\Delta$.

The tensor T_{ab} defined above is the *Geroch tensor* and satisfies the following property, the proof of which the reader can see [69],

Theorem 1.1. Geroch, '76

Let S be a two-dimensional surface. Then there is a unique symmetric tensor T_{ab} that satisfies

$$\text{If } \rho_{ab} = \frac{R}{2}q_{ab} - T_{ab}, \text{then} \quad D_{[a}\rho_{b]c} = 0 \tag{1.87}$$

Proof. □

In a two dimensional metric, (1.87) is equivalent to

$$D_a R = -2D^b T_{ab}, \tag{1.88}$$

which can be verified taking $T_{ab} = 2\left(D_a\psi D_b\psi + D_a D_b\psi\right)^{TF}$ and R as in (1.86).

Finally, we present the following result concerning the tangent space in $\mathcal{S}_{GBMS}$, regarding the variations of the metric.

Proposition 1.1. Any variation δq_{ab} can be written as $\delta_W q_{ab}$, with W^a a vector field.

Proof. From the formula for q^{ϕ}_{ab}, we have that the infinitesimal transformation (for any diffeomorphism) is

$$q^{\epsilon\phi}_{ab} = 2\delta R^{(1)}_{(\epsilon\phi)}\delta_a{}^C\delta_b{}^D q_{CD} + \epsilon\frac{\partial\phi^C}{\partial x^a}\delta_b{}^D q_{CD} + \epsilon\frac{\partial\phi^D}{\partial x^b}\delta_a{}^C q_{CD}, \tag{1.89}$$

where $R^{(1)}_{(\epsilon\phi)}$ is the $R^{(1)}$ coefficient for the diffeomorphism $\epsilon\phi$. Taking $\epsilon \to 0$, the raising and lowering of index can be done in the metric q_{CD},

$$\delta_{\phi}q_{AB} = \left(2\delta R^{(1)}_{(\epsilon\phi)} + \frac{\partial\phi^C}{\partial x^A} + \frac{\partial\phi^D}{\partial x^B}\right)q_{CD}. \tag{1.90}$$

Next,

$$R^{(1)}_{(\epsilon\phi)} = 1 + \epsilon \left(\frac{1}{2} \partial_A \phi^A + \frac{1}{2} \phi^A \partial_A q_{CD} \right) \tag{1.91}$$

Finally, given a general metric q_{ab} and a general variation, generated by diffeomorphism ϕ, we define $W^a := \phi^a$ and

$$\delta_\phi q_{ab} = \mathcal{L}_W q_{ab} + 2\alpha_W q_{ab} = \delta_W q_{ab} \tag{1.92}$$

□

The above proposition tells us that there is a map

$$X : \mathcal{F} \to \mathrm{Diff}(S^2), \tag{1.93}$$

from the jet bundle formed by the manifold and $\mathcal{S}_{GBMS}$ to the diffeomorphisms on the sphere (which is embedded in $\mathscr{I}$), and that the differential of the map takes a variation δq_{ab} and gives the vector field W^a. For each tangent in the solution space, there is an associated diffeomorphism, and therefore we will have in general a field-dependent expressions for the variations. Observe also that the directions contain in the tangent subspaces to $\mathcal{S}_{BMS}$ are in the kernel of the differential of the map X. This map will play a central role in chapter 3.

1.5.4 Diff(S^2)-covariant derivative

Schematically, from (1.71) and (1.71) we have the following structure on our phase space,

$$\Gamma_{GBMS} := \bigcup_{q_{ab} := \sqrt{q} = \sqrt{\mathring{q}}} \Gamma_{q_{ab}}, \tag{1.94}$$

where each $\Gamma_{q_{ab}}$ is the orbit of the solution with fixed q_{ab} under the Lorentz symmetries generated by the CKV fields. A general diffeomorphism will change the metric, and therefore change the "fibre" in the above definition. The idea of this section is to construct a *covariant derivative* that allows to write equations covariantly on the fibres.

First, we have to define what does it means to be covariant under the Diff(S^2) group. We say that a $u-$independent tensor $T^{a_1\cdots}_{b_1\cdots}$ on the celestial sphere is *Diff(S^2) k-covariant* if it satisfies the transformation rule,

$$\delta_V T^{a_1\cdots}_{b_1\cdots} = \mathcal{L}_V T^{a_1\cdots}_{b_1\cdots} + k\alpha T^{a_1\cdots}_{b_1\cdots}, \tag{1.95}$$

where V is the generator of the diffeomorphism, and $\alpha = \frac{1}{2} D_a V^a$. As one can see for a Diff(S^2) k-covariant scalar field, the standard covariant derivative associated to the metric q_{ab} does not preserves the covariance of a field: if $\delta_V \phi = (\mathcal{L}_V + k\alpha)\phi$, then

$$\delta_V D_a \phi = (\mathcal{L}_V + k\alpha) D_a \phi + k\phi D_a \alpha \tag{1.96}$$

As the first step towards the definition of the new covariant derivative, for ψ in (1.84) we have,

$$\delta_V \psi = \mathcal{L}_V \psi - \alpha, \tag{1.97}$$

which can be seen by identifying $\delta_V(e^{2\psi}\overset{\circ}{q}_{ab})$ with $\delta_V q_{ab}$. Then, we define for scalars the Diff$(S^2)-$covariant derivative $\bar{D}$ as follows

$$\bar{D}_a \phi := D_a \phi + k\phi D_a \psi. \tag{1.98}$$

From this definition, we can construct iteratively the action on higher order tensors. The Christoffel symbols for the conformal change $q^{\psi}_{ab} = e^{2\psi} q_{ab}$ are in the general formula for a tensor $T^{a_1 \dots}_{b_1 \dots}$,

$$\bar{D}_c T^{a_1 \dots}_{b_1 \dots} = \partial_c T^{a_1 \dots}_{b_1 \dots} - \tilde{\Gamma}^{d}_{cb_1} T^{a_1 \dots}_{d \dots} + \dots + \tilde{\Gamma}^{a_1}_{cd} T^{d \dots}_{b_1 \dots} + \dots + k D_c \psi T^{a_1 \dots}_{b_1 \dots}, \tag{1.99}$$

where $\tilde{\Gamma}^c_{ab} = \Gamma^c_{ab} - 2D_{(a}\psi \delta^c_{b)} + q_{ab} D^c \psi$. Given a Diff$(S^2)$ $k-$covariant tensor, a straightforward computation now gives

$$\delta_V \bar{D}_c T^{a_1 \dots}_{b_1 \dots} = (\mathcal{L}_V + k\alpha) T^{a_1 \dots}_{b_1 \dots}. \tag{1.100}$$

Several properties for $\bar{D}$ are enumerated below.

1. The weight of the product of tensors is the sum of the weights: $\bar{D}$ satisfies Leibniz rule with weight k.

2. $\bar{D}_a q_{bc} = 0$.

3. $[\bar{D}_a, \bar{D}_b]\omega_c = \bar{R}_{abc}{}^{d} \omega_d$, with

$$\bar{R}_{abcd} = \bar{R} q_{a[b} q_{c]d}, \quad \bar{R} = R + 2\Delta\psi. \tag{1.101}$$

4. $\bar{D}_a \bar{R} = 0$. This is a restatement of the Uniformization Theorem for 2 dimensions: any Riemannian metric is conformally equivalent to $\overset{\circ}{q}$, and therefore the curvature is Diff(S^2)-covariantly constant.

5. The covariantized vector field divergence, $\bar{\alpha} = \frac{1}{2}\bar{D}_a V^a$, satisfies

$$\bar{\alpha} = -\delta_V \psi \tag{1.102}$$

$$(\bar{D}_a \bar{D}_b \bar{\alpha})^{TF} = -\frac{1}{2}(\delta_V T_{ab})^{TF} \tag{1.103}$$

1.5.5 Phase space

In this section we give a precise definition for (1.94) by extending (1.69) to non-Bondi frames. The basic idea is to perform a "change of variables" such that we can split the full phase space in a Diff(S^2) covariant way. First, we need to understand the magnetic condition for $\hat{C}_{ab}$ in terms of $\bar{D}$. From the equation for $\delta_V \hat{C}_{ab}$, we have that $\hat{C}^{\pm}$ are Diff(S^2) (-1)-covariant tensors, then

$$\bar{D}_{[a} \bar{D}^c \hat{C}^{\pm}_{b]c} = D_{[a} D^c \hat{C}^{\pm}_{b]c} - \frac{1}{2} T^c_{[a} \hat{C}^{\pm}_{b]c}. \tag{1.104}$$

Then, the vanishing of the left hand side is the covariantize version of the magnetic condition in Bondi frame.

Let us redefine each fiber $\Gamma_{q_{ab}}$ as follows,

$$\Gamma_{q_{ab}} = \{\hat{C}_{ab}| \quad q^{ab}\hat{C}_{ab} = 0, \quad \partial_u \hat{C}_{ab} \overset{u\to\pm\infty}{=} O(1/|u|^{2+\epsilon}), \quad \bar{D}_{[a}\bar{D}^c\hat{C}^{\pm}_{b]c} \overset{u\to\pm\infty}{=} 0\}, \tag{1.105}$$

as the "vertical" phase space corresponding to each metric q_{ab} such that $\sqrt{q} = \sqrt{\mathring{q}}$. Then, (1.94) is well-defined as a set, ready to admit a symplectic structure.

1.6 Charge Algebra

In this section we give expressions for the supertranslation and superrotation that are compatible with a symplectic structure on (1.94). We have two charges to compute: the ones associated to the supertranslation, and the ones associated to superrotations.

1.6.1 GBMS charges

To make easy the reading, let us define the following tensors,

$$\overset{0}{N}_{ab} := \int_{\mathbb{R}} \partial_u \hat{C}_{ab} du \tag{1.106}$$

$$\overset{1}{N}_{ab} := \int_{\mathbb{R}} u\partial_u \hat{C}_{ab} du \tag{1.107}$$

$$\overset{0}{S}{}^f_{ab} := -2\bar{D}_a\bar{D}_b f^{TF} \tag{1.108}$$

$$\overset{1}{S}{}^V_{ab} := [-4\bar{D}_a\bar{D}_b\alpha + \bar{D}_{(a}\bar{D}^c\delta_V q_{b)c} - \frac{\bar{R}}{2}\delta_V q_{ab}]^{TF} \tag{1.109}$$

Supermomentum

In the original introduction of the general frames, (e.g. [69], [5]), special care is taken to ensure frame-independence. In particular,the Ashtekar-Streubel expression for the supermomentum is valid in any frame, given the correct decay in the u^0 part of the shear C_{ab}. Therefore, the supermomentum in a general non-Bondi frame takes the form,

$$P_f = \int_{\mathscr{I}} \partial_u \hat{C}^{ab}\delta_f \hat{C}_{ab}\sqrt{q}du d^2x, \tag{1.110}$$

where it should be noted that the difference with (1.57) is that we are not taking the linear in u part in the shear. Observe that this integral converges. We can split the contributions in (1.57) given by the *hard* and *soft* parts,

$$P_f^{hard} := \int_{\mathscr{I}} \partial_u \hat{C}^{ab} f \partial_u \hat{C}_{ab}\sqrt{q}du d^2x, \tag{1.111}$$

$$P_f^{soft} := \int_{S^2} \overset{0}{N}_{ab}\overset{0}{S}{}^f_{ab}\sqrt{q}d^2x, \tag{1.112}$$

where the terms hard and soft come from the soft theorems: quadratic and linear terms in C_{ab} respectively.

Super angular momentum

The proposal in [21], [22] for an asymptotic GBMS symmetry provided a candidate for the super angular momentum in the Bondi frame can be covariantized to be defined in all of the set Γ,

$$J_V = \int_{\mathscr{I}} \partial_u \hat{C}^{ab} \delta_V \hat{C}_{ab} \sqrt{q} du d^2x + \int_{S^2} \overset{1}{N}{}^{ab} \overset{1}{S}{}^V_{ab} \sqrt{q} d^2x. \quad (1.113)$$

Thus, we have a candidate for the super angular momentum in Γ. As with the supermomentum, we can split the hard and soft contributions,

$$J_V^{hard} := \int_{\mathscr{I}} \partial_u \hat{C}^{ab} \delta_V \hat{C}_{ab} \sqrt{q} du d^2x, \quad (1.114)$$

$$J_V^{soft} := \int_{S^2} \overset{1}{N}_{ab} \overset{1}{S}{}^V_{ab} \sqrt{q} d^2x, \quad (1.115)$$

Poisson structure

We have defined the charges P_f and J_V on Γ_{GBMS}. Now, we should verify that they satisfy indeed the structure of a Poisson algebra. Consider a large gauge transformation parametrized by λ, and let us denote by Q_λ its associated charge. If there exists a symplectic form Ω on Γ such that

$$\delta Q_\lambda = \Omega(\delta, \delta_\lambda) \quad (1.116)$$

then the charges must obey the following identity,

$$\delta_{\lambda_2} Q_{\lambda_1} = \Omega(\delta_{\lambda_2}, \delta_{\lambda_1}) = -\delta_{\lambda_1} Q_{\lambda_2}. \quad (1.117)$$

Therefore, for the charge candidates to be consistent with a symplectic structure on Γ (and therefore for the Poisson structure to close), several consistency checks are in order:

1. $\delta_{f'} P_f = 0, \quad \forall f, f' \in C^\infty(S^2)$.
2. $\delta_V P_f = -P_{V(f)}, \quad \forall V \in \mathcal{X}(S^2), f \in C^\infty(S^2)$.
3. $\delta_f J_V = P_{V(f)}, \quad \forall V \in \mathcal{X}(S^2), f \in C^\infty(S^2)$.
4. $\delta_{V'} J_V = J_{[V,V']},, \quad \forall V, V' \in \mathcal{X}(S^2)$.

After a little work, it can be shown 1, 2 and 4 are satisfied for the expressions we have for P_f and J_V (see Appendix D in [32]).

Condition 3, however, is not true. A direct computation gives

$$\delta_f J_V = P_{V(f)} + \int_{S^2} \overset{0}{N}{}^{ab} \left((\mathcal{L}_V - \alpha) \overset{0}{S}{}^f_{ab} - f \overset{1}{S}{}^V_{ab} - \overset{0}{S}{}^{V(f)}_{ab} \right) \sqrt{q} d^2x. \quad (1.118)$$

In [32] we propose a way to fix this non-closure of the algebra, by adding certain boundary field term. The boundary term will be generated by a boundary field that ultimately depend on q_{ab}, since is the new "dynamical" degree of freedom which depart from the standard BMS analysis. First, let us define

$$K(f,V) := \int_{S^2} \overset{0}{N}{}^{ab} \left((\mathcal{L}_V - \alpha)\overset{0}{S}{}^{f}_{ab} - f\overset{1}{S}{}^{V}_{ab} - \overset{0}{S}{}^{V(f)}_{ab} \right) \sqrt{q} d^2x \tag{1.119}$$

This term is the non-CKV generalization of the Barnich-Troessaert (BT) extension term [29]. We will discuss the comparison with BT charges in the next chapter. Since it is a non zero term, we see that condition 3 does not apply for the expressions we have for P_f and J_V. As it will turn out, an extra term for the super angular momentum will be obtained from $K(f,V)$, in order for the equation to hold.

After a lengthy computation, (Appendix C in [32]), one can show ,

$$K(f,V) = -\delta_f J_V^{\partial\mathscr{I}} + \mathrm{mag}(f,V), \tag{1.120}$$

where

$$J_V^{\partial\mathscr{I}} = \int_{\partial\mathscr{I}} (V^a \hat{C}^{bc} D_c \hat{C}_{ab} + \frac{3}{2}\bar{\alpha}\hat{C}^{ab}\hat{C}_{ab})\sqrt{q}d^2x, \tag{1.121}$$

$$\mathrm{mag}(f,V) = -4\int_{S^2} \overset{0}{N}{}^{ab} \left(\bar{D}_{(a}\bar{D}^c(\bar{D}_{[b)} f V_{c]} - \frac{1}{2} f \bar{D}_{[b)} V_{c]}) \right)^{TF} \sqrt{q} d^2x. \tag{1.122}$$

where $\partial\mathscr{I} := \mathscr{I}^+_- \cup \mathscr{I}^+_+$. Since the magnetic condition for $\hat{C}_{ab}$ (1.69) implies

$$D^{[c} D_a \overset{0}{N}{}^{b]a} = 0, \tag{1.123}$$

the term $\mathrm{mag}(f,V)$ vanishes. The term $\delta_f J_V^{\partial\mathscr{I}}$ is a total variation of a boundary term, which can be absorbed in the left hand side, as a redefinition $J_V \mapsto J_V + J_V^{\partial\mathscr{I}}$, and condition 3 now is satisfied. Condition 4 is true, as can be easily verified that

$$\delta_{V'} J_V^{\partial\mathscr{I}} = J_{[V,V']}^{\partial\mathscr{I}}. \tag{1.124}$$

In the proof of the next proposition we show another way of writing $J_V^{\partial\mathscr{I}}$.

Proposition 1.2. $J_V^{\partial\mathscr{I}}$ vanishes for global CKV.

Proof. Observe that

$$\overset{0}{N}_{ab} = \hat{C}^+_{ab} - \hat{C}^-_{ab}, \tag{1.125}$$

then we have $K(f,V) = K^+(f,V) - K^-(f,V)$

$$K^{\pm}(f,V) := \int_{S^2} \hat{C}^{\pm}_{ab} \left((\mathcal{L}_V - \alpha)\overset{0}{S}{}^{f}_{ab} - f\overset{1}{S}{}^{V}_{ab} - \overset{0}{S}{}^{V(f)}_{ab} \right) \sqrt{q} d^2x. \tag{1.126}$$

We also have $J_V^{\partial\mathscr{I}} = J_V^{\partial\mathscr{I}+} - J_V^{\partial\mathscr{I}-}$,

$$J_V^{\partial\mathscr{I}\pm} = \int_{S^2} (V^a \hat{C}^{\pm bc} D_c \hat{C}^{\pm}_{ab} + \frac{3}{2}\bar{\alpha}\hat{C}^{\pm ab}\hat{C}^{\pm}_{ab})\sqrt{q}d^2x, \tag{1.127}$$

and identity (1.120) transfers to $K^{\pm}(f, V) = -\delta_f J_V^{\partial\mathscr{I}\pm}$. Since $J_V^{\partial\mathscr{I}\pm}$ is quadratic in $C^{\pm}$, we have

$$J_V^{\partial\mathscr{I}\pm} = \frac{1}{2}\delta_f J_V^{\partial\mathscr{I}\pm}|_{f=C^{\pm}} = -\frac{1}{2}K^{\pm}(C^{\pm}, V). \tag{1.128}$$

Finally,

$$J_V^{\partial\mathscr{I}} = -\frac{1}{2}\left(K^+(\hat{C}^+, V) - K^-(\hat{C}^-, V)\right) \tag{1.129}$$

□

Finally, we have a new expression for the super angular momentum, given by

$$\boxed{J_V = J_V^{hard} + J_V^{soft} + J_V^{\partial\mathscr{I}},} \tag{1.130}$$

while the supermomentum remains the same,

$$\boxed{P_f = P_f^{hard} + P_f^{soft}.} \tag{1.131}$$

The charge algebra formed by this charges is a closed Poisson algebra, with no central extension.

1.7 Symplectic form

We want to construct a symplectic form on Γ_{GBMS} such that (i) it is consistent with the definition of the charges, (1.116),

$$\delta P_f = \Omega(\delta, \delta_f), \quad \forall \delta \in T\Gamma, \tag{1.132}$$

$$\delta J_V = \Omega(\delta, \delta_V), \quad \forall \delta \in T\Gamma, \tag{1.133}$$

and (ii) when evaluated on the subspaces $T\Gamma_{q_{ab}} \subset T\Gamma$ it reduces to the symplectic form in each Ashtekar-Streubel (AS) phase space $\Gamma_{q_{ab}}$ (1.71), given by

$$\Omega(\delta, \delta')\Big|_{T\Gamma_{q_{ab}} \times T\Gamma_{q_{ab}}} = \int_{\mathscr{I}} (\delta\partial_u \hat{C}^{ab} \wedge \delta' \hat{C}_{ab})\sqrt{q}dud^2x \tag{1.134}$$

Our starting point is to write

$$\Omega = \Omega^{hard} + \Omega^{soft} + \Omega^{\partial\mathscr{I}}, \tag{1.135}$$

where we take as the hard part the AS expression,

$$\Omega^{hard} := \int_{\mathscr{I}} (\delta\partial_u \hat{C}^{ab} \wedge \delta' \hat{C}_{ab})\sqrt{q}dud^2x, \tag{1.136}$$

and $\Omega^{soft}, \Omega^{\partial\mathscr{I}}$ are yet to be determined. By using condition (1.132), we obtain

$$\Omega^{hard}(\delta, \delta_f) = \delta P_f^{hard} + \int_{S^2} \delta \overset{0}{N}{}^{ab} \overset{0}{S}{}^f_{ab} \sqrt{q} d^2x, \tag{1.137}$$

while from (1.133),

$$\Omega^{hard}(\delta, \delta_V) = \delta J_V^{hard}. \tag{1.138}$$

Then, it suffices to find $\Omega^{\partial\mathscr{I}}, \Omega^{soft}$ such that

$$(\Omega^{\partial\mathscr{I}} + \Omega^{soft})(\delta, \delta_f) = \int_{S^2} \overset{0}{N}{}^{ab} \delta \overset{0}{S}{}^f_{ab} \sqrt{q} d^2x, \tag{1.139}$$

$$(\Omega^{\partial\mathscr{I}} + \Omega^{soft})(\delta, \delta_V) = \delta J_V^{soft} + \delta J_V^{\partial\mathscr{I}} \tag{1.140}$$

Assume that equation (1.140) splits accordingly,

$$\Omega^{\cdots}(\delta, \delta_V) = \delta J_V^{\cdots}, \tag{1.141}$$

where ... denotes *soft* or $\partial\mathscr{I}$. We will also assume that each $\Omega^{\cdots}$ is the exterior derivative in solution space of a symplectic potential $\theta^{\cdots}$ [4], that satisfy

1. Compatibility with δ_V, e.g. [37], $\theta^{\cdots}(\delta_V) = J_V^{\cdots}$, and
2. $\delta_V \theta^{\cdots}(\delta) + \theta^{\cdots}([\delta, \delta_V]) = 0$.

By inspecting the formula for J_V^{soft} and $\overset{1}{S}{}^V_{ab}$, we have a candidate for θ^{soft} as follows: if we define

$$\overset{1}{S}_{ab}(\delta) := [2\delta T_{ab} + \bar{D}_{(a}\bar{D}^c \delta q_{b)c} - \frac{\bar{R}}{2}\delta q_{ab}]^{TF}, \tag{1.142}$$

we see that $\overset{1}{S}_{ab}(\delta_V) = \overset{1}{S}{}^V_{ab}$. Then,

$$\theta^{soft}(\delta) := \int_{\partial\mathscr{I}} \overset{1}{N}{}^{ab} \overset{1}{S}_{ab}(\delta)\sqrt{q}d^2x \tag{1.143}$$

For the remaining component, $\theta^{\mathscr{I}}(\delta)$, we have to rewrite the new term on the angular momentum, using (1.70),

$$J_V^{\partial\mathscr{I}} = \int_{\partial\mathscr{I}} (V^a \overset{0}{S}{}^{C\ bc} D_c \hat{C}_{ab} + \frac{3}{2}\bar{\alpha}\hat{C}^{ab} \overset{0}{S}{}^C_{ab})\sqrt{q}d^2x, \tag{1.144}$$

which after some algebra can be written as

$$J_V^{\partial\mathscr{I}} = \int_{\partial\mathscr{I}} \hat{C}^{ab}(\delta_V \overset{0}{S}{}^C_{ab} - \overset{0}{S}{}^{\delta_V C}_{ab} - C\overset{1}{S}(\delta_V))\sqrt{q}d^2x. \tag{1.145}$$

[4] $\Omega(\delta_1, \delta_2) = \delta_1\theta(\delta_2) - \delta_2\theta(\delta_1) - \theta([\delta_1, \delta_2])$

Then, we define

$$\theta^{\partial\mathscr{I}}(\delta) := \int_{\partial\mathscr{I}} \hat{C}^{ab}(\delta \overset{0}{S}{}^{C}_{ab} - \overset{0}{S}{}^{\delta C}_{ab} - C\overset{1}{S}(\delta))\sqrt{q}d^2x. \tag{1.146}$$

We have completed condition 1. For condition 2, we refer the reader to Proposition 1.1. By setting $\delta q_{ab} = \delta_W q_{ab}$ for some vector field W on the celestial sphere and using (1.124) one can show condition 2 to hold.

Finally, we have to check that 1.132 did not get spoiled after the new definition for $\Omega^{\cdots}$. This follows again from Proposition 1.1, the variation formula $\delta_f J_V = P_{V(f)}$ and the fact that $\theta^{\cdots}$ vanishes if evaluated on variations with $\delta q_{ab} = 0$.

We can write the symplectic potential and form as the sum of two contributions, one from the bulk $\mathscr{I}$ and one from the boundary $\partial\mathscr{I}$,

$$\theta = \theta^{\mathscr{I}} + \theta^{S^2}, \quad \Omega = \Omega^{\mathscr{I}} + \Omega^{S^2}, \tag{1.147}$$

where

$$\theta^{S^2}(\delta) := \theta^{soft}(\delta) + \theta^{\partial\mathscr{I}}(\delta) = \int_{S^2}(p^{ab}\delta q_{ab} + \Pi^{ab}\delta T_{ab})\sqrt{q}d^2x, \tag{1.148}$$

$$\Omega^{S^2}(\delta,\delta') := \Omega^{soft}(\delta,\delta') + \Omega^{\partial\mathscr{I}}(\delta,\delta') = \int_{S^2}(\delta p^{ab} \wedge \delta' q_{ab} + \delta\Pi^{ab} \wedge \delta' T_{ab})\sqrt{q}d^2x, \tag{1.149}$$

with

$$p^{ab} = D^{(a}D_c\overset{1}{N}{}^{b)c} - \frac{R}{2}\overset{1}{N}{}^{ab} + (\text{quadratic in } \hat{C}^{ab})\Big|_{\partial\mathscr{I}}, \quad \Pi^{ab} = 2\overset{1}{N}{}^{ab} + \frac{1}{2}CC^{ab}\Big|_{\partial\mathscr{I}}, \tag{1.150}$$

where the quadratic terms in $\hat{C}^{ab}$ can be obtained from (1.146). When evaluating on a variation such that $\delta q_{ab} = 0$, we have that the S^2 contributions vanish. This is a remarkable aspect of this expressions, but it is not a surprise: if we take a CKV field, the orbits are $\Gamma_{q_{ab}}$, and therefore we see only the AS structure on each fibre. In chapter 3 we will understand this boundary fields in a more general context, the covariant phase space formalism.

Asymptotic symmetries in Einstein-Yang-Mills theory

In the previous chapter the GBMS extension to BMS was considered, and the charge algebra was computed, imposing a well-defined Poisson structure. The new charges contain an extra boundary term, coming from the arbitrary diffeomorphism freedom on the sphere.

In this chapter, we study the following natural question: can GBMS be realized by coupling gravity with a gauge theory, such in Einstein-Yang-Mills theory (EYM)? In particular, adding an extra field (the gauge potential $\mathcal{A}_\mu$ in EYM) implies extra symmetries coming from the symmetries in the extra piece of the Lagrangian, [70]. In Minkowski flat spacetime, the large gauge transformations that act on $\mathcal{A}_\mu$ satisfy a closed infinite dimensional algebra [19,26], and they have leading charges associated to them which make up for a infinite dimensional Poisson charge algebra.

2.1 Asymptotic structure

In this section we review the main properties of EYM. A more detailed exposition can be found in [70] and references therein. The strategy is the same as in the previous chapter, so the details are skipped.

2.1.1 Symmetries of EYM

Given a Lorentzian manifold (M, g), a Lie group G, the Lie algebra $\mathfrak{g} := \text{Lie}(G)$ and a $\mathfrak{g}$-valued 1-form $\mathcal{A}_\mu$, the EYM Lagrangian is the following

$$\mathscr{L}[g, \mathcal{A}] = R[g]\sqrt{g} - \text{Tr}(\mathcal{F}[\mathcal{A}] \wedge *\mathcal{F}[\mathcal{A}]), \tag{2.1}$$

where Tr is an invariant non-degenerate metric in $\mathfrak{g}$, $*$ is the Hodge-dual of the metric g, and

$$\mathcal{F}[\mathcal{A}] := d_{\mathcal{A}}(\mathcal{A}), \quad d_{\mathcal{A}} := d + [\mathcal{A}, \cdot] \tag{2.2}$$

are the curvature of the connection $\mathcal{A}$ and the covariant derivative associated to $\mathcal{A}$, respectively. We will use $\mathcal{D}_\mu = \nabla_\mu + [\mathcal{A}_\mu, \cdot]$ as the covariant derivative using index notation. By the properties of the Hodge-dual, we can rewrite (2.1) as follows,

$$\mathscr{L}[g, \mathcal{A}] = \left(R[g] - \frac{1}{4}\mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \right) \sqrt{g}, \tag{2.3}$$

A *symmetry* ϕ acting on $\mathscr{L}$ is defined such that it satisfies,

$$\mathscr{L}[\phi * g, \phi * \mathcal{A}] = \mathscr{L}[g, \mathcal{A}] \tag{2.4}$$

We have two different symmetries acting on this lagrangian:

- Diffeomorphism invariance: given a diffeomorphism generator ξ, its infinitesimal action is given by
$$\delta_\xi g = \mathcal{L}_\xi g, \quad \delta_\xi \mathcal{A} = \mathcal{L}_\xi \mathcal{A}, \tag{2.5}$$
- G-invariance: given a finite gauge transformation generated by the infinitesimal parameter Λ,
$$\delta_\Lambda g = 0, \quad \delta_\Lambda \mathcal{A} = d_\mathcal{A}\Lambda, \tag{2.6}$$

The symmetries acting on the unconstrained (before fixing any particular gauge) fields form a symmetry algebra, which can be computed through its representation on the fields g and $\mathcal{A}$. The variation algebra is given by,

$$[\delta_{\xi_1,\Lambda_1}, \delta_{\xi_2,\Lambda_2}] = \delta_{\hat{\xi},\hat{\Lambda}}, \tag{2.7}$$

with

$$\hat{\xi} = [\xi_1, \xi_2]_{\mathrm{Lie}}, \tag{2.8}$$
$$\hat{\Lambda} = \mathcal{L}_{\xi_1}\Lambda_2 - \mathcal{L}_{\xi_2}\Lambda_1 + [\Lambda_1, \Lambda_2]_\mathfrak{g}, \tag{2.9}$$

where $[,]_\mathfrak{g}$ denotes the Lie bracket on $\mathfrak{g}$. This equations have an algebra realization that corresponds to the group

$$G_{EYM} := Diff(M) \ltimes G, \tag{2.10}$$

where the bracket between two elements is given by

$$[(\xi_1, \Lambda_1), (\xi_2, \Lambda_2)]_{EYM} = (\hat{\xi}, \hat{\Lambda}) \tag{2.11}$$

2.1.2 Gauge fixing and residual gauge transformations

For the metric field, we consider the Bondi gauge, as in (1.2). The choice that we will take for the vector potential is the following,

$$\mathcal{A}_r = 0 \tag{2.12}$$

since it will prove to be more easy treat when computing the residual gauge symmetries [70]. We will discussed the harmonic gauge below.

Once the gauge is fixed, we are left with the residual gauge transformations. As in the previous chapter, the residual gauge due to diffeomorphisms are the ones generated by the following vectors,

$$\mathcal{L}_\xi g_{rr} = \mathcal{L}_\xi g_{ra} = 0, \quad g^{ab}\mathcal{L}_\xi g_{ab} = 0. \tag{2.13}$$

As we saw in section 1.2, the vector fields satisfying the above equations are given by

$$\xi^u = F(u, x^a), \tag{2.14}$$

$$\xi^a = V^a(u, x^a) - \partial_b F \int_r^\infty e^{2\beta} g^{ab} dz, \tag{2.15}$$

$$\xi^r = -\frac{r}{2}\left(\nabla^g_a \xi^a - \partial_b F U^b + \frac{1}{2} F g^{-1}\partial_u g\right), \tag{2.16}$$

The residual gauge symmetries for the vector potential satisfy the equation

$$\partial_r \Lambda = \nabla_b \xi^u g^{ab} e^{2\beta} A_a \tag{2.17}$$

which has immediate solution,

$$\Lambda = \lambda(u, x^a) - \partial_b F \int_r^\infty e^{2\beta} g^{ab} \mathcal{A}_b dz, \tag{2.18}$$

Observe that the diffeomorphism generators are present in (2.18): this generally implies a non-trivial field dependence in the residual gauge symmetries.

When the gauge parameters are field-dependent, equation (2.7) is no longer valid, and we have the modified bracket [70],

$$[\delta_{\xi_1,\Lambda_1}, \delta_{\xi_2,\Lambda_2}] = \delta_{[(\xi_1,\Lambda_1),(\xi_2,\Lambda_2)]^*_{EYM}}, \tag{2.19}$$

where

$$[(\xi_1, \Lambda_1), (\xi_2, \Lambda_2)]^*_{EYM} = (\hat{\xi}, \hat{\Lambda}) - \delta_{\xi_1,\Lambda_1}(\xi_2, \Lambda_2) + \delta_{\xi_2,\Lambda_2}(\xi_1, \Lambda_1). \tag{2.20}$$

Harmonic gauge

For completeness, let us discuss the residual gauge symmetries in the harmonic gauge,

$$\nabla^\mu A_\mu = 0. \tag{2.21}$$

This gauge is the one we are taking in chapter 4 and chapter 5, suitable for perturbative calculations. By imposing the harmonic gauge on a variation, we obtain

$$\nabla^\mu \left(\mathcal{L}_\xi \mathcal{A}_\mu + \mathcal{D}_\mu \Lambda\right) = 0, \tag{2.22}$$

which after some computational massaging results,

$$\mathcal{A}_\nu(-R_\mu{}^\nu\xi^\mu + \nabla^\mu\nabla_\mu\xi^\nu) + 2\nabla^{(\mu}\xi^{\nu)}(\nabla_\nu\mathcal{A}_\mu) + \nabla^\mu\nabla_\mu\Lambda + [\mathcal{A}_\mu, \nabla^\mu\Lambda] = 0. \tag{2.23}$$

Observe that equation (2.23) gives us a nice characterization of the BMS algebra in the harmonic gauge: for asymptotically Killing fields, where $\nabla_{(\mu}\xi_{\nu)} \to 0$ as $r \to +\infty$, the terms in the first brackets and the second term vanish as $r \to +\infty$, since the equation in brackets is the trace of the Killing equation. Thus, asymptotically both terms vanish and the remaining terms are the same as the harmonic gauge compatibility in Minkowski, (4.19), which we will see in more detail later in chapter 4.

The leading order behaviour that we are studying for the YM sector is equivalent in both gauges, as equation (2.23) can be solved *after* we impose the fall offs and prove to have the same leading form.

2.1.3 Solution Space and Phase space

The equations of motion are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \quad T_{\mu\nu} := 2\mathrm{Tr}(\mathcal{F}_{\mu\sigma}\mathcal{F}_\nu{}^\sigma) - \frac{1}{2}g_{\mu\nu}\mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}) \tag{2.24}$$

$$\nabla^\mu\mathcal{F}_{\mu\nu} + [\mathcal{A}^\mu, \mathcal{F}_{\mu\nu}] = 0 \tag{2.25}$$

On the gravity side, since the stress-energy tensor $T_{\mu\nu}$ is traceless, equation (2.24) implies $R = 0$, and then is equivalent to

$$R_{\mu\nu} = T_{\mu\nu} \tag{2.26}$$

The boundary conditions that we are taking in the gravity sector are the ones defined by Γ_{GBMS}, (1.72), giving the same parametrization of the solution space. The decays for the vector potential are taken to be compatible with the radiative space for Yang-Mills in Minkowski,

$$\mathcal{A}_u = o(1), \quad \mathcal{A}_r = 0, \quad \mathcal{A}_a = O(1). \tag{2.27}$$

On the Yang-Mills side,through the equations of motion, it can be proven (in Einstein-Maxwell case see e.g. [71, 72]) that the components of $\mathcal{A}$ in an r-expansion can be calculated from the leading term of the angular part, denoted by A_a,

$$\mathcal{A}_a(u,x) = A_a(u,x) + o(1) \tag{2.28}$$

Next, regarding the $u-$decays for and A_a, we take "tree level fall offs" [73],

$$\partial_u A_a(u,x) = O(1/|u|^\infty). \tag{2.29}$$

This decay allows a finite limit when reaching $u \to \pm\infty$,

$$A_a^{\pm}(x) := \lim_{u\to\pm\infty} A_a(u,x) \tag{2.30}$$

This provides a natural radiative structure on $\mathscr{I}$ for YM, with phase space

$$\Gamma_{YM}^{rad} := \{A_a(u,x)\big|\partial_u A_a(u,x) = O(1/|u|^{\infty})\} \tag{2.31}$$

In this chapter we aim to obtain a symplectic structure on the product space,

$$\Gamma_{EYM} = \Gamma_{GBMS} \times \Gamma_{YM}^{rad}, \tag{2.32}$$

which conforms the EYM phase space.

Finally, by imposing the fall off (2.27) on the variations acting on $\mathcal{A}$, we obtain

$$\partial_u \lambda = 0, \tag{2.33}$$

so we can take λ as a free $\mathfrak{g}-$valued function on the celestial sphere.

Finally, the symmetry group depends on four free functions on the sphere, given by f, V^a, λ, where f and V^a parametrizes the diffeomorphisms and λ the large gauge transformations on the Yang-Mills sector. In the next section we will study the algebra on this group.

2.1.4 Asymptotic symmetries algebra

The first step towards the symplectic structure is to analyse the action of variations on Γ_{EYM}. Let us call δ_f, δ_V and δ_λ the variations generated by the functions f, V and λ. By taking the leading order on (2.5) and (2.6), we obtain the following expressions.

1. Supertranslations: the action on Γ_{GBMS} is given by,

$$\delta_f q_{ab} = \delta_f T_{ab} = 0, \quad \delta_f \hat{C}_{ab} = f\partial_u \hat{C}_{ab} - 2D_a D_b f^{TF} + fT_{ab}, \tag{2.34}$$

   while its action on A_a is

$$\delta_f A_a = f\partial_u A_a \tag{2.35}$$

2. Superrotations: the action on Γ_{GBMS} is given by,

$$\delta_V q_{ab} = \mathcal{L}_V q_{ab} - 2\alpha q_{ab}, \quad \delta_V \hat{C}_{ab} = \mathcal{L}_V \hat{C}_{ab} + \alpha \hat{C}_{ab} + u\alpha \hat{N}_{ab}, \tag{2.36}$$

$$\delta_V T_{ab} = \mathcal{L}_V T_{ab} - 2D_a D_b \alpha^{TF}, \tag{2.37}$$

   and on A_a,

$$\delta_V A_a = u\alpha\partial_u A_a + \mathcal{L}_V A_a. \tag{2.38}$$

3. Large gauge transformations (LGT): the action on the gravity side vanishes,

$$\delta_\lambda q_{ab} = \delta_\lambda \hat{C}_{ab} = \delta_\lambda T_{ab} = 0 \tag{2.39}$$

but on Γ^{rad}_{YM} is the usual,

$$\delta_\lambda A_a = \mathcal{D}_a \lambda, \tag{2.40}$$

Just for reference, we will need the contravariant version of some of the above identities. In the case of the upper index tensors, the variations are,

$$\delta_V \hat{C}^{ab} = \mathcal{L}_V \hat{C}^{ab} + \alpha u \partial_u \hat{C}^{ab} + 3\alpha \hat{C}^{ab} \tag{2.41}$$

$$\delta_V A^a = \mathcal{L}_V A^a + u\alpha \partial_u A^a + 2\alpha A^a \tag{2.42}$$

From the action of the symmetries we see that the symmetry group acting on (2.32), let us call EYM_4, is a semi-direct product of $GBMS_4$ and G^∞, where G^∞ is the group of symmetries generated by the large gauge transformations (the Λ's) in the Yang-Mills sector. This semi-direct property translates to the Lie algebra, $\mathfrak{eym}_4$,

$$\mathfrak{eym}_4 = \mathfrak{gbms}_4 \ltimes \mathfrak{g}^\infty, \tag{2.43}$$

where $\mathfrak{g}$ is the Lie algebra of G. The mixed variations are:

$$[\delta_f, \delta_\lambda] = 0, \quad [\delta_V, \delta_\lambda] = \delta_{-\mathcal{L}_V \lambda} \tag{2.44}$$

We can write the algebra in a more compact form, as in (2.7),

$$[\delta_{(f_1,V_1,\lambda_1)}, \delta_{(f_2,V_2,\lambda_2)}] = \delta_{(\hat{f},\hat{V},\hat{\lambda})}, \tag{2.45}$$

with

$$\hat{f} = \mathcal{L}_{V_1} f_2 - \alpha_1 f_2 - (1 \leftrightarrow 2) \tag{2.46}$$

$$\hat{V} = [V_1, V_2]_{\text{Lie}} \tag{2.47}$$

$$\hat{\lambda} = [\lambda_1, \lambda_2]_{\mathfrak{g}} - \mathcal{L}_{V_1}\lambda_2 + \mathcal{L}_{V_2}\lambda_1 \tag{2.48}$$

Observe that since we reduce the parametrization of the asymptotic group to independent functions, the modified bracket coincides with the usual bracket (e.g., $\delta_f \lambda = 0$).

2.2 Charges

The symplectic form in the YM radiative space Γ^{rad}_{YM} is given by

$$\Omega^{rad}_{YM}(\delta, \delta') = \int_{\mathscr{I}} \text{Tr}(\delta \partial_u A^a \wedge \delta A_a) \sqrt{q} du d^2 x, \tag{2.49}$$

which is the YM version of the radiative spaces introduced in [4]. The standard charge for the radiative phase space Γ^{rad}_{YM} is given by

$$Q_\lambda = \int_{\mathscr{I}} \mathrm{Tr}[\partial_u A^a \mathcal{D}_a \lambda] \sqrt{q} du d^2x. \tag{2.50}$$

By integrating by parts and along $\mathscr{I}$, this expression is also equal to a boundary term,

$$Q_\lambda = \int_{\partial\mathscr{I}} \mathrm{Tr}[\lambda \mathcal{D}_a A^a] \sqrt{q} du d^2x. \tag{2.51}$$

In the previous chapter we found the complete expressions for the GBMS charges, P_f and J_V, imposing the compatibility with the Poisson algebra structure on the phase space. We will proceed with the same strategy in here, allowing extra terms in the charges,

$$\begin{aligned} P_f &= P_f^{GBMS} + P_f^{YM}, & (2.52)\\ J_V &= J_V^{GBMS} + J_V^{YM}, & (2.53)\\ Q_\Lambda &= Q_\lambda^{YM} + Q_\lambda^{gr}, & (2.54) \end{aligned}$$

where P_f^{YM} and J_V^{YM} are the contributions of the YM field to the supermomentum and superrotation, and Q_λ^{gr} is the (a priori possible) pure gravitational contribution to the Yang-Mills charge. By imposing that the brackets between the charges resemble the variation algebra (2.45), conditions on the extra terms will be obtained. The steps are as follows.

1. $\delta_V Q_\lambda = \{Q_\lambda, J_V\}$: candidate for J_V^{YM}
2. $\delta_f Q_\lambda = \{P_f, J_V\}$: candidate for P_f^{YM}.
3. GBMS Poisson-algebra for P_f^{YM} and J_V^{YM}.

Of course, we can naturally set $Q_\lambda^{gr} = 0$, since otherwise we would have a non-trivial Yang-Mills charge from Minkowski in a pure GBMS context.

2.2.1 $\{Q_\lambda, J_V\}$

The algebra has to satisfy,

$$\{Q_\lambda, J_V\} = \delta_V Q_\lambda = -Q_{\mathcal{L}_V \lambda}, \tag{2.55}$$

and also

$$\{Q_\lambda, J_V\} = -\delta_\lambda J_V = -\delta_\lambda J_V^{YM}, \tag{2.56}$$

since $\delta_\lambda J_V^{GBMS} = 0$. By a straightforward computation, the first equations hold,

$$\delta_V Q_\lambda = -\int_{\mathscr{I}} \mathrm{Tr}[\partial_u A^a \mathcal{D}_a(\mathcal{L}_V \lambda)] \sqrt{q} du d^2x = -Q_{\mathcal{L}_V \lambda}, \tag{2.57}$$

where we use $\mathcal{L}_V \sqrt{q} = 2\alpha\sqrt{q}$ and $\mathcal{L}_V(d\lambda) = d\mathcal{L}_V \lambda$ (a consequence of Cartan formula).

The expression for the symplectic form (2.49) provides us with a clear ansatz for J_V^{YM},

$$J_V^{YM} := \int_{\mathscr{I}} \mathrm{Tr}[\partial_u A^a \delta_V A_a]\sqrt{q}dud^2x, \tag{2.58}$$

which is quadratic in A_a. Apply δ_λ,

$$\delta_\lambda J_V^{YM} = \int_{\mathscr{I}} \mathrm{Tr}[\partial_u A^a(\mathcal{D}_a\mathcal{L}_V\lambda)]\sqrt{q}dud^2x = Q_{\mathcal{L}_V\lambda}, \tag{2.59}$$

and therefore the first equations are satisfied.

2.2.2 $\{Q_\lambda, P_f\}$

Next, we look for a candidate for P_f^{YM}, by imposing

$$\{Q_\lambda, P_f\} = \delta_f Q_\lambda = 0. \tag{2.60}$$

and

$$\{Q_\lambda, P_f\} = -\delta_\lambda P_f = -\delta_\lambda P_f^{YM}, \tag{2.61}$$

A straightforward computation gives

$$\delta_f Q_\lambda = \int_{\mathscr{I}} f\mathrm{Tr}[\partial_u^2 A^a \mathcal{D}_a\lambda + \partial_u A^a[\partial_u A_a, \lambda]]\sqrt{q}dud^2x = 0. \tag{2.62}$$

Thus, $P_f^{YM} \in \mathrm{Ker}(\delta_\lambda)$, when δ_λ viewed as an operator. As in the previous case, an ansatz can be used from (2.49),

$$P_f^{YM} := \int_{\mathscr{I}} \mathrm{Tr}[\partial_u A^a \delta_f A_a]\sqrt{q}dud^2x = \int_{\mathscr{I}} f\mathrm{Tr}[\partial_u A^a \partial_u A_a]\sqrt{q}dud^2x. \tag{2.63}$$

With this definition in mind, we take the variation δ_λ,

$$\delta_\lambda P_f^{YM} = \int_{\mathscr{I}} 2f\mathrm{Tr}[\partial_u A^a \partial_u D_a\lambda]\sqrt{q}dud^2x = 0. \tag{2.64}$$

2.2.3 Closure of J_V^{YM} and P_f^{YM}

Now that we have the candidates for the new J_V and P_f, we must verify that the subalgebra generated by them closes on $\mathfrak{eym}_4$. In particular, we will see that the set of phase space functions generated by J_V^{YM} and P_f^{YM} satisfies the equations

$$\{J_V^{YM}, P_f^{YM}\} \equiv \delta_V P_f^{YM} = -P_{V(f)}^{YM}, \tag{2.65}$$

$$\{J_V^{YM}, P_f^{YM}\} \equiv -\delta_f J_V^{YM} = -P_{V(f)}^{YM}, \tag{2.66}$$

$$\{P_{f_1}^{YM}, P_{f_2}^{YM}\} = 0, \quad \{J_{V_1}^{YM}, J_{V_2}^{YM}\} = J_{[V_1,V_2]}^{YM} \tag{2.67}$$

so that the full algebra spanned by $\{J_V, P_f\}$ closes. The easiest one is the abelian part,

$$\delta_{f_1} P^{YM}_{f_2} = 2\int_{\mathscr{I}} f\mathrm{Tr}[\partial_u A^a \partial_u^2 A_a]\sqrt{q}dud^2x = \int_{\partial\mathscr{I}} f\mathrm{Tr}[\partial_u A^a \partial_u A_a]\sqrt{q}dud^2x\bigg|_{u=\pm\infty} = 0, \tag{2.68}$$

by the decays on $A_a(u,x)$. For the mixed variation, we have,

$$\begin{aligned}\delta_f J^{YM}_V &= \int_{\mathscr{I}} \delta_f \mathrm{Tr}[\partial_u A^a(u\alpha\partial_u A_a + \mathcal{L}_V A_a)]\sqrt{q}dud^2x \\ &= \int_{\mathscr{I}} \mathrm{Tr}[(\mathcal{L}_V f - f\alpha)\partial_u A^a \partial_u A_a]\sqrt{q}dud^2x + \int_{\partial\mathscr{I}} \mathrm{Tr}[fu\alpha\,(\partial_u A^a\partial_u A_a) + f\partial_u A^a \mathcal{L}_V A_a]\sqrt{q}d^2x, \\ &= P^{YM}_{V(f)},\end{aligned} \tag{2.69}$$

where the last integral vanishes again due to the decays in A_a. Next we verify the other mixed variation,

$$\begin{aligned}\delta_V P^{YM}_f &= \int_{\mathscr{I}} f\delta_V \mathrm{Tr}[\partial_u A^a \partial_u A_a]\sqrt{q}dud^2x \\ &= -\int_{\mathscr{I}} (\mathcal{L}_V f - \alpha f)\mathrm{Tr}[\partial_u A^a \partial_u A_a]\sqrt{q}dud^2x + \int_{\partial\mathscr{I}} u\alpha f\mathrm{Tr}[\partial_u A^a\partial_u A_a]\sqrt{q}dud^2x \\ &= -P^{YM}_V\end{aligned} \tag{2.70}$$

Finally, after some computations (using the same techniques as above), the bracket $\{J^{YM}_{V_1}, J^{YM}_{V_2}\}$ can be checked,

$$\{J^{YM}_{V_1}, J^{YM}_{V_2}\} = \delta_{V_1} J^{YM}_{V_2} = J^{YM}_{[V_1,V_2]}. \tag{2.71}$$

2.3 Symplectic form

In this section we derive a symplectic structure on Γ_{EYM} such that the charges act canonically on the phase space. The symplectic form in the YM radiative space Γ^{rad}_{YM} is given by (2.49). Such expression is valid for any metric, as long as the space is asymptotically locally Minkowski [4]. Thus, is valid for any metric on each Γ_q from the split (1.94). The natural candidate for the Γ_{EYM} symplectic form is

$$\Omega_{EYM} = \Omega_{GBMS} + \Omega^{rad}_{YM} \tag{2.72}$$

Since the pure gravitational parts of P_f and J_V are compatible with Ω_{GBMS}, and $\Omega_{GBMS}(\cdot, \delta_\lambda) = 0$, we only need have to verify the following identities

$$\Omega^{rad}_{YM}(\delta, \delta_f) = \delta P^{YM}_f, \quad \Omega^{rad}_{YM}(\delta, \delta_V) = \delta J^{YM}_V, \tag{2.73}$$

which we already take as an ansatz for each extra term, (2.58) and (2.63).

Finally, observe that this identities are compatible with condition 1 in section 1.7,

$$\theta^{YM}(\delta_V) = J^{YM}_V \tag{2.74}$$

We have arrived at the main result of this chapter: to show that GBMS can be coupled to other asymptotic symmetries groups, as it is the case of large gauge transformations for Yang-Mills (associated to the leading charges). A natural questions is whether GBMS can be coupled consistently with *extensions* of asymptotic symmetries groups. This discussion will be delayed until the end of chapter 4, where we will understand the extension of the asymptotic symmetries in YM.

Relations with the Covariant Phase Space Extensions

This chapter will provide the connection between our results in chapter 1 and the ones from covariant phase space formalism, [31,40,41,43,46,54], trying to understand the extra boundary terms arising when extending from BMS to GBMS.

The natural framework for the covariant phase space formalism is the *jet bundle*, where fields and their derivatives are sections of a fiber bundle over the spacetime M, [74]. We provide a review of Cartan calculus on this spaces in chapter A. We encourage the reader who is not familiar with these topics to look at the appendix.

The focus of this chapter in on the concept of charge in general relativity, and the different prescriptions one can take. First, we present the Noether charges and Iyer-Wald charges, which are going to be used throughout this chapter. After a discussion of the charges given by Ashtekar-Streubel ([4], [5]), we review the charges given by Compere, Fiorucci and Ruzziconi ([31]). Then, we proceed to use the extensions in the covariant phase spaces to explain the "extra" boundary terms that arise when the boundary metric is also a dynamical field.

3.1 Motivation

As we presented in chapter 1, when extending BMS to $GBMS$ we are introducing an extra field in the phase space, which in the symplectic form (1.147) appear explicitly in the Ω^{S^2} term as q_{ab}, the metric on the celestial sphere.

Covariant phase space methods are useful in the construction of charges (e.g. [12,37,38]), and provide a formal framework easy to generalize in various context, such as gauge theories and gravity. In [39], Freidel and Donnelly proposed a general procedure to associate a gauge-invariant classical phase space to a spatial slice with boundary by introducing new degrees of freedom on the boundary. This is precisely the situation in GBMS: the boundary, $\partial\mathscr{I}$, which can be thought as the celestial sphere, contains the new dynamical field q_{ab}. In other words, the presence of the boundary promoted some pure gauge transformations to physical degrees of freedom. Their proposal was to consider a counter-term, involving fields on the boundary, that cancels the non-covariance of the symplectic potential. In the case of gravity, these fields on the boundary are embedding of a neighbourhood of the Cauchy slice to $\mathcal{M}$. In [54], Speranza constructed a prescription to couple the embedding map to the theory, by writing the Lagrangian in terms of the pull back fields.

Recent works ([40–46]) have emphasized the existence of an universal symmetry group, called the *extended corner symmetry group*, with many properties such as maximality [44]. Roughly speaking, the gravitational symmetries on a corner have the following structure, [44],

$$(\mathrm{Diff}((S^2) \ltimes GL(2,\mathbb{R})) \ltimes (\mathbb{R}^2)^{S^2} \tag{3.1}$$

Each term corresponds to a different symmetry acting on the corner surface: $\mathrm{Diff}(S^2)$ are the diffeomorphisms on the sphere, $GL(2,\mathbb{R})$ are the surface boosts and $(\mathbb{R}^2)^{S^2}$ are the translations on along both outgoing normals. In the case of null infinity, its boundary $\partial\mathscr{I} = \mathscr{I}^+_+ - \mathscr{I}^+_-$, located at $u = \pm\infty$, is invariant under such (super) translations, so the whole group is *surface preserving*, [39], and reduces to

$$(\mathrm{Diff}((S^2) \ltimes \mathcal{W}) \ltimes \mathfrak{s}, \tag{3.2}$$

where we can identify the superrotations, $\mathcal{W}$ is the Weyl group (rescalings) and $\mathfrak{s}$ is the infinite dimensional abelian group of supertranslations.

In [46] Freidel showed that by dressing the Lagrangian by an embedding field, the symplectic form is modified in such a way that allows for a canonical representation of supertranslations. In [53], Ciambelli, Leigh and Pai showed that working with an embedding mapping parametrizing the extension of the phase space, nonzero charges can be integrated for *all* diffeomorphisms, giving a representation of the extended corner symmetry, without any central extension.

The charges (1.130) and (1.131) are integrable (from the symplectic form (1.147)), have no central extension and are compatible with the leading and sub-leading charges given by the soft theorems ([22]). Therefore, the natural question is whether they are a representation of the extended corner algebra. As we will see, $J^{\partial\mathscr{I}}_V$ can be understood as a necessary term coming from the surface preserving subalgebra of the extended corner algebra [39, 43].

3.2 Charges in General Relativity

In this section we present the notion of Noether charges and Iyer-Wald charges.

In this chapter, we will use the following notation:

- d, ι and $\mathcal{L}$ are the exterior derivative, interior product and Lie derivative on $\mathcal{M}$.
- δ, I and $\mathfrak{L}$ are the exterior derivative, interior product and Lie derivative on solution space.
- We denote the tangent vector fields to solution spaces as *characteristics*, and use the letter q (instead of the usual "$\delta\phi$" for some field ϕ).

- Given a diffeomorphism ξ, it generates a flow on solution space and therefore it has associated a characteristic tangent to solutions. We denote such the characteristic as $\hat{\xi}$.

For more details, we refer the reader to chapter A and the references provided there.

3.2.1 Noether Charges

The Einstein-Hilbert Lagrangian,

$$\mathscr{L}[g] = R[g]d\mu_g, \tag{3.3}$$

gives the following symplectic potential,

$$\Theta[q] = \int_\Sigma \theta^\mu[q]dS_\mu = -\int_\Sigma \sqrt{-g}g^{\mu\lambda}\left(\nabla^\nu(\mathfrak{L}_q g)_{\lambda\nu} - g^{\nu\rho}\nabla_\lambda(\mathfrak{L}_q g)_{\nu\rho}\right)dS_\mu, \tag{3.4}$$

where we use $\mathfrak{L}_q g$ as the usual "δg". By evaluating the symplectic potential on a $\hat{\xi}$ ([12]) we can compute the Noether-Wald charge,

$$Q_\xi = \theta[\hat{\xi}] = 2\sqrt{-g}(\nabla^{[\mu}\xi^{\nu]})(d^{n-2}x)_{\mu\nu}, \tag{3.5}$$

which are the Komar integrands ([75]). For instance, one of the applications of these charges is the well known *Smarr formula* for an isolated horizon, [76],

$$M = \frac{1}{4\pi}\kappa A + 2\Omega J, \tag{3.6}$$

where κ and Ω are the surface gravity and the angular velocity of the black hole horizon.

3.2.2 Iyer-Wald Charges

In terms of the Noether charge and the symplectic potential, the Iyer-Wald charges [12] are given by the integration of the following codimension 2 form,

$$k_\xi = \delta Q_\xi - Q_{\delta\xi} - \iota_\xi\theta. \tag{3.7}$$

We call the form k the *Iyer-Wald form* associated to ξ. For the Noether charges as in (3.5), the expression for the Iyer-Wald form is the following, [12,58],

$$k_\xi = 2\sqrt{-g}\left(\xi^\mu\nabla_\rho h^{\nu\rho} - \xi^\mu\nabla^\nu h + \xi_\rho\nabla^\nu h^{\mu\rho} + \frac{1}{2}h\nabla^\nu\xi^\mu - h^{\rho\nu}\nabla_\rho\xi^\nu\right)(d^{n-2}x)_{\mu\nu}, \tag{3.8}$$

where $h_{\mu\nu} := \delta g_{\mu\nu}$. The charge is denoted as

$$\boxed{\not\delta H_\xi = \int_{\partial\Sigma} k_\xi} \tag{3.9}$$

which defines the Iyer-Wald surface charge. The symbol $\not\delta$ makes explicitly the generally non-integrable character of the charge. We can split the Iyer-Wald charges in terms of the integrable and non-integrable terms,

$$\not\delta H_\xi = \int_{\partial\Sigma} \delta Q_\xi - Q_{\delta\xi} - \iota_\xi\theta = \delta\left(\int_{\partial\Sigma} Q_\xi\right) + \Xi_\xi, \tag{3.10}$$

where Ξ_ξ is a $1-$form in solution space,

$$\Xi_\xi = -\int_{\partial\mathscr{I}} Q_{\delta\xi} + \iota_\xi\theta. \tag{3.11}$$

We have two alternatives regarding this non-integrable term:

1. Define a new field, on the boundary, that restores the integrability (cf. subsection A.2.4).
2. Define stronger boundary condition so that the non-integrable part vanishes.

We will see below examples of both solutions. First, let us define the embedding mappings.

3.2.3 Embedding Maps

To express the field-dependence of the embeddings, take a map $\mathcal{G} : \mathcal{S} \to C^\infty(\mathscr{I} \hookrightarrow \mathcal{M})$, from the solution space to the set of smooth embeddings $\phi : \mathscr{I} \hookrightarrow \mathcal{M}$.

In a field-independent case, the map is constant, the exterior Lie derivative $\mathfrak{L}$ commutes with embedding pullbacks and therefore integrations are independent of the field (which is the formal statement that "the variation goes inside the integral"). But in the case of a field-dependent embeddings, the pullback and $\mathfrak{L}$ do not commute (see [43], [53], [45]). Consider q a characteristic, tangent to solution space $\mathcal{S}$, and denote by $\mathcal{G}^*$ the pullback of the elements in $\mathrm{Im}(\mathcal{G})$ (we abuse a little bit notation here to ease the formulas). Since q is tangent to $\mathcal{S}$, it defines a flow Φ^q_t such that

$$\left.\frac{d}{dt}\Phi^q_t(\phi^i)\right|_{t=0} = q(\phi^i), \quad \phi^i \in \mathcal{S}. \tag{3.12}$$

This flow defines a flow on $\mathrm{Im}(\mathcal{G}) \subset C^\infty(\mathscr{I} \hookrightarrow \mathcal{M})$, through $\mathcal{G}$. Then, the exterior derivative of $\mathcal{G}$ in field space, when contracted with a characteristic, gives a tangent vector on $C^\infty(\mathscr{I} \hookrightarrow \mathcal{M})$, which is a vector field on $\mathscr{I}$, generating a diffeomorphism of $\mathscr{I}$:

$$\mathcal{S} \overset{\mathcal{G}}{\to} C^\infty(\mathscr{I} \hookrightarrow \mathcal{M}) \overset{X}{\to} \mathrm{Diff}(\mathscr{I}), \tag{3.13}$$

where $X(\phi) := \psi \circ e_{\mathscr{I}}^{-1}$, where $e_{\mathscr{I}}$ is some reference embedding [1]. We can compute the differential of the map $X \circ \mathcal{G}$ evaluated on q. First, the differential of $\mathcal{G}$ on q is

$$\delta\mathcal{G}[q] = \frac{d}{dt}\mathcal{G}(\Phi^q_t)\ |_{t=0}\ . \tag{3.14}$$

[1]We are taking $e_{\mathscr{I}}|_{\mathcal{M}}$ as equal to $e_{\mathscr{I}}$ to avoid extra symbols in the formulas.

$$\delta(X \circ \mathcal{G})[q] = \frac{d}{dt}\mathcal{G}(\Phi_t^q) \circ e_{\mathscr{I}}^{-1}\Big|_{t=0}. \tag{3.15}$$

This is the differential evaluated at q, based on a general point ϕ in $\mathcal{S}$.

A couple of formulas from the appendix are going to be used in the following. The first one is the commutation of $\mathcal{G}^*$ with the exterior derivative in solution space,

$$\delta\mathcal{G}^*\alpha = \mathcal{G}^*(\delta\alpha + \mathcal{L}_{\chi_\mathcal{G}}\alpha), \tag{3.16}$$

where $\chi(q;\mathcal{G})$ is a vector field generating an infinitesimal diffeomorphism on Diff$(\mathscr{I})$,

$$\chi(q;\mathcal{G}) := \frac{d}{dt}\mathcal{G}(\Phi_t^q)\mathcal{G}(\Phi_0^q)^{-1}\Big|_{t=0}, \tag{3.17}$$

By formula (3.15), the above equation can be written as the Maurer-Cartan form for the embedding with respect to a fixed field configuration, $e_{\mathscr{I}}$, and therefore we have

$$\chi(q;\mathcal{G}) =: \chi_\mathcal{G}[q] = \delta(X \circ \mathcal{G}) \circ (X \circ \mathcal{G})^{-1}[q], \tag{3.18}$$

which defines a $1-$form $\chi_\mathcal{G}$ in solution space.

The second formula is the extended symplectic potential, computed from the definition of the action as

$$S[\mathscr{L},\mathcal{G}] = \int_R \mathscr{L}[\mathcal{G}^*_{bulk}\phi], \tag{3.19}$$

where $\mathcal{G}_{\text{bulk}}$ is the extension of $\mathcal{G}$ to the bulk. The explicit dependence on the map $\mathcal{G}$ is what makes the corresponding phase space enlarge: we are including the embedding as the extra dynamical variable. The symplectic potential can be computed, resulting

$$\theta_{cov} = \theta + \iota_{\chi_\mathcal{G}}\mathscr{L} + dQ_{\chi_\mathcal{G}}. \tag{3.20}$$

In the case of gravity, on-shell we have $\mathscr{L} \equiv 0$, so

$$\theta_{cov} = \theta^{grav} + dQ_{\chi_\mathcal{G}}, \tag{3.21}$$

where θ^{grav} is the usual symplectic potential (which, in the case of GBMS, comes from a renormalization, [31]). The extended symplectic form for gravity is then,

$$\Omega^{grav}_{cov} := \int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(\delta\theta^{grav}) + \int_{\partial\mathscr{I}} \mathcal{G}^*(\iota_{\chi_\mathcal{G}}\theta^{grav} + \delta Q_{\chi_\mathcal{G}} + \mathcal{L}_{\chi_\mathcal{G}}Q_{\chi_\mathcal{G}}). \tag{3.22}$$

The first term is the usual symplectic form, while the extra boundary terms are the new extra terms due to the inclusion of $\mathcal{G}$ as dynamical variable. The last two terms in the boundary term are part of the corner ambiguity, [46,54], depending on which prescription one uses. Our prescription, the same as [54], reflects the covariance of the action with respect to the embedding map. We will discuss this later in section 3.5.

3.3 Ashtekar-Streubel/Barnich-Troessaert charge computation

In [4], the symplectics potential and form for the radiative phase space in general relativity were defined on $\mathscr{I}$,

$$\Theta_{rad} = \int_{\mathscr{I}} N_{ab}\delta C^{ab}\sqrt{q}dud^2x, \quad \Omega_{rad} = \int_{\mathscr{I}} \delta N_{ab} \wedge \delta' C^{ab}\sqrt{q}dud^2x, \tag{3.23}$$

We can compute the *integrated charge flux* along $\mathscr{I}$, which is the integral of the Noether current. As we will see, it can be regarded as a boundary term after discarding boundary terms by imposing stronger fall offs on the fields. This is one of the main differences between the integrated version of the charges and the corner/surface: for certain restrictive fall offs, the expressions coincide.

For instance, the fall offs for C_{ab} in [4] in the radiative space are

$$C_{ab} := C^{\pm}_{ab}(x^a) + O(1/|u|^{\epsilon}), \quad \epsilon > 0, \tag{3.24}$$

and imply that the charge flux is *integrable*, in the following sense: the equations

$$\not{\delta} P_f = I_{\hat{\xi}_f}\Omega, \quad \not{\delta} J_V = I_{\hat{\xi}_V}\Omega, \tag{3.25}$$

defines Noether currents that are integrable, and their expressions are

$$P^{rad}_f = \int_{\mathscr{I}} N_{ab}\mathfrak{L}_{\hat{\xi}_f} C_{ab}\sqrt{q}dud^2x, \tag{3.26}$$

$$J^{rad}_V = \int_{\mathscr{I}} N_{ab}\mathfrak{L}_{\hat{\xi}_V} C_{ab}\sqrt{q}dud^2x, \tag{3.27}$$

which are integrated along $\mathscr{I}$.

As a pedagogical example, let us take the computation for P_f in the radiative phase space with the fall offs (3.24) in two different ways: form the Iyer-Wald perspective, and from the flux perspective. At the end, both results coincide.

- From a corner charge perspective,

$$\begin{aligned} I_{\hat{\xi}_f}\Omega &= -\int_{\mathscr{I}} \left(\mathfrak{L}_{\hat{\xi}_f} N_{ab}\delta C^{ab} - \delta N_{ab}\mathfrak{L}_{\hat{\xi}_f} C^{ab}\right)\sqrt{q}dud^2x \\ &= -\delta\left(\int_{\mathscr{I}} 8f\partial_u M\sqrt{q}dud^2x\right) - \int_{\partial\mathscr{I}} fN_{ab}\delta C^{ab}\sqrt{q}d^2x, \end{aligned} \tag{3.28}$$

  where
$$\partial_u M = -\frac{1}{8}N_{ab}N^{ab} + \frac{1}{4}D_a D_b N^{ab}, \tag{3.29}$$

  the Bondi mass aspect. The charge is automatically splitted into integrable and non-integrable parts on the boundary $\partial\mathscr{I}$.

- On the other hand, using the flux perspective,

$$\begin{aligned} I_{\hat{\xi}_f}\Omega &= \int_{\mathscr{I}} \left(\delta(N_{ab}\mathfrak{L}_{\hat{\xi}_f}C^{ab}) - N_{ab}\delta\mathfrak{L}_{\hat{\xi}_f}C^{ab} - \mathfrak{L}_{\hat{\xi}_f}N_{ab}\delta C^{ab}\right)\sqrt{q}dud^2x \qquad (3.30)\\ &= \int_{\mathscr{I}} \delta(N_{ab}\mathfrak{L}_{\hat{\xi}_f}C^{ab})\sqrt{q}dud^2x - \int_{\partial\mathscr{I}} fN_{ab}\delta C^{ab}\sqrt{q}d^2x, \end{aligned}$$

  which gives exactly the same result, but the starting point of the procedure is to directly go to the integrable part of the expression (3.26).

The same computation can be carried out for super angular momentum, although it is more involved due to the handling of the terms of $\partial_u N^a$, (1.29). As we mention, there are several prescriptions for the definition of the Bondi angular momentum aspect, (1.24), (1.25), which in principle could lead to different expressions for the charges. In the case of GBMS, we will see that the covariant prescription (the one in [60]) is more convenient.

The non-integrable part in BMS is given by ([29]),

$$\Xi_{BMS} = \int_{\partial\mathscr{I}} \left((f + u\alpha)N_{ab}\delta C^{ab}\right)\sqrt{q}d^2x, \qquad (3.31)$$

which can be seen immediately as the integral of $-\iota_\xi\theta$ in (3.10) (cf. and (A.110)),

$$\Xi_{BMS} = -\int_{\partial\mathscr{I}} \iota_\xi\theta_{rad} \qquad (3.32)$$

Let us explore both options regarding the non-integrable term, cf. subsection 3.2.2. In view of [39,43] , if we have a non trivial embedding (a non constant map $\mathcal{G}$), then we can construct the extra boundary term that restores integrability. In our case, since $\delta q_{ab} = 0$ both for ξ_f and ξ_V, we see that the embeddings are trivial. This is in contrast to the case where the gravitational subsystem is in the bulk of $\mathcal{M}$, [39], where supertranslation act moving the boundaries in a non-canonically way, and therefore a precise study has to be done [46]. In our case, supertranslation only act on the constant mode of the shear, cf. (1.49).

That leaves us with the second option: stronger boundary conditions. Imposing the AF conditions of chapter 1, and the condition

$$\lim_{u\to\pm\infty} N_{ab} = 0, \qquad (3.33)$$

we are setting the term Ξ_{BMS} to zero, while allowing non-trivial dynamics. Consistency conditions with the convergence of the integrals, together with the previous limit implies the radiative fall offs,

$$C_{ab}(u,x) = C^{\pm}_{ab}(x) + O(1/u^{1+\epsilon}). \qquad (3.34)$$

By imposing this decay for C_{ab}, we re obtain BMS action.

3.4 Compère-Fiorucci-Ruzziconi charge computation

First in [29] for the EBMS and then in [31] for the GBMS, the charges corresponding to the symplectic form integrated along the hypersurface $\mathcal{g}^+_\imath \cup \mathcal{U}^+ \cup \mathcal{U}^-$, with $\mathcal{g}^+_\imath = \{r = \imath\}, \mathcal{U}^- = \{u = u_-\}$ and $\mathcal{U}^+ = \{u = u_+\}$, for certain values $\imath, u^\pm$. Taking the limit $\imath \to +\infty$, $\mathcal{g}^+_\imath$ becomes a subset of $\mathscr{I}^+$.

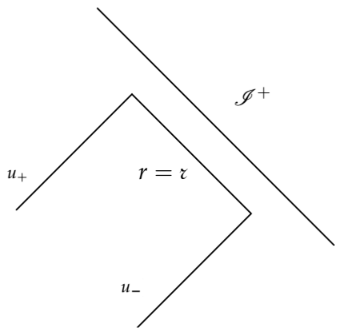

Figure 3.1: Hypersurface $\mathcal{g}^+_\imath \cup \mathcal{U}^+ \cup \mathcal{U}^-$ for the calculation for the BT symplectic form.

As we saw in section 1.5, the symplectic structure diverges in the limit $r \to +\infty$, since we have contribution due to $\delta q_{ab} \neq 0$. After a suitable renormalization, the variational principle implies that a balance between the symplectic potential fluxes on $\mathcal{U}^\pm$ and $\mathcal{g}^+_\imath$ must be exact. A symplectic form is given in $\mathcal{g}^+_\imath$. First, the variation of the action is computed, having three contributions,

$$\delta S = \int_{\mathcal{U}^-} \theta^u_{in} dr d^2x + \int_{\mathscr{I}^+_r} \bar{\theta}_{flux} du d^2x + \int_{\mathcal{U}^+} \theta^u_{out} dr d^2x, \tag{3.35}$$

with

$$\bar{\theta}_{flux} = \sqrt{q}\left(\frac{1}{2}N_{ab}\delta C^{ab} - \frac{1}{4}R(q)C_{ab}\delta q^{ab} + U_b D_a \delta q^{ab}\right), \tag{3.36}$$

$$\theta^u_{in} = \theta^u_{out} = O(r^{-2}). \tag{3.37}$$

Assuming no incoming gravitational radiation from $\mathscr{I}^-$, and ignoring the corner symmetries for the moment (which will be the main subject of the next chapters for gravitation, Yang-Mills and Maxwell), we have that (on-shell) only remains the central term in the above expansion,

$$\Theta_{flux} = \int_{\mathscr{I}^+} \sqrt{q}\left(\frac{1}{2}N_{ab}\delta C^{ab} - \frac{1}{4}R(q)C_{ab}\delta q^{ab} + U_b D_a \delta q^{ab}\right) du d^2x. \tag{3.38}$$

The symplectic form is then

$$\Omega = \delta\Theta_{flux}, \tag{3.39}$$

from which the charges are computed as the Iyer-Wald charges, using the formula (3.7). The final result is the following,

$$\delta\!\!\!/ H_\xi = \delta(Q_\xi) + \Xi_\xi, \tag{3.40}$$

where the integrable term is given by,

$$Q_\xi = 2\int_{\partial\mathscr{I}} \left(4FM + 2V^a N_a + \frac{1}{16}Y^a D_a(C_{bc}C^{bc})\right)\sqrt{q}d^2x, \tag{3.41}$$

while the non-integrable term is

$$\Xi_\xi = 2\int_{\partial\mathscr{I}} \left(\frac{1}{2}F(N^{ab} + \frac{1}{2}q^{ab}R(q))\delta C_{ab} - 2\partial_{(a}FU_{b)}\delta q^{ab} - FD_{(a}U_{b)}\delta q^{ab} - \frac{1}{4}D_c D^c F C_{ab}\delta q^{ab}\right)\sqrt{q}d^2x, \tag{3.42}$$

where $F = f + u\alpha$. The first contribution is the integrable Noether charge, while the non-integrable part is equivalent to the following term [31],

$$\Xi_\xi = 2\int_{\partial\mathscr{I}} \iota_{\xi^*}\theta_{flux}. \tag{3.43}$$

Together, they can be written compactly as

$$I_{\delta_\xi}\Omega_{\mathscr{I}} = \int_{\mathscr{I}} \left(d\iota_{\xi^*}\theta_{flux} + \delta dQ_\xi\right), \tag{3.44}$$

where the $Q_{\delta\xi}$ term in (3.7) gives a vanishing contribution at leading order. Observe that this is the same structure as we have in (A.110).

The fluxes associated to the infinitesimal charges $\not\delta H_\xi$ on $\mathscr{I}^+$ are given by

$$F_\xi := \int_{\mathscr{I}^+} \partial_u \not\delta H_\xi du \tag{3.45}$$

Of course, this integral has to be defined with care, since it is not guaranteed to be finite. In fact, as it is explained in [31], some prescription have to be taken for the boundary conditions for C_{ab}. For example, if we consider the flux balance in $\mathscr{I}^+$ [7] we have the Bondi mass formula at fixed u,

$$\frac{d}{du}\mathcal{M} = -\frac{1}{4}\int_{S^2} N_{ab}N^{ab}d^2x. \tag{3.46}$$

This is the well-known *leakage* of energy through $\mathscr{I}^+$ is the responsible for the non-integrability [58], a thermodynamical-like identity used also in other contexts, such as in emergent gravity (see e.g. [77]).

Finally, regarding the Poisson charge algebra, the presence of central extensions in other formulations of the charges, such as in [31] and [29], is a generic result given by the Charge Representation Theorem, both in the integrable case ([78,79]) as well as in the non-integrable case ([58]).

In [31], the authors defined a new bracket,

$$\{H_{\xi_1}, H_{\xi_2}\}_* = \delta_{\xi_2}H_{\xi_1} + \Xi_{\xi_2}[\delta_{\xi_1}], \tag{3.47}$$

which is isomorphic to the Lie algebra of diffeomorphism up to a field-dependent 2 cocycle:

$$\{H_{\xi_1}, H_{\xi_2}\}_* = H_{[\xi_1,\xi_2]_*} + K_{\xi_1,\xi_2}, \tag{3.48}$$

with

$$K_{\xi_1,\xi_2} = \int_{S_\infty} \left(\frac{1}{2} F_2 D_a F_1 D^a R[q] + \frac{1}{2} C^{bc} T_2 D_b D_c D_d V_1^d \right) du d^2x - (1 \leftrightarrow 2). \tag{3.49}$$

3.5 GBMS computation - Review

In chapter 1 we constructed a symplectic form on Γ^{GBMS} such that the charges (fluxes in the BT nomenclature) P_f and J_V act canonically.

We obtained a symplectic structure θ^{GBMS} that can be split into three contributions,

$$\theta^{GBMS} = \theta^{hard} + \theta^{soft} + \theta^{\partial \mathscr{I}}, \tag{3.50}$$

where

$$\theta^{hard} = \int_{\mathscr{I}} N^{ab} \delta \hat{C}_{ab} \sqrt{q} du d^2x \tag{3.51}$$

$$\theta^{soft} = \int_{\partial \mathscr{I}} \overset{1}{N}{}^{ab} \overset{1}{S}_{ab}(\delta) \sqrt{q} d^2x, \tag{3.52}$$

$$\theta^{\partial \mathscr{I}} = -\frac{1}{2} \int_{\partial \mathscr{I}} \hat{C}^{ab} (\delta \overset{0}{S}{}^{C}_{ab} - \overset{0}{S}{}^{\delta C}_{ab} - C \overset{1}{S}(\delta)) \sqrt{q} d^2x. \tag{3.53}$$

The soft term is linear in $\overset{1}{N}{}^{ab}$, and corresponds to those found in [22] and [31], while the boundary term contain quadratic in C terms. It is important to remark that these terms are non-local, since it has been shown [80] that there cannot be a symplectic structure at $\mathscr{I}$ that supports the action of GBMS, if it is constructed from a local and covariant symplectic current. This non-locality comes in T_{ab}, which depends non-locally on q_{ab}, and on $C^{\pm}$, which depends non-locally on C_{ab}.

As we mentioned in subsection 3.2.3, the extended symplectic potential is given by (3.20), [43],

$$\Theta_{cov} := \int_{\mathbb{R} \times S^2} \mathcal{G}^*_{\mathscr{I}} (\theta_{rad} + dQ_{\chi_{\mathcal{G}}}), \tag{3.54}$$

which leads to the symplectic form, (3.22)

$$\Omega_{cov} := \int_{\mathbb{R} \times S^2} \omega_{rad} + \int_{S^2} \mathcal{G}^*_{\partial \mathscr{I}} (\iota_{\chi_{\mathcal{G}}} \theta_{rad} + \delta Q_{\chi_{\mathcal{G}}} + \mathcal{L}_{\chi_{\mathcal{G}}} Q_{\chi_{\mathcal{G}}}). \tag{3.55}$$

The first term is the usual radiative term, which we already analysed in section 3.3. The corner term involves only evaluations on $\chi_{\mathcal{G}}$, which given a characteristic q it gives the vector field on $\mathscr{I}$ generating the corresponding embedding. Therefore, the variations that will be relevant in the calculation are those for which $\delta q_{ab} \neq 0$, so we can assume $f = 0$. By Proposition (1.1), any variation δq_{ab} can be written as $\delta_W q_{ab}$, so this implies that any characteristic q such that $\chi_{\mathcal{G}}$ is non-zero is associated to a diffeomorphism generator. Then, without loss of generality, we can evaluate the symplectic form on two characteristics $\hat{\xi}_V, \hat{\xi}_W$.

- Take (A.112), and contract with two generators, ξ_V and ξ_W:

$$I_{\hat{\xi}_V} I_{\hat{\xi}_W} \left(\int_{\partial \mathscr{I}} \mathcal{G}^*(\iota_{\chi_{\mathcal{G}}} \theta^{flux}) \right) = \int_{\partial \mathscr{I}} \mathcal{G}^*(\iota_{\chi_{\mathcal{G}}[\hat{\xi}_W]} \theta^{flux}[\hat{\xi}_V] - \iota_{\chi_{\mathcal{G}}[\hat{\xi}_V]} \theta^{flux}[\hat{\xi}_W]). \tag{3.56}$$

The embedding vector $\chi_{\mathcal{G}}[\hat{\xi}_W] = -\xi_W$, as a tangent vector on $\mathscr{I}$, has two directions with respect to the celestial sphere: two tangential ∂_a and one transversal ∂_u. When computing $\iota_{\chi_{\mathcal{G}}[\hat{\xi}_W]} \theta^{flux}[\hat{\xi}_V]$, we are contracting with the component along ∂_u, giving

$$\int_{\partial \mathscr{I}} \iota_{\chi_{\mathcal{G}}[\hat{\xi}_W]} \theta^{flux}[\hat{\xi}_V] \sqrt{q} d^2x - (V \leftrightarrow W) = \int_{\partial \mathscr{I}} \alpha_W u N^{ab} \delta_V \hat{C}_{ab} \sqrt{q} d^2x - (V \leftrightarrow W). \tag{3.57}$$

By the fall-offs on $\hat{C}_{ab}$, this term vanishes. This implies that the first term gives no contribution.

- For the last two terms, we use identities $\chi_{\mathcal{G}}[\hat{\xi}_V] = -\xi_V$,

$$\begin{aligned} I_{\hat{\xi}_V} I_{\hat{\xi}_W} \left(\delta Q_{\chi_{\mathcal{G}}} + \mathcal{L}_{\chi_{\mathcal{G}}} Q_{\chi_{\mathcal{G}}} \right) &= \mathfrak{L}_{\hat{\xi}_V} Q_{\chi_{\mathcal{G}}[\hat{\xi}_W]} + \mathcal{L}_{\chi_{\mathcal{G}}[\hat{\xi}_V]} Q_{\chi_{\mathcal{G}}[\hat{\xi}_W]} - (V \leftrightarrow W) &(3.58)\\ &= (\mathfrak{L}_{\hat{\xi}_V} - \mathcal{L}_{\xi_V}) Q_{\chi_{\mathcal{G}}[\hat{\xi}_W]} - (V \leftrightarrow W), &(3.59)\end{aligned}$$

which is the *anomaly operator*, see e.g. [45,81–83]. Observe that in our case, when evaluated on phase space coordinates,

$$(\mathfrak{L}_{\hat{\xi}_V} - \mathcal{L}_{\xi_V}) \hat{C}^{\pm}_{ab} = -\alpha C^{\pm}_{ab}, \quad (\mathfrak{L}_{\hat{\xi}_V} - \mathcal{L}_{\xi_V}) q_{ab} = -2\alpha q_{ab}, \tag{3.60}$$

the anomaly operator is the Diff(S^2)-weight times the identity!

In BMS, the Noether charge is given by (3.41), which after taking $f = 0$ (since $\delta_f q_{ab} = 0$) and integrating by parts, results

$$Q_{\xi_W} = \int_{\partial \mathscr{I}} (4u\alpha_W M^{\pm} + 2W^a N^{\pm}_a - \frac{1}{8} \alpha_W (\hat{C}^{\pm}_{bc} \hat{C}^{\pm cb})) \sqrt{q} d^2x. \tag{3.61}$$

By integrating by parts on the sphere, the first two terms in the integrand can be written as

$$2W^a \left(N^{\pm}_a - u D_a M^{\pm} \right) \tag{3.62}$$

This resembles the prescription due to Hawking, Perry and Strominger for the Bondi angular momentum aspect, (1.25), although it involves extra terms. Nevertheless, when computing (3.59), we see that N^{HPS}_a is indeed the "covariant quantity".

The only non trivial term in the last contribution in (3.61) is

$$\frac{1}{8} \mathcal{L}_V \alpha_W (\hat{C}^{\pm}_{bc} \hat{C}^{\pm cb}) - (V \leftrightarrow W) = \frac{1}{8} \alpha_{[V,W]} (\hat{C}^{\pm}_{bc} \hat{C}^{\pm cb}). \tag{3.63}$$

For the computation regarding the first two terms, we have to use equations (1.29) and

(1.28). The result is

$$(\mathfrak{L}_{\hat{\xi}_V} - \mathcal{L}_{\xi_V})(4u\alpha_W M^{\pm} + 2W^a N_a^{\pm}) - (V \leftrightarrow W) = \mathcal{L}_V W^a \hat{C}^{\pm\, bc} D_c \hat{C}_{ab}^{\pm} + \frac{23}{8}\alpha_{[V,W]}(\hat{C}_{bc}^{\pm}\hat{C}^{\pm cb}) \tag{3.64}$$

Observe then that the sum of this boundary contribution is exactly $J^{\partial\mathscr{I}}_{[V,W]}$, cf. (1.127).

Putting it all together, after using $N_a^+ - N_a^- = \int \partial_u N_a$, (1.29), (1.25) and some rearrangements of terms, we arrive at

$$I_{\hat{\xi}_V} I_{\hat{\xi}_W}(\Omega_{cov}) = J^{soft}_{[V,W]} - 4\int_{\mathscr{I}} \mathcal{L}_V W^a \partial_u N_a^{hard,HPS} \sqrt{q}\, du d^2x, \tag{3.65}$$

which, in turn, satisfy

$$-4\int_{\mathscr{I}} V^a \partial_u N_a^{HPS} \sqrt{q}\, du d^2x = J_V^{hard} + J_V^{\partial\mathscr{I}} \tag{3.66}$$

The extra term, $J_V^{\partial\mathscr{I}}$, can now be understood as a necessary term coming from the surface preserving subalgebra of the extended corner algebra. The notion of covariance with respect to the extended corner symmetry for N_a^{HPS} is directly related to its definition in terms of the Riemann tensor, [60],

$$\lim_{r\to\infty} r^3 R_{arru} = N_a^{HPS} + u\partial_a M. \tag{3.67}$$

Part II

Phase space extensions in Gauge Theories

Subleading charges and $O(r)$ extensions in Yang Mills

Inspired by the BMS generalization due to a relaxation on the metric fall-offs, it was proposed in [24] that large $O(r)$ gauge symmetries can explain similar tree-level gauge theory sub-leading formula [20]. Whereas the proposal was initially established in the context of massless scalar electrodynamics, its validity was later extended to more general charged matter [84], higher dimensions, and non-abelian gauge fields [67]. In these investigations, however, there was no explicit description of the underlying phase space where the symmetries act. In particular, it was not possible to calculate the algebra of charges.

Since the seminal work of Strominger [19] it has been understood that the leading soft gluon factorization can be understood as a conservation law associated to large $O(r^0)$ gauge symmetries (see also [85–89]). On the other hand, the symmetry interpretation of the subleading factorization is more subtle: The asymptotic charges are known thanks to the work of Lysov, Pasterski and Strominger (LPS) [20],[1] but it is unclear what the underlying symmetry algebra is.

In the present chapter we review the improvements on this situation as it is shown in [73]. As in the gravitational example, we proceed by first identifying the appropriate "kinematical" fields that allow for $O(r)$ gauge symmetries. There is however a major difference between the gravitational and gauge theory cases: Whereas superrotations form a closed algebra, $O(r)$ gauge symmetries do not, since their commutator is generically $O(r^2)$. In fact, once $O(r)$ gauge transformations are allowed, one is forced to include $O(r^n)$ ones for all positive integers n. In order to avoid this proliferation, we will work in an approximation where the $O(r)$ gauge symmetries are linearized, thus effectively setting to zero the higher order terms. This restricted setting still allows for interesting structure, in particular regarding the algebra between $O(r^0)$ and $O(r)$ gauge symmetries. We hope our approximation describes a truncation of an underlying (tree-level) non-linear structure. The hope is based on (i) the results of [50] (and also the next chapter) imply, in the abelian case, a one-to-one correspondence between $O(r^n)$ large gauge charges and

[1] The work [20] is in the abelian context but it admits a direct generalization to the non-abelian case; see e.g. [84].

tree-level sub^{n-1}−leading formulas [48, 49], and (ii) the recently discovered [90, 91] infinite dimensional chiral algebra obeyed by tree-level (conformally) soft gluons of a given helicity.

4.1 Radiative phase spaces

We consider pure classical Yang-Mills theory with a matrix group G in 4d flat spacetime. We denote by $\mathfrak{g}$ the Lie algebra, $[,]$ its Lie bracket, $\mathcal{A}_\mu$ the $\mathfrak{g}$-valued gauge connection and

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu], \tag{4.1}$$

the field strength. The field equations are

$$\mathcal{D}^\mu \mathcal{F}_{\mu\nu} = \nabla^\mu \mathcal{F}_{\mu\nu} + [\mathcal{A}^\mu, \mathcal{F}_{\mu\nu}] = 0, \tag{4.2}$$

with ∇_μ and $\mathcal{D}_\mu$ denoting the metric and gauge covariant derivatives respectively. Local gauge transformations are parametrized by $\mathfrak{g}$-valued functions Λ as

$$\delta_\Lambda \mathcal{A}_\mu = \mathcal{D}_\mu \Lambda = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda]. \tag{4.3}$$

The "bulk" symplectic form is

$$\Omega^{\text{bulk}} = -\int_\Sigma dS_\mu \text{Tr}(\delta \mathcal{F}^{\mu\nu} \wedge \delta \mathcal{A}_\nu), \tag{4.4}$$

where Tr is the matrix trace and the integral is taken over any Cauchy slice Σ. The symplectic form can be used to obtain canonical charges associated with symmetries. In particular, for the gauge symmetry (4.3) one has

$$Q^{\text{bulk}}_\Lambda = -\int_\Sigma dS_\mu \partial_\nu \text{Tr}(\Lambda \mathcal{F}^{\mu\nu}), \tag{4.5}$$

where the charge satisfies $\delta Q^{\text{bulk}}_\Lambda = \Omega^{\text{bulk}}(\delta, \delta_\Lambda)$.

To describe the gauge field near future null infinity, we employ retarded coordinates (r, u, x^a), where r is the radial coordinate, $u = t - r$ the retarded time, and x^a coordinates on the celestial sphere. The flat spacetime metric takes the form (1.4). Since the round metric on the celestial sphere is fixed, we will denote it simply as q_{ab}. The "bulk" gauge field $\mathcal{A}_\mu$ induces a gauge field A_a at null infinity,

$$A_a(u, x) = \lim_{r\to\infty} \mathcal{A}_a(r, u, x), \tag{4.6}$$

that is unconstrained by the field equations and thus plays the role of free data. We will work under the assumption of "tree-level" $u \to \pm\infty$ fall-offs, in which the u-derivative of the asymptotic gauge field decays faster than any power of $1/|u|$. Schematically,

$$\partial_u A_a(u, x) = O(1/|u|^\infty), \tag{4.7}$$

consistent with a $O(\omega^0)$ subleading behavior in the $\omega \to 0$ frequency expansion.[2] This still allows for non-trivial asymptotic values of A_a at $u = \pm\infty$,

$$A_a^{\pm}(x) := \lim_{u\to\pm\infty} A_a(u,x). \tag{4.8}$$

The gauge field near null infinity can be determined in terms of $A_a(u,x)$ by solving the field equations (see e.g. [19, 92] and section 4.2). We denote by Γ^{rad} the resulting space of gauge fields and write schematically

$$\Gamma^{\text{rad}} \approx \{A_a(u,x)\}. \tag{4.9}$$

Under standard fall-offs, the bulk symplectic form (4.4) can be evaluated on the surface $\Sigma \to \mathscr{I}^+$, leading to the symplectic form

$$\Omega^{\text{rad}} = \int_{\mathscr{I}^+} \text{Tr}(\delta\partial_u A^a \wedge \delta A_a) du d^2x, \tag{4.10}$$

where $A^a \equiv q^{ab}A_b$ and the determinant $\sqrt{q}$ is implicit in the d^2x measure. We refer to the pair $(\Gamma^{\text{rad}}, \Omega^{\text{rad}})$ as the *radiative phase space*. It is the YM version of the Maxwell and gravity radiative phase spaces introduced in [4].

We denote by D_a the gauge-covariant derivative at null infinity,

$$D_a := \partial_a + [A_a,], \tag{4.11}$$

and use ∂_a to denote the sphere-covariant derivative compatible with q_{ab}, i.e. $\partial_c q_{ab} = 0$.

4.2 YM field near null infinity

We will work in harmonic gauge

$$\nabla^\mu \mathcal{A}_\mu = 0, \tag{4.12}$$

although we expect our results to be valid for more general gauge choices. Starting from the standard $O(r^{-1})$ free field fall-offs, one is lead to the following asymptotic expansion:

$$\begin{aligned}
\mathcal{A}_r &= \tfrac{1}{r^2}(\ln r \overset{0,\ln}{A}_r + \overset{0}{A}_r) + \tfrac{1}{r^3}(\ln r \overset{1,\ln}{A}_r + \overset{1}{A}_r) + o(r^{-3}), \\
\mathcal{A}_u &= \tfrac{\ln r}{r} \overset{0,\ln}{A}_u + \tfrac{1}{r^2}(\ln^2 r \overset{1,\ln^2}{A}_u + \ln r \overset{1,\ln}{A}_u + \overset{1}{A}_u) + o(r^{-2}), \\
\mathcal{A}_a &= A_a + \tfrac{1}{r}(\ln r \overset{1,\ln}{A}_a + \overset{1}{A}_a) + o(r^{-1}),
\end{aligned} \tag{4.13}$$

where all coefficients are functions of u and x^a, and $o(1/r^n)$ denotes quantities decaying faster than $1/r^n$ as $r \to \infty$. We show in appendix A of [73] that (4.13) is consistent with the field equations and the harmonic gauge condition. It is also showed that the $r \to \infty$ expansion of both equations leads to a hierarchy of equations that can be recursively solved to determine

[2] Condition (4.7) is in fact stronger than what we strictly need in this paper. In order to get a $O(\omega^0)$ subleading behavior it suffices to require $\partial_u A_a(u,x) = O(1/|u|^{2+\epsilon})$. We however keep (4.7) as it represents the fall-offs compatible with an all-order power expansion in ω, as available for tree-level amplitudes.

the coefficients in (4.13) in terms of the free data A_a (modulo integration constants that can be specified by boundary conditions in u).

The field strength is found to have the following leading $r \to \infty$ behavior,[3]

$$\mathcal{F}_{ru} = r^{-2} F_{ru} + o(r^{-2}), \quad \mathcal{F}_{ra} = r^{-2} F_{ra} + o(r^{-2}), \quad \mathcal{F}_{ua} = F_{ua} + o(1), \quad \mathcal{F}_{ab} = F_{ab} + o(1). \tag{4.14}$$

From (4.1) and (4.13) one has

$$F_{ua} = \partial_u A_a, \quad F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]. \tag{4.15}$$

F_{ua} plays the role of asymptotic transverse chromo-electric field and F_{ab} is the curvature of A_a when viewed as a 2d gauge connection on the celestial sphere. The remaining leading components in (4.14) are determined by the asymptotic field equations,

$$\partial_u F_{ru} + D^a F_{ua} = 0, \tag{4.16}$$

$$-2\partial_u F_{ra} + D_a F_{ru} + D^b F_{ba} = 0, \tag{4.17}$$

We shall later integrate these equation using the boundary conditions

$$\lim_{u \to +\infty} F_{ru}(u,x) = 0, \quad \lim_{u \to +\infty} F_{ra}(u,x) = 0. \tag{4.18}$$

In analogy to the abelian case [11, 18, 50], we interpret (4.18) as due to the absence of massive colored fields. Similar $u \to +\infty$ boundary conditions may also hold for other coefficients of the field strength, but (4.18) will suffice for our purposes.

4.2.1 Residual large gauge symmetries

In order for gauge symmetries (4.3) to be compatible with the harmonic gauge, they must satisfy

$$\nabla^\mu \delta_\Lambda \mathcal{A}_\mu = \nabla^\mu \mathcal{D}_\mu \Lambda = \Box \Lambda + [\mathcal{A}_\mu, \nabla^\mu \Lambda] = 0. \tag{4.19}$$

This introduces a field-dependence on residual gauge parameters. For the moment, we notice that the commutator of field-dependent gauge transformations can be written as (see e.g. [92]),

$$[\delta_\Lambda, \delta_{\Lambda'}]\mathcal{A}_\mu = \delta_{[\Lambda,\Lambda']^*} \mathcal{A}_\mu, \tag{4.20}$$

where the modified bracket is defined as

$$[\Lambda, \Lambda']^* := [\Lambda, \Lambda'] + \delta_\Lambda \Lambda' - \delta_{\Lambda'} \Lambda, \tag{4.21}$$

where $\delta_\Lambda \Lambda'$ is the change in Λ' under a gauge transformation δ_Λ due to its non-trivial dependence on the gauge field. One can verify that $[\Lambda, \Lambda']^*$ satisfies (4.19) provided Λ and Λ' do

[3] The gauge field (4.13) appears to introduce logarithmic terms that are overleading to those displayed in (4.14). These however vanish due to the field equations.

so.

We will be interested in large gauge parameters with leading behavior $O(r^0)$ and $O(r^1)$. We denote these two types of parameters by

$$\Lambda^0_\lambda(r,u,x) \overset{r\to\infty}{=} \lambda(x)+\cdots \quad (4.22)$$
$$\Lambda^1_\varepsilon(r,u,x) \overset{r\to\infty}{=} r\varepsilon(x)+\cdots\,. \quad (4.23)$$

As it is showed in Appendix B in [73], the coefficients $\lambda(x)$ and $\varepsilon(x)$ are the "free data" for the gauge parameters and the dots represent subleading terms that can be determined by solving (4.19). Notice that only the $O(r^0)$ gauge parameters are compatible with the radiative fall-offs (4.13). For those one can show that[4]

$$[\Lambda^0_\lambda,\Lambda^0_{\lambda'}]^* = \Lambda^0_{[\lambda,\lambda']}. \quad (4.24)$$

In section 4.3 we will present a relaxation of the radiative fall-offs that admit $O(r)$ gauge symmetries to first order in the parameter $\varepsilon(x)$. This will allow us to compute the bracket (4.21) between Λ^0_λ and Λ^1_ε.

4.2.2 Review of known asymptotic charges

$O(r^0)$ gauge transformations $\delta_{\Lambda^0_\lambda}\mathcal{A}_\mu$ induce an action on the free data A_a that we denote by δ^0_λ and is given by

$$\delta^0_\lambda A_a = D_a\lambda = \partial_a\lambda + [A_a,\lambda]. \quad (4.25)$$

One can verify this action is symplectic wrt Ω^{rad} (4.10) and satisfies

$$\Omega^{\text{rad}}(\delta,\delta^0_\lambda) = \delta Q^{0,\text{rad}}_\lambda \quad (4.26)$$

with

$$Q^{0,\text{rad}}_\lambda = \int_{\mathscr{I}^+} \text{Tr}\big(\partial_u A^a D_a\lambda\big)dud^2x. \quad (4.27)$$

An alternative way to obtain this charge is to evaluate the bulk expression (4.5) for $\Lambda=\Lambda^0_\lambda$ and $\Sigma\to\mathscr{I}^+$. Since (4.5) is a total derivative, this results in a pure boundary term (see e.g. [26])

$$Q^{0,\text{rad}}_\lambda = \int_{\mathscr{I}^+_-} \text{Tr}\big(\lambda(x)F_{ru}(u=-\infty,x)\big)d^2x, \quad (4.28)$$

where $\mathscr{I}^+_- \approx S^2$ is the $u=-\infty$ boundary of $\mathscr{I}^+$. The equality between (4.28) and (4.27) follows from the field equation (4.16) and the boundary condition (4.18).

[4] It is easy to verify that in this case the leading term of the bracket (4.21) is given by the ordinary bracket. Since the leading term determines all subleading terms via the gauge parameter equation (4.19), one concludes both sides of (4.24) are equal.

LPS charges

The sub-leading soft gluon factorization formula takes the same form as its abelian counterpart, with color factors replacing abelian charges [93]. Since we are dealing with pure YM theory, the external colored states are just gluons. The corresponding creation/annihilation operators are proportional to the negative/positive energy components of the Fourier transformed asymptotic gauge field,

$$\hat{A}_a(\omega, x) = \int_{-\infty}^{\infty} du e^{i\omega u} A_a(u, x). \tag{4.29}$$

The non-abelian version of the LPS charges are parametrized by Lie-algebra valued sphere vector fields Y^a according to

$$Q_Y = Q_Y^{\text{soft}} + Q_Y^{\text{hard}} \tag{4.30}$$

where

$$Q_Y^{\text{soft}} = 2 \lim_{\omega \to 0} \partial_\omega \left(\omega \int d^2x \text{Tr}\big(Y^z \partial_z^2 \hat{A}_{\bar{z}}(\omega, x) + Y^{\bar{z}} \partial_{\bar{z}}^2 \hat{A}_z(\omega, x)\big) \right) \tag{4.31}$$

(z and $\bar{z}$ are stereographic coordinates on the celestial sphere) and Q_Y^{hard} is defined by[5]

$$[Q_Y^{\text{hard}}, \hat{A}_a(\omega, x)]_{\text{op}} = \delta_Y \hat{A}_a(\omega, x), \tag{4.32}$$

$$\delta_Y \hat{A}_a := [\partial_a Y^a \partial_\omega - \omega^{-1} \mathcal{L}_Y, \hat{A}_a] = [\partial_a Y^a, \partial_\omega \hat{A}_a] - \omega^{-1}\big([Y^b, \partial_b \hat{A}_a] + [\partial_a Y^b, \hat{A}_b]\big). \tag{4.33}$$

Following [20], one can use these definitions to obtain expressions of the charges in terms of the radiative data $A_a(u, x)$. One finds

$$Q_Y^{\text{soft}} = -2 \int du d^2x\, u \text{Tr} Y^z \partial_z^2 \partial_u A_{\bar{z}} + c.c, \tag{4.34}$$

$$Q_Y^{\text{hard}} = \int du d^2x\, u \text{Tr}\big(\partial_a Y^a J_u - Y^a \partial_u J_a\big), \tag{4.35}$$

where

$$J_u := [A^a, \partial_u A_a], \quad J_z := 2q^{z\bar{z}}[A_z, \partial_z A_{\bar{z}}]. \tag{4.36}$$

The first and second term in (4.35) correspond to the first and second term in (4.33).[6] As in [20], the factors J_u and J_z are related to the $O(r^{-2})$ components of the spacetime current, which in our case is just the pure YM "current" $\mathcal{J}_\nu = -\nabla^\mu[\mathcal{A}_\mu, \mathcal{A}_\nu] - [\mathcal{A}^\mu, \mathcal{F}_{\mu\nu}]$.

4.3 Extended phase space and $O(r)$ charge algebra

In this section we present an extension of the radiative phase space that supports linearized $O(r)$ large gauge symmetries.

Whereas the standard radiative space Γ^{rad} is parametrized by gauge fields $A_a(u, x)$ at null in-

[5] We use $[,]_{\text{op}}$ to denote operator commutators in order to distinguish them from the Lie algebra brackets $[,]$. We have absorbed a factor of i in the definition of δ_Y; the action of the hard charge is given by i times (4.33).

[6] The J_a term in (4.35) differs by a total u-derivative from the expression in [20]. Our prescription ensures convergence of the u integral under the assumed fall-offs (4.7).

finity, the extended space Γ^{ext} will include an extra scalar field $\phi(x)$ that can be interpreted as the Goldstone mode associated to $O(r)$ large gauge symmetries, similar to other known instances of asymptotic symmetries [26]. This is also reminiscent of the Stückelberg procedure [94,95], which reinstates a broken local symmetry in the action of some field theory via the introduction of additional fields transforming non-linearly (similar to our ϕ). The Stückelberg procedure consists of performing a transformation of the non-invariant action (for example a gauge theory with a mass term), and then promoting the parameters to new fields.

First, we present the extended space and the corresponding action of $O(r^0)$ and $O(r)$ large gauge symmetries, denoted respectively by δ^0_λ and δ^1_ε. Next, we aim to identify the corresponding charges Q^0_λ and Q^1_ε. Rather than attempting a first-principles derivation of such structure (which would require a subtle renormalization procedure as in [31,96]), we seek to obtain the charges from a set of consistency conditions. The conditions are presented in subsection 4.3.2, and the corresponding charges are derived in subsection 4.3.3 and subsection 4.3.4. Finally, by demanding the charges to arise from a symplectic structure, we obtain in section subsection 4.3.5 a candidate symplectic form on Γ^{ext}. This allows us to realize the $O(r)$ symmetry algebra obtained in subsection 4.3.1 as a Poisson bracket charge algebra. This "reverse-logic" approach of "charges before symplectic structure" is inspired by our previous analysis in the gravitational case reviewed in chapter 1.

4.3.1 Extended space and $O(r)$ variation algebra

We would like to minimally relax the radiative fall-offs described in section 4.2 so as to allow for $O(r)$ gauge transformations. A natural way to proceed is to apply all possible $O(r)$ gauge transformations to these radiative fields. As we previously discussed, in the YM case this procedure cannot be done consistently without allowing for higher order $O(r^n)$ gauge transformations. As a first step, let us consider a linearized enlargement along the $O(r)$ gauge direction:

$$\Gamma^{\text{ext}} := \{\tilde{\mathcal{A}}_\mu = \mathcal{A}_\mu + \mathcal{D}_\mu \Lambda^1_\phi, \quad \mathcal{A}_\mu \in \Gamma^{\text{rad}}, \quad \phi \in C^\infty(S^2)\}. \tag{4.37}$$

Since Γ^{rad} is parametrized by fields $A_a(u,x)$, the extended space is parametrized by pairs[7]

$$\Gamma^{\text{ext}} \approx \{(A_a(u,x), \phi(x))\}. \tag{4.38}$$

By construction, the space (4.37) supports the action of $O(r)$ gauge transformations (4.23). In the parametrization (4.38), the action is simply given by

$$\delta^1_\varepsilon A_a = 0, \quad \delta^1_\varepsilon \phi = \varepsilon. \tag{4.39}$$

We emphasize that we are working to first order in ϕ and ε. All our expressions should be understood to hold modulo $O(\phi^2)$, $O(\varepsilon^2)$ and $O(\phi\varepsilon)$ terms.

[7] In the analogy with the gravitational case, ϕ would correspond to a sphere diffeomorphism labeling the different superrotation sectors, see e.g. [31]. Unlike the gravitational case, we linearize the finite gauge transformation $\mathcal{A}_\mu \to e^{\Lambda^1_\phi}\mathcal{A}_\mu e^{-\Lambda^1_\phi} + e^{\Lambda^1_\phi}\partial_\mu e^{-\Lambda^1_\phi} \approx \mathcal{A}_\mu + \mathcal{D}_\mu\Lambda^1_\phi + O(\phi^2)$.

We next need to specify how $O(r^0)$ gauge transformations act on Γ^{ext}. In the parametrization (4.38) we define

$$\delta^0_\lambda A_a = D_a\lambda, \quad \delta^0_\lambda \phi = [\phi, \lambda], \tag{4.40}$$

leading to an algebra of variations

$$[\delta^0_\lambda, \delta^0_{\lambda'}] = \delta^0_{[\lambda,\lambda']}, \quad [\delta^1_\varepsilon, \delta^0_\lambda] = \delta^1_{[\varepsilon,\lambda]}, \quad [\delta^1_\varepsilon, \delta^1_{\varepsilon'}] = 0. \tag{4.41}$$

We take (4.41) as the defining relations for the (linearized) $O(r)$ large gauge symmetry algebra. It can be shown ([73]) that this algebra follows from the bracket (4.21) between $O(r^0)$ and $O(r)$ gauge parameters.

4.3.2 Conditions on $O(r)$ asymptotic charges

Since we do not yet know the symplectic structure on Γ^{ext}, we will find the charges by imposing certain conditions we expect them to satisfy. Our requirements for the charges are:

1. $Q^0_\lambda|_{\Gamma^{\text{rad}}} = Q^{0,\text{rad}}_\lambda$
2. Q^1_ε is compatible with the tree-level subleading soft gluon factorization
3. $\delta^0_\lambda Q^1_\varepsilon + \delta^1_\varepsilon Q^0_\lambda = 0$
4. $\delta^0_\lambda Q^1_\varepsilon = -Q^1_{[\varepsilon,\lambda]}$

The first condition requires that when Q^0_λ is restricted to $\Gamma^{\text{rad}} \subset \Gamma^{\text{ext}}$, one recovers the standard expression (4.27) for the radiative phase space $O(r^0)$ charge (which is known to encode the leading soft gluon factorization). As we shall discuss, the second condition fixes the dependence of Q^1_ε on $A_a(u,x)$ up to (hard) quadratic order. The third is a necessary condition for the existence of a Poisson bracket realization of the symmetries. The last condition, probably the least well-motivated one, requires the charges to reproduce the variation algebra (4.41) without extension terms.

Our strategy to obtain the charges is as follows. It turns out that conditions 1 and 3 uniquely fix Q^0_λ in terms of $Q^{0,\text{rad}}_\lambda$ and Q^1_ε, once the latter is known. The most difficult part is then to find Q^1_ε satisfying conditions 2 and 4.

4.3.3 Q^1_ε

Condition 2 can be restated as the condition that the Ward identity generated by Q^1_ε is compatible with the one generated by the LPS charge Q_Y. In the abelian case, it was shown in [24] that Q_Y can be understood in terms of an $O(r)$ large gauge charge and its magnetic dual, by splitting the vector field Y^a into "electric" and "magnetic" components

$$Y_a = \frac{1}{2}(\partial_a \varepsilon + \epsilon_a{}^b \partial_b \mu), \tag{4.42}$$

where $\varepsilon(x)$ and $\mu(x)$ are interpreted as the $O(r)$ coefficients of large gauge (and dual gauge) parameters. A first guess could then be to set $Q^1_\varepsilon = Q_{Y_a=\partial_a\varepsilon/2}$. This however does not satisfy the gauge covariance property required by condition 4. We shall correct this initial guess so that the resulting charge satisfies 4 without affecting its compatibility with the tree-level soft gluon theorem. We will proceed in two stages: First "covariantize" Q_Y and then consider a gauge covariant version of the splitting (4.42).

It is easy to verify that the expression of Q_Y given in Eqs. (4.34), (4.35) is not gauge covariant at null infinity, i.e.

$$\delta^0_\lambda Q_Y \neq -Q_{[Y,\lambda]}. \tag{4.43}$$

Notice however that since Q_Y was read off from a tree-level soft theorem, it only captures terms at most quadratic in $A_a(u,x)$ ([73]). That is, Q_Y should be understood as giving the $O(A)$ and $O(A^2)$ parts of an asymptotic charge that may contain higher order terms. In addition, there can also be $O(A^2)$ "soft" contributions that do not affect the single soft theorem (but which could leave an imprint in the double-soft behavior). Given this freedom, we now explore the possibility of completing Q_Y into a gauge-covariant charge.

A natural way to proceed is to look for an expression of the charge in terms of the field strength, as in the rewriting of Q^0_λ given in Eq. (4.28). The starting point are the asymptotic field equations (4.17), (4.16) that relates the $O(r^{-2})$ components of $\mathcal{F}_{ra}$ and $\mathcal{F}_{ru}$, given the boundary condition (4.18). Explicitly,

$$F_{ru} = \partial^a A^+_a - \partial^a A_a + \int_u^\infty J_{u'} du', \tag{4.44}$$

where $A^+_a(x) = A_a(u = +\infty, x)$ and J_u is given in (4.36).

From (4.15) one finds that $F_{ra} = O(u)$ as $u \to -\infty$. The coefficient of the $O(u)$ factor is determined by the $O(1)$ coefficient of the asymptotic value of the last two terms in (4.15). One can then write an expression for the finite part of the $u \to -\infty$ asymptotic value of F_{ra}, out of which the charge candidate is defined:

$$\begin{aligned} Q^{\text{cov}}_Y &:= \lim_{u\to-\infty} \int d^2x \text{Tr} Y^a \left(2F_{ra} - u(D_a F_{ru} + D^b F_{ba})\right), & (4.45)\\ &= \int dud^2xu\text{Tr}Y^a \partial_u (D_a F_{ru} + D^b F_{ba}), & (4.46)\end{aligned}$$

where to get the second equality we relied on the $u \to \infty$ boundary conditions (4.18) to express the charge as a total u-derivative, and used the field equation (4.15) to simplify the resulting expression.[8] By construction, the charge expression (4.46) is gauge covariant, i.e. it satisfies $\delta^0_\lambda Q^{\text{cov}}_Y = -Q^{\text{cov}}_{[Y,\lambda]}$.

We now discuss its relation with the LPS charge Q_Y. By direct computation ([73]), it can be establish the following identity,

$$Q^{\text{cov}}_Y = Q_Y + \frac{1}{2}\int dud^2xu\text{Tr}(\partial_a Y_b - \partial_b Y_a)\partial_u[A^a, A^b] + \cdots, \tag{4.47}$$

[8]Consistency of (4.18) with (4.17) requires that $\lim_{u\to\infty} F_{ab} = 0$.

where the dots indicate terms that do not affect the tree-level, single-soft gluon behavior. The second term in (4.47) is however incompatible with the subleading soft gluon theorem (since the Ward identity would include also (4.33), which already captures the tree level subleading factors) and thus presents an obstruction for the covariantization of Q_Y. Fortunately, such term is absent for purely "electric" vector fields $Y_a = \partial_a \varepsilon$, which, as described earlier, are the ones relevant for $O(r)$ large gauge charges.[9]

We finally address the non-covariance in the decomposition (4.42). A first guess is to write $Y_a = D_a\varepsilon = \partial_a\varepsilon + [A_a, \varepsilon]$. This however introduces unwanted quadratic terms in (4.47) that would spoil the compatibility of the charge with the soft theorem. To avoid this problem, we consider a gauge covariant derivative associated to the $u \to -\infty$ asymptotic value of A_a,

$$Y_a = D_a^- \varepsilon := \partial_a \varepsilon + [A_a^-, \varepsilon]. \tag{4.48}$$

With this definition, the quadratic terms introduced in (4.47) are "soft" and hence do not affect the single soft theorem. We thus define the $O(r)$ large gauge charge as

$$Q^1_\varepsilon := Q^{\text{cov}}_{Y_a = D_a^- \varepsilon/2} = \int d^2x \text{Tr}(\varepsilon \pi), \tag{4.49}$$

where

$$\pi(x) := -\frac{1}{2} \int_{-\infty}^{\infty} du u \partial_u D_a^- (D^a F_{ru} + D_b F^{ba}), \tag{4.50}$$

is a function of $A_a(u, x)$, due to Eqs. (4.15), (4.44), (4.48). Notice that the charge is independent of the ϕ direction in Γ^{ext} (4.38). This is because we are working to order $O(\varepsilon) = O(\phi)$ and Q^1_ε is already first order in ε.

By construction π is gauge covariant, in the sense that

$$\delta^0_\lambda \pi = [\pi, \lambda]. \tag{4.51}$$

This immediately implies that Q^1_ε satisfies the desired covariance property

$$\delta^0_\lambda Q^1_\varepsilon = -Q^1_{[\varepsilon,\lambda]}. \tag{4.52}$$

We conclude by emphasizing that our definition of Q^1_ε does not follow uniquely from requirements 2 and 4 above. For instance, one could consider a different prescription for the covariant gradient in (4.48), or use a different field-strength component as a starting point (e.g. $\mathcal{F}_{ru}$ instead of $\mathcal{F}_{ra}$, which lead to identical expressions only in the abelian case). All choices would lead to an expression of the form (4.49) with slightly different versions of $\pi(x)$. It may be that higher order relations omitted in this work (like those that would follow from the commutation between two $O(r)$ charges) could further constrain, and perhaps single out, the form of $\pi(x)$. The discussion in the following sections however is insensitive to the specific form of

[9]Eq. (4.47) appears to be in conflict with the interpretation of Q_Y as a sum of electric *and* magnetic $O(r)$ large gauge charges [24]. This may be related with known obstructions for a non-abelian extension of electric-magnetic abelian duality [97]. See [98] for a recent discussion of non-abelian magnetic charges at null infinity.

$\pi(x)$ and only uses the covariance property (4.51).

4.3.4 Q^0_λ

We now discuss the extension of $Q^{0,\text{rad}}_\lambda$ to Γ^{ext}. Given condition 4 is satisfied, condition 3 can be written as

$$\delta^1_\varepsilon Q^0_\lambda = Q^1_{[\varepsilon,\lambda]}. \tag{4.53}$$

Since $\delta^1_\varepsilon A_a = 0$ and $\delta^1_\varepsilon \phi = \varepsilon$, the simplest extension of $Q^{0,\text{rad}}_\lambda$ that is compatible with (4.53) is

$$Q^0_\lambda = Q^{0,\text{rad}}_\lambda + Q^1_{[\phi,\lambda]}. \tag{4.54}$$

In fact, this is the unique solution to conditions 1 and 3 (for a given Q^1_ε). To see why, consider a different extension $\tilde{Q}^0_\lambda$ and write it as

$$\tilde{Q}^0_\lambda = Q^0_\lambda + K_\lambda, \tag{4.55}$$

for some function K_λ on Γ^{ext}. Condition 3 then implies

$$\delta^1_\varepsilon K_\lambda = 0. \tag{4.56}$$

Given the action of δ^1_ε (4.39), it follows that K_λ must be independent of ϕ. Thus, K_λ must vanish in order to ensure that $\tilde{Q}^0_\lambda|_{\phi=0} = Q^{0,\text{rad}}_\lambda$.

It is interesting to note that due to the gauge covariance of both terms in (4.54) it follows that

$$\delta^0_\lambda Q^0_{\lambda'} = -Q^0_{[\lambda',\lambda]}. \tag{4.57}$$

Together with (4.52), this implies the proposed charges Q^0_λ and Q^1_ε reproduce the total variation algebra (4.41).

4.3.5 Extended symplectic form and charge algebra

We finally present a symplectic form Ω^{ext} on Γ^{ext} that is compatible with the charges, in the sense that

$$\delta Q^0_\lambda = \Omega^{\text{ext}}(\delta, \delta^0_\lambda), \quad \delta Q^1_\varepsilon = \Omega^{\text{ext}}(\delta, \delta^1_\varepsilon). \tag{4.58}$$

Given the second condition in (4.58) and the form (4.49) of Q^1_ε we are lead to define

$$\Omega^{\text{ext}} := \Omega^{\text{rad}} + \int d^2x \text{Tr}(\delta\pi \wedge \delta\phi) \tag{4.59}$$

where we recall that

$$\Omega^{\text{rad}} = \int dud^2x \text{Tr}(\delta\partial_u A^a \wedge \delta A_a). \tag{4.60}$$

Indeed, since $\delta^1_\varepsilon A_a = 0$ (and consequently $\delta^1_\varepsilon \pi = 0$) the only non-trivial contribution to $\Omega^{\text{ext}}(\delta, \delta^1_\varepsilon)$ is

$$\Omega^{\text{ext}}(\delta, \delta^1_\varepsilon) = \int d^2x \text{Tr}(\delta\pi\delta^1_\varepsilon\phi) = \int d^2x \text{Tr}(\delta\pi\varepsilon) = \delta \int d^2x \text{Tr}(\pi\varepsilon) = \delta Q^1_\varepsilon. \tag{4.61}$$

We can now verify that (4.59) satisfies the first condition in (4.58):

$$\begin{aligned}
\Omega^{\text{ext}}(\delta, \delta^0_\lambda) &= \Omega^{\text{rad}}(\delta, \delta^0_\lambda) + \int d^2x \text{Tr}(\delta\pi\delta^0_\lambda\phi - \delta^0_\lambda\pi\delta\phi) && (4.62)\\
&= \delta Q^{0,\text{rad}}_\lambda + \int d^2x \text{Tr}(\delta\pi[\phi, \lambda] - [\pi, \lambda]\delta\phi) && (4.63)\\
&= \delta Q^{0,\text{rad}}_\lambda + \delta Q^1_{[\phi,\lambda]} = \delta Q^0_\lambda. && (4.64)
\end{aligned}$$

With this symplectic form we can finally realize the relations (4.52), (4.53) and (4.53) as a Poisson bracket algebra,[10]

$$\{Q^0_\lambda, Q^0_{\lambda'}\} = Q^0_{[\lambda,\lambda']}, \quad \{Q^0_\lambda, Q^1_\varepsilon\} = Q^1_{[\lambda,\varepsilon]}. \tag{4.65}$$

Remark 4.1. Following the discussion at the end of chapter 2, constructing an extension of EYM such that GBMS is coupled with the leading and subleading extension in YM poses some non-trivial issues to be resolved. In particular, the commutation between δ^1_ϵ and δ_f: by the action of δ_f on A_a, see (2.35), we have, schematically,

$$\delta_f Q^1_\epsilon = Q^0_{\hat{\mu}} \tag{4.66}$$

for some $\hat{\mu}$. This implies a non-trivial commutation,

$$[\delta^1_\epsilon, \delta_f] = \delta^0_{\hat{\mu}}. \tag{4.67}$$

We left for future works the understanding of this structure.

[10] The Poisson bracket between two functions F and G is given by $\{F, G\} = \Omega^{\text{ext}}(X_G, X_F)$ where X_F is the symmetry transformation generated by F, i.e. $\Omega^{\text{ext}}(\delta, X_F) = \delta F$.

Infinite hierarchy of asymptotic charges in Electrodynamics

In this chapter we focus on large gauge transformations beyond $O(1)$ for Electrodynamics.

In the case of Quantum Electrodynamics (QED), it was shown in [48] and [49] that for tree level amplitudes there exist an infinite number of soft theorems, each of them implying a conservation law for the tree level scattering process. Weinberg's soft photon theorem corresponds to the first level in the hierarchy, while Low's sub-leading soft photon theorem [15,99] corresponds to the second level.

The conserved quantities found in [19] for the S-matrix constitute the first level in an infinite hierarchy of soft theorems leading to conserved charges. A first approach was done in [51], where Seraj showed that at spatial infinity there are an infinite number of conserved quantities, proportional to the multipole moments, and generated by specific large gauge transformations of order $O(r^n)$. In [50], Campiglia and Laddha showed that for tree level scattering, and restricting the radiative data space to a suitable subset, there exists an infinite tower of conservation laws such that at each level there is an infinite dimensional family of conserved charges Q^n_ϵ, labelled by functions on the sphere. The authors also presented evidence that the Ward identities associated to the level n of the charges are equivalent to sub-n soft photon theorems, along with the conservation laws within the classical theory. The non-abelian case is substantially harder, since the charges up to level n of the hierarchy do not form a close algebra, as in the abelian case. In [73] it is suggested a first step towards a classical derivation of the charge hierarchy in the non-abelian case. Some recent developments in celestial holography using Operator Product Expansion (OPE) tools [90,100,101] seem to be promising avenues in the study of asymptotic symmetries and the role of CCFT in flat holography for Yang-Mills and gravity. Also, in [102], the incorporation of logarithmic charges at spatial infinity has been done.

The r-expansion in retarded coordinates of the LGT's at the bulk establishes a hierarchy of charges at the asymptotic region. $O(1)$ LGT's correspond to leading charges (for instance, by imposing a constant LGT we obtain the total electric charge of the system, [26]), while $O(r)$ LGT corresponds to sub-leading charges, see [20,24]. In

The results of the previous chapter impose the following question: can the infinite tower of charges associated to sub^n-leading soft photon/gluons theorems be canonically derived within the classical theory? Two main issues arise when studying $O(r^n)$ LGT's in Maxwell and Yang-Mills. First, the divergent formulas for the charges when calculated from the usual phase space structure, as was shown in [24]. In particular, the expressions for the symplectic form evaluated on a LGT at level n (and therefore the charges) diverge both in the $t \to +\infty$ and $u \to -\infty$ limits. Second, for Yang-Mills, the representation of the variation algebra on the phase space has to be well defined. Otherwise, the non-linear terms automatically enter in every variation.

In this chapter we provide a renormalization procedure that allows to compute the charges given in [50] from first principles. First, in section section 5.2, we review the computation for leading and subleading charges, which was done in the previous chapter for Yang-Mills. The difference this time is that we explicitly compute the symplectic form. Following ideas from [96], in section 5.3 we show that there exist suitable boundary and corner terms for the symplectic form that renormalise the divergences, while not changing the dynamics of the fields. we define a subset of the radiative space and an extended phase space that contains all LGT's up to arbitrary order. This extended space is provided with a sympletic structure, that allows us to calculate the electric-type charges. In section 5.4, by incorporating the duality symmetry, the magnetic analogue of the electric hierarchy is also presented, as well as the full electromagnetic charge algebra.

5.1 Preliminaries

5.1.1 Radiative phase space

For definiteness we consider a massless charged scalar field ϕ coupled to the Maxwell field $\mathcal{A}_\mu$ in Minkowski spacetime, but our analysis can be generalized to the situation with massive charged fields or Fermions. The standard lagrangian is given by,

$$\mathscr{L}[\mathcal{A}_\mu] = -\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \mathcal{D}_\mu\phi\overline{\mathcal{D}^\mu\phi}, \tag{5.1}$$

and satisfying the field equations,

$$\begin{aligned} \nabla^\nu \mathcal{F}_{\mu\nu} &= j_\mu, & (5.2)\\ \mathcal{D}_\mu \mathcal{D}^\nu \phi &= 0, & (5.3) \end{aligned}$$

where $j_\mu = ie\phi\overline{\mathcal{D}_\mu\phi} + c.c.$, with $\mathcal{D}_\mu\phi := \partial_\mu\phi - ie\mathcal{A}_\mu\phi$, the gauge covariant derivative and ∇ the metric covariant derivative. In retarded coordinates, Maxwell equations are

$$\begin{aligned} r^2 j_r &= -\partial_r(r^2\mathcal{F}_{ru}) + \partial^a\mathcal{F}_{ra}, & (5.4)\\ r^2 j_u &= -\partial_r(r^2\mathcal{F}_{ru}) + \partial_u(r^2\mathcal{F}_{ru}) + \partial^a\mathcal{F}_{ua}, & (5.5)\\ j_a &= \partial_r(\mathcal{F}_{ua} - \mathcal{F}_{ra}) + \partial_u\mathcal{F}_{ra} + \frac{1}{r^2}\partial^b\mathcal{F}_{ab}, & (5.6) \end{aligned}$$

where ∂_a denotes the covariant derivative on the sphere compatible with q_{ab}, as in the previous chapter. Bianchi identities , $0 = \partial_{[a}\mathcal{F}_{bc]}$, are the integrability conditions for the electromagnetic strength tensor: there exists a one-form $\mathcal{A}_\mu$ such that $F_{\mu\nu} = \partial_{[\mu}\mathcal{A}_{\nu]}$. We will work in the harmonic gauge, $\nabla^\mu \mathcal{A}_\mu = 0$, which in this particular coordinates implies

$$r^2\partial_u\mathcal{A}_u + \partial_r(r^2\mathcal{A}_r) + r^2\partial^a\mathcal{A}_a = 0. \tag{5.7}$$

The usual fall off for the electromagnetic tensor are (see [26] and [73]):

$$\mathcal{F}_{ru} = \frac{1}{r^2}F^{(-2)}_{ru} + o(r^{-2}), \quad \mathcal{F}_{ar} = o(r^{-1}), \quad \mathcal{F}_{au} = O(1), \quad \mathcal{F}_{ab} = F^{(0)}_{ab} + o(1). \tag{5.8}$$

The fal-off for the scalar field is,

$$\phi = \frac{\varphi}{r} + o(r^{-1}). \tag{5.9}$$

These expressions imply the following fall offs on the charge current:

$$j_u = \frac{j_u^{(-2)}}{r^2} + o(r^{-2}), \quad j_a = \frac{j_A^{(-2)}}{r^2} + o(r^{-2}), \quad j_r = \frac{j_r^{(-2)}}{r^4} + o(r^{-4}). \tag{5.10}$$

Fall offs for $\mathcal{A}_\mu$ compatible with the expansion above and the harmonic gauge condition are:

$$\mathcal{A}_a = A^{(0)}_a + o(1), \quad \mathcal{A}_u = A_u^{(-1,\ln)}\frac{\ln r}{r} + o(r^{-1}), \quad \mathcal{A}_r = o(r^{-1}). \tag{5.11}$$

The previous asymptotic behaviours are consistent with the field equations and the harmonic gauge condition, as can be computed also form the previous chapter, by taking an abelian group G. Moreover, using Maxwell equations, the scalar field equation, Bianchi identities and the harmonic gauge condition, we can solve all the components of the electromagnetic tensor and the scalar field in terms of $A^{(0)}_a$ and $\phi^{(-1)}$ (see appendix A of [73] for Yang-Mills case). These functions are the free data for the Maxwell field and the scalar field, respectively.

The hypothesis of "tree-level" decays for A_a in the limits $u \to \pm\infty$,

$$\partial_u A_a(u, x^1, x^2) = O(1/|u|^\infty), \tag{5.12}$$

that is, its decay is faster than that of any power $1/|u|^n$, implies the following fall offs for the radiative data of a generic solution of Maxwell's equations, ([4])

$$F^{(-2)}_{ru}(u, x^1, x^2) = F^{-2,0}_{ru}(x^1, x^2) + O(1/|u|^\infty). \tag{5.13}$$

For the massless field we assume no "soft" charged particles,

$$\varphi(u, x^1, x^2) = O(1/|u|^\infty). \tag{5.14}$$

Our radiative phase space is thus defined in terms of the functions A_a and φ,

$$\Gamma^{\text{rad}} = \{(A_A(u,x^a), \varphi(u,x^a)) : \partial_u A_a(u,x^1,x^2), \varphi(u,x^1,x^2) = O(1/|u|^\infty)\}. \tag{5.15}$$

5.1.2 *u*-expansions for fields

From Maxwell equations and Bianchi identities, we can obtain recursive formulas for the coefficients in the $\mathcal{F}_{ru}$ expansion in r and u. By Bianchi identity $\partial_{[a}\mathcal{F}_{ru]} = 0$, contracting with ∂^a and the first two Maxwell equations, we have

$$\Delta\mathcal{F}_{ru} + \partial_r(\partial_r(r^2\mathcal{F}_{ru}) - 2r^2\partial_u\mathcal{F}_{ru}) = r^2\partial_u j_r - \partial_r(r^2 j_u) \tag{5.16}$$

where Δ denotes the Laplacian operator on the sphere. We assume that $\mathcal{F}_{ru}$ can be expanded in an r-series [1], $\mathcal{F}_{ru} = \frac{1}{r^2}\sum_{k=0}^{\infty}\frac{F_{ru}^{(-2-k)}}{r^k}$, and substituting in (5.16),

$$2(k+1)\partial_u F_{ru}^{(-2-k-1)} + (\Delta + k(k+1))\, F_{ru}^{(-2-k)} = \partial_u j_r^{(-2-k)} + k j_r^{(-2-k)}. \tag{5.17}$$

From the assumed fall off (5.13), and equation (5.17), it is clear that the behaviour of $F_{ru}^{(-2-n)}$ in the limit $u \to -\infty$ is

$$F_{ru}^{(-2-n)} = \sum_{j=0}^{n} u^j F_{ru}^{(-2-n,j)}(x^a) + r_n(u,x^a), \tag{5.18}$$

where each of the $\mathcal{F}_{ru}^{(-2-n,0)}(x^a)$ is a function on the sphere, and r_n some function with an $O(1/u^\infty)$ decay (analogous expansion can be done in the limit $u \to +\infty$). We can solve order by order recursively in terms of the current and this free functions. As a reference, the full expression for $\mathcal{F}_{ru}$ is

$$r^2\mathcal{F}_{ru} = \sum_{k\geq 0}\frac{1}{r^k}\sum_{j=0}^{k} u^j F_{ru}^{(-2-k,j)}(x^a) + O(1/u^\infty). \tag{5.19}$$

5.1.3 Variation space

We now turn to the large gauge transformations (LGT) on the variation space. The usual formulas for the gauge symmetries,

$$\mathcal{A}_\mu \mapsto \mathcal{A}_\mu + \partial_\mu\epsilon, \quad \phi \mapsto e^{-ie\epsilon}\phi \tag{5.20}$$

establish the following action for variations of the fields,

$$\delta_\epsilon\mathcal{A}_\mu = \partial_\mu\epsilon, \quad \delta_\epsilon\phi = -ie\epsilon\phi. \tag{5.21}$$

The variations allowed in our radiative phase space are those that are tangent to $\mathcal{F}_0$, i.e., that

[1] We discrad olyhomogeneous terms, i.e. $\log^m(r)/r^n$, although a generic analysis would contain such terms. We left for future works the inclusion of such terms.

maintain the fall offs of the fields. By the definition of finite symmetry, given a gauge symmetry generator ϵ we see that $\partial_\mu \epsilon$ must have the same fall offs as $\mathcal{A}_\mu$:

$$\partial_a \epsilon = O(1), \quad \partial_u \epsilon = o(1), \quad \partial_r \epsilon = o(r^{-1}) \tag{5.22}$$

We study the global symmetries as arising from the residual LGT, and by the choice of harmonic gauge, are solutions to the wave equation,

$$\Box \epsilon = 0. \tag{5.23}$$

This equation can be solved up to order $O(r^{-1})$ (see Appendix A in [24]),

$$\epsilon(u, r, x^a) = \epsilon_0(x^a) + O(\ln(r)/r). \tag{5.24}$$

5.1.4 Higher order LGT

We are interested in relating higher orders in r LGTs with the charges that arise from sub$^n-$leading soft photons theorems. The usual mode expansion reasoning in the soft theorem derivation suggests that for a sub$^n-$leading soft photon we need to look for a LGT Λ whose $O(1)$ in the $r-$expansion behaves as u^n. This asymptotic behaviour of the gauge generator must be compatible with the harmonic gauge, and therefore, implies an $O(r^n)$ leading behaviour, as we show below by solving $\Box\Lambda = 0$.

Consider the following $r-$expansion for a $O(r^n)$ large gauge parameter,

$$\Lambda(u, x^a) = r^n \epsilon^{(n)} + \sum_{k=0}^{n-1} r^k \epsilon^{(k)} + \frac{\ln r}{r} \epsilon^{ln} + O(r^{-1}), \tag{5.25}$$

where $\epsilon^{(i)} = \epsilon^{(i)}(u, x^1, x^2)$. We have $\Box\Lambda = 0$, which in retarded coordinates reads,

$$\begin{aligned} 0 \quad = \quad & -6r^{n-1}\partial_u \epsilon^{(n)} + \sum_{k=-1}^{n-2} r^k \left(\Delta\epsilon^{(k+2)} - 2(k+2)\partial_u \epsilon^{(k+1)} + (k+2)(k+3)\epsilon^{(k+2)} \right) \\ & + \frac{\ln r}{r^3} \Delta\epsilon^{(ln)} + \frac{2}{r^2}(\Delta\epsilon^{(0)} - \partial_u \epsilon^{(ln)}) + \frac{1}{r^3}\epsilon^{(ln)} + ... \end{aligned} \tag{5.26}$$

The first term in (5.26) imply that $\epsilon^{(n)}$ is a free function on the sphere. Next, we have a recursive equation between the successive coefficients:

$$2(k+1)\partial_u \epsilon^{(k)} = \Delta\epsilon^{(k+1)} + (k+1)(k+2)\epsilon^{(k+1)} \tag{5.27}$$

Integrating (5.27) and fixing each integration constant to zero in each step gives a LGT of order $O(r^n)$ generated by $\epsilon \equiv \epsilon^{(n)}$, which we will call Λ^n_ϵ. If the integration constants are non-zero, each one of them will be a free S^2 function that contribute linearly with a LGT of corresponding order:

$$\Lambda_\alpha = \Lambda^n_{\epsilon_n} + \Lambda^{n-1}_{\epsilon_{(n-1)}} + ..., \tag{5.28}$$

where $\alpha = \{\epsilon_j\}_j$ is the sequence of integration constants ϵ_j in the equation (5.27), that are free S^2-functions, each one generating an $O(r^j)$ LGT. We will call a LGT "pure" if there is only one free function generating it. When using the notation Λ^m_f, subscripts indicate the generating function or sequence of functions, and superscripts indicate the leading term in the $r-$expansion, if the generating function is not a sequence.

Some remarks are in order. First, one implication of equation (5.27) for a pure $O(r^n)$ LGT is the following property:

$$\epsilon^{(n-1)} = O(u), \quad ..., \quad \epsilon^{(k)} = O(u^{n-k}) \tag{5.29}$$

This shows that the order $O(r^n)$ is necessary for a u^n asymptotic behaviour at order $O(r^0)$ for the LGT, as was stated at the beginning of the section. Second, the term $\ln(r)/r$ is needed for the $O(r^0)$ to be consistent, otherwise we would get $\Delta\epsilon^{(0)} = 0$, and since we are in a sphere, that would give a trivial function. Third, the non-trivial fact that equation (5.27) resembles the form of equation (5.17), but it presents crucial differences in the constants multiplying the functions. This similarity between the recursive expressions is useful when showing the Ward identity equivalence with the sub$^n-$ soft theorems [50].

5.2 Leading and Subleading charges

In this section we review the phase space construction and the symplectic charges in the case where the large gauge transformation are $O(r)$. We leave the renormalization procedure for the next section, focusing exclusively in the first step of the phase space extension and in the recovery of the charges via the symplectic form.

5.2.1 Linearly extended phase space

The usual phase space, (5.15), contains the physical information regarding the leading order charges, restricted to $O(r^0)$ LGT. Their usual expressions are ([19], [26]):

$$Q_{\epsilon_0} = \int_{S^2} \epsilon_0 \int_{\mathbb{R}} \partial_u F^{(-2)}_{ru} dud^2x, \tag{5.30}$$

where ϵ_0 is a function on the sphere, and d^2x contains $\sqrt{\mathring{q}}$ implicitly. As soon as we lift the condition on the LGT order, the fall offs (5.11) are not preserved by an $O(r^1)$ LGT (through its action (5.20)) and therefore the variations are no longer tangent to the radiative phase space $\mathcal{F}_0$, but rather have another direction. We expand the phase space in this direction by first defining an extended version of the potential sector in (5.15). Consider the following space:

$$\Gamma^{\text{lin}} = \Gamma^{\text{rad}} \times \{\psi(x^1, x^2) : \psi \in C^\infty(S^2)\} \tag{5.31}$$

We define the electromagnetic potential as $\hat{\mathcal{A}}_\mu = \mathcal{A}_\mu + \partial_\mu \Lambda^1_\psi$, where $\mathcal{A}_\mu$ is the vector potential that has A_a as initial data (from section section 5.1) and Λ^1_ψ is the pure $O(r)$ LGT generated by ψ. Observe that this definition is indeed consistent, since $\partial_{[\mu}\partial_{\nu]}\Lambda^1_\psi = 0$ and thus a makes no contribution to the electromagnetic tensor, i.e. $\hat{\mathcal{F}} = \mathcal{F}$. This is in sharp contrast with the previous chapter, where $\delta\mathcal{F}_{\mu\nu} = [\mathcal{F}_{\mu\nu}, \Lambda]$. Observe also that the harmonic gauge condition is trivially satisfied for the extended electromagnetic potential [2].

Given a general $O(r)$ LGT, $\Lambda_{\{\epsilon_1,\epsilon_0\}}$, the variations generated by it on Γ^{lin} are splitted in terms of the S^2 free functions ϵ_1 and ϵ_0 corresponding to order $O(r)$ and order $O(1)$ in the $r-$expansion respectively (see (5.28)):

$$\Lambda_{\{\epsilon_1,\epsilon_0\}} = r\epsilon_1 + (\epsilon_0 + u\frac{1}{2}(\Delta+2)\epsilon_1) + o(r^0) \tag{5.32}$$

The action on the phase space Γ^{lin} comes from the identity $\delta_{\Lambda_{\{\epsilon_1,\epsilon_0\}}}\hat{\mathcal{A}}_\mu = \partial_\mu \Lambda_{\{\epsilon_1,\epsilon_0\}}$, which after the splitting reads:

$$\delta_{\Lambda_{\{\epsilon_1,\epsilon_0\}}} A_a = \partial_a \epsilon_0, \tag{5.33}$$

$$\delta_{\Lambda_{\{\epsilon_1,\epsilon_0\}}} \psi = \epsilon_1. \tag{5.34}$$

In the massless field sector, allowing a $O(r)$ LGT implies also a change in the massless field ϕ. The equations of motion are invariant under the simultaneous change

$$\mathcal{A}_\mu \mapsto \hat{\mathcal{A}} = \mathcal{A}_\mu + \partial_\mu \Lambda^1_\psi, \quad \phi \mapsto \hat{\phi} = e^{ie\Lambda^1_\psi}\phi \tag{5.35}$$

Since the finite gauge symmetry involves a product $e^{-ie\Lambda^1_\psi}\phi$, we can define an extended field $\hat{\phi} = e^{-ie\Lambda^1_\psi}\phi$, where ψ is the free S^2 function now generating a phase for the scalar field, while ϕ is the massless field with the usual fall off, with $\varphi \in \Gamma^{\text{rad}}$ as free data. The covariant gauge derivative is given by

$$\hat{\mathcal{D}}_\mu := \partial_\mu - ie\hat{\mathcal{A}}_\mu, \tag{5.36}$$

from where we have that the new current $\hat{j}_\mu$ maintain its original form,

$$\hat{j}_\mu = ie\hat{\phi}(\hat{\mathcal{D}}_\mu\hat{\phi})^* + c.c. = ie\phi(\mathcal{D}_\mu\phi)^* + c.c., \tag{5.37}$$

The consistency of the action of the $O(r)$ LGT action on $\hat{\phi}$ with the splitting of the extended phase space implies

$$\delta_{\Lambda_{\{\epsilon_1,\epsilon_0\}}}\varphi = ie\epsilon_0\varphi. \tag{5.38}$$

5.2.2 Calculation of leading and subleading charges

Consider the Lagrangian (5.1), in our extended phase space we have the usual symplectic potential current,

[2] It is left for future works to study the phase space extension in more general gauges, and whether it changes the structure.

$$\theta^{\mu}(\delta) = \sqrt{g}\left(\hat{\mathcal{F}}^{\mu\nu}\delta\hat{\mathcal{A}}_{\nu} + \hat{\mathcal{D}}^{\mu}\hat{\phi}\delta\bar{\hat{\phi}} + c.c.\right), \tag{5.39}$$

and the symplectic current by taking the exterior derivative in the phase space,

$$\omega^{\mu}(\delta,\delta') = \delta\theta^{\mu}(\delta') - \delta'\theta^{\mu}(\delta) - \theta([\delta,\delta']) \tag{5.40}$$

Given a Cauchy slice $\Sigma_t = \{t = cnt\}$, the symplectic form is obtained by integrating the symplectic current over Σ_t, $\Omega(\delta,\delta') = \int_{\Sigma_t}\omega^{\mu}(\delta,\delta')dS_{\mu}$. We evaluate it on a variation generated by a general LGT $(\Lambda_{\epsilon_1,\epsilon_0})$ and an admissible variation (denoted by δ), obtaining an expression for the charge,

$$\delta Q_{\Lambda_{\epsilon_1,\epsilon_0}} = \Omega(\delta,\delta_{\Lambda_{\epsilon_1,\epsilon_0}}) = \int_{\Sigma_t}\omega^{\mu}(\delta,\delta_{\Lambda_{\epsilon_1,\epsilon_0}})dS_{\mu}. \tag{5.41}$$

As it was shown in [24], one could find the leading and subleading charges (consistent with the Ward identities) by taking the limit $t = r + u \to +\infty$ at constant u,

$$Q_{\Lambda_{\epsilon_1,\epsilon_0}} = \lim_{t\to\infty}\int_{\Sigma_t}(\partial_r - \partial_u)(r^2\Lambda_{\epsilon_1,\epsilon_0}\hat{\mathcal{F}}_{ru})dx^2du, \tag{5.42}$$

and considering the finite part in the limit. By counting orders in t, it is straightforward to see that the expression (5.42) contains divergent terms and therefore in general the definition of the charge at the limit is ill-defined. In what follows we drop the hat $\hat{}$ in $\mathcal{F}_{ru}$, since is the same field as in the radiative space.

As we previously mentioned, we can define a procedure to renormalize the symplectic potential and get rid of the divergent terms in (5.42), for any arbitrary higher order $O(r^n)$. This will be the content of the next section, while in the remainder of this section we motivate the renormalization in the particular case of the extension for $n = 1$.

Since we can trace back the divergences to the symplectic potential, due to varying with $\delta_{\Lambda_{\epsilon_1,\epsilon_0}}$, our starting point is to compute the symplectic potential on the hypersurfaces Σ_t,

$$\theta^{t}(\delta) = \sqrt{q}\left(r^2\mathcal{F}_{ru}(\delta\mathcal{A}_r - \delta\mathcal{A}_u) + q^{bc}\mathcal{F}_{ub}\delta\mathcal{A}_c\right) + \sqrt{q}(\partial_r - \partial_u)(r^2\mathcal{F}_{ru}\delta\Lambda^1_{\psi}), \tag{5.43}$$

where we did not write the total derivative $r^2 D_c(\sqrt{q}q^{bc}\mathcal{F}_{ub}\delta\Lambda^1_{\psi})$, since it vanishes after integration on Σ_t. The first term can be regarded as the radiative phase space symplectic potential, θ^t_0, while the second term is the new extended term, which we will call θ^t_1. The term $\theta^t_0(\delta)$ will contribute to the symplectic form (by integrating by parts and using the equations of motion) as usual,

$$\omega^t_0(\delta,\delta') = \sqrt{q}q^{bc}\delta\mathcal{F}_{ub}\wedge\delta'\mathcal{A}_c + \sqrt{q}r^2\delta\mathcal{F}_{ru}\wedge\delta'(\mathcal{A}_r - \mathcal{A}_u), \tag{5.44}$$

The term $\theta^t_1(\delta)$ presents the divergence: the action of ∂_u on $\delta\Lambda^1_{\psi}$ leaves an $O(r)$ term, which in turns imply a t factor when changing variables from (u, r, x^1, x^2) to (t, r, x^1, x^2). In the next section we give a systematic approach for the renormalization of such terms. For now, we assume that we can discard the divergent term and that the expression we obtain has also a finite limit $u \to -\infty$. We find the following expression for the renormalization of $\theta_1^t(\delta)$,

$$\theta_1^{ren,t}(\delta) = \sqrt{q}\left(D^a j_a^{(0)} - \frac{u}{2}\Delta\partial_u F_{ru}^{(-2)}\right)\delta\psi, \tag{5.45}$$

where *ren* stands for "renormalized". The symplectic current is splitted then,

$$\omega^{ren,t}(\delta,\delta') = \omega_0^t(\delta,\delta') + \omega_1^{ren,t}(\delta,\delta'), \tag{5.46}$$

with $\omega_1^{ren,t}$ the exterior derivative of $\theta_1^{ren,t}$ in the solution space. Thus, the total symplectic form on $\mathscr{I}^+$ is well defined (by taking $t \to +\infty$) ,

$$\Omega^{ren}(\delta,\delta') = \int_{\mathscr{I}} \omega^{ren}(\delta,\delta') = \int_{\mathscr{I}} \omega_0(\delta,\delta') + \int_{S^2} \sqrt{q}(\delta F_{ru}^{(-3,0)} \wedge \delta'\psi). \tag{5.47}$$

The last term comes form the value of $F_{ru}^{(-3,0)}$ in (5.18), which can be seen as the value of the following limit (see [50] for details):

$$F_{ru}^{(-3,0)} = \lim_{u\to-\infty} F_{ru}^{(-3)} - uF_{ru}^{(-3,1)} = \int_{\mathbb{R}} \left(D^a j_a^{(0)} - \frac{u}{2}\Delta\partial_u F_{ru}^{(-2)}\right) du, \tag{5.48}$$

where the contribution from $u = +\infty$ in integral zero due to the absence of massive charges ($F_{ru}^{(2)}(u = +\infty, x^1, x^2) = 0$). Since $\partial_u F_{ru}^{(-2)}$ decays faster than any polynomial in u, the above integral is convergent. Observe that $F_{ru}^{(-3,0)}$ is the canonical conjugate to ψ.

Next, we compute the leading and subleading charges. Taking δ' to be a large gauge transformation, and δ any arbitrary admissible variation (compatible with $\mathcal{F}_1$), we calculate the charge associated to any LGT $\Lambda_{\{\epsilon_1,\epsilon_0\}}$ by the equation (5.41). Since $F_{\mu\nu}$ is invariant under $\delta_{\Lambda_{\{\epsilon_1,\epsilon_0\}}}$ and $\Lambda_{\{\epsilon_1,\epsilon_0\}}$ is not affected by δ [3], the calculation is straightforward,

$$Q_{\Lambda_{\{\epsilon_1,\epsilon_0\}}} = \int_{S^2} \sqrt{q}\left(\epsilon_0 F_{ru}^{(-2,0)} + \epsilon_1 F_{ru}^{(-3,0)}\right) dx^2 =: Q^0_{\epsilon_0} + Q^1_{\epsilon_1}, \tag{5.49}$$

where we also used (5.18) in the radiative space sector, and $Q^i_{\epsilon_i}$, with $i = 0, 1$, denotes the leading and subleading charges, respectively.

The charge $Q^0_{\epsilon_0}$ is the usual for a $O(1)$ gauge, while the second term is the one obtained in [20], [24], [50]. In both cases, we obtained "corner" charges, in the sense that they depend on the values of the fields at the boundary of $\mathscr{I}$.

5.3 Tower of asymptotic charges

In this section, we derive an infinite hierarchy of charges from a symplectic form in an extended phase space that contains sufficient degrees of freedom to allow for $O(r^n)$ LGT's, for arbitrary n. Certain difficulties in the definition of the symplectic potential arise, in particular the appearance of several divergent integrals, as was shown in the previous section. The renormalization

[3] This again is in contrast with the non-Abelian case, where the harmonic gauge condition implies a field dependent LGT's.

procedure we apply is based on [96].

First, we define the extended phase space and show the type of divergences we have, both in the $t \to +\infty$ and $u \to +\infty$ limits inside the expresion (5.42). Then, we proceed to prescribe a renormalization on the symplectic potential that will lead to the correct expression for the charges, while the symplectic form remains finite.

5.3.1 Extended phase space and charges

Let $\mathcal{S}$ be the space of sequences $\{\psi_i\}_{i>0}$ of functions $\psi_i : S^2 \to \mathbb{R}$ such that only finitely many are non-zero [4]. Given a sequence $\Psi \in \mathcal{S}$, we define the LGT associated to the sequence as

$$\Lambda_\Psi := \sum_{i>0} \Lambda^i_{\psi_i} \tag{5.50}$$

where each $\Lambda^i_{\psi_i}$ is a pure $O(r^i)$ LGT associated to a_i, in the sense of section (5.1). Observe that the sum is finite for every $\Psi \in \mathcal{S}$. We define the extended phase space as the following set,

$$\Gamma^\infty = \Gamma^{\text{rad}} \times \mathcal{S}, \tag{5.51}$$

with the extended electromagnetic potential and scalar field are defined as

$$\hat{\mathcal{A}}_\mu = \mathcal{A}_\mu + \partial_\mu \Lambda_\Psi, \quad \hat{\phi} = e^{ie\Lambda_\Psi}\phi, \tag{5.52}$$

where $\mathcal{A}_\mu$ and ϕ are the vector potential and the scalar field generated by the free data $(A_a, \varphi) \in \Gamma^{\text{rad}}$.

The admissible variations δ of this phase space are such that when acting on the degrees of freedom parametrized by Ψ, it satisfies $\delta\Psi \in \mathcal{S}$. This property is not restrictive regarding the variations, as we will see below.

Given a sequence $\varepsilon = \{\epsilon_0, \epsilon_1, ..., \epsilon_i, ...\}$ of free S^2 functions, such that $\{\epsilon_i\}_{i>0} \in \mathcal{S}$, consider the LGT associated to it, $\Lambda_\varepsilon = \Lambda^0_{\epsilon_0} + \sum_{i>0} \Lambda^i_{\epsilon_i}$. The variation generated by this LGT acts on Γ^∞ by acting in $\mathcal{A}_\mu$ with its $O(r^0)$ free function and by acting on α on each sequence term,

$$\delta_{\Lambda_\varepsilon} A_A = \partial_A \epsilon_0, \quad \delta_{\Lambda_\varepsilon}\varphi = ie\epsilon_0\varphi, \quad \delta_{\Lambda_\varepsilon}\Psi = \{\epsilon_i\}_{i>0} \tag{5.53}$$

This structure is the same as in the previous section, extended to contain any order in the $r-$expansion.

We can write the full symplectic potential, equation (5.39), and proceed in the same way as in the previous section, obtaining the expression (5.43), but with Λ_Ψ in place of Λ^1_ψ, and split the symplectic potential in the radiative phase space contribution and the extended part, given by

$$\theta^t_\infty(\delta) = \sqrt{q}(\partial_r - \partial_u)(r^2 \mathcal{F}_{ru}\delta\Lambda_\Psi), \tag{5.54}$$

[4] In what follows we assume that the sequences of functions have this property, unless stated otherwise

where the ∞ stands for the extension to all orders in r.

Given δ and Λ_Ψ, let us calculate the symplectic potential evaluated at δ. Consider the integral,

$$\Theta_{\Sigma_t,\infty}(\delta) = \int_{\Sigma_t} \sqrt{q}(\partial_r - \partial_u)(r^2 \mathcal{F}_{ru} \delta\Lambda_\Psi) dx^2 du, \tag{5.55}$$

and observe that the term inside the integral is divergent in the limit $t \to +\infty$ with the same order as the highest power of r in $\delta\Lambda_\alpha$. Our aim in this section is to understand better this integral. For brevity let us call

$$\rho_k(\delta) = \sum_{i=k}^{+\infty} F_{ru}^{(-2+k-i)} \delta\Lambda_\Psi^{(i)}, \tag{5.56}$$

where $\delta\Lambda_\Psi^{(i)}$ is the coefficient corresponding to r^i in the $r-$expansion of $\delta\Lambda_\Psi$. $\rho_k(\delta)$ is thus the $O(r^k)$ coefficient in the expansion of the term inside the brackets. Upon direct computation and substituting $r = t - u$

$$\Theta_{\Sigma_t,\infty}(\delta) = \int_{\Sigma_t} \sqrt{q} \sum_{k=1}^{\infty} \left(k r^{k-1} \rho_k(\delta) - r^k \partial_u \rho_k(\delta) \right) dx^2 du = \sum_{j=0}^{\infty} t^j \int_{\Sigma_t} \theta_j(\delta) dx^2 du, \tag{5.57}$$

for some $t-$independent functions $\theta_j(\delta)$. This gives us a $t-$expansion of the symplectic potential. Assuming we can throw the divergences away (details in the next subsection), we are left with the $O(t^0)$ term, which satisfies the identity

$$\begin{aligned} \Theta_\infty^{\mathscr{I}}(\delta) &:= \lim_{t\to+\infty} \Theta_{t,\infty}(\delta) = \int_{\mathscr{I}} \sqrt{q} \sum_{k=1}^{\infty} \left(k(-u)^{k-1} \rho_k(\delta) - (-u)^k \partial_u \rho_k(\delta) \right) dx^2 du \\ &= -\int_{\mathscr{I}} \partial_u \left(\sqrt{q} \sum_{k=1}^{\infty} (-u)^k \rho_k(\delta) \right) dx^2 du, \end{aligned} \tag{5.58}$$

which gives us a boundary term. The charges associated to higher order LGT can be directly computed using the identity $\delta Q_{\Lambda_\epsilon} = \Omega_\infty^{\mathscr{I}}(\delta, \delta_{\Lambda_\epsilon})$,

$$Q_\epsilon = \int_{\mathscr{I}} \partial_u \left(\sum_{k=1}^{\infty} (-u)^k \rho_k(\delta_{\Lambda_\epsilon}) \right) du d^2x. \tag{5.59}$$

When evaluating the term in the brackets in the last line of (5.58) at $u = +\infty$, we use the hypothesis that $\mathcal{F}_{ru} = 0$ at $\mathscr{I}_+^+$. When evaluating at $\mathscr{I}_-^+$, we run into divergences. Since the general behaviour of $\rho_k(\delta)$ admitted by (5.13) and (5.16) near spatial infinity is polynomial in u plus a $O(1/|u|^\infty)$ remainder, we have that the above expression for $\Theta_t(\delta)$ is not well defined. By keeping only the $O(u^0)$ in $\rho_0(\delta)$,

$$\Theta_\infty^{\mathscr{I}}(\delta) = \int_{S^2} \sqrt{q} \sum_{i=1}^{\infty} F_{ru}^{(-2-i,0)} \delta\psi_i dx^2 du, \tag{5.60}$$

where $F_{ru}^{(-2-i,0)}$ are the $O(u^0)$ of $F_{ru}^{(-2-i)}$. The renormalization procedure of the next subsection will address the previous two divergences: the t divergence from the limit to $\mathscr{I}$, and the u divergences in the integrals over $\mathscr{I}$.

Regarding the concrete expressions of the charges, the order $O(u^0)$ of Q_{Λ_ε} in equation (5.59) is the same as the subleading charges presented in [50]. This is equivalent to prove that the $O(u^0)$ coefficient of Q_{Λ_ε} is $\sum_{k=0}^{+\infty}\int_{S^2}\epsilon_k F_{ru}^{-2-k,0}d^2x$. Remember that we take the $u-$decay in the remainder functions r_i in equation (5.18) as faster than any polynomic decay. Therefore, inspecting the expressions for $\Lambda_\varepsilon^{(k)}$ and F_{ru}^{-2+k-i}, we see that each $\rho_k(\delta_{\Lambda_\varepsilon})$ has at least order u^0, therefore the term in the sum contributes with at least u^k. The only term with a possible u^0 order is thus $\rho_0(\delta_{\Lambda_\varepsilon})$,

$$\rho_0(\delta_{\Lambda_\varepsilon}) = \sum_{i=1}^{\infty}\Lambda_\varepsilon^{(k)}F_{ru}^{(-2+k)}. \tag{5.61}$$

Again, a close inspection in the $u-$ expansion of the functions shows that the order u^0 is given by the sum of the products $\epsilon_k F_{ru}^{(-2-k,0)}$.

5.3.2 Regularization procedure

In this subsection, following [96], we will renormalise the symplectic potential for QED in the extended phase space in order to eliminate the divergences. The idea is to write the higher order terms in the t component of the symplectic potential as a boundary plus corner terms, substract them and obtain a finite expression in the $t\to\infty$ limit as well as in $u\to\pm\infty$.

From the first variation of the Lagrangian (5.1), we have

$$\delta\mathcal{L} = E^\mu\delta\hat{A}_\mu + E\delta\hat{\phi} + \partial_\mu\theta^\mu(\delta) \tag{5.62}$$

where E^μ and E are the field equations for $\hat{A}_\mu$ and the massless scalar, respectively. By taking the retarded coordinates u, t, x^1, x^2 on Minkowski space time, we write the previous equation on-shell and obtain an equation for $\partial_t\theta^t(\delta)$

$$\partial_t\theta^t(\delta) = \delta\mathcal{L} - \partial_u\theta^u(\delta) - D_a\theta^a(\delta) \tag{5.63}$$

We will assume that all the functions have t and u expansions around $t=+\infty$ and $u=\pm\infty$, as is the case for $F_{ru}^{(2)}$, A_a and φ (equations (5.13), (5.11), and (5.9)).

Consider the derivation of the divergent part of the symplectic potential done in the previous section, but now applied to our extended phase space:

$$\theta^\mu(\delta) = \sqrt{q}r^2\left(\mathcal{F}^{\mu\nu}\delta\hat{A}_\nu) + \overline{\hat{\mathcal{D}}^\mu\hat{\phi}}\delta\hat{\phi} + c.c.\right). \tag{5.64}$$

Remember that we use (u, r, x^1, x^2) coordinates to integrate, and then take the limit $t\to+\infty$ at fixed u. The general form for the symplectic potential is,[5]

$$\theta^t(\delta) = Y_0(\delta)(u,t,x^a) + \sum_{i=1}^{\infty} t^i Y_i(\delta)(u,x^a). \tag{5.65}$$

where $Y_0(\delta)(u,t,,x^a)$ is such that $\lim_{t\to+\infty} Y_0(\delta)(u,t,,x^a)$ is a well defined function, $Y_0(\delta)(u,x^a)$. We introduce the renormalized symplectic potential as $\theta^t_{ren} := \theta^t - H_{ren}$, where H_{ren} is such that

$$\partial_t\theta^t(\delta) - \partial_t H_{ren}(\delta) = K(\delta)(u,t,x^a), \tag{5.66}$$

where K is such that its limit when $t \to +\infty$ vanishes. In general K and H_{ren} are not uniquely determined by the previous equation. The natural prescription for H_{ren} to resolve the divergences is the following,

$$H_{ren}(\delta) = \sum_{i=1}^{+\infty} t^i Y_i(\delta)(u,x^a) + C(\delta)(u,x^a), \tag{5.67}$$

where $C(u,x^a)$ is a function to be determined. Observe that H_{ren} has the same order than θ^t in the t expansion, and that the divergences in the t parameter are cancelled, so θ^t_{ren} converges in the limit $t \to +\infty$. The coefficients Y_i are obtained from the integration of the terms in the variation of the lagrangian and the total derivative of the symplectic potential in equation (5.63), on $\{t = cnt\}$ surfaces, directly by the t expansion.

Therefore, we can prescribe

$$Y_i(\delta) = \text{Finite part}\left(\lim_{t\to+\infty}\frac{1}{t^i}\left(\delta\mathcal{L} - \partial_u\theta^u(\delta) - D_a\theta^a(\delta)\right)\right), \tag{5.68}$$

for each i. Observe in (5.68) that each Y_i can be written as a total derivative plus a total variation.

By taking the free function C to be a total derivative, $C = \partial_u X^u + D_a X^a$, we can add the last term in (5.68) to obtain a new total derivative term. Then, the renormalized symplectic potential has the form

$$\theta^t_{ren}(\delta) := \theta^t(\delta) + \partial_\nu \Upsilon^{t\nu}(\delta) + \delta\Xi^t = P(\delta)(u,t,x^a) \tag{5.69}$$

where Υ and Ξ are calculated from Y_i, X^u_i and X^a_i directly, and P is at most $O(t^0)$ in the $t-$expansion. This symplectic potential does not contain divergences in the limit $t \to \infty$. The general form of the symplectic potential will be changing the upper index t by a 4d index μ. We have that $\Upsilon^{\mu\nu} = -\Upsilon^{\nu\mu}$, by definition of "corner terms" (see [96]). Without any loss of generality, we can define $\Upsilon^{jl} = 0$, for j,l running in the set $\{u,x^a\}$, since these terms are not uniquely defined and do not affect the renormalization of θ^t. Therefore, we have a well defined limit

$$\theta^{\mathscr{I}}_{ren}(\delta)(u,x^a) := \lim_{t\to+\infty}\theta^t_{ren}(\delta)(t,u,x^a) = Y_0(\delta)(u,x^a) - C(\delta)(u,x^a) \tag{5.70}$$

We have still at our disposal the function $C(u,x^1,x^2)$ (the only condition we imposed so far is that it is a total derivatives), which can be determined by imposing a finite limit when

[5] In the following equations we write the explicit dependence of the functions on variations and coordinates.

$u \to -\infty$ for the symplectic potential.

As it was shown in the previous subsection, under general LGT's the $O(t^0)$ of the symplectic potential has $O(u^N)$ terms, and therefore θ^t_{ren} will in general have an expansion in powers of u, starting in some u^N (corresponding to the highest power in δ or α), the coefficients of the expansion depending in general on which limit we are computing, $u \to \pm\infty$. We consider the following u expansion for $Y_0(\delta)$ near $u = \pm\infty$,

$$Y_0(\delta)(u, x^a) \overset{u\to\pm\infty}{=} R_{Y_0}(\delta)(u, x^a) + \sum_{k=1}^{\infty} u^k Y^{\pm}_{0,k}(\delta)(x^a), \tag{5.71}$$

where $\partial_u R_{Y_0}(u, x^a) = O(1/|u|^\infty)$. This condition comes from the tree level assumption on the soft theorems, and implies in particular that the limits when $u \to \pm\infty$ are in principle different,

$$R^{\pm}_{Y_0}(\delta)(x^a) := \lim_{u\to\pm\infty} R_{Y_0}(\delta)(u, x^a). \tag{5.72}$$

By inserting (5.71) in (5.70), we have

$$\theta^{\mathscr{I}}_{ren}(\delta) = R_{Y_0}(\delta)(u, x^a) + \sum_{k=1}^{\infty} u^k Y^{\pm}_{0,k}(\delta)(x^a) - \partial_u X^u(\delta)(u, x^a) - D_a X^a(\delta)(u, x^a), \tag{5.73}$$

and immediately we can find functions X^u, X^a such that their expansions around $u = \pm\infty$ renormalise the limits of the symplectic potential. For X^u we find,

$$X^u_{\pm}(\delta)(u, x^a) = \sum_{k=1}^{\infty} \frac{1}{k+1} u^{k+1} Y^{\pm}_{0,k}(\delta)(x^a), \tag{5.74}$$

while for X^a we have,

$$D_a X^a(\delta)(u, x^a) = \begin{cases} R^-_{Y_0}(\delta)(x^a) + O(1/|u|^\infty) & \text{when } u \to -\infty \\ R^+_{Y_0}(\delta)(x^a) + O(1/|u|^\infty) & \text{when } u \to +\infty \end{cases} \tag{5.75}$$

Finally, the symplectic potential density gives a finite result upon integration on $\mathscr{I}$, due to the fall offs of R_{Y_0}.

5.3.3 Electric-like charge algebra

The previous renormalization procedure adjust exactly all the divergences, while maintaining the same convergent terms discussed in subsection 5.3.1. The expression for the renormalized symplectic potential is therefore:

$$\Theta_{ren}(\delta) = \int_{\mathscr{I}^+} \theta_0(\delta) du dx^2 + \int_{S^2} \sum_{i=1}^{\infty} F_{ru}^{(-2-i,0)} \delta a_i dx^2 \tag{5.76}$$

where θ_0 is the usual symplectic potential in Γ^{rad}. The symplectic form is the exterior derivative (in the extended phase space) of the symplectic potential:

$$\Omega_{ren}(\delta,\delta') = \int_{\mathscr{I}} \omega_0(\delta,\delta')dudx^2 + \int_{S^2} \sum_{i=1}^{\infty} \delta F_{ru}^{(-2-i,0)} \wedge \delta' a_i dx^2 \tag{5.77}$$

Now, all three ingredients in the charge calculation are well defined and finite: the limit $t \to +\infty$, the integration on $\mathscr{I}$ and the series.

We are now in position to show the full hierarchy of charges for arbitrary $O(r^n)$ LGT in QED. The electric charges associated to a LGT Λ_ε can be calculated form (5.77), substituting the sequence $\{\epsilon_i\}$ in the identity $\delta Q_\varepsilon = \Omega_{ren}(\delta, \delta_{\Lambda_\varepsilon})$,

$$Q_\varepsilon = \sum_{j=0}^{\infty} \int_{S^2} \sqrt{q}\epsilon_j F_{ru}^{-2-j,0} dx^2 \tag{5.78}$$

where we are using that $\mathcal{F}_{\nu\mu}$ is invariant under $\delta_{\Lambda_\varepsilon}$. This expression is the same as the one obtained in [50]. Observe that the full algebra of charges is abelian:

$$\{Q_{\varepsilon_1}, Q_{\varepsilon_2}\} = 0, \quad \forall \varepsilon_1, \varepsilon_2 \tag{5.79}$$

5.4 Duality extension of tower of asymptotic charges

In the previous sections we treated only the electric part of Maxwell theory, renormalizing the symplectic potential in the extended phase space to contain the sub^n-leading charges in a natural framework. In this section, we extend the phase space (again) in order to include the magnetic freedom, *á la* Freidel-Pranzetti, as in [103]. This type of extensions are analogous to those we previously study in chapter 3, has been thoroughly studied in recent years in several contexts: electromagnetic duality (e.g. [102,104,105]), BF theories ([106]) and under more general structures ([53]). Throughout this section we are using form notation, without writing the indexes explicitly, in order to ease the notation. Also, we are considering no extra fields.

Electromagnetism posses a *duality symmetry*, which can be characterized as follows: the Lagrangian for the theory is

$$\mathscr{L}[\mathcal{F}] = \frac{1}{2} * \mathcal{F} \wedge \mathcal{F}, \tag{5.80}$$

where $\wedge$ is the wedge product in the space of p-forms on Minkowski space $\mathcal{M}$ and $*$ is the Hodge dual operator, $* : \Omega^p(\mathcal{M}) \to \Omega^{4-p}(\mathcal{M})$, in $\mathcal{M}$. This operator satisfies

$$* * \alpha = (-1)^{p(4-p)+1}\alpha, \tag{5.81}$$

where the extra $+1$ in the exponent comes from the signature of the metric in Minkowski space. Therefore, taking $p = 2$ and applying $*$ to F in (5.80), we have

$$\mathscr{L}[*\mathcal{F}] = -\frac{1}{2} * \mathcal{F} \wedge \mathcal{F}, \tag{5.82}$$

and critical points in both actions would be the same.

The first step towards this extension is to consider the standard radiative phase space duality extension. On each Σ_t, we have the Freidel-Pranzetti extension for the symplectic form, [103],

$$\Omega(\delta,\delta') = \int_{\Sigma_t} \delta\mathcal{A}\wedge\delta'\star\mathcal{F} + \int_{S^2}\delta a_0\wedge\delta' B_0 \tag{5.83}$$

where $\star$ is the Hodge dual in the hypersurface, $a_0 \overset{S^2}{=} \mathcal{A} + d\phi_0$ is the electric boundary gauge field, and B_0 is the magnetic boundary gauge field. ϕ_0 is the *edge mode*, which extends the phase space, $(\mathcal{A}, a_0)$, which now contains this boundary field. We see that the symplectic form now contains a corner term, motivated by the gauge invariance of the theory.

To make the connection with our past sections definition for $\mathcal{A}$, we have

$$\mathcal{A}_{new} + d\phi = \mathcal{A}_{old}, \tag{5.84}$$

where *old* refers to the $\mathcal{A}$ used in the previous sections, and *new* is the one in the present section. In particular, the expressions for curvature tensor and the charges are still valid. Observe that ϕ can be though as a zero-order extension, using the same idea as the previous sections: extending the vector potential with a large gauge symmetry.

We distinguish between symmetries that leave fixed the bulk variable $\mathcal{A}$, and symmetries that act only on the boundary. In the previous sections, we use this differentiation when defining the extension to higher order LGT, where $\delta_{\Lambda_\epsilon}$ only acts on $A_a(u,x)$ through the first component. In the present section, as it was done in [103], we are isolating the bulk from the boundary action on the ϵ_0 variation, in order to have a well define canonical action that includes the duality symmetry, and such that the symplectic potential is invariant under the gauge transformation of the fields.

We are working in $\mathscr{I}^+$, so in (5.83) we take $t\to+\infty$. The "bulk" part now is $\mathcal{A}$ along $\mathscr{I}$, while the boundaries are $\mathscr{I}^+_\pm$, with topology S^2. The values at the boundary are not independent, since the boundary symmetries act simultaneously on both $\mathscr{I}^+_\pm$ (i.e., they are independent of u). Under a gauge transformation generated by G, both the bulk and the corner fields transform,

$$\delta_G(\mathcal{A}, a_0, B_0) = (dG, -dG, 0), \tag{5.85}$$

so the variation δ_G is indeed gauge, in the sense that has a vanishing charge $\Omega(\delta,\delta_G) = 0$ on shell. The electric (magnetic) symmetry $\delta_{\epsilon_0}(\delta_{\lambda_0})$ acts only on the electric (magnetic) boundary field,

$$\delta_{\epsilon_0}(\mathcal{A}, a_0, B_0) = (0, d\epsilon_0, 0), \quad \delta_{\lambda_0}(A, a_0, B_0) = (0,0,d\lambda_0), \tag{5.86}$$

where $d\lambda_0$ is locally but not globally exact (such as in the standard examples of a charge in the $z-$axis, see section V in [103]). Observe that on-shell, upon acting with G, we obtain the identity

$$dB_0 = \star\mathcal{F}, \tag{5.87}$$

which, on $\mathscr{I}^+$, is $dB_0 = F_{ru}^{(-2,0)} dx^2$.

Our extended phase space of section 5.3.1 adapts well to the construction given above to the duality extension. The gauge transformation Λ_α is the "bulk" potential, generated by the boundary fields in the sequence α, in a hierarchy graded by the correspondent power of r. Therefore, we can extend directly as

$$\Omega_{ren}(\delta,\delta') = \int_{\mathscr{I}} [\delta\mathcal{A}\wedge\delta'\star\mathcal{F}]_{ren} + \int_{S^2}\delta a_0\wedge\delta' B_0 + \int_{S^2}\sum_{k=1}^{\infty}\delta dB_k\delta' a_k, \tag{5.88}$$

where here a_k are functions on the sphere, a_0 is a $1-$form [6], and *ren* indicates that is the renormalized term, given by (5.77). We define the action of a gauge transformations G (of order r^n arbitrary) as

$$\delta_G\mathcal{A} = dG, \quad \delta_G a_0 = dG_0, \quad \delta_G a_j = G_j, \quad \delta_G B_j = 0, j\geq 1. \tag{5.89}$$

Evaluating the sympletic form in δ_G,

$$\Omega_{ren}(\delta,\delta_G) = -\delta\left(\int_{\mathscr{I}} [dG\wedge\star\mathcal{F}]_{ren} + \int_{S^2}\delta dG_0\wedge\delta' B_0 + \int_{S^2}\sum_{k=0}^{\infty} G_k\delta dB_k\right), \tag{5.90}$$

which on-shell and after integrating by parts, we obtain (after the renormalization, allowing variations δ such that δA has order higher than r^0 before taking the limit $t\to+\infty$)

$$dB_k = F_{ru}^{(-2-k,0)}. \tag{5.91}$$

This equality establishes the value of the magnetic boundary gauge field as the field strength functions.

Finally, we will denote the magnetic variations acting on B_k's as $\lambda = \{\lambda_i\}_{i\geq 0}$, in the same fashion as we define the LGT generators. Electric (magnetic) variations act as follows on the extended phase space variables,

$$\delta_{\epsilon_k}\mathcal{A} = 0, \quad \delta_{\epsilon_k} a_0 = \delta_{0k} d\epsilon_k, \quad \delta_{\epsilon_k} a_j = \delta_{kj}\epsilon_k, \quad \delta_{\epsilon_k} B_j = 0, \quad k\geq 0, j\geq 1 \tag{5.92}$$

$$\delta_{\lambda_k}\mathcal{A} = 0, \quad \delta_{\lambda_k} a_j = 0, \quad \delta_{\lambda_k} B_j = \delta_{kj} d\lambda_k, \quad k,j\geq 0, \tag{5.93}$$

where $d\lambda_k$ is locally but not globally defined, and δ_{ij} is Kroenecker delta.

5.4.1 Charges and dual charges and their algebra

By computing $\Omega_{ren}(\delta,\delta_{\Lambda_\epsilon})$ and $\Omega_{ren}(\delta,\delta_{\Lambda_\lambda})$, we obtain the electric (denoted as Q) and magnetic (denoted as $\tilde{Q}$) charges,

[6] a_0 is not generally a gradient.

$$Q_\varepsilon = \sum_{k=0}^{\infty} \int_{S^2} \epsilon_k dB_k \tag{5.94}$$

$$\tilde{Q}_\lambda = \sum_{k=0}^{\infty} \int_{S^2} a_k d^2\lambda_k, \tag{5.95}$$

where the first integral gives directly 5.78, thanks to 5.91, and the last integral does no vanish due to the failure of $d\lambda$ to be globally exact.

Finally, we have can compute the charge algebra. As the electric charges, the magnetic charges $\tilde{Q}_\lambda$ are abelian,

$$\{\tilde{Q}_\lambda, \tilde{Q}_{\lambda'}\} = \delta_\lambda \sum_{k=0}^{\infty} \int_{S^2} a_k d^2\lambda'_k = 0. \tag{5.96}$$

A non-trivial component of algebra is given by the mixed Poisson bracket (as shown by Hosseinzadeh, Seraj and Sheikh-Jabbari in [104], and by Freidel and Pranzetti in [103]),

$$\{\tilde{Q}_\varepsilon, \tilde{Q}_\lambda\} = \delta_\varepsilon \sum_{k=0}^{\infty} \int_{S^2} a_k d^2\lambda_k = \sum_{k=0}^{\infty} \int_{S^2} \epsilon_k d^2\lambda_k =: c_k \tag{5.97}$$

This term shows that the boundary duality symmetry algebra posses a hierarchy of central charges, $\{c_k\}_{k\geq 0}$. Recent developments ([102]) showed that these central charges also appear when considering charges associated to $O(\ln r)$ large gauge transformations. We leave to future works to analyse in detail the central extensions in the context of soft theorems and Ward identities.

Extension to all orders in r for SD Yang-Mills

In this chapter, we show the proposal in [107] for an extension to all orders in Yang-Mills theory. This extension allows, a perturbative-like approach to the computation of the variation algebra.

The natural question, after chapter 4 and chapter 5, is whether it is possible to extend the phase space in Yang-Mills to account for all the symmetries coming form the sub$^n-$leading/Ward identity equivalences. These symmetries will correspond to symmetry parameters with increasingly divergent behaviour at null infinity coming form the radial expansion. The present chapter aims to answer this question, by giving a definition of the extension and showing that a well-defined algebra of variations can be constructed, restricting to the self-dual sector (SD) of the theory.

The self dual sector is a consistent truncation of the theory, which it is in its own right a wide research area for several reasons: it is an integrable theory [108], making it approachable with the standard tools from Integrable Systems, it presents a rich structure of symmetries (e.g. [109, 110]) and it is deeply connected with complex and conformal geometries (e.g. [111]). In particular, we are interested in the field-independence of the large gauge transformations using light-cone gauge, [109], where the phase space extension can be defined straightforwardly.

6.1 Light-cone gauge in the self-dual sectors of YM and gravity

We will follow the conventions of [109] and use light-cone coordinates $(U, V, Z, \bar{Z})$, related to Cartesian coordinates X^μ via:

$$U = \frac{X^0 - X^3}{\sqrt{2}}, \quad V = \frac{X^0 + X^3}{\sqrt{2}}, \quad Z = \frac{X^1 + iX^2}{\sqrt{2}}, \quad \bar{Z} = \frac{X^1 - iX^2}{\sqrt{2}}. \tag{6.1}$$

It is useful to introduce the notation:

$$x^i := (U, \bar{Z}), \quad y^\alpha := (V, Z). \tag{6.2}$$

which splits space-time into two $2d$ subspaces. The Minkowski metric is then

$$ds^2 = 2\eta_{i\alpha}dx^i dy^\alpha = -2dUdV + 2dZd\bar{Z}. \tag{6.3}$$

and we introduce the anti-symmetric "area elements" of the $2d$ subspaces:

$$\Omega_{ij}dx^i \wedge dx^j = dU \wedge d\bar{Z} - d\bar{Z} \wedge dU \tag{6.4}$$
$$\Pi_{\alpha\beta}dy^\alpha \wedge dy^\beta = dV \wedge dZ - dZ \wedge dV \tag{6.5}$$

which act as inverses of each other:

$$\Omega_i{}^\alpha \Pi_\alpha^j = \delta_i^j \tag{6.6}$$
$$\Pi_\alpha^i \Omega_i{}^\beta = \delta_\alpha^\beta. \tag{6.7}$$

6.1.1 Self-Dual Yang-Mills

Consider a YM field $\mathcal{A}_\mu$ with field strength

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu], \tag{6.8}$$

The dual is defined as

$$* \mathcal{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\mathcal{F}_{\rho\sigma}. \tag{6.9}$$

The field strength is self-dual provided

$$* \mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}. \tag{6.10}$$

Then, working in the light-cone gauge $\mathcal{A}_U = 0$, the self-dual sector can be described via (see [109, 110, 112–114])

$$\mathcal{A}_i = 0, \quad \mathcal{A}_\alpha = \Pi_\alpha^i \partial_i \Phi \tag{6.11}$$

with $\Pi_{\alpha\beta}$ as in (6.5) and where the Lie algebra valued scalar satisfies

$$\Box\Phi = \Pi^{ij}[\partial_i\Phi, \partial_j\Phi] \tag{6.12}$$

It is easy to see that the Harmonic gauge is satisfied automatically

$$\partial^\mu \mathcal{A}_\mu = \eta^{i\alpha}\partial_i \mathcal{A}_\alpha = \Pi^{ij}\partial_i\partial_j\Phi = 0 \tag{6.13}$$

making use of the anti-symmetry of Π. It also follows that

$$D^\mu \mathcal{A}_\mu = 0, \qquad \text{with} \quad D_\mu \mathcal{A}_\nu = \partial_\mu \mathcal{A}_\nu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{6.14}$$

In this paper, we will be be considering the family of gauge transformations

$$\delta_\Lambda \mathcal{A}_\mu = D_\mu \Lambda = \partial_\mu \Lambda + [\mathcal{A}_\mu, \Lambda,], \tag{6.15}$$

which preserve the light-cone gauge condition (6.11). It is easy to see [109] that these are parametrised by

$$\partial_i \Lambda = 0 \implies \Lambda = \Lambda(y) \tag{6.16}$$

We note that this subset of the gauge transformations also automatically preserves the gauge condition (6.14). This is one of the crucial simplifications that follow from starting with the more restrictive light-cone gauge, instead of (6.14). Had we taken (6.14) as our starting point and attempted to solve for Λ (for example as a perturbation series in the coordinate V), we would obtain a field-dependent expression for Λ (see chapter 4 and [73] for details), which would have significantly increased the difficulty of calculations in the next section.

6.1.2 Phase space for SDYM and fields near infinity

In this section we make the connection with the asymptotic symmetries of chapter 4. We will study fields near future null infinity, $\mathscr{I}^+$, so we are switching to coordinates adapted to it. The natural choice are Bondi-type coordinates $(r, u, z, \bar{z})$ given by [109]

$$r = V, \quad ,z = \frac{Z}{V}, \quad \bar{z} = \frac{\bar{Z}}{V}, \quad u = U - \frac{Z\bar{Z}}{V}. \tag{6.17}$$

In these coordinates, the Minkowski metric reads

$$ds^2 = -2du dr + 2r^2 dz d\bar{z}, \tag{6.18}$$

where we see that by taking the conformally rescaled metric $\frac{1}{r^2}ds^2$, we have a well defined (degenerate) metric on $\mathscr{I}^+$ given by $2dzd\bar{z}$.[1]

The radiative data of any massless scalar field (in particular of Φ) is given by

$$\Phi(r, u, z, \bar{z}) = \frac{\Phi_{\mathscr{I}}(u, z, \bar{z})}{r} + \mathcal{O}(r^{-2}), \tag{6.19}$$

where $\varphi_{\mathscr{I}}$ can be regarded as defined *at* $\mathscr{I}$.

Let us use the notation

$$\mathcal{A}_z(r, u, z, \bar{z}) \overset{r\to\infty}{=} A_z(u, z, \bar{z}) + \mathcal{O}(r^{-1}) \tag{6.20}$$

Then, considering equations (6.11) in Bondi coordinates (6.17), we have

$$\mathcal{A}_z = r\partial_u \Phi, \quad \mathcal{A}_{\bar{z}} = 0, \tag{6.21}$$

[1] See e.g. Section 4 of [55]. for a derivation of this metric from general assumptions about asymptotic flatness.

which implies (by (6.19) and (6.20))

$$A_z = \partial_u \Phi_{\mathscr{I}}, \quad A_{\bar{z}} = 0 \tag{6.22}$$

The *radiative phase space* will then be given by $\phi_{\mathscr{I}}$ as free data,

$$\Gamma^{rad} = \{\Phi_{\mathscr{I}}(u, z, \bar{z})\}. \tag{6.23}$$

Recall that the gauge parameters depend only on y coordinates, $\Lambda(y)$, which in this case are (r, z). Therefore, we can assume

$$\Lambda = \sum_{n=-\infty}^{+\infty} \Lambda^{(n)}(z) r^n, \tag{6.24}$$

as an $r-$expansion for the gauge parameter. Unlike in the general case, [73], the coefficients $\Lambda^{(n)}$ are field-independent since no further restriction has been imposed.

Variations on Γ^{rad}

Next, we study the space of variations and its action, by taking the ones that preserve the fall offs of the fields in Γ^{rad}. Take Λ to be the generator of the gauge transformations. Its expansion in r generally is given by,

$$\Lambda = \sum_{n=-\infty}^{+\infty} \Lambda^{(n)}(z) r^n, \tag{6.25}$$

where $\Lambda^{(n)} = \Lambda^{(n)}(z)$, since the variations must be gauge preserving (see (6.16)). To ensure convergence, for the rest of the paper we will assume that only finitely many of the terms with power $n > 0$ are non-zero (in reminiscence of section 5.3). Then the variations of $\mathcal{A}_z$ is

$$\sum_{n=-\infty}^{\infty} \delta_\Lambda \mathcal{A}_z^{(n)} r^n = \sum_{n=-\infty}^{+\infty} \left(\partial_z \Lambda^{(n)} + \sum_{k=0}^{\infty} \left[\mathcal{A}_z^{(n-k)}, \Lambda^{(k)} \right] \right) r^n \tag{6.26}$$

The fall-off for $\mathcal{A}_z$ given in (6.22) is preserved by any Λ of the form,

$$\Lambda = \sum_{n=-\infty}^{0} \Lambda^{(n)}(z) r^n, \tag{6.27}$$

which for $\Lambda^{(0)} \neq 0$ are the large gauge transformations, [19].

6.2 Yang-Mills extension to all orders

Consider the following extension, which is the finite analogue of (4.37),

$$\Gamma^{\text{ext}}_{\infty,\text{YM}} := \{\hat{\mathcal{A}}_\alpha = e^{-\Psi} \mathcal{A}_\alpha e^{\Psi} + e^{-\Psi} \partial_\alpha e^{\Psi}, \quad \mathcal{A}_\alpha \in \Gamma^{rad}, \quad \Psi = \sum_{n=1}^{+\infty} r^n \Psi^{(n)}(z)\}. \tag{6.28}$$

where the coefficients $\{\Psi^{(n)}(z)\}_{n\geq 1}$ are taken such that only finitely many of them are non-zero, and $\alpha \in \{r, z\}$. Now, we will consider the most general large gauge transformations, as in (6.29), so let us consider

$$\Lambda = \Lambda^{(0)}(z) + \sum_{n=1}^{+\infty} \Lambda^{(n)}(z) r^n \tag{6.29}$$

We will derive an expression for $\delta_\Lambda \Psi$ using the consistency of the gauge condition, i.e., *the extended gauge field* $\hat{\mathcal{A}}_\alpha$ *transforms as a gauge field in the extended space*,

$$\delta_\Lambda \hat{\mathcal{A}}_\alpha = \hat{D}_\alpha \Lambda. \tag{6.30}$$

Note that it is still sufficient to work to linear order in the transformation parameter Λ[2]. Then, making use of the expression for a general derivation acting on the exponential map,

$$\delta e^X = e^X \mathbb{O}_X(\delta X), \quad \delta e^{-X} = -\mathbb{O}_X(\delta X) e^{-X} \tag{6.31}$$

with

$$\mathbb{O}_X := \frac{1 - e^{-ad_X}}{ad_X} = \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(k+1)!} (ad_X)^k, \tag{6.32}$$

where $ad_X(Y) = [X, Y]$, the left hand side of (6.30) results,

$$\begin{aligned} \text{lhs } (6.30) \;=\; & e^{-\Psi} \delta \mathcal{A}_\alpha e^{\Psi} + e^{-\Psi} \mathcal{A}_\alpha e^{\Psi} \mathbb{O}_\Psi(\delta\Psi) - \mathbb{O}_\Psi(\delta\Psi) e^{-\Psi} \mathcal{A}_\alpha e^{\Psi} \\ & + \delta\left(\mathbb{O}_\Psi(\partial_\alpha \Psi)\right). \end{aligned} \tag{6.33}$$

The first line is

$$e^{-\Psi}\left(\delta \mathcal{A}_\alpha + \left[A_\alpha, e^{\Psi} \mathbb{O}_\Psi(\delta\Psi) e^{-\Psi}\right]\right) e^{\Psi}, \tag{6.34}$$

while the second line can be written as

$$e^{-\Psi}\left(\partial_\alpha\left(e^{\Psi} \mathbb{O}_\Psi(\delta\Psi) e^{-\Psi}\right)\right) e^{\Psi}, \tag{6.35}$$

using the identity $[\mathbb{O}_\Psi(\partial_\alpha\Psi), \mathbb{O}_\Psi(\delta\Psi)] = \delta(\mathbb{O}_\Psi(\partial_\alpha\Psi)) - \partial_\alpha(\mathbb{O}_\Psi(\delta\Psi))$. Similarly, the right hand side of (6.30) is

$$\text{rhs } (6.30) = e^{-\Psi}\left(D_\alpha\left(e^{\Psi} \Lambda e^{-\Psi}\right)\right) e^{\Psi} \tag{6.36}$$

Finally,

$$e^{-\Psi}\left(\delta A_\alpha + D_\alpha\left(e^{\Psi} \mathbb{O}_\Psi(\delta_\Lambda \Psi) e^{-\Psi}\right)\right) e^{\Psi} = e^{-\Psi}\left(D_\alpha(e^{\Psi} \Lambda e^{-\Psi})\right) e^{\Psi}. \tag{6.37}$$

This equation, as it stands, is valid everywhere. Following chapter 4, we need to write (6.37) in the phase space variables, and prescribe the action of the variation on $A_z^{(0)}$. We define the same

[2] In principle, we could consider these transformations to all orders in Λ, however this is not necessary for applications to soft theorems.

transformation rule for $A_z^{(0)}$ as in the first term in equation (4.40), as expected:

$$\delta_\Lambda A_z^{(0)} = D_z^{(0)} \Lambda^{(0)} \tag{6.38}$$

where the expansion of Λ is given in (6.29). The interesting part follows when considering the variation of Ψ, which turns out to satisfy a remarkably simple constraint, at all orders in r, as can be read off directly from (6.37), upon subtracting (6.38) in both sides:

$$\mathcal{O}_\Psi(\delta_\Lambda \Psi) = \Lambda - e^{-\Psi} \Lambda^{(0)} e^{\Psi}. \tag{6.39}$$

Formally, we can write

$$[\delta_\Lambda \Psi]^{(n)} = \left(\mathcal{O}_\Psi^{-1} (\Lambda - e^{-\Psi} \Lambda^{(0)} e^{\Psi}) \right)^{(n)}. \tag{6.40}$$

where $^{(n)}$ denotes the coefficient of r^n in the $r-$expansion. We show below that the operator $\mathcal{O}$ is indeed invertible, working order by order in Ψ and the transformations Λ. Let $\delta^{[m]} X$ denote the variation of X at order m in Ψ, and $T^{[m]}$ the component of order Ψ^m in the expression T.[3] Then, in equation (6.40) we have two expansions, one in r and on in Ψ, where each equation reads,

$$\left[\delta_\Lambda^{[m]} \Psi \right]^{(n)} = \left(\mathcal{O}_\Psi^{-1} (\Lambda - e^{-\Psi} \Lambda^{(0)} e^{\Psi}) \right)^{[m],(n)}. \tag{6.41}$$

First, observe that since Ψ is of orders r^1 and higher, then the equations in (6.41) are meaningful only when $n \geq m$, since otherwise a Ψ^m term would start at least at order r^m.

Then, we have a natural "induction step" to solve the inverse operator both in r and in Ψ. By the definition of $\mathcal{O}_\Psi$ we have

$$\begin{aligned} \mathcal{O}_\Psi(\delta_\Lambda \Psi) &= \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(k+1)!} \left(ad_\Psi \right)^k (\delta_\Lambda \Psi) \\ &= \delta_\Lambda \Psi - \frac{1}{2} [\Psi, \delta_\Lambda \Psi] + \frac{1}{6} [\Psi, [\Psi, \delta_\Lambda \Psi]] - \frac{1}{24} [\Psi, [\Psi, [\Psi, \delta_\Lambda \Psi]]] + ..., \end{aligned} \tag{6.42}$$

Then we can invert recursively (6.39) in powers of Ψ, finding the expressions for the successive order n inverses,

$$\overset{0}{\delta} = \delta^{[0]} \tag{6.43}$$

$$\overset{1}{\delta} = \delta^{[0]} + \delta^{[1]} \tag{6.44}$$

$$... =$$

$$\overset{n}{\delta} = \delta^{[0]} + ... + \delta^{[n]} \tag{6.45}$$

For $[n] = 0$,

$$\delta_\Lambda^{[0]} \Psi = \Lambda - \Lambda^{(0)} \tag{6.46}$$

[3]This is to distinguish it from the expansion in r, which we have denoted by $X^{(n)}$.

This variation is analogous to the second term in (4.39). For $[n] = 1$, we get

$$\delta_{\Lambda}^{[1]}\Psi - \frac{1}{2}[\Psi, \delta_{\Lambda}^{[0]}\Psi] = [\Psi, \Lambda^{(0)}] \tag{6.47}$$

Making use of (6.46), this becomes

$$\delta_{\Lambda}^{[1]}\Psi = \frac{1}{2}[\Psi, \Lambda + \Lambda^{(0)}] \tag{6.48}$$

The equation (6.48) is the analogous to the second term in (4.40), as expected. The *order 1 inverse* of (6.39) is then

$$\overset{1}{\delta}\Psi = \Lambda - \Lambda^{(0)} + \frac{1}{2}[\Psi, \Lambda + \Lambda^{(0)}], \tag{6.49}$$

which is analogous to the variation defined in chapter 4. Now we can press on to higher orders. For example, at $[n] = 2$ we have

$$\delta_{\Lambda}^{[2]}\Psi - \frac{1}{2}[\Psi, \delta_{\Lambda}^{[1]}\Psi] + \frac{1}{6}[\Psi, [\Psi, \delta_{\Lambda}^{[0]}\Psi]] = -\tfrac{1}{2}[\Psi, [\Psi, \Lambda^{(0)}]] \tag{6.50}$$

Then, plugging in (6.46) and (6.48), we get

$$\delta_{\Lambda}^{[2]}\Psi = \frac{1}{12}[\Psi, [\Psi, \Lambda - \Lambda^{(0)}]] \tag{6.51}$$

This will allow us to extract the transformation of Ψ at $O(r^2)$. The rule generalises, in the sense that if we want the transformation to order n in r, we must first construct it to at least order n in Ψ (this follows from the fact that the expansion of Ψ starts at $O(r)$, see (6.28)). One can then proceed recursively and construct the transformation to all orders in Ψ and r.

The important property of this expansion in Ψ is that each term preserves the $\mathfrak{g}$-structure on the variation algebra. As an example, take two $O(r)$ LGT, let say Λ_1 and Λ_2. Then,

$$[\Lambda_1, \Lambda_2] = r^2[\Lambda_1^{(1)}, \Lambda_2^{(1)}] + r([\Lambda_1^{(1)}, \Lambda_2^{(0)}] + [\Lambda_1^{(0)}, \Lambda_2^{(1)}]) + ..., \tag{6.52}$$

and we want that the variations have the correct representation on the phase space functions,

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}]\Psi = \delta_{[\Lambda_1, \Lambda_2]}\Psi. \tag{6.53}$$

Let us compute the variation on $\Psi^{(2)}$ for $\overset{1}{\delta}$:

$$\begin{aligned}
\overset{1}{\delta}_{\Lambda_1}\Psi^{(1)} &= \Lambda^{(1)} + [\Psi^{(1)}, \Lambda_1^{(0)}] \\
\overset{1}{\delta}_{\Lambda_1}\Psi^{(2)} &= [\Psi^{(2)}, \Lambda_1^{(0)}] + \frac{1}{2}[\Psi^{(1)}, \Lambda_1^{(1)}] \\
\overset{1}{\delta}_{\Lambda_2}\overset{1}{\delta}_{\Lambda_1}\Psi^{(2)} &= [[\Psi^{(2)}, \Lambda_2^{(0)}] + \frac{1}{2}[\Psi^{(1)}, \Lambda_2^{(1)}], \Lambda_1^{(0)}] + \frac{1}{2}[\Lambda_2^{(1)} + [\Psi^{(1)}, \Lambda_2^{(0)}], \Lambda_1^{(1)}] \\
[\overset{1}{\delta}_{\Lambda_1}, \overset{1}{\delta}_{\Lambda_2}]\Psi^{(2)} &= [\Lambda_1^{(1)}, \Lambda_2^{(1)}] + \frac{1}{2}[\Psi^{(1)}, [\Lambda_1^{(0)}, \Lambda_2^{(1)}] + [\Lambda_1^{(1)}, \Lambda_2^{(0)}]] + [\Psi^{(2)}, [\Lambda_1^{(0)}, \Lambda_2^{(0)}]]
\end{aligned}$$

and

$$\overset{1}{\delta}_{[\Lambda_1,\Lambda_2]}\Psi^{(2)} = [\Lambda_1^{(1)}, \Lambda_2^{(1)}] + \frac{1}{2}[\Psi^{(1)}, [\Lambda_1, \Lambda_2]^{(1)}] + [\Psi^{(2)}, [\Lambda_1, \Lambda_2]^{(0)}], \tag{6.54}$$

where $[\Lambda_1, \Lambda_2]^{(1)} = [\Lambda_1^{(0)}, \Lambda_2^{(1)}] + [\Lambda_1^{(1)}, \Lambda_2^{(0)}]$. Now, the term $\Psi^{(2)}$ and $[\Psi^{(1)}, \delta\Psi^{(1)}]$ are of the same order in r, but the latter is not *seen* by the order 1 inverse. Therefore, we need second order corrections in the variation for the algebra to have the correct representation up to quadratic order. A straightforward computation shows that the *order 2 inverse* of (6.39), $\overset{2}{\delta}$, satisfies indeed the above equation for $\Psi^{(2)}$. The higher the order of Ψ we want to compute, the higher order inverses we should consider to satisfy (6.53).

If we take two large gauge transformations with parameters,

$$\Lambda_1 = \sum_{n=-\infty}^{N_1} \Lambda_1^{(n)} r^n, \quad \Lambda_2 = \sum_{n=-\infty}^{N_2} \Lambda_2^{(n)} r^n, \tag{6.55}$$

then

$$[\Lambda_1, \Lambda_2] = r^{N_1+N_2}[\Lambda_1^{(N_1)}, \Lambda_2^{(N_2)}] + O(r^{N_1+N_2-1}). \tag{6.56}$$

We solve (6.39) up to order $N_1 + N_2$, and then we have the correct algebra of variations (modulus higher orders).

This extension, then, allows us to compute the variation algebra consistently with the large gauge transformations algebra. We hope this could provide the first step to the computation of the sub^n-leading charges.

Outlook

In this thesis we presented the study of phase space extensions for gravity and gauge theories. The precise contributions are the following:

- Construction of a phase space of gravity at null infinity where superrotations act canonically, chapter 1.
- The study of the GBMS group coupled to Yang-Mills large gauge symmetries, chapter 2.
- Linearized extension in Yang-Mills that contains large gauge transformations associated to the leading and the subleading charges, chapter 4.
- Extension of the phase space in Maxwell at all order, revealing the infinite hierarchy of sub^n-leading charges, chapter 5.
- Proposal for an extension to all orders in the self-dual sector of Yang-Mills, providing a perturbative-like approach to the computation of the variation algebra, chapter 6.

In recent years, this area has been studied under the perspective of multiple fronts coming together: asymptotic symmetries for radiative data (e.g. [5,7,9]), infrared triangle (e.g. [22,26, 27]), covariant phase space formalism (e.g. [12,39,43]) and double copy mappings (e.g. [109]). The interplay between the different approaches have come to be harmonious, complementing each other in different parts of the problems. From a large scale perspective, understanding the extension of (classical) phase spaces in gravity is crucial for the quantization of the theory. Within this broad confluence of techniques and ideas, the structure of symmetries is the main guiding star in the pursuing of a quantum gravity theory. This indicates, once again, that gravity *is* a geometric theory above all else, even the adjective quantum.

The study of asymptotic symmetries has many branches currently under intense research or still emerging in their own right. Below we put our findings in a broader context, summarizing some of the research lines.

- The weak-strong duality between gauge theories and gravitational theories, started by the AdS/CFT conjecture, remains to be understood for most spacetimes. Asymptotic symmetries are intrinsic to holography, e.g. [27, 115], and the covariant phase space formalism is the natural ground on which to study them. The algebraic properties in different backgrounds and extensions, and their connection to soft theorems and the S-matrix, could in principle led to new insights of a more general holographic correspondence with the dual field theory.

- The problem of the phase space of compact regions in General Relativity is as old as the theory itself. It is related to the non-locality of the energy of the gravitational field. In this context, several attempts to obtain a symmetry group canonically realisable, e.g. [39–45, 53,54,116]. These works established the existence of the *corner symmetry group*, which can be represented canonically on the gravitational phase space. The extension of the corner symmetry group aims to includes supertranslations, which poses a series of difficulties [46].

  The asymptotic symmetries have showed to contain this extended corner symmetry, e.g. [5,31], and the properties of the structure of the charges has been thoroughly studied in the past decades. The common tool between the asymptotic symmetry group and that of finite boundaries is the covariant phase space formalism. In chapter 3 we use such tool to understand the proposal made in [32] for the superrotation charge. A natural path that stems from this study is regarding higher order large gauge transformations. Subsubleading theorems [16,25,33] seem to corresponde to $O(r)$ large gauge diffeomorphisms. It would be interesting to extend the present phase space supporting the GBMS action to include such diffeomorphisms.

- Our work in the asymptotic structure of symmetries for YM is inspired in part from the asymptotic symmetries in gravity. In particular, the above mentioned problem of the $O(r)$ large gauge diffeomorphisms.

  The first order approximation presented in chapter 4 should be the realization of a higher order symmetry algebra, at least within the tree-level theory. The first step towards this general symmetry algebra is presented in chapter 6, where we provide a realization for the algebra of variations in the Self Dual sector of the theory. In particular, it can be shown from equation (6.51) that the second order includes linearized $O(r^2)$ gauge transformations and $O(r)$ ones at second order. The former would be related to the (partial) subsubleading soft gluon factorization [48].

  In this context, it would be important to make contact with the "celestial" 2d CFT approach to symmetries [100,115,117–122], which naturally incorporates higher order factorization formulas satisfying a rich algebraic structure [90,123,124].

- The contact between gravity and gauge theories takes a particular clear form in the self-dual sector, where the so called *double copy* has been proposed, see e.g. [109,110,125–128]. The basic idea is to construct a map from gravity to two copies of YM. Among the features included in this map, symmetries would be a central part. Infinitesimal symmetries has been included non-perturbatively in [109], while in [107] is proposed the extension to finite transformations, defining the map also at null infinity, see chapter 6. It would be interesting to consider the double copy map between charges. This could in principle bring more information regarding the higher orders symmetries in gravity.

- And since it could not be missing in a thesis on gravity, black holes are among the most intriguing objects where applying the study of asymptotic symmetries could bring new physics (e.g. [60, 129–135]). In particular, symmetries in the near horizon geometry for gravity and/or fields coupled could provide insights regarding the universal properties of the black hole entropy [135].

Covariant Phase Space Formalism

A.1 Covariant Phase Space

In this section we will fix nomenclature. Most of the differentiable structure on the spacetime was used in the previous chapters, but we present it here in the more general context of the jet bundle.

A.1.1 Differentiable structure on $\mathcal{M}$

$\mathcal{M}$ will be the spacetime $n-$manifold, with differentiable structure [1]. The tangent bundle $T\mathcal{M}$ is the set of pairs (x, v), where the tangent vectors $v \in T_x\mathcal{M}$ can be decomposed in a local basis $\{\partial_\mu\}$, while the cotangent bundle $T^*\mathcal{M}$ is the set of pairs (x, ω), where ω is a $1-$form, which can be decomposed in a local co-basis $\{dx^\mu\}$. Vector fields are differentiable sections of $T\mathcal{M}$, denoted as $\Gamma(T\mathcal{M})$, and one-forms are differentiable sections $T^*\mathcal{M}$, denoted $\Gamma(T^*\mathcal{M})$.

The tensorial algebra is given by the collection $\mathcal{T}\mathcal{M}$ of differentiable sections in each of the bundles $T^kT^{*l}\mathcal{M}$ for $k, l \geq 0$,

$$\mathcal{T}\mathcal{M} := \bigoplus_{k,l\geq 0} \Gamma(T^kT^{*l}\mathcal{M}). \tag{A.1}$$

This algebra is graded by the tensor type (k, l), and posses a product, $\otimes$, such that the grades are additive with respect to it.

We can also define a derivation on $\mathcal{T}\mathcal{M}$, by considering each tensor of type (k, l) as a differentiable multilinear map from $\Gamma^k(T\mathcal{M}) \times \Gamma^l(T^*\mathcal{M})$ to $\mathbb{R}$. Fixed $\xi \in \Gamma(T\mathcal{M})$ we can define the *interior product* ι_ξ as the operator

$$\iota_\xi : T^kT^{*l}\mathcal{M} \to T^kT^{*l-1}\mathcal{M}, \tag{A.2}$$

$$(\iota_v(T))\,(\xi_1, ..., \xi_{l-1}, \omega_1, ..., \omega_k) = T(\xi, \xi_1, ..., \xi_{l-1}, \omega_1, ..., \omega_k), \quad \forall \xi_i \in \Gamma(T\mathcal{M}), \omega_j \in \Gamma(T^*\mathcal{M}) \tag{A.3}$$

So far, the differentiable structure of $\mathcal{M}$ has only played a role in defining the sections, but

[1] In these first sections, for the sake of generality, we take the dimension of the base spacetime manifold to be generic, $n \geq 2$

not the algebraic properties of $\mathcal{T}\mathcal{M}$. First, we can compare the neighbouring values of the tensors by using the diffeomorphisms in $\mathcal{M}$: given $\phi : \mathcal{M} \to \mathcal{M}$ a diffeomorphism, $y = \phi(x)$ and x two points in $\mathcal{M}$, and $T \in \Gamma(T^k T^{*l}\mathcal{M})$, the pullback ϕ^* and pushforward ϕ_* of a vector field are defined as

$$\phi^*\xi(y) = d\phi_x(\xi(x)), \quad \phi_*\xi(x) = d\phi_y^{(-1)}(\xi(y)) \equiv (\phi^{-1})^*\xi(x), \tag{A.4}$$

respectively (for diffeomorphisms the identity $\phi_* = (\phi^*)^{-1}$ holds). For one-forms, we have to define in terms of evaluations with vector fields, so the natural definition is

$$\phi^*\omega(y) = \omega \circ \phi_*(x), \quad \phi_*\omega(x) = \omega \circ \phi^*(y) \tag{A.5}$$

This definitions extend to a general tensor,

$$(\phi^*(T))_y(\xi_1(y), ..., \xi_l(y), \omega_1(y), ..., \omega_k(y)) = T_x(\phi_*(\xi_1)(x), ..., \phi_*(\xi_l)(x), \phi_*(\omega_1)(x), ..., \phi_*(\omega_k)(x)) \tag{A.6}$$

and analogous formula for the pushforward $\phi_* T$.

Once that we can "transport" tensors along $\mathcal{M}$, a notion of differentiation can be defined on the tensorial algebra: the *Lie derivative*. Given ξ a vector field on $\mathcal{M}$, it generates a one parameter family of diffeomorphisms $\phi_t : \mathcal{M} \to \mathcal{M}$ by writting the differential equations (in local coordinates)

$$\dot{\phi_t(x)} \mid_{t=0} = \xi(x), \quad \phi_0 = id_{\mathcal{M}} \tag{A.7}$$

Then, we define

$$\mathcal{L}_\xi T(x) = \lim_{\epsilon \to 0} \frac{\phi^*_{-\epsilon} T(\phi_\epsilon(x)) - T(x)}{\epsilon}. \tag{A.8}$$

Observe that both tensors in the numerator above are evaluated at the same point. Also, the Lie derivative preserve the tensor valence.

Finally, the last ingredient in the exterior calculus is the exterior algebra. A $k-$form is a totally antisymmetric $k-$covariant tensor. We denote by $\Omega^k(\mathcal{M})$ the set of all the $k-$forms. The exterior algebra is given by

$$\Omega(\mathcal{M}) := \sum_{k=0}^{n} \Omega^k(\mathcal{M}), \tag{A.9}$$

since a totally antisymmetric tensor has at most order n. The algebra structure is given by the *wedge* product $\wedge$, such that each $\omega \in \Omega^1(\mathcal{M})$ is Grassmann odd with respect to it,

$$\alpha \wedge \beta = (-1)^l \beta \wedge \alpha, \quad \alpha \in \Omega^k(\mathcal{M}), \beta \in \Omega^l(\mathcal{M}) \tag{A.10}$$

The exterior derivative d is a derivation in the exterior algebra defined as the unique $\mathbb{R}-$linear map $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ that satisfies

$$df(\xi) = \xi(f), \quad f \in \Omega^0(\mathcal{M}) \tag{A.11}$$
$$d^2 f = 0, \quad f \in \Omega^0(\mathcal{M}) \tag{A.12}$$
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta, \alpha \in \Omega^k(\mathcal{M}), \beta \in \Omega^l(\mathcal{M}) \tag{A.13}$$

This map induces the De Rham exact sequence,

$$C^\infty(\mathcal{M}) \cong \Omega^0(\mathcal{M}) \to \Omega^1(\mathcal{M}) \to ... \to \Omega^n(\mathcal{M}) \to 0, \tag{A.14}$$

where the operator d, the *exterior derivative*, increases the grade of the form.

This ends the presentation of the main operators: the interior product, ι, the exterior derivative, d, and the Lie derivative. Observe that we do not use the metric at all: this structure comes only with the manifold differentiable structure.

Derivations on the Exterior algebra

We can think about the operators $\mathcal{L}, \iota$ and d with a more general approach: as derivations on the graded commutative algebra $\Omega(\mathcal{M}) = \bigoplus_{k\in\mathbb{Z}} \Omega^k(\mathcal{M})$, (we refer the reader to [136]), where $\Omega^k(\mathcal{M}) = 0$ if $k < 0$ or $k > n$.

Let us denote by $\mathrm{Der}_k\Omega(\mathcal{M})$ the space of all graded derivations of degree k, i.e., all linear mappings $D : \Omega(\mathcal{M}) \to \Omega(\mathcal{M})$ with $D(\Omega^l(\mathcal{M})) \subset \Omega^{l+k}(\mathcal{M})$ that also satisfy the graded Leibnitz rule,

$$D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + (-1)^{kl} \alpha \wedge D(\beta). \tag{A.15}$$

For example, given a vector field ξ, we have

$$\mathcal{L}_\xi \in \mathrm{Der}_0\Omega(\mathcal{M}), \quad \iota_\xi \in \mathrm{Der}_{-1}\Omega(\mathcal{M}), \quad d \in \mathrm{Der}_1\Omega(\mathcal{M}), \tag{A.16}$$

so the sets $\mathrm{Der}_k\Omega(\mathcal{M})$ are non-empty.

Then (Chapter 8, [136]), $\mathrm{Der}\Omega(\mathcal{M}) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Der}_k\Omega(\mathcal{M})$ is a graded Lie algebra with graded commutator

$$[D_1, D_2]_{\mathrm{Der}} := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1 \tag{A.17}$$

As a remarkable consequence of the above result, we have *Cartan's magic formula*, given by the following identity (which the reader might as well memorize!),

$$\mathcal{L} = [\iota, d]_{\mathrm{Der}}, \tag{A.18}$$

which, in view of the derivation graded algebra, it reads,

$$\mathcal{L} = \iota d + d\iota. \tag{A.19}$$

A.1.2 Integration and embeddings

As an application of the above formulas, consider an embedding $\phi_m : S_m \to \mathcal{M}$, where S_m is an $m-$manifold, $m < n$. Consider $\alpha \in \Omega^m(\phi(S_m))$. As a top form, we can integrate α on $\phi(S_m)$, or by pulling it back to S_m, we have

$$\int_{\phi(S_m)} \alpha = \int_{S_m} \phi^*(\alpha) \tag{A.20}$$

Consider a vector field ξ on $\phi(S_m)$ such that it generates a diffeomorphism flow $\psi_t : \phi(S_m) \to \phi(S_m)$. Then, $\psi_t \circ \phi : S_m \to M$ is a family of embeddings. Then, we can define the function

$$f_\xi[\alpha](t) := \int_{\psi_t \circ \phi(S_m)} \alpha = \int_{S_m} \phi^* \psi_t^*(\alpha), \tag{A.21}$$

whose derivative at $t = 0$ is

$$f'_\xi[\alpha](0) = \int_{S_m} \phi^*(\mathcal{L}_\xi \alpha). \tag{A.22}$$

We can distinguish two sub cases of the formula above,

- If we integrate an exact form, $d\beta$, then

$$f'_\xi(0) = \int_{S_m} \phi^*(\mathcal{L}_\xi d\beta) = \int_{S_m} \phi^*(d\iota_\xi d\beta) = \int_{\partial S_m} \phi_\partial^*(\iota_\xi d\beta), \tag{A.23}$$

  where $\phi_\partial : \partial S_m \to \partial\phi(S_m)$ is the boundary embedding.

- For a top form,

$$f'_\xi(0) = \int_{S_m} \phi^*(\mathcal{L}_\xi \alpha) = \int_{S_m} \phi^*(d\iota_\xi \alpha) = \int_{\partial S_m} \phi_\partial^*(\iota_\xi \alpha), \tag{A.24}$$

Observe that f' is linear in α and tensorial in ξ. This is, fixed α, the map

$$\xi \to f'_\xi[\alpha](0), \tag{A.25}$$

can be thought as a $1-$form. Let us call this 1-form χ_α. In particular, we can compute its exterior derivative,

$$d\chi_\alpha(\xi_1, \xi_2) = \int_{S_m} \phi^*(\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2}\alpha - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1}\alpha - \mathcal{L}_{[\xi_1,\xi_2]}\alpha) = 0 \tag{A.26}$$

so is an exact form. This formula is the "horizontal" version (i.e. on $\mathcal{M}$) of the second variation formula that we will study in the next section.

A.1.3 Exterior bi-algebra in the Jet bundle

The next ingredient to construct the jet bundle is the jet space, $\mathcal{J}$, which is the collection of all the fields and their (symmetrized) derivatives [2], schematically denoted as $(\phi_{(\mu)})$. These are the

[2] The derivatives are symmetrized by definition of a *jet*: it is the family of Taylor expansions for a set of functions at a given point. Therefore, only take part the symmetric components $\partial_{(\mu_1} ... \partial_{\mu_k)} f$

"coordinates" of the jet space.

We will denote as *characteristics* the sections of the tangent bundle $T\mathcal{G}$ [3], to avoid confusion with the vectors in the base manifold. We usually will use the letter q to indicate characteristics. The dual basis for the sections of the cotangent bundle $T^*\mathcal{G}$ is the set $\{\delta\phi^i_{(\mu)}\}_{(\mu)}$. The equation that defines the dual vectors in finite dimensions has an analogous equation in $\mathcal{G}$,

$$d_v x^\mu = v^\mu \quad \mapsto \quad \delta_q \phi^i = q^i, \tag{A.27}$$

which also satisfies $\delta_q \phi^i_{(\mu)} = \partial_{(\mu)} q^i$. With tangent and cotangent section we can construct

In complete analogy with the exterior algebra in $\mathcal{M}$, we can define the interior product, the exterior derivative and the Lie derivative for an exterior algebra in $\mathcal{G}$.

- Interior product I: given a characteristic q, the contraction of any contravariant tensor $F \in \mathcal{T}\mathcal{F}$ with q is defined as the interior product,

$$I_q F = \sum_{(\mu)} \partial_{(\mu)} q^i \frac{\partial F}{\partial \delta\phi^i_{(\mu)}}, \tag{A.28}$$

  in analogy with $\iota_v = v^\mu \frac{\partial}{\partial dx^\mu}$. This operator lowers the covariant valence of the tensors.

- Exterior derivative δ: Given a function $F : \mathcal{G} \to \mathbb{R}$,

$$\delta F = \sum_{(\mu)} \delta\phi^i_{(\mu)} \frac{\partial F}{\partial \phi^i_{(\mu)}} \tag{A.29}$$

  in the same fashion as $d = dx^\mu \partial_\mu$.

  In general, an abuse of notation is made when treating variations, as we did in the last two chapters: is common to talk about $\{\delta\phi^i_{(\mu)}\}_{(\mu)}$ as the *variation*, tangent in the field space, and not as the basis on which we *express* the value of the variation. To see this, consider the Taylor expansion of the fields at a given point x_0, which is a scalar in the jet space,

$$\phi(x_0) + \xi^\mu \phi_\mu(x_0) + \xi^\mu \xi^\nu \phi_{(\mu\nu)}(x_0) + \ldots \tag{A.30}$$

  By applying (A.29) on the real functions $\phi_\mu(x_0), \phi_{(\mu\nu)}(x_0), \ldots : \mathcal{G} \to \mathbb{R}$, which is the evaluation of the field at the point x_0, observe that,

$$\delta\phi(x_0) + \xi^\mu \delta\phi_\mu(x_0) + \xi^\mu \xi^\nu \delta\phi_{(\mu\nu)}(x_0) + \ldots, \tag{A.31}$$

  where the notational abuse is already manifest: one consider the $\delta\phi$ as an infinitesimal perturbation of ϕ,

$$\phi \to \phi + \delta\phi. \tag{A.32}$$

[3] We will assume that the sections are *sufficiently smooth* for all of our purposes. The study of differentiable structures in $\mathcal{G}$ is beyond the scope of this thesis.

Finally, starting with the exterior derivative for scalar function we can construct the exterior algebra in the jet space, by defining the Grassmann odd elements as $\{\delta\phi^i_{(\mu)}\}_{(\mu)}$ with the product $\wedge$. This procedure gives us $\Omega^1(\mathcal{J}), \Omega^2(\mathcal{J}), \ldots$.

- Lie derivative $\mathfrak{L}$: changes in coordinates for the jet space are transformations of the fields, and are parametrized by the characteristics via (A.27). On the other hand, a characteristic generates a diffeomorphism flow on the jet bundle, which can be used to define the Lie derivative in the field space. For example, given a characteristic (remember they are the tangent vectors)

$$\mathfrak{L}_q \phi^i = q^i, \tag{A.33}$$

Instead of defining it through a diffeomorphism of the jet space onto itself, we will provide a more algebraic definition (equivalent to the one involving the flow). For any tensor $F : \mathfrak{T}\mathcal{F} \to \mathbb{R}$ the Lie derivative in field space is given by

$$\mathfrak{L}_q F = \sum_{(\mu)} \partial_{(\mu)} q^i \frac{\partial F}{\partial \phi^i_{(\mu)}} + \partial_{(\mu)} \delta q^i \frac{\partial F}{\partial \delta \phi^i_{(\mu)}} \tag{A.34}$$

Observe that Cartan's magic formula still holds for the field space exterior algebra, by definition of the Lie derivative

$$\mathfrak{L}_q = \delta I_q + I_q \delta. \tag{A.35}$$

We can define $\mathrm{Der}\Omega(\mathcal{J})$ in the same way we define $\mathrm{Der}\Omega(\mathcal{M})$, and have the graded Lie algebra structure on it. $\mathfrak{L}, I, \delta$ have degrees $0, -1, 1$ respectively.

Finally, the *jet bundle* $\mathcal{F}$ is the set of pairs $\{x, \phi^i_{(\mu)}\}$ where $\phi^i_{(\mu)} \in \mathcal{J}$ and $x \in \mathcal{M}$. These local coordinates give $\mathcal{F}$ the local trivialization $\mathcal{M} \times \mathcal{J}$.

The key idea behind the jet bundle structure is to have the values of the fields along with their derivatives, together with the base manifold, in a unique covariant object. Operators along $\mathcal{M}$ are called horizontal, such as $\iota, \mathcal{L}, d$, while operators along $\mathcal{J}$ are called vertical, such as $I, \mathfrak{L}, \delta$.

This double structure that we arrive at is called *bicovariant Cartan calculus*. As we will see, introducing "dynamical symmetries" allows for the horizontal and vertical operators to mix, [39], [43], [53]. This will be more clear in the following sections.

A.1.4 Symmetries and Symplectic structure

In this section we review the definitions of symmetries in the covariant phase space formalism.

The fields that we are taking as the building blocks of $\mathcal{J}$ will be the solutions to the field equations, provided some boundary conditions. Relaxing the boundary conditions imply an enlargement of the field space, and in consequence of the jet bundle, while tighter boundary conditions reduce the jet bundle. We denote by $\mathcal{S}$ the solution field space, and observe that it is a submanifold of the field space corresponding to all the possible fields, so all the definitions above are valid on solution space by taking the pullback via the embedding of $\mathcal{S}$ into $\mathcal{J}$.

The starting point in the discussion of a (classical) field theory is the Lagrangian density, $\mathscr{L}[\phi]$, a top form in $\mathcal{M}$ that depends on the field configurations, whose integral is the action,

$$S = \int_{\mathcal{M}} \mathscr{L}, \quad \mathscr{L} : \mathcal{G} \to \Omega^n(\mathcal{M}). \tag{A.36}$$

For example, in the case of General Relativity or Yang-Mills,

$$\mathscr{L}^{GR}[g] = (R[g] - 2\Lambda) d\mu_g, \tag{A.37}$$

$$\mathscr{L}^{YM}[A_\mu] = -\frac{1}{2}\mathrm{Tr}(F \wedge \star F), \tag{A.38}$$

where $R[g]$ is the scalar curvature of g, Λ is the cosmological constant, and $F = d_A A$ is the field strength.

Definition A.1. We say that a characteristic tangent to $\mathcal{G}$ is a *symmetry* of a lagrangian theory, with lagrangian $\mathscr{L}$, if

$$\mathfrak{L}_q \mathscr{L} = dB_q, \tag{A.39}$$

where B_q is a $(n-1)$-form. In other words, the action is invariant under the flow generated by q in $\mathcal{F}$.

The next is one of the possible definitions of gauge symmetries. Later we will see an equivalent one.

Definition A.2. Given the group of symmetries for the lagrangian $\mathscr{L}$, there will be a subset that is parametrized by free spacetime functions $\{\lambda_i\}_{i=1}^{n_g}$. These are the *gauge symmetries* of the theory.

Symplectic potential and Symplectic form

As a scalar function $S : \mathcal{G} \to \mathbb{R}$, we can take its exterior derivative in field space,

$$\delta S = \delta \int_R \mathscr{L}, \tag{A.40}$$

where the domain R is fixed (there is no field dependent construction on it, or, equivalently, the embedding $i : R \to M$ is constant with respect to the fields), so

$$\delta S = \int_R \sum_{(\mu)} \delta\phi^i_{(\mu)} \frac{\partial \mathscr{L}}{\partial \phi^i_{(\mu)}}. \tag{A.41}$$

Since $\delta\phi^i_{(\mu)} = \partial_{(\mu)}(\delta\phi^i)$, for some $\delta\phi^i$ coordinate functions on $\mathcal{F}$, we can integrate by parts using Leibniz rule,

$$\delta S = \int_R \sum_{(\mu)} (-1)^{|\mu|} \delta\phi^i \partial_{(\mu)} \frac{\partial \mathscr{L}}{\partial \phi^i_{(\mu)}} + \int_{\partial R} \theta(\phi), \tag{A.42}$$

where θ is an $(n-1)-$form in $\mathcal{M}$, and a $1-$form in $\mathcal{G}$. We take ∂R as a spacelike or null hypersurface. This field space 1-form is known as *presymplectic potential*.

The principle of least action, that is, that the physical solutions for the fields are given when $\delta S = 0$, needs certain prescription for the second term integral, the boundary term. Here is where the boundary conditions enter the field. By fixing sufficiently fast decay rates for the fields as they approach ∂R, we can set to zero the second term and obtain the *Euler-Lagrange equations* for the theory,

$$\sum_{(\mu)}(-1)^{|\mu|}\partial_{(\mu)}\frac{\partial \mathscr{L}}{\partial \phi^i_{(\mu)}} = 0. \tag{A.43}$$

Solving these equations provide us with the solution space, $\mathcal{S}$. Observe that the least action principle states that the field space gradient of the action vanishes, and thus the solutions are "critical" points of the action. We often will say that a property is satisfied "on-shell" if it is satisfied when pullbacked to the submanifold $\mathcal{S} \subset \mathcal{G}$.

Now we study the second term: it is a 1-form in $\mathcal{F}$, which we will denote $\Theta[\phi]$. We can compute its exterior derivative, known as the *presymplectic form*,

$$\Omega := \delta\Theta, \quad \omega := \delta\theta \tag{A.44}$$

the second equality being in the case where R is fixed. Now, since

$$\delta\mathscr{L} = E + d\theta, \tag{A.45}$$

where E are the equations (A.43), we see that

$$0 = \delta^2\mathscr{L} = \delta E + d\delta\theta = \delta E + d\omega, \tag{A.46}$$

and therefore on shell we have that $d\omega = 0$, which implies that the symplectic form is conserved.

We can give a more precise definition for the gauge symmetries using the presymplectic form,

Definition A.3. Given a characteristic q, we say it generates a gauge symmetry if and only if

$$\Omega[q_1, q] = 0, \quad \forall q \in \Gamma(T\mathcal{G}). \tag{A.47}$$

When R is not fixed, equation A.41 is no longer true, since now we have a field-dependent embedding $\psi_n : R \to M$, which have a non-vanishing variation, such as in the subsection A.1.2. We will review the correct expressions for the case of Diffeomorphism covariance in general relativity, and the gauge covariance in Yang-Mills theory.

A.1.5 Diffeomorphism covariance

Consider ξ vector field on $\mathcal{M}$. As we see before, it defines a family of diffeomorphisms $\psi_t : M \to \mathcal{M}$. This, in turn, via $\phi^i \circ \psi_t$, generates a curve in field space $\mathcal{G}$. By taking the derivative with respect to t, we obtain a characteristic, denoted by $\hat{\xi}$

$$\hat{\xi} := \frac{d}{dt}\phi^i \circ \psi_t\Big|_{t=0} \tag{A.48}$$

This characteristic is the associated one to the vector field ξ.

Definition A.4. Covariance and Semi-covariance under diffeomorphisms

- A general *covariant theory* is such that for an arbitrary field space form θ, it is satisfied,

$$\mathfrak{L}_{\hat{\xi}}\theta = \mathcal{L}_{\xi}\theta + I_{\delta\xi}\theta \tag{A.49}$$

- A *semi-covariant theory* is such that the variation of the lagrangian $\mathscr{L}$ under a diffeomorphism is a total derivative,

$$\mathfrak{L}_{\hat{\xi}}\mathscr{L} = d\ell_{\xi}, \tag{A.50}$$

  for some local function ℓ_{ξ}.

For a diffeomorphism covariant theory, the lagrangian $\mathscr{L}$ satisfies,

$$\delta_{\xi}\mathscr{L} = d(\iota_{\xi}\mathscr{L}), \tag{A.51}$$

since it is a scalar in field space.

Covariance for a theory can be broken by gauge fixing or boundaries, as is the case in BMS: by fixing Bondi gauge, we allow certain diffeomorphism to act non trivially in the solution space, since they have non-zero charges associated. These are the so called *large gauge transformations*. Nevertheless, the celestial sphere metric in $\mathscr{I}$ is still fixed, since $\delta_f q_{ab} = \delta_V q_{ab} = 0$.

When taking the action of $GBMS$, as we saw, the celestial metric on $\mathscr{I}$ is not fixed. This imply that the diffeomorphisms come with extra information: the embedding of some fixed metric, say $\mathring{q}_{ab}$, on the new $\mathscr{I}$. In terms of the section A.1.2, when taking $\mathfrak{L}_{\hat{\xi}}$ of an integral, the *domain* is moving also! The extra terms due to this behaviour prove to contain dynamical information regarding the fields. We will explore this in the rest of this section.

A.1.6 Embedding maps

To express the field-dependence of the embeddings, take a map $\mathcal{G} : \mathcal{S} \to C^\infty(m \hookrightarrow M)$, from the solution space to the set of smooth embeddings $\phi : m \hookrightarrow M$, where m is some $m-$manifold, $m \leq n$.

In a field-independent case, the map is constant, and the Lie derivative $\mathfrak{L}$ commutes with embedding pullbacks and therefore integrations are independent of the field. But in the case of a field-dependent embeddings, the pullback and $\mathfrak{L}$ do not commute (see [43], [53], [45]). Consider q a characteristic, tangent to solution space $\mathcal{S}$, and denote by $\mathcal{G}^*$ the pullback of the elements in $\mathrm{Im}(\mathcal{G})$ (we abuse a little bit notation here to ease the formulas). Since q is tangent to $\mathcal{S}$, it defines a flow Φ_t^q such that

$$\frac{d}{dt}\Phi_t^q(\phi^i)\Big|_{t=0} = q(\phi^i), \quad \phi^i \in \mathcal{S}. \tag{A.52}$$

This flow defines a flow on $\mathrm{Im}(\mathcal{G}) \subset C^\infty(m \hookrightarrow M)$, through $\mathcal{G}$. Then, the exterior derivative of $\mathcal{G}$ in field space, when contracted with a characteristic, gives a tangent vector on $C^\infty(m \hookrightarrow M)$, which is a vector field on m, generating a diffeomorphism of m:

$$\mathcal{S} \xrightarrow{\mathcal{G}} C^\infty(m \hookrightarrow M) \xrightarrow{X} \mathrm{Diff}(m), \tag{A.53}$$

where $X(\phi) := \psi \circ e_m^{-1}$, where e_m is some reference embedding [4]. We can compute the differential of the map $X \circ \mathcal{G}$ evaluated on q. First, the differential of $\mathcal{G}$ on q is

$$\delta\mathcal{G}[q] = \frac{d}{dt}\mathcal{G}(\Phi_t^q)\ |_{t=0}\ . \tag{A.54}$$

$$\delta(X \circ \mathcal{G})[q] = \frac{d}{dt}\mathcal{G}(\Phi_t^q) \circ e_m^{-1}\Big|_{t=0}. \tag{A.55}$$

This is the differential evaluated at q, based on a general point ϕ in $\mathcal{S}$.

Variations of integrals

Consider some $\alpha[\phi] \in \Omega^m(M)$, a field-dependent form (such as a Lagrangian). Its integral on the image of m by the embeddings is given by

$$S[\alpha] = \int_m \mathcal{G}^*\alpha. \tag{A.56}$$

We can compute the exterior Lie derivative of $S : \mathcal{S} \to \mathbb{R}$, using the result in section A.1.2 and that α depends on the fields, and arrive at the result,

$$\mathfrak{L}_q\mathcal{G}^*\alpha = \mathcal{G}^*(\mathfrak{L}_q\alpha + \mathcal{L}_{\chi(q;\mathcal{G})}\alpha), \tag{A.57}$$

where $\chi(q;\mathcal{G})$ is a vector field (analogous to ξ in (A.22)) generating an infinitesimal diffeomorphism on $\mathrm{Diff}(m)$,

$$\chi(q;\mathcal{G}) := \frac{d}{dt}\mathcal{G}(\Phi_t^q)\mathcal{G}(\Phi_0^q)^{-1}\Big|_{t=0}, \tag{A.58}$$

By formula (A.55), the above equation can be written as the Maurer-Cartan form for the embedding with respect to a fixed field configuration, e_m, and therefore we have

$$\chi(q;\mathcal{G}) =: \chi_{\mathcal{G}}[q] = \delta(X \circ \mathcal{G}) \circ (X \circ \mathcal{G})^{-1}[q], \tag{A.59}$$

[4] We are taking $e_m|_{\mathcal{M}}$ as equal to e_m to avoid extra symbols in the formulas.

which defines a 1−form $\chi_{\mathcal{G}}$ in $\mathcal{F}$. Since (A.57) is valid for any characteristic q, we can stablish the identity

$$\delta \mathcal{G}^* \alpha = \mathcal{G}^*(\delta\alpha + \mathcal{L}_{\chi_{\mathcal{G}}}\alpha), \tag{A.60}$$

where the operator $\mathcal{L}_{\chi_{\mathcal{G}}}$ is a new derivation in $\mathcal{F}$: it acts on $\mathcal{M}$ forms as a Lie derivative and on $\mathcal{F}$ as an exterior derivative, and the vector on which the Lie derivative is computed is $\chi_{\mathcal{G}}[q]$. Thus, $\mathcal{L}_{\chi_{\mathcal{G}}}$ has degree $(0,1)$ in the bicovariant Cartan structure of $\mathcal{F}$.

The same change in degree occur for the operator $I_{\chi_{\mathcal{G}}}, \mathfrak{L}_{\chi_{\mathcal{G}}}$ in field space and $\iota_{\chi_{\mathcal{G}}}$ in base space. The list below summarize the degrees of each operator,

Operator	M	$\mathcal{G}$
d	+1	0
δ	0	+1
ι	-1	0
I	0	-1
$\mathcal{L}$	0	0
$\mathfrak{L}$	0	0
$\iota_{\chi_{\mathcal{G}}}$	-1	+1
$I_{\chi_{\mathcal{G}}}$	0	0
$\mathcal{L}_{\chi_{\mathcal{G}}}$	0	+1
$\mathfrak{L}_{\chi_{\mathcal{G}}}$	0	+1

The graded Lie algebra structure of the derivations is still valid, and Cartan's formula is valid on the new operators,

$$\mathfrak{L}_{\chi_{\mathcal{G}}} = [I_{\chi_{\mathcal{G}}}, \delta] \tag{A.61}$$

Second variation formula for the embedding map

For the embeddings, we had in section A.1.2 formula (A.26), which states that χ_a is a closed form. We will see that in the general case of $\chi_{\mathcal{G}}$, it is not closed in $\mathcal{F}$.

Theorem A.1. The curvature operator associated to the Maurer-Cartan form vanishes

$$R[\mathcal{G}] := \delta\chi_{\mathcal{G}} + \frac{1}{2}[\chi_{\mathcal{G}}, \chi_{\mathcal{G}}] = 0 \tag{A.62}$$

Proof. Consider (A.60). This is a $(p,1)$−form in $\mathcal{F}$, in the sense that α is a p−form in $\mathcal{M}$ and δ is the exterior derivation on the field space. On taking the exterior derivative of the right hand side of (A.60), we need the formula for the exterior derivative of a general 1−form β in field space,

$$(\delta\beta)[q_1, q_2] = q_1(\beta[q_2]) - q_2(\beta[q_1]) - \beta[[q_1, q_2]], \quad \forall q_1, q_2 \text{ tangents to field space.} \tag{A.63}$$

Then,

$$0 = \delta^2 \mathcal{G}^* \alpha[q_1, q_2] = \left(\delta \mathcal{G}^* (\delta \alpha + \mathcal{L}_{\chi_{\mathcal{G}}} \alpha)\right)[q_1, q_2], \tag{A.64}$$

and therefore

$$\begin{aligned} 0 &= \left(\delta \mathcal{G}^* (\delta \alpha + \mathcal{L}_{\chi_{\mathcal{G}}} \alpha)\right)[q_1, q_2] \qquad (A.65) \\ &= \mathfrak{L}_{q_1} \left(\mathcal{G}^* (\mathfrak{L}_{q_2} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha) \right) - \mathfrak{L}_{q_2} \left(\mathcal{G}^* (\mathfrak{L}_{q_1} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_1]} \alpha) \right) - \mathcal{G}^* (\mathfrak{L}_{[q_1,q_2]} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[[q_1,q_2]]} \alpha) \qquad (A.66) \end{aligned}$$

We use the identity (A.57) on each term, the first term being,

$$\mathcal{G}^* \left(\mathfrak{L}_{q_1} (\mathfrak{L}_{q_2} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha) - \mathcal{L}_{\chi_{\mathcal{G}}[q_1]} ((\mathfrak{L}_{q_2} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha)) \right) \tag{A.67}$$

Regrouping everything, and since $\mathcal{G}$ is tensorial, we can take its argument to be zero,

$$\begin{aligned} 0 &= \mathfrak{L}_{q_1} (\mathfrak{L}_{q_2} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha) + \mathcal{L}_{\chi_{\mathcal{G}}[q_1]} ((\mathfrak{L}_{q_2} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha)) - (1 \leftrightarrow 2) \qquad (A.68) \\ & -\mathfrak{L}_{[q_1,q_2]} \alpha - \mathcal{L}_{\chi_{\mathcal{G}}[[q_1,q_2]]} \alpha. \qquad (A.69) \end{aligned}$$

Since $\mathfrak{L}_{q_1} \mathfrak{L}_{q_2} - \mathfrak{L}_{q_2} \mathfrak{L}_{q_1} = \mathfrak{L}_{[q_1,q_2]}$, the terms corresponding to $\delta^2 \alpha$ vanish, as they should. Next, we have the terms,

$$\mathcal{L}_{\chi_{\mathcal{G}}[q_1]} \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha - (1 \leftrightarrow 2) = \mathcal{L}_{[\chi_{\mathcal{G}}[q_1], \chi_{\mathcal{G}}[q_2]]} \alpha, \tag{A.70}$$

which in terms of forms in field space, we have

$$([\chi_{\mathcal{G}}, \chi_{\mathcal{G}}]_{Lie})[q_1, q_2] = [\chi_{\mathcal{G}}[q_1], \chi_{\mathcal{G}}[q_2]]_{Lie} - [\chi_{\mathcal{G}}[q_2], \chi_{\mathcal{G}}[q_1]]_{Lie} = 2[\chi_{\mathcal{G}}[q_1], \chi_{\mathcal{G}}[q_2]]_{Lie}, \tag{A.71}$$

where we explicitly write *Lie* under the bracket. The above equation is extremely important for curvature formula to be valid.

Next, we have the terms

$$\mathfrak{L}_{q_1} \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_1]} \mathfrak{L}_{q_2} \alpha - (1 \leftrightarrow 2) - \mathcal{L}_{\chi_{\mathcal{G}}[[q_1,q_2]]} \alpha, \tag{A.72}$$

where we can use that $\mathfrak{L}_{q_1} = \mathcal{L}[q_1^\vee] + I_{\chi_{\mathcal{G}}[q]}$, where $q_1^\vee$ is the vector field that generates locally a diffeomorphism in $\mathrm{Diff}(M)$. Therefore, since all the function in which $\mathfrak{L}$ is acting are field space scalars, we have

$$\mathcal{L}_{[q_1^\vee]} \mathcal{L}_{\chi_{\mathcal{G}}[q_2]} \alpha + \mathcal{L}_{\chi_{\mathcal{G}}[q_1]} \mathcal{L}_{[q_2^\vee]} \alpha - (1 \leftrightarrow 2) - \mathcal{L}_{\chi_{\mathcal{G}}[[q_1,q_2]]} \alpha = \mathcal{L}_{[q_1^\vee, \chi_{\mathcal{G}}[q_2]] - [q_2^\vee, \chi_{\mathcal{G}}[q_1]] - \chi_{\mathcal{G}}[[q_1,q_2]]} \alpha. \tag{A.73}$$

The vector in the last Lie derivative is exactly

$$(\delta \chi_{\mathcal{G}})[q_1, q_2] \tag{A.74}$$

Finally, putting everything back together,

$$0 = \left(\mathcal{L}_{\delta \chi_{\mathcal{G}} + \frac{1}{2}[\chi_{\mathcal{G}}, \chi_{\mathcal{G}}]} \alpha \right) [q_1, q_2], \quad \forall q_1, q_2, \tag{A.75}$$

which proves (A.62).

□

A.2 Symplectic structure and Extensions

In this section we will construct the charges associated to the covariant phase space that we review in the previous section, applied to the particular case of GBMS action. This section follows the expositions in [43, 45, 54, 58].

We have $\mathscr{I}^+$ as the codimension 1 manifold where we integrate the forms. Its boundary is the union $\mathscr{I}^+_- \cup \mathscr{I}^+_+ =: \partial\mathscr{I}$. We take as the base manifold m the celestial sphere $S_\infty \cong S^2$. A map $\mathcal{G} : \mathcal{S} \to C^\infty(m \times \mathbb{R} \hookrightarrow M)$ can be restricted to a map $\mathcal{G}_{\partial\mathscr{I}} : \mathcal{S} \to C^\infty((m) \hookrightarrow M)$ trivially. Also, we can extend $\mathcal{G}$ to the bulk, $\mathcal{G}_{bulk}$, since the subleading terms of the diffeomorphism generators are field dependent.

A.2.1 Noether theorems and charges

Theorem A.2. Noether's First Theorem Given a Lagrangian $\mathscr{L}$ defined on (M, g), there is a bijection between (the equivalence classes of) global symmetries and (the equivalence classes of) conserved codimension 1 forms J, known as Noether currents.

The content of this theorem states that after applying a Lie derivative of the Lagrangian with a characteristic $\hat{\xi}$, associated to a diffeomorphism generator ξ, we have

$$\mathfrak{L}_{\hat{\xi}} \mathscr{L} = I_{\hat{\xi}} \delta \mathscr{L} = I_{\hat{\xi}} E + I_{\hat{\xi}} d\theta, \tag{A.76}$$

with E the Euler-Lagrange set of equations in form notation. In the case of a covariant theory,

$$d(\iota_{\xi} \mathscr{L}) = I_{\hat{\xi}} E + I_{\hat{\xi}} d\theta, \tag{A.77}$$

and therefore we have

$$0 = I_{\hat{\xi}} E + d(I_{\hat{\xi}} \theta - \iota_{\xi} \mathscr{L}) =: d(S_{\xi} + j_{\xi}), \tag{A.78}$$

where

$$j_{\xi} = I_{\hat{\xi}} \theta - \iota_{\xi} \mathscr{L}, \quad dS_{\xi} = I_{\hat{\xi}} E, \tag{A.79}$$

are the *Noether current* and the *Noether weakly vanishing current*, respectively. The fact that S_{ξ} *can* be defined is the content of Noether's Second theorem, which we will not prove here,

Theorem A.3. Noether's second theorem

$$\delta_\xi \phi \frac{\delta \mathscr{L}}{\delta \phi} = dS_\xi \left[\frac{\delta L}{\delta \phi}\right], \tag{A.80}$$

where S_ξ is the Noether weakly vanishing current.

Proof. □

The idea behind this theorem is that, upon imposing the boundary condition and finding a family of diffeomorphisms that preserve them, we can obtain identities within the equations of motions. For example, in vacuum General Relativity, [12],

$$S_\xi = -2(d^{n-1}x)_\mu \sqrt{-g} G^{\mu\nu} \xi_\nu, \tag{A.81}$$

while for Einstein-Maxwell theory ([137]),

$$S_{\xi,\lambda} = -2(d^{n-1}x)_\mu \sqrt{-g} \left(\nabla_\nu F^{\nu\mu}(\xi^\rho A_\rho + \lambda) G^{\mu\nu} \xi_\nu\right) \tag{A.82}$$

For vacuum gravity, on shell we have $\mathscr{L} = 0$, and $E = 0$, so covariance immediatly implies

$$j_\xi = I_\xi \theta. \tag{A.83}$$

Observe that this is what we call *charge density*: is the integral of the Noether flux along $\mathscr{I}$ is the charge

$$H_\xi := \int_{\mathscr{I}} j_\xi \tag{A.84}$$

The corner charges come from realizing that

$$dj_\xi = 0, \tag{A.85}$$

and therefore, since $\mathscr{I}$ is simply connected, there exists a 1-form in field space and a 2-form in the base manifold Q_ξ such that

$$j_\xi = dQ_\xi \tag{A.86}$$

In relating to chapter 1, chapter 2, chapter 3, we have,

$$P_f := H_{\xi_f}, \quad J_V := H_{\xi_V}, \quad Q_\lambda := H_\lambda, \tag{A.87}$$

where the last charge comes from the $G-$invariance in Yang-Mills. In the presence of boundary conditions, the assumption $\delta_\xi = \mathcal{L}_\xi + I_\xi$ is no longer true and appears the concept of *anomaly* [45],

$$\Delta_\xi := \delta_\xi - \mathcal{L}_\xi - I_{\delta\xi}, \tag{A.88}$$

where $\delta\xi$ denotes the variation of ξ due to the field dependence. This plays an important role in chapter 3.

A.2.2 Symplectic potential and symplectic form

We have two prescriptions regarding the extended phase space structure containing $\chi_{\mathcal{G}}$ and the fields ϕ. They are equivalent, the only difference being a boundary term in the extended symplectic form. The first prescription, see e.g. [39], defines as the *extended action* the following functional,

$$S[\mathscr{L},\mathcal{G}] = \int_R \mathcal{G}^*_{bulk}\mathscr{L}[\phi], \tag{A.89}$$

where $\mathscr{L}[\phi]$ is the lagranngian of the theory, depending on the fields ϕ. Let us start with the first variation of the action,

$$\delta S = \int_R \delta(\mathcal{G}^*_{bulk}\mathscr{L}), \tag{A.90}$$

where R is the region enclosed by two $t = cnt$ hypersurfaces, $\Sigma_{t_1}, \Sigma_{t_2}$. By (A.60), and keeping in mind that $\delta\mathscr{L} = E + d\theta$,

$$\delta S = \int_R \mathcal{G}^*_{bulk}(E) + \int_{\partial R} \mathcal{G}^*_{\partial R}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L}). \tag{A.91}$$

Then, the covariant symplectic potential is given by

$$\theta_{cov} = \theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L}. \tag{A.92}$$

The second prescription, see e.g. [43,54,138], defines as the extended action as follows,

$$S[\mathscr{L},\mathcal{G}] = \int_R \mathscr{L}[\mathcal{G}^*_{bulk}\phi], \tag{A.93}$$

which, in other words, is the same as in (A.90) but $\mathscr{L}$ is evaluated at the *dressed fields* $\mathcal{G}^*_{bulk}\phi$. Now, when computing the first variation,

$$\delta S = \int_R E\{\mathcal{G}^*_{bulk}\phi\}\delta\mathcal{G}^*_{bulk}\phi + \int_{\partial R} \theta\{\delta\mathcal{G}^*_{\partial R}\phi\}. \tag{A.94}$$

where $\theta\{\delta\mathcal{G}^*_{\partial R}\phi\}$ means that θ is the solution space $1-$form evaluated at the variation of the *dressed field* $\delta\mathcal{G}^*_{\partial R}\phi$ By (A.60), the last term is

$$\theta\{\delta\mathcal{G}^*_{\partial R}\phi\} = \theta\{\mathcal{G}^*_{\partial R}(\delta\phi + \mathcal{L}_{\chi_{\mathcal{G}}}\phi)\} = \left(\theta + I_{\chi_{\mathcal{G}}}\theta\right)\{\mathcal{G}^*_{\partial R}\phi\}, \tag{A.95}$$

where now they are evaluated at the dressed variation of the solutions. By definitions (A.79) and (A.86), we can have,

$$\delta S = \int_R \mathcal{G}^*_{bulk}(E) + \int_{\partial R} \mathcal{G}^*_{\partial R}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + dQ_{\chi_{\mathcal{G}}}), \tag{A.96}$$

Then, the covariant symplectic potential is given by

$$\theta_{cov} = \theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + dQ_{\chi_{\mathcal{G}}}. \tag{A.97}$$

Observe that the difference between (A.92) and (A.97) is the boundary term

$$dQ_{\chi_{\mathcal{G}}}. \tag{A.98}$$

This ambiguity can be partially fixed by imposing that θ_{cov} do annihilate infinitesimal diffeomorphisms [138]. We will use the second prescription to relate our results in chapter 1 with those in the literature regarding extended corner symmetries, e.g. [45,46,54].

Let us now proceed with the computation of the symplectic form on $\mathscr{I}^+$. By defining

$$\Theta^t_{cov} := \int_{\Sigma_t} \mathcal{G}^*_{\mathscr{I}}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + dQ_{\chi_{\mathcal{G}}}), \tag{A.99}$$

we have a symplectic potential on each Cauchy surface. In particular, the limit $t \to +\infty$ at fixed u allows us to take $\Sigma_t \to \mathscr{I}^+$. As it was shown in [31] for the gravity case, the limit can be taken upon certain renormalization procedure, adding counter terms to cancel the divergences. Then, we can take the limit and integrate on $\mathscr{I}$, by taking the pull back with $\mathcal{G}_{\mathscr{I}}$,

$$\Theta_{cov} := \int_{m\times\mathbb{R}} \mathcal{G}^*_{\mathscr{I}}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + dQ_{\chi_{\mathcal{G}}}) = \int_{m\times\mathbb{R}} \mathcal{G}^*_{\mathscr{I}}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + Q_{\chi_{\mathcal{G}}}), \tag{A.100}$$

Assuming we have a well defined Θ_{cov},

$$\Omega_{cov} := \int_{\mathscr{I}} \delta\mathcal{G}^*_{\mathscr{I}}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L}), \tag{A.101}$$

which by (A.60),

$$\Omega_{cov} := \int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}\left(\delta(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + dQ_{\chi_{\mathcal{G}}}) + \mathcal{L}_{\chi_{\mathcal{G}}}(\theta + \iota_{\chi_{\mathcal{G}}}\mathscr{L} + dQ_{\chi_{\mathcal{G}}})\right), \tag{A.102}$$

which, through (A.62), can be show to be [53], [43],

$$\Omega_{cov} := \int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(\delta\theta) + \int_{\partial\mathscr{I}} \mathcal{G}^*\left(\iota_{\chi_{\mathcal{G}}}\theta + \frac{1}{2}\iota_{\chi_{\mathcal{G}}}\iota_{\chi_{\mathcal{G}}}\mathscr{L} + \delta Q_{\chi_{\mathcal{G}}} + \mathcal{L}_{\chi_{\mathcal{G}}}Q_{\chi_{\mathcal{G}}}\right), \tag{A.103}$$

In the gravity case, on solutions we have $\mathscr{L} = 0$, so we have

$$\Omega^{grav}_{cov} := \int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(\delta\theta^{grav}) + \int_{\partial\mathscr{I}} \mathcal{G}^*(\iota_{\chi_{\mathcal{G}}}\theta^{grav} + \delta Q_{\chi_{\mathcal{G}}} + \mathcal{L}_{\chi_{\mathcal{G}}}Q_{\chi_{\mathcal{G}}}), \tag{A.104}$$

A.2.3 Iyer-Wald charges

In [137], after renormalizing the symplectic potential on a segment on $\mathscr{I}$, they computed the Iyer-Wald charge, given by the following covariant phase space theorem, [139],

Theorem A.4. Fundamental theorem of covariant phase space formalism

By contracting ω with δ_ξ, there exists a unique (up to total derivatives) $(n-2,1)-$form k_ξ that satisfies

$$I_{\delta_\xi}\omega = dk_\xi. \tag{A.105}$$

In terms of the Noether charge, we have

$$k_\xi = \delta Q_\xi - Q_{\delta\xi} - \iota_\xi\theta + d(\cdot). \tag{A.106}$$

We call the form k the *Iyer-Wald codimension 2 form* associated to ξ.

Proof. □

We arrive at the identity

$$\boxed{\not{\delta} H_\xi = \int_{\partial\Sigma} k_\xi} \tag{A.107}$$

which defines the Iyer-Wald surface charge.

A.2.4 Integrability and Poisson algebra

Starting from the expression (A.106), we see two possible sources of non-integrability,

- Field-dependence: the term $Q_{\delta\xi}$ contains the charge associated to the infinitesimal diffeomorphism generator $\delta\xi$, which in general will not be vanishing, do to the fact that we are fixing boundary conditions. The introduction of correcting field-dependent diffeomorphism that cancel this term has been studied in the past years (e.g. [140], [132], [78]).

- Symplectic flux: the term $\iota_\xi\theta$ implies

$$\delta \int_{\partial\Sigma} k_\xi = - \int_{\partial\Sigma} \iota_\xi\omega, \tag{A.108}$$

  and therefore we have the obstruction given by the symplectic flux at the boundary. This is known as the *leaky boundary conditions*, where some amount of the charges pervades through the boundary.

Leaky boundary conditions ([58]) can be split into two categories: Kinematical and Dynamical. The kinematical part of the flux $\iota_\xi\omega$ affects the changes the intrinsic boundary degrees of freedom, such as the celestial 2-metric q_{ab}, while the radiative degrees of freedom remains fixed. Such is the case of 3d-gravity, where the Weyl tensor vanishes (and therefore no gravitons are present) but the asymptotic diffeomorphisms can change the leading order of the metric [79, 141–145].

In the case of general relativity, the dynamical part of the flux $\iota_\xi\omega$ is the responsible for the change in the Weyl tensor. We show now see how the embedding map formulation allow us to restore the integrability of the charges, using the first prescription, (A.90) (since it differ by a boundary term with respect to (A.93), the integrability result will be the same).

In gravity, the lagrangian vanishes on-shell (since it is proportional to the scalar curvature) and therefore Noether charge for a field-independent diffeomorphism generator ξ is

$$dQ_\xi = I_{\hat{\xi}}\theta, \tag{A.109}$$

therefore,

$$I_{\hat{\xi}}\Omega^{grav} = I_{\hat{\xi}}\left(\int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(\delta\theta^{grav})\right) = \int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(d\iota_\xi\theta^{grav} - \delta dQ_\xi). \tag{A.110}$$

We see that we have an integrable term and a non-integrable term, which we can write schematically as

$$\not{\delta} H_\xi = \iota_\xi\theta - \delta Q_\xi. \tag{A.111}$$

If we add the boundary term $\mathcal{G}^*(\iota_{\chi_\mathcal{G}}\theta^{grav})$ from (A.104), and compute its contraction with $\hat{\xi}$,

$$I_{\hat{\xi}}\left(\int_{\partial\mathscr{I}} \mathcal{G}^*(\iota_{\chi_\mathcal{G}}\theta^{grav})\right) = \int_{\mathscr{I}} \mathcal{G}^*(\iota_{\chi_\mathcal{G}[\hat{\xi}]}\theta^{grav} - \iota_{\chi_\mathcal{G}}\theta^{grav}[\hat{\xi}]) \tag{A.112}$$

Observe that $\chi_\mathcal{G}[\hat{\xi}] = -\xi$, since is (minus) the generator of the diffeomorphism on the surface $(\partial\mathscr{I})$, and

$$\iota_{\chi_\mathcal{G}}\theta^{grav}[\hat{\xi}] = \iota_{\chi_\mathcal{G}} dQ_\xi. \tag{A.113}$$

Then,

$$I_{\hat{\xi}}\Omega^{grav}_{cov} = \int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(d\iota_\xi\theta^{grav} - \delta dQ_\xi) + \int_{\partial\mathscr{I}} \mathcal{G}^*(-\iota_\xi\theta^{grav} - \iota_{\chi_\mathcal{G}} dQ_\xi) \tag{A.114}$$

which, by virtue of (A.60) applied to Q_ξ,

$$I_{\hat{\xi}}\Omega^{grav}_{cov} = -\delta\int_{\mathscr{I}} \mathcal{G}^*_{\mathscr{I}}(dQ_\xi). \tag{A.115}$$

Then, when considering the corner contribution, the charges are integrable.

Finally, [53], one can show that using Ω^{grav}_{cov} the charges are canonically represented,

$$\{H_{\xi_1}, H_{\xi_2}\} = H_{[\xi_1,\xi_2]} \tag{A.116}$$

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