Diameter constrained network reliability: exact evaluation by factorization and bounds.

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Abstract:

Consider a network where the links are subject to random, independent failures. The diameter constrained network reliability parameter R(G,K,D) measures the probability that the set K of terminals of the network are linked by operational paths of length less or equal to D. This parameter generalizes the classical network reliability, allowing to reflect performance objectives that restrict the maximum length of a path in the network. This is the case, for example, when the transmissions between every two terminal nodes in the subset K are required to experience a maximum delay D.T (where T is the delay experienced at a single node or link); then the probability that after random failures of the communication links, the surviving network meets the maximum delay requirement is the diameter constrained reliability R(G,K,D).

This paper defines the diameter constrained network reliability, and gives a formulation in terms of events corresponding to the operation of the (length constrained) paths of the network. Based on this formulation, the exact value of the diameter constrained reliability is derived, for the special case where K={s,t} and the upper bound D of the path length is 2. For other values of K and D an exact evaluation algorithm based on a factorization approach is proposed. As this algorithm has exponential worst case complexity, upper and lower bounds for K={s,t} are developed, which in some cases may be used instead of the exact value.

Key words: system reliability, diameter constraints, graph theory, factorization.

1. Introduction

The system under study is an undirected, connected communication (or transport) network G = (V, E, K) consisting of a set of nodes V, a set of connecting links E and a set of terminals K (a fixed subset of the node-set V). Nodes do not fail, but each link *e* is assigned an independent probability of failure q_e (called link unreliability). In the classical reliability measure, the network is supposed to work when all the terminal nodes can be connected using the operational links (i.e, when there is an operational path between each pair of nodes in K). This is a random event, which has probability R(G). The problem of evaluating R(G) or its complement, Q(G) = 1-R(G), is usually called the K-terminal reliability problem. There is a vast literature on this problem; the works by Colbourn (1987) and Rubino (1998) provide good starting points. One of the fundamental results in this area is that the exact evaluation of the K-terminal reliability problem R(G) is an NP-hard problem Ball(1979). This is also true for the special cases of K={s,t} (source-terminal or two-terminal reliability, see Valiant (1979)) and K=V (all-terminal reliability, Provan and Ball (1983)).

There are many situations where it is not enough that the terminal nodes are connected, but is also necessary that the length of the connecting paths (measured by the number of links) is smaller or equal to a given upper bound. This is the case for example where at each node (or at each link) there is a transmission delay T, and the total communication time between two terminals must be less than D times this delay; then it is necessary that the operational paths which connect the terminals have at most length D. There is currently much research in this area, which has been mainly oriented to deterministic models, which do not take into account reliability measures. For example, the current work by Gouveia and Magnanti (2000) on finding diameter-constrained minimum cost spanning trees, and by Kortsarz and Peleg (1999) on diameter-constrained minimum cost for Steiner trees, can be used as the basis for network design when all components are perfect and there is no failure possibility. Another approach is to find families of large graphs with given degree and diameter (see for instance Gomez et al (1999)); also here the components of the network are supposed to operate perfectly without failures. Other interesting line of work (exemplified by Farago et al (1999) and by Nikoletseas et al (2000)) is to study families of random graphs (representing networks with independent, equiprobable edge faults), and investigate different asymptotic properties of these families, among which the connectivity and the diameter of the surviving networks.

In this work we present the Diameter-Constrained Reliability measure as a generalization of the classical K-terminal reliability problem, taking into account path length restrictions in the definition of the model (diameter is the maximum of the distances between pairs of nodes of a network). In the Diameter-Constrained Reliability problem, the network is up if for each pair of nodes in K, there is a path of length no greater than D which connects them. We denote by R(G,K,D) (the diameter-constrained reliability of G) the probability of this event. In particular, if G has *n* nodes, the diameter-constrained reliability R(G,K,*n*-1) corresponds to the classical K-terminal reliability R(G).

In next section we give a formulation for the Diameter-Constrained Reliability in terms of the paths of the network. Sections 3 and 4 discuss computationally efficient exact formulas for the cases where D=1 (trivial case) and where K={s,t} and D=2. Section 5 presents exact recursive formulas, for the general case, based on a factoring theorem. Section 6 presents recursive bounds for the case where K={s,t}. Finally, Section 7 presents conclusions and future work.

2. A path-based formulation for Diameter Constrained Reliability

In this section, we develop a formulation for the Diameter Constrained Reliability based on the paths of the network. We will use the following notation:

- G = (V, E, K): an undirected network topology
- V ={1,...,n}: the node-set of G
- E={e_1,...,e_m}: the link-set of G
- K: the terminal set of G
- m,n,k: the number of [links, nodes,terminals] of G
- diam(G): the diameter of G, i.e. the maximum distance between two any nodes of the graph
- x_e: state of the link e; we take state 1 when the link e is up (operational), and state 0 when it is down (failed). Sometimes we will denote a link by its extremities, for example e=(s,t); in this case x_{st} will be an alternative notation for the state of link e.
- r_e=Pr(x_e=1): operating probability of link e.
- q_e=Pr(x_e=0)=1-r_e: failure probability of link e

Suppose we have a network G = (V, E, K), and we want to compute its diameter constrained reliability R(G,K,D), for a given value of D. For any pair of nodes s, t belonging to K, we define P_{st}(d) as the set of paths between s and t, of length at most d. We will denote by E(p) the event in which all the links in p are operational. The probability of event E(p) is the probability of finding all the links composing the path p in operational state; by the hypothesis of independence between the states of the links, it can be computed as the product of the links reliabilities: $\Pr(E(p)) = \prod_{e\in p} r_e$.

The diameter-constrained reliability measure can then be expressed in function of these events:

$$R(G, K, D) = \Pr\left(\bigcap_{s,t\in K}\left(\bigcup_{p\in Pst(D)} E(p)\right)\right).$$

Unfortunately, the events corresponding to the paths are neither independent from each other, nor disjoint, so this formula is not suitable for direct computation of the reliability measure. Nevertheless, one important consequence is that for any link e$\in$E, if it is not used in any path between terminals of length at most D, then the link is irrelevant for computing R(G,K,D). In this case, it is possible to reduce the graph, eliminating e, and obtaining a new network G-e which will have the same reliability as the original one: R(G,K,D)=(G-e,K,D).

If the network has maximum degree δ, a brute force approach can identify all the relevant paths by using a simple DFS procedure in time O((k(k-1)/2×δ^D).

3. Exact formulas for the trivial case: D=1

When D=1, the only relevant links are those among the nodes of K (as only paths with at most one link will be considered). In this case, if the the subgraph of G induced by K is not a clique, then the reliability R(G,K,D) = 0. Suppose then that the subgraph of G induced by K is a clique (i.e, for all s,t $\in$ K, there is a link e=(s,t) in E). In this case, the network is operational if and only if all the links of this clique are operational:

$$R(G,K,D) = \prod_{s,t\in K} r_{s,t} \ .$$

4. Exact formulas for source-terminal Diameter Constrained Reliability with D=2

We will now look at the case K = {s,t} (source-terminal diameter-constrained reliability), and D=2. In this case, an application of the “disjoint products” technique (standard for classical reliability problems, see for example Colbourn(1987) p.21), will enable us to find a formulation for R(G,{s,t},2) which can be efficiently computed.

We start by observing that in this case, the only relevant paths are those belonging to $P_{st}(2) = \{(s,t)\}\cup\{(s,i,t),\ i\in V\setminus\{s,t\},\ (s,i)\in E, (i,t)\in E\}$. In order to simplify the formulation, we will suppose that for any pair of nodes (i,j), the link (i,j) exists, eventually with reliability 0. We will order arbitrarily the paths in $P_{st}(2)$ (starting with the single link path p=(s,t)), and we will say that $p_1 < p_2$ if p_1 precedes p_2. We can then write

$$R(G,\{s,t\},2) = \Pr\left(\bigcup_{p\in Pst(2)} E(p)\right) = \Pr\left(\bigcup_{p\in Pst(2)} \left(E(p)\bigcap\left(\bigcap_{q<p}\overline{E(q)}\right)\right)\right)$$

where $\overline{E(q)}$ is the complement of event $E(q)$. Observing that the events $E(p)\bigcap\left(\bigcap_{q<p}\overline{E(q)}\right)$ are disjoint (by construction), we have

$$R(G,\{s,t\},2) = \sum_{p\in Pst(2)} \Pr\left(E(p)\bigcap\left(\bigcap_{q<p}\overline{E(q)}\right)\right).$$

What is more, as the links states are independent, and there are no links repeated among the paths in $P_{st}(2)$, the events involved in each term of the sum are independent, which allows us to write

$$R(G,\{s,t\},2) = \sum_{p\in Pst(2)} \Pr(E(p)) \prod_{q<p} \Pr\left(\overline{E(q)}\right).$$

Finally, as each event E(p) is the intersection of the independent events corresponding to the operational state of the composing links (which are independent), we can substitute the terms Pr{E(p)} by their values (keeping in mind that the first path has only one link, and the subsequent ones have two links):

$$R(G,\{s,t\},2) = r_{st} + (1-r_{st}) \sum_{i\in V\setminus\{s,t\}} r_{si}r_{it} \prod_{j<i,\, j\in V\setminus\{s,t\}} (1-r_{sj}r_{jt}).$$

This formula can be evaluated very efficiently (in time linear in the number of nodes of the network). Unfortunately, it is not easy to generalize the formula to other values of D and K. The problem arises because in the general case, the paths involved have links in common, and then the corresponding events are not independent, and their intersection cannot be computed as efficiently; Boolean algebra methods are applicable in this case, sometimes combined with heuristics for reordering the terms of the general expression.

5. Recursive factoring formulas for Diameter Constrained Reliability

This section is devoted to the evaluation of R(G,K,D) for arbitrary K and D. We introduce some definitions first. Let there be a network G=(V,E,K):

- Link deletion: for a given link e, G-e denotes the network with node-set V and link-set derived from E by removing link e. The target set of G-e is equal to K.

- Link contraction: for a given link e=(u,v), G|e denotes the network derived from G by setting the link reliability of e to 1 (link e cannot fail). The target set of G|e is equal to K.

The factoring formula is based on considering the two possible states of an arbitrary link e, which is non perfect (subject to failures). If link e is failed, it can be deleted from the network; if it is operating, it can be contracted. As the state of link e is independent from the state of the other components of the network, we have the following expression (sometimes known as the Factoring Theorem) :

$$R(G,K,D) = r_e R(G \mid e, K, D) + (1 - r_e) R(G - e, K, D)\,.$$

If we repeat the operation for the remaining links (those whose state has not yet been fixed) in the resulting networks, the process corresponds to a state enumeration algorithm. Nevertheless, it is very often possible to stop the recursion earlier; when we delete a link e, it may be the case that the resulting network G-e does not have any possible paths between a pair s,t of lenght bounded by D, independently from the state of the other links, and then has reliability 0; also, when contracting a link e, it may be the case that the terminals become connected by paths made only of perfect links, and the network will always operate independently from the failure of the remaining links, and has then reliability 1. In those cases, it is needless to continue the recursion.

This formula, complemented with the previous observations, can be used as the basis of a recursive algorithm. In order to check whether to continue or to stop the recursive calls, a number of auxiliary structures and precomputed data will be handy. First of all, we will suppose that we have previously computed the set of paths of length at most D between s and t, $P_{st}(D)$, for all the pairs of terminals s,t in K (using for example the procedure mentioned in Section 2); we will denote by P(D) the union of all the sets $P_{st}(D)$. We will define the following auxiliary data structures:

- np_{st}: the number of paths of length at most D between s and t in the network being considered.

- $links_p$: the number of non-perfect links (links e such that $r_e < 1$) in path p, for every $p \in P(D)$.

- $feasible_p$: this is a flag, which has value False when the path is not longer feasible, i.e. it includes a link wich is failed; and True otherwise.

- $connected_{st}$: this is a flag, which has value True when s and t are connected by a perfect path of length at most D and False otherwise.

- connectedpairs: this is the number of connected pairs of terminals (those between which there is a perfect path of length at most D).

- P(e): the set of paths of P(D) which include link e.

The following pseudocode corresponds to the proposed factoring procedure. We have followed C language convention for including comments (which are bracketed between /* */):

```
                         Procedure FACTO(G,D)

Input:
•  network G=(V,E,K); path length bound D;
•  auxiliary structures P_st(D), P(e), np_st, links_p, feasible_p, connected_st, connectedpairs
Output: diameter-constrained reliability R(G,K,D)

Initialization:
        /* RContract: partial result corresponding to contract branch */
        /* RDelete: partial result corresponding to delete branch */
        Initialize RContract=0
        Initialize Rdelete=0
        Select an arbitrary link e such that 0<r_e<1

"Contract" branch:
        Compute G|e
        /* Now we consider all feasible paths which include selected link e*/
        For each p=(s,...,t) ∈ P(e) such that feasible_p = True do
            Decrement the number of non-perfect links of p:  links_p=links_p-1
            If connected_st = False and links_p =0 Then  /*p includes only perfect links */
                                          /* there is now a perfect path between s and t*/
                connected_st = True
                connectedpairs = connectedpairs+1
                If connectedpairs = n(n-1)/2) then /*all pairs of terminal are connected*/
                                   /* by perfect paths; the network is always operational*/
                    Rcontract= 1
                    Goto to "Delete" branch
                Endif
            Endif
        EndFor
        Call the procedure recursively: RContract=FACTO( G|e,D)

"Delete" branch:
        Compute G-e
        /* Again we consider all feasible paths which include selected link e*/
        For each p = (s,...,t)∈ P(e) such that that feasible_p =True do
            feasible_p =False   /* Now  p is unfeasible, because link e is deleted */
            np_st=np_st -1  /*Decrement the number of feasible paths between s and t*/
            If np_st= 0 Then /*the network G-e can no longer be operational*/
                RDelete = 0
                Goto to Final Computations
            Endif
        Endfor
        Call the procedure recursively: RDelete=FACTO(G-e,D)

Final computations:
        R(G,K,D) = r_e×RContract + (1-r_e)×RDelete
        Return R(G,K,D)
```

This code can be made more efficient, by checking out some conditions before considering the different paths (for example, when a pair of terminals are already connected by a perfect path, it is no longer productive to consider other paths between them). The algorithm will work correctly with any choice of link e; a general open question is to study the influence of this choice on the running time of the algorithm (or equivalently on the size of the recursion tree).

A quick worst case complexity analysis of the algorithm can be done observing that the number of recursive calls (i.e, the size of the recursion tree) will be at most 2^m, with m being the number of links of the network. At each call, the more costly operation is the the iteration over all the paths which include the selected link e. If the network has maximum degree δ, there are at most $k(k-1)/2\times\delta^D$ relevant paths; as we work with simple networks, the maximum degree is always less than the number of nodes of the network, $\delta<n$. We have then worst case complexity of order $O(2^m k^2 \delta^D)$, exponential in the number of links m and in the bound of the path length, D.

6. Recursive bounds for source-terminal Diameter Constrained Reliability

As exact evaluation of the R(G,K,D) measure is a very costly task, in some cases, it may be enough to know lower and upper bounds to its value. In this section we develop a set of recursive bounds for the special case of source-terminal Diameter Constrained Reliability (i.e, K={s,t}).

Let G=(V,E,{s,t}) the network under study, and D the path length bound. Then we have that the network is operating if either the terminals s and t are directly connected by an operational link (i.e, if the link e=(s,t) is working) or if there is a node i such that the link (s,i) is working, and the node i is connected to the node t by a path of lenght at most D-1. We can then write down:

$$R(G,\{s,t\},D) = \Pr\left((x_{st}=1) \bigcup \left(\bigcup_{i\in V} (x_{si}=1) \bigcap \left(\bigcup_{p\in Pit(D-1)} E(p) \right) \right) \right),$$

and using the fact that the link (s,t) does not appear in the other involved paths, we have

$$R(G,\{s,t\},D) = \Pr(x_{st}=1) + (1-\Pr(x_{st}=1))\Pr\left(\bigcup_{i\in V-\{s,t\}} (x_{si}=1) \bigcap \left(\bigcup_{p\in Pit(D-1)} E(p) \right) \right) =$$

$$= r_{st} + (1-r_{st})\Pr\left(\bigcup_{i\in V-\{s,t\}} (x_{si}=1) \bigcap \left(\bigcup_{p\in Pit(D-1)} E(p) \right) \right).$$

To obtain our upper bounds, we use the fact that the probability of the union of a number of events is always less or equal to the sum of the probabilities of the events themselves:

$$R(G,\{s,t\},D) \le r_{st} + (1-r_{st}) \sum_{i\in V-\{s,t\{} \Pr\left((x_{si}=1) \bigcap \left(\bigcup_{p\in Pit(D-1)} E(p) \right) \right).$$

We observe that $R(G-s,\{i,t\},D-1) = \Pr\left(\bigcup_{p\in Pit(D-1)} E(p) \right)$, and that the event of the operation or failure of link (s,i) is independent from the event of operation or failure of the network G-s with terminals set {i,t}; finally we obtain

$$R(G,\{s,t\},D) \le r_{st} + (1-r_{st}) \sum_{i\in V-\{s,t\}} r_{si} R(G,\{i,t\},D-1)$$

which gives an upper bound for the reliability of network G and diameter D in terms of the reliability of at most n-2 smaller networks, with diameter D-1. In turn, the reliability of those networks can be bounded above by the same formula.

We now turn to the problem of obtaining a lower bound on the reliability. For this, we will observe that for any two events A and B, it is always the case that $A \supseteq (A \cap B)$, which in turn implies that $\Pr(A) \geq \Pr(A \cap B)$,; as a particular case, the following equation holds:

$$(x_{si}=1) \bigcap \left(\bigcup_{p \in Pit(D-1)} E(p) \right) \supseteq (x_{si}=1) \bigcap \left(\bigcap_{j<i,\, j \in V-\{s,t\}} (\overline{x_{sj}=1}) \right) \bigcap \left(\bigcup_{p \in Pit(D-1)} E(p) \right).$$

Substituting in previous equations, we have that

$$R(G,\{s,t\},D) \geq r_{st} + (1-r_{st}) \Pr\left(\bigcup_{i \in V-\{s,t\}} (x_{si}=1) \bigcap \left(\bigcap_{j<i,\, j \in V-\{s,t\}} (\overline{x_{sj}=1}) \right) \bigcap \left(\bigcup_{p \in Pit(D-1)} E(p) \right) \right).$$

Now we can profit from the fact that (by their construction) these events are disjoinct; as the probability of the union of disjoinct events is the sum of the probability of the individual events, we obtain

$$R(G,\{s,t\},D) \geq r_{st} + (1-r_{st}) \sum_{i \in V-\{s,t} \Pr\left((x_{si}=1) \bigcap \left(\bigcap_{j<i,\, j \in V-\{s,t\}} (\overline{x_{sj}=1}) \right) \bigcap \left(\bigcup_{p \in Pit(D-1)} E(p) \right) \right).$$

We now take advantage of the fact that the links (s,i) and (s,j) are independent from the other paths involved in the construction, and we make the same observation as in the case of the upper bound, in order to arrive at the final formulation:

$$R(G,\{s,t\},D) \geq r_{st} + (1-r_{st}) \sum_{i \in V-\{s,t} r_{si} \left(\prod_{j<i,\, j \in V-\{s,t\}} (1-r_{sj}) \right) R(G-s,\{i,t\},D-1).$$

The computation of these bounds has computational complexity of order $O(\delta^D)$, which is an important improvement from the complexity of the exact computation, especially for small values of the path lenght upper bound D.

7. Conclusions

In this paper, we have presented a new reliability measure, called the diameter constrained network reliability R(G,K,D). As discussed, this parameter generalizes the classical network reliability (which can be seen as the special case where D=n-1), allowing to reflect more stringent performance objectives that restrict the maximum length of a path in the network.

We have developed exact algorithms (of exponential order) for the general case, and we have also given more efficient computation methods for the special case where K={s,t} and D=2. There is much open work in this area. In the general case, the problem is NP-hard (because it includes as particular cases the classical network reliability problems, which are known to belong to this complexity class), which almost precludes hope in finding polinomial complexity algorithms. Nevertheless, the complexity of the problem for particular cases remains to be studied (for example, the case K={s,t} and D=2 is shown in this paper to be of linear complexity; it is an open problem to determine the complexity for general K and D=2).

Another approach is finding bounds. We have given upper and lower bounds for K={s,t}, which (albeit much faster than exact reliability computation), still take exponential (in D) computation time. There is need to develop more efficient (for example, polinomial) and more general (holding for arbitrary K, for example) bounding methods.

It should also be mentioned that Monte Carlo simulation methods, including variance reduction techniques such as those employed by Cancela and El Khadiri (1998) and Cancela and Urquhart (2000) could be used to estimate the diameter constrained network reliability; this may be the technique which holds most promise for large networks of medium reliability.

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