Earthquakes and graftings of hyperbolic surface laminations.

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Abstract: We study compact hyperbolic surface laminations. These are a generalization of closed hyperbolic surfaces which appear to be more suited to the study of Teichmüller theory than arbitrary non-compact surfaces. We show that the Teichmüller space of any non-trivial hyperbolic surface lamination is infinite dimensional. In order to prove this result, we study the theory of deformations of hyperbolic surfaces, and we derive what we believe to be a new formula for the derivative of the length of a simple closed geodesic with respect to the action of grafting. This formula complements those derived by McMullen in [23], in terms of the Weil-Petersson metric, and by Wolpert in [33], for the case of earthquakes.

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1 - Introduction.

1.1 - Teichmüller theory of laminations. Laminations, which have various applications in the study of hyperbolic dynamics (c.f. [20], [21], [22] and [29]), are an extension of foliations where the ambient space is no longer assumed to be a smooth manifold.* A hyperbolic surface lamination is a lamination in which all leaves are Riemann surfaces of hyperbolic type, that is, Riemann surfaces which are uniformized by the Poincaré disk. Such laminations arise quite frequently (see, for example, [17]), and the property of a given compact surface lamination being hyperbolic is *topological* in the sense that it does not depend on the laminated conformal structure chosen (see [10]).

Hyperbolic surface laminations also appear to possess a better structured Teichmüller theory than arbitrary non-compact hyperbolic surfaces. Indeed, in [3], [4] and [5], natural constructions of the Teichmüller space of a hyperbolic surface of infinite topological type are defined using pants decompositions, complex structures and length spectra. However, the authors then show that each of these different methods may yield a different space. On the other hand, the known natural constructions of the Teichmüller space of a compact hyperbolic surface lamination are all equivalent.

In [29] and [30], Sullivan defines the Teichmüller space of a hyperbolic surface lamination to be the set of transversally continuous conformal structures modulo leafwise diffeorphisms which are leafwise isotopic to the identity. An alternative description using leafwise complex structures is described by Moore & Schochet in [24]. In [10] (c.f. also [32]), Candel proves that every hyperbolic surface lamination carries a unique leafwise metric in its conformal class (see Section 4.1) which allows the Teichmüller space of compact hyperbolic surface laminations to be studied as the space of hyperbolic leafwise metrics modulo leafwise diffeomorphisms which are leafwise isotopic to the identity. It is this framework that we will adopt in the sequel.

1.2 - Previous results. Little is currently known about the general theory of the Teichmüller space of a given compact hyperbolic surface lamination. In [30], Sullivan shows that this space is a *Banach manifold* carrying a natural complex structure with respect to which it is biholomorphic to an open subset of the space of leafwise holomorphic quadratic differentials. In [14], motivated by Ghys' construction of non-constant meromorphic functions on hyperbolic laminations using Poincaré series (see [17]), Deroin proves

Theorem 1.1, Deroin [14]

If a hyperbolic surface lamination contains a simply connected leaf, then its Teichmüller space is infinite dimensional.

Beyond these general results, the authors are only aware of three specific cases in which the Teichmüller space of a compact hyperbolic surface lamination is understood. The first is that of the family of laminations, discussed in [17] and [30], associated to expanding maps of the unit circle. The second is that of Sullivan's universal solenoid, obtained as the inverse limit of finite coverings of a closed hyperbolic surface, whose Teichmüller space was computed by Šarić in [26]. The third is that of the Hirsch foliation, defined as the

* A brief review of the fundamentals of the theory of laminations is given in Section 4.1.

quotient of the stable foliation of the hyperbolic attractor of Smale's solenoidal map, whose Teichmüller space was computed by the first author in collaboration with Lessa in [8].

1.3 - Main results. Our main result completes that of Deroin. A hyperbolic surface lamination will be said to be *trivial* whenever it consists of a finite union of closed surfaces. We show

Theorem 1.2

The Teichmüller space of a non-trivial hyperbolic surface lamination is infinite dimensional.

We will explain presently how Theorem 1.2 follows immediately from Deroin's result and the second main result of this paper. Our approach takes advantage of the topology of the leaves to generate an arbitrarily large number of independent movements in Teichmüller space via perturbations of Candel's hyperbolic leafwise metric. In order to state the result, we introduce some notation. First, given a complete hyperbolic surface Σ and a simple closed curve γ, let $[\gamma]$ denote the free homotopy class in which γ lies and let $l([\gamma],g)$ denote the infimal length of curves in this class with respect to the metric g. Next, given a surface lamination X, its leafwise topology is defined to be the topology generated by all open subsets of leaves of this lamination (see Section 4.2). In particular, a subset Y of X is compact in this topology if and only if it is a finite union of compact subsets of leaves. We prove

Theorem 1.3

Let X be a compact hyperbolic surface lamination. Suppose that for every subset Y of X which is compact in the leafwise topology there exists a simple, closed leafwise geodesic γ in X not intersecting Y. Then there exists an infinite sequence $(\gamma_m)_{m\in\mathbb{N}}$ of simple, closed leafwise geodesics in X such that, for every $m\in\mathbb{N}$ and for every finite sequence $a_1,...,a_m\in\mathbb{R}$, there exists a smooth family of leafwise hyperbolic metrics $(g_t)_{t\in]-\epsilon,\epsilon[}$ such that, for all $1\leq i\leq m$,

$$\frac{\partial}{\partial t}Log(l([\gamma_i],g_t))\bigg|_{t=0}=a_i.$$

In particular, the Teichmüller space of X is infinite dimensional.

Theorem 1.2 follows from Theorems 1.1 and 1.3 by an argument used in [2] and [6]. Indeed, it was proven independently by Epstein, Millett & Tischler in [16] and by Hector in [19] that the generic leaf of a compact lamination has trivial holonomy. Consequently, if a hyperbolic surface lamination has no simply connected leaf, then there exists a simple, closed geodesic inside a leaf with trivial holonomy. By Reeb's stability theorem, this geodesic has a neighbourhood trivially laminated by annuli. It then follows by the transverse continuity of the leafwise metric and the persistence of closed geodesics under perturbations of hyperbolic metrics that each of these annuli also contains a simple, closed geodesic. Since the lamination is non-trivial, this yields sufficient simple, closed leafwise geodesics for the hypotheses of Theorem 1.3 to be satisfied and Theorem 1.2 follows.

However, there are laminations that can be treated simultaneously by the methods of both theorems. Indeed, the laminations to which Deroin's theorem applies but not ours are precisely those for which all leaves are simply connected except for finitely many leaves which are of finite topological type. Although examples of such laminations were constructed in [7] for 3-dimensional ambient spaces, we believe this condition to be quite restrictive, as this is the case for foliations. Indeed, in [7] it is shown that if all but a finite number of leaves a smooth, minimal foliation are simply connected, and if all the remaining leaves are of finite topological type, then all the leaves of the foliation are in fact planes. It then follows by the result [25] of Rosenberg that the ambient manifold is a 3-dimensional torus and the foliation is by parabolic planes.

1.4 - Graftings and Earthquakes. Our proof closely follows the ideas developed by the first author in collaboration with Lessa in [8], where Fenchel-Nielsen coordinates were used to parametrize the Teichmüller space of the Hirsch foliation. In the general case, the existence of such coordinates cannot be guaranteed, and we thus make use of the curves given in the hypotheses of Theorem 1.3 in order to define surgeries of the leafwise metric which vary *independently* the lengths of an arbitrarily large number of curves.

The surgeries that we use are generalisations of graftings, which we recall are defined as follows (c.f. [15], [23] and [27]). Given a simple, closed geodesic γ in a marked, hyperbolic surface Σ and a positive real number t, the *grafting of length* t of Σ along γ is defined to be the marked, hyperbolic surface obtained by cutting Σ along γ, inserting a flat cylinder of length t, and multiplying the metric of the resulting surface by a suitable conformal factor. A related surgery operation is that of earthquakes. For any real number t, the right earthquake of Σ along γ is defined to be the marked, hyperbolic surface obtained by rotating the right hand side of γ a distance t in the positive direction of this geodesic. The resulting marked, hyperbolic surfaces will be denote by $G(\gamma,\Sigma)(t)$ and $E(\gamma,\Sigma)(t)$ respectively.

Our main result essentially consists in extending the grafting surgery to the framework of laminations, which we believe to be of independent interest. It seems unlikely that there exist a canonical way of extending graftings to laminations in general. Furthermore, although it is straightforward to extend to the entire lamination a grafting along a simple, closed geodesic with trivial holonomy, it is less clear how this can be done for geodesics with more general holonomy. Our construction yields extensions of graftings along arbitrary simple, closed geodesics. Furthermore, although these extensions are non-canonical, they can always be chosen so that their supports are contained in arbitrarily small neighbourhoods of the geodesic in question (c.f. Lemma 4.13).

Theorem 1.3 then follows by carefully estimating the effect of grafting on the lengths of all other simple, closed leafwise geodesics of the lamination. It is unsurprising and perfectly consistent with known results of hyperbolic geometry that the effect of a grafting should decay exponentially with distance from the locus of surgery. However, our calculations are simplified with the help of the following exact formula, which complements those derived by McMullen in [23] and Wolpert in [33] (see also [15] and [27]) and which, to our knowledge, has not previously appeared in the litterature. Using the notation introduced before the statement of Theorem 1.3, we show

Theorem 1.4

For every pair (γ,γ') *of simple, closed geodesics in* Σ*,*

$$\left.\frac{\partial}{\partial t}l([\gamma'],G(\gamma,\Sigma)(t))\right|_{t=0} = \sum_{x\in\gamma\cap\gamma'} sin(\theta_x) + \int_\gamma\int_{\gamma'} K(p,q)dl_pdl_q. \qquad (1)$$

where $K(x,y)$ is the Green's kernel over Σ of the operator

$$L := \Delta - 2,$$

and, for all $x \in \gamma\cap\gamma'$, θ_x denotes the angle that γ makes with γ' at the point x.

This result is proven in Theorem 3.1, below.

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2 - Analytic preliminaries.

2.1 - Green's functions with point singularities. Let Σ be a complete hyperbolic surface without cusps. Let Δ be its Laplace-Beltrami operator. Define

$$L := \Delta - 2. \qquad (2)$$

This operator is, up to a change of sign, the linearisation about the hyperbolic metric of the curvature operator (c.f. [12] and [13])

$$\kappa_\phi := -e^{-2\phi}\big(\Delta\phi - \kappa_0\big). \qquad (3)$$

It is common to study its properties using general elliptic theory. In the present context, however, it is simpler to determine explicit formulae for its inverses in terms of Green's functions. Recall first that, considered as an operator acting on distributions, L has trivial kernel in the space $C^2_{\text{bdd}}(\Sigma)$ of bounded, twice-differentiable functions over Σ as well as in the space $L^1(\Sigma)$ of integrable functions over Σ. For all $x \in \Sigma$, the *Green's function* of L over Σ with singularity at x is defined to be the unique function $K_x \in L^1(\Sigma)$ such that

$$LK_x\text{dArea} = \delta_x, \qquad (4)$$

in the distributional sense, where dArea denotes the area form of Σ, and δ_x denotes the *Dirac delta distribution* with singularity at x, that is

$$\langle\delta_x, g\rangle := f(x).$$

The *Green's kernel* of L over Σ is defined such that, for all $x \neq y \in \Sigma$,

$$K(x,y) := K_x(y).$$

Lemma 2.1

For all $x \in \mathbb{H}^2$, the Green's function K_x of L over $\mathbb{H}^2$ with singularity at x is

$$K_x(y) = \frac{1}{2\pi} - \frac{1}{2\pi} arccoth(cosh(r(y)))cosh(r(y)), \tag{5}$$

where $r(y)$ here denotes the distance in $\mathbb{H}^2$ from y to x.

Proof: By uniqueness, K_x is invariant under rotation about x and is therefore a function of r only. Since L is given in polar coordinates of $\mathbb{H}^2$ about x by

$$Lu = u_{rr} + \coth(r)u_r + \frac{1}{\sinh^2(r)}u_{\theta\theta} - 2u,$$

K_x is a solution of the equation

$$F_{rr} + \coth(r)F_r - 2F = 0.$$

We verify by inspection that the function $F(r) := \cosh(r)$ is a solution and a second, linearly independent solution is then obtained using the Wronskian. The function K_x is the unique linear combination of these two solutions which also satisfies

$$\operatorname*{Lim}_{r\rightarrow\infty} e^{2r}K_x(r) = -\frac{2}{3\pi},$$

and

$$\operatorname*{Lim}_{r\rightarrow 0} r\frac{\partial}{\partial r}K_x(r) = \frac{1}{2\pi}.$$

These properties imply that K_x is an element of $L^1(\mathbb{H}^2)$ which solves (4), and the result follows. □

Corollary 2.2

The Green's kernel $K(x,y)$ of L over $\mathbb{H}^2$ has the following properties.

(1) For all $x \neq y$, $K(x,y) < 0$;

(2) for all $x \neq y$, $K(x,y) = K(y,x)$; and

(3) for all $R > 0$, there exists $C > 0$ such that for $d(x,y) > R$,

$$|K(x,y)| \leq Ce^{-2d(x,y)}. \tag{6}$$

In particular,

Lemma 2.3

Let Σ be a hyperbolic surface. For all $R>0$, there exists $C>0$ with the property that

(1) if f is a twice-differentiable function such that both f and Lf are bounded; and

(2) if $x\in\Sigma$ is such that $B_r(x)\cap Supp(Lf)=\emptyset$ for some $r\geq R$,

then

$$|f(x)|\leq Ce^{-r}\|Lf\|_{L^\infty}. \tag{7}$$

Proof: It suffices to consider the case where $\Sigma=\mathbb{H}^2$. Denote $g:=Lf$. Since L has trivial kernel over $C^2_{\mathrm{bdd}}(\mathbb{H}^2)$, we have

$$f(x)=\int_{\mathbb{H}^2}K(x,y)g(y)\mathrm{dArea}_y.$$

However, in polar coordinates (r,θ) of $\mathbb{H}^2$ about x,

$$\mathrm{dArea}=\sinh(r)drd\theta,$$

and the result follows by Item (3) of Corollary 2.2. □

2.2 - Green's functions with geodesic singularities. Let Σ be a complete hyperbolic surface without cusps and let γ be a simple, closed geodesic in Σ. The *Green's function* of L over Σ with singularity along γ is defined to be the unique function $K_\gamma\in L^1(\Sigma)$ such that

$$LK_\gamma\mathrm{dArea}=\delta_\gamma \tag{8}$$

in the distributional sense, where δ_γ denotes the *Dirac delta distribution* with singularity along γ, that is

$$\langle\delta_\gamma,f\rangle:=\int_\gamma f(x)dl_x.$$

Using Fubini's Theorem, we readily verify

Lemma 2.4

Let Σ be a hyperbolic surface without cusps. For every simple, closed geodesic γ in Σ, the Green's function K_γ of L over Σ with singularity along γ is given by

$$K_\gamma(x)=\int_\gamma K(x,y)dl_y.$$

In order to derive more explicit estimates for K_γ, we consider the following special case. An *hourglass* is defined to be a complete, connected hyperbolic surface containing a unique simple closed geodesic, that is, a complete hyperbolic annulus not conformal to the pointed disk.

Lemma 2.5

Let Σ be an hourglass with simple, closed geodesic γ. The Green's function K_γ of L over Σ with singularity along γ is given by

$$K_\gamma(x) = -\frac{1}{\pi} + \frac{1}{\pi}arccot(sinh(r(x)))sinh(r(x)), \tag{9}$$

where $r(x)$ here denotes the distance in Σ from x to γ.

Proof: By uniqueness, K_γ is invariant under rotation along and reflection about γ and is therefore a function of r only. In Fermi coordinates of Σ about γ, L is given by

$$Lu = u_{rr} + \mathrm{tanh} r u_r + \frac{1}{\cosh^2(r)}u_{tt} - 2u,$$

where t here denotes a path-length parameter of γ. It follows that K_γ is a solution of the equation

$$F_{rr} + \tanh(r)F_r - 2F = 0.$$

We verify by inspection that the function $F(r) := \sinh(r)$ is a solution and a second, linearly independent solution is then obtained using the Wronskian. The function K_γ is the unique linear combination of these two functions which also satisfies

$$\mathop{\mathrm{Lim}}_{r\rightarrow\infty} e^{2r}K_\gamma(r) = -\frac{2}{3\pi},$$

and

$$\mathop{\mathrm{Lim}}_{r\rightarrow 0^+} \frac{\partial}{\partial r}K_\gamma(r) - \mathop{\mathrm{Lim}}_{r\rightarrow 0^-} \frac{\partial}{\partial r}K_\gamma(r) = 1.$$

These properties imply that K_γ is an element of $L^1(\Sigma)$ which solves (8), and the result follows. $\square$

Corollary 2.6

Let Σ be an hourglass with simple, closed geodesic γ. The Green's function K_γ of L over Σ with singularity along γ has the following properties.

(1) For all x, $K_\gamma(x) < 0$;

(2) for all $x \in \gamma$, $K_\gamma(x) = -\frac{1}{\pi}$; and

(3) there exists a constant $C > 0$, which does not depend on Σ such that, for all x,

$$|K_\gamma(x)| \leq Ce^{-2d(\gamma,x)}. \tag{10}$$

Lemma 2.7

Let Σ be a complete hyperbolic surface without cusps. Let γ be a simple, closed geodesic in Σ. Let $\hat{\Sigma}$ be an hourglass with unique simple, closed geodesic $\hat{\gamma}$. Let $\pi : \hat{\Sigma} \to \Sigma$ be a local isometry whose restriction to $\hat{\gamma}$ defines an isometry onto γ. The Green's functions K_γ and $K_{\hat{\gamma}}$ are related by

$$K_\gamma(x) = \sum_{\pi(\hat{x})=x} K_{\hat{\gamma}}(\hat{x}). \tag{11}$$

Proof: It suffices to prove that the sum is locally uniformly absolutely convergent. However, for $x \in \Sigma$, the *orbital counting function* of x in $\hat{\Sigma}$ is defined by

$$N(x)(R) := \#\pi^{-1}(\{x\}) \cap B_R(\hat{\gamma}),$$

where $B_R(\hat{\gamma})$ denotes the tubular neighbourhood of radius R about $\hat{\gamma}$ in $\hat{\Sigma}$. By comparing areas, we obtain

$$N(x)(R) \leq \frac{l\mathrm{sinh}(R + r_{\mathrm{inj}})}{\mathrm{sinh}^2(r_{\mathrm{inj}}/2)},$$

where l here denotes the length of γ and r_{inj} denotes the injectivity radius of Σ about x. Local uniform convergence follows from this and Item (3) of Corollary 2.6. This completes the proof. □

Corollary 2.8

Let Σ be a complete hyperbolic surface without cusps. Let γ be a simple, closed geodesic in Σ. The Green's function K_γ of L over Σ with singularity along γ has the following properties.

(1) For all x, $K_\gamma(x) < 0$; and

(2) for all $x \in \gamma$, $K_\gamma(x) \leq -1/\pi$.

Remark: In fact, equality holds in the second relation at a single point if and only if Σ is an hourglass.

The following estimate will play a key role in the sequel.

Lemma 2.9

For all $\epsilon > 0$, there exists $C > 0$ with the property that if Σ is a complete hyperbolic surface without cusps, if γ is a simple, closed geodesic in Σ, if K_γ is the Green's function of L over Σ with singularity along γ, and if the injectivity radius of Σ about every point of γ is bounded below by ϵ, then, for all $x \in \Sigma$,

$$|K_\gamma(x)| \leq Ce^{-d(\gamma,x)}. \tag{12}$$

Proof: Let $\pi : \mathbb{H}^2 \to \Sigma$ be the canonical projection. Let $\hat{x}$ be an element of $\pi^{-1}(\{x\})$. Let $\mathcal{G}$ be the set of complete geodesics in $\mathbb{H}^2$ which are preimages of γ under π. The *orbital counting function* $N(\gamma, x) : [0, \infty[\to \mathbb{R}$ of γ with respect to x is given by

$$N(\gamma, x)(R) := \#\left\{\hat{\gamma} \in \mathcal{G} \mid \hat{\gamma} \cap B_R(\hat{x}) \neq \emptyset\right\}.$$

By comparing areas, we obtain

$$N(\gamma,x)(R)\leq\frac{\sinh^2((R+\epsilon)/2)}{\sinh^2(\epsilon/2)}.$$

By Lemmas 2.5 and 2.7,

$$K_\gamma(x)=\sum_{\hat{\gamma}\in\mathcal{G}}\bigg(-\frac{1}{\pi}+\frac{1}{\pi}\mathrm{arccot}(\sinh(d(\hat{\gamma},\hat{x})))\sinh(d(\hat{\gamma},\hat{x}))\bigg).$$

It follows by Item (3) of Corollary 2.6 that, for some $C>0$,

$$\begin{aligned}|K_\gamma(x)|&\leq C\sum_{\hat{\gamma}\in\mathcal{G}}e^{-2d(\hat{\gamma},\hat{x})}\\&\leq C\int_0^{e^{-2d(\gamma,x)}}N(\gamma,x)\bigg(-\frac{1}{2}\ln(t)\bigg)dt\\&\leq\frac{Ce^\epsilon}{4\sinh^2(\epsilon/2)}e^{-d(\gamma,x)},\end{aligned}$$

as desired. □

3 - Graftings and earthquakes.

3.1 - Graftings and earthquakes. Let Σ be a complete, marked hyperbolic surface without cusps. Let g denote its metric. We recall the notation introduced in Section 1.3. Given a simple, closed geodesic γ in Σ, let $[\gamma]$ denote the free homotopy class in which it lies. Let $l([\gamma],\Sigma)$ denote the infimal length with respect to g amongst all curves in $[\gamma]$. Recall (see [9]) that $[\gamma]$ contains no other geodesics and that $l([\gamma],\Sigma)$ is realised by γ.

Recall that, given $t>0$, the *grafting* of length t of Σ along γ, which we denote by $G(\gamma,\Sigma)(t)$, is defined to be the unique complete, marked hyperbolic surface obtained by cutting Σ along γ, inserting a cylinder of length t, and multiplying the metric of the resulting surface by a suitable conformal factor. In this section, we show

Theorem 3.1

Let Σ be a complete, marked hyperbolic surface without cusps. Let γ be a simple, closed geodesic in Σ, and for all $t\geq 0$, denote $\Sigma_t:=G(\gamma,\Sigma)(t)$. We have,

$$\frac{\partial}{\partial t}Log(l([\gamma],\Sigma_t))\bigg|_{t=0}\leq-\frac{1}{\pi}. \tag{13}$$

Furthermore, for all $\epsilon>0$, there exists $C>0$, which does not depend on Σ, such that, if the injectivity radius of Σ is bounded below by $\epsilon>0$, then, for every other simple, closed geodesic γ' not intersecting γ,

$$\bigg|\frac{\partial}{\partial t}Log(l([\gamma'],\Sigma_t))\bigg|_{t=0}\bigg|\leq Ce^{-d(\gamma,\gamma')}. \tag{14}$$

Theorem 3.1 will be proven at the end of the following section. We first show how graftings are also constructed as smooth non-conformal perturbations of g. For completeness, we also study earthquakes, as their analysis is almost identical, and this will allow us to recover the result [33] of Wolpert. As outlined in Section 1.3, given $t \in \mathbb{R}$, the (right) *earthquake* of length t of Σ along γ of length t, denoted by $E(\gamma, \Sigma)(t)$, is defined to be the unique marked hyperbolic surface obtained by cutting Σ along γ and rotating the right hand side of γ by a distance t in the positive direction of this geodesic. Since the resulting metric is automatically hyperbolic, there is no need to multiply by a conformal factor in this case. This definition is independent of the orientation of γ chosen, but depends on the orientation of the ambient surface. The left earthquake of (Σ, g) along γ of length t is defined in a similar manner and is readily shown to be equal to $E(\gamma, \Sigma)(-t)$. Reversing the orientation of Σ interchanges left and right earthquakes.

Let $R > 0$ be such that the tubular neighbourhood $B_R(\gamma)$ of radius R about γ is isometric to $\gamma \times]-R, R[$ furnished with the twisted product metric

$$\cosh^2(r)dt^2 + dr^2. \tag{15}$$

For $\phi, \psi \in C_0^\infty(]-R, R[)$, define the metric $g_{\phi,\psi}$ over this annulus by

$$g_{\phi,\psi} := (\cosh(r)dt - \phi(r)dr)^2 + e^{2\psi(r)}dr^2, \tag{16}$$

and extend $g_{\phi,\psi}$ to a smooth metric over the whole of Σ by setting it equal to g over the complement of this annulus.

Lemma 3.2

(1) For all ψ, $(\Sigma, g_{0,\psi})$ is conformally equivalent to $G(\gamma, \Sigma)(t)$, where

$$t = \int_{-R}^{R} \frac{e^{\psi(r)} - 1}{cosh(r)} dr. \tag{17}$$

(2) For all ϕ, $(\Sigma, g_{\phi,0})$ is conformally equivalent to $E(\gamma, \Sigma)(t)$, where

$$t = \int_{-R}^{R} \frac{\phi(s)}{cosh(s)} ds. \tag{18}$$

Remark: The proof of Lemma 3.2 uses the concept of conformal module (see [1]). Recall that every annulus A of hyperbolic type is conformally equivalent to $S^1 \times]0, M[$ for a unique $M \in]0, \infty]$. For our purposes, the *conformal module* of A is defined to be equal to this number.

Proof: Consider first the map

$$\Psi : N_R(\gamma) \to S^1 \times]0, M[; (t, r) \mapsto \frac{2\pi}{l} \left(t, \int_{-R}^{r} \frac{1}{\cosh(s)} dr \right),$$

where l here denotes the length of γ and M is to be determined. We verify that Φ is conformal with respect to the metric g over $B_R(\gamma)$ and the product metric over $S^1\times]0,M[$. Thus, if we denote by $M(B_R(\gamma),g)$ the conformal module of this cylinder, then

$$M(B_R(\gamma),g)=\frac{2\pi}{l}\int_{-R}^{R}\frac{1}{\cosh(s)}ds.$$

The conformal module of the cylinder $(B_R(\gamma),g_{0,\psi})$ is likewise given by

$$M(B_R(\gamma),g_{0,\psi})=\frac{2\pi}{l}\int_{-R}^{R}\frac{e^{\psi(s)}}{\cosh(s)}ds.$$

On the other hand, for all t, the conformal module of the cylinder $(\gamma\times]0,t[,dt^2+dr^2)$ is

$$M(\gamma\times]0,t[,dt^2+dr^2)=\frac{2\pi t}{l}.$$

Item (1) now follows upon comparing these three conformal modules. Finally, for $\phi\in C_0^\infty(]-R,R[)$, define

$$\Phi_\phi:B_R(\gamma)\rightarrow B_R(\gamma);(t,r)\mapsto\left(t-\int_{-R}^{R}\frac{\phi(s)}{\cosh(s)}dr,r\right).$$

Since Φ_ϕ defines an isometry from $(B_R(\gamma),g_{\phi,0})$ to $(B_R(\gamma),g)$, Item (2) follows, and this completes the proof. □

3.2 - First order variations of curvature. Let Σ be a complete, marked hyperbolic surface without cusps and let g denote its metric. Let γ be a simple, closed geodesic in Σ. For suitable $R>0$ and for $\phi,\psi\in C_0^2(]-R,R[)$, let $g_{\phi,\psi}$ denote the complete metric obtained by perturbing g in a neighbourhood of γ as in Section 3.1.

For $\phi,\psi\in C_0^\infty(]-R,R[)$ and for $f\in C^2(\Sigma)$, denote

$$g_{\phi,\psi,f}:=e^{2f}g_{\phi,\psi},\tag{19}$$

and let $\kappa_{\phi,\psi,f}$ denote the curvature function of this metric. The operator κ defines a smooth functional from a neighbourhood of $(0,0,0)$ in $C_0^{2,\alpha}(]-R,R[)^2\times C^{2,\alpha}(\Sigma)$ into $C^{0,\alpha}(\Sigma)$. Furthermore, for all f, we have (c.f. [13]),

$$\kappa_{0,0,f}=-e^{-2f}\big(\Delta f+1\big).\tag{20}$$

In particular, considered as a smooth map between open subsets of Banach spaces, the partial derivative of κ at $(0,0,0)$ with respect to the third component is given by

$$D_3\kappa_{0,0,0}f=-Lf,\tag{21}$$

where L is the operator introduced in Section 2.1. The existence of a Green's kernel ensures that L defines a linear isomorphism from $C^{2,\alpha}(\Sigma)$ into $C^{0,\alpha}(\Sigma)$ (c.f. [18]). The Implicit Function Theorem then yields

Lemma 3.3

For all $\phi,\psi\in C^{2,\alpha}_0(]-R,R[)$, there exists $\epsilon>0$ and a smooth function $f(\phi,\psi):]-\epsilon,\epsilon[\rightarrow C^{2,\alpha}(\Sigma)$ such that, for all s,

$$\kappa_{s\phi,s\psi,f(\phi,\psi)(s)}=-1.$$

Let γ' be another simple, closed geodesic in Σ. For all (ϕ,ψ,f), let $l_{\phi,\psi,f}([\gamma'])$ denote the infimal length of curves in $[\gamma']$ with respect to the metric $g_{\phi,\psi,f}$.

Lemma 3.4

For all $(\phi,\psi)\in C^{2,\alpha}_0(]-R,R[)$,

$$\begin{aligned}\frac{\partial}{\partial s}l_{s\phi,s\psi,f(\phi,\psi)(s)}([\gamma'])\bigg|_{s=0} &= -\int_{\gamma'}(\phi\circ r)cosh^2(r)\frac{dt}{dl'}\frac{dr}{dl'}dl'+\int_{\gamma'}(\psi\circ r)\left(\frac{dr}{dl'}\right)^2dl'\\ &\qquad+\int_{\gamma'}\frac{\partial}{\partial s}f(\phi,\psi)(s)\bigg|_{s=0}dl',\end{aligned}\tag{22}$$

where (t,r) are the coordinates of $B_R(\gamma)$ given in Section 3.1.

Remark: Since $(\psi\circ r)$ and $(\phi\circ r)$ are supported in $B_R(\gamma)$, the integrands of the first two terms on the right-hand side of (22) are non-trivial only along those segments of γ' which lie inside this tubular neighbourhood. In particular, since r is smooth over these segments, these terms are indeed well-defined.

Proof: For all (ϕ,ψ,f) and for all $\eta\in C^{2,\alpha}(\gamma',]-\delta,\delta[)$, let $l'_{\phi,\psi,f,\eta}$ denote the length of the graph of η over γ' with respect to the metric $g_{\phi,\psi,f}$. This is a smooth functional whose partial derivatives with respect to the first three components at zero are

$$\begin{aligned}D_1l_{0,0,0,0}\phi &= -\int_{\gamma'}(\phi\circ r)\cosh^2(r)\frac{dt}{dl'}\frac{dr}{dl'}dl',\\ D_2l_{0,0,0,0}\psi &= \int_{\gamma'}(\psi\circ r)\left(\frac{dr}{dl'}\right)^2dl',\ \text{and}\\ D_3l_{0,0,0,0}f &= \int_{\gamma'}fdl'.\end{aligned}$$

Since γ' is a critical point of the length functional for g, its partial derivative with respect to the fourth component at zero vanishes. Since geodesics in hyperbolic surfaces are stable under small perturbations, upon decreasing ϵ if necessary, there exists a smooth function $\eta:]-\epsilon,\epsilon[\rightarrow C^{2,\alpha}(\gamma',]-\delta,\delta[)$ such that, for all s, the graph of $\eta(s)$ is the unique geodesic in $[\gamma']$ which realises the infimal length with respect to the metric $g_{s\phi,s\psi,f(\phi,\psi)(s)}$ amongst curves in this class. The result now follows by the chain rule. □

Lemma 3.5

Over $B_R(\gamma)$, the curvature of $g_{\phi,\psi,0}$ is given by

$$\kappa_{\phi,\psi,0} = e^{-2\psi(r)}\psi'(r)tanh(r) - e^{-2\psi(r)}. \tag{23}$$

Proof: Consider first a general metric h over Σ. Let (e_1, e_2) be a local orthonormal moving frame of h. Recall (c.f. [12] and [28]) that, for each i and for every vector field ξ,

$$\nabla^h_\xi e_i = -\alpha_h(\xi)J^h e_i,$$

where J^h denotes the complex structure of h, ∇^h denotes its Levi-Civita covariant derivative and the *connection form* α_h is given by

$$\alpha_h(\xi) := h([e_1, e_2], \xi). \tag{24}$$

Recall that the curvature of h is then given by

$$\kappa_h = d\alpha_h(e_1, e_2). \tag{25}$$

Consider now the local orthonormal moving frame given by

$$\begin{aligned} e_1 &:= \frac{1}{\cosh(r)}\partial_t, \\ e_2 &:= e^{-\psi(r)}\partial_r + \frac{\phi(r)e^{-\psi(r)}}{\cosh(r)}\partial_t. \end{aligned}$$

We compute

$$[e_1, e_2] = e^{-\psi(r)}\tanh(r)e_1,$$

and the result now follows by (25). □

Consider now sequences (R_m), (ϕ_m) and (ψ_m) such that,

(1) $(R_m) \downarrow 0$ as m tends to infinity;

(2) $\phi_m, \psi_m \in C^\infty_0(] - R_m, R_m[)$;

(3) $\int_{-R_m}^{R_m} \cosh(s)\psi_m(s)ds = 1$; and

(4) $(\|\psi_m\|_{L^1})$ is uniformly bounded independent of m.

For all m, denote

$$\tilde{\kappa}_m := \frac{\partial}{\partial s}\kappa_{s\phi_m,s\psi_m,0}\bigg|_{s=0}. \tag{26}$$

Lemma 3.6

For any Lipschitz function f,

$$\mathop{Lim}_{m\to\infty}\int_\Sigma f\tilde{\kappa}_m dArea = \int_\gamma f dl. \tag{27}$$

Remark: In other words, $\tilde{\kappa}_m$dArea converges towards δ_γ in the distributional sense as m tends to infinity.

Proof: Indeed, for all m and for all sufficiently small s, denote $g_{m,s} := g_{s\phi_m,s\psi_m,0}$ and $\kappa_{m,s} := \kappa_{s\phi_m,s\psi_m,0}$. By (23),

$$\kappa_{m,s}(r) = s\psi_m'(r)\text{tanh}(r)e^{-2s\psi_m(r)} - e^{-2s\psi_m(r)}.$$

Differentiating this relation with respect to s at $s=0$ yields

$$\tilde{\kappa}_m(r) = 2\psi_m(r) + \psi_m'(r)\text{tanh}(r).$$

Bearing in mind Property (3) of the sequence (ψ_m), upon integrating by parts we obtain, for Lipschitz f,

$$\begin{aligned}\int_{-R_m}^{R_m} f(t,r)\tilde{\kappa}_m(r)\text{cosh}(r)dr &= 2\int_{-R_m}^{R_m} f(t,r)\psi_m(r)\text{cosh}(r)dr\\ &\qquad + \int_{-R_m}^{R_m} f(t,r)\psi_m'(r)\text{sinh}(r)dr\\ &= f(t,0) + \int_{-R_m}^{R_m}\big(f(t,r)-f(t,0)\big)\psi_m(r)\text{cosh}(r)dr\\ &\qquad - \int_{-R_m}^{R_m}\psi_m(r)\text{sinh}(r)f_r(t,r)dr,\end{aligned}$$

where the derivative of f with respect to r is taken in the distributional sense. However, for all m,

$$\left|\int_{-R_m}^{R_m}\big(f(t,r)-f(t,0)\big)\psi_m(r)\text{cosh}(r)dr\right| \leq 2R_m\text{cosh}(R_m)\|\psi_m\|_{L^1}[f]_1,\ \text{and}$$
$$\left|\int_{-R_m}^{R_m}\psi_m(r)\text{sinh}(r)f'(r)dr\right| \leq \text{sinh}(R_m)\|\psi_m\|_{L^1}[f]_1,$$

where $[f]_1$ here denotes the Lipschitz seminorm of f. The result follows upon integrating with respect to t and letting m tend to infinity. $\square$

Theorem 3.7

For every pair (γ,γ') of simple, closed geodesics in Σ,

$$\frac{\partial}{\partial t}l([\gamma'],G(\gamma,\Sigma)(t))\bigg|_{t=0} = \sum_{x\in\gamma\cap\gamma'} sin(\theta_x) + \int_\gamma\int_{\gamma'}K(p,q)dl_pdl_q,\ \text{and} \tag{28}$$

$$\frac{\partial}{\partial t}l([\gamma'],E(\gamma,\Sigma)(t))\bigg|_{t=0} = \sum_{x\in\gamma\cap\gamma'} cos(\theta_x), \tag{29}$$

where $K(x,y)$ is the Green's kernel of L over Σ and, for all $x\in\gamma\cap\gamma'$, θ_x denotes the angle that γ makes with γ' at the point x.

Remark: Equation (29) was first determined by Wolpert in [33].

Remark: By convention, the sums on the right-hand sides of (28) and (29) are taken to be zero when $\gamma=\gamma'$.

Remark: If $\gamma\neq\gamma'$, then these two geodesics intersect transversally, so that the above sums are finite and thus indeed well-defined.

Proof: Let $\psi\in C_0^\infty(]-R,R[)$ be a positive function such that

$$\int_{-R}^R\frac{\psi(r)}{\cosh(r)}dr=1.$$

By Lemma 3.2, up to first order in t, $(\Sigma,g_{0,t\psi})$ is conformally equivalent to $G(\gamma,\Sigma)(t)$. For all t, let $f_t:=f(t)\in C^{2,\alpha}(\Sigma)$ be the unique function such that $g_{0,t\psi,f(t)}$ is hyperbolic. Then, by Lemma 3.4,

$$\frac{\partial}{\partial t}l([\gamma'],G(\gamma,\Sigma)(t))\bigg|_{t=0} = \int_{\gamma'}(\psi\circ r)(x)\left(\frac{dr}{dl'}\right)^2dl'_x+\int_{\gamma'}\frac{\partial}{\partial t}f(x)\bigg|_{t=0}dl'_x.$$

Letting R tend to zero in the first integral yields

$$\operatorname*{Lim}_{R\rightarrow0}\int_{\gamma'}\psi(r)\left(\frac{dr}{dl'}\right)^2dl'_x=\sum_{x\in\gamma\cap\gamma'}\frac{dr}{dl'}(x)=\sum_{x\in\gamma\cap\gamma'}\sin(\theta_x).$$

On the other hand, for all x,

$$\frac{\partial}{\partial t}\kappa_{0,t\psi,0}(x)\bigg|_{t=0}-L\frac{\partial}{\partial t}f_t(x)\bigg|_{t=0}=\frac{\partial}{\partial t}\kappa_{0,t\psi,f(t)}\bigg|_{t=0}=0,$$

so that,

$$\frac{\partial}{\partial t}f_t(x)\bigg|_{t=0}=\int_\Sigma K(x,y)\tilde{\kappa}_\psi(y)dArea_y,$$

where

$$\tilde{\kappa}_\psi = \frac{\partial}{\partial t}\kappa_{0,t\psi,0}(y)\bigg|_{t=0}.$$

Thus, bearing in mind Fubini's theorem, Item (2) of Corollary 2.2 and Lemma 2.4,

$$\begin{aligned}\int_{\gamma'}\frac{\partial}{\partial t}f_t(x)\bigg|_{t=0}dl'_x &= \int_{\gamma'}\int_\Sigma K(x,y)\tilde{\kappa}_\psi(y)dArea_y dl'_x\\ &= \int_\Sigma\int_{\gamma'}K(y,x)\tilde{\kappa}_\psi(y)dl'_x dArea_y\\ &= \int_\Sigma K_{\gamma'}(y)\tilde{\kappa}_\psi(y)dArea_y.\end{aligned}$$

Since $K_{\gamma'}$ is Lipschitz, it follows by Lemma 2.4 again and Lemma 3.6 that

$$\operatorname*{Lim}_{R\rightarrow 0}\int_{\gamma'}\frac{\partial}{\partial t}f_t(x)\bigg|_{t=0}dl_x = \int_\gamma K_{\gamma'}(x)dl_x = \int_\gamma\int_{\gamma'}K(x,y)dl'_y dl_x,$$

and the first relation follows. The second relation is proven in an analogous manner, and this completes the proof. □

Proof of Theorem 3.1: When $\gamma' = \gamma$ or when γ' is disjoint from γ, the first term on the right hand side of (28) vanishes, so that, bearing in mind Lemma 2.4,

$$\frac{\partial}{\partial t}l([\gamma'],G(\gamma,\Sigma)(t))\bigg|_{t=0} = \int_{\gamma'}\int_\gamma K(x,y)dl_x dl_y = \int_{\gamma'}K_\gamma(x)dl_x.$$

Thus, by Item (2) of Corollary 2.8,

$$\frac{\partial}{\partial t}l([\gamma],G(\gamma,\Sigma)(t))\bigg|_{t=0} \leq -\int_\gamma\frac{1}{\pi}dl_x = -\frac{l([\gamma],\Sigma)}{\pi},$$

and the first result follows. Likewise, with C as in Item (3) of Corollary 2.8,

$$\left|\frac{\partial}{\partial t}l([\gamma'],G(\gamma,\Sigma)(t))\bigg|_{t=0}\right| \leq C\int_{\gamma'}e^{-d(\gamma,\gamma')}dl_x = Cl([\gamma'],\Sigma)e^{-d(\gamma,\gamma')},$$

and the second result follows. This completes the proof. □

4 - Laminations and hyperbolic perturbations.

4.1 - Riemannian laminations. We now recall basic definitions and results of the theory of laminations, referring the reader to [11] for a thorough introduction. Let X be a topological space. A d-dimensional *laminated chart* of X is defined to be a pair (U,T,Φ) where U is an open subset of X, T is a topological space and $\Phi : U \rightarrow]-1,1[^d\times T$

is a homeomorphism. Given two laminated charts $(U_i,T_i,\Phi_i)_{i\in\{1,2\}}$, the transition map $\alpha_{21}:\Phi_1(U_1\cap U_2)\rightarrow\Phi_2(U_1\cap U_2)$ is defined by

$$\alpha_{21}:=\Phi_2\circ\Phi_1^{-1}.$$

This map is said to be of class C^∞_l whenever every point of $\Phi_1(U_1\cap U_2)$ has a neighbourhood Ω of the form

$$\Omega:=]x_1-\epsilon,x_1+\epsilon[\times...\times]x_d-\epsilon,x_d+\epsilon[\times S$$

over which

$$\alpha_{21}(x,t)=(\phi_{21}(x,t),\tau_{21}(t)),$$

where

(1) τ_{21} is a homeomorphism onto its image;

(2) for all t, $\phi_{21}(\cdot,t)$ is a smooth diffeomorphism onto its image; and

(3) $\phi_{21}(\cdot,t)$ varies continuously with t in the C^∞_{loc} sense.

A *lamination* is defined to be a separable, metrizable space X furnished with an atlas $\mathcal{A}$ of laminated charts all of whose transition maps are of class C^∞_l.

Given a lamination X, a laminated chart (U,T,Φ) of X and a point t of T, we call the set $\Phi^{-1}(]-1,1[^d\times\{t\})$ a *plaque* of the chart. Plaques glue together to yield a partition of X into smooth, d-dimensional manifolds, called *leaves* of the lamination. For all $x\in X$, the leaf passing through x will be denoted by Σ_x. Given two laminations X and Y and $k\in\mathbb{N}\cup\{\infty\}$, $C^k_l(X,Y)$ is defined to be the space of all continuous functions $f:X\rightarrow Y$ which restrict to C^k functions from leaves to leaves and which, in addition vary continuously in the C^k_{loc} sense as the plaques vary in any given laminated chart. Functions in $C^\infty_l(X,Y)$ are said to be *leafwise smooth*. Leafwise smooth maps are the morphisms of the category of laminations.

Since every finite-dimensional manifold naturally carries the structure of a lamination consisting of a single leaf, the theory of laminations may be viewed as an extension of the theory of manifolds. Standard constructions of manifold theory then carry over to the theory of laminations with appropriate modifications. In particular, vector bundles over laminations are defined in the natural manner and, given a lamination X, the *tangent bundle* TX of X is defined to be the vector bundle whose fibre at the point $x\in X$ is the tangent space to the leaf Σ_x at this point. The cotangent bundle and other tensor bundles over X are likewise defined in the natural manner.

Given a lamination X and $k\in\mathbb{N}\cup\{\infty\}$, let $C^k_l(X)$ denote the space of C^k_l functions from X into the trivial lamination $\mathbb{R}$. We recall

Theorem 4.1, Candel [10]

For every open cover $(U_i)_{i\in I}$ of X, there exists a locally finite, leafwise smooth partition of unity $(\chi_j)_{j\in J}$ of X subordinate to this cover.

When, in addition, X is compact, leafwise smooth partitions of unity serve to furnish $C^k_l(X)$ with a canonical Banach space structure as follows. First, let (U,T,Φ) be a laminated chart of X and, for $f\in C^k_l(]-1,1[^d\times T)$, define

$$\|f\|_{C^k_l}:=\mathop{\text{Sup}}_{t\in T}\|f(\cdot,t)\|_{C^k}.$$

Next, let $(U_\alpha, T_\alpha, \Phi_\alpha)_{\alpha\in A}$ be a finite atlas of X by laminated charts, let $(\chi_\alpha)_{\alpha\in A}$ be a leafwise smooth partition of unity of X subordinate to this atlas and, for $f \in C^k_l(X)$, define

$$\|f\|_{C^k_l} := \sum_{\alpha\in A} \|(f\chi_\alpha)\circ\Phi_\alpha^{-1}\|_{C^k_l}. \tag{30}$$

Up to uniform equivalence, the norm (30) is independent of the atlas and partition of unity chosen, and thus defines a canonical Banach space structure over $C^k_l(X)$. For all (k,α), the Hölder norm $\|\cdot\|_{C^{k,\alpha}_l}$ is defined in a similar manner, and the Hölder space $C^{k,\alpha}_l(X)$ is defined to be the Banach space of all functions $f \in C^k_l(X)$ such that

$$\|f\|_{C^{k,\alpha}_l(X)} < \infty.$$

Finally, given any vector bundle EX over X, for all (k,α), the Banach space of $C^{k,\alpha}_l$ sections of EX over X is defined in an analogous manner and will be denoted by $\Gamma^{k,\alpha}_l(EX)$.

A *leafwise metric* over X is defined to be a leafwise smooth, positive-definite section of the bundle $\mathrm{Symm}(TX)$ of symmetric, bilinear forms over X. In view of Theorem 4.1, leavewise metrics always exist and are constructed in the same way as in the classical theory of riemannian manifolds. Given a leafwise metric g, the pair (X,g) will be called a *riemannian lamination*. For all x, the restriction of g to the leaf Σ_x will be denoted by g_x.

A *hyperbolic surface lamination* is a riemannian lamination all of whose leaves are complete hyperbolic surfaces. In [10], Candel characterises hyperbolic surface laminations in terms of the conformal classes of its leaves. Recall first that two leafwise metrics g and g' are said to be *conformally equivalent* whenever there exists a leafwise smooth function u such that

$$g' = e^{2u}g.$$

Theorem 4.2, Candel [10]

Let (X,g) be a compact riemannian surface lamination. If every leaf of (X,g) is of hyperbolic type, then there exists a unique leafwise smooth metric $\tilde{g}$ in the conformal class of g with respect to which every leaf is complete and hyperbolic.

A closer examination of Candel's proof yields

Lemma 4.3, Candel [10]

Let $\mathcal{G}^{k,\alpha}_l(X)$ denote the space of negatively curved, leafwise smooth metrics of class $C^{k,\alpha}_l$ over X. Let $\mathcal{H} : \mathcal{G}^{k,\alpha}_l(X) \to \mathcal{G}^{k,\alpha}_l(X)$ be such that, for all g, $\mathcal{H}(g)$ is the unique leafwise smooth hyperbolic metric given by Theorem 4.2. Then $\mathcal{H}$ is smooth as a map between Banach manifolds.

4.2 - Elementary differential topology of laminations. Let X be a compact riemannian lamination. We gather here various elementary properties of X which will be required in the sequel. First, the *leafwise distance* over X is defined by

$$d_l(x,y) := \operatorname*{Inf}_{\gamma} l_g(\gamma), \tag{31}$$

where γ varies over all tangential curves from x to y and $l_g(\gamma)$ denotes the length of γ with respect to g. In particular, the leafwise distance between two points is finite if and only if they both lie on the same leaf. This distance function defines the *leafwise topology* of X, which is the smallest topology containing all open subsets of leaves of X. A sequence (x_m) of points in X converges to the point x_∞ in this topology if and only if (x_m) converges to x_∞ in the ambient topology of X and, for sufficiently large m, x_m also lies in the same leaf as x_∞. A subset Y of X is compact with respect to this topology if and only if it consists of a finite union of compact subsets of leaves.

Lemma 4.4

Let (U,T,Φ) be a laminated chart of X. Then T is separable, metrizable and locally compact.

Proof: Indeed, being a subset of a separable, metrizable space, U is also separable and metrizable. Since $T=\{0\}\times T$ is homeomorphic to a subset of U, separability and metrizability follow. It remains to prove local compactness. However, choose $t\in T$. Denote $x:=\Phi^{-1}(\{0\})$. Since x is metrizable, there exists a neighbourhood V of x in X such that $\overline{V}\subseteq U$. Since X is compact, so too is $\overline{V}$. Since Φ is a homeomorphism, $\Phi(V)$ is open and $\overline{\Phi V}=\Phi(\overline{V})$ is compact. Let W be a neighbourhood of t in T such that $\{0\}\times W\subseteq\Phi(V)$. Then $\overline{W}$ is compact, and the result follows. □

Lemma 4.5

For all $x\in X$, there exists $C>0$ and a laminated chart of X containing x whose plaques have volume bounded below by $1/C$ and diameter bounded above by C.

Remark: By compactness of X, B may even be chosen independent of x.

Proof: Let (U,T,Φ) be a laminated chart of X about x. Let $t\in T$ be such that $\Phi(x)$ is contained in the plaque $]-1,1[^d\times\{t\}$. Let V be a neighbourhood of t in T with compact closure. □

Lemma 4.6

Let $Y\subseteq X$ be compact in the leafwise topology. For all $x\in X$, there exists a laminated chart (U,T,Φ) of X about x such that $U\cap Y$ is contained in, at most, a single plaque.

Proof: We suppose that X is furnished with a leafwise metric g. Let (U,T,Φ) be a laminated chart of X about x whose plaques have volume bounded below by $1/C$, say, and diameter bounded above by C, say. Since Y is compact in the leafwise topology, there exist $y_1,...,y_m\in Y$ and $R>0$ such that

$$Y\subseteq\bigcup_{i=1}^{m}B_R(y_i),$$

where, for each i, $B_R(y_i)$ here denotes the ball of radius R about y_i in X with respect to the leafwise metric. If V is the volume of the set

$$\bigcup_{i=1}^{m} B_{R+C}(y_i),$$

then (U, T, Φ) contains at most $\lfloor CV \rfloor$ plaques which intersect Y non-trivially where, for all λ, $\lfloor \lambda \rfloor$ denotes the greatest integer less than or equal to λ. The result now follows upon reducing T if necessary. $\square$

Lemma 4.7

Let $Y \subseteq X$ be compact in the leafwise topology. Let Z be the (finite) union of all leaves which intersect Y non-trivially. There exists a neighbourhood Ω of Y in X and a leafwise smooth function $\pi : \Omega \to Z$ such that, for all $y \in Y$, $\pi(y) = y$.

Proof: It suffices to consider the case where Y is contained in a single leaf Σ, say. By compactness, Y is covered by a finite family $(U_i, T_i, \Phi_i)_{1 \leq i \leq m}$ of laminated charts. By Lemma 4.6, we may suppose that, for all i,

$$\Phi_i(Y \cap U_i) \subseteq]-1, 1[^d \times \{t_i\},$$

for some $t_i \in T_i$. For all i, denote

$$V_i := \Phi_i^{-1}(]-1, 1[^d \times \{t_i\})/$$

By definition,

$$Y \subseteq \bigcup_{i=1}^{m} V_i.$$

For all i, upon reducing U_i slightly if necessary, we may suppose that V_i is relatively compact as a subset of Σ. By Lemma 4.6 again, upon reducing each U_i further if necessary, we may suppose that, for all i and for all j,

$$V_i \cap U_j = V_i \cap V_j.$$

For all $i \in \{1, ..., m\}$, let

$$p_{i,1} :]-1, 1[^d \times T_i \to]-1, 1[^d \text{ and}$$
$$p_{i,2} :]-1, 1[^d \times T_i \to T_i$$

be the projections onto the first and second factors respectively. For all i, define $\pi_i : U_i \to \Sigma$ by

$$\pi_i(x) = \Phi_i^{-1}(p_{i,1}(\Phi_i(x)), t_i).$$

Observe that, for all i, π_i is leafwise smooth and, for all $y \in V_i$, $\pi_i(y) = y$.

The open set Ω and the function π will be constructed by induction as follows. For all i, define

$$W_i := V_1 \cup ... \cup V_i.$$

Suppose that, for some i, we have constructed an open set Ω_i and a leafwise smooth map $\tilde{\pi}_i : \Omega_i \to \Sigma$ such that $W_i \subseteq \Omega_i$ and $\tilde{\pi}_i(y) = y$ for all $y \in W_i$. Let O be a neighbourhood of the diagonal in $\Sigma \times \Sigma$ consisting of pairs (p, q) for which there exists a unique length-minimising geodesic $\gamma_{pq} : [0, 1] \to \Sigma$ such that $\gamma_{pq}(0) = p$ and $\gamma_{pq}(1) = q$. Define the smooth map $G : O \times [0, 1] \to \Sigma$ by

$$G(p, q, t) := \gamma_{pq}(t).$$

Upon reducing Ω_i and U_{i+1} if necessary, we may suppose that $\Omega_i = \tilde{\pi}_i^{-1}(W_i)$ and that $U_{i+1} = \pi_{i+1}^{-1}(V_{i+1})$. By Theorem 4.1, there exists a leafwise smooth partition of unity (f_i, g_i) of $\Omega_i \cup U_{i+1}$ subordinate to the cover (Ω_i, U_{i+1}). Define the open subsets $\Omega_{i+1,1} \subseteq \Omega_i$, $\Omega_{i+1,2} \subseteq U_{i+1}$ and $\Omega_{i+1,3} \subseteq \Omega_i \cap U_{i+1}$ by

$$\begin{aligned}
\Omega_{i+1,1} &:= \Omega_i \setminus \mathrm{Supp}(g_i), \\
\Omega_{i+1,2} &:= U_{i+1} \setminus \mathrm{Supp}(f_i), \text{ and} \\
\Omega_{i+1,3} &:= (\tilde{\pi}_i, \pi_{i+1})^{-1}(O).
\end{aligned}$$

Denote

$$\Omega_{i+1} := \Omega_{i+1,1} \cup \Omega_{i+1,2} \cup \Omega_{i+1,3}.$$

We claim that $W_{i+1} := W_i \cup V_{i+1} \subseteq \Omega_{i+1}$. Indeed, $W_i \setminus \mathrm{Supp}(g) \subseteq \Omega_{i+1,1}$ and $V_{i+1} \setminus \mathrm{Supp}(f) \subseteq \Omega_{i+1,2}$. By construction,

$$W_i \cap \mathrm{Supp}(g) \subseteq W_i \cap U_{i+1} \subseteq V_{i+1},$$

so that $\tilde{\pi}$ and π_{i+1} are both defined and are equal over this set. Likewise,

$$V_{i+1} \cap \mathrm{Supp}(f) \subseteq V_{i+1} \cap \Big(\bigcup_{j=1}^{i} U_j \Big) = V_{i+1} \cap \Big(\bigcup_{j=1}^{i} V_j \Big) \subseteq W_i,$$

so that $\tilde{\pi}$ and π_{i+1} again are both defined and are equal over this set. It follows that

$$W_i \cap \mathrm{Supp}(g), V_{i+1} \cap \mathrm{Supp}(f) \subseteq \Omega_{i+1,3},$$

so that $W_i \cup V_{i+1} \subseteq \Omega_{i+1}$, as asserted.

Define $\pi_{i+1} : \Omega_{i+1} \to \Sigma$ by

$$\pi_{i+1}(x) := \begin{cases} \tilde{\pi}_i(x) & \text{if } x \in \Omega_{i+1,1}, \\ \pi_{i+1}(x) & \text{if } x \in \Omega_{i+1,2}, \text{ and} \\ G(\tilde{\pi}_i(x), \pi_{i+1}(x), 1 - f_i(x)) & \text{if } x \in \Omega_{i+1,3}. \end{cases}$$

We readily verify that π_{i+1} is well-defined and, for all $y \in W_{i+1}$, $\tilde{\pi}_{i+1}(y) = y$. Furthermore, since this function is leafwise smooth over each of $\Omega_{i+1,1}$, $\Omega_{i+1,2}$ and $\Omega_{i+1,3}$, it is leafwise smooth over the whole of Ω_{i+1}. Furthermore, $W_{i+1} \subseteq \Omega_{i+1}$ and,

The induction process is initiated with $W_1 := V_1$, $\Omega_1 := U_1$ and $\tilde{\pi}_1 := \pi_1$ and the result follows upon setting $\Omega := \Omega_m$ and $\pi := \tilde{\pi}_m$. $\square$

Lemma 4.8

Let $\Omega \subseteq Y \subseteq X$ be such that Ω is open in the leafwise topology and Y is compact in the leafwise topology. There exists an open subset $\hat{\Omega}$ of X such that $\hat{\Omega} \cap Y = \Omega$.

Proof: Let x be a point of Ω. Let (U, T, Φ) be a laminated chart of X containing x. By Lemma 4.6, we may suppose that the only plaque of U which intersects Y non-trivially is the plaque containing x. Upon reducing U if necessary, we may suppose furthermore that this plaque is contained entirely in Ω. The result follows upon taking the union of all such laminated charts. □

Lemma 4.9

Let $Y \subseteq \Omega \subseteq X$ be such that Y is compact in the leafwise topology and Ω is open in the leafwise topology. There exists a sequence $(C_k)_{k\in\mathbb{N}}$ of positive constants such that if $f \in C_0^\infty(\Omega)$ is supported in Y and if $\hat{\Omega}$ is an open subset of X containing Ω, then there exists a leafwise smooth function $\tilde{f} \in C_l^\infty(X)$ supported in $\hat{\Omega}$ such that

(1) for all $x \in Y$, $\tilde{f}(x) = f(x)$; and

(2) for all k,

$$\|\tilde{f}\|_{C_l^k} \leq C_k \|f\|_{C^k}.$$

Proof: Suppose first that Y is contained in a single plaque of a laminated chart (U, T, Φ) and that this plaque is contained in Ω. By Lemma 4.6, we may suppose that

$$\Phi(Y) \subseteq]-1,1[^d \times \{t\},$$

for some $t \in T$. Let

$$p_1 :]-1,1[^d \times T \to]-1,1[^d \text{ and}$$
$$p_2 :]-1,1[^d \times T \to T$$

be the projections onto the first and second factors respecively. Define $\pi : U \to \Sigma$ by

$$\pi(x) := \Phi^{-1}(p_1(\Phi(x)), t).$$

Let d be a metric of T. Let $\chi \in C^\infty(\mathbb{R}, [0,1])$ be such that $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. For $\lambda > 0$, define $\chi_\lambda : U \to \mathbb{R}$ by

$$\chi_\lambda(x) := \chi(d(p_2(\Phi(x)), t)/\lambda).$$

Define $\tilde{f}_\lambda$ by

$$\tilde{f}_\lambda(x) := (f \circ \pi)(x)\chi_\lambda(x).$$

Since T is locally compact, there exists $\epsilon > 0$ such that $Z := \Phi^{-1}(\Phi(Y) \times \overline{B}_\epsilon(t))$ is a compact, subset of X, where $B_\epsilon(t)$ denotes the ball of radius ϵ about t in T with respect to the metric d. Upon reducing ϵ if necessary, we may suppose in addition that $Z \subseteq \hat{\Omega}$. For sufficiently large λ, $\text{Supp}(\tilde{f}_\lambda) \subseteq Z \subseteq \hat{\Omega}$. Furthermore, by compactness, for all k, there exists C_k such that, for all such λ,

$$\|\tilde{f}_\lambda\|_{C_l^k} \leq \|f \circ \pi|_Z\|_{C_l^k} \leq C_k \|f\|_{C^k}.$$

Finally, by compactness again, Y is contained in finitely many such laminated charts, and the general case follows using a finite C_l^∞ partition of unity. □

4.3 - Hyperbolic perturbations. Before proving our main result concerning the infinite dimensionality of Teichmüller space, we return to the case of a single complete, hyperbolic surface Σ without cusps. Let g denote its hyperbolic metric. We review the perturbation theory of hyperbolic metrics over this surface, adopting a formalism similar to that of [31]. Let Σ be a complete hyperbolic surface without cusps. Let A be a smooth, bounded section of $\mathrm{End}(T\Sigma)$ which is symmetric with respect to g. For sufficiently small t, denote

$$g_t := g((\mathrm{Id}+tA)\cdot,(\mathrm{Id}+tA)\cdot), \tag{32}$$

and let $\kappa_t:\Sigma\to\mathbb{R}$ be the curvature function of this metric. A is said to be a *hyperbolic perturbation* whenever

$$\kappa_t = -1+\mathrm{O}(t^2). \tag{33}$$

Given a hyperbolic perturbation A, for every simple, closed geodesic γ in Σ, define

$$\Delta([\gamma],A) := \frac{\partial}{\partial t}\mathrm{Log}(l([\gamma],h_t))\bigg|_{t=0},$$

where $l([\gamma],h_t)$ is as in Section 1.3. We readily obtain

Lemma 4.10

If γ is parametrised by arc length, then

$$\Delta([\gamma],A) = \frac{1}{l([\gamma],g)}\int_\gamma\langle\dot\gamma,A\dot\gamma\rangle dl.$$

Since the curvature operator is a second order, non-linear partial differential operator, (33) can be rewritten as a linear differential condition on A. In particular, the space of hyperbolic perturbations is a vector space. It will be helpful for what follows to determine this condition explicitly. To this end, recall first that the *divergence* of an endomorphism field B with respect to g is the 1-form defined by

$$(\nabla\cdot B)(\cdot) = \sum_{i=1}^2\langle e_i,(\nabla_{e_i}B)(\cdot)\rangle, \tag{34}$$

where (e_1,e_2) is a local orthonormal frame of g. Likewise, the *divergence* of a 1-form β with respect to g is the function defined by

$$\nabla\cdot\beta = \sum_{i=1}^2(\nabla_{e_i}\beta)(e_i). \tag{35}$$

Lemma 4.11

Let A be a smooth, bounded, symmetric section of End$(T\Sigma)$. A is a hyperbolic perturbation if and only if

$$\nabla\cdot\nabla\cdot(JAJ) + \mathrm{Tr}(A) = 0, \tag{36}$$

where J denotes the complex structure of g.

Proof: As in the proof of Lemma 3.5, we use the formalism of moving frames. Let (e_1, e_2) denote a local orthonormal moving frame of g. For all sufficiently small t, denote $B_t := \mathrm{Id} + tA$ and

$$g_t := g(B_t\cdot, B_t\cdot),$$

and let κ_t denote the curvature of g_t. For all such t, a locally orthonormal frame of g_t is given by $(B_t^{-1}e_1, B_t^{-1}e_2)$. Using (24), we show that the connection form of this frame is

$$\alpha_t(X) = \alpha_0(X) - \mathrm{Det}(B_t^{-1})g(B_t\nabla\cdot(B_tJ), X).$$

Differentiating at $t = 0$ then yields

$$\frac{\partial}{\partial t}\alpha_t(X)\bigg|_{t=0} = -g(\nabla\cdot(AJ), X),$$

so that

$$\frac{\partial}{\partial t}d\alpha_t(e_1, e_2)\bigg|_{t=0} = \nabla\cdot\nabla\cdot(JAJ).$$

It follows by (25) that

$$\frac{\partial}{\partial t}\kappa_t\bigg|_{t=0} = \nabla\cdot\nabla\cdot(JAJ) - \kappa_g\mathrm{Tr}(A),$$

and the result follows since $\kappa_g = -1$. $\square$

4.4 - Infinite dimensionality of Teichmüller space. Let X be a compact hyperbolic surface lamination and denote its leafwise metric by g. By compactness, the injectivity radius of every leaf is bounded below by $\epsilon > 0$, say, so that no leaf of X has cusps. Recall (c.f. [29] and [30]) that the Teichmüller space of X is defined to be the space of transversally continuous conformal structures over X modulo leafwise diffeomorphisms of X which are leafwise isotopic to the identity. Recall, furthermore, that [30], Sullivan shows that this space naturally carries the structure of a Banach manifold. In this section, we show

Theorem 4.12

Suppose that, for every $K\subseteq X$ which is compact in the leafwise topology, there exists a simple, closed geodesic γ in X such that

$$\gamma\cap K=\emptyset.$$

Then there exists a sequence (γ_m) of simple, closed geodesics in X such that, for every finite sequence $a_1,...,a_m\in\mathbb{R}$, there exists a smooth family $(g_t)_{t\in]-\epsilon,\epsilon[}$ of leafwise smooth hyperbolic metrics such that, for all $1\leq i\leq m$,

$$\frac{\partial}{\partial t}Log(l([\gamma_i],g_t))\bigg|_{t=0}=a_i.$$

In particular, the Teichmüller space of X is infinite dimensional.

Theorem 4.12 is proven by a straightforward induction argument. The induction step is provided by

Lemma 4.13

Suppose that for every $K\subseteq X$ which is compact in the leafwise topology, there exists a simple, closed geodesic γ in X such that

$$\gamma\cap K=\emptyset.$$

For every finite set G of simple, closed geodesics in X, and for all $\epsilon>0$, there exists a simple, closed geodesic γ in X and a hyperbolic perturbation A such that

$$\Delta([\gamma],A)<-\frac{1}{2\pi},$$

and, for all $\gamma'\in G$,

$$|\Delta([\gamma'],A)|<\epsilon.$$

Proof: Let $R_1>0$ be a positive number to be determined presently. Let γ be a simple, closed geodesic in X such that, for all $\gamma'\in G$,

$$d_l(\gamma,\gamma')>2R_1.$$

Let Σ be the leaf containing γ. Let G_0 be the subset of G consisting of those geodesics that are contained in Σ and let G_1 consist of all other geodesics of G.

Let $R<R_1$ be such that the tubular neighbourhood $B_R(\gamma)$ of radius R about γ is isometric to $\gamma\times]-R,R[$ furnished with the twisted product metric

$$\cosh^2(r)dt^2+dr^2.$$

Let $\psi \in C_0^\infty(]-R/2,R/2[)$ be a positive function such that

$$\int_{-R}^{R}\frac{\psi(r)}{\cosh(r)}dr = 1.$$

Define the endomorphism field A_1 over $B_R(\gamma)$ by

$$g(A_1\cdot,\cdot) = \psi(r)dr^2,$$

and extend it to a smooth endomorphism field over the whole of Σ by setting it equal to zero outside this annulus. Let $u_1 : \Sigma \to \mathbb{R}$ be the unique bounded, smooth function such that

$$(\Delta - 2)u_1 = \nabla\cdot\nabla\cdot(JA_1J) + \mathrm{Tr}(A_1),$$

so that

$$A_2 := A_1 + u_1\mathrm{Id},$$

is a hyperbolic perturbation. By Theorem 3.1 and Lemma 3.2,

$$\Delta([\gamma],A_2) \leq -\frac{1}{\pi},$$

and, provided $R_1 > 0$ is sufficiently large, for all $\gamma' \in G_0$,

$$|\Delta([\gamma'],A_2)| < \frac{\epsilon}{2}.$$

Let $R_2 > R_1$ be another positive constant to be determined presently. Let $\chi \in C_0^\infty(B_{3R/4}(\gamma),[0,1])$ be such that χ is equal to 1 over $\overline{B}_{R/2}(\gamma)$. By Lemma 4.7, there exists a neighbourhood Ω_1 of $\overline{B}_R(\gamma)$ in X and a leafwise smooth function $\pi : \Omega_1 \to \Sigma$ such that, for all $x \in \overline{B}_R(\gamma)$, $\pi(x) = x$. By Lemma 4.8, there exists another open subset Ω_2 of X such that

$$\Omega_2 \cap \big(B_{R_2}(\gamma) \cup \bigcup_{\gamma'\in G} B_{R_2}(\gamma')\big) = B_R(\gamma).$$

Set $\Omega := \Omega_1 \cap \Omega_2$. By Lemma 4.9, there exists $C_1 > 0$, independent of R_2 and Ω, and a leafwise smooth function $\tilde{\chi} : X \to [0,1]$ such that

(1) $\mathrm{Supp}(\tilde{\chi}) \subseteq \Omega$,

(2) $\tilde{\chi}(x) = \chi(x)$ for all $x \in B_R(\gamma)$, and

(3) $\|\tilde{\chi}\|_{C^2_l} \leq C_1$.

Define the leafwise smooth endomorphism field A_3 over X by

$$A_3 := \left\{\begin{array}{ll}\tilde{\chi}(x)(\pi^*A_1)(x) & \text{if } x \in \Omega \text{ and}\\ 0 & \text{if } x \notin \mathrm{Supp}(\tilde{\chi})\end{array}\right.,$$

and let $u_3 : \Sigma \to \mathbb{R}$ be the unique bounded, leafwise smooth function such that

$$Lu_3 = \nabla\cdot\nabla\cdot(JA_3J) + \mathrm{Tr}(A_3),$$

so that

$$A_4 := A_3 + u_3\mathrm{Id},$$

is a hyperbolic perturbation. For $\gamma' \in G_0 \cup \{\gamma\}$, we have

$$\begin{aligned}\mathrm{Supp}(A_3 - A_1) \cap B_{R_2}(\gamma') = \emptyset \text{ and}\\ \mathrm{Supp}\big((\Delta - 2)(u_3 - u_1)\big) \cap B_{R_2}(\gamma') = \emptyset.\end{aligned}$$

It follows from the first relation that, near γ', $A_4 - A_2$ only depends on $u_3 - u_1$. It then follows by Lemmas 2.3 and 4.10 that there exists $C_2 > 0$, independent of R_2, such that

$$|\Delta([\gamma'], A_4) - \Delta([\gamma'], A_2)| \leq C_2 e^{-R_2}.$$

Likewise, for $\gamma' \in G_1$,

$$\begin{aligned}\mathrm{Supp}(A_3) \cap B_{R_2}(\gamma') = \emptyset \text{ and}\\ \mathrm{Supp}\big(L(u_3)\big) \cap B_{R_2}(\gamma') = \emptyset,\end{aligned}$$

so that, upon increasing C_2 if necessary, we have

$$|\Delta([\gamma'], A_4)| \leq C_2 e^{-R_2}.$$

The result now follows upon setting R_2 sufficiently large. □

Lemma 4.14

For every leafwise smooth hyperbolic perturbation A over X there exists $\epsilon > 0$ and a smooth family $(g_t)_{t \in]-\epsilon,\epsilon[}$ of leafwise metrics over X such that

$$\frac{\partial}{\partial t} g_t \bigg|_{t=0} = 2g(A\cdot, \cdot).$$

Proof: Indeed, for sufficiently small t, denote $B_t := \mathrm{Id} + tA$ and

$$h_t := g(B_t\cdot, B_t\cdot).$$

By Lemma 4.3, there exists $\epsilon > 0$ and a smooth family $(u_t)_{t \in]-\epsilon,\epsilon[}$ of leafwise smooth functions such that, for all t, the metric

$$g_t := e^{2u_t} h_t$$

is hyperbolic. Consider now a leaf Σ of X. By Lemma 4.11, over Σ,

$$L\,\frac{\partial}{\partial t} u_t \bigg|_{t=0} = \nabla \cdot \nabla \cdot (JAJ) + \mathrm{Tr}(A) = 0.$$

However, since Σ is without cusps, L has trivial kernel in the space $C^2_{\text{bdd}}(\Sigma)$ of bounded, twice differentiable functions over this surface, so that, over Σ,

$$\frac{\partial}{\partial t}u_t\bigg|_{t=0} = 0.$$

Since Σ is arbitrary, this relation in fact holds over the whole of X, so that, since $u_0 = 0$,

$$\frac{\partial}{\partial t}g_t\bigg|_{t=0} = \frac{\partial}{\partial t}h_t\bigg|_{t=0} = 2g(A\cdot,\cdot),$$

as desired. $\square$

Proof of Theorem 4.12: Using Lemma 4.13, we show by induction that there exist sequences (ϵ_m), (B_m) of positive numbers, a sequence (γ_m) of simple, closed geodesics in X, and a sequence (A_m) of hyperbolic perturbations such that

(1) for all m,

$$\epsilon_{m+1} \leq \frac{1}{(4\pi)^{m+1}(m+1)!B_1\cdots B_m};$$

(2) for all m,

$$\Delta([\gamma_m], A_m) \leq -\frac{1}{2\pi};$$

(3) for all m, and for all $n < m$,

$$|\Delta([\gamma_n], A_m)| < \epsilon_m;\ \text{and}$$

(4) for all m, and for every simple, closed geodesic γ,

$$|\Delta([\gamma], A_m)| < B_m.$$

Consider now a finite sequence $a_1, ..., a_m \in \mathbb{R}$, and define the $m \times m$ matrix M by

$$M_{ij} := \Delta([\gamma_i], A_j).$$

The above estimates yield,

$$|\text{Det}(M)| \geq \frac{1}{(4\pi)^m},$$

so that M is invertible. There therefore exists $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that, for all $1 \leq i \leq m$,

$$\Delta\big([\gamma_i], \sum_{j=1}^m \alpha_j A_j\big) = a_i.$$

Setting

$$A := \sum_{j=1}^m \alpha_j A_j,$$

the result follows by Lemma 4.14. $\square$

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