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An algorithm for the serial capacitated economic lot-sizing problem with non-speculative costs and stationary capacities

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***Abstract*.** We address the serial capacitated economic lot-sizing problem under particular assumptions on the costs and the capacity pattern. We prove that when the involved costs are non-speculative with respect to the transfer to future periods and the capacity pattern is stationary for all levels, the optimal plan for each level can be obtained independently in $O(T^3)$ time. This leads to an $O(T^{3L})$ algorithm for the problem with L levels.

***Keywords*:** Capacitated Economic Lot-Sizing Problem; Inventory Control.

1. Introduction

We analyze the serial extension of the capacitated economic lot-sizing problem for a single item. The problem can be stated as follows. There is a customer demand for a single product, which is known in advance for each one of the periods over a finite planning horizon. Demand must be satisfied on time, either by distributing new items from a retailer in the last level or by using previously stocked items in the customer storage. In turn, the requirements of the retailer in level ℓ are satisfied on time by a retailer in an upstream level $(\ell - 1)$, until reaching the supplier/producer at the first level. We point out that backlogging demand is not allowed at any level. The numbers of units produced and distributed are limited by maximum values, known in advance for each period and level, and can be stocked, if they are not used to serve the requirements. Costs are incurred when a positive amount is produced or distributed in a certain period and for carrying stocked items from a period to the next. All costs and values mentioned above are dynamic, i.e., possibly different for each period, activity and level. The objective is determine the activity quantities (i.e., production and distribution quantities) for each level and period of the planning horizon in order to meet the customer demand

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requirements on time, fulfilling the capacity constraints and minimizing all the involved costs. We refer to this problem as the Serial Capacitated Economic Lot-Sizing Problem, and briefly as SCLSP.

Problems such as described above occur in a wide range of practical situations, e.g. when there is a coordinated supply chain management for both production and distribution activities. It is known today that coordinating these decisions, rather than considering each one of them in isolation, can cut costs for all participants in the supply chain, even in the case of different participants. Perspectives and benefits of coordinated supply chain management are further described in Arshinder et al. [3]. Some real-word examples of these chains are mentioned in Kaminsky and Simchi-Levi [16] for the pharmaceutical industry and in van Hoesel et al. [25] for the third-party logistics industry. A recent innovation in this direction is the Vendor-Managed Resupply strategy, in which the supplier manages the inventory replenishment of its customers, ensuring that no stock-out occurs (Campbell et al. [7] and [8]; Jans and Degraeve [15]).

We point out that in this paper we consider capacity constraints for all levels, i.e. supplier and retailers. The SCLSP is NP-hard in general since the single-level case of the problem, named CLSP, is NP-hard not only in the general case but even for many special cases (Florian et al. [14]; Bitran and Yanasse [5]). On the other hand, the CLSP can be solved in polynomial time for concave cost functions and stationary capacity-pattern. Therefore, we analyze the SCLSP under these common assumptions. As we will see later, we need to impose an additional assumption on the cost functions in order to provide an efficient algorithm for the problem. We refer to this assumption as *non-speculative motives with respect to the transfer*, i.e., it is profitable to transfer the production or distribution quantity from one period to another future period in the planning horizon. Adding this assumption on the costs, we are able to develop an algorithm of $O(T^3)$ time for each level, thus improving the running-time of the Florian and Klein [13] algorithm of $O(T^4)$ for this particular case of the CLSP. This improvement is based on the introduction of a new kind of capacity constrained sequences that we refer as *ascending capacity constrained sequences*. For the SCLSP with $L > 1$ number of levels, the running-time of the proposed algorithm is $O(T^{3L})$.

The remainder of this paper is organized as follows. In the next section we provide the literature review. A detailed description of the SCLSP, along with the mathematical formulation and an analysis of the problem are given in Section 3. In Section 4 we provide certain properties related to the form of the SCLSP optimal solutions under the particular assumptions described above. The algorithm for the SCLSP is provided and shown in Section 5. Finally, Section 6 concludes the paper with some suggestions for future research.

2. Literature Review

The single-level economic lot-sizing problem with capacity constraints is commonly referred as CLSP in the literature. The CLSP is NP-hard in the general case, and even for special cases on the cost structures and/or the capacity pattern (Bitran and Yanasse [5]; Florian et al. [14]). On the other hand, for the case of concave cost functions and stationary capacity (i.e., equal capacity upper-bound for each period) Florian and Klein [13] developed a polynomial-time algorithm of $O(T^4)$ based on a dynamic programming approach. The effectiveness of the proposed algorithm is based on the fact that there exists an optimal production plan composed only by capacity constrained sequences, i.e., each sequence has at most one period with positive production less than capacity. An improved algorithm of $O(T^3)$ time was then suggested by van Hoesel and Wagelmans [26] for the particular case of linear holding inventory costs. Bitran and Yanasse [2] developed polynomial-time algorithms for several particular cases of the CLSP. The authors introduced a notation for classifying the different capacitated lot-sizing problems, distinguishing the form of the set-up and variable production costs, the inventory holding costs and the capacity pattern, respectively. Recent works dealing with this classification are Chung and Lin [11], van den Heuvel and Wagelmans [24], and Chen et al. [9]. For NP-hard cases of the CLSP, van Hoesel and Wagelmans [27] were the first to provide a fully polynomial approximation scheme (FPTAS). Later, faster implementations have been presented by Chubanov et al. [10] and Ng et al. [19] for certain particular cases on the costs and the capacity pattern. For surveys on the CLSP, we refer to Karimi et al. [17] and Brahimi et al. [6].

While the literature related to the lot-sizing problems is very large, papers dealing with both capacity constraints and more than one level, are scarce. Kaminsky and Simchi-Levi [16] analyzed a two-stage supply chain, with production capacity constraints in both stages and with costs consisting of a set-up charge and variable unit costs. By means of non-speculative assumptions on the costs, the authors were able to reduce the original problem to an equivalent single-stage model. For this equivalent problem, the authors provided an algorithm of $O(T^4)$ time. In the case of concave cost functions for the transportation and stationary capacity pattern, they were able to provide an algorithm of $O(T^8)$. In van Hoesel et al. [25] a serial lot-sizing problem is investigated with capacity constraints only in the first level, i.e., the production level. They provided a network flow formulation for the case of concave cost functions, and derived several properties related to the form of the optimal solutions. Based on a dynamic programming approach similar to that of Florian et al. [14], they also provided a polynomial time algorithm of $O(T^7)$ time for the problem. The authors showed that the algorithm can be easily adapted to cover positive initial inventories. For the general case of L levels and stationary capacity, a pseudopolynomial-time algorithm of $O(LT^{2L+3})$ time is developed. Finally, real polynomial-time algorithms are provided for the multi-level case with both fixed-charge transportation cost without speculative motives and linear transportation costs. We note that one of the three directions proposed for future research by the authors is the inclusion of capacity constraints on the distribution levels.

There are many papers, as the seminal works of Zangwill [28] and Crowston and Wagner [12], that were not considered in this review for one or more of the following reasons: 1) capacity constraints are not considered on both production and distribution simultaneously; 2) demand is assumed deterministic and stationary; 3) more restrictive cost functions are employed. We refer the reader to Robinson et al. [23] and Ben-Daya et al. [4] for recent overviews of this kind of problems. We especially wish to emphasize recent works dealing with the capacitated economic lot-sizing problem in different and important directions out of the scope of this present paper. Anily and Tzur [2] and Absi and Kedad-Sidhoum [1] consider multi-item and capacity vehicles for the distribution activity; Mitra [18] and Pan et al. [20] analyze the problem with product returns.

3. Problem Statement

We consider a dynamic multi-level and serial inventory system of a single item, with capacity constraints for all levels and a finite planning horizon of length $T > 0$. For each period $t = 1,...,T$, there is a known customer demand $D_t \geq 0$ which must be satisfied on time by a retailer in the last level, which in turn is served by another retailer in the preceding level until reaching the supplier in the first level. Backlogging demand is not allowed at any level. There are costs for carrying on activities (production or distribution) and for storing positive quantities at each period and level. Henceforth, we assume that all the cost functions are concave on the closed interval $[0,+\infty)$, and equal to zero when their argument is zero or negative. For ease of exposition, we refer as production quantities to the quantities of the first level, and distribution quantities for the quantities of the other levels. In turn, we refer as activity quantities to the quantities of any level. Finally, it is assumed that the initial inventory and the lead-time for the activities is equal to zero for all levels. The objective is determine the optimal quantities for all levels and periods, in order to meet the periodic demand requirements on time and fulfilling the capacity constraints. We refer to the problem as the Serial Capacitated Economic Lot-Sizing Problem, SCLSP. The components of the problem are summarized below:

- $T > 0$: Planning horizon length, with $T < +\infty$.
- $L > 0$: Number of levels, with $L < +\infty$.
- $D_t \geq 0$: Number of items demanded by the customer in period *t*, with $t = 1,\ldots,T$.
- $x_t^\ell \geq 0$: Number of items produced or distributed in period *t*, with $\ell = 1,\ldots,L$ and $t = 1,\ldots,T$.
- $0 < C_t^\ell < +\infty$: The capacity constraint in period *t* and level ℓ, with $\ell = 1,\ldots,L$ and $t = 1,\ldots,T$.
- $y_t^\ell \geq 0$: Inventory level of items during period *t* at level ℓ, with $\ell = 1,\ldots,L$ and $t = 0,1,\ldots,T$.
- $f_t^\ell(\cdot)$: Function cost for the activity at level ℓ and period *t*, with $\ell = 1,\ldots,L$ and $t = 1,\ldots,T$.
- $h_t^\ell(\cdot)$: Holding function cost at level ℓ for period *t*, with $\ell = 1,\ldots,L$ and $t = 1,\ldots,T$.

The problem described above can be formulated as the following Mixed Integer Linear Programming (MILP) problem:

$$\min \sum_{\ell=1}^{L} \sum_{t=1}^{T} \left\{ f_t^\ell(x_t^\ell) + h_t^\ell(y_t^\ell) \right\} \tag{P}$$

subject to :

$$y_t^\ell = y_{t-1}^\ell + x_t^\ell - x_t^{\ell+1}, \quad \forall\, \ell = 1,\ldots,L-1, \forall\, t = 1,\ldots,T \tag{1}$$

$$y_t^L = y_{t-1}^L + x_t^L - D_t, \quad \forall\, t = 1,\ldots,T \tag{2}$$

$$y_0^\ell = 0, \quad \forall\, \ell = 1,\ldots,L \tag{3}$$

$$x_t^\ell \le C_t^\ell, \quad \forall\, \ell = 1,\ldots,L, \forall\, t = 1,\ldots,T \tag{4}$$

$$x_t^\ell, y_t^\ell \ge 0, \quad \forall\, \ell = 1,\ldots,L, \forall\, t = 1,\ldots,T \tag{5}$$

Constraints (1) and (2) are the inventory equilibrium equations for the first (L – 1) levels and the final level L, respectively. Constraint (3) establishes that the initial inventory quantity must be zero for all levels. The capacity constraint for each level is given by (4). Finally, constraint (5) states the set of possible values for the decision variables.

Note that we can replace the decision variables y_t^ℓ and y_t^L by $(x_{1t}^\ell - x_{1t}^{\ell+1})$ and $(x_{1t}^L - D_{1t})$ respectively, where x_{ij}^ℓ denotes the accumulated activity quantities and D_{ij} the accumulated demand quantities between periods i and j with $1 \le i \le j \le T$ and $\ell = 1,\ldots,L$. Therefore, the problem formulated above reduces to the problem of finding feasible plans $x^\ell = (x_1^\ell,\ldots,x_T^\ell)$ for each level ℓ, in order to provide a complete solution $s = (x^1,\ldots,x^\ell,\ldots,x^L)$ of minimum cost of (P). We also note that the set of feasible solutions is not empty if and only if the accumulated demand of the first t periods does not exceed the accumulated capacities over these periods for any level ℓ. Formally:

$$\sum_{i=1}^{t} C_i^\ell \ge \sum_{i=1}^{t} D_i, \quad \forall\, \ell = 1,\ldots,L, \forall\, t = 1,\ldots,T \tag{6}$$

Therefore, from now on we assume that expression (6) is fulfilled. Since the objective function of (P) is a concave function and the constraints (1) – (5) define a closed bounded convex set, there is an optimal solution of the SCLSP that is an extreme point of this set. On the other hand, without loss of generality we can assume that the different plans of a feasible solution $s = (x^1,\ldots,x^\ell,\ldots,x^L)$ of the SCLSP are composed by subplans $S_{ij}^\ell = (x_i^\ell,\ldots,x_j^\ell)$ called *sequences* such that $y_i^\ell = y_j^\ell = 0$ and $y_t^\ell > 0$, for all t in $0 \le i < t < j \le T$ and $\ell = 1,\ldots,L$. Periods i and j are commonly referred as *regeneration points*.

For the CLSP, Florian and Klein [13] showed that the extreme-point solutions are composed only by sequences for which the production quantities of the periods are zero or equal to the capacity, except in at most one period, which is called the *fractional period*. This kind of sequences are known as *capacity constrained sequences*. Based on this property, the authors proposed an $O(T^4)$ algorithm for solving the CLSP under the assumption of stationary capacity-pattern. Unfortunately, this property is not true for the SCLSP with more than one level, as we will see further in Section 4.3. This means that for the SCLSP in general, i.e. with any number of levels, it may be that the plans of an extreme-point solution are not composed only by capacity constrained sequences. Hence, it is unlikely that we can develop an efficient algorithm for solving the SCLSP in general. However, in the following section we introduce an additional assumption on the cost functions through which we can conclude that there is an optimal solution of the SCLSP such that all the plans are composed only by a new kind of capacity constrained sequences.

4. The optimal solutions of the SCLSP under particular assumptions

In this section we present some properties related to the optimal solutions of the SCLSP under particular assumptions on the cost functions and the capacity pattern. We say that the cost functions of the SCLSP are *non-speculative with respect to the transfer* if it is profitable to transfer all or part of the production or distribution quantity from one positive period to another future period in the planning horizon. Formally, the cost functions are non-speculative with respect to the transfer when the expressions below are fulfilled for all $1 \le \ell \le L$ and $1 \le k \le t \le T$.:

$$\begin{aligned} f_k^\ell(a) + \sum_{i=k}^{t-1} h_i^\ell(b_i) + \sum_{i=k}^{t-1} h_i^{\ell-1}(d_i) \ge \\ \sum_{i=k}^{t-1} h_i^\ell(b_i - a) + \sum_{i=k}^{t-1} h_i^{\ell-1}(d_i + a) + f_t^\ell(a), \quad \text{with } a > 0 \end{aligned} \tag{7.1}$$

$$\begin{aligned} f_k^\ell(a) + \sum_{i=k}^{t-1} h_i^\ell(b_i) + \sum_{i=k}^{t-1} h_i^{\ell-1}(d_i) + f_t^\ell(c) \ge \\ f_k^\ell(a-\varphi) + \sum_{i=k}^{t-1} h_i^\ell(b_i - \varphi) + \sum_{i=k}^{t-1} h_i^{\ell-1}(d_i + \varphi) + f_t^\ell(c+\varphi), \quad \text{with } a,c > 0, 0 < \varphi \le a \end{aligned} \tag{7.2}$$

To interpret these conditions, note that in (7.1) all the production is transferred from one period to other future period that was inactive, while in (7.2) all or part is transferred forward between two positive periods. Expressions (7.1) and (7.2) are fulfilled in different settings of practical interest; one example is when all the costs involved are stationary and defined as follows:

$$f_t^\ell(x) = K^\ell + c^\ell x, \text{ with } K^\ell > 0, c^\ell \geq 0 \text{ and } 1 \leq \ell \leq L \tag{8}$$

$$h_t^\ell(y) = h^\ell.y, \text{ with } h^\ell \geq 0 \text{ and } 1 \leq \ell \leq L \tag{9}$$

$$h^\ell \geq h^{\ell-1} \geq 0, \text{ with } 1 \leq \ell \leq L \text{ and } h^0 = 0 \tag{10}$$

We note that constraint (10) can be justified as value is added in the lower levels of the supply chain (Kaminsky and Simchi-Levi [16]).

Proposition 1. Assume that the cost functions of the SCLSP are non-speculative with respect to the transfer. Then, the solutions of the SCLSP for which the plans of the different levels consist only of capacity constrained sequences are *dominant*, i.e., given a feasible solution of the SCLSP for which there is at least one plan that is not composed by capacity constrained sequences, we can determine a new feasible solution with all plans composed only by capacity constrained sequences with at most the same cost as the original.

Proof: Let $s = \left(x^1,\dots,x^\ell,\dots,x^L\right)$ be a feasible solution of the SCLSP. Without loss of generality, suppose that all the plans of *s* are composed only by capacity constrained sequences except by x^ℓ with only one sequence $S_{\alpha\beta}^\ell$ which is not capacity constrained. In order to determine a new feasible solution $z = \left(x^1,\dots,x^{\ell-1},z^\ell,x^{\ell+1},\dots,x^L\right)$ composed only by capacitated constrained sequences we proceed as follows. For each consecutive pair of periods *i* and *j* of $S_{\alpha\beta}^\ell$, with $\alpha \leq i < j \leq \beta$, such that $0 < x_i^\ell < C_i^\ell$ and $0 < x_j^\ell < C_j^\ell$, define $\varepsilon = \min\left\{x_i^\ell, y_i^\ell, y_{i+1}^\ell,\dots,y_{j-1}^\ell, C_j^\ell - x_j^\ell\right\}$. The new plan for level ℓ is obtained as follows:

$$\begin{cases} z_i^\ell = x_i^\ell - \varepsilon & (11) \\ z_j^\ell = x_j^\ell + \varepsilon & (12) \\ z_t^\ell = x_t^\ell, \quad i < t < j & (13) \end{cases}$$

The new plan z^ℓ consists only of capacity constrained sequences, since at least one of the three following cases is fulfilled: 1) $z_i^\ell = 0$; 2) $z_j^\ell = C_j^\ell$; or 3) $y_t^\ell = 0$ for some *t* in $i \leq t < j$. We note that if case 1) is fulfilled, i.e. $z_i^\ell = 0$, the original sequence $S_{\alpha\beta}^\ell$ is decomposed into two new capacity constrained sequences $S^{\ell}_{\alpha i}$ and $S^{\ell}_{i\beta}$. As we are transferring all or part of the activity quantity from period *i* to the future period *j*, by (7.1) and (7.2) we have that the total cost of the new solution *z* is less or equal than the original solution *s*. Thereby, we have constructed other feasible solution with at most the same cost as the original, where in addition all the plans are composed only by capacity constrained sequences. ■

Proposition 1 states that we can focus on those SCLSP solutions composed only by capacity constrained sequences for all levels in order to find an optimal solution, whenever the costs involved are non-speculative with respect to the transfer. Although this last assumption reduces the number of solutions that we must consider, it is not enough to develop an efficient algorithm. As we mentioned earlier, the CLSP is NP-hard for the case of arbitrary capacity patterns (Florian et al. [14]). Nevertheless, the CLSP is polynomial-time solvable in the case of stationary capacity and concave cost functions (Florian and Klein [13]). Hence, from this point on, for the remainder of the paper we consider a stationary capacity-pattern for all levels, i.e. $C^{\ell}_{t} = C^{\ell}$ for all $\ell = 1,\dots,L$ and $t = 1,\dots,T$.

If we add the assumption of stationary capacity, the result of Proposition 1 can be extended to a new kind of sequences. We say that a sequence is an *ascending capacity constrained sequence* (ACC sequence) whenever the period with a positive quantity below capacity, if it exists, is the first among the positive periods in the sequence.

Proposition 2. Assume that the cost functions of the SCLSP are non-speculative with respect to the transfer and the capacity pattern is stationary, i.e. $C^{\ell}_{t} = C^{\ell}$, for all levels $\ell = 1,\dots,L$ and periods $t = 1,\dots,T$. Then, the solutions of the SCLSP with plans composed only by ACC sequences are dominant.

Proof. Consider a feasible solution $s = \left(x^{1},\dots,x^{\ell},\dots,x^{L}\right)$ of the SCLSP with plans composed only by capacity constrained sequences. By Proposition 1 we know that this kind of solutions exists. Without loss of generality, suppose that there is only one level ℓ with only one sequence $S^{\ell}_{\alpha\beta}$ that is not an ACC sequence, i.e., there are at least two consecutive periods *i* and *j* such that $C^{\ell} = x^{\ell}_{i} > x^{\ell}_{j} > 0$ with $\alpha \le i < j \le \beta$ and $1 \le \ell \le L$.

Then, we can determine a new feasible solution *z* equal to *s* except by the plan of level ℓ, defining $\varepsilon = \min\{y_i^\ell, y_{i+1}^\ell, \ldots, y_{j-1}^\ell, C^\ell - x_j^\ell\}$ and by means of expressions (11) – (13) given in Proposition 1. Then, at least one of the two following cases is fulfilled: 1) $z_j^\ell = C^\ell$; or 2) $y_t^\ell = 0$, for some *t* in $i < t < j$. If case 1) is fulfilled, then the activity quantity of period *i* in the new solution is below capacity, i.e., $z_i^\ell < C^\ell$. If period *i* is not the first positive period in the sequence, we determine a new ε for period *i* and the immediately previous period *k* of the sequence such that $C^\ell = x_k^\ell > x_i^\ell > 0$, with $\alpha \le k < i \le \beta$. If case 2) is fulfilled, we note that the sequence $S_{\alpha\beta}^\ell$ has been decomposed into two new sequences $S_{\alpha t}^\ell$ and $S_{t\beta}^\ell$ for some *t* in $i \le t < j$. Note that sequence $S_{t\beta}^\ell$ is an ACC sequence, since all the positive periods are at capacity. We also note that period *i* may be the only period below capacity in the ACC sequence $S_{\alpha t}^\ell$. If this is the case, we proceed as we explained for case 1) for period *i* and the immediately previous period *k* of the sequence. Since we are always transferring a production or distribution quantity from one period to another future period in the sequence, and we are assuming the costs are non-speculative with respect to the transfer, by (7.1) and (7.2) the total cost of the new solution *z* is less or equal than the original solution *s*. Thereby, we have constructed another feasible solution with at most the same cost as the original with all plans composed only by ACC sequences for all levels. ■

5. The algorithm for the SCLSP under particular assumptions

In this section we describe a recursive algorithm suggested for optimally solving the SCLSP under the assumptions that the costs are non-speculative with respect to the transfer and that the capacity-pattern is stationary. By Proposition 2 of Section 4, we know that there is an optimal solution of the SCLSP that is composed only by ACC sequences. Hence, we provide the algorithm in terms of this kind of sequences. The algorithm has four inputs: the level to process ℓ; the requirements vector $R = (r_1, \ldots, r_T)$; the on-going solution $X = (x^{\ell+1}, \ldots, x^L)$; and the best solution up to the moment denoted by S_{opt}. If the level to be processed is the first one, i.e., $\ell = 1$, then a new solution is constructed and evaluated in order to determine the optimal. Otherwise, Ψ_R^ℓ, the set of all feasible plans composed only by ACC sequences for the current level ℓ and requirements R, is determined. Then, for each plan x^{ℓ} of Ψ_R^{ℓ}, the algorithm is recursively invoked with a level equal to $\ell - 1$, the requirements x^{ℓ}, the new on-going solution $X = (x^{\ell}, x^{\ell+1}, \ldots, x^{L})$ and S_{opt}.

```
SCLSP_Alg(ℓ, R, X, S_opt):
  If ℓ = 1 {
    Determine Ψ_R^1;
    For each x^1 ∈ Ψ_R^1 {
      If F(x^1 ⊕ X) < F(S_opt) {
        S_opt = x^1 ⊕ X;
      };
    };
  }
  Else {
    Determine Ψ_R^{ℓ-1};
    For each x^{ℓ-1} ∈ Ψ_R^{ℓ-1} {
      X = x^{ℓ-1} ⊕ X;
      SCLSP_Alg(ℓ-1, x^{ℓ-1}, X, S_opt);
    };
  };
```

Figure 1: Algorithm for solving the SCLSP

A sketch of the algorithm, named SCLSP_Alg, is given in Figure 1. In the pseudo-code presented, $F(\cdot)$ is the cost of a SCLSP solution, with $F(\{\}) = \infty$, and the operator $\oplus$ takes a vector x and a vector of vectors $(y_1, \ldots, y_n)$ and returns the vector of vectors $(x, y_1, \ldots, y_n)$. In order to solve a SCLSP instance we need to compute $\text{SCLSP\_Alg}(L+1, D, \{\}, \{\})$. Assuming that the time for computing the cost functions is constant, the running time of the algorithm depends on the number of plans at each level that we must consider. We analyze this issue in the following section.

5.1. Determining the plans composed only by ACC sequences

First we note that all the feasible ACC sequences that we must consider at each level can be determined in $O(T^2)$ time, as it is argued in Florian et al. [14]. Secondly, by Florian and Klein [13], we know that for any capacity constrained sequence $S_{ij}^{\ell} = (x_i^{\ell}, x_{i+1}^{\ell}, \ldots, x_j^{\ell})$, there are K periods at capacity, at most one positive period below capacity and the remaining periods equal to zero, with $x_i^{\ell} + \ldots + x_j^{\ell} = R_{ij} = K.C + \varepsilon$, where R is the requirement vector imposed for the consecutive level $(\ell + 1)$ or the demand vector

D if $\ell = L$. The authors proposed an acyclic network in order to find the feasible capacity constrained sequences between any pair of periods, which results in a $O(T^2)$ time procedure. Nevertheless, in order to determine the ACC sequences, we can develop a simpler and faster procedure, as we show below.

For a given pair of periods *i* and *j* at any level ℓ, we must determine an ACC sequence $A_{ij}^{\ell} = (x_i^{\ell},...,x_j^{\ell})$ fulfilling the requirements $R_{ij} = (R_i,...,R_j)$ with $R_{ij} = K.C + \varepsilon$. Without loss of generality, we assume that $R_i > 0$. If $0 < \varepsilon < C$, then it must be $x_i^{\ell} = \varepsilon$, otherwise $x_i^{\ell} = C^{\ell}$, and the remaining positive periods at capacity. The next one will be the earliest period *t* such that $R_{it} > x_i^{\ell}$, with $i < t \leq j$. We continue until all the *K* positive periods at capacity have been reached. If $x_i^{\ell} = \varepsilon < R_i$ or for some period *t* we have that $R_{it} = x_{it}^{\ell}$, then there is not a feasible ACC sequence between periods *i* and *j*. As we are producing or distributing as late as possible, by (7.1) the ACC sequence obtained is optimal. Note that there is at most only one ACC sequence between any pair of periods.

With a short analysis we can see that the procedure described above is linear in the number of periods and independent of the requirements. Thus, the set of all feasible plans composed only by ACC sequences, Ψ_R^{ℓ}, can be determined in $O(T^3)$ for each level $\ell = 1,...,L$ and any vector of requirements *R*. Therefore, the running time of the algorithm SCLSP_Alg is $O(T^{3L})$.

5.2. An example of applying the algorithm

We show the algorithm suggested above by solving the following SCLSP instance of three levels and four periods, i.e. $L = 3$ and $T = 4$. Cost functions, demand and capacities values are based on those given in Florian and Klein [13] for the CSLP. The cost functions are defined as follows:

$$f_t^{\ell}(x_t) = \delta(x_t)(6 - t)\ell + [5 + 2(3 - \ell)]x_t, \quad t = 1,2,3,4,\ \ell = 1,2,3 \qquad (14)$$

$$h_t^{\ell}(x_t) = tx_t, \quad t = 1,2,3,4,\ \ell = 1,2,3 \qquad (15)$$

where $\delta(x) = 1$ if $x > 0$ and $\delta(x) = 0$ otherwise. We note that the costs are non-speculative with respect to the transfer, since (7.1) and (7.2) are fulfilled. The vector demand is

$D = (3,6,8,3)$ and the capacity constraints are $C^1 = 11$, $C^2 = 9$ and $C^3 = 7$ for periods 1, 2 and 3 respectively. We must compute $\text{SCLSP\_Alg}(4,(3,6,8,3),\{\},\{\})$. The algorithm starts by processing the last level, i.e. $\ell = L = 3$. The only plans to be considered in this case are (6,7,7,0) and (3,7,7,3) which correspond to the plans composed by the sequences $\{A_{04}\}$ and $\{A_{01}, A_{13}, A_{34}\}$, respectively. Each of these plans for level 3 are the requirements for level 2, which in turn are the requirements for level 1. In Table 1 we show the different sets of plans composed only by ACC sequences for each one of the levels that the algorithm explores. The cost of the corresponding SCLSP solution is provided in the last column.

$\ell = 3, C^3 = 7$	$\ell = 2, C^2 = 9$	$\ell = 1, C^1 = 11$	Cost
(6,7,7,0)	(6,7,7,0)	(9,11,0,0)	586
		(6,7,7,0)	596
(3,7,7,3)	(8,9,0,3)	(9,11,0,0)	585
		(8,9,0,3)	587
	(3,8,9,0)	(9,11,0,0)	588
		(11,0,9,0)	**563**
		(3,8,9,0)	591
	(3,7,7,3)	(9,11,0,0)	594
		(6,11,0,3)	594
		(10,0,10,0)	566
		(10,0,7,3)	575
		(3,7,10,0)	595
		(3,7,7,3)	604

Table 1. The ACC sequences for the example and its costs

The optimal SCLSP solution found by the algorithm is $(x^1, x^2, x^3) = ((11,0,9,0),(3,8,9,0),(3,7,7,3))$ with an optimal value equal to 563 (values marked in bold in Table 1). We note that the solution $(x^1, x^2, x^3) = ((11,0,9,0),(4,7,9,0),(4,7,6,3))$ is also optimal. However, the plans of levels 2 and 3 are not composed by ACC sequences. As we pointed out in Section 3, the extreme-point solutions of the SCLSP are not always composed by capacity constrained sequences.

6. Conclusions and future research

In this paper we analyze the serial capacitated economic lot-sizing problem of a single-item and finite planning-horizon, briefly SCLSP. We show that when the costs are non-speculative with respect to the transfer and the capacity pattern is stationary, we can optimally solve the SCLSP with *L* levels in $O(T^{3L})$ time by means of a recursive algorithm. Thus, we improve the running time of the algorithm of Florian and Klein [13] for the single-level case from $O(T^4)$ to $O(T^3)$ for this particular case. The algorithm is supported by the fact that there is an optimal solution of the SCLSP whose plans are composed exclusively by a new kind of sequences that we named ascending constrained capacity sequences (ACC sequences). For this kind of capacity constrained sequences, the only fractional period, if it exists, is the first one of all the positive periods of the sequence.

Extensions of this work may be based on relaxing one or more of the assumptions like not allowing backlogging, supposing instantaneous lead-times or zero initial inventory levels. Other interesting extensions to be tackled are the cases of multi-item and/or multi-customer. For the multi-item case, a possible starting point may be to relax the assumptions on the costs structure considered by Anily and Tzur [2]. For the multi-customer case, several works have appeared in the literature, but with more restrictive assumptions that those assumed in the present paper. We are also interested in extending this work by considering product returns and remanufacturing (Piñeyro and Viera [21] and [22]). These future researches may also include other cost functions assumptions or different capacity patterns, e.g. speculative motives or nondecreasing capacities. We note that for the case of speculative motives it is not clear how to transfer the production or distribution quantities from one period to another one in order to obtain a feasible sequence with only one fractional period. On the other hand, it is interesting to explore if there is a more efficient algorithm in the case of simpler functions, e.g., linear cost functions.

With respect to the solution method, it may be interesting to propose and evaluate heuristics procedures for the SCLSP that allow to reduce the computing time, while maintaining good quality of the solution.

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