Optimal stopping of Brownian motion with broken drift

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Abstract

We solve an optimal stopping problem where the underlying diffusion is Brownian motion on \mathbf{R} with a positive drift changing at zero. It is assumed that the drift μ_1 on the negative side is smaller than the drift μ_2 on the positive side. The main observation is that if $\mu_2 - \mu_1 > 1/2$ then there exists values of the discounting parameter for which it is not optimal to stop in the vicinity of zero where the drift changes. However, when the discounting gets bigger the stopping region becomes connected and contains zero. This is in contrast with results concerning optimal stopping of skew Brownian motion where the skew point is for all values of the discounting parameter in the continuation region.

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1 Introduction

1.1 Motivation. Optimal stopping problems for one-dimensional diffusions is a much studied topic and there exists a variety of methods for finding solutions. During recent decade some interest has been focused on cases where the underlying diffusion has exceptional points such as: sticky points; skew points; discontinuities in the infinitesimal variance and/or drift. The main observation in the presence of sticky points is that the smooth fit does not in general hold (see for instance Crocce [8], Crocce and Mordecki [9], and Salminen and Ta [22]). If the diffusion has skew points then it is possible that in the vicinity of such a point it is not optimal to stop for any value of the discounting parameter, as was found by Álvarez and Salminen [1] and also by Presman [19]. Explicit solutions of optimal stopping problems when the underlying diffusion has discontinuous coefficients have not, to our best knowledge, been encountered in the literature. Many of the methods to solve optimal stopping problems do not, however, exclude such diffusions. This is, in particular, the case in the approach based on excessive functions, see, e.g., Salminen [21], Dayanik and Karatzas [10], Christensen and Irle [5], and Crocce and Mordecki [9]. Also the approach via variational inequalities and free boundary problems, see Lamberton and Zervos [12] and Ruschendorf and Urusov [20], does not seem to require continuity of the diffusion coefficients. However, our main motivation for the present work is to study how the stopping set changes as a function of the discounting parameter when, in our case, the drift is discontinuous.

1.2 Diffusions and the optimal stopping problem. Consider a non-terminating and regular one-dimensional (or linear) diffusion $X = \{X_t : t \ge 0\}$ in the sense of Itô and McKean [11] (see also Borodin and Salminen [4]). The state space of X is denoted by I, an interval of the real line **R** with left endpoint $a = \inf I$ and right endpoint $b = \sup I$, where $-\infty \le a < b \le \infty$. The notations m and S are used for the speed measure and the scale function, respectively, of X. Moreover, for $r \ge 0$ let φ_r and ψ_r denote the decreasing and increasing, respectively, fundamental solutions of the generalized ODE (see [4] II.10 p.18)

$$\frac{d}{dm}\frac{d}{dS}u = ru.$$
(1)

Let \mathbb{P}_x stand for the probability measure associated with X when starting from x, and by \mathbb{E}_x the corresponding mathematical expectation. Denote by \mathcal{M} the set of all stopping times with respect to $\{\mathcal{F}_t : t \geq 0\}$, the usual augmentation of the natural filtration generated by X. Given a non-negative lower semicontinuous reward function $g: \mathbf{I} \to \mathbf{R}$ and a discount factor $r \geq 0$, consider the optimal stopping problem consisting of finding a function V_r and a stopping time $\tau^* \in \mathcal{M}$, such that

$$V_r(x) = \mathbb{E}_x[e^{-r\tau^*}g(X_{\tau^*})] = \sup_{\tau \in \mathcal{M}} \mathbb{E}_x[e^{-r\tau}g(X_{\tau})].$$
(2)

The value function V_r and the optimal stopping time τ^* constitute the solution of the problem.

1.2 References. After the classical works of McKean [13], Taylor [24] and Merton [14] there has been, in recent times, an important effort to characterize optimal stopping problems with one sided solutions, i.e. such that the optimal stopping time is a threshold stopping time, usually of the form

$$\tau^* = \inf\{t \ge 0 \colon X(t) \ge x^*\},\$$

for some critical x^* . In this situation, we say that the problem has a *one-sided* solution. Villeneuve [25] gives sufficient conditions to have threshold optimal strategies, Arkin [2] gives necessary and sufficient conditions for Itô diffusions with C^2 payoffs functions to have one sided solutions, whereas Arkin and Slastnikov [3] and Crocce and Mordecki [9] give also necessary and sufficient conditions in different and more general diffusion frameworks. For more general Markov processes Mordecki and Salminen [15], Christensen et al. [7], and Christensen and Irle [6] propose verification results for one sided solutions, but also for problems where the optimal stopping time is of the form

$$\tau^* = \inf\{t \ge 0 \colon X(t) \notin (x_*, x^*)\}.$$

In this second situation is it is said that the problem has a *two sided* solution. For general reference of the theory of optimal stopping, see the books by Shiryaev [23] and by Peskir and Shiryaev [18].

1.3 Present study. In this paper we are interested to understand situations where the stopping region is disconnected due to the behavior of the underlying stochastic process and not due to the properties of the payoff function. Such a case has been found in [1] with skew Brownian motion as the underlying. The idea is to study the shape of the continuation set as a function of the discount parameter r. To depart, we consider a model where the solution of the optimal stopping problem is trivial in cases r = 0 (no stopping) and $r = \infty$ (immediate stopping), and describe the stopping set as a function of r as r increases from 0 to ∞ . When isolated continuation intervals appear when r increases we say that a *shape transition* occurs. Such intervals are called *bubbles*, for the definition, see Section 2

The rest of the paper is organized as follows. In Section 2 we present some preliminary general results mainly on optimal stopping for diffusions. In Section 3 the Brownian motion with broken drift as the solution of a stochastic differential equation is introduced and its main characteristics are analyzed. In Section 4 we solve the optimal stopping problem of the Brownian motion with broken drift with reward $x \mapsto (1 + x)^+$. In particular, it is seen that if the discontinuity in the drift is big enough a shape transition in the continuation region occurs.

2 Preliminary results

From the general theory of Markovian stopping problems, the optimal stopping time τ^* in (2), if such a time exists, can be characterized (see Theorem 3, Section 3.3 in [23]) as the first entrance time into the stopping set

$$\mathbf{S}(r) = \{ x \in \mathbf{I} \colon V_r(x) = g(x) \},\tag{3}$$

where we have indicated the dependence on the discounting parameter $r \ge 0$. The complement of $\mathbf{S}(r)$, i.e., $\mathbf{C}(r) = \mathbf{I} \setminus \mathbf{S}(r)$ is called the continuation set.

Proposition 1. Let $0 \le r_1 < r_2$ be two discounting parameters and consider the corresponding OSPs as given in (2). Then

$$V_{r_1}(x) \ge V_{r_2}(x) \quad \text{for all } x \in \mathbf{I},\tag{4}$$

and

$$\mathbf{S}(r_1) \subseteq \mathbf{S}(r_2). \tag{5}$$

Proof. Let τ be a fixed stopping time. Then

$$e^{-r_1\tau}g(X_{\tau}) \ge e^{-r_2\tau}g(X_{\tau}),$$

and taking the suprema yields (4). Using the characterization of the stopping set in (3) we obtain now that if $x \in \mathbf{S}(r_1)$ then $x \in \mathbf{S}(r_2)$ resulting to (5).

We recall next the smooth fit theorem from [21], [17], [22]. To fix ideas and to focus on the case studied below, the theorem is formulated here for a left boundary point of the stopping region S(r).

Theorem 1. Let z be a left boundary point of $\mathbf{S}(r)$, i.e., $[z, z + \varepsilon_1) \subset \mathbf{S}(r)$ and $(z - \varepsilon_2, z) \subset \mathbf{C}(r)$ for some positive ε_1 and ε_2 . Assume that the reward function g and the fundamental solutions φ_r and ψ_r are differentiable at z. Then the value function V_r in (2) is differentiable at z and it holds V'(z) = g'(z).

Since the value function V is bigger than the reward g on the continuation set C(r) we introduce the following terminology.

Definition 1. A bounded open intervall $(x_1, x_2) \subseteq \mathbf{C}(r)$ is called a bubble if $x_1, x_2 \in \mathbf{S}(r)$.

In [1] it is seen that for skew Brownian motion and a large class of reward functions one can find a lower bound r_0 for the discounted parameter r such that for all $r \ge r_0$ there is a bubble (containing the skew point). This is, in particular, true for the reward function $g(x) = (1+x)^+$, i.e., no matter how big is the discounting it is not optimal to stop at the skew point. In the present paper we study the appearance and the disappearance of a bubble for a Brownian motion with positive drift changing at the origin. We define this process - which we call a Brownian motion with broken drift - in the next section.

To make the presentation more self-contained, we display a result from [1], see Lemma 2 therein, which is used to verify that a candidate solution of OSP (2) is indeed the value function. This is essentially Corollary on p. 124 in [23].

Proposition 2. Let $A \subset \mathcal{I}$ be a nonempty Borel subset of \mathcal{I} and

$$H_A = \inf\{t \ge 0 : X_t \in A\}.$$

Assume that the function

$$\hat{V}(x) := \mathbb{E}_x \left[e^{-r H_A} g(X_{H_A}) \right]$$

is *r*-excessive and dominates g. Then \hat{V} coincides with the value function of OSP (2) and H_A is an optimal stopping time.

The following technical result is needed in the proof of Proposition 8.

Proposition 3. Let $h_{r_1}, r_1 > 0$, be an r_1 -excessive function and $\{h_r; r_1 \le r < r_2\}, r_1 < r_2$, a sequence of functions such that h_r is r-excessive, Assume that $h_r \le h_{r_1}$ for $r \ge r_1$ and $\lim_{r\uparrow r_2} h_r(x) =: h_{r_2}(x)$ exists for all x. Then h_{r_2} is r_2 -excessive.

Proof. Consider

$$\mathbb{E}_x \left[e^{-r_2 t} h_{r_2}(X_t) \right] = \mathbb{E}_x \left[\lim_{r \uparrow r_2} e^{-rt} h_r(X_t) \right]$$
$$= \lim_{r \uparrow r_2} \mathbb{E}_x \left[e^{-rt} h_r(X_t) \right]$$
$$\leq \lim_{r \uparrow r_2} h_r(x)$$
$$= h_{r_2}(x),$$

where in the second step we use the dominated convergence theorem which is applicable since $e^{-rt}h_r \le e^{-r_1t}h_{r_1}$ and

$$\mathbb{E}_x\left[\mathrm{e}^{-r_1t}h_{r_1}(X_t)\right] \le h_{r_1}(x) < \infty.$$

3 Brownian motion with broken drift

Consider a diffusion

$$X(t) = x + \int_0^t \mu(X(s))ds + W(t),$$

where

$$\mu(x) = \begin{cases} \mu_1, & \text{ for } x < 0, \\ \mu_2, & \text{ for } x \ge 0, \end{cases}$$

and $0 \le \mu_1 < \mu_2$. The speed measure of this diffusion is given by

$$m(dx) = \begin{cases} 2e^{2\mu_1 x} dx, & \text{for } x < 0, \\ 2e^{2\mu_2 x} dx, & \text{for } x > 0, \end{cases}$$

while the scale function is

$$S(x) = \begin{cases} \frac{1}{2\mu_1} (1 - e^{-2\mu_1 x}), & \text{for } x < 0, \\ \frac{1}{2\mu_2} (1 - e^{-2\mu_2 x}), & \text{for } x \ge 0. \end{cases}$$

We call the diffusion $\{X_t : t \ge 0\}$ a Brownian motion with broken drift, and remark that in the literature one can also find a diffusion called the Brownian motion with alternating drift (also the bang-bang Brownian motion), see [4] p. 128 and references therein. Notice that the scale function is differentiable everywhere with the derivative

$$S'(x) = \begin{cases} e^{-2\mu_1 x}, & \text{for } x < 0, \\ e^{-2\mu_2 x}, & \text{for } x \ge 0. \end{cases}$$

We find next the fundamental solutions for r > 0. Introduce

$$\begin{aligned} \lambda_1^- &= -\sqrt{\mu_1^2 + 2r - \mu_1} < 0, \qquad \lambda_1^+ = \sqrt{\mu_1^2 + 2r - \mu_1} > 0, \\ \lambda_2^- &= -\sqrt{\mu_2^2 + 2r} - \mu_2 < 0, \qquad \lambda_2^+ = \sqrt{\mu_2^2 + 2r} - \mu_2 > 0. \end{aligned}$$

The decreasing fundamental solution is

$$\varphi_r(x) = \begin{cases} A_1 \exp(\lambda_1^- x) + A_2 \exp(\lambda_1^+ x), & \text{for } x < 0, \\ \exp(\lambda_2^- x), & \text{for } x \ge 0, \end{cases}$$
(6)

where the constants A_1 and A_2 are determined so that φ_r is differentiable at 0. Hence,

$$A_1 = \frac{\lambda_1^+ - \lambda_2^-}{\lambda_1^+ - \lambda_1^-} = \frac{\lambda_1^+ - \lambda_2^-}{2\sqrt{\mu_1^2 + 2r}} > 0, \qquad A_2 = \frac{\lambda_2^- - \lambda_1^-}{\lambda_1^+ - \lambda_1^-} = \frac{\lambda_2^- - \lambda_1^-}{2\sqrt{\mu_1^2 + 2r}} < 0.$$

Observe that $A_1 + A_2 = 1$. Analogously, the increasing solution is

$$\psi_r(x) = \begin{cases} \exp(\lambda_1^+ x), & \text{for } x < 0, \\ B_1 \exp(\lambda_2^+ x) + B_2 \exp(\lambda_2^- x), & \text{for } x \ge 0 \end{cases}$$
(7)

with

$$B_1 = \frac{\lambda_1^+ - \lambda_2^-}{\lambda_2^+ - \lambda_2^-} = \frac{\lambda_1^+ - \lambda_2^-}{2\sqrt{\mu_2^2 + 2r}} > 0, \qquad B_2 = \frac{\lambda_2^+ - \lambda_1^+}{\lambda_2^+ - \lambda_2^-} = \frac{\lambda_2^+ - \lambda_1^+}{2\sqrt{\mu_2^2 + 2r}} < 0.$$

The above stated properties $A_1 > 0, B_1 > 0$, and $A_2 < 0$ are easily seen from the explicit expressions for $\lambda_i^+, \lambda_i^-, i = 1, 2$. For $B_2 < 0$ notice that $\mu \mapsto \sqrt{\mu^2 + 2r} - \mu$ is decreasing when $\mu > 0$. In Figure 1 we have visualized φ_r and ψ_r when $r = 3, \mu_1 = 1, \mu_2 = 10$.



Figure 1: The functions ψ_r (left) and φ_r (right), for r = 3, $\mu_1 = 1$, $\mu_2 = 10$.

4 Solution of the optimal stopping problem

We analyze the optimal stopping problem (2) for the broken-drift diffusion introduced above and the reward function $g(x) = (1 + x)^+$, i.e.,

$$\sup_{\tau \in \mathcal{M}} \mathbb{E}_x \left[e^{-r\tau} g(X_\tau) \right].$$
(8)

The main issue is to show that there are values on μ_1 and μ_2 such that, for some values on r, the continuation region is disconnected and contains 0, i.e., the point where the drift changes. However, letting here r to increase makes the continuation

region connected and then 0 is in the stopping set. This situation is different from the one studied in [1] where the skew point 0 is for all values of r and the skewness parameter $\beta > 1/2$ in the continuation set.

Clearly, it follows from the assumption $0 \le \mu_1 < \mu_2$ that $X(t) \to +\infty$ as $t \to +\infty$. Consequently, to make OSP (2) non-trivial we assume that r > 0. Because g(x) = 0 for $x \le -1$ it holds $(-\infty, -1) \subset \mathbf{C}(r)$. Notice also that the smooth fit theorem applies for all values of r, i.e., the value function meets the reward smoothly at every boundary point between $\mathbf{C}(r)$ and $\mathbf{S}(r)$.

For the analysis to follow, we define (cf. [21])

$$G_{-}(x) := \left(\psi_{r}'(x)(x+1) - \psi_{r}(x)\right) / S'(x)$$
(9)

$$G_{+}(x) := \left(\varphi_{r}(x) - (1+x)\varphi_{r}'(x)\right) / S'(x), \tag{10}$$

and their derivatives with respect to the speed measure for x > -1 and $x \neq 0$ are given by

$$G'_{-}(x) = m(x)\frac{d}{dm}G_{-}(x) = m(x)\psi_{r}(x)\left(r(1+x) - \frac{d}{dm}\frac{d}{dS}(1+x)\right)$$
$$= m(x)\psi_{r}(x)\begin{cases} r(1+x) - \mu_{1}, & x < 0, \\ r(1+x) - \mu_{2}, & x > 0, \end{cases}$$
(11)

and, similarly,

$$G'_{+}(x) = m(x)\varphi_{r}(x) \begin{cases} \mu_{1} - r(1+x), & x < 0, \\ \mu_{2} - r(1+x), & x > 0, \end{cases}$$
(12)

where we have used that the fact that φ_r and ψ_r solve the differential equation

$$\frac{d}{dm}\frac{d}{dS}u = ru$$

The functions G_{-} and G_{+} are used to check the excessivity of a proposed value function. An alternative way is to evoke the generalization of the Ito formula developed in Peskir [16].

Proposition 4. In case $0 < r \le \mu_1 \le \mu_2$ the continuation region for OSP (8) is given by

$$\mathbf{C}(r) = (-\infty, c),$$

where c = c(r) > 0 is the unique solution of the equation

$$\psi_r'(x)(x+1) - \psi_r(x) = 0. \tag{13}$$

Proof. We show first that equation (13) has a unique positive solution. For this consider for x > -1 the function G_- defined in (9). Since S'(x) > 0 for all x the claim is equivalent with the statement that G_- has a unique positive zero. In fact, we prove a bit more; namely that G_- attains the global minimum at $x_0 := (\mu_2 - r)/r > 0$, is negative and decreasing for $x \le x_0$, is increasing for $x > x_0$, and has, therefore, a unique zero. Analyzing G'_- as given in (11), it is straightforward to deduce, since $0 < r \le \mu_1 \le \mu_2$, the claimed properties of G_- . Let

$$H_c := \inf\{t : X_t \ge c\},\$$

where c is the unique solution of (13), and define

$$\widehat{V}(x) := \mathbb{E}_x \left[e^{-rH_c} g(X_{H_c}) \right] = \begin{cases} \frac{\psi_r(x)}{\psi_r(c)} g(c), & x \le c, \\ g(x), & x \ge c. \end{cases}$$
(14)

If \widehat{V} is an *r*-excessive majorant of *g* it follows from Proposition 3 that \widehat{V} is the value function of OSP (8). The excessivity can be checked with the method based on the representation theory of excessive functions (cf. [21] Section 3). This boils down to study for $x \neq -1$ the functions

$$I_V(x) := \left(\psi_r'(x)\widehat{V}(x) - \psi_r(x)\widehat{V}'(x)\right) / S'(x), \tag{15}$$

$$D_V(x) := \left(\varphi_r(x)\widehat{V}'(x) - \widehat{V}(x)\varphi_r'(x)\right) / S'(x).$$
(16)

Clearly, $I_V(x) = 0$ for $x \le c$ and increasing for x > c. Notice that $I_V = G_$ on $[c, +\infty)$. Studying the derivative (with respect to the speed measure) of D_V it is easily seen that D_V is positive and decreasing to 0 on $[c, +\infty)$. Consequently, I_V and D_V induce a (probability) measure which represent \hat{V} proving that \hat{V} is *r*-excessive. To prove that \hat{V} is a majorant of *g* consider for -1 < x < c

$$\widehat{V}(x) \ge g(x) \quad \Leftrightarrow \quad \frac{\psi_r(x)}{g(x)} \ge \frac{\psi_r(c)}{g(c)}$$

The right hand side of this equivalence holds since the derivative of $x \mapsto \psi_r(x)/g(x)$ is G_- which is negative for -1 < x < c, as is shown above. \Box

In case $\mu_1 = \mu_2$ it is well-known (see [24], and [21] where the problem is solved using the representation theory of the excessive functions) that $\mathbf{S}(r) = [c, +\infty)$ with c = c(r) as in Proposition 4, i.e.,

$$c = \frac{1}{\lambda_1^+} - 1.$$

Consequently, it is expected that if μ_2 is relatively close to μ_1 the stopping region is of this form for all values of the discounting parameter r; in other words, there is no bubble. This is indeed the case and the exact formulation is as follows.

Proposition 5. In case $0 \le \mu_1 \le \mu_2 \le \mu_1 + \frac{1}{2}$ the continuation region for the OSP (8) is given by

$$\mathbf{C}(r) = (-\infty, c)$$

where c = c(r) is the unique solution of equation (13):

$$\psi_r'(x)(x+1) - \psi_r(x) = 0$$

As r increases from 0 to $+\infty$ then c(r) decreases monotonically from $+\infty$ to -1. In particular, c(r) = 0 for $r = \mu_1 + \frac{1}{2}$.

Proof. If $r \leq \mu_1$ (and $\mu_1 > 0$) the statement is the same as in Proposition 4. We assume next that $r \geq \mu_2$. The proof in this case is very similar to the proof of Proposition 4. It can be proved that G_- attains the global minimum at $x_1 := (\mu_1 - r)/r < 0$, is negative and decreasing for $x \leq x_1$, is increasing for $x > x_1$, and has, therefore, a unique zero. Consequently, this root can be taken to be an optimal stopping point c = c(r) and the analogous function \widehat{V} as in (14) can be proved to be the value of OSP (8). Finally, assume $\mu_1 < r < \mu_2$. In this case, $G_$ has a local maximum at 0, which is negative since

$$G_{-}(0) = \psi'_{r}(0) - \psi_{r}(0) = \lambda_{1}^{+} - 1 < 0 \quad \Leftrightarrow \quad r < \mu_{1} + 1/2.$$

Hence, equation (13) has a unique positive root and the proof can be completed as in the previous cases. \Box

Finally, we study the situation $0 \le \mu_1 < \mu_1 + \frac{1}{2} < \mu_2$. The main observation is that there exists a bounded interval such that if r is in this interval then the continuation set has a bubble. The first result concerns the localization of a possible bubble, and is perhaps intuitively obvious. Anyway, we present its proof since the result is needed when proving Proposition 8 below which characterizes the continuation (and the stopping) set in the present case.

Proposition 6. Assume r > 0 is such that $(c_1(r), c_2(r))$ is a bubble. Then it holds that $0 \in [c_1(r), c_2(r))$, and there exists at most one bubble.

Proof. Let G_- and G_+ be given as in (9) and (10), respectively. Since g, φ_r and ψ_r are differentiable everywhere we may apply (4.7) Theorem in [21] p. 95 to deduce that $c_1 = c_1(r)$ and $c_2 = c_2(r)$ satisfy

$$G_{-}(c_1) = G_{-}(c_2),$$

 $G_{+}(c_1) = G_{+}(c_2).$

Moreover, G_{-} and G_{+} are positive, non-decreasing and non-increasing , respectively, on $(c_1 - \varepsilon, c_1] \cup [c_2, c_2 + \varepsilon) \subseteq \mathbf{S}(r)$ for some $\varepsilon > 0$. Studying these explicit expressions of G'_{-} and G'_{+} given in (11) and (12), respectively, we conclude that necessarily $c_1 \leq 0 < c_2$, as claimed.

Recall that the principle of smooth fit holds for our stopping problem. Hence, it is enlightening to investigate which "good" candidates satisfying the smooth fit principle cannot be value functions since they fail to be excessive. The following result shows that for $\mu_1 + \frac{1}{2} \le r < \mu_2$ there exist smooth fit (at 0) functions which are harmonic on \mathbf{R}_- but which are not *r*-excessive.

Proposition 7. For $r \ge \mu_1 + \frac{1}{2}$ there exist A and B such that the function

$$F(x) := \begin{cases} A \exp(\lambda_1^+ x) + B \exp(\lambda_1^- x), & x \le 0, \\ 1 + x, & x \ge 0, \end{cases}$$
(17)

satisfies the principle of smooth fit at 0, i.e., F'(0-) = F'(0+) = 1. The function F is r-harmonic (and positive) on $(-\infty, 0)$ but not r-excessive if $r < \mu_2$. For $r < \mu_1 + \frac{1}{2}$ the coefficient B is negative and the function $F(x) \to -\infty$ as $x \to -\infty$ (and the function is not r-excessive).

Proof. We study only the case $r = r_0 := \mu_1 + \frac{1}{2}$ and leave the details of the other cases to the reader. In this case $\lambda_1^+ = \sqrt{\mu_1^2 + 2r} - \mu_1 = 1$, and, obviously,

$$F(x) := \begin{cases} \exp(x), & x \le 0, \\ 1+x, & x \ge 0, \end{cases}$$

satisfies smooth fit at 0. Consequently, F is r_0 -harmonic (and positive) on $(-\infty, 0)$ and it remains to prove that F is not r_0 -excessive. For this, consider the representing function (this corresponds G_- in (9))

$$x \mapsto (\psi'_{r_0}(x)F(x) - \psi_{r_0}(x)F'(x)) / S'(x).$$

The claim is that this function is not non-decreasing. Indeed, take derivative with respect to the speed measure to obtain

$$\frac{d}{dm} \left(\left(\psi_{r_0}'(x)F(x) - \psi_{r_0}(x)F'(x) \right) / S'(x) \right) \\
= F(x) \frac{d}{dm} \frac{d}{dS} \psi_{r_0}(x) - \psi_{r_0}(x) \frac{d}{dm} \frac{d}{dS} F(x) \\
= \psi_{r_0}(x) \begin{cases} 0, & x < 0, \\ r_0(1+x) - \mu_2, & x > 0. \end{cases}$$

Since $r_0 < \mu_2$ this derivative is negative, e.g., for small positive x-values; therefore, F is not r_0 -excessive.

Remark 1. From Proposition 5 and 8 we may conclude that if $r \ge \mu_2$ ($r \le \mu_1 + 1/2$) then the problem is one-sided and the optimal stopping point is negative (positive). Notice that the smooth fit function F in (17) could be excessive for $r \ge \mu_2$ but since the optimal stopping point is negative F is not the smallest excessive majorant of the reward.

Next proposition can be seen as our main result concerning OSP (8). It is proved that if $r \in (\mu_1 + 1/2, \mu_2)$ but is "close to" μ_2 then $\mathbf{C}(r)$ has a bubble. However, the bubble disappears when r becomes bigger than μ_2 or tends to $\mu_1 + 1/2$. We give a complete description of $\mathbf{C}(r)$ although there is some overlap with Proposition 5.

Proposition 8. In case $0 \le \mu_1 < \mu_1 + \frac{1}{2} < \mu_2$ there exists $r_0 \in (\mu_1 + 1/2, \mu_2)$ with the following properties:

(i) If $r \in [r_0, \mu_2)$ the continuation region is given by

$$\mathbf{C}(r) = (-\infty, c_1) \cup (c_2, c_3),$$

where $c_i = c_i(r)$, i = 1, 2, 3, are such that $c_3 > 0 \ge c_2 \ge c_1 > -1$. In *particular, for* $r = r_0$ *it holds* $c_1 = c_2 < 0$.

(ii) If $r \ge \mu_2$ the continuation region is given by

$$\mathbf{C}(r) = (-\infty, c_{-}),$$

where $c_{-} = c_{-}(r) < 0$ is the unique solution of (13). In particular, for $r = \mu_2$

$$c_{-}(\mu_{2}) = \frac{1}{\lambda_{1}^{+}(\mu_{2})} - 1 = (\sqrt{\mu_{1}^{2} + 2\mu_{2}} + \mu_{1} - 2\mu_{2})/2\mu_{2} < 0.$$
(18)

(iii) If $r < r_0$ the continuation region is given by

$$\mathbf{C}(r) = (-\infty, c_+),$$

where $c_+ = c_+(r) > 0$ is the unique solution of (13).

Proof. The proof of (ii) is as the proof of Proposition 5 when $r \ge \mu_2$. Notice, however, that in the present case c(r) < 0 for all $r \ge \mu_2$. We consider next (iii) in case $r \le \mu_1 + \frac{1}{2}$. Studying the derivative of G_- and the value of G_- at zero it is seen, as in the proof of Proposition 4, that equation $G_-(x) = 0$ has for $r < \mu_1 + \frac{1}{2}$ one (and only one) root $\rho = \rho(r) > 0$. In case $r = \mu_1 + \frac{1}{2}$ there are two roots $\rho_1 = \rho_1(r) = 0$ and $\rho_2 = \rho_2(r) > 0$. Proceeding as in the proof of Proposition 4 it is seen that the stopping region is as claimed with $c_+ = \rho$ if $r < \mu_1 + \frac{1}{2}$ and $c_+ = \rho_2$ if $r = \mu_1 + \frac{1}{2}$. Assume now that there does not exist a bubble for any $r \in [\mu_1 + \frac{1}{2}, \mu_2]$. Then for all $r \in [\mu_1 + \frac{1}{2}, \mu_2]$ we can find c = c(r) such that $\mathbf{S}(r) = [c, +\infty)$. Knowing that c(r) > 0 for $r = \mu_1 + \frac{1}{2}$ and c(r) < 0 for $r = \mu_2$ we remark first there does not exists r such that c(r) = 0. Indeed, by Theorem 1 the value should satisfy the smooth fit principle at 0 but from Proposition 7 we know that such functions are not r-excessive. Next, using $\mathbf{S}(r_1) \subseteq \mathbf{S}(r_2)$ for $r_1 < r_2$ (cf. Proposition 1) it is seen that $r \mapsto c(r)$ is non-increasing, and has, hence, left and right limits. Consquently, there exists a unique point \hat{r} such that

$$\hat{c}_+ := \lim_{r \uparrow \hat{r}} c(r) > 0$$
 and $\hat{c}_- := \lim_{r \downarrow \hat{r}} c(r) < 0.$

Under the assumption that there is no bubble the value function is of the form given in (14), i.e.,

$$V_{r}(x) = \begin{cases} \psi_{r}(x) \frac{1+c(r)}{\psi_{r}(c(r))}, & x \leq c(r), \\ 1+x, & x \geq c(r). \end{cases}$$

$$= \mathbb{E}_{x} \left(e^{-rH_{c}} \left(1+X_{H_{c}} \right) \right), \qquad (19)$$

where $H_c := \inf\{y : X_t \ge c(r)\}$. Letting in (19) $r \uparrow \hat{r}$ yields by Proposition 3 an \hat{r} -excessive function which by Proposition 2 is the value of the corresponding OSP (8). Similarly, letting $r \downarrow \hat{r}$ yields an \hat{r} -excessive function which should also be the value of the same OSP. However, the functions are clearly different and since the value is unique we have reached a contradiction showing that there exists at least one bubble. Evoking Proposition 6 completes the proof.

Remark 2. The fact that there is a bubble when $r < \mu_2$ but "close" to μ_2 would also follow if we can prove that G_- has a unique negative zero. Notice that $G_$ in this case is not monotone around 0. This would then imply the exisistence of a bubble if we can verify that the local minimum on $(0, +\infty)$ is positive. However, we have not been able to show this. Numerical calculations with some parameter values give evidence that the local minimum on $(0, +\infty)$ is indeed positive.

Remark 3. For $r < r_0$ we have the value function (cf. (14))

$$V_r(x) = \begin{cases} \psi_r(x) \frac{1+c_+}{\psi_r(c_+)}, & x \le c_+, \\ 1+x, & x \ge c_+. \end{cases}$$
(20)

Since $c_+ > 0$ it holds $V_r(0) > g(0)$, and, hence, $(1 + c_+)/\psi_r(c_+) > 1$. Consequently,

$$V_r'(0) = \lambda_1^+ \frac{1+c_+}{\psi_r(c_+)} > 1,$$

because also $\lambda_1^+ > 1$ for $r > \mu_1 + \frac{1}{2}$. Moreover, $V'_r(x) \to 0$ as $x \to -\infty$, and there exists a unique point a = a(r) such that $V'_r(a) = 1$. From Proposition 8 we

know that the bubble appears as r increases and takes the value r_0 . Therefore, we may describe the value function V_{r_0} to be of the form in (20) satisfying the smooth fit at $c_+(r_0) > 0$ and also at another point $a(r_0) < 0$ which is a tangent point with the reward.

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References

- Alvarez E., L. and Salminen, P.: Timing in the Presence of Directional Predictability: Optimal Stopping of Skew Brownian Motion. *Mathematical Meth*ods of Operations Research, Vol. 86, pp. 377-400, 2017.
- [2] Arkin, V. I.: Threshold strategies in optimal stopping problem for onedimensional diffusion processes. *Theory Prob. Appl.*, Vol. 59, pp. 311–319, 2015.
- [3] Arkin, V. and Slastnikov, A.: On the threshold strategies in optimal stopping problems for diffusion processes. J. Appl. Probab., Vol. 54, pp. 963–969, 2017.
- [4] Borodin, A.N. and Salminen, P.: Handbook of Brownian motion—Facts and formulae, 2nd. ed., 2nd corr. print., Birkhäuser Verlag, 2015.
- [5] Christensen, S. and Irle, A.: A harmonic function technique for the optimal stopping of diffusions. *Stochastics*, Vol. 83, pp. 347–363, 2011.
- [6] Christensen, S. and Irle, A.: A general method for finding the optimal threshold in discrete time. ArXiv, 1710.08250v1, 2017
- [7] Christensen, S. and Salminen, P. and Ta, B. Q.: Optimal stopping of strong Markov processes. *Stoch. Process. Appl.* Vol. 123, pp. 138–1159, 2013.
- [8] Crocce, F.: Optimal Stopping for Strong Markov Processes: Explicit solutions and verification theorems for diffusions, multidimensional diffusions, and jumpprocesses, Phd Thesis. Universidad de la República (arXiv:1405.7539), 2014.
- [9] Crocce, F. and Mordecki, E.: Explicit solutions in one-sided optimal stopping problems for one-dimensional diffusions. *Stochastics*, Vol. 86, pp. 491–509, 2014.
- [10] Dayanik, S. and Karatzas, I.: On the optimal stopping problem for onedimensional diffusions. *Stochastic Processes and their Applications*, Vol. 107, pp. 173–212, 2003.
- [11] Itô, K. and McKean, H. P.: Diffusion Processes and their Sample Paths, Springer-Verlag, Berlin, 1974.
- [12] Lamberton, D. and Zervos, M.: On the optimal stopping of a onedimensional diffusion. *Electron. J. Probab.*, Vol. 18, pp. 1–49, 2013.
- [13] McKean, Jr. H.P.: Appendix: A free boundary problem for the heat equation arising from a problem in Mathematical Economics. *Industrial Management Review*, Vol. 6, pp. 32–39, 1965.
- [14] Merton, R.C.: Theory of rational option pricing. Bell J. Econom. Manag. Sci., Vol. 4, pp. 141–183, 1973.

- [15] Mordecki, E. and Salminen, P.: Optimal stopping of Hunt and Lévy processes. *Stochastics*, Vol 79, pp. 233–251, 2007.
- [16] Peskir, G.: A change-of-variable formula with local time on curve. J. Theor. Probab., Vol. 18, pp. 499–535, 2005.
- [17] Peskir, G.: Principle of smooth fit and diffusions with angles. *Stochastics*, Vol. 79, pp. 293–302, 2007.
- [18] Peskir, G. and Shiryaev, A.N.: *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2006.
- [19] Presman E.: Solution of the optimal stopping problem of one-dimensional diffusion based on a modification of payoff function. *Honor of Yuri V. Prokhorov, Prokhorov and contemporary probability theory. Springer Proceedings in Mathematics and Statistics,* Vol. 33, pp 371403, 2012.
- [20] Rüschendorf, L. and Urusov, M.A.: On a class of optimal stopping stopping problems for diffusions with discontinuous coefficients. *Ann. App. Probab.*, Vol. 18, pp. 847878, 2008.
- [21] Salminen, P.: Optimal stopping of one-dimensional diffusions. *Math. Nachr.* Vol. 124, pp. 85–101, 1985
- [22] Salminen, P. and Ta, B. Q. : Differentiability of excessive functions of onedimensional diffusions and the principle of smooth fit. *Banach Center Publications*, Vol. 104, pp. 181–199, 2015.
- [23] Shiryaev, A.N.: Optimal Stopping Rules. Springer-Verlag, New York, 1978.
- [24] Taylor, H.: Optimal stopping in a Markov process. Ann. Math. Stat. Vol. 39, pp. 1333-1344, 1968.
- [25] Villeneuve, S. : On the threshold strategies and smooth-fit principle for optimal stopping problems. J. Appl. Prob., Vol. 44, pp. 181–198. 2007.