ON THE K-THEORY OF $\mathbb{Z}$-CATEGORIES.

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Abstract. We relate the notions of Noetherian, regular coherent and regular n-coherent category for $\mathbb{Z}$-linear categories with finite objects with the corresponding notions for unital rings. We use this relation to obtain a vanishing negative $\mathbb{K}$-theory of $\mathbb{Z}$-linear categories.

1. Introduction

Let $\mathcal{C}$ be a small $\mathbb{Z}$-linear category. Associated to $\mathcal{C}$ there exists a ring

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{a,b\in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(a,b).$$

With the natural sum and multiplication, $\mathcal{A}(\mathcal{C})$ is a ring with with local units, which is unital if and only if ob$\mathcal{C}$ is finite. There exists a weak equivalence between the spectrum of the algebraic K-theory of $\mathcal{C}$ and the spectrum of the algebraic K-theory of $\mathcal{A}(\mathcal{C})$, see [7, Sec. 4.2]. Thus the K-theory groups of $\mathcal{C}$ and $\mathcal{A}(\mathcal{C})$ coincides.

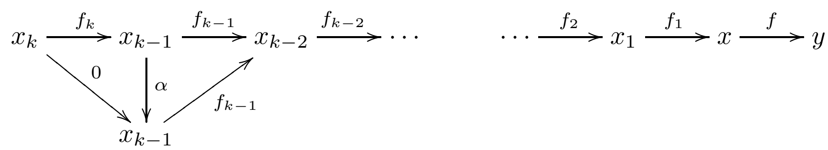
We consider the category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ of contravariant functors from $\mathcal{C}$ to Ab. Using Yoneda Lemma we embed $\mathcal{C}$ into $\mathrm{Fun}(\mathcal{C}^{op}\,\mathrm{Ab})$ with the purpose to do homological constructions in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ which a prior make no sense in $\mathcal{C}$. Yoneda Lemma is used to define Noetherian additive categories or regular coherent additive categories, see [3]. Following this idea we extend in Section 3 the notion of regular n-coherence from rings with unit to small $\mathbb{Z}$-linear categories.

As is shown in [5, Corollary 2.14] an object $F : \mathcal{C}^{op} \to \mathrm{Ab}$ is of type $\mathcal{FP}_n$ if and only if there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

where P_i is finitely generated and projective for every $0 \le i \le n$. We say that $\mathcal{C}$ is right n-coherent if the category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ is n-coherent in the sense of [5, Definition 4.6]. In other words $\mathcal{C}$ is right n-coherent if and only if the objects of type $\mathcal{FP}_n$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ coincide with the objects of type $\mathcal{FP}_\infty$. If $\mathcal{C}$ is right n-coherent, we say that $\mathcal{C}$ is *regular* if every object $F : \mathcal{C}^{op} \to \mathrm{Ab}$ of type $\mathcal{FP}_n$ has a finite projective dimension. We prove in Proposition 2.7 that this homological property of $\mathcal{C}$ also holds for $\mathcal{C}_\oplus$. We show in Proposition 2.9 that an additive category $\mathcal{C}$ is right regular n-coherent if and only if the following conditions hold in $\mathcal{C}$:

i) Every morphism in $\mathcal{C}$ with a pseudo $(n-1)$-kernel has a pseudo n-kernel.
ii) For every morphism $f : x \to y$ in $\mathcal{C}$ with pseudo ∞-kernel there exists $k \in \mathbb{N}$ and $\alpha : x_{k-1} \to x_{k-1}$ making the following diagram commute:



We see in Proposition 3.5 that the category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ is equivalent to Mod-$\mathcal{A}(\mathcal{C})$, where Mod-$\mathcal{A}(\mathcal{C})$ denotes the category of unital right modules. Let $\mathcal{C}$ be a Noetherian or an n-coherent category. Is the ring $\mathcal{A}(\mathcal{C})$ also Noetherian or n-coherent? We answer this question in the case that $\mathcal{A}(\mathcal{C})$ has unit or equivalently when $\mathcal{C}$ has finite objects. We prove in Proposition 3.9 that $\mathcal{C}$ is a Noetherian (n-coherent or regular n-coherent) $\mathbb{Z}$-linear category with finite objects if and only if $\mathcal{A}(\mathcal{C})$ is a Noetherian (strong n-coherent or n-regular and strong n-coherent) ring with unit.

A new way to obtain information about the K-theory of a $\mathbb{Z}$-linear category is obtained. In Section 4 we prove that if $\mathcal{D} = \mathcal{C}$, $\mathcal{D} = \mathcal{C}_\oplus$ or $\mathcal{D} = \mathrm{colim}_{f\in F}\, \mathcal{C}_f$ with $\mathcal{C}$ or $\mathcal{C}_f$ a regular $\mathbb{Z}$-linear category with finite object then $K_i(\mathcal{D}) = 0$, $\forall i < 0$. We also prove that if $\mathcal{D} = \mathcal{C}$, $\mathcal{D} = \mathcal{C}_\oplus$ or $\mathcal{D} = \mathrm{colim}_{f\in F}\, \mathcal{C}_f$ with $\mathcal{C}$ or $\mathcal{C}_f$ a regular coherent $\mathbb{Z}$-linear category with finite object then $K_{-1}(\mathcal{D}) = 0$. In Proposition 4.6 we obtain a generalization of [10, Thm 3.2].

2. Modules over $\mathbb{Z}$-linear categories

A *$\mathbb{Z}$-linear category* is a category $\mathcal{C}$ such that for every two objects $a, b \in \mathcal{C}$, the set of morphisms $\hom_{\mathcal{C}}(a, b)$ is an abelian group, and for any other object $c \in \mathcal{C}$, the composition

$$\hom_{\mathcal{C}}(b, c) \times \hom_{\mathcal{C}}(a, b) \to \hom_{\mathcal{C}}(a, c)$$

is a bilinear map. Throughout this paper we assume that $\mathbb{Z}$-linear categories $\mathcal{C}$ are small, i.e. the collection of objects is a set. A $\mathbb{Z}$-linear category is *additive* if it has an initial object and finite products. We consider the free additive category $\mathcal{C}_\oplus$ as follow. The objects of $\mathcal{C}_\oplus$ are finite tuples of objects in $\mathcal{C}$. A morphism from $\mathbf{a} = (a_1, \cdots, a_k)$ to $\mathbf{c} = (c_1, \cdots, c_m)$ for $a_i, c_j \in \mathcal{C}$ is given by $m \times k$ matrix of morphisms in $\mathcal{C}$ (the composition is given by the usual row-by-column multiplication of matrices),

- $\mathrm{ob}\mathcal{C}_\oplus = \{(c_1, \cdots, c_k) : c_i \in \mathcal{C}, k \in \mathbb{N}\}$
- $\hom_{\mathcal{C}_\oplus}(\mathbf{a}, \mathbf{c}) = \prod_{i=1}^{k} \prod_{j=1}^{m} \hom_{\mathcal{C}}(a_i, c_j)$.

There is an obvious embedding $\mathcal{C} \to \mathcal{C}_\oplus$ which maps objects and morphisms to their associated 1-tuple. If $\mathcal{C}$ is a $\mathbb{Z}$-linear category then $\mathcal{C}_\oplus$ is a small additive category.

The *idempotent completion* $\mathrm{Idem}(\mathcal{C}_\oplus)$ of $\mathcal{C}_\oplus$ is defined to be the following small additive category.

- $\mathrm{ob}(\mathrm{Idem}(\mathcal{C}_\oplus)) = \{(\mathbf{c}, p) :\ c \in \mathrm{ob}\mathcal{C}_\oplus,\ p : \mathbf{c} \to \mathbf{c} \text{ such that } p^2 = p\}$
- $\hom_{\mathrm{Idem}(\mathcal{C}_\oplus)}((\mathbf{c}_1, p_1), (\mathbf{c}_2, p_2)) = \{w : \mathbf{c}_1 \to \mathbf{c}_2 \text{ such that } w = p_2 w p_1\}$.

By construction $\mathcal{C} \simeq \mathcal{C}_\oplus$ if $\mathcal{C}$ is additive and $\mathcal{C}_\oplus \simeq \mathrm{Idem}(\mathcal{C}_\oplus)$ if idempotents split in $\mathcal{C}_\oplus$. Recall the additive category $\mathcal{C}_\oplus$ is equivalent to $\mathrm{Idem}(\mathcal{C}_\oplus)$ if and only if every idempotent has a kernel.

Example 2.1. Given a ring R, consider $\mathcal{C} = \underline{R}$ the category which has one object $\star$ and $\hom_{\mathcal{C}}(\star, \star) = R$. The multiplication on R gives the composition on $\underline{R}$. The category $\mathcal{C}_\oplus$ is the category whose objects are natural numbers $m > 0$ and the morphisms are the matrices with coefficients in R, $\hom_{\mathcal{C}_\oplus}(m,n) = M_{n\times m}(R)$.

Example 2.2. Let R be an associative ring with unit. If $\mathcal{C}$ is the category of finitely generated free R-modules, then $\mathrm{Idem}(\mathcal{C})$ is equivalent to the category of finitely generated projective R-modules.

2.1. **Pseudo n-kernels and pseudo n-cokernels.** Given a $\mathbb{Z}$-linear category $\mathcal{C}$ we recall that a *pseudo kernel* of a morphism $f : x \to y$ in $\mathcal{C}$ is a morphism $g : k \to x$ with $f \circ g = 0$, such that for any morphism $h : c \to x$ with $f \circ h = 0$, there exists $t : c \to k$ with $g \circ t = h$. Equivalently, a morphism $g : k \to x$ in $\mathcal{C}$ is said to be a pseudo kernel of f if, for any $c \in \mathrm{ob}\mathcal{C}$, the following sequence of abelian groups is exact

$$\hom_{\mathcal{C}}(c,k) \to \hom_{\mathcal{C}}(c,x) \to \hom_{\mathcal{C}}(c,y).$$

Pseudo-kernels have been introduced by Freyd [11] as weak kernels. *Pseudo-cokernels* are pseudo kernels in $\mathcal{C}^{op}$. By [15, Corollary 1.1] the categories $\mathcal{C}$, $\mathcal{C}_\oplus$ and $\mathrm{Idem}(\mathcal{C}_\oplus)$ all have pseudo kernels or they don't. Let us remark that any triangulated or abelian category has pseudo-kernels and pseudo-cokernels.

Let $n \geq 1$ and $f : x \to y$ be a morphism in $\mathcal{C}$. Following [6], we say that f has a *pseudo n-kernel* if there exists a chain of morphisms

$$x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} x_{n-2} \to \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact

$$\hom_{\mathcal{C}}(-,x_n) \xrightarrow{f_{n*}} \cdots \to \hom_{\mathcal{C}}(-,x_1) \xrightarrow{f_{1*}} \hom_{\mathcal{C}}(-,x) \xrightarrow{f_*} \hom_{\mathcal{C}}(-,y).$$

We denote the pseudo n-kernel by $(f_n, f_{n-1}, \cdots, f_1)$. The case $n = 1$ gives us the classic pseudo-kernels. For convenience, we let $x_0 := x$. Furthermore, any morphism f in $\mathcal{C}$ will be assumed to be a pseudo 0-kernel of itself. We say that f has a *pseudo ∞-kernel* if there exists a chain of morphisms

$$\cdots \to x_{n+1} \xrightarrow{f_{n+1}} x_n \xrightarrow{f_n} x_{n-1} \to \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact

$$\cdots \to \hom_{\mathcal{C}}(-,x_{n+1}) \xrightarrow{f_{n+1*}} \hom_{\mathcal{C}}(-,x_n) \xrightarrow{f_{n*}} \cdots \xrightarrow{f_{1*}} \hom_{\mathcal{C}}(-,x) \xrightarrow{f_*} \hom_{\mathcal{C}}(-,y).$$

Pseudo n-cokernels are defined as pseudo n-kernels in $\mathcal{C}^{op}$.

2.2. **Categories of Additive Functors.** The category of abelian groups will be denoted by Ab. For any $\mathbb{Z}$-linear category $\mathcal{C}$, we define a *left $\mathcal{C}$-module* as a functor $F : \mathcal{C} \to \mathrm{Ab}$. We consider natural transformations as morphisms of $\mathcal{C}$-modules. Define a *right $\mathcal{C}$-module* as an functor $F : \mathcal{C}^{op} \to \mathrm{Ab}$. In these categories limits and colimits of functors are defined objectwise. Denote by $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ the category of right $\mathcal{C}$-modules. This category is cocomplete and abelian. If c is an object of $\mathcal{C}$ then there is the corresponding representable functor $\hom_{\mathcal{C}}(-,c) : \mathcal{C}^{op} \to \mathrm{Ab}$.

Lemma 2.3. *(Yoneda Lemma) Let $\mathcal{C}$ be any $\mathbb{Z}$-linear category. Take $c \in \mathcal{C}$ and $F : \mathcal{C}^{op} \to Ab$. Then there is a natural identification*

$$\hom_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Ab})}(\hom_{\mathcal{C}}(-,c), F(-)) \cong F(c).$$

By Yoneda Lemma, the family $\{\hom_{\mathcal{C}}(-,c)\}_{c\in\mathcal{C}}$ is a generating set of finitely generated projective in $\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Ab})$. A module $M \in \mathrm{Fun}(\mathcal{C}^{op},\mathrm{Ab})$ is *free* if it is isomorphic to $\bigoplus_{i\in I}\hom_{\mathcal{C}}(-,a_i)$. It is free and finitely generated if I is finite.

Let R be a ring and $\underline{R}$ be the $\mathbb{Z}$-linear category defined in Example 2.1. Note that

$$R\text{-Mod} \cong \mathrm{Fun}(\underline{R},\mathrm{Ab})$$
$$\text{Mod-}R \cong \mathrm{Fun}(\underline{R}^{op},\mathrm{Ab}).$$

2.3. **Finitely n-presented objects and n-coherent categories.** Let $n \geq 1$ be a positive integer. Following [5, Definition 2.1] we say that an object $F:\mathcal{C}^{op}\to Ab$ is *finitely n-presented* if the functors $\mathrm{Ext}^i_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Ab})}(F,-)$ preserves direct limits for all $0 \leq i \leq n-1$. Denote by $\mathcal{FP}_0$ to the set of finitely generated objects, then M is an object of type $\mathcal{FP}_0$ if there exists a collection of objects $\{c_j : j \in J\}$ in $\mathcal{C}$ for some finite set J and an epimorphism $\bigoplus_{j\in J}\hom_{\mathcal{C}}(-,c_j)\to M$. A functor $F:\mathcal{C}^{op}\to \mathrm{Ab}$ is of type $\mathcal{FP}_\infty$ if it is of type $\mathcal{FP}_n$ for all $n \geq 0$.

Recall that a *Grothendieck category* is a cocomplete abelian category, with a generating set and with exact direct limits. A Grothendieck category is *locally finitely generated (presented)* if it has a set of finitely generated (presented) generators. In other words, each object is a direct union (limit) of finitely generated (presented) objects. A Grothendieck category is *locally type* $\mathcal{FP}_n$ [5, Definition 2.3], if it has a generating set consisting of objects of type $\mathcal{FP}_n$.

By [13, Example 3.2] any finitely generated projective object is of type $\mathcal{FP}_n$ for all $n \geq 0$. Then the functor category $\mathrm{Fun}(\mathcal{C}^{op}\,\mathrm{Ab})$ is a locally type $\mathcal{FP}_\infty$ Grothendieck category. By the [5, Corollary 2.14] an object $F:\mathcal{C}^{op}\to\mathrm{Ab}$ is of type $\mathcal{FP}_n$ if and only if there exists an exact sequence

$$P_n \to \cdots \to P_1 \to P_0 \to F \to 0$$

where P_i is finitely generated and projective for every $0 \leq i \leq n$.

Recall from [5, Definition 4.1] an object $F:\mathcal{C}^{op}\to Ab$ is *n-coherent* if satisfies the following conditions:

(1) F is of type $\mathcal{FP}_n$.
(2) For each subobject $S \subseteq F$ such that S is type $\mathcal{FP}_{n-1}$ then S is also of type $\mathcal{FP}_n$.

Definition 2.4. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category and $n \geq 0$. We say that $\mathcal{C}$ is right (left) *n-coherent* if every object $F:\mathcal{C}^{op}\to\mathrm{Ab}$ ($F:\mathcal{C}\to\mathrm{Ab}$) of type $\mathcal{FP}_n$ is n-coherent.

We say that $\mathcal{C}$ is right n-coherent if the category $\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Ab})$ is n-coherent in the sense of [5, Definition 4.6]. Thus by [5, Theorem 4.7], $\mathcal{C}$ is right n-coherent if and only if the objects of type $\mathcal{FP}_n$ in $\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Ab})$ coincide with the objects of type $\mathcal{FP}_\infty$. In particular, an additive category $\mathcal{C}$ is Noetherian in the sense of [3, Definition 5.2] if and only if it is 0-coherent. Note that for $1 \leq n \leq \infty$, if $\mathcal{C}$ is any small additive category, by [6, Proposition 5.4], the following conditions are equivalent:

1) $\mathcal{C}$ is right n-coherent.
2) If a morphism in $\mathcal{C}$ has a pseudo $(n-1)$-kernel, then it has a pseudo n-kernel.

Definition 2.5. Let $n \geq 0$ and $\mathcal{C}$ be a right n-coherent $\mathbb{Z}$-linear category. We say that $\mathcal{C}$ is *regular* if every object $F : \mathcal{C}^{op} \to \mathrm{Ab}$ of type $\mathcal{FP}_n$ has a finite projective dimension.

Let $\mathcal{C}$ be a small additive category then $\mathcal{C}$ is regular coherent in the sense of [3, Definition 5.2] if and only if it is right regular 1-coherent.

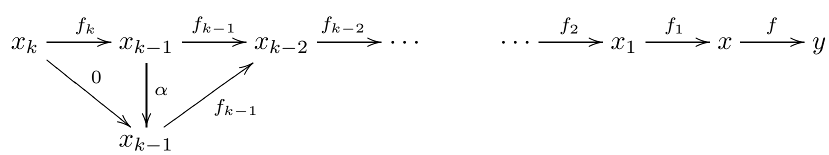
Example 2.6. Let $\mathcal{C}$ be a small additive category and $n \geq 1$.

I. **Additive category with kernels.** By a result due to Auslander [2, Theorem 2.2.b] a small additive category $\mathcal{C}$ with kernels is 1-coherent and every object of type $\mathcal{FP}_1$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ has projective dimension at most 2. Then $\mathcal{C}$ is right regular 1-coherent.

II. **Von Neumann regular categories.** We recall that $\mathcal{C}$ is called von Neumann regular if for any morphism $f : a \to b$ in $\mathcal{C}$ there exists a morphism $g : b \to a$ such that $fgf = f$. By [4, Corollary 8.1.3] $\mathcal{C}$ is right regular 1-coherent.

III. **Locally finitely presented categories.** An object $c \in \mathcal{C}$ is finitely presented if the functor $\hom_{\mathcal{C}}(c, -)$ preserves direct limits. The category $\mathcal{C}$ is locally finitely presented if every directed system of objects and morphisms has a direct limit, the class of finitely presented objects of $\mathcal{C}$ is skeletally small and every object of $\mathcal{C}$ is the direct limit of finitely presented objects. Then by [12, Lemma 2.2], every locally finitely presented category is left 1-coherent.

IV. **n-hereditary categories**. Suppose the following two conditions hold in $\mathcal{C}$:
   (a) Every morphism in $\mathcal{C}$ with a pseudo $(n-1)$-kernel has a pseudo n-kernel.
   (b) For every morphism $f : x \to y$ in $\mathcal{C}$ with pseudo n-kernel $(f_n, \cdots, f_1)$, there exists an endomorphism $\alpha : x_{n-1} \to x_{n-1}$ making the following diagram commute:



   By [6, Theorem 5.5], $\mathcal{C}$ is right n-coherent and every object of type $\mathcal{FP}_1$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ has projective dimension less than or equal 1. Therefore, $\mathcal{C}$ is right regular n-coherent.

Due to [15, Lemma 1.1, 1.2] we have the following equivalences of categories

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab}) \simeq \mathrm{Fun}(\mathcal{C}_{\oplus}^{op}, \mathrm{Ab}) \simeq \mathrm{Fun}(\mathrm{Idem}(\mathcal{C}_{\oplus}^{op}), \mathrm{Ab})$$

It implies that $\mathcal{C}$, $\mathcal{C}_{\oplus}$ and $\mathrm{Idem}(\mathcal{C}_{\oplus})$ are Morita equivalents. In particular, we obtain the following result.

Proposition 2.7. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. The following are equivalent:*

(1) *$\mathcal{C}$ is right regular n-coherent.*
(2) *$\mathcal{C}_{\oplus}$ is right regular n-coherent.*
(3) *$\mathrm{Idem}(\mathcal{C}_{\oplus})$ is right regular n-coherent.*

Let R be a ring with unit. A finitely n-presented right R-module M is n-coherent if every finitely $(n-1)$-presented submodule $N \subseteq M$ is finitely n-presented. The ring R is right n-coherent if R is n-coherent as a right R-module (i.e. if each $(n-1)$-presented ideal of R is n-presented). We say that R is right strong n-coherent if each finitely n-presented right R-module is $(n+1)$-presented. A strong n-coherence ring is equivalent to a n-coherence ring for $n = 1$, but it is an open question for $n \geq 2$. A coherent ring is a 1-coherent ring (strong 1-coherent ring) and it is regular if and only if every finitely presented module has finite projective dimension. Motivated by this we introduce in [10, Definition 2.9] the definition of n-regular ring. Let $n \geq 1$, a ring R is called right *n-regular* if each finitely n-presented right R-module has finite projective dimension.

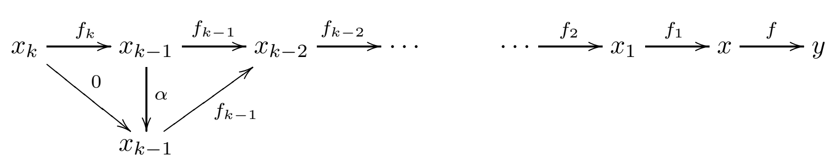
Corollary 2.8. *Let R be a ring with unit and $n \geq 1$. Then the following are equivalent.*

1. *The ring R is right strong n-coherent or right n-regular and strong n-coherent respectively;*
2. *The additive category $\underline{R}_{\oplus}$ is right n-coherent or right regular n-coherent respectively;*
3. *The additive category $Idem(\underline{R}_{\oplus})$ is right n-coherent or right regular n-coherent respectively.*

Let $\mathcal{C}$ be a small additive category. By [3, Lemma 5.8], $\mathcal{C}$ is right Noetherian if and only if each object c has the following property. Consider any directed set I and collections of morphisms $\{f_i : a_i \to c\}_{i\in I}$ with c as target such that $f_i \subseteq f_j$ holds for $i \leq j$, then there exists $i_0 \in I$ with $f_i \subseteq f_{i_0}$ for all $i \in I$. Our aim is to find out intrinsic condition of $\mathcal{C}$ which guarantees that $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ is regular n-coherent.

Proposition 2.9. *Let $\mathcal{C}$ be a small additive category and $n \geq 1$. The following are equivalent*

1. *$\mathcal{C}$ is right regular n-coherent.*
2. *The following conditions hold in $\mathcal{C}$:*
   - i) *Every morphism in $\mathcal{C}$ with a pseudo $(n-1)$-kernel has a pseudo n-kernel.*
   - ii) *For every morphism $f : x \to y$ in $\mathcal{C}$ with pseudo ∞-kernel there exists $k \in \mathbb{N}$ and $\alpha : x_{k-1} \to x_{k-1}$ making the following diagram commute:*



Proof. $(1 \Rightarrow 2)$ Suppose that $\mathcal{C}$ is right regular n-coherent. First, we note that (i) is clear by [6, Prop 5.4]. Now suppose that $f : x \to y$ is a morphism in $\mathcal{C}$ with a pseudo ∞-kernel $(\cdots, f_3, f_2, f_1)$. Here, we let $f_0 := f$. Thus $\mathrm{coker}(f_*)$ is of type $\mathcal{FP}_\infty$ in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ because there exists an exact sequence of the form

$$\cdots \xrightarrow{f_{2*}} \hom_{\mathcal{C}}(-, x_1) \xrightarrow{f_{1*}} \hom_{\mathcal{C}}(-, x) \xrightarrow{f_*} \hom_{\mathcal{C}}(-, y) \to \mathrm{coker}(f_*) \to 0.$$

There exists $k \in \mathbb{N}$ such that $\mathrm{coker}(f_*)$ has projective dimension $\leq k$. It implies that $\ker(f_{k-2*}) = \mathrm{im}(f_{k-1*})$ is projective, and therefore $\ker(f_{k-1*}) = \mathrm{im}(f_{k*})$ is projective too. Consider

$$\begin{array}{ccccc} \cdots \to \hom_{\mathcal{C}}(-,x_k) & \xrightarrow{f_{k*}} & \hom_{\mathcal{C}}(-,x_{k-1}) \to \cdots \\ & \searrow^{\sigma} & \uparrow{\scriptstyle\iota} \\ & & \mathrm{im}(f_{k*}) \end{array}$$

where $\iota : \mathrm{im}(f_{k*}) \hookrightarrow \hom_{\mathcal{C}}(-,x_{k-1})$ and $\sigma : \hom_{\mathcal{C}}(-,x_k) \twoheadrightarrow \mathrm{im}(f_{k*})$ are the canonical morphisms. There exists $\iota' : \mathrm{im}(f_{k*}) \to \hom_{\mathcal{C}}(-,x_k)$ and $\sigma' : \hom_{\mathcal{C}}(-,x_{k-1}) \to \mathrm{im}(f_{k*})$ such that $\sigma \circ \iota' = \mathrm{id}_{\mathrm{im}(f_{k*})}$ and $\sigma' \circ \iota = \mathrm{id}_{\mathrm{im}(f_{k*})}$. By Yoneda Lemma and using the same techniques [6, Theorem 5.5] there exists $h : x_{k-1} \to x_k$ in $\mathcal{C}$ such that $h_* : \hom_{\mathcal{C}}(-,x_{k-1}) \to \hom_{\mathcal{C}}(-,x_k)$ satisfy $h_* \circ f_{k*} = \iota' \circ \sigma$. The morphism $\alpha := \mathrm{id}_{x_{k-1}} - f_k \circ h$ satisfies the desired condition.

($2 \Rightarrow 1$) Suppose that the affirmation (2) is satisfied for $n \geq 1$. Using the condition (2-i), we deduce that $\mathcal{C}$ is right n-coherent [6, Prop 5.4] and thus $\mathcal{FP}_n = \mathcal{FP}_\infty$. Now, for each $F : \mathcal{C}^{op} \to Ab$ of type $\mathcal{FP}_n$ we get an exact sequence of the form

$$\cdots \to \hom_{\mathcal{C}}(-,x_n) \to \cdots \to \hom_{\mathcal{C}}(-,x_1) \xrightarrow{f_{1*}} \hom_{\mathcal{C}}(-,x) \xrightarrow{f_*} \hom_{\mathcal{C}}(-,y) \to F \to 0$$

where $f : x \to y$ is a morphism in $\mathcal{C}$. It implies that f has a pseudo ∞-kernel, and therefore, there is $k \in \mathbb{N}$ and an endomorphism $\alpha : x_{k-1} \to x_{k-1}$ making the following diagram commute:

$$\begin{array}{ccccc} x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} \\ & \searrow^{0} & \downarrow{\scriptstyle\alpha} & \nearrow_{f_{k-1}} & \\ & & x_{k-1} & & \end{array}$$

Next, we show that $\mathrm{im}(f_{k-1*}) = \ker(f_{k-2*})$ is a projective functor. Consider

$$\begin{array}{ccccc} \hom_{\mathcal{C}}(-,x_k) & \xrightarrow{f_{k*}} & \hom_{\mathcal{C}}(-,x_{k-1}) & \xrightarrow{f_{k-1*}} & \hom_{\mathcal{C}}(-,x_{k-2}) \\ & \searrow^{0} & \downarrow{\scriptstyle\alpha_*} \quad \searrow^{\sigma} \ \mathrm{im}(f_{k-1*}) \ \nearrow^{\iota} & & \Vert \\ & & \hom_{\mathcal{C}}(-,x_{k-1}) & \xrightarrow{f_{k-1*}} & \hom_{\mathcal{C}}(-,x_{k-2}) \end{array}$$

where $\sigma : \hom_{\mathcal{C}}(-,x_{k-1}) \to \mathrm{im}(f_{k-1*})$ and $\iota : \mathrm{im}(f_{k-1*}) \to \hom_{\mathcal{C}}(-,x_{k-2})$ are the canonical natural transformations. Note that $\mathrm{im}(f_{k-1*}) = \mathrm{coker}(f_{k*})$, then there exists unique natural transformation $t : \mathrm{im}(f_{k-1*}) \to \hom_{\mathcal{C}}(-,x_{k-1})$ such that $t \circ \sigma = \alpha_*$. Moreover, applying the same techniques [6, Theorem 5.5] we have

$$\iota \circ \mathrm{id}_{\mathrm{im}(f_{k-1*})} \circ \sigma = \iota \circ \sigma = f_{k-1*} = (f_{k-1} \circ \alpha)_* = f_{k-1*} \circ \alpha_* = f_{k-1*} \circ t \circ \sigma = \iota \circ \sigma \circ t \circ \sigma$$

which implies that

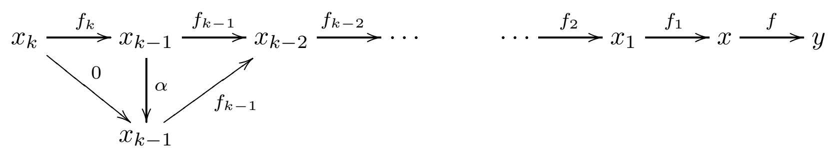
$$\mathrm{id}_{\mathrm{im}(f_{k-1*})} = \sigma \circ \ t.$$

Then σ is a split epimorphism, and therefore, $\mathrm{im}(f_{k-1*})$ is projective. □

Following [3] the category $\mathcal{C}$ is right *regular* if it is right Noetherian and right regular coherent. This should not be confused with von Neumann regular.

Corollary 2.10. *Let $\mathcal{C}$ be a small additive category. The following are equivalent*

(1) *$\mathcal{C}$ is right regular.*
(2) *The following conditions hold in $\mathcal{C}$:*
   i) *Every object c in $\mathcal{C}$ has the following property. Consider any directed set I and collections of morphisms $\{f_i : a_i \to c\}_{i\in I}$ with c as target such that $f_i \subseteq f_j$ holds for $i \leq j$. Then there exists $i_0 \in I$ with $f_i \subseteq f_{i_0}$ for all $i \in I$.*
   ii) *For every morphism $f : x \to y$ in $\mathcal{C}$ with pseudo ∞-kernel there exists $k \in \mathbb{N}$ and $\alpha : x_{k-1} \to x_{k-1}$ making the following diagram commute:*



In [3] another regularity in introduced due to bad behavior of regularity with respect to infinity product. Let R be a ring with unit. The ring R is *l-uniformly regular coherent*, if every finitely presented R-module M admits a l-dimensional finite projective resolution, i.e. there exists an exact sequence

$$0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0$$

such that each P_i is finitely generated and projective. This notion is generalized to additive categories in [3, Section 6]. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category and $l \geq 1$. We say that $\mathcal{C}$ is right *l-uniformly regular coherent*, if every object $F : \mathcal{C}^{op} \to Ab$ of type $\mathcal{FP}_1$ admits a l-dimensional finite projective resolution, i.e. there exists an exact sequence

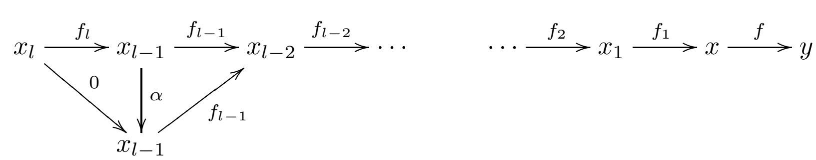
$$0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to F \to 0$$

such that each P_i is finitely generated and projective.

The equivalence $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab}) \simeq \mathrm{Fun}(\mathcal{C}^{op}_{\oplus}, \mathrm{Ab})$ implies that $\mathcal{C}$ is right l-uniformly regular coherent if and only if $\mathcal{C}_{\oplus}$ is right l-uniformly regular coherent. Note that, if $\mathcal{C}$ is right 1-coherent and every object $F : \mathcal{C}^{op} \to Ab$ of type $\mathcal{FP}_1$ has a projective dimension $\leq l$ then $\mathcal{C}$ is right l-uniformly regular coherent.

Corollary 2.11. *Let $l \geq 1$ and let $\mathcal{C}$ be a small additive category. Suppose that $\mathcal{C}$ is right 1-coherent. Then, the following are equivalent:*

(1) *$\mathcal{C}$ is right l-uniformly regular coherent.*
(2) *For every morphism $f : x \to y$ in $\mathcal{C}$ there exists $l \in \mathbb{N}$, an pseudo l-kernel $(f_l, f_{l-1}, \cdots, f_1)$ of f and $\alpha : x_{l-1} \to x_{l-1}$ making the following diagram commute:*



3. The ring $\mathcal{A}(\mathcal{C})$ and the $\mathbb{Z}$-linear category $\mathcal{C}$

In this section we study the relation between some properties of a $\mathbb{Z}$-linear category $\mathcal{C}$ with the properties of a ring $\mathcal{A}(\mathcal{C})$ associated to it. We prove the categories $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ and Mod-$\mathcal{A}(\mathcal{C})$ are equivalent.

3.1. **The ring $\mathcal{A}(\mathcal{C})$.** Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. Recall from [7]

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{a,b\in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(a,b). \tag{3.1}$$

If $f \in \mathcal{A}(\mathcal{C})$ write $f_{a,b}$ for the component in $\hom_{\mathcal{C}}(b,a)$. The following multiplication law

$$(fg)_{a,b} = \sum_{c\in \mathrm{ob}\mathcal{C}} f_{a,c}g_{c,b} \tag{3.2}$$

makes $\mathcal{A}(\mathcal{C})$ into an associative ring, which is unital if and only if $\mathrm{ob}\mathcal{C}$ is finite. Whatever the cardinal of $\mathrm{ob}\mathcal{C}$ is, $\mathcal{A}(\mathcal{C})$ is always a ring with *local units*, i.e. a filtering colimit of unital rings.

3.2. **The $\mathbb{Z}\mathcal{C}$-modules.** Recall that M is a unital right $\mathcal{A}(\mathcal{C})$-module if $M \cdot \mathcal{A}(\mathcal{C}) = M$. Consider Mod-$\mathcal{A}(\mathcal{C})$ the category of unital right $\mathcal{A}(\mathcal{C})$-modules. Let us define functors

$$\mathcal{S}(-) : \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab}) \to \text{Mod-}\mathcal{A}(\mathcal{C}) \qquad (-)_{\mathcal{C}} : \text{Mod-}\mathcal{A}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$$

Let $M \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$

$$\mathcal{S}(M) = \bigoplus_{a\in \mathrm{ob}\mathcal{C}} M(a)$$

Let $N \in$ Mod-$\mathcal{A}(\mathcal{C})$

$$N_{\mathcal{C}} : \mathcal{C}^{op} \to \mathrm{Ab} \qquad a \mapsto N \cdot \mathrm{id}_a\,.$$

Lemma 3.3. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category then*

$$\mathcal{A}(\mathcal{C}_{\oplus}) \cong \bigoplus_{n=1,m=1}^{\infty} M_{n\times m}(\mathcal{A}(\mathcal{C})).$$

Proof. It is straightforward from the definition. □

Lemma 3.4. *If N is a unital right $\mathcal{A}(\mathcal{C})$-module then*

$$\bigoplus_{a\in \mathrm{ob}\mathcal{C}} N \cdot \mathrm{id}_a = N.$$

Proof. For every $a \in \mathrm{ob}\mathcal{C}$ we have $N \cdot \mathrm{id}_a \subseteq N$ then $\bigoplus_{a\in\mathrm{ob}\mathcal{C}} N \cdot \mathrm{id}_a \subseteq N$. Let $n \in N$, because N is unital $N = N \cdot \mathcal{A}(\mathcal{C})$ then $n = \sum_{i=1}^{i=m} n_i \cdot f_i$ with $n_i \in N$ and $f_i \in \hom_{\mathcal{C}}(a_i, b_i)$. Let $I = \{a \in \mathrm{ob}\mathcal{C} : a = a_i, \text{for some } i = 1, \dots, m\}$ then

$$n = \sum_{i=1}^{i=m} n_i \cdot f_i = (\sum_{i=1}^{i=m} n_i \cdot f_i) \cdot (\sum_{a\in I} \mathrm{id}_a) = n \cdot \sum_{a\in I} \mathrm{id}_a$$

We conclude $N \subseteq \bigoplus_{a\in\mathrm{ob}\mathcal{C}} N \cdot \mathrm{id}_a$. □

Proposition 3.5. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category then*

$$\mathcal{S}(-) : \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab}) \to \text{Mod-}\mathcal{A}(\mathcal{C}) \qquad (-)_{\mathcal{C}} : \text{Mod-}\mathcal{A}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$$

are an equivalence of categories.

Proof. Let $N \in \text{Mod-}\mathcal{A}(\mathcal{C})$ and $M \in \text{Fun}(\mathcal{C}^{op}, \text{Ab})$ then

$$S(N_{\mathcal{S}}) = \bigoplus_{a \in \text{ob}\mathcal{C}} N_{\mathcal{C}}(a) = \bigoplus_{a \in \text{ob}\mathcal{C}} N \cdot \text{id}_a = N$$

$$(S(M))_{\mathcal{C}}(c) = S(M) \cdot \text{id}_c = \bigoplus_{a \in \text{ob}\mathcal{C}} M(a) \cdot \text{id}_c = M(c) \qquad \forall c \in \text{ob}\mathcal{C}$$

□

The abelian structure of $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ comes from the abelian structure in Ab. A sequence $M \xrightarrow{f} N \xrightarrow{g} R$ is exact in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ if for each object $c \in \mathcal{C}$ the sequence $M(c) \xrightarrow{f(c)} N(c) \xrightarrow{g(c)} R(c)$ is exact in Ab.

Proposition 3.6. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category then*

$$\mathcal{S}(-) : \text{Fun}(\mathcal{C}^{op}, \text{Ab}) \to \text{Mod-}\mathcal{A}(\mathcal{C}) \qquad (-)_{\mathcal{C}} : \text{Mod-}\mathcal{A}(\mathcal{C}) \to \text{Fun}(\mathcal{C}^{op}, \text{Ab})$$

are exact functors.

Proof. Let $M \xrightarrow{f} N \xrightarrow{g} R$ be an exact sequence in $\text{Mod-}\mathcal{A}(\mathcal{C})$. Let us prove $M_{\mathcal{C}} \xrightarrow{f_{\mathcal{C}}} N_{\mathcal{C}} \xrightarrow{g_{\mathcal{C}}} R_{\mathcal{C}}$ is exact in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ showing $M_{\mathcal{C}}(a) \xrightarrow{f_{\mathcal{C}}(a)} N_{\mathcal{C}}(a) \xrightarrow{g_{\mathcal{C}}(a)} R_{\mathcal{C}}(a)$ is exact for every object a in $\mathcal{C}$. By functoriality $\text{im}(f_{\mathcal{C}}(a)) \subseteq \ker(g_{\mathcal{C}}(a))$. Let $n \cdot \text{id}_a \in \ker(g_{\mathcal{C}}(a))$ then

$$g_{\mathcal{C}}(a)(n \cdot \text{id}_a) = g(n) \cdot \text{id}_a = g(n \cdot \text{id}_a) = 0$$

then $n \cdot \text{id}_a \in \ker(g) = \text{im}(f)$. There exists $m \in M$ such that $f(m) = n \cdot \text{id}_a$ then

$$f_{\mathcal{C}}(a)(m \cdot \text{id}_a) = f(m \cdot \text{id}_a) = f(m) \cdot \text{id}_a = (n \cdot \text{id}_a) \cdot \text{id}_a = n \cdot \text{id}_a$$

then $n \cdot \text{id}_a \in \text{im}(f_{\mathcal{C}}(a))$. We conclude $(-)_{\mathcal{C}}$ is exact.

We proceed to show that S is exact. Let $M \xrightarrow{f} N \xrightarrow{g} R$ be an exact sequence in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$. Consider

$$S(M) = \bigoplus_{a \in \text{ob}\mathcal{C}} M(a) \xrightarrow{S(f)} S(N) = \bigoplus_{a \in \text{ob}\mathcal{C}} N(a) \xrightarrow{S(g)} S(R) = \bigoplus_{a \in \text{ob}\mathcal{C}} R(a)$$

Similarly as above, let $\sum_{a \in \mathcal{C}} x_a \in \ker S(g)$ then

$$\begin{aligned} S(g)(\textstyle\sum_{a \in \mathcal{C}} x_a) = \textstyle\sum_{a \in \mathcal{C}} g(a)(x_a) = 0 \quad &\Rightarrow \quad g(a)(x_a) = 0 \qquad \forall x_a \in N(a) \\ &\Rightarrow \quad x_a \in \ker g(a) = \text{im}\, f(a) \qquad \forall x_a \in N(a) \\ &\Rightarrow \quad \exists y_a \in M(a) \text{ such that } f(a)(y_a) = x_a \end{aligned}$$

□

Corollary 3.7. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category.*

(1) *If $p : M \to N$ is an epimorphism in* $\text{Mod-}\mathcal{A}(\mathcal{C})$ *then $p_{\mathcal{C}} : M_{\mathcal{C}} \to N_{\mathcal{C}}$ is an epimorphism in* $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$.
(2) *If $\pi : M \to N$ is an epimorphism in* $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ *then $\mathcal{S}(\pi) : \mathcal{S}(M) \to \mathcal{S}(N)$ is an epimorphism in* $\text{Mod-}\mathcal{A}(\mathcal{C})$.
(3) $(M \oplus N)_{\mathcal{C}} = M_{\mathcal{C}} \oplus N_{\mathcal{C}}$ *in* $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$.
(4) $\mathcal{S}(M \oplus N) = \mathcal{S}(M) \oplus \mathcal{S}(N)$ *in* $\text{Mod-}\mathcal{A}(\mathcal{C})$.

Let A be a ring with local units. From [17] we recall that an A-module is *quasi-free* if it is isomorphic to a direct sum of modules of the form $e \cdot A$ with $e^2 = e$, $e \in A$. Quasi-free modules over a ring with local units plays the same role as free modules over a ring with unit. Also recall that M is a finitely generated module if and only if it is an image of a finitely generated quasi-free module. A finitely generated module M is projective if and only if it is a direct summand of a finitely generated quasi-free modules. In this paper we work with $A = \mathcal{A}(\mathcal{C})$ and we say that M is a quasi-free right $\mathcal{A}(\mathcal{C})$-module if it is isomorphic to a finite sum of modules $\mathrm{id}_a \cdot\mathcal{A}(\mathcal{C})$.

Lemma 3.8. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category.*

1. *If F is a free finitely generated module in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ then $S(F)$ is a quasi-free finitely generated $\mathcal{A}(\mathcal{C})$-module.*
2. *If P is a projective finitely generated module then $S(P)$ is projective and finitely generated.*
3. *If M is a quasi-free finitely generated $\mathcal{A}(\mathcal{C})$-module then $M_{\mathcal{C}}$ is a finitely generated free module.*
4. *If P is a projective finitely generated $\mathcal{A}(\mathcal{C})$-module then $P_{\mathcal{C}}$ is projective and finitely generated module.*

Proof. (1) Consider I a finite subset of object in $\mathcal{C}$ such that $F = \bigoplus_{b\in I} \hom_{\mathcal{C}}(-, b)$, then

$$S(F) = \bigoplus_{b\in I} S(\hom_{\mathcal{C}}(-, b)) = \bigoplus_{b\in I} \mathrm{id}_b \cdot\mathcal{A}(\mathcal{C})$$

then $S(F)$ is a quasi-free finitely generated $\mathcal{A}(\mathcal{C})$-module.

(2) If P is a projective and finitely generated module there exist Q such that $P \oplus Q = F$ with F a free module. Then $S(P) \oplus S(Q) = S(F)$ with $S(F)$ quasi-free and finitely generated, we conclude $S(P)$ is projective.

(3) If M is a quasi-free finitely generated $\mathcal{A}(\mathcal{C})$-module then $M = \bigoplus_{b\in I} \mathrm{id}_b \cdot\mathcal{A}(\mathcal{C})$ with I a finite set. Note

$$M_{\mathcal{C}}(a) = M \cdot \mathrm{id}_a = (\bigoplus_{b\in I} \mathrm{id}_b \cdot\mathcal{A}(\mathcal{C})) \cdot \mathrm{id}_a = \bigoplus_{b\in I} \hom_{\mathcal{C}}(a, b)$$

Then

$$M_{\mathcal{C}} = \bigoplus_{b\in I} \hom_{\mathcal{C}}(-, b)$$

is a free finitely generated module in $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$.

(4) If P is a projective finitely generated $\mathcal{A}(\mathcal{C})$-module there exist Q such that $P\oplus Q = F$ is a quasi-free finitely generated $\mathcal{A}(\mathcal{C})$-module, then $P_{\mathcal{C}}\oplus Q_{\mathcal{C}} = F_{\mathcal{C}}$ then $P_{\mathcal{C}}$ is projective and finitely generated module.

□

Proposition 3.9. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category with finite objects and $n \geq 1$.*

1. *The category $\mathcal{C}$ is right Noetherian if and only if $\mathcal{A}(\mathcal{C})$ is right Noetherian.*
2. *The category $\mathcal{C}$ is right n-coherent if and only if $\mathcal{A}(\mathcal{C})$ is right strong n-coherent.*
3. *The category $\mathcal{C}$ is regular n-coherent if and only if $\mathcal{A}(\mathcal{C})$ is right n-regular and strong n-coherent.*

Proof. (1) Let M be a finitely generated right $\mathcal{A}(\mathcal{C})$-module and N be a submodule. We have an epimorphism

$$\mathcal{A}(\mathcal{C}) \oplus \ldots \oplus \mathcal{A}(\mathcal{C}) \to M$$

then by Corollary 3.7 the following is an epimorphism

$$\mathcal{A}(\mathcal{C})_{\mathcal{C}} \oplus \ldots \oplus \mathcal{A}(\mathcal{C})_{\mathcal{C}} \to M_{\mathcal{C}}$$

As $\mathcal{A}(\mathcal{C})_{\mathcal{C}} = \bigoplus_{b \in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(-, b)$ we obtain that $M_{\mathcal{C}}$ is finitely generated.

$$\mathcal{A}(\mathcal{C})_{\mathcal{C}} \oplus \ldots \oplus \mathcal{A}(\mathcal{C})_{\mathcal{C}} = \bigoplus_{j \in J \ b_j \in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(-, b_j)$$

Because $\mathcal{C}$ is Noetherian then $N_{\mathcal{C}}$ is finitely generated. We have an epimorphism

$$\bigoplus_{i \in I} \hom_{\mathcal{C}}(-, a_i) \to N_{\mathcal{C}}$$

then

$$\bigoplus_{i \in I} \mathcal{S}(\hom_{\mathcal{C}}(-, a_i)) \to \mathcal{S}(N_{\mathcal{C}}) = N.$$

Consider the projection

$$p_i : \mathcal{A}(\mathcal{C}) \to \mathcal{S}(\hom_{\mathcal{C}}(-, a_i)) = \bigoplus_{c \in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(c, a_i)$$

Taking $n = \#I$ we obtain an epimorphism

$$\mathcal{A}(\mathcal{C})^n \to \bigoplus_{i \in I} \mathcal{S}(\hom_{\mathcal{C}}(-, a_i)) \to N,$$

then N is finitely generated.

Conversely if $M \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ is finitely generated let us show that every subobject is also finitely generated. Take N a submodule of M. There is an epimorphism

$$\bigoplus_{i \in I} \hom_{\mathcal{C}}(-, a_i) \to M$$

then we have an epimorphism

$$\bigoplus_{i \in I, c \in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(c, a_i) = \bigoplus_{i \in I} S(\hom_{\mathcal{C}}(-, a_i)) \to S(M).$$

We obtain that $S(N)$ is a submodule of $S(M)$ which is is finitely generated, then $S(N)$ is also finitely generated and $S(N)_{\mathcal{C}} = N$ is finitely generated.

(2) Let M be a finitely n-presented right $\mathcal{A}(\mathcal{C})$-module. Consider $m_0, m_1, \ldots, m_n \in \mathbb{N}$ such that

$$\mathcal{A}(\mathcal{C})^{m_n} \to \mathcal{A}(\mathcal{C})^{m_{n-1}} \to \ldots \to \mathcal{A}(\mathcal{C})^{m_1} \to \mathcal{A}(\mathcal{C})^{m_0} \to M \to 0$$

is exact. By Proposition 3.6 the following is also an exact sequence

$$\mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_n} \to \mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_{n-1}} \to \ldots \to \mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_1} \to \mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_0} \to M_{\mathcal{C}} \to 0$$

As $\mathcal{A}(\mathcal{C})_{\mathcal{C}} = \bigoplus_{b \in \mathrm{ob}\mathcal{C}} \hom_{\mathcal{C}}(-, b)$ we obtain that $M_{\mathcal{C}}$ is of type $\mathcal{FP}_n$. Because $\mathcal{C}$ is n-coherent there exist an exact sequence

$$\cdots \to P_{n+1} \to P_n \to \ldots \to P_1 \to P_0 \to M_{\mathcal{C}} \to 0$$

with P_i projective and finitely generated. Then

$$\cdots \to S(P_{n+1}) \to S(P_n) \to \ldots \to S(P_1) \to S(P_0) \to M \to 0$$

is exact and by Lemma 3.8 $S(P_i)$ is projective and finitely generated. Then $\mathcal{A}(\mathcal{C})$ is a strong n-coherent ring.

Conversely, if $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ is of type $\mathcal{FP}_n$ then $S(F)$ is an $\mathcal{A}(\mathcal{C})$-module of the type $\mathcal{FP}_n$. As $\mathcal{A}(\mathcal{C})$ is a strong n-coherent ring there exists P_i projective finitely generated $\mathcal{A}(\mathcal{C})$-modules such that

$$\ldots \to P_n \to \cdots \to P_1 \to P_0 \to S(F) \to 0$$

Then

$$\ldots \to (P_n)_{\mathcal{C}} \to \cdots \to (P_1)_{\mathcal{C}} \to (P_0)_{\mathcal{C}} \to F \to 0$$

where $(P_i)_{\mathcal{C}}$ are projective and finitely generated by Lemma 3.8(4).

(3) Let M be a finitely n-presented right $\mathcal{A}(\mathcal{C})$-module. We can note from the previous item that $M_{\mathcal{C}}$ is of type $\mathcal{FP}_n$. Because $\mathcal{C}$ is n-regular there exist an exact sequence

$$0 \to P_k \to P_{k-1} \to \ldots \to P_1 \to P_0 \to M_{\mathcal{C}} \to 0$$

with P_i projective and finitely generated. Then

$$0 \to S(P_k) \to S(P_{k-1}) \to \ldots \to S(P_1) \to S(P_0) \to M \to 0$$

is exact and by Lemma 3.8 $S(P_i)$ is projective and finitely generated. Then $\mathcal{A}(\mathcal{C})$ is a n-regular and strong n-coherent ring. The conversely is similar. □

Example 3.10. Let consider some examples of ℤ-linear categories with finite objects.

(1) Let R be a ring and $G = \mathbb{Z}_n$. Consider $\tilde{R} = \frac{R[t]}{<t^n>}$. The category $\mathcal{C}_{\tilde{R}}$ is the category with n objects and

$$\hom_{\mathcal{C}_{\tilde{R}}}(p, q) = \tilde{R}_{q-p} = R$$

Note $\mathcal{A}(\mathcal{C}_{\tilde{R}}) = M_{n\times n}(R)$. If R is a Noetherian ring, then $\mathcal{A}(\mathcal{C}_{\tilde{R}})$ is also Noetherian. By Proposition 3.9, then $\mathcal{C}_{\tilde{R}}$ is Noetherian.

(2) We recall from [8] that a ring is said to be (n, d)-ring is every n-presented R-module has projective dimension at most d. Remark that if $n \le n'$ and $d \le d'$, then every (n, d)-ring is also a (n', d')-ring.

Let R, S be a finite direct sum of fields and $\mathcal{C}$ be the ℤ-linear category with two objects a and b such that $\hom_{\mathcal{C}}(a, b) = \hom_{\mathcal{C}}(b, a) = 0$, $\hom_{\mathcal{C}}(a, a) = R$ and $\hom_{\mathcal{C}}(b, b) = S$. Notice $\mathcal{A}(\mathcal{C}) = R \oplus S$, by [8, Theorem 1.3 (i)] $\mathcal{A}(\mathcal{C})$ is a $(0, 0)$-ring and hence a Noetherian and regular coherent ring.

(3) Let G be a finite commutative group. An associative ring R graded by G is

$$R = \bigoplus_{g\in G} R_g$$

such that the multiplication satisfies $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. A (left) graded module over R is an R-module M together with a decomposition $M = \bigoplus_{g\in G} M_g$ such that $R_g M_h \subseteq M_{g+h}$. We denote by R-GrMod the category of graded R-modules. The category $\mathcal{C}_R$ is the ℤ-linear category whose set of objects is $\{g : g \in G\}$ and whose morphism groups are given by $\hom_{\mathcal{C}_R}(g,h) = R_{h-g}$. By [9, Lemma 2.2] there is an equivalence between R-GrMod and the additive functor category $\mathrm{Fun}(\mathcal{C}_R, \mathrm{Ab})$.

(4) Let G be a group and $\mathcal{F}$ be a finite family of subgroups of G, closed by conjugation and closed by taking subgroups. Let $\mathcal{C} = \mathrm{Or}_{\mathcal{F}}(G)$ be the orbit category

$$\mathrm{ob}\mathcal{C} = \{G/H : H \in \mathcal{F}\}$$
$$\hom_{\mathcal{C}}(G/H, G/K) = \{f : G/H \to G/K : f(gsH) = gf(sH)\}$$

4. K-theory of $\mathbb{Z}$-linear categories

Let $\mathcal{B}$ be a small abelian category, in [16] M. Schlichting prove that $K_{-1}(\mathcal{B}) = 0$. When $\mathcal{B}$ is in addition Noetherian, then $K_i(\mathcal{B}) = 0$, $\forall i < 0$. It was a question if the hypothesis of Noetherian was really necessary. In [16] the conjecture that if $\mathcal{B}$ is a small abelian category then $K_i(\mathcal{B}) = 0\ \forall i < 0$ was stated. In [14], A. Neeman shows a counterexample of this conjecture. In [1] the authors prove that if $\mathcal{B}[t_1, \ldots, t_n]$ is abelian for every $n \in \mathbb{N}$ then $K_i(\mathcal{B}) = 0\ \forall i < 0$. A similar result [3, Theorem 11.1] claims that if $\mathcal{A}$ is an additive category such that $\mathcal{A}[t_1, \ldots, t_n]$ is coherent regular for every $n \in \mathbb{N}$ then $K_i(\mathcal{A}) = 0\ \forall i < 0$. Finally [3, Corollary 11.2] state that if $\mathcal{A}$ is a regular (Noetherian and coherent regular) additive category then $K_i(\mathcal{A}) = 0$ $\forall i < 0$.

In this section we have a result of vanishing negative K-theory of $\mathbb{Z}$-linear categories. Recall from [7, Section 4] the definition of the K-theory spectrum of a $\mathbb{Z}$-linear category $\mathcal{C}$, the K-theory spectrum of the ring $\mathcal{A}(\mathcal{C})$ and the map

$$\varphi : K(\mathcal{C}) \to K(\mathcal{A}(\mathcal{C})) \tag{4.1}$$

which is a natural equivalence in $\mathcal{C}$, see [7, Proposition 4.2.8].

Theorem 4.2. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category with finite objects.*

(1) *If $\mathcal{C}$ is right regular then $K_i(\mathcal{C}) = 0\ \forall i < 0$.*
(2) *If $\mathcal{C}$ is right regular coherent then $K_{-1}(\mathcal{C}) = 0$.*

Proof. If $\mathcal{C}$ is regular, then $\mathcal{A}(\mathcal{C})$ is regular by Proposition 3.9. By fundamental theorem of K-theory $K_i(\mathcal{A}(\mathcal{C})) = 0\ \forall i < 0$. We conclude

$$K_i(\mathcal{C}) \simeq K_i(\mathcal{A}(\mathcal{C})) = 0 \qquad \forall i < 0.$$

If $\mathcal{C}$ is regular coherent then $\mathcal{A}(\mathcal{C})$ is a regular coherent ring. By [1, Theorem 3.30] we obtain $K_{-1}(\mathcal{A}(\mathcal{C})) = 0$ then

$$K_{-1}(\mathcal{C}) \simeq K_{-1}(\mathcal{A}(\mathcal{C})) = 0.$$

□

Corollary 4.3. *Let $\mathcal{D} = \mathcal{C}_{\oplus}$ with $\mathcal{C}$ be a $\mathbb{Z}$-linear category with finite objects.*

(1) *If $\mathcal{C}$ is right regular then $K_i(\mathcal{D}) = 0\ \forall i < 0$.*
(2) *If $\mathcal{C}$ is right regular coherent then $K_{-1}(\mathcal{D}) = 0$.*

Definition 4.4. A $\mathbb{Z}$-linear category $\mathcal{C}$ is right *AF-regular* if there is $\{\mathcal{C}_f\}_{f \in F}$ a direct system of right regular $\mathbb{Z}$-linear categories with finite objects such that

$$\mathcal{C} = \operatorname*{colim}_{f \in F} \mathcal{C}_f$$

Similarly we say that $\mathcal{C}$ is right *AF-Noetherian* (*AF-regular coherent*) if

$$\mathcal{C} = \operatorname*{colim}_{f \in F} \mathcal{C}_f$$

with $\mathcal{C}_i$ right Noetherian (regular coherent) $\mathbb{Z}$-linear categories with finite objects.

Theorem 4.5. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category.*

(1) *If $\mathcal{C}$ is right AF-regular then $K_i(\mathcal{C}) = 0$ $\forall i < 0$.*
(2) *If $\mathcal{C}$ is right AF-regular coherent then $K_{-1}(\mathcal{C}) = 0$.*

Proof. If $\mathcal{C} = \operatorname{colim}_{f\in F} \mathcal{C}_f$ then $K_i(\mathcal{C}) = \operatorname{colim}_{f\in F} K_i(\mathcal{C}_f)$. The rest of the proof follows from Theorem 4.2. □

Using [10, Thm 3.2] and Proposition 3.9 we obtain the following result.

Proposition 4.6. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category with finite objects. Suppose that $\mathcal{C}$ is right regular n-coherent. Then*

$$K_i(\mathcal{C}) \simeq K_i(\mathcal{FP}_n(\mathcal{A}(\mathcal{C}))) \quad i \geq 0.$$

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