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ALGEBRAIC kk-THEORY AND THE KH-ISOMORPHISM CONJECTURE

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Abstract. We relate the Davis-Lück homology with coefficients in Weibel's homotopy K-theory to the equivariant algebraic kk-theory using homotopy theory and adjointness theorems. We express the left hand side of the assembly map for the KH-isomorphism conjecture introduced by Bartels-Lück in terms of equivariant algebraic kk-groups.

1. Introduction

Algebraic kk-theory was introduced in [8] in order to show how methods from K-theory of operator algebras can be applied in a completely algebraic setting. The definition of algebraic kk-theory was motivated by the works [9] and [17] on Kasparov's KK-theory introduced in [21]. An equivariant version of the algebraic kk-theory was introduced in [13]. We can write a dictionary between Kasparov's KK-theory and algebraic kk-theory. Kasparov's KK-theory of separable C^*-algebras is the major tool in noncommutative topology. It is a common generalization both of topological K-homology and topological K-theory as an additive bivariant functor. Let A and B be separable C^*-algebras. The groups $KK_*(A,B)$ are defined such that

$$KK_*(\mathbb{C},B)\cong K_*^{top}(B) \qquad \text{and} \qquad KK^*(A,\mathbb{C})\cong K^*_{hom}(A).$$

Here $K_*^{top}(B)$ denotes the topological K-theory groups of B and $K^*_{hom}(A)$ the K-homology groups of A. The Kasparov groups $KK(A,B)=KK_0(A,B)$ for separable C^*-algebras A and B form the set of morphisms $A\to B$ in a category KK. The composition in KK is given by the Kasparov product. The category KK admits a triangulated category structure and there is a canonical functor k : $C^*\text{-Alg}\to KK$ which is homotopy invariant, C^*-stable and split-exact. Moreover, k is universal for these properties. Similarly algebraic kk-theory is a bivariant K-theory on Alg_ℓ, the category of algebras over a commutative ring ℓ. For ℓ-algebras A and B the groups $kk_*(A,B)$ are defined such that

$$kk_*(\ell,A)\cong \mathrm{KH}_*(A).$$

Here $\mathrm{KH}_*(A)$ denotes Weibel's homotopy K-theory groups. We can consider a triangulated category kk whose objects are the ℓ-algebras and whose morphisms are the elements of $kk(A,B)=kk_0(A,B)$. There exists a canonical functor j :

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$\mathrm{Alg}_\ell \to kk$ such that j is stable with respect to matrices, polynomial homotopy invariant and excisive. Moreover, j is universal for these properties.

Let G be a (discrete) group. The equivariant Kasparov's KK-theory is a bivariant K-theory defined on G-C^*-Alg, the category of separable C^*-algebras with an action by automorphisms of G. There exists a triangulated category KK^G and a functor $j : G\text{-}C^*\text{-Alg} \to KK^G$ that is universal for the properties: stable with respect to compact operators on $\ell^2(G \times \mathbb{N})$, continuous homotopy invariant and split exact. The equivariant algebraic kk-theory is a bivariant K-theory defined in $G\mathrm{Alg}_\ell$, the category of ℓ-algebras with an action of G. There exists a functor $j : G\mathrm{Alg}_\ell \to kk^G$ universal for the properties: G-stable, polynomial homotopy invariant and excisive. In the algebraic setting there are versions of the Green-Julg Theorem, the adjointness between induction and restriction functors and the Baaj-Skandalis duality. Let K be a finite group whose order is invertible in ℓ and let $A \in K\mathrm{Alg}_\ell$. The Green-Julg Theorem gives us a natural isomorphism

$$\psi_{GJ} : kk^K_*(\ell, A) \xrightarrow{\cong} kk_*(\ell, A \rtimes K) = \mathrm{KH}_*(A \rtimes K). \tag{1.1}$$

Let $H \leq G$. The adjunction between the induction and restriction functors gives us a natural isomorphism

$$\psi_{IR} : kk^G_*(\mathrm{Ind}^G_H(B), A) \xrightarrow{\cong} kk^H_*(B, \mathrm{Res}^H_G(A)), \qquad B \in H\mathrm{Alg}_\ell, A \in G\mathrm{Alg}_\ell. \tag{1.2}$$

Let H be a finite subgroup of G with $\frac{1}{|H|} \in \ell$. Notice that

$$\mathrm{Ind}^G_H(\ell) = \ell^{(G/H)} = \bigoplus_{G/H} \ell.$$

Combining the isomorphisms (1.1) and (1.2) we obtain an isomorphism

$$\psi : kk^G_*(\ell^{(G/H)}, A) \xrightarrow{\cong} \mathrm{KH}_*(A \rtimes H).$$

Can we express $\mathrm{KH}_*(A \rtimes G)$ as a kk^G-group? If G is finite with $\frac{1}{|G|} \in \ell$ the answer is positive; see (1.1). In this paper we go deep into this question. We consider the KH-isomorphism conjecture introduced in [3] and we write the left hand side of this conjecture in terms of kk^G-groups. One motivation to do this is to formulate in the algebraic context techniques used in some proofs of the Baum-Connes conjecture with coefficients.

The *Baum-Connes conjecture with coefficients* is formulated in [4]. Let G be a discrete group and let A be a separable G-C^*-algebra. Let $E_{\mathcal{F}in}(G)$ denote a model for the classifying space of G with respect to the family of finite subgroups. The Baum-Connes property is true for the pair (G, A) if the *assembly map* $\mu_{G,A}$,

$$\mu_{G,A} : KK^G_*(C_0(E_{\mathcal{F}in}(G)), A) = \operatorname*{colim}_X KK^G_*(C_0(X), A) \to K^{top}_*(A \rtimes G), \tag{1.3}$$

is an isomorphism; here X runs over the directed set of G-compact subsets of $E_{\mathcal{F}in}(G)$. For more details about the Baum-Connes conjecture see [16].

We are looking for an assembly map similar to (1.3) with KH instead of K^{top} and algebraic kk^G-groups instead of Kasparov's KK^G-groups.

For discrete groups, the Baum-Connes conjecture can also be formulated using homotopy theory through the Davis-Lück approach [11]. Write Sp for the category of (simplicial) spectra. Let $\mathrm{Or}(G)$ denote the orbit category of G: its objects are G/H with H a subgroup of G and its morphisms are G-equivariant maps $f : G/H \to G/K$. Recall that an $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ is a functor $\mathbf{E} : \mathrm{Or}(G) \to \mathrm{Sp}$.

In [11], an equivariant homology theory is constructed for each $\mathrm{Or}(G)$-spectrum $\mathbf{E}$. The homology theory $H^G(-;\mathbf{E})$ in the category of G-simplicial sets $\mathbb{S}^G$ is defined as follows:

$$H^G(-;\mathbf{E}):\mathbb{S}^G\to \mathrm{Sp},\qquad H^G(X;\mathbf{E})=\mathrm{map}_G(-,X)_+\otimes_{\mathrm{Or}(G)}\mathbf{E}(-). \tag{1.4}$$

Let $\mathcal{F}$ be a family of subgroups of G and let $E_{\mathcal{F}}G$ be a model of the classifying space of G with respect to $\mathcal{F}$, i.e. a G-simplicial set such that

$$(E_{\mathcal{F}}G)^H=\begin{cases}\text{contractible} & \text{if} \quad H\in\mathcal{F},\\ \emptyset & \text{if} \quad H\notin\mathcal{F}.\end{cases}$$

Let $p:E_{\mathcal{F}}(G)\to\star$ be the projection to the point. The *assembly map* associated to $(G,\mathcal{F},\mathbf{E})$ is the map

$$H^G(p):H^G(E_{\mathcal{F}}(G);\mathbf{E})\to H^G(\star;\mathbf{E}). \tag{1.5}$$

We remark that in the original construction of [11] topological spectra are used instead of simplicial spectra. In our work we use that Sp is a combinatorial category (see Lemma C.2) and that holds for simplicial spectra. In [7, Section 2] it is proved that Top^G is Quillen equivalent to $\mathbb{S}^G$ and that it is equivalent to work with assembly maps in the topological or in the simplicial setting. If $\mathbf{E}$ is the topological spectrum K_A^{top} and $\mathcal{F}=\mathcal{F}in$ then the assembly map (1.5) coincides with (1.3), see [22].

The KH-isomorphism conjecture was introduced in [3, Section 7]; see [23, Section 15.3] for the status of this conjecture. In this paper we use a particular $\mathrm{Or}(G)$-spectrum $\mathbf{E}$ that we proceed to describe. For a G-algebra B and $G/H\in\mathrm{Or}(G)$, we define an algebra $\mathcal{R}(B\rtimes G/H)$ that has the properties of being natural in G/H and kk^G-equivalent to $B\rtimes H$; see Sections 3.1 and 3.2. Let $\mathbb{K}$ be the spectrum representing kk-theory defined by Garkusha [14]; see Section C.2 for details. The KH-isomorphism conjecture states that for $\mathcal{F}=\mathcal{F}in$ and

$$\mathbf{E}:\mathrm{Or}(G)\to\mathrm{Sp},\quad \mathbf{E}(G/H)=\mathbb{K}(\ell,\mathcal{R}(B\rtimes G/H)), \tag{1.6}$$

the assembly map (1.5) is an isomorphism. Note that

$$\pi_*(\mathbf{E}(G/H))=\pi_*(\mathbb{K}(\ell,\mathcal{R}(B\rtimes G/H))=kk_*(\ell,\mathcal{R}(B\rtimes G/H))=\mathrm{KH}_*(B\rtimes H).$$

Let $\mathbb{K}^G$ be the spectrum representing kk^G-theory defined in [25]; see Section C.2 for details. The main theorem of this work is Theorem 6.6:

Theorem 1.7 (cf. Theorem 6.6)**.** *Let G be a group such that $\frac{1}{|H|}\in\ell$ for every finite subgrup $H\leq G$. Let B be a G-algebra and $\mathbf{E}$ be the $\mathrm{Or}(G)$-spectrum defined in (1.6). Then*

$$H^G(E_{\mathcal{F}in}G;\mathbf{E})\cong\operatorname*{colim}_X\mathbb{K}^G(\ell^{(X)},B)$$

where X runs over the $(G,\mathcal{F}in)$-finite subcomplexes of $E_{\mathcal{F}in}G$.

The proof of Theorem 1.7 has two parts. Let $\mathrm{Or}(G,\mathcal{F}in)$ be the full subcategory of $\mathrm{Or}(G)$ whose objects are G/H with $H\in\mathcal{F}in$. We consider the $\mathrm{Or}(G,\mathcal{F}in)$-spectrum

$$\mathbf{F}:\mathrm{Or}(G,\mathcal{F}in)\to\mathrm{Sp},\qquad \mathbf{F}(G/H)=\mathbb{K}^G(\ell^{(G/H)},B),$$

which provides an homology theory on $(G,\mathcal{F}in)$-complexes, see details in [11]. A $(G,\mathcal{F}in)$-*complex* is a G-simplicial set that is built from cells of the form $G/H\times\Delta^n$

with $H \in \mathcal{F}in$. The first step in the proof of Theorem 1.7 is to prove that for every $(G, \mathcal{F}in)$-complex Y we obtain:

$$\boxed{H^G(Y; \mathbf{E}) \cong H^G(Y; \mathbf{F})} \tag{1.8}$$

In order to prove the above, we construct in Theorem 5.35 a zig-zag of $\mathrm{Or}(G, \mathcal{F}in)$-spectra:

$$\mathbf{E}(G/H) = \mathbb{K}(A, \mathcal{R}(B \rtimes G/H)) \dashrightarrow \mathbb{K}^G(A^{(G/H)}, B) = \mathbf{F}(G/H)$$

The second step is to prove that

$$\boxed{H^G(Y; \mathbf{F}) = \operatorname{colim}_X \mathbb{K}^G(\ell^{(X)}, B)} \tag{1.9}$$

where Y is a $(G, \mathcal{F}in)$-complex and $X \subseteq Y$ is $(G, \mathcal{F}in)$-finite. This is proved in Lemma 6.4.

The paper is organized as follows. In Section 2 we recall the adjointness theorems from [13]. Upon composing (1.1) and (1.2) we get an isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \tag{1.10}$$

for $A \in \mathrm{Alg}$, $B \in G\mathrm{Alg}_\ell$ and H a finite subgroup of G such that $\frac{1}{|H|} \in \ell$. Along this section we give explicit descriptions of the unit and counit of this adjunction; this is summarized in Proposition 2.10. In Section 3 we define a triangulated functor $\mathcal{R}(- \rtimes G/H) : kk^G \to kk$ that is naturally isomorphic to the crossed product with H. This allows us to replace the right-hand side of (1.10) by $kk(A, \mathcal{R}(B \rtimes G/H))$ and consider it as a covariant functor on $\mathrm{Or}(G)$. In Section 4, we prove that the isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, \mathcal{R}(B \rtimes G/H)) \tag{1.11}$$

is natural in G/H; see Theorem 4.9. Moreover, we provide an explicit description for the counit of this adjunction in Lemma 4.6. Section 5 is the technical core of this work and is devoted to lifting the isomorphism (1.11) to a natural weak equivalence of spectra. By Lemma 4.6, we can describe the isomorphism (1.11) as the composite of the morphisms in the zig-zag (5.2). Upon replacing kk by $\mathbb{K}$ and kk^G by $\mathbb{K}^G$ we obtain a zig-zag:

$$\begin{tikzcd}
\mathbb{K}(A, \mathcal{R}(B \rtimes G/N)) \arrow[r, "(-)^{(G/N)}"] & \mathbb{K}^G(A^{(G/N)}, [\mathcal{R}(B \rtimes G/N)]^{(G/N)}) \\
\mathbb{K}^G(A^{(G/N)}, M_G B) & \mathbb{K}^G(A^{(G/N)}, \mathcal{R}\left[(B \rtimes G/N)^{(G/N)}\right]) \arrow[l, "\mathcal{R}(\zeta_{G/N})"'] \arrow[u, "\wr"']
\end{tikzcd} \tag{1.12}$$

Here the technical difficulties arise:

(1) How to consider the spectra on the right column as covariant functors $\mathrm{Or}(G, \mathcal{F}in) \to \mathrm{Sp}$ (or replace them by ones)?
(2) Once the previous question has been addressed, are the morphisms in (1.12) natural in G/N?

These issues are explained with greater detail in Section 5.1. In Section 5.2 we introduce a more convenient notation, rewriting (1.12) as:

$$J(t) \xrightarrow{(-)^{(t)}} M_t(t,t) \xleftarrow[\sim]{\psi} L_t(t,t) \xrightarrow{\mathcal{R}(\zeta_t)} \mathbb{K}^G(A^{(t)}, M_G B) \tag{1.13}$$

In Sections 5.3 and 5.4 we show how defining $\mathrm{Or}(G,\mathcal{F}in)$-spectra as objecwise coends provides an answer to (1). In Section 5.5 we define a morphism φ of $\mathrm{Or}(G,\mathcal{F}in)$-spectra that can be thought of as a convenient analogue to the morphism $(-)^{(t)}$ in (1.13). In Section 5.6 we introduce a model structure on categories of bifunctors that will allow us to build models for the homotopy coends of certain morphisms of bifunctors. In Section 5.7 we gather the previous results and give an answer to (2) in Theorem 5.35. In Section 6 we finally prove Theorem 1.7 and show that the left-hand side of the assembly map for the KH-isomorphism conjecture can be expressed in terms of equivariant algebraic kk-theory groups.

2. Adjoint theorems revisited

Throughout this text, ℓ will denote a commutative ring with unit and $\otimes$ will denote the tensor product over ℓ. We will refer to ℓ-algebras simply as *algebras*. An algebra with an action of a group G will be called a *G-algebra*. The following lemma will be useful later on.

Lemma 2.1 ([6, Proposition 2.2.6]). *Let $\mathcal{D}$ be a category and let $F : G\mathrm{Alg}_\ell \to \mathcal{D}$ be an M_2-stable functor. Let $B \subseteq C$ be an inclusion of G-algebras and let $V \in C$ be an invertible element such that $VB, BV^{-1} \subseteq B$ and $g \cdot V = V$ for all $g \in G$. Then the formula $\phi^V(b) = VbV^{-1}$ defines a G-algebra homomorphism $\phi^V : B \to B$ such that $F(\phi^V) = \mathrm{id}_{F(B)}$.*

Proof. It is easily verified that $g \cdot V^{-1} = V^{-1}$ for all $g \in G$ and that ϕ^V defines a G-algebra homomorphism. The rest of the proof of [6, Proposition 2.2.6] carries over verbatim. □

If X is a set, we will write M_X for the algebra of finite matrices with coefficients in ℓ that are indexed over $X \times X$. Let S be a G-set and let $|S|$ be its underlying set. We will write M_S for the ℓ-algebra $M_{|S|}$ endowed with the G-action defined by $g \cdot e_{s,t} = e_{g\cdot s, g\cdot t}$. For any $B \in G\mathrm{Alg}_\ell$, we have an ℓ-algebra isomorphism $R_{S,B} : (M_S \otimes B) \rtimes G \to M_{|S|} \otimes (B \rtimes G)$ defined by:

$$R_{S,B}((e_{s,t} \otimes b) \rtimes g) = e_{s,g^{-1}t} \otimes (b \rtimes g)$$

This isomorphism is clearly natural in S with respect to injective morphisms of G-sets and natural in B with respect to ℓ-algebra homomorphisms.

Let $G_+ = G \coprod \{*\}$ and let $\iota : \ell \to M_{G_+}$ (respectively $\iota' : M_G \to M_{G_+}$) be the morphism induced by the inclusion $\{*\} \subset G$ (resp. $G \subset G_+$). Let $B \in G\mathrm{Alg}_\ell$. Recall from [13, (4.1.3)] that we have the following zig-zag of isomorphisms in kk^G:

$$B \xrightarrow[\cong]{\iota_*} M_{G_+} \otimes B \xleftarrow[\cong]{(\iota')_*} M_G \otimes B \tag{2.2}$$

Let G be a group and let $H \subseteq G$ be a finite subgroup of order n such that $\frac{1}{n} \in \ell$. By [13, Theorem 5.2.1] and [13, Theorem 6.14] we have an adjunction isomorphism

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \tag{2.3}$$

for any $A \in \mathrm{Alg}_\ell$ and any $B \in G\mathrm{Alg}_\ell$. For $A \in \mathrm{Alg}_\ell$, let $\varphi_A : A \to A^{(G/H)} \rtimes H$ be the algebra homomorphism defined by:

$$\varphi_A(a) = a\chi_H \rtimes \frac{1}{n}\sum_{h\in H} h$$

Put $\eta_A = j(\varphi_A) \in kk(A, A^{(G/H)} \rtimes H)$. We will show that η_A is a unit for the adjunction (2.3). For $B \in G\mathrm{Alg}_\ell$, let $\psi_B : (B \rtimes H)^{(G/H)} \to M_G \otimes B$ be the G-algebra homomorphism defined by:

$$\psi_B\left((b \rtimes h)\chi_{wH}\right) = \sum_{p \in wH} e_{p,ph} \otimes p(b) \tag{2.4}$$

Let $\varepsilon_B \in kk^G((B \rtimes H)^{(G/H)}, B)$ be the following composite in kk^G, where the isomorphism on the right is given by the zig-zag (2.2):

$$(B \rtimes H)^{(G/H)} \xrightarrow{j^G(\psi_B)} M_G \otimes B \cong B \tag{2.5}$$

We will show that ε_B is a counit for the adjunction (2.3).

Lemma 2.6. *For any $B \in G\mathrm{Alg}_\ell$ we have $(\varepsilon_B \rtimes H) \circ \eta_{B \rtimes H} = \mathrm{id}_{B \rtimes H}$ in kk.*

Proof. It is easily verified that the following diagram in kk commutes, where all the arrows are isomorphisms:

$$\begin{tikzcd}
(M_G \otimes B) \rtimes H \arrow[r, "(\iota')_*"] \arrow[d, "(R_{G,B})_*"'] & (M_{G_+} \otimes B) \rtimes H \arrow[d, "(R_{G_+,B})_*"'] & B \rtimes H \arrow[l, "\iota_*"'] \arrow[dl, bend left, "\iota_*"] \\
M_{|G|} \otimes (B \rtimes H) \arrow[r, "(\iota')_*"'] & M_{|G_+|} \otimes (B \rtimes H) &
\end{tikzcd}$$

Thus, the isomorphism $(M_G \otimes B) \rtimes H \cong B \rtimes H$ in kk induced by the zig-zag (2.2) equals the composite:

$$(M_G \otimes B) \rtimes H \xrightarrow{j(R_{G,B})} M_{|G|} \otimes (B \rtimes H) \xleftarrow{j(e_{1,1} \otimes ?)} B \rtimes H$$

To prove the lemma, it will be enough to show that the composite

$$B \rtimes H \xrightarrow{\varphi_{B \rtimes H}} (B \rtimes H)^{(G/H)} \rtimes H \xrightarrow{\psi_B \rtimes H} (M_G \otimes B) \rtimes H \xrightarrow{R_{G,B}} M_{|G|} \otimes (B \rtimes H)$$

and the inclusion $e_{1,1} \otimes ? : B \rtimes H \to M_{|G|} \otimes (B \rtimes H)$ induce the same morphism in kk. Let Γ be the algebra of matrices with coefficients in $\tilde{B} \rtimes H$ indexed by $G \times G$ that have only finitely many nonzero coefficients in each column and each row. Notice that Γ is a unital algebra that contains $M_{|G|} \otimes (B \rtimes H)$ as a subalgebra. Let $V = \sum_{g \in G} e_{g,g} \otimes (1 \rtimes g) \in \Gamma$. We have:

$$\begin{aligned}
[R_{G,B} \circ (\psi_B \rtimes H) \circ \varphi_{B \rtimes H}](b \rtimes h) &= \frac{1}{n} \sum_{p,q \in H} e_{p,q} \otimes (p(b) \rtimes phq^{-1}) \\
&= V \left(\frac{1}{n} \sum_{p,q \in H} e_{p,q} \otimes (b \rtimes h) \right) V^{-1}
\end{aligned}$$

Moreover, $\frac{1}{n} \sum_{p,q \in H} e_{p,q}$ is a conjugate of $e_{1,1}$ in $M_{|G|}$; see [13, Remark 3.1.11]. By [6, Proposition 2.2.6], we have

$$j[R_{G,B} \circ (\psi_B \rtimes H) \circ \varphi_{B \rtimes H}] = j(e_{1,1} \otimes ?) : B \rtimes H \to M_{|G|} \otimes (B \rtimes H),$$

as we wanted to prove. $\square$

Lemma 2.7. *For any $A \in \mathrm{Alg}_\ell$ we have $\varepsilon_{A^{(G/H)}} \circ \left[(\eta_A)^{(G/H)} \right] = \mathrm{id}_{A^{(G/H)}}$ in kk^G.*

Proof. It is easily verified that the composite

$$A^{(G/H)} \xrightarrow{(\eta_A)^{(G/H)}} \left(A^{(G/H)} \rtimes H\right)^{(G/H)} \xrightarrow{\varepsilon_{A^{(G/H)}}} M_G \otimes A^{(G/H)} \xrightarrow{\iota'} M_{G_+} \otimes A^{(G/H)}$$

is given by:

$$a\chi_{wH} \mapsto \frac{1}{n} \sum_{p,q \in H} e_{wp,wq} \otimes a\chi_{wH} \tag{2.8}$$

To prove the lemma, it suffices to show that the above formula induces the same morphism as $\iota : A^{(G/H)} \to M_{G_+} \otimes A^{(G/H)}$, $\iota(a\chi_{wH}) = e_{*,*} \otimes a\chi_{wH}$, upon applying $j^G : G\mathrm{Alg}_\ell \to kk^G$. Let $\tilde{A}$ be the unitalization of A and let $\Gamma_{G_+}(\tilde{A})$ be the set of those matrices with coefficients in $\tilde{A}$, indexed over $G_+ \times G_+$, that have finitely many nozero coefficients in each row and each column. Then $\Gamma_{G_+}(\tilde{A})$ is a G-algebra with the usual matrix multiplication and the G-action defined by $(g \cdot a)_{x,y} := a_{g^{-1}\cdot x, g^{-1}\cdot y}$. Moreover, we have inclusions of G-ℓ-algebras as follows:

$$M_{G_+} \otimes A^{(G/H)} \subseteq \left(M_{G_+} \otimes A\right)^{G/H} \subseteq \left(\Gamma_{G_+}(\tilde{A})\right)^{G/H}$$

The G-action on the right is described by

$$(g \cdot f)(tH) := g \cdot f(g^{-1}tH),$$

where $f : G/H \to \Gamma_{G_+}(\tilde{A})$ is a function and $g \in G$. We will show that there exists an invertible $V \in (\Gamma_{G_+}(\tilde{A}))^{G/H}$ such that the following diagram commutes, where the horizontal morphism is given by (2.8):

$$\begin{tikzcd} A^{(G/H)} \arrow[r] \arrow[dr, "\iota"'] & M_{G_+} \otimes A^{(G/H)} \arrow[d, "\phi^V"] \\ & M_{G_+} \otimes A^{(G/H)} \end{tikzcd} \tag{2.9}$$

Once this is done, the result will follow by Lemma 2.1 since $j^G : G\mathrm{Alg}_\ell \to kk^G$ is M_2-stable. For $wH \in G/H$, define:

$$V_{wH} := \sum_{x \in wH} \left(\tfrac{n-1}{n} e_{x,x} + e_{*,x} + e_{x,*}\right) - \sum_{x,y \in wH} \tfrac{1}{n} e_{x,y} + \sum_{g \in G \setminus wH} e_{g,g} \in \Gamma_{G_+}(\tilde{A})$$

Note that $g \cdot V_{wH} = V_{gwH}$ for all $g \in G$. We can picture V_{wH} as a block diagonal matrix having an identity block in the coordinates corresponding to $g \in G \setminus wH$, and the following block in the coordinates corresponding to elements of $wH \cup \{*\}$:

$$\begin{array}{c} \\ wH\left\{\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right. \\ \\ \end{array} \begin{array}{c} \overbrace{\hspace{14em}}^{wH} \\ \left(\begin{array}{ccccc:c} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & 1 \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & 1 \\ \vdots & & \ddots & & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} & -\frac{1}{n} & 1 \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{n-1}{n} & 1 \\ \hdashline 1 & 1 & \cdots & 1 & 1 & 0 \end{array}\right) \end{array}$$

It is easily verified that V_{gH} is invertible with inverse given by:

$$V_{wH}^{-1} := \sum_{x\in wH}\left(\tfrac{n-1}{n}e_{x,x} + \tfrac{1}{n}e_{*,x} + \tfrac{1}{n}e_{x,*}\right) - \sum_{x,y\in wH}\tfrac{1}{n}e_{x,y} + \sum_{g\in G\setminus wH} e_{g,g} \in \Gamma_{G_+}(\tilde{A})$$

Again, we can think of V_{wH}^{-1} as a block diagonal matrix having an identity block in the coordinates corresponding to $g \in G \setminus wH$, and the following block in the coordinates corresponding to elements of $wH \cup \{*\}$:

$$\begin{array}{c} \overbrace{\hphantom{\tfrac{n-1}{n}\quad -\tfrac{1}{n}\quad \cdots \quad -\tfrac{1}{n}\quad -\tfrac{1}{n}}}^{wH} \\ wH\left\{\left(\begin{array}{ccccc|c} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & \frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & \frac{1}{n} \\ \vdots & & \ddots & & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} & -\frac{1}{n} & \frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{n-1}{n} & \frac{1}{n} \\ \hline \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} & 0 \end{array}\right)\right. \end{array}$$

Define $V^{\pm 1} : G/H \to \Gamma_{G_+}(\tilde{A})$ by $V^{\pm 1}(wH) := V_{wH}^{\pm 1}$. Then $V, V^{-1} \in (\Gamma_{G_+}(\tilde{A}))^{G/H}$ are mutual inverses and

$$V(M_{G_+} \otimes A^{(G/H)}), (M_{G_+} \otimes A^{(G/H)})V^{-1} \subseteq (M_{G_+} \otimes A^{(G/H)}).$$

Moreover, $g \cdot V = V$ for all $g \in G$. An easy computation shows that the triangle (2.9) commutes. This finishes the proof. $\square$

Proposition 2.10 (cf. [13, Theorems 5.2.1 and 6.14])**.** *Let G be a group and let $H \subseteq G$ be a finite subgroup of order n such that $\frac{1}{n} \in \ell$. Then the morphisms $\eta_A \in kk(A, A^{(G/H)} \rtimes H)$ and $\varepsilon_B \in kk^G((B \rtimes H)^{(G/H)}, B)$ defined above are respectively the unit and the counit of an adjunction:*

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \tag{2.11}$$

Proof. It follows immediately from Lemmas 2.6 and 2.7. $\square$

3. Crossed product with G/H

We want to show that the adjunction (2.11) is natural in G/H. The first thing to do is to replace the right-hand side by an actual functor on $\mathrm{Or}(G, \mathcal{F}in)$.

3.1. Non unital ℓ-linear categories.

Definition 3.1. A *non-unital ℓ-linear category* $\mathcal{C}$ consists of:

(1) a set of objects $\mathrm{ob}\,\mathcal{C}$,
(2) an ℓ-module $\mathcal{C}(x, y)$ for every $x, y \in \mathrm{ob}\,\mathcal{C}$, and
(3) ℓ-module homomorphisms

$$\circ : \mathcal{C}(y, z) \otimes \mathcal{C}(x, y) \to \mathcal{C}(x, z) \tag{3.2}$$

for every $x, y, z \in \mathrm{ob}\,\mathcal{C}$, that are associative in the obvious way.

Non-unital ℓ-linear categories with only one object can be identified with (non-unital) ℓ-algebras. In the sequel, we will refer to non-unital ℓ-linear categories simply as *ℓ-linear categories*.

Definition 3.3. Let $\mathcal{C}$ and $\mathcal{D}$ be ℓ-linear categories. An ℓ-linear functor $F : \mathcal{C} \to \mathcal{D}$ consists of a function $F : \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$ together with ℓ-module homorphisms

$$F_{x,y} : \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y)) \tag{3.4}$$

that are compatible with the composition.

We will write Cat_ℓ for the category whose objects are ℓ-linear categories and whose morphisms are ℓ-linear functors.

Let $\mathcal{C}$ and $\mathcal{D}$ be ℓ-linear categories. The *tensor product* $\mathcal{C} \otimes \mathcal{D}$ is the ℓ-linear category with objects $\operatorname{ob}(\mathcal{C}) \otimes \operatorname{ob}(\mathcal{D})$ and such that:

$$(\mathcal{C} \otimes \mathcal{D})((c, d), (\tilde{c}, \tilde{d})) = \mathcal{C}(c, \tilde{c}) \otimes \mathcal{D}(d, \tilde{d})$$

Composition is defined in the usual way, using the composition laws in $\mathcal{C}$ and $\mathcal{D}$ and the commutativity of the tensor product of ℓ-modules.

We proceed to recall some definitions from [7, Section 3]. Let $\mathcal{C}$ be an ℓ-linear category. Put:

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{x,y \in \operatorname{ob}\mathcal{C}} \mathcal{C}(x, y)$$

If $f \in \mathcal{A}(\mathcal{C})$, write $f_{y,x}$ for its component in $\mathcal{C}(x, y)$. Then $\mathcal{A}(\mathcal{C})$ is an ℓ-algebra with multiplication given by

$$(gf)_{y,x} = \sum_{z \in \operatorname{ob}\mathcal{C}} g_{y,z} \circ f_{z,x}$$

Example 3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be ℓ-linear categories. It is easily verified that:

$$\mathcal{A}(\mathcal{C} \otimes \mathcal{D}) \cong \mathcal{A}(\mathcal{C}) \otimes \mathcal{A}(\mathcal{D})$$

Example 3.6. Let $A \in \mathrm{Alg}_\ell$, let G be a group and let H be a subgroup. We can regard $\mathcal{A}\left(A \rtimes \mathcal{G}^G(G/H)\right)$ as a subalgebra of $M_{|G/H|}(A \rtimes G)$ using the inclusion that sends $a \rtimes g \in \mathrm{Hom}_{A \rtimes \mathcal{G}^G(G/H)}(uH, vH)$ to $e_{vH,uH} \cdot (a \rtimes g)$.

A problem with $\mathcal{A}(\mathcal{C})$ is that it is not natural with respect to all ℓ-linear functors, but only with respect to those that are injective on objects; see [7, p.1231]. To fix this, one defines the ℓ-algebra $\mathcal{R}(\mathcal{C})$ [7, Section 3.4]. If M is an ℓ-module, write $T(M) = \oplus_{n \geq 1} M^{\otimes n}$ for the unaugmented tensor algebra. Put:

$$\mathcal{R}(\mathcal{C}) = T\left(\mathcal{A}(\mathcal{C})\right) / \langle \{ g \otimes f - g \circ f : f \in \mathcal{C}(x, y),\, g \in \mathcal{C}(y, z),\, x, y, z \in \operatorname{ob}\mathcal{C} \} \rangle$$

This defines a functor $\mathcal{R} : \mathrm{Cat}_\ell \to \mathrm{Alg}_\ell$.

Remark 3.7. Let G be a group. One can define a G-category as an ℓ-linear category $\mathcal{C}$ such that the hom-modules $\mathcal{C}(x, y)$ are G-modules and the composition law (3.2) is G-equivariant, endowing $\mathcal{C}(y, z) \otimes \mathcal{C}(x, y)$ with the diagonal G-action. If $\mathcal{C}$ and $\mathcal{D}$ are G-categories, a G-functor $F : \mathcal{C} \to \mathcal{D}$ is an ℓ-linear functor such that the morphisms (3.4) are G-equivariant. These definitions give rise to a category $G\mathrm{Cat}_\ell$ whose objects are G-categories and whose morphisms are G-functors.

If $\mathcal{C}$ is a G-category, then $\mathcal{A}(\mathcal{C})$ and $\mathcal{R}(\mathcal{C})$ are G-algebras in a natural way. Thus, we have a functor $\mathcal{R} : G\mathrm{Cat}_\ell \to G\mathrm{Alg}_\ell$.

Example 3.8. Let $\mathcal{C}$ be an ℓ-linear category and $D \in \mathrm{Alg}_\ell$. We claim that there is a natural morphism:

$$\mathcal{R}(\mathcal{C} \otimes D) \to \mathcal{R}(\mathcal{C}) \otimes D$$

To see this, first note that there is an ℓ-linear functor $\mathcal{C}\to\mathcal{R}(\mathcal{C})$ that takes $f\in\mathcal{C}(x,y)$ to the class in $\mathcal{R}(\mathcal{C})$ of $f\in\mathcal{C}(x,y)\subseteq\mathcal{A}(\mathcal{C})\subseteq T(\mathcal{A}(\mathcal{C}))$. Upon tensoring this functor with D and then applying $\mathcal{R}(-)$, we get the desired morphism:

$$\mathcal{R}(\mathcal{C}\otimes D)\to\mathcal{R}(\mathcal{R}(\mathcal{C})\otimes D)=\mathcal{R}(\mathcal{C})\otimes D \tag{3.9}$$

If $\mathcal{C}$ is a G-category and D is a G-algebra, then (3.8) is a morphism of G-algebras.

Is $\mathcal{C}$ is an ℓ-linear category, there is a morphism $p:\mathcal{R}(\mathcal{C})\to\mathcal{A}(\mathcal{C})$ induced by multiplication in $\mathcal{A}(\mathcal{C})$.

Lemma 3.10 (cf. [7, Lemma 3.4.3])**.** *Let $\mathcal{C}$ be an ℓ-linear category (respectively a G-category). Then the morphism*

$$p:\mathcal{R}(\mathcal{C})\to\mathcal{A}(\mathcal{C})$$

induces an isomorphism in kk (resp. in kk^G).

Proof. The proof of [7, Lemma 3.4.3] carries on verbatim in this setting. □

Corollary 3.11. *Let $\mathcal{C}\in\mathrm{Cat}_\ell$ and $D\in\mathrm{Alg}_\ell$ (resp. $\mathcal{C}\in G\mathrm{Cat}_\ell$ and $D\in G\mathrm{Alg}_\ell$). Then the morphism*

$$\mathcal{R}(\mathcal{C}\otimes D)\to\mathcal{R}(\mathcal{C})\otimes D$$

of Example 3.8 is an isomorphism in kk (resp. in kk^G).

Proof. It is easily verified that the following diagram commutes in Alg_ℓ (resp. in $G\mathrm{Alg}_\ell$):

$$\begin{tikzcd}
\mathcal{R}(\mathcal{C}\otimes D) \arrow[r] \arrow[d, "p"'] & \mathcal{R}(\mathcal{C})\otimes D \arrow[d, "p\otimes\mathrm{id}_D"] \\
\mathcal{A}(\mathcal{C}\otimes D) \arrow[r, "\cong"] & \mathcal{A}(\mathcal{C})\otimes D
\end{tikzcd}$$

Indeed, it suffices to check commutativity on the generators $f\otimes d$, and this is immediate. The result follows from Lemma 3.10. □

3.2. **Crossed product with G/H.** Fix $G/H\in\mathrm{Or}(G)$ and write $-\rtimes G/H$ for the functor $-\rtimes\mathcal{G}^G(G/H):G\mathrm{Alg}_\ell\to\mathrm{Cat}_\ell$. We claim that the composite functor

$$G\mathrm{Alg}_\ell\xrightarrow{-\rtimes G/H}\mathrm{Cat}_\ell\xrightarrow{\mathcal{R}}\mathrm{Alg}_\ell\xrightarrow{j}kk \tag{3.12}$$

factors through $j^G:G\mathrm{Alg}_\ell\to kk^G$. To prove this, it suffices to show that it is excisive, homotopy invariant and G-stable [13, Theorem 4.1.1]. Homotopy invariance and G-stability follow easily from the following.

Lemma 3.13. *Let H be a subgroup of G and let $uH\in G/H$. Write ${}^uH$ for the conjugate uHu^{-1}. Then the composite functor in (3.12) is naturally isomorphic to:*

$$A\mapsto j(\mathrm{Res}_G^{{}^uH}(A)\rtimes{}^uH). \tag{3.14}$$

Proof. To ease notation, we will omit $\mathrm{Res}_G^{{}^uH}$ throughout the proof and write $A\rtimes{}^uH$ instead of $\mathrm{Res}_G^{{}^uH}(A)\rtimes{}^uH$. Let $B\in G\mathrm{Alg}_\ell$. Consider $B\rtimes{}^uH\subseteq B\rtimes G/H$ as the full subcategory whose only object is uH. Upon applying $\mathcal{R}$ to this inclusion, we get an algebra homomorphism:

$$B\rtimes{}^uH=\mathcal{R}(B\rtimes{}^uH)\longrightarrow\mathcal{R}\left(B\rtimes G/H\right)$$

Let ν_{uH} be the image of this morphism in kk. Clearly, ν_{uH} is a natural transformation from (3.14) to the composite (3.12). To finish the proof, we will show that

ν_{uH} is an isomorphism. We claim that there is an isomorphism α that fits in the following commutative diagram in kk:

$$\begin{tikzcd}
B \rtimes {}^uH \arrow[r, "\nu_{uH}"] \arrow[ddr, bend right, "\cdot e_{uH,uH}"'] & \mathcal{R}(B \rtimes G/H) \arrow[d, "\simeq"', "p"] \\
& \mathcal{A}(B \rtimes G/H) \arrow[d, "\simeq"', "\alpha"] \\
& M_{|G/H|}(B \rtimes {}^uH)
\end{tikzcd}$$

Here, the bent arrow is induced by the inclusion into the (uH, uH)-coefficient and it is an isomorphism by matrix invariance. It follows that ν_{uH} is an isomorphism too. The isomorphism α is constructed as in the proof of [7, Lemma 3.2.6]. More precisely, let $s : G/H \to G$ be a section of the projection such that $s(uH) = u$. Write $\hat{g} = s(gH)$ for $g \in G$. For $b \in B$ and $g \in \hom_{\mathcal{G}^G(G/H)}(sH, tH)$ put:

$$\alpha(b \rtimes g) = e_{tH,sH} \cdot (u\hat{t}^{-1}(b) \rtimes u\hat{t}^{-1} g \hat{s} u^{-1})$$

It is straightforward to verify that this formula defines an isomorphism of algebras $\alpha : \mathcal{A}(B \rtimes G/H) \to M_{|G/H|}(B \rtimes {}^uH)$. □

Corollary 3.15 (cf. [13, Proposition 5.1.2 and Section 6])**.** *The composite functor* (3.12) *is homotopy invariant and G-stable.*

Proof. Write $F : G\mathrm{Alg}_\ell \to kk$ for the functor $F = j(\mathrm{Res}^H_G(-) \rtimes H)$. By Lemma 3.13 it suffices to show that F is homotopy invariant and G-stable. By definition, F is the composite:

$$G\mathrm{Alg}_\ell \xrightarrow{\mathrm{Res}^H_G} H\mathrm{Alg}_\ell \xrightarrow{j(-\rtimes H)} kk$$

The functor $j(- \rtimes H)$ is homotopy invariant and H-stable by [13, Proposition 5.1.2]. The functor Res^H_G is easily seen to send homotopic morphisms in $G\mathrm{Alg}_\ell$ to homotopic morphisms in $H\mathrm{Alg}_\ell$. It follows that F is homotopy invariant. Recall the definition of G-stable functor from [13, Section 3.1]. Let $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ be G-modules by locally finite automorphisms such that $\mathrm{card}(B_i) \leq \mathrm{card}(\mathbb{N})$ for $i = 1, 2$ and let A be a G-algebra. Then $(\mathcal{W}_1, B_1)$ and $(\mathcal{W}_2, B_2)$ are H-modules by locally finite automorphisms and thus

$$\mathrm{Res}^H_G(\mathrm{End}^F_\ell(\mathcal{W}_1)) \otimes \mathrm{Res}^H_G A \to \mathrm{Res}^H_G(\mathrm{End}^F_\ell(\mathcal{W}_1 \oplus \mathcal{W}_2)) \otimes \mathrm{Res}^H_G A$$
$$f \mapsto \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$$

becomes an isomorphism upon applying $j(- \rtimes H)$. It follows that F is G-stable. □

We still have to show that the composite (3.12) is excisive.

Construction 3.16. Let $\mathcal{E} : A \to B \to C$ be an extension in $G\mathrm{Alg}_\ell$ (that splits in $G\mathrm{Mod}_\ell$). We will to construct a triangle in kk as follows, that is natural with respect to morphisms of extensions:

$$\Omega(\mathcal{R}(C \rtimes G/H)) \xrightarrow{\partial_{\mathcal{E} \rtimes G/H}} \mathcal{R}(A \rtimes G/H) \longrightarrow \mathcal{R}(B \rtimes G/H) \longrightarrow \mathcal{R}(C \rtimes G/H)$$

Let $\nu_H : j(\mathrm{Res}^H_G(-) \rtimes H) \to j\mathcal{R}(- \rtimes G/H)$ be the natural isomorphism of Lemma 3.13. Note that $j(\mathrm{Res}^H_G(-) \rtimes H)$ is excisive, since Res^H_G sends extensions in $G\mathrm{Alg}_\ell$ to extensions in $H\mathrm{Alg}_\ell$ and $j(- \rtimes H) : H\mathrm{Alg}_\ell \to kk$ is excisive [13, Proposition 5.1.2]. To simplify notation, we omit explicit mention to j and Res^H_G. For example, we write $A \rtimes H$ instead of $j(\mathrm{Res}^H_G(A) \rtimes H)$. We have the following commutative diagram of solid arrows in kk, where the top row is a triangle:

$$\begin{tikzcd}
\Omega(C \rtimes H) \arrow[r, "\partial_{\mathcal{E}\rtimes H}"] \arrow[d, "\Omega(\nu_H)", "\simeq"'] & A \rtimes H \arrow[r] \arrow[d, "\nu_H", "\simeq"'] & B \rtimes H \arrow[r] \arrow[d, "\nu_H", "\simeq"'] & C \rtimes H \arrow[d, "\nu_H", "\simeq"'] \\
\Omega(\mathcal{R}(C \rtimes G/H)) \arrow[r, dashed] & \mathcal{R}(A \rtimes G/H) \arrow[r] & \mathcal{R}(B \rtimes G/H) \arrow[r] & \mathcal{R}(C \rtimes G/H)
\end{tikzcd}$$

Define $\partial_{\mathcal{E}\rtimes G/H} : \Omega(\mathcal{R}(C \rtimes G/H)) \to \mathcal{R}(A \rtimes G/H)$ to be the dashed arrow that makes the left square commute. Then the bottom row becomes a triangle too. This triangle is clearly natural with respect to the extension $\mathcal{E}$. These morphisms $\partial_{\mathcal{E}\rtimes G/H}$ make the composite (3.12) into an excisive homology theory.

Proposition 3.17. *Let H be subgroup of G. Then there exists a unique triangulated functor $\overline{- \rtimes G/H} : kk^G \to kk$ that makes the following diagram commute:*

$$\begin{tikzcd}
G\mathrm{Alg}_\ell \arrow[r, "\mathcal{R}(-\rtimes G/H)"] \arrow[d, "j^G"'] & \mathrm{Alg}_\ell \arrow[d, "j"] \\
kk^G \arrow[r, "\overline{-\rtimes G/H}"'] & kk
\end{tikzcd}$$

Moreover, for every extension $\mathcal{E} : A \to B \to C$ in $G\mathrm{Alg}_\ell$ and every $uH \in G/H$, the following square in kk commutes

$$\begin{tikzcd}
\Omega(C \rtimes {}^uH) \arrow[r, "\partial_{\mathcal{E}\rtimes {}^uH}"] \arrow[d, "\Omega(\nu_{uH})"', "\simeq"] & A \rtimes {}^uH \arrow[d, "\nu_{uH}", "\simeq"'] \\
\Omega(\mathcal{R}(C \rtimes G/H)) \arrow[r, "\partial_{\mathcal{E}\rtimes G/H}"] & \mathcal{R}(A \rtimes G/H)
\end{tikzcd} \tag{3.18}$$

Proof. The functor composite functor (3.12) is homotopy invariant and G-stable by Corollary 3.15. Moreover, endowed with the morphisms $\partial_{\mathcal{E}\rtimes G/H}$ defined in Construction 3.16, it becomes an excisive homology theory. Then the existence of $\overline{- \rtimes G/H} : kk^G \to kk$ follows from [13, Theorem 4.1.1].

To prove the assertion about (3.18), consider the following diagram in kk:

$$\begin{tikzcd}
\Omega(C \rtimes {}^uH) \arrow[r, "\partial_{\mathcal{E}\rtimes {}^uH}"] \arrow[d, "\Omega(\nu_{uH})"', "\simeq"] & A \rtimes {}^uH \arrow[d, "\nu_{uH}", "\simeq"'] \\
\Omega(\mathcal{R}(C \rtimes G/H)) \arrow[r, "\partial_{\mathcal{E}\rtimes G/H}"] \arrow[d, "\Omega(\nu_H)^{-1}"', "\simeq"] & \mathcal{R}(A \rtimes G/H) \arrow[d, "(\nu_H)^{-1}", "\simeq"'] \\
\Omega(C \rtimes H) \arrow[r, "\partial_{\mathcal{E}\rtimes H}"] & A \rtimes H
\end{tikzcd} \tag{3.19}$$

The bottom square commutes by definition of $\partial_{\mathcal{E}\rtimes G/H}$; see Construction 3.16. Thus, the commutativity of (3.18) is equivalent to that of the outer square in (3.19). We will see that the latter commutes since it is induced by a morphism of extensions in Alg_ℓ. For $D \in G\mathrm{Alg}_\ell$, let $\varphi_u : D \rtimes {}^uH \to D \rtimes H$ be the algebra homomorphism defined by $\varphi_u(a \rtimes g) = u^{-1}(a) \rtimes u^{-1}gu$. We have a morphism of extensions

$$\begin{tikzcd}
A \rtimes {}^uH \arrow[r] \arrow[d, "\varphi_u"] & B \rtimes {}^uH \arrow[r] \arrow[d, "\varphi_u"] & C \rtimes {}^uH \arrow[d, "\varphi_u"] \\
A \rtimes H \arrow[r] & B \rtimes H \arrow[r] & C \rtimes H
\end{tikzcd}$$

that induces a morphism of triangles in kk. In particular, there is a commutative square in kk as follows:

$$\begin{tikzcd}
\Omega(C \rtimes {}^uH) \arrow[r, "\partial_{\mathcal{E} \rtimes {}^uH}"] \arrow[d, "\Omega j(\varphi_u)"'] & A \rtimes {}^uH \arrow[d, "j(\varphi_u)"] \\
\Omega(C \rtimes H) \arrow[r, "\partial_{\mathcal{E} \rtimes H}"] & A \rtimes H
\end{tikzcd}$$

This square turns out to be the outer square of (3.19). To see this, it suffices to show that $j(\varphi_u) = (\nu_H)^{-1} \circ \nu_{uH}$ or, equivalently, that

$$j(p) \circ \nu_H \circ j(\varphi_u) = j(p) \circ \nu_{uH} \tag{3.20}$$

where $p : \mathcal{R}(D \rtimes G/H) \longrightarrow \mathcal{A}(D \rtimes G/H)$. Each side of (3.20) is induced by an algebra homomorphism $D \rtimes {}^uH \to \mathcal{A}(D \rtimes G/H)$. We will show that both morphisms are conjugate in $M_{|G/H|}(D \rtimes G)$—regarding $\mathcal{A}(D \rtimes G/H)$ as a subalgebra of $M_{|G/H|}(D \rtimes G)$ with the inclusion of Example 3.6. A straightforward verification shows that the left- and the right-hand sides of (3.20) are induced, respectively, by the algebra homomorphisms λ and ρ defined by:

$$\begin{aligned}
\lambda(d \rtimes g) &= e_{H,H} \cdot (u^{-1}(d) \rtimes u^{-1}gu) \\
\rho(d \rtimes g) &= e_{uH,uH} \cdot (d \rtimes g)
\end{aligned}$$

Now put:

$$V = \sum_{vH \in G/H} e_{uvH,vH} \cdot (1 \rtimes u) \in \Gamma_{|G/H|}(\tilde{D} \rtimes G) \supset M_{|G/H|}(D \rtimes G)$$

It is easily seen that $\rho = V \cdot \lambda \cdot V^{-1}$. This proves the equality (3.20) and concludes the proof of the proposition. □

Proposition 3.21. *Let G be a group. Then every morphism $G/H \to G/K$ in $\mathrm{Or}(G)$ induces a (graded) natural transformation $\overline{- \rtimes G/H} \to \overline{- \rtimes G/K}$ of functors $kk^G \to kk$. Moreover, these assemble into a functor $- \rtimes - : kk^G \times \mathrm{Or}(G) \to kk$.*

Proof. Let $f : G/H \to G/K$ be a morphism in $\mathrm{Or}(G)$. Clearly, f induces a natural transformation $j \circ \mathcal{R}(- \rtimes G/H) \to j \circ \mathcal{R}(- \rtimes G/K)$ of functors $G\mathrm{Alg}_\ell \to kk$. We will prove that this natural transformation is compatible with the excisive homology theory structures. More precisely, let $\mathcal{E} : A \to B \to C$ be an extension in $G\mathrm{Alg}_\ell$. We will prove that the following square in kk commutes:

$$\begin{tikzcd}
\Omega(\mathcal{R}(C \rtimes G/H)) \arrow[r, "\partial_{\mathcal{E} \rtimes G/H}"] \arrow[d, "f_*"'] & \mathcal{R}(A \rtimes G/H) \arrow[d, "f_*"] \\
\Omega(\mathcal{R}(C \rtimes G/K)) \arrow[r, "\partial_{\mathcal{E} \rtimes G/K}"] & \mathcal{R}(A \rtimes G/K)
\end{tikzcd} \tag{3.22}$$

Supose for a moment that this square commutes. Put $\mathscr{A} = (kk)^I$ where I is the interval category. Then f induces a functor $\nu_f : G\mathrm{Alg}_\ell \to \mathscr{A}$ defined by:

$$D \mapsto (f_* : \mathcal{R}(D \rtimes G/H) \to \mathcal{R}(D \rtimes G/K))$$

The commutativity of (3.22) implies that ν_f is a homotopy invariant and G-stable δ-functor [27, Definition 10.6]. Thus, it factors uniquely through kk^G by universal property of kk^G; cf. [27, Theorem 10.15]:

$$\begin{tikzcd} G\mathrm{Alg}_\ell \arrow[r, "j^G"] \arrow[dr, bend right, "\nu_f"'] & kk^G \arrow[d, dashed, "\exists!\, \bar\nu_f"] \\ & \mathscr{A} \end{tikzcd}$$

The functor $\bar\nu_f$ corresponds to the desired natural transformation. Let us now show that (3.22) commutes. The morphism $f : G/H \to G/K$ is determined by $f(H) = uK$ for some $u \in G$ with $H \subseteq uKu^{-1}$. We have the following commutative square of ℓ-linear categories:

$$\begin{tikzcd} A \rtimes G/H \arrow[r, "f_*"] & A \rtimes G/K \\ A \rtimes H \arrow[u] \arrow[r, "\mathrm{incl}"] & A \rtimes uKu^{-1} \arrow[u] \end{tikzcd}$$

The bottom arrow is an inclusion of algebras. The left and right arrows are the inclusions of the full subcategories whose only objects are H and uK, respectively. Upon applying $j \circ \mathcal{R} : \mathrm{Cat}_\ell \to kk$ the vertical arrows become isomorphisms and we get:

$$f_* = \nu_{uK} \circ j(\mathrm{incl}) \circ (\nu_H)^{-1} : \mathcal{R}(A \rtimes G/H) \longrightarrow \mathcal{R}(A \rtimes G/K)$$

Thus, the commutativity of (3.22) is equivalent to that of the outer square in the following diagram in kk:

$$\begin{tikzcd} \Omega(\mathcal{R}(C \rtimes G/H)) \arrow[r, "\partial_{\mathcal{E} \rtimes G/H}"] \arrow[d, "\Omega(\nu_H)^{-1}"', "\cong"] & \mathcal{R}(A \rtimes G/H) \arrow[d, "(\nu_H)^{-1}", "\cong"'] \\ \Omega(C \rtimes H) \arrow[r, "\partial_{\mathcal{E} \rtimes H}"] \arrow[d, "\Omega j(\mathrm{incl})"'] & A \rtimes H \arrow[d, "j(\mathrm{incl})"] \\ \Omega(C \rtimes uKu^{-1}) \arrow[r, "\partial_{\mathcal{E} \rtimes uKu^{-1}}"] \arrow[d, "\Omega(\nu_{uK})"', "\cong"] & A \rtimes uKu^{-1} \arrow[d, "\nu_{uK}", "\cong"'] \\ \Omega(\mathcal{R}(C \rtimes G/K)) \arrow[r, "\partial_{\mathcal{E} \rtimes G/K}"] & \mathcal{R}(A \rtimes G/K) \end{tikzcd}$$

Here, the bottom and top squares commute by Proposition 3.17. The middle square commutes because it fits into the morphism of triangles induced by the inclusion $H \subseteq uKu^{-1}$. $\square$

4. A natural adjunction

Let G be a group and let H be a finite subgroup of G. Recall from Lemma 3.13 that there is a natural isomorphism $\nu_H : - \rtimes H \to \overline{- \rtimes G/H}$ of functors $kk^G \to kk$. If $\frac{1}{|H|} \in \ell$, we have isomorphisms

$$kk^G(A^{(G/H)}, B) \cong kk(A, B \rtimes H) \cong kk(A, \mathcal{R}(B \rtimes G/H)), \tag{4.1}$$

where the isomorphism on the right is induced by ν_H and the one on the left is that of Proposition 2.10. In other words, there is an adjunction:

$$(-)^{(G/H)} : kk \rightleftarrows kk^G : \overline{- \rtimes G/H} \tag{4.2}$$

We will show that this adjunction is natural in G/H. For the rest of this paper, we assume that G satisfies the following property:

$$\tfrac{1}{|H|} \in \ell \text{ for every } H \in \mathcal{F}in \tag{4.3}$$

We will prove that for every morphism $f : G/H \to G/K \in \mathrm{Or}(G, \mathcal{F}in)$ the following square commutes:

$$\begin{tikzcd} kk(A, \mathcal{R}(B \rtimes G/H)) \arrow[r, "\cong"] \arrow[d, "f_*"'] & kk^G(A^{(G/H)}, B) \arrow[d, "f_*"] \\ kk(A, \mathcal{R}(B \rtimes G/K)) \arrow[r, "\cong"] & kk^G(A^{(G/K)}, B) \end{tikzcd} \tag{4.4}$$

The commutativity of this square is not obvious a priori since the middle term of (4.1) is not a functor on $\mathrm{Or}(G, \mathcal{F}in)$. Let $\alpha \in kk(A, \mathcal{R}(B \rtimes G/H))$ and $f : G/H \to G/K$ in $\mathrm{Or}(G, \mathcal{F}in)$. The commutativity of (4.4) is equivalent to that of the outer square in the following diagram in kk, where $\epsilon_{G/H}$ is the counit of (4.2):

$$\begin{tikzcd} A^{(G/H)} \arrow[r, "\alpha^{(G/H)}"] & {[\mathcal{R}(B \rtimes G/H)]^{(G/H)}} \arrow[r, "\epsilon_{G/H}"] & B \\ A^{(G/K)} \arrow[u, "A^{(f)}"] \arrow[r, "\alpha^{(G/K)}"'] & {[\mathcal{R}(B \rtimes G/H)]^{(G/K)}} \arrow[u, "f^*"] \arrow[r, "f_*"'] & {[\mathcal{R}(B \rtimes G/K)]^{(G/K)}} \arrow[u, "\epsilon_{G/K}"'] \end{tikzcd} \tag{4.5}$$

The square on the left clearly commutes. The commutativity of the square on the right follows easily from the following result.

Lemma 4.6. *Let G be a group satisfying (4.3). Then:*

(1) *For every $G/H \in \mathrm{Or}(G, \mathcal{F}in)$ there is a G-functor*

$$\zeta_{G/H} : (B \rtimes G/H)^{(G/H)} \to M_G \otimes B \tag{4.7}$$

that sends $(b \rtimes g)\chi_{sH} \in \hom(uH, vH) = B \otimes \ell[vHu^{-1}]^{(G/H)}$ to:

$$\sum_{p \in sHv^{-1}} e_{p,pg} \otimes p(b)$$

(2) *For any $f : G/H \to G/K$ in $\mathrm{Or}(G, \mathcal{F}in)$, the following square in $G\mathrm{Cat}_\ell$ commutes:*

$$\begin{tikzcd} (B \rtimes G/H)^{(G/H)} \arrow[r, "\zeta_{G/H}"] & M_G \otimes B \\ (B \rtimes G/H)^{(G/K)} \arrow[u, "f^*"] \arrow[r, "f_*"] & (B \rtimes G/K)^{(G/K)} \arrow[u, "\zeta_{G/K}"'] \end{tikzcd} \tag{4.8}$$

(3) *The counit $\epsilon_{G/H}$ of the adjunction (4.2) fits into the following commutative diagram in kk^G:*

$$\begin{tikzcd} \left[\mathcal{R}\left(B \rtimes G/H\right)\right]^{(G/H)} \arrow[ddr, bend right, "\epsilon_{G/H}"'] & \mathcal{R}\left[(B \rtimes G/H)^{(G/H)}\right] \arrow[l, "Cor.\ 3.11"', "\cong"] \arrow[d, "\mathcal{R}(\zeta_{G/H})"] \\ & M_G \otimes B \arrow[d, "zig\text{-}zag", "\cong"'] \\ & B \end{tikzcd}$$

Proof. Let us verify that $\zeta_{G/H}$ as defined in (1) is compatible with composition. Let $f_1 = (b_1 \rtimes g_1)\chi_{sH} \in \mathrm{Hom}(uH, vH)$ and $f_2 = (b_2 \rtimes g_2)\chi_{tH} \in \mathrm{Hom}(vH, wH)$. We have

$$f_2 \circ f_1 = \delta_{sH,tH}(b_2 g_2(b_1) \rtimes g_2 g_1)\chi_{tH}$$

where $\delta_{sH,tH}$ is Kronecker's delta. Then:

$$\zeta_{G/H}(f_2 \circ f_1) = \delta_{sH,tH} \sum_{p \in tHw^{-1}} e_{p,pg_2g_1} \otimes p(b_2 g_2(b_1))$$

On the other hand, we have:

$$\begin{aligned} \zeta_{G/H}(f_2)\zeta_{G/H}(f_1) &= \left(\sum_{q \in tHw^{-1}} e_{q,qg_2} \otimes q(b_2)\right)\left(\sum_{p \in sHv^{-1}} e_{p,pg_1} \otimes p(b_1)\right) \\ &= \sum_{\substack{p \in sHv^{-1} \\ q \in tHw^{-1}}} e_{q,qg_2} e_{p,pg_1} \otimes q(b_2)p(b_1) \\ &= \delta_{sH,tH} \sum_{q \in tHw^{-1}} e_{q,qg_2g_1} \otimes q(b_2)(qg_2)(b_1) \end{aligned}$$

The appearance of Kronecker's delta in the last line is explained as follows: $qg_2 \in tHv^{-1}$ and $p \in sHv^{-1}$ can be equal if and only if $sH = tH$. This shows that $\zeta_{G/H}$ is indeed a well-defined functor.

Let us prove (2). The square commutes on objects since $M_G \otimes B$ has only one object. A morphism $f : G/H \to G/K$ is determined by $f(H) = xK$ with x such that $H \subseteq xKx^{-1}$. Let $(b \rtimes g)\chi_{sK} \in \mathrm{Hom}(uH, vH)$. We have:

$$\begin{aligned} \zeta_{G/K}\left(f_*\left((b \rtimes g)\chi_{sK}\right)\right) &= \zeta_{G/K}\left((b \rtimes g)\chi_{sK}\right) \\ &= \sum_{p \in sK(vx)^{-1}} e_{p,pg} \otimes p(b) \end{aligned}$$

On the other hand, we have:

$$\begin{aligned}\zeta_{G/H}\left(f^*\left((b\rtimes g)\chi_{sK}\right)\right) &= \zeta_{G/H}\left(\sum_{tH\in f^{-1}(sK)}(b\rtimes g)\chi_{tH}\right)\\ &= \sum_{tH\in f^{-1}(sK)}\ \sum_{p\in tHv^{-1}} e_{p,pg}\otimes p(b)\\ &= \sum_{p\in sKx^{-1}v^{-1}} e_{p,pg}\otimes p(b)\end{aligned}$$

Here, the last equality follows from the fact that sKx^{-1} is the disjoint union of $f^{-1}(sK)$. This finishes the proof of (2).

Let us prove (3). It follows from (4.1) that the counit $\epsilon_{G/H}$ of the adjunction (4.2) equals $\varepsilon_H\circ\left[(\nu_H)^{-1}\right]^{(G/H)}$, where ε_H is the counit of (2.11). Recall the definition of ε_H from (2.5). Then $\epsilon_{G/H}$ equals the following composite in kk:

$$[\mathcal{R}(B\rtimes G/H)]^{(G/H)}\xrightarrow{\left[(\nu_H)^{-1}\right]^{(G/H)}}(B\rtimes H)^{(G/H)}\xrightarrow[(2.4)]{\psi}M_G\otimes B\xrightarrow[\cong]{\text{zig-zag}}B$$

The statement in (3) now follows from the commutativity of the following diagram, which is straightforward — indeed, the diagram commutes in Alg_ℓ:

$$\begin{tikzcd}
{[\mathcal{R}(B\rtimes G/H)]^{(G/H)}} & & \mathcal{R}\left[(B\rtimes G/H)^{(G/H)}\right] \arrow[ll, "\text{Cor. 3.11}"'] \arrow[d, "\mathcal{R}(\zeta_{G/H})"] \\
(B\rtimes H)^{(G/H)} \arrow[u, "(\nu_H)^{(G/H)}"] \arrow[urr] \arrow[rr, "\psi"] & & M_G\otimes B
\end{tikzcd}$$

Here the diagonal morphism is the one induced by the inclusion

$$(B\rtimes H)^{(G/H)}\subseteq(B\rtimes G/H)^{(G/H)}$$

as the full subcategory on the object H. This finishes the proof. □

We collect the main results of this section in the following theorem.

Theorem 4.9. *Let G be a group satisfying (4.3) and let H be a finite subgroup of G. We have an adjunction*

$$(-)^{(G/H)}:kk\rightleftarrows kk^G:\overline{-\rtimes G/H}$$

$$kk^G(A^{(G/H)},B)\cong kk(A,\mathcal{R}(B\rtimes G/H))$$

that is natural in $G/H\in\mathrm{Or}(G,\mathcal{F}in)$. Moreover, the counit of this adjunction is described by Lemma 4.6.

5. Lifting the adjunction to spectra

5.1. **The primitive zig-zag.** Let $A\in\mathrm{Alg}_\ell$, $B\in G\mathrm{Alg}_\ell$ and $G/N\in\mathrm{Or}(G,\mathcal{F}in)$. By Theorem 4.9 we have an adjunction isomorphism

$$kk^G(A^{(G/N)},B)\cong kk(A,B\rtimes G/N) \tag{5.1}$$

that is natural in G/N. Let $\mathbb{K}$ and $\mathbb{K}^G$ be, respectively, the spectra representing kk-theory and kk^G-theory. These were defined in [14] and in [25]; see Section C.2 for details. We would like to lift the isomorphism (5.1) to a natural weak equivalence of spectra:

$$\mathbb{K}(A, \mathcal{R}(B \rtimes G/N)) \dashrightarrow \mathbb{K}^G(A^{(G/N)}, B)$$

Here, we want the dashed arrow to represent a zig-zag of $\mathrm{Or}(G, \mathcal{F}in)$-spectra inducing the isomorphism (5.1) upon taking homotopy groups. As a starting point, let us recall how to obtain this adjunction. By Lemma 4.6 (3), the isomorphism (5.1) equals the following composite:

$$\begin{tikzcd}
kk(A, \mathcal{R}(B \rtimes G/N)) \arrow[r, "(-)^{(G/N)}"] & kk^G(A^{(G/N)}, [\mathcal{R}(B \rtimes G/N)]^{(G/N)}) \\
kk^G(A^{(G/N)}, M_G B) \arrow[d, "\cong"'] & kk^G(A^{(G/N)}, \mathcal{R}\left[(B \rtimes G/N)^{(G/N)}\right]) \arrow[l, "\mathcal{R}(\zeta_{G/N})", "(4.7)"'] \arrow[u, "\text{Cor. } 3.11"', "\cong"] \\
kk^G(A^{(G/N)}, M_{G_+} B) & kk^G(A^{(G/N)}, B) \arrow[l, "\cong"']
\end{tikzcd} \tag{5.2}$$

The last two morphisms are easily lifted to spectra. Indeed, the G-stability zig-zag (2.2) induces a zig-zag of weak equivalences that is clearly natural in G/N:

$$\mathbb{K}^G(A^{(G/N)}, M_G B) \xrightarrow{\sim} \mathbb{K}^G(A^{(G/N)}, M_{G_+} B) \xleftarrow{\sim} \mathbb{K}^G(A^{(G/N)}, B)$$

Lifting the rest of (5.2) is somewhat more delicate. If we simply replace groups by spectra, we get:

$$\begin{tikzcd}
\mathbb{K}(A, \mathcal{R}(B \rtimes G/N)) \arrow[r, "(-)^{(G/N)}"] & \mathbb{K}^G(A^{(G/N)}, [\mathcal{R}(B \rtimes G/N)]^{(G/N)}) \\
\mathbb{K}^G(A^{(G/N)}, M_G B) & \mathbb{K}^G(A^{(G/N)}, \mathcal{R}\left[(B \rtimes G/N)^{(G/N)}\right]) \arrow[l, "\mathcal{R}(\zeta_{G/N})", "(4.7)"'] \arrow[u, "\text{Cor. } 3.11"', "\wr"]
\end{tikzcd} \tag{5.3}$$

This is what we call the *primitive zig-zag*. While the spectra on the left are covariant functors on $\mathrm{Or}(G, \mathcal{F}in)$, this is not the case for those on the right. We should start by replacing the latter by covariant functors on $\mathrm{Or}(G, \mathcal{F}in)$ if we expect a zig-zag that is natural in G/N. In the following sections—taking the primitive zig-zag as our model—we proceed to construct a zig-zag of spectra that depends covariantly on G/N and that induces the isomorphism (5.1) upon taking homotopy groups.

5.2. **Notation and preliminary definitions.** To ease notation, the category $\mathrm{Or}(G, \mathcal{F}in)$ will be denoted by $\mathcal{O}$ for the rest of this section. Its objects—the orbit spaces corresponding to finite subgroups of G—will be denoted by letters r, s and t.

Let $\mathcal{C}$ be a category and Γ be a small category. We will write $B(\Gamma, \mathcal{C})$ for the category $\mathcal{C}^{\Gamma^{op} \times \Gamma}$ of bifunctors $\Gamma^{op} \times \Gamma \to \mathcal{C}$. Let $f : \Gamma \to \Lambda$ be a functor between small categories. Then we can restrict along f either of the variables of a bifunctor $\Lambda^{op} \times \Lambda \to \mathcal{C}$, or both of them, as shown by the following commutative diagram of categories:

$$\begin{tikzcd}
B(\Lambda,\mathcal{C}) \arrow[r, "f^*"] \arrow[d, "f^*"'] \arrow[dr, "f^\star"] & \mathcal{C}^{\Lambda^{op}\times\Gamma} \arrow[d, "f^*"] \\
\mathcal{C}^{\Gamma^{op}\times\Lambda} \arrow[r, "f^*"] & B(\Gamma,\mathcal{C})
\end{tikzcd}$$

Here, f^* denotes restriction of one of the variables (either the covariant or the contravariant one) and $f^\star$ denotes restriction of both variables.

Define functors $J \in \mathrm{Sp}^{\mathcal{O}}$ and $M, L \in B(\mathcal{O},\mathrm{Sp})^{\mathcal{O}}$ by

$$\begin{aligned} J(t) &:= \mathbb{K}(A, \mathcal{R}(B \rtimes t)) \\ L_t(s,r) &:= \mathbb{K}^G(A^{(t)}, \mathcal{R}[(B\rtimes r)^{(s)}]) \\ M_t(s,r) &:= \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B\rtimes r)]^{(s)}) \end{aligned} \tag{5.4}$$

for $r,s,t \in \mathcal{O}$. The kk^G-equivalence $\mathcal{R}[(B\rtimes r)^{(s)}] \to [\mathcal{R}(B\rtimes r)]^{(s)}$ of Corollary 3.11 induces, upon applying $\mathbb{K}^G(A^{(t)}, -)$, a natural transformation $\psi : L \to M$ that is an objecwise weak equivalence of spectra. With this notation, the primitive zig-zag (5.3) becomes:

$$J(t) \xrightarrow{(-)^{(t)}} M_t(t,t) \underset{\sim}{\xleftarrow{\psi}} L_t(t,t) \xrightarrow{\mathcal{R}(\zeta_t)} \mathbb{K}^G(A^{(t)}, M_G B) \tag{5.5}$$

5.3. **Coends enter the game.** Fix $t \in \mathcal{O}$. The commutativity of (4.8) suggests that the morphism induced by $\mathcal{R}(\zeta_t)$ in (5.5) could be replaced by a morphism from a certain coend, as we proceed to explain. If $f : r \to s$ is a morphism in $\mathcal{O}$, the following square commutes by Lemma 4.6 (2):

$$\begin{tikzcd}
(B\rtimes r)^{(r)} \arrow[r, "\zeta_r"] & M_G \otimes B \\
(B\rtimes r)^{(s)} \arrow[u, "f^*"] \arrow[r, "f_*"] & (B\rtimes s)^{(s)} \arrow[u, "\zeta_s"']
\end{tikzcd}$$

Upon applying $\mathbb{K}^G(A^{(t)}, \mathcal{R}(-))$, we get a commutative diagram:

$$\begin{tikzcd}
L_t(r,r) \arrow[r, "\zeta_r"] & \mathbb{K}^G(A^{(t)}, M_G B) \\
L_t(s,r) \arrow[u, "f^*"] \arrow[r, "f_*"] & L_t(s,s) \arrow[u, "\zeta_s"']
\end{tikzcd} \tag{5.6}$$

For reasons that will become clear later on (see Remark 5.16 and Lemma 5.30) we will take coends over the slice category $\mathcal{O}_{/t}$ of orbit spaces over t. We will denote by u_t the forgetful functor $\mathcal{O}_{/t} \to \mathcal{O}$. Let now $f : \alpha \to \beta$ be a morphism in $\mathcal{O}_{/t}$, where $\alpha : r \to t$ and $\beta : s \to t$. Then (5.6) equals the square:

$$\begin{tikzcd}
{[(u_t)^\star L_t]\,(\alpha,\alpha)} \arrow[r, "\zeta_r"] & \mathbb{K}^G(A^{(t)}, M_G B) \\
{[(u_t)^\star L_t]\,(\beta,\alpha)} \arrow[u, "f^\star"] \arrow[r, "f_*"] & {[(u_t)^\star L_t]\,(\beta,\beta)} \arrow[u, "\zeta_s"']
\end{tikzcd}$$

By the universal property of the coend, there is a unique morphism ζ making the following triangle commute for all objects $\alpha : r \to t$ of $\mathcal{O}_{/t}$:

$$\begin{tikzcd} \int^{\mathcal{O}_{/t}} (u_t)^\star L_t \arrow[r, "\zeta"] & \mathbb{K}^G(A^{(t)}, M_G B) \\ {[(u_t)^\star L_t]}(\alpha, \alpha) \arrow[u, "can_\alpha"] \arrow[ur, bend right, "\zeta_r"'] & \end{tikzcd} \tag{5.7}$$

Here the vertical morphism is the structural morphism into the coend corresponding to α. In the next section we will prove that ζ depends covariantly on $t \in \mathcal{O}$.

5.4. **Defining $\mathcal{O}$-spectra as objectwise coends.** Let us show that the morphisms ζ of (5.7) assemble, for varying t, into a morphism of $\mathcal{O}$-spectra. We first prove some preliminary lemmas.

Lemma 5.8. *Let $\mathcal{C}$ be a cocomplete category, $f : \Gamma \to \Lambda$ be a functor between small categories and $T \in B(\Lambda, \mathcal{C})$. Then there is a unique morphism*

$$\int^\Gamma f^\star T \longrightarrow \int^\Lambda T \tag{5.9}$$

making the following square in $\mathcal{C}$ commute, for every object γ of Γ:

$$\begin{tikzcd} \int^\Gamma f^\star T \arrow[r] & \int^\Lambda T \\ (f^\star T)(\gamma, \gamma) \arrow[u, "can_\gamma"] \arrow[r, equal] & T(f(\gamma), f(\gamma)) \arrow[u, "can_{f(\gamma)}"'] \end{tikzcd}$$

Here the vertical arrows are the structural morphisms into the coends. Moreover, (5.9) is natural in T.

Proof. This is immediate from the universal property of coends. $\square$

Remark 5.10. For composable functors $\Gamma \xrightarrow{f} \Lambda \xrightarrow{g} \Sigma$ and $T \in B(\Sigma, \mathcal{C})$, the morphisms (5.9) clearly fit into the following commutative triangle:

$$\begin{tikzcd} \int^\Gamma (g \circ f)^\star T = \int^\Gamma f^\star g^\star T \arrow[r] \arrow[dr, bend right] & \int^\Lambda g^\star T \arrow[d] \\ & \int^\Sigma T \end{tikzcd}$$

Lemma 5.11. *Let $\mathcal{C}$ be a cocomplete category and Γ be a small category. Then there is a functor $\mathscr{C} : B(\Gamma, \mathcal{C})^\Gamma \to \mathcal{C}^\Gamma$ described as follows.*

(1) *Let V be an object of $B(\Gamma, \mathcal{C})^\Gamma$, $t \in \Gamma \mapsto V_t \in B(\Gamma, \mathcal{C})$. For $t \in \Gamma$, we have:*

$$\mathscr{C}(V)(t) = \int^{\Gamma_{/t}} (u_t)^\star V_t$$

For a morphism $f : t \to t'$ in Γ, $\mathscr{C}(V)(f)$ equals the composite:

$$\int^{\Gamma_{/t}} (u_t)^\star V_t \xrightarrow{V_f} \int^{\Gamma_{/t}} (u_t)^\star V_{t'} = \int^{\Gamma_{/t}} f^\star (u_{t'})^\star V_{t'} \xrightarrow{(5.9)} \int^{\Gamma_{/t'}} (u_{t'})^\star V_{t'}$$

(2) *For a morphism $h : V \to W$ in $B(\Gamma, \mathcal{C})^\Gamma$, the natural transformation $\mathscr{C}(h)$ has components:*

$$\int^{\Gamma_{/t}} (u_t)^\star V_t \xrightarrow{\int^{\Gamma_{/t}} (u_t)^\star h_t} \int^{\Gamma_{/t}} (u_t)^\star W_t$$

Proof. The fact that the equalities in (1) indeed define a functor $\mathscr{C}(V) \in \mathcal{C}^\Gamma$ boils down to the naturality of (5.9) and Remark 5.10. The fact that $\mathscr{C}(h)$ is indeed a natural transformation follows as well from the naturality of (5.9). □

Lemma 5.12. *The morphism ζ defined by (5.7) is a morphism of $\mathcal{O}$-spectra.*

Proof. The codomain of ζ is clearly an $\mathcal{O}$-spectrum. Its domain is an $\mathcal{O}$-spectrum as well; indeed, it is $\mathscr{C}(t \mapsto L_t)$ where $\mathscr{C}$ is the functor of Lemma 5.11. Let $f : t \to t'$ be a morphism in $\mathcal{O}$. We claim that following square commutes:

$$\begin{tikzcd}
\int^{\mathcal{O}_{/t}} (u_t)^\star L_t \arrow[r, "\zeta"] \arrow[d, "f"'] & \mathbb{K}^G(A^{(t)}, M_G B) \arrow[d, "f"] \\
\int^{\mathcal{O}_{/t'}} (u_{t'})^\star L_{t'} \arrow[r, "\zeta"] & \mathbb{K}^G(A^{(t')}, M_G B)
\end{tikzcd}$$

Indeed, by the universal property of the coend, it suffices to show that the square commutes when precomposed with the structural morphisms

$$[(u_t)^\star L_t]\,(\alpha, \alpha) \to \int^{\mathcal{O}_{/t}} (u_t)^\star L_t$$

for every object $\alpha : r \to t$ of $\mathcal{O}_{/t}$. Upon precomposing with the latter we get the following square, that clearly commutes:

$$\begin{tikzcd}
\mathbb{K}^G(A^{(t)}, \mathcal{R}\left[(B \rtimes r)^{(r)}\right]) \arrow[r, "\zeta_r"] \arrow[d, "f"'] & \mathbb{K}^G(A^{(t)}, M_G B) \arrow[d, "f"] \\
\mathbb{K}^G(A^{(t')}, \mathcal{R}\left[(B \rtimes r)^{(r)}\right]) \arrow[r, "\zeta_r"] & \mathbb{K}^G(A^{(t')}, M_G B)
\end{tikzcd}$$

This proves the lemma. □

5.5. **The morphism φ.** In this section we define a morphism φ that will be part of the natural zig-zag of Theorem 5.35. Let $\delta : \mathrm{Sp}^{\mathcal{O}} \to B(\mathcal{O}, \mathrm{Sp})$ be the functor that adds a constant contravariant variable, defined by

$$\delta_F(s, r) = F(r) \tag{5.13}$$

for $F \in \mathrm{Sp}^{\mathcal{O}}$ and $r, s \in \mathcal{O}$. Let $(u_t)^* : B(\mathcal{O}, \mathrm{Sp}) \to \mathrm{Sp}^{(\mathcal{O}_{/t})^{op} \times \mathcal{O}}$ be the restriction of the contravariant variable along the forgetful functor. Recall the definitions of $J \in \mathrm{Sp}^{\mathcal{O}}$ and $M_t \in B(\mathcal{O}, \mathrm{Sp})$ from (5.4). Define a morphism

$$\varphi^\sharp : (u_t)^* \delta_J \to (u_t)^* M_t \tag{5.14}$$

as follows. For objects r of $\mathcal{O}$ and $\alpha : s \to t$ of $\mathcal{O}_{/t}$, let

$$\varphi^\sharp_{(\alpha, r)} : [(u_t)^* \delta_J]\,(\alpha, r) \to [(u_t)^* M_t]\,(\alpha, r) \tag{5.15}$$

be the following composition:

$$\mathbb{K}(A, \mathcal{R}(B \rtimes r)) \xrightarrow{(-)^{(t)}} \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(t)}) \xrightarrow{\alpha^*} \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)})$$

It is easily verified that $\varphi^\sharp$ is a natural transformation of bifunctors. Indeed, for morphisms $f : r \to r'$ in $\mathcal{O}$ and $g : \alpha \to \alpha'$ in $\mathcal{O}_{/t}$, the following diagrams commute:

$$\begin{tikzcd}
 & \mathbb{K}(A, \mathcal{R}(B \rtimes r)) \arrow[d, "(-)^{(t)}"] \arrow[ddl, bend right, "\varphi^\sharp_{(\alpha,r)}"'] \arrow[ddr, bend left, "\varphi^\sharp_{(\alpha',r)}"] & \\
 & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(t)}) \arrow[dl, "\alpha^*"'] \arrow[dr, "(\alpha')^*"] & \\
\mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)}) & & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s')}) \arrow[ll, "g^*"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{K}(A, \mathcal{R}(B \rtimes r)) \arrow[r, "f_*"] \arrow[d, "(-)^{(t)}"] \arrow[dd, bend right=60, "\varphi^\sharp_{(\alpha,r)}"'] & \mathbb{K}(A, \mathcal{R}(B \rtimes r')) \arrow[d, "(-)^{(t)}"'] \arrow[dd, bend left=60, "\varphi^\sharp_{(\alpha,r')}"] \\
\mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(t)}) \arrow[r, "f_*"] \arrow[d, "\alpha^*"] & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r')]^{(t)}) \arrow[d, "\alpha^*"'] \\
\mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)}) \arrow[r, "f_*"] & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r')]^{(s)})
\end{tikzcd}$$

Remark 5.16. The definition of the components of $\varphi^\sharp$ (5.15) makes use of the structural morphism $\alpha : s \to t$ and it is not clear how to define $\varphi^\sharp$ as a morphism $\delta_J \to M_t$ in $B(\mathcal{O}, \mathrm{Sp})$. This is one of the reasons that motivated our choice of $\mathcal{O}_{/t}$ as the indexing category for coends.

Construction 5.17. By Remark 5.25 we have, for each $t \in \mathcal{O}$, a pair of adjoint functors:

$$(u_t)_! : \mathrm{Sp}^{(\mathcal{O}_{/t})^{op} \times \mathcal{O}} \rightleftarrows B(\mathcal{O}, \mathrm{Sp}) : (u_t)^* \tag{5.18}$$

Let $f : t \to t'$ be a morphism in $\mathcal{O}$ and write $f^* : \mathrm{Sp}^{(\mathcal{O}_{/t'})^{op} \times \mathcal{O}} \to \mathrm{Sp}^{(\mathcal{O}_{/t})^{op} \times \mathcal{O}}$ for the restriction of the contravariant variable along $f : \mathcal{O}_{/t} \to \mathcal{O}_{/t'}$. Note that $f^* \circ (u_{t'})^* = (u_t)^*$ and consider the following diagram of solid arrows for $F \in \mathrm{Sp}^{\mathcal{O}}$ and $H \in B(\mathcal{O}, \mathrm{Sp})$:

$$\begin{tikzcd}
B(\mathcal{O}, \mathrm{Sp})\left((u_{t'})_!(u_{t'})^*\delta_F, H\right) \arrow[r, "\cong"] \arrow[d, dashed] & \mathrm{Sp}^{(\mathcal{O}_{/t'})^{op} \times \mathcal{O}}\left((u_{t'})^*\delta_F, (u_{t'})^*H\right) \arrow[d, "f^*"] \\
B(\mathcal{O}, \mathrm{Sp})\left((u_t)_!(u_t)^*\delta_F, H\right) \arrow[r, "\cong"] & \mathrm{Sp}^{(\mathcal{O}_{/t})^{op} \times \mathcal{O}}\left((u_t)^*\delta_F, (u_t)^*H\right)
\end{tikzcd}$$

Let the dashed arrow complete the diagram to a commutative square. By the Yoneda Lemma, the dashed arrow is induced by precomposition with a unique morphism $(u_t)_!(u_t)^*\delta_F \to (u_{t'})_!(u_{t'})^*\delta_F$. The latter, for varying f, assemble into a functor $\mathcal{O} \to B(\mathcal{O}, \mathrm{Sp})$, $t \mapsto (u_t)_!(u_t)^*\delta_F$. This construction is, moreover, clearly natural in F, so that we get a functor:

$$\begin{tikzcd}
\mathrm{Sp}^{\mathcal{O}} \arrow[r] & B(\mathcal{O}, \mathrm{Sp})^{\mathcal{O}} \\
F \arrow[r, mapsto] & (t \mapsto (u_t)_!(u_t)^*\delta_F)
\end{tikzcd}$$

Remark 5.19. Let $F \in \mathrm{Sp}^{\mathcal{O}}$. Later on, it will be useful to have an explicit description of the functor $\mathcal{O} \to B(\mathcal{O}, \mathrm{Sp})$, $t \mapsto (u_t)_!(u_t)^*\delta_F$ mentioned in Construction 5.17. Let us first describe the bifunctor $(u_t)_!(u_t)^*\delta_F \in B(\mathcal{O}, \mathrm{Sp})$ for fixed $t \in \mathcal{O}$. For $r, s \in \mathcal{O}$ we have:

$$[(u_t)_!(u_t)^*\delta_F](s,r) = \coprod_{\alpha \in \mathcal{O}(s,t)} [(u_t)^*\delta_F](\alpha, r) = \coprod_{\mathcal{O}(s,t)} F(r)$$

For a morphism $f : r \to r'$ in $\mathcal{O}$, the induced morphism $f_* = [(u_t)_!(u_t)^*\delta_F](s,f)$ equals the morphism:

$$\coprod F(f) : \coprod_{\mathcal{O}(s,t)} F(r) \to \coprod_{\mathcal{O}(s,t)} F(r')$$

For a morphism $g : s' \to s$ in $\mathcal{O}$, the induced morphism $g^* = [(u_t)_!(u_t)^*\delta_F](g,r)$ is the unique morphism making the following triangle commute for all $\beta \in \mathcal{O}(s,t)$:

$$\begin{tikzcd}
F(r) \arrow[r, "can_\beta"] \arrow[dr, bend right, "can_{\beta\circ g}"'] & \coprod_{\mathcal{O}(s,t)} F(r) \arrow[d, "g^*"] \\
 & \coprod_{\mathcal{O}(s',t)} F(r)
\end{tikzcd}$$

Now let $h : t \to t'$ be a morphism in $\mathcal{O}$. Then the components of the induced natural transformation $h_* : (u_t)_!(u_t)^*\delta_F \to (u_{t'})_!(u_{t'})^*\delta_F$ are the unique morphisms making the following triangle commute for all $\beta \in \mathcal{O}(s,t)$:

$$\begin{tikzcd}
F(r) \arrow[r, "can_\beta"] \arrow[dr, bend right, "can_{h\circ\beta}"'] & \coprod_{\mathcal{O}(s,t)} F(r) \arrow[d, "h_*"] \\
 & \coprod_{\mathcal{O}(s,t')} F(r)
\end{tikzcd}$$

Lemma 5.20. *Fix $t \in \mathcal{O}$ and let $\varphi^\sharp : (u_t)^*\delta_J \to (u_t)^*M_t$ be the morphism in $\mathrm{Sp}^{(\mathcal{O}_{/t})^{op}\times\mathcal{O}}$ defined in (5.14). Under the adjunction (5.18), $\varphi^\sharp$ corresponds to a morphism $\varphi : (u_t)_!(u_t)^*\delta_J \to M_t$ in $B(\mathcal{O}, \mathrm{Sp})$. Explicitly, for $r, s \in \mathcal{O}$, the component $\varphi_{(s,r)} : [(u_t)_!(u_t)^*\delta_J](s,r) \to M_t(s,r)$ is the unique morphism making the following triangle commute for every $\alpha \in \mathcal{O}(s,t)$:*

$$\begin{tikzcd}
\coprod_{\mathcal{O}(s,t)} J(r) \arrow[r, "\varphi_{(s,r)}"] & M_t(s,r) \\
J(r) \arrow[u, "can_\alpha"] \arrow[ur, bend right, "\varphi^\sharp_{(\alpha,r)}"'] &
\end{tikzcd}$$

Then the latter, for varying t, assemble into a morphism in $B(\mathcal{O}, \mathrm{Sp})^{\mathcal{O}}$—where the domain of φ is considered as an object of $B(\mathcal{O}, \mathrm{Sp})^{\mathcal{O}}$ as explained in Construction 5.17.

Proof. Let $f : t \to t'$ be a morphism in $\mathcal{O}$. We have to show that the following square in $B(\mathcal{O}, \mathrm{Sp})$ commutes:

$$\begin{tikzcd} (u_t)_!(u_t)^*\delta_J \arrow[r, "\varphi"] \arrow[d, "f_*"'] & M_t \arrow[d, "f_*"] \\ (u_{t'})_!(u_{t'})^*\delta_J \arrow[r, "\varphi"] & M_{t'} \end{tikzcd}$$

By Remark 5.19, for $r, s \in \mathcal{O}$, we have $[(u_t)_!(u_t)^*\delta_J](s,r) = \coprod_{\mathcal{O}(s,t)} J(r)$. Thus, it suffices to show that the following square commutes for $r, s \in \mathcal{O}$:

$$\begin{tikzcd} \coprod\limits_{\mathcal{O}(s,t)} J(r) \arrow[r, "\varphi"] \arrow[d, "f_*"'] & M_t(s,r) \arrow[d, "f_*"] \\ \coprod\limits_{\mathcal{O}(s,t')} J(r) \arrow[r, "\varphi"] & M_{t'}(s,r) \end{tikzcd}$$

By the universal property of the coproduct it suffices to show that this square commutes when precomposed with every structural morphism into the coproduct in the upper left corner. Let $\alpha \in \mathcal{O}(s,t)$. Upon precomposing the latter square with $can_\alpha : J(r) \to \coprod_{\mathcal{O}(s,t)} J(r)$ we get:

$$\begin{tikzcd} J(r) \arrow[r, "\varphi^\sharp_{(\alpha,r)}"] \arrow[dr, bend right, "\varphi^\sharp_{(f\circ\alpha,r)}"'] & M_t(s,r) \arrow[d, "f_*"] \\ & M_{t'}(s,r) \end{tikzcd}$$

Unravelling the definition of $\varphi^\sharp$, this triangle becomes:

$$\begin{tikzcd} \mathbb{K}(A, \mathcal{R}(B \rtimes r)) \arrow[r, "(-)^{(t)}"] \arrow[d, "(-)^{(t')}"'] & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(t)}) \arrow[r, "\alpha^*"] \arrow[d, "f_*"] & \mathbb{K}^G(A^{(t)}, [\mathcal{R}(B \rtimes r)]^{(s)}) \arrow[d, "f_*"] \\ \mathbb{K}^G(A^{(t')}, [\mathcal{R}(B \rtimes r)]^{(t')}) \arrow[r, "f^*"] & \mathbb{K}^G(A^{(t')}, [\mathcal{R}(B \rtimes r)]^{(t)}) \arrow[r, "\alpha^*"] & \mathbb{K}^G(A^{(t')}, [\mathcal{R}(B \rtimes r)]^{(s)}) \end{tikzcd}$$

The square on the right clearly commutes. The one on the left commutes by Lemma C.6. □

5.6. **Model structure on categories of bifunctors.** Let $\mathcal{C}$ be a model category and let Γ be a small category. In this section, we endow $B(\Gamma, \mathcal{C})$ with a model structure that will allow us to build models for the *homotopy coends* of certain morphisms of bifunctors. We will need these later on to define the morphisms going backwards in the zig-zag of $\mathcal{O}$-spectra of Theorem 5.35.

Proposition 5.21 ([24, Proposition A.2.8.2])**.** *Let $\mathcal{C}$ be a combinatorial model category and let Γ be a small category. Then there exist two combinatorial model structures on $\mathcal{C}^\Gamma$:*

- *The* injective model structure*, denoted $\mathcal{C}^\Gamma_{\mathrm{inj}}$, where weak equivalences and cofibrations are defined objectwise.*

- *The* projective model structure, *denoted* $\mathcal{C}^{\Gamma}_{\text{proj}}$, *where weak equivalences and fibrations are defined objectwise.*

We will always consider $B(\Gamma, \mathcal{C})$ as a model category with the structure $(\mathcal{C}^{\Gamma^{op}}_{\text{inj}})^{\Gamma}_{\text{proj}}$ whenever this structure exists. The model structure on $B(\Gamma, \mathrm{Sp})$ exists for any Γ by Lemma C.2 and Proposition 5.21.

Theorem 5.22 ([2, Theorem 4.1])**.** *Let $\mathcal{C}$ be a model category and Γ be a small category such that the model structure on $B(\Gamma, \mathcal{C})$ exists. Then the functor*

$$\int^{\Gamma} : B(\Gamma, \mathcal{C}) \to \mathcal{C} \tag{5.23}$$

is a left Quillen functor.

Proof. This is [2, Theorem 4.1]; we sketch the proof for completeness. For $c \in \mathcal{C}$, define $R(c) : \Gamma^{op} \times \Gamma \to \mathcal{C}$ by:

$$R(c)(s, r) := \prod_{\alpha \in \Gamma(r,s)} c$$

For morphisms $f : r \to r'$ and $g : s' \to s$ in Γ, let f_* and g^* be the unique morphisms making the following diagrams commute for all $\alpha \in \Gamma(r', s)$ and all $\beta \in \Gamma(r, s')$:

$$\begin{tikzcd} R(c)(s,r) \arrow[r, "f_*"] \arrow[dr, bend right, "can_{\alpha\circ f}"'] & R(c)(s,r') \arrow[d, "can_\alpha"] \\ & c \end{tikzcd} \qquad \begin{tikzcd} R(c)(s,r) \arrow[r, "g^*"] \arrow[dr, bend right, "can_{g\circ\beta}"'] & R(c)(s',r) \arrow[d, "can_\beta"] \\ & c \end{tikzcd}$$

It is easily verified that $R : \mathcal{C} \to \mathcal{C}^{\Gamma^{op}\times\Gamma}$ is right adjoint to $\int^{\Gamma} : \mathcal{C}^{\Gamma^{op}\times\Gamma} \to \mathcal{C}$. Thus, proving the theorem is equivalent to showing that

$$R : \mathcal{C} \to B(\Gamma, \mathcal{C})$$

preserves fibrations and trivial fibrations. Let $c \to c'$ be a (trivial) fibration in $\mathcal{C}$. To prove that $R(c) \to R(c')$ is a (trivial) fibration in $B(\Gamma, \mathcal{C}) = (\mathcal{C}^{\Gamma^{op}}_{\text{inj}})^{\Gamma}_{\text{proj}}$, it suffices to show that $R(c)(-, r) \to R(c')(-, r)$ is a (trivial) fibration in $\mathcal{C}^{\Gamma^{op}}_{\text{inj}}$ for every $r \in \Gamma$. But the latter holds since, for every r, there is a Quillen adjunction

$$\mathrm{ev}_r : \mathcal{C}^{\Gamma^{op}}_{\text{inj}} \rightleftarrows \mathcal{C} : R(-)(-, r)$$

where ev_r is the evaluation at r [2, Corollary 2.3 (iii)]. Indeed, this adjunction is Quillen since ev_r clearly preserves cofibrations and trivial cofibrations. □

Lemma 5.24 ([24, Proposition A.2.8.7])**.** *Let $\mathcal{C}$ be a model category and let $f : \Gamma \to \Lambda$ be a functor. Then the restriction functor $f^* : \mathcal{C}^{\Lambda} \to \mathcal{C}^{\Gamma}$ fits into the following Quillen adjunctions, whenever the model structures in question exist:*

(1) $f^* : \mathcal{C}^{\Lambda}_{\text{inj}} \rightleftarrows \mathcal{C}^{\Gamma}_{\text{inj}} : f_*$
(2) $f_! : \mathcal{C}^{\Gamma}_{\text{proj}} \rightleftarrows \mathcal{C}^{\Lambda}_{\text{proj}} : f^*$

Remark 5.25. Let $\mathcal{C}$ be a category with small coproducts and let Γ be a small category. Fix $t \in \Gamma$ and let $u_t : \Gamma_{/t} \to \Gamma$ be the forgetful functor. Then there is an adjunction:

$$(u_t)_! : \mathcal{C}^{(\Gamma_{/t})^{op}} \rightleftarrows \mathcal{C}^{\Gamma^{op}} : (u_t)^*$$

Moreover, the pushforward functor $(u_t)_!$ can be explicitely described as follows. For $s \in \Gamma$, we have:

$$[(u_t)_! F]\,(s) = \coprod_{\alpha \in \mathrm{Hom}(s,t)} F(\alpha)$$

For a morphism $g : s' \to s$ in Γ, $[(u_t)_! F]\,(g)$ is the unique morphism making the following squares commute, where the vertical arrows are the structural morphisms into the coproducts:

$$\begin{tikzcd}
\coprod\limits_{\alpha \in \mathrm{Hom}(s,t)} F(\alpha) \arrow[r, "{[(u_t)_! F](g)}"] & \coprod\limits_{\alpha' \in \mathrm{Hom}(s',t)} F(\alpha') \\
F(\beta) \arrow[u, "can_\beta"] \arrow[r, "F(g)"] & F(\beta \circ g) \arrow[u, "can_{\beta \circ g}"']
\end{tikzcd}$$

Lemma 5.26 (cf. [24, Lemma A.2.8.10])**.** *Let $\mathcal{C}$ be a model category, Γ be a small category, fix $t \in \Gamma$, and let $u_t : \Gamma_{/t} \to \Gamma$ be the forgetful functor from the slice category. Then the following adjunctions are Quillen adjunctions, whenever the model structures in question exist:*

(1) $(u_t)^* : \mathcal{C}^{\Gamma}_{\mathrm{proj}} \rightleftarrows \mathcal{C}^{\Gamma_{/t}}_{\mathrm{proj}} : (u_t)_*$
(2) $(u_t)_! : \mathcal{C}^{(\Gamma_{/t})^{op}}_{\mathrm{inj}} \rightleftarrows \mathcal{C}^{\Gamma^{op}}_{\mathrm{inj}} : (u_t)^*$

Proof. To prove (2) is a Quillen adjunction, let us show that $(u_t)_!$ preserves cofibrations and trivial cofibrations. Recall from Remark 5.25 that, for $F \in \mathcal{C}^{(\Gamma_{/t})^{op}}$ and $s \in \Gamma$, we have:

$$[(u_t)_! F]\,(s) = \coprod_{\alpha \in \Gamma(s,t)} F(\alpha)$$

Let $\eta : F \to F'$ be a morphism in $\mathcal{C}^{(\Gamma_{/t})^{op}}$. For $s \in \Gamma$, $(u_t)_!(\eta)(s)$ is the coproduct of the morphisms:

$$\{\eta(\alpha) : F(\alpha) \to F'(\alpha)\}_{\alpha \in \Gamma(s,t)}$$

If η is a cofibration (resp. a trivial cofibration) in $\mathcal{C}^{(\Gamma_{/t})^{op}}_{\mathrm{inj}}$, the latter are cofibrations (resp. trivial cofibrations) in $\mathcal{C}$ and, thus, $(u_t)_!(\eta)(s)$ is again a cofibration (resp. trivial cofibration) in $\mathcal{C}$. Since this holds for every $s \in \Gamma$, it follows that $(u_t)_!(\eta)$ is a cofibration (resp. a trivial cofibration) in $\mathcal{C}^{\Gamma^{op}}_{\mathrm{inj}}$. The proof of (1) is dual to that of (2), using the fact that

$$[(u_t)_* F]\,(s) = \prod_{\alpha \in \Gamma(t,s)} F(\alpha)$$

for $F \in \mathcal{C}^{\Gamma_{/t}}$ and $s \in \Gamma$. □

Lemma 5.27 ([24, Remark A.2.8.6])**.** *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a Quillen adjunction of combinatorial model categories and let Γ be a small category. Then composition with F and U determines a Quillen adjunction*

$$F^{\Gamma} : \mathcal{C}^{\Gamma} \rightleftarrows \mathcal{D}^{\Gamma} : U^{\Gamma}$$

with respect to either the injective or the projective model structures.

Lemma 5.28. *Let $\delta : \mathrm{Sp}^{\mathcal{O}}_{\mathrm{proj}} \to B(\mathcal{O}, \mathrm{Sp})$ be the functor defined in (5.13). Then δ is a left Quillen functor.*

Proof. Let const : $\mathrm{Sp} \to \mathrm{Sp}_{\mathrm{inj}}^{\mathcal{O}^{op}}$ be the functor that sends a spectrum E to the constant functor on E. Then const is left Quillen since it clearly sends cofibrations (resp. trivial cofibrations) to cofibrations (resp. trivial cofibrations) and it has a right adjoint (taking limit). Note that we have:

$$\delta = (\mathrm{const})^{\mathcal{O}} : \mathrm{Sp}_{\mathrm{proj}}^{\mathcal{O}} \to \left[\mathrm{Sp}_{\mathrm{inj}}^{\mathcal{O}^{op}}\right]_{\mathrm{proj}}^{\mathcal{O}} = B(\mathcal{O}, \mathrm{Sp})$$

Then δ is left Quillen by Lemma 5.27. $\square$

Lemma 5.29. *Let $\mathcal{C}$ be a combinatorial model category, Γ be a small category, fix $t \in \Gamma$, and let $u_t : \Gamma_{/t} \to \Gamma$ be the forgetful functor from the slice category. Then the restriction on both variables*

$$(u_t)^\star : B(\Gamma, \mathcal{C}) \to B(\Gamma_{/t}, \mathcal{C})$$

is a left Quillen functor.

Proof. Consider the following commutative square:

$$\begin{tikzcd}
B(\Gamma,\mathcal{C}) = (\mathcal{C}_{\mathrm{inj}}^{\Gamma^{op}})_{\mathrm{proj}}^{\Gamma} \arrow[r, "(u_t)^*"] \arrow[d, "(u_t)^*"'] \arrow[dr, "(u_t)^\star"] & (\mathcal{C}_{\mathrm{inj}}^{\Gamma^{op}})_{\mathrm{proj}}^{\Gamma_{/t}} \arrow[d, "(u_t)^*"] \\
\left[\mathcal{C}_{\mathrm{inj}}^{(\Gamma_{/t})^{op}}\right]_{\mathrm{proj}}^{\Gamma} \arrow[r, "(u_t)^*"] & \left[\mathcal{C}_{\mathrm{inj}}^{(\Gamma_{/t})^{op}}\right]_{\mathrm{proj}}^{\Gamma_{/t}} = B(\Gamma_{/t},\mathcal{C})
\end{tikzcd}$$

The horizontal morphisms are left Quillen by Lemma 5.26 (1) and the vertical morphisms are left Quillen by Lemma 5.24 and Lemma 5.27. Then $(u_t)^\star$ is left Quillen as well, for being the composite of left Quillen functors. $\square$

5.7. **The natural zig-zag.** In this section we finally construct a zig-zag of $\mathcal{O}$-spectra inducing (5.1) upon taking homotopy groups. We begin with the following lemma, that shows that every $\mathcal{O}$-spectrum can be canonically described as an objectwise coend.

Lemma 5.30. *Let $\delta : \mathrm{Sp}_{\mathrm{proj}}^{\mathcal{O}} \to B(\mathcal{O}, \mathrm{Sp})$ be the functor defined in (5.13). For $F \in \mathrm{Sp}^{\mathcal{O}}$ and $t \in \mathcal{O}$, the structural morphisms into the coends*

$$F(t) = [(u_t)^\star \delta_F](\mathrm{id}_t, \mathrm{id}_t) \longrightarrow \int^{\mathcal{O}_{/t}} (u_t)^\star \delta_F \tag{5.31}$$

are isomorphisms. Moreover, these are natural in $t \in \mathcal{O}$ and in $F \in \mathrm{Sp}^{\mathcal{O}}$.

Proof. Fix $t \in \mathcal{O}$. Since $(u_t)^\star \delta_F$ is constant in the contravariant variable, we have:

$$\int^{\mathcal{O}_{/t}} (u_t)^\star \delta_F \cong \operatorname*{colim}_{\alpha \in \mathcal{O}_{/t}} F(u_t(\alpha))$$

Since id_t is a final object of $\mathcal{O}_{/t}$, the structural morphism

$$F(t) = F(u_t(\mathrm{id}_t)) \longrightarrow \operatorname*{colim}_{\alpha \in \mathcal{O}_{/t}} F(u_t(\alpha))$$

is an isomorphism. Combining the above we get the desired isomorphism:

$$F(t) \xrightarrow{\cong} \int^{\mathcal{O}_{/t}} (u_t)^\star \delta_F$$

It is easily verified that this is natural in t and in F. $\square$

Lemma 5.32. *Let $F \in \mathrm{Sp}^{\mathcal{O}}$ and fix $t \in \mathcal{O}$. Let $(u_t)_!$ and $(u_t)^*$ be the functors that form the adjunction (5.18). Then there is a morphism of bifunctors $(u_t)^\star\delta_F \to (u_t)^\star(u_t)_!(u_t)^*\delta_F$ described as follows. For objects $\alpha : r \to t$ and $\beta : s \to t$ of $\mathcal{O}_{/t}$, the component corresponding to the pair (β, α) is the structural morphism into the coproduct corresponding to β:*

$$[(u_t)^\star\delta_F](\beta,\alpha) = F(r) \xrightarrow{can_\beta} \coprod_{\mathcal{O}(s,t)} F(r) = [(u_t)^\star(u_t)_!(u_t)^*\delta_F](\beta,\alpha) \tag{5.33}$$

Moreover, upon taking coend we get a morphism of spectra

$$\int^{\mathcal{O}_{/t}} (u_t)^\star\delta_F \longrightarrow \int^{\mathcal{O}_{/t}} (u_t)^\star(u_t)_!(u_t)^*\delta_F \tag{5.34}$$

that is natural in t.

Proof. The fact that the morphisms (5.33) are natural in α and β is easily verified using the explicit description of the bifunctor $(u_t)_!(u_t)^*\delta_F$ given in Remark 5.19.

Let us now prove that (5.34) is natural in t. Let $h : t \to t'$ be a morphism in $\mathcal{O}$. By the universal property of the coend, it suffices to show that the outer square in the following diagram commutes for every $\alpha : r \to t$ in $\mathcal{O}_{/t}$:

$$\begin{tikzcd}
F(r) = [(u_t)^\star\delta_F](\alpha,\alpha) \arrow[dr, "can_\alpha"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \int^{\mathcal{O}_{/t}} (u_t)^\star\delta_F \arrow[r, "(5.34)"] \arrow[d, "h_*"'] & \int^{\mathcal{O}_{/t}} (u_t)^\star(u_t)_!(u_t)^*\delta_F \arrow[d, "h_*"] \\
& \int^{\mathcal{O}_{/t'}} (u_{t'})^\star\delta_F \arrow[r, "(5.34)"] & \int^{\mathcal{O}_{/t'}} (u_{t'})^\star(u_{t'})_!(u_{t'})^*\delta_F
\end{tikzcd}$$

Unravelling the definitions of h_* (see Lemma 5.11 and Remark 5.19), it is straightforward to verify that both ways from $F(r)$ to $\int^{\mathcal{O}_{/t'}} (u_{t'})^\star(u_{t'})_!(u_{t'})^*\delta_F$ in the diagram above equal the composite:

$$F(r) \xrightarrow{can_{h\circ\alpha}} \coprod_{\mathcal{O}(r,t')} F(r) \xrightarrow{can_{h\circ\alpha}} \int^{\mathcal{O}_{/t'}} (u_{t'})^\star(u_{t'})_!(u_{t'})^*\delta_F$$

This finishes the proof. □

Theorem 5.35. *Let $q : Q \xrightarrow{\sim} \mathrm{id}$ and $\bar{q} : \bar{Q} \xrightarrow{\sim} \mathrm{id}$ be, respectively, cofibrant replacements in $B(\mathcal{O}, \mathrm{Sp})$ and $\mathrm{Sp}^{\mathcal{O}}_{\mathrm{proj}}$. Then we have a zig-zag of $\mathcal{O}$-spectra as follows:*

$$\begin{tikzcd}
\int^{\mathcal{O}_{/t}} (u_t)^\star\delta_J & \int^{\mathcal{O}_{/t}} (u_t)^\star\delta_{\bar{Q}J} \arrow[l, "\bar{q}"', "\sim"] \arrow[r, "(5.34)"] & \int^{\mathcal{O}_{/t}} (u_t)^\star(u_t)_!(u_t)^*\delta_{\bar{Q}J} \\
\int^{\mathcal{O}_{/t}} (u_t)^\star Q(M_t) & \int^{\mathcal{O}_{/t}} (u_t)^\star Q(u_t)_!(u_t)^*\delta_J \arrow[l, "\varphi"'] & \int^{\mathcal{O}_{/t}} (u_t)^\star Q(u_t)_!(u_t)^*\delta_{\bar{Q}J} \arrow[l, "\bar{q}"'] \arrow[u, "q"', "\wr"] \\
\int^{\mathcal{O}_{/t}} (u_t)^\star Q(L_t) \arrow[u, "\psi", "\wr"'] \arrow[r, "q"] & \int^{\mathcal{O}_{/t}} (u_t)^\star L_t \arrow[r, "\zeta"] & \mathbb{K}(A^{(t)}, M_G B)
\end{tikzcd}$$

Moreover, upon identifying

$$\mathbb{K}(A, \mathcal{R}(B \rtimes t)) = J(t) \overset{(5.31)}{\cong} \int^{\mathcal{O}_{/t}} (u_t)^\star \delta_J$$

and then taking homotopy groups, this zig-zag induces the isomorphism (5.1)*.*

Proof. Let us first show that the morphisms appearing in the zig-zag are indeed natural in t. For (5.34) this is part of Lemma 5.32 and for ζ it is Lemma 5.12. The rest of the morphisms are natural in t because they result from aplying the functor $\mathscr{C}$ of Lemma 5.11 to appropriate morphisms in $B(\mathcal{O}, \mathrm{Sp})^{\mathcal{O}}$.

The morphisms in the zig-zag labeled with $\sim$ are indeed weak equivalences by Ken Brown's Lemma [18, Lemma 1.1.12]: they result from applying a left Quillen functor to an appropriate weak equivalence between cofibrant objects. Here we use that the functors δ, $(u_t)^\star$ and $\int^{\mathcal{O}_{/t}}$ are left Quillen by Lemmas 5.28, 5.29 and Theorem 5.22 respectively.

The fact that the zig-zag induces (5.1) upon taking homotopy groups follows from the commutativity of the following diagram of spectra.

$$
\begin{tikzcd}
\boxed{\mathbb{K}(A, \mathcal{R}(B \rtimes t))} \arrow[r, equal] \arrow[dd, "can_{\mathrm{id}_t}"] \arrow[ddddd, bend right, "(-)^{(t)}"'] & J(t) \arrow[r, "can_{\mathrm{id}_t}", "\cong \text{ by Lem. 5.30}"'] & \int^{\mathcal{O}/t} (u_t)^\star \delta_J \\
& (\bar{Q}J)(t) \arrow[u, "\bar{q}", "\wr"'] \arrow[r, "can_{\mathrm{id}_t}", "\cong \text{ by Lem. 5.30}"'] \arrow[d, "can_{\mathrm{id}_t}"'] & \int^{\mathcal{O}/t} (u_t)^\star \delta_{\bar{Q}J} \arrow[u, "\bar{q}", "\wr"'] \arrow[d, "(5.34)"'] \\
\coprod_{\gamma \in \mathcal{O}(t,t)} J(t) \arrow[ddd, "\varphi"] & \coprod_{\gamma \in \mathcal{O}(t,t)} (\bar{Q}J)(t) \arrow[l, "\bar{q}"'] \arrow[r, "can_{\mathrm{id}_t}"] & \int^{\mathcal{O}/t} (u_t)^\star (u_t)_! (u_t)^* \delta_{\bar{Q}J} \\
& \left[Q(u_t)_! \delta_{\bar{Q}J}\right](t,t) \arrow[u, "q", "\wr"'] \arrow[r, "can_{\mathrm{id}_t}"] \arrow[d, "\bar{q}"'] & \int^{\mathcal{O}/t} (u_t)^\star Q (u_t)_! (u_t)^* \delta_{\bar{Q}J} \arrow[u, "q", "\wr"'] \arrow[d, "\bar{q}"'] \\
& \left[Q(u_t)_! \delta_J\right](t,t) \arrow[uul, bend left, "q", "\sim"'] \arrow[r, "can_{\mathrm{id}_t}"] \arrow[d, "\varphi"'] & \int^{\mathcal{O}/t} (u_t)^\star Q (u_t)_! (u_t)^* \delta_J \arrow[d, "\varphi"'] \\
M_t(t,t) & (QM_t)(t,t) \arrow[l, "q", "\sim"'] \arrow[r, "can_{\mathrm{id}_t}"] & \int^{\mathcal{O}/t} (u_t)^\star Q(M_t) \\
& (QL_t)(t,t) \arrow[u, "\psi", "\wr"'] \arrow[r, "can_{\mathrm{id}_t}"] \arrow[d, "q"'] & \int^{\mathcal{O}/t} (u_t)^\star Q(L_t) \arrow[u, "\psi", "\wr"'] \arrow[d, "q"'] \\
& L_t(t,t) \arrow[uul, bend left, "\psi", "\sim"'] \arrow[r, "can_{\mathrm{id}_t}"] \arrow[dr, bend right, "\zeta_t"'] & \int^{\mathcal{O}/t} (u_t)^\star L_t \arrow[d, "\zeta"] \\
& & \boxed{\mathbb{K}^G(A^{(t)}, M_G B)}
\end{tikzcd}
$$

Upon taking homotopy groups, the morphisms labeled with $\sim$ in the diagram above become isomorphisms and can be inverted. The zig-zag equals the morphism in the top row followed by the composite of the morphisms in the rightmost column. By Lemma 4.6 (3), the composite of the bent morphisms on the left equals, upon taking homotopy groups, the isomorphism (5.1). □

6. Equivariant homology with coefficients in kk^G-theory

Lemma 6.1. *Let A be an algebra, B be a G-algebra and X be a $(G, \mathcal{F}in)$-complex. Then there exists a morphism of spectra*

$$\alpha_X : H^G(X; \mathbb{K}^G(A^{(-)}, B)) \to \mathbb{K}^G(A^{(X)}, B) \tag{6.2}$$

that is natural in X and that is an isomorphism for $X = G/H$.

Proof. Let $\mathbf{E} : \mathrm{Or}(G, \mathcal{F}in) \to \mathrm{Sp}$ be a functor and X be a $(G, \mathcal{F}in)$-complex. By Lemma A.4, Lemma A.1 and the fact that coends and smash products commute with colimits, we have:

$$\begin{aligned}
\boxed{H^G(X;\mathbf{E})} &= \int^{G/H} X_+^H \wedge \mathbf{E}(G/H) \\
&\cong \int^{G/H} \Big(\operatorname*{colim}_{G/K\times\Delta^n\downarrow X} G/K\times\Delta^n\Big)_+^H \wedge \mathbf{E}(G/H) \\
&\cong \int^{G/H} \operatorname*{colim}_{G/K\times\Delta^n\downarrow X} (G/K\times\Delta^n)_+^H \wedge \mathbf{E}(G/H) \\
&\cong \operatorname*{colim}_{G/K\times\Delta^n\downarrow X} \Delta^n_+ \wedge \int^{G/H} (G/K)_+^H \wedge \mathbf{E}(G/H) \\
&\cong \operatorname*{colim}_{G/K\times\Delta^n\downarrow X} \Delta^n_+ \wedge H^G(G/K;\mathbf{E}) \\
&\cong \boxed{\operatorname*{colim}_{G/K\times\Delta^n\downarrow X} \Delta^n_+ \wedge \mathbf{E}(G/K)}
\end{aligned}$$

Thus, to construct the morphism (6.2) it suffices to define compatible morphisms

$$\Delta^n_+ \wedge \mathbb{K}^G(A^{(G/K)},B) \to \mathbb{K}^G(A^{(X)},B) \tag{6.3}$$

for every $f : G/K\times\Delta^n \to X$. Define (6.3) as the composite:

$$\Delta^n_+ \wedge \mathbb{K}^G(A^{(G/K)},B) \xrightarrow{(C.4)} \mathbb{K}^G(A^{(G/K\times\Delta^n)},B) \xrightarrow{f_*} \mathbb{K}^G(A^{(X)},B)$$

The compatibility of these morphisms is immediate from the naturality of (C.4) in G/K. To see that the induced morphism (6.2) is an isomorphism for $X = G/H$, note that in this case the identity of $X \cong G/H\times\Delta^0$ is a final object among those $G/K\times\Delta^n \downarrow X$. Then, in this case, taking colimit over $G/K\times\Delta^n\downarrow X$ boils down to evaluating at the final object $\mathrm{id}_{G/H\times\Delta^0}$ and the result follows from Lemma C.3. $\square$

Lemma 6.4. *Let A be an algebra and B be a G-algebra. For every $(G,\mathcal{F}in)$-complex Y there is a natural weak equivalence of spectra:*

$$H^G(Y;\mathbb{K}^G(A^{(-)},B)) \xrightarrow{\sim} \operatorname*{colim}_{\substack{X\subseteq Y\\ (G,\mathcal{F}in)\text{-finite}}} \mathbb{K}^G(A^{(X)},B)$$

Proof. For every $(G,\mathcal{F}in)$-complex Y we have that

$$Y \cong \operatorname*{colim}_{\substack{X\subseteq Y\\ (G,\mathcal{F}in)\text{-finite}}} X. \tag{6.5}$$

From Lemma 6.1 there exist morphims of spectra α_X that are natural in X. Since homology commutes with filtered colimits, we have a morphism of spectra induced by the α_X:

$$\beta_Y = \operatorname{colim}\alpha_X : H^G(Y;\mathbb{K}^G(A^{(-)},B)) \to \operatorname*{colim}_{\substack{X\subseteq Y\\ (G,\mathcal{F}in)\text{-finite}}} \mathbb{K}^G(A^{(X)},B)$$

Let us prove that the morphism β_Y is a weak equivalence. It suffices to show that α_X is a weak equivalence for every $(G,\mathcal{F}in)$-finite X. We are going to prove the latter following these steps:

(a) $X = G/H$ for $H \in \mathcal{F}in$
(b) $X = G/H \times \Delta^n$ for $H \in \mathcal{F}in$ and $n \in \mathbb{N}$
(c) $X = \bigsqcup_{i=1}^{m} G/H_i \times \Delta^n$ for $H_i \in \mathcal{F}in$ and $m, n \in \mathbb{N}$
(d) X any $(G,\mathcal{F}in)$-finite complex

The first step is proved in Lemma 6.1. If $X = G/H \times \Delta^n$ we obtain the following diagram:

$$\begin{tikzcd}
H^G(G/H \times \Delta^n; \mathbb{K}^G(A^{(-)}, B)) \arrow[r] \arrow[d, "\pi_*"'] & \mathbb{K}^G(A^{(G/H\times\Delta^n)}, B) \arrow[d, "\pi_*"] \\
H^G(G/H; \mathbb{K}^G(A^{(-)}, B)) \arrow[r] & \mathbb{K}^G(A^{(G/H)}, B)
\end{tikzcd}$$

The left vertical arrow is an equivalence by homotopy invariance of the equivariant homology theory. The right vertical arrow is a weak equivalence because $A^{(G/H\times\Delta^n)} \cong A^{(G/H)} \otimes A^{\Delta^n}$ and by homotopy invariance of $kk^G_*(-, B)$.

Consider $X = \bigsqcup_{i=1}^{m} G/H_i \times \Delta^n$ and write $\mathbf{E} = \mathbb{K}^G(A^{(-)}, B)$. Note that this step follows from the previous one because:

$$\begin{aligned}
H^G(X; \mathbf{E}) &\cong \bigoplus_{i=1}^{n} H^G_*(G/H_i \times \Delta^n; \mathbf{E})) \\
kk^G(A^{(X)}, B) &\cong \bigoplus_{i=1}^{n} kk^G_*(A^{(G/H_i\times\Delta^n)}, B) \\
\alpha_X &= \oplus \alpha_{G/H_i\times\Delta^n}
\end{aligned}$$

Let us assume that α_X is a weak equivalence for every $(G,\mathcal{F}in)$-finite X complex of dimension $n-1$. Consider X obtained as the following pushout diagram:

$$\begin{tikzcd}
\bigsqcup_{i=1}^{n} G/H_i \times \partial\Delta^n \arrow[r] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\
\bigsqcup_{i=1}^{n} G/H_i \times \Delta^n \arrow[r] & X
\end{tikzcd}$$

By Lemma B.2, upon applying the functor $A^{(-)}$ we obtain the following Milnor square:

$$\begin{tikzcd}
A^{(\bigsqcup_{i=1}^{n} G/H_i\times\partial\Delta^n)} & A^{(X_{n-1})} \arrow[l] \\
A^{(\bigsqcup_{i=1}^{n} G/H_i\times\Delta^n)} \arrow[u, two heads] & A^{(X)} \arrow[l] \arrow[u, two heads]
\end{tikzcd}$$

Then

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^G_*(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n, \mathbf{E}) \arrow[r, "\alpha_{\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n}"] \arrow[d] & kk^G_*(A^{(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n)}, B) \arrow[d] \\
H^G_*(\bigsqcup_{i=1}^n G/H_i \times \Delta^n, \mathbf{E}) \oplus H^G_*(X_{n-1}, \mathbf{E}) \arrow[r, "\alpha_{\bigsqcup_{i=1}^n G/H_i \times \Delta^n} \oplus \alpha_{X_{n-1}}"] \arrow[d] & kk^G_*(A^{(\bigsqcup_{i=1}^n G/H_i \times \Delta^n)}, B) \oplus kk^G_*(A^{(X_{n-1})}, B) \arrow[d] \\
H^G_*(X, \mathbf{E}) \arrow[r, "\alpha_X"] \arrow[d] & kk^G_*(A^{(X)}, B) \arrow[d] \\
H_{*-1}(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n, \mathbf{E}) \arrow[r] \arrow[d] & kk^G_{*-1}(A^{(\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n)}, B) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

The left column is given by Mayer-Vietoris sequence for $H^G(-;\mathbf{E})$ and the right column is given by excision of kk^G; see Lemma B.5. By induction, $\alpha_{\bigsqcup_{i=1}^n G/H_i \times \partial\Delta^n}$ and $\alpha_{X_{n-1}}$ are isomorphisms. By step (c), $\alpha_{\bigsqcup_{i=1}^n G/H_i \times \Delta^n}$ is an isomorphism. We conclude by the Five Lemma that α_X is an isomorphism. □

Theorem 6.6. *Let G be a group satisfying (4.3). Let A be an algebra, B be a G-algebra and Y be a $(G, \mathcal{F}in)$-complex. Then there is a natural isomorphism*

$$H^G_*(Y; \mathbb{K}(A, \mathcal{R}(B \rtimes -))) \cong \operatorname*{colim}_{\substack{X \subseteq Y \\ (G,\mathcal{F}in)\text{-finite}}} kk^G_*(A^{(X)}, B)$$

induced by a natural zig-zag of spectra.

Proof. Define $\mathbf{E}, \mathbf{F} : \mathrm{Or}(G, \mathcal{F}in) \to \mathrm{Sp}$ by:

$$\mathbf{E}(G/N) = \mathbb{K}(A, \mathcal{R}(B \rtimes G/N))$$
$$\mathbf{F}(G/N) = \mathbb{K}^G(A^{(G/N)}, B)$$

By Theorem 5.35, there is a zig-zag of $\mathrm{Or}(G, \mathcal{F}in)$-spectra

$$\mathbf{E} \dashrightarrow \mathbf{F} \tag{6.7}$$

that induces the isomorphism (5.1) upon taking homotopy groups. Moreover, the morphisms in (6.7) going backwards are objectwise weak equivalences of spectra. Let Y be a $(G, \mathcal{F}in)$-complex. After applying $H^G(Y; -)$ to (6.7) we get a zig zag of spectra:

$$H^G(Y; \mathbf{E}) \dashrightarrow H^G(Y; \mathbf{F}) \tag{6.8}$$

Here the morphisms going backwards are weak equivalences by [11, Lemma 4.6]. Upon taking homotopy groups, we get an homomorphism:

$$H^G_*(Y; \mathbf{E}) \longrightarrow H^G_*(Y; \mathbf{F}) \tag{6.9}$$

This morphism is clearly natural in Y. Moreover, we claim that it is an isomorphism. The assertion holds for $Y = G/N$ since, in this case, it is the isomorphism (5.1). Now continue as in the proof of Lemma 6.4: the case $Y = G/H \times \Delta^n$ follows from homotopy invariance and the general case follows from excision considering the skeletal filtration of Y; see also [11, Theorem 6.3 (2)]. Finally, combine (6.9) with Lemma 6.4:

$$H_*^G(Y;\mathbb{K}(A,\mathcal{R}(B\rtimes -)))\xrightarrow[\cong]{(6.9)} H_*^G(Y;\mathbb{K}^G(A^{(-)},B) \xrightarrow[\cong]{} \operatorname*{colim}_{\substack{X\subseteq Y\\ (G,\mathcal{F}in)\text{-finite}}} kk_*^G(A^{(X)},B)$$

This finishes the proof. $\square$

Appendix A. G-Simplicial sets

Let G be a group. We recall some definitions and properties of the G-simplicial sets. A *G-simplicial set* X is a simplicial set with a (left) action of G. We write $\mathbb{S}^G$ for the category of G-simplicial sets with equivariant morphisms. Every G-simplicial set X has a skeletal filtration such that the n-skeleton $\operatorname{sk}_n X$ is obtained from $\operatorname{sk}_{n-1} X$ upon attaching cells of the form $G/H \times \Delta^n$ for $H \in \operatorname{Or}(G)$. Let $\mathcal{F}$ be a nonempty family of subgroups of G closed under conjugation and subgroups — we are interested in the family $\mathcal{F}in$ of finite subgroups. A G-simplicial set X is called a *$(G,\mathcal{F})$-complex* if X can be built from cells of the form $G/H \times \Delta^n$ with $H \in \mathcal{F}$. The $(G,\mathcal{F})$-complexes are the cofibrant objects for a certain model structure on $\mathbb{S}^G$; see [7, Proposition 2.3]. A $(G,\mathcal{F})$-complex X is called *$(G,\mathcal{F})$-finite* if X can be built from a finite number of cells of the form $G/H \times \Delta^n$ with $H \in \mathcal{F}$. In the rest of this section we gather some technical results that are used in Section 6.

Lemma A.1. *Let G be a group and let $\mathbb{S}_c \subset \mathbb{S}$ denote the full subcategory of connected simplicial sets. Let $G/H, G/K \in \operatorname{Or}(G)$ and $X, Y \in \mathbb{S}_c$. Then there is a natural isomorphism:*

$$\hom_{\mathbb{S}^G}(G/H \times X, G/K \times Y) \cong \hom_{\operatorname{Or}(G)}(G/H, G/K) \times \hom_{\mathbb{S}}(X, Y)$$

In other words, the full subcategory of $\mathbb{S}^G$ whose objects are $G/H \times X$ with $G/H \in \operatorname{Or}(G)$ and $X \in \mathbb{S}_c$ is equivalent to the product category $\operatorname{Or}(G) \times \mathbb{S}_c$.

Proof. Let $f : G/H \times X \to G/K \times Y$ be a morphism in $\mathbb{S}^G$. By connectedness, there is a unique coset uK such that there exists a dashed arrow completing the following diagram into a commutative square:

$$\begin{tikzcd} \coprod_{G/H} X \arrow[r, "f"] & \coprod_{G/K} Y \\ X \arrow[u, "can_H"] \arrow[r, dashed, "h"] & Y \arrow[u, "can_{uK}"'] \end{tikzcd}$$

Call the dashed arrow h. Moreover, it follows from the equivariance of f that $g(tH) = tuK$ defines a morphism $g : G/H \to G/K$. Conversely, every pair $(g,h) \in \hom_{\operatorname{Or}(G)}(G/H, G/K) \times \hom_{\mathbb{S}}(X,Y)$ defines a unique G-equivariant morphism f

making the following squares commute for all t:

$$\begin{tikzcd} \coprod_{G/H} X \arrow[r, "f"] & \coprod_{G/K} Y \\ X \arrow[u, "can_{tH}"] \arrow[r, "h"] & Y \arrow[u, "can_{g(tH)}"'] \end{tikzcd}$$

It is easily verified that both constructions are mutually inverse. □

Lemma A.2. *Let $\mathcal{F}$ be a family of subgroups of G and let $\psi : Y \to X$ be a morphism of G-simplicial sets. If X is a $(G, \mathcal{F})$-complex then Y is a $(G, \mathcal{F})$-complex too.*

Proof. Let $\sigma \in Y_n$; it is easily verified that $\mathrm{Stab}(\sigma) \subseteq \mathrm{Stab}(\psi(\sigma)) \in \mathcal{F}$. □

Lemma A.3. *Let G be an infinite group, let $H \subseteq G$ be a subgroup, let X be a $(G, \mathcal{F}in)$-complex and let K be a finite simplicial set. Then every G-equivariant morphism $\psi : G/H \times K \to X$ is proper, i.e. $\psi^{-1}(L)$ is a finite simplicial set for every finite sub-simplicial set $L \subseteq X$.*

Proof. First of all notice that $H \in \mathcal{F}in$ by Lemma A.2. Let $L \subseteq X$ be a finite simplicial set and suppose that $\psi^{-1}(L)$ is not finite. Then there is an infinite number of non-degenerate simplices in $\psi^{-1}(L) \subseteq G/H \times K$. Since every non-degenerate simplex of $G/H \times K$ has dimension $\leq d := \dim K$, there exists $0 \leq p \leq d$ such that there is an infinite number of non-degenerate p-simplices in $\psi^{-1}(L)$. Let $\{g_i, i \in I\} \subseteq G$ be a system of representatives for the cosets in G/H; notice that I is infinite because H is finite. Every non-degenerate p-simplex of $G/H \times K$ is of the form $(g_i H, \sigma)$ for some $i \in I$ and some non-degenerate p-simplex σ of K. Since K has finitely many non-degenerate p-simplices, there exist a non-degenerate p-simplex σ of K and an infinite subset $J \subseteq I$ such that $\{(g_i H, \sigma), i \in J\} \subseteq \psi^{-1}(L)$. Then

$$\{\psi(g_i H, \sigma), i \in J\} \subseteq L_p.$$

Since L_p is a finite set, replacing J by a smaller but still infinite subset, we can assume without loss of generality that there is $\tau \in L_p$ such that $\psi(g_i H, \sigma) = \tau$ for every $i \in J$. Fix $i_0 \in J$. Then

$$g_{i_0}\psi(H, \sigma) = \psi(g_{i_0} H, \sigma) = \tau = \psi(g_i H, \sigma) = g_i \psi(H, \sigma)$$

for every $i \in J$ and it follows that $\{g_i^{-1} g_{i_0}, i \in J\} \subseteq \mathrm{Stab}(\psi(H, \sigma)) \in \mathcal{F}in$; this is a contradiction since J is infinite. □

Lemma A.4. *Let G be a group, $K \subset G$ be a subgroup and X be a G-simplicial set. Then:*

$$X^K \cong \operatorname*{colim}_{G/H \times \Delta^n \downarrow X} (G/H \times \Delta^n)^K$$

Proof. There is a natural morphism:

$$\operatorname*{colim}_{G/H \times \Delta^n \downarrow X} (G/H \times \Delta^n)^K \to X^K \tag{A.5}$$

Let us prove that it is surjective. Let $\sigma \in (X^K)_p = (X_p)^K$ and put $H := \mathrm{Stab}(\sigma)$; note that we have $K \subset H$. Let $f : G/H \times \Delta^p \to X$ be the G-equivariant morphism determined by $(H, \iota_p) \mapsto \sigma$. Since $(H, \iota_p) \in (G/H \times \Delta^p)^K$, we have that $\sigma =$

$f^K(H,\iota_p)$, showing that σ is in the image of (A.5). We still have to prove that (A.5) is injective. Let us first show that every p-simplex of

$$\operatorname*{colim}_{G/H\times\Delta^n\downarrow X}(G/H\times\Delta^n)^K \tag{A.6}$$

is represented by one of the form $(L,\iota_p)\in(G/L\times\Delta^p)^K$. Let $f:G/H\times\Delta^n\to X$ be a G-equivariant morphism and let (gH,τ) be a p-simplex of $(G/H\times\Delta^n)^K$. Then $K\subset gHg^{-1}$ and $\tau=\tau_*(\iota_p)$ for some non decreasing function $\tau:[p]\to[n]$. The commutativity of the following triangle implies that (gH,τ) and (gH,ι_p) represent the same simplex of (A.6):

$$\begin{tikzcd}
(gH,\tau) & G/H\times\Delta^n \arrow[r, "f"] & X \\
(gH,\iota_p) \arrow[u, mapsto] & G/H\times\Delta^p \arrow[u, "\mathrm{id}\times\tau_*"] \arrow[ur, bend right, "f\circ(\mathrm{id}\times\tau_*)"'] &
\end{tikzcd}$$

Write $L:=gHg^{-1}$ and note that there is a G-equivariant bijection $\beta:G/H\to G/L$ determined by $\beta(H)=g^{-1}L$. The commutativity of the following triangle implies thata (gH,ι_p) and (L,ι_p) represent the same simplex of (A.6):

$$\begin{tikzcd}
(gH,\iota_p) \arrow[d, mapsto] & G/H\times\Delta^p \arrow[r, "f\circ(\mathrm{id}\times\tau_*)"] \arrow[d, "\beta\times\mathrm{id}"', "\cong"] & X \\
(L,\iota_p) & G/L\times\Delta^p \arrow[ur, bend right] &
\end{tikzcd}$$

Now let (L,ι_p) and (M,ι_p) represent two p-simplices of (A.6) having the same image $\sigma\in(X^K)_p$ under (A.5). Put $S:=\operatorname{Stab}(\sigma)$ and note that $L,M\subset S$. The commutativity of the following diagram shows that (L,ι_p) and (M,ι_p) represent the same simplex of (A.5):

$$\begin{tikzcd}
(L,\iota_p) \arrow[d, mapsto] & G/L\times\Delta^p \arrow[d] \arrow[dr, bend left] & \\
(S,\iota_p) & G/S\times\Delta^p \arrow[r] & X \\
(M,\iota_p) \arrow[u, mapsto] & G/M\times\Delta^p \arrow[u] \arrow[ur, bend right] &
\end{tikzcd}$$

This finishes the proof. ◻

Appendix B. A Mayer-Vietoris sequence in bivariant algebraic K-theory

Definition B.1. A *Milnor square of G-algebras* is a pullback square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
C \arrow[r] & D
\end{tikzcd}$$

where f is surjective and has a G-linear section.

Lemma B.2. *Let A be an algebra. Let X be a finite $(G,\mathcal{F}in)$-complex and let X^n be the n-skeleton of X. Since X is a $(G,\mathcal{F}in)$-complex there is a pushout*

$$\begin{tikzcd} \coprod_{i=1}^r G/H_i \times \partial\Delta^n \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\ \coprod_{i=1}^r G/H_i \times \Delta^n \arrow[r] & X^n \end{tikzcd} \tag{B.3}$$

of G-simplicial sets with $H_i \in \mathcal{F}in$ for all i. Then all the morphisms appearing in (B.3) are proper and this diagram induces a Milnor square of G-algebras:

$$\begin{tikzcd} A^{\left(\coprod_{i=1}^r G/H_i \times \partial\Delta^n\right)} & A^{(X^{n-1})} \arrow[l] \\ A^{\left(\coprod_{i=1}^r G/H_i \times \Delta^n\right)} \arrow[u] & A^{(X^n)} \arrow[l] \arrow[u] \end{tikzcd} \tag{B.4}$$

Proof. The vertical morphisms in (B.3) are proper because they are inclusions of simplicial sets; the horizontal morphisms in (B.3) are proper by Lemma A.3. Then we can apply $A^{(-)}$ to get a commutative diagram of G-algebras like (B.4); it is easily verified that this is a pullback of G-algebras. Write $i : \partial\Delta^n \to \Delta^n$ for the inclusion. By [8, Lemma 3.1.2 and Proposition 3.1.3], the morphism $i^* : A^{\Delta^n} \to A^{\partial\Delta^n}$ admits a linear section. Then the left vertical morphism in (B.4) admits a G-linear section, since it identifies with

$$\bigoplus_i \mathrm{id}_{\ell^{(G/H_i)}} \otimes i^* : \bigoplus_i \ell^{(G/H_i)} \otimes A^{\Delta^n} \longrightarrow \bigoplus_i \ell^{(G/H_i)} \otimes A^{\partial\Delta^n}$$

by [7, 9.3.4]. $\square$

Lemma B.5. *Let E be a G-algebra. Then every Milnor square of G-algebras*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

induces a long exact Mayer-Vietoris sequence:

$$kk_*^G(D,E) \longrightarrow kk_*^G(B,E) \oplus kk_*^G(C,E) \longrightarrow kk_*^G(A,E) \longrightarrow kk_{*-1}^G(D,E)$$

Proof. It follows from excision in kk^G [13, Theorem 4.1.1] and from the argument explained in [10, Theorem 2.41]. $\square$

Appendix C. The model category of spectra and spectra representing bivariant K-theory

C.1. **The stable model category of spectra.** In this section we recall the definition of the stable model category of spectra and discuss some of its properties.

Definition C.1 ([5, Definition 2.1])**.** A *spectrum* X is a sequence of pointed simplicial sets $X^0, X^1, X^2, \ldots$ together with pointed morphisms $S^1 \wedge X^n \to X^{n+1}$ for all n, called *bonding maps*. Here, $S^1 = \Delta^1/\partial\Delta^1$. A *morphism of spectra* $f : X \to Y$

is a sequence of pointed morphisms $f^n : X^n \to Y^n$ that commute with the bonding maps:

$$\begin{tikzcd} S^1 \wedge X^n \arrow[r] \arrow[d, "S^1\wedge f^n"'] & X^{n+1} \arrow[d, "f^{n+1}"] \\ S^1 \wedge Y^n \arrow[r] & Y^{n+1} \end{tikzcd}$$

We write Sp for the category of spectra and morphisms of spectra.

We endow Sp with its stable model structure, which we proceed to describe; see [5, Section 2] and [19, Section 3] for details:

- Let X be a spectrum and let $m \in \mathbb{Z}$. The *m-th stable homotopy group* of X is defined as:
$$\pi_m(X) = \operatorname*{colim}_k \pi_{m+k}(X^k)$$
A morphism of spectra $f : X \to Y$ is a *weak equivalence* if $\pi_m(f)$ is an isomorphism for all $m \in \mathbb{Z}$.
- A morphism of spectra $f : X \to Y$ is a *fibration* if $f^n : X^n \to Y^n$ is a fibration of simplicial sets for all n.
- A morphism of spectra is a *cofibration* if it has the left lifting property with respect to trivial fibrations.

Lemma C.2. *The stable model structure on* Sp *is combinatorial.*

Proof. Recall from [12, Definition 2.1] that a model category is *combinatorial* if it is cofibrantly generated and its underlying category is locally presentable. The stable model structure on Sp is cofibrantly generated by [19, Definition 3.3 and Corollary 3.5]. We have to show that the underlying category is locally presentable. We claim that Sp is locally *finitely* presentable. By [1, Theorem 1.11], to prove this claim we have to show that Sp has a strong generator formed by finitely presentable objects. By [1, Example 1.12], the set $\{\Delta^m : m \geq 0\}$ is a strong generator for $\mathbb{S}$ formed by finitely presentable objects. Since the forgetful functor $\mathbb{S}_* \to \mathbb{S}$ commutes with filtered colimits, the latter is easily seen to imply that $\{\Delta^m_+ : m \geq 0\}$ is a strong generator for $\mathbb{S}_*$ formed by finitely presentable objects. For $n \geq 0$, let $F_n : \mathbb{S}_* \to \mathrm{Sp}$ be the left adjoint to evaluation at n. Explicitely, for a pointed simplicial set X, let $F_n(X)$ be the spectrum whose level k is $(S^1)^{\wedge(k-n)} \wedge X$ if $k \geq n$ and $*$ otherwise. The bonding maps are the obvious ones. It is easily verified that $\{F_n(\Delta^m_+) : m, n \geq 0\}$ is a strong generator for Sp formed by finitely presentable spectra. □

C.2. **Bivariant K-theory spectra.** In this section we recall the definitions of spectra representing kk-theory [14, Theorem 9.8] and kk^G-theory [25, Theorem 5.3.11].

Let $\mathcal{C}$ denote either Alg_ℓ or $G\mathrm{Alg}_\ell$. For two objects A and B of $\mathcal{C}$, the *bivariant K-theory space* of the pair (A, B) [26, Definition 4.10] is defined as the fibrant simplicial set:

$$\mathscr{K}(A, B) := \operatorname*{colim}_n \Omega^n Ex^\infty \mathrm{Hom}_{\mathcal{C}}(J^n A, B^\Delta)$$

This definition is equivalent to the original one given in [14, Section 4]. By [14, Theorem 5.1] there is a natural isomorphism of simplicial sets:

$$\mathscr{K}(A, B) \cong \Omega \mathscr{K}(JA, B)$$

Thus, we have an Ω-spectrum $\mathbb{K}(A,B)$ defined by the sequence:

$$\mathscr{K}(A,B),\ \mathscr{K}(JA,B),\ \mathscr{K}(J^2A,B),\ \dots$$

The spectrum $\mathbb{K}(A,B)$ (denoted by $\mathbb{K}^{\mathrm{unst}}(A,B)$ in [14]) represents a universal bivariant K-theory introduced by Garkusha that is excisive, homotopy invariant but matrix-unstable [14, Comparison Theorem B]. Different matrix-stabilizations can be perfomed in order to obtain spectra representing kk- and kk^G-theories:

(1) *Stabilization by finite matrices.* For two objects A and B of $\mathcal{C}$ put

$$\mathbb{K}_f(A,B) := \operatorname*{colim}_n \mathbb{K}(A, M_nB)$$

where the transition maps are induced by the inclusion $M_nB \to M_{n+1}B$ into the upper left corner. These spectra represent a universal bivariant K-theory that is excisive, homotopy invariant and stable by finite matrices [14, Theorem 9.8]; see [14, Section 9] and [25, Section 5.1] for details.

(2) *Stabilization by finite matrices indexed on an infinite set.* Let $\mathcal{X}$ be an infinite set. For two objects A and B of $\mathcal{C}$ put [25, Definition 5.2.21]:

$$\mathbb{K}_{\mathcal{X}}(A,B) := \mathbb{K}_f(A, M_{\mathcal{X}}B)$$

The spectra $\mathbb{K}_{\mathcal{X}}(A,B)$ represent a universal bivariant K-theory that is excisive, homotopy invariant and $M_{\mathcal{X}}$-stable [25, Theorem 5.2.22]. For any $\mathcal{X}$, Weibel's homotopy K-theory KH is the functor represented by the base ring ℓ [25, Theorem 5.2.20]. For $\mathcal{X}=\mathbb{N}$, this theory coincides with kk.

(3) *G-stabilization.* Let G be a group and let $\mathcal{X} = G\times\mathbb{N}$. For two G-algebras A and B put:

$$\mathbb{K}^G(A,B) := \mathbb{K}_{\mathcal{X}}(M_GA, M_GB)$$

These spectra represent kk^G-theory [25, Theorem 5.3.11].

Lemma C.3 (cf. [25, Section 4.4])**.** *Let $\mathbb{S}_f\subset\mathbb{S}$ denote the full subcategory of finite simplicial sets. Let G be a group, $\mathcal{X}$ be an infinite set, $G/K\in\mathrm{Or}(G,\mathcal{F}in)$, $A,B\in G\mathrm{Alg}_\ell$ and $S\in\mathbb{S}_f$. Then, for $\mathbb{E}\in\{\mathbb{K},\mathbb{K}_f,\mathbb{K}_{\mathcal{X}},\mathbb{K}^G\}$, there is a morphism of spectra*

$$S_+\wedge\mathbb{E}(A^{(G/K)},B)\to\mathbb{E}(A^{(G/K\times S)},B) \tag{C.4}$$

that is natural in A, B, G/K and S. Moreover, for $S=\Delta^0$ this is an isomorphism.

Proof. It suffices to prove the Lemma for $\mathbb{E}=\mathbb{K}$: the case $\mathbb{E}=\mathbb{K}_f$ follows from this upon taking colimit over the inclusions $M_nB\to M_{n+1}B$ and the rest of the cases follow from the latter upon replacing A and B with appropriate matrix algebras.

Let us prove the case $\mathbb{E}=\mathbb{K}$. We will define (C.4) levelwise. At level $p\geq 0$, we have to define a morphism of simplicial sets:

$$S_+\wedge\mathscr{K}(J^p(A^{(G/K)}),B)\to\mathscr{K}(J^p(A^{(G/K\times S)}),B) \tag{C.5}$$

Let us describe it in dimension $q\geq 0$. A q-simplex of $S_+\wedge\mathscr{K}(J^p(A^{(G/K)}),B)$ is represented by a pair (σ,α) where σ is a q-simplex of S and α is a q-simplex of $\mathscr{K}(J^p(A^{(G/K)}),B)$. Let α be represented by a G-algebra homomorphism

$$\alpha : J^{p+v}(A^{(G/K)})\to B_r^{(I^v\times\Delta^q,\partial I^v\times\Delta^q)}$$

for some $v, r \geq 0$. Then the morphism (C.5) sends the pair (σ, α) to the q-simplex of $\mathscr{K}(J^p(A^{(G/K\times S)}), B)$ represented by the following composite in $G\mathrm{Alg}_\ell$:

$$\begin{tikzcd}
J^{p+v}\left(A^{(G/K\times S)}\right) \arrow[d, equal] \\
J^{p+v}\left(\left(A^{(G/K)}\right)^S\right) \arrow[d, "\mathrm{clas}"] \\
\left[J^{p+v}\left(A^{(G/K)}\right)\right]^S \arrow[d, "\alpha_*"] \\
\left[B_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}\right]^S \arrow[d, "\sigma^*"] \\
\left[B_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}\right]^{\Delta^q} \arrow[d, "\mu"] \\
B_r^{(I^v\times\Delta^q\times\Delta^q, \partial I^v\times\Delta^q\times\Delta^q)} \arrow[d, "\mathrm{diag}^*"] \\
B_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}
\end{tikzcd}$$

This clearly defines a morphism (C.5) that is natural in A, B, G/K and S. Let us now show that (C.5) is an isomorphism for $S = \Delta^0$. First note that the classifying map

$$\mathrm{clas} : J^{p+v}\left(\left(A^{(G/K)}\right)^{\Delta^0}\right) \to \left[J^{p+v}\left(A^{(G/K)}\right)\right]^{\Delta^0}$$

is an isomorphism. Moreover, it follows from the naturality of μ that the composite $\mathrm{diag}^* \circ \mu \circ \sigma^*$ equals the obvious isomorphism:

$$\left[B_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}\right]^{\Delta^0} \xrightarrow{\cong} B_r^{(I^v\times\Delta^q, \partial I^v\times\Delta^q)}$$

Together, these observations imply that, for $S = \Delta^0$ and making the obvious identifications, the morphism (C.5) is the identity of $\mathscr{K}(J^p(A^{(G/K)}), B)$. This finishes the proof. $\square$

Lemma C.6. *Let A and B be two G-algebras and let $f : C \to D$ be a morphism of G-algebras. Then the following square of spectra commutes:*

$$\begin{tikzcd}
\mathbb{K}^G(A, B) \arrow[r, "-\otimes C"] \arrow[d, "-\otimes D"'] & \mathbb{K}^G(A\otimes C, B\otimes C) \arrow[d, "f_*"] \\
\mathbb{K}^G(A\otimes D, B\otimes D) \arrow[r, "f^*"] & \mathbb{K}^G(A\otimes C, B\otimes D)
\end{tikzcd}$$

Proof. Unravelling the definitions of the spectra $\mathbb{K}^G$, $\mathbb{K}_{\mathcal{X}}$ and $\mathbb{K}_f$, it suffices to show that the following square commutes:

$$\begin{tikzcd} \mathbb{K}(A,B) \arrow[r, "-\otimes C"] \arrow[d, "-\otimes D"'] & \mathbb{K}(A\otimes C, B\otimes C) \arrow[d, "f_*"] \\ \mathbb{K}(A\otimes D, B\otimes D) \arrow[r, "f^*"] & \mathbb{K}(A\otimes C, B\otimes D) \end{tikzcd}$$

At level $p \geq 0$ the latter is the following square of simplicial sets:

$$\begin{tikzcd} \mathscr{K}(J^pA,B) \arrow[r, "-\otimes C"] \arrow[d, "-\otimes D"'] & \mathscr{K}(J^p(A\otimes C), B\otimes C) \arrow[d, "f_*"] \\ \mathscr{K}(J^p(A\otimes D), B\otimes D) \arrow[r, "f^*"] & \mathscr{K}(J^p(A\otimes C), B\otimes D) \end{tikzcd} \tag{C.7}$$

Let $q \geq 0$. Write $B_r^{S^n\times\Delta^q}$ instead of $B_r^{(I^n\times\Delta^q,\partial\Delta^n\times\Delta^q)}$ to ease notation. Let α be a q-simplex of $\mathscr{K}(J^pA,B)$, represented by an algebra homomorphism $\alpha : J^{p+n}A \to B_r^{S^n\times\Delta^q}$ for some $n, r \geq 0$. Consider the following commutative diagram of algebras:

$$\begin{tikzcd} J^{p+n}(A\otimes C) \arrow[r, "\mathrm{clas}"] \arrow[d, "J^{p+n}(\mathrm{id}\otimes f)"] & J^{p+n}(A)\otimes C \arrow[r, "\alpha\otimes\mathrm{id}"] \arrow[d, "J^{p+n}(\mathrm{id})\otimes f"] & B_r^{S^n\times\Delta^q}\otimes C \arrow[r, equal] \arrow[d, "\mathrm{id}\otimes f"] & (B\otimes C)_r^{S^n\times\Delta^q} \arrow[d, "(\mathrm{id}\otimes f)_r^{S^n\times\Delta^q}"] \\ J^{p+n}(A\otimes D) \arrow[r, "\mathrm{clas}"] & J^{p+n}(A)\otimes D \arrow[r, "\alpha\otimes\mathrm{id}"] & B_r^{S^n\times\Delta^q}\otimes D \arrow[r, equal] & (B\otimes D)_r^{S^n\times\Delta^q} \end{tikzcd}$$

The composite of the top morphisms followed by the rightmost vertical morphism represents the q-simplex $f_*(\alpha\otimes C)$ of $\mathscr{K}(J^p(A\otimes C), B\otimes D)$. The leftmost vertical morphism followed by the composite of the bottom morphisms represents the q-simplex $f^*(\alpha\otimes D)$ of $\mathscr{K}(J^p(A\otimes C), B\otimes D)$. The commutativity of the diagram shows that $f_*(\alpha\otimes C) = f^*(\alpha\otimes D)$ and, thus, that (C.7) commutes. $\square$

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