# Generalized BMS charge algebra 

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#### Abstract

It has been argued that the symmetries of gravity at null infinity should include a $\operatorname{Diff}\left(S^{2}\right)$ factor associated to diffeomorphisms on the celestial sphere. However, the standard phase space of gravity does not support the action of such transformations. Building on earlier work by Laddha and one of the authors, we present an extension of the phase space of gravity at null infinity on which $\operatorname{Diff}\left(S^{2}\right)$ acts canonically. The Poisson brackets of supertranslation and $\operatorname{Diff}\left(S^{2}\right)$ charges reproduce the generalized BMS algebra introduced in [1].


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## I. INTRODUCTION

Since the work of Bondi, van der Burg, Metzner [2] and Sachs [3] it is known that the asymptotic Killing symmetries of asymptotically flat spacetimes at null infinity are generated by supertranslations $\xi_{f}$ (labeled by functions $f$ on the sphere) and Lorentz rotations $\xi_{V}$ (labeled by conformal Killing vector (CKV) fields $V^{a}$ on the sphere), forming together the BMS algebra

$$
\begin{equation*}
\left[\xi_{f}, \xi_{f^{\prime}}\right]=0, \quad\left[\xi_{V}, \xi_{f}\right]=\xi_{V(f)}, \quad\left[\xi_{V}, \xi_{V^{\prime}}\right]=\xi_{\left[V, V^{\prime}\right]} \tag{1.1}
\end{equation*}
$$

Here $\left[V, V^{\prime}\right]$ is the Lie bracket of sphere vector fields and

$$
\begin{equation*}
V(f):=V^{a} \partial_{a} f-\frac{1}{2} D_{a} V^{a} f \tag{1.2}
\end{equation*}
$$

This algebra can be thought of as a generalization of the Poincare algebra, with translations replaced by the infinite dimensional abelian algebra of supertranslations.

As described by Ashtekar and Streubel (AS) [4, 5], the gravitational field at null infinity has a natural phase space structure that allows to associate canonical charges to BMS symmetries. ${ }^{1}$ These are the supermomenta $P_{f}$ and angular momenta $\stackrel{\circ}{J}_{V}$, with Poisson brackets (PBs) reproducing the BMS algebra

$$
\begin{equation*}
\left\{P_{f}, P_{f^{\prime}}\right\}=0, \quad\left\{\circ_{V}, P_{f}\right\}=P_{V(f)}, \quad\left\{\circ_{V}, \circ_{V^{\prime}}\right\}=\check{J}_{\left[V, V^{\prime}\right]} \tag{1.3}
\end{equation*}
$$

Many years after these foundational works, the subject of gravitational symmetries at null infinity experienced two major revisions. Firstly, Barnich and Troessaert (BT) [7] studied an extension of the BMS algebra in which the vector fields $V^{a}$ are allowed to have poles, thus enlarging the 6 dimensional algebra of global CKVs on the sphere into the infinite dimensional algebra of local CKVs. Whereas the BT extended BMS algebra has the same form as (1.1), its associated charge algebra exhibits an extension term in the bracket between $P_{f}$ and $J_{V}$ [8]. Secondly, Strominger and collaborators [9, 10] showed how BMS can be understood as a symmetry of the gravitational S-matrix, identifying the corresponding supertranslation Ward identities with Weinberg's soft graviton theorem [11]. These two fronts came together in the work 12], where a subleading soft graviton factorization [13] was identified as a Ward identity of BT superrotations. Since then, there appeared many

[^1]further developments on soft factorization and asymptotic symmetries, see the reviews [1417] and references therein.

In [1], Laddha and one of the authors argued that, from the perspective of the subleading soft graviton theorem, it is more natural to consider a different generalization of the BMS algebra, one in which $V^{a}$ is allowed to be an arbitrary smooth vector field on the sphere. The defining relations for this Generalized BMS (GBMS) algebra are again given by (1.1), and from the subleading soft graviton theorem one can identify a candidate for 'super angular momentum' $J_{V}$.

It is then natural to ask if there exists an underlying phase space on which $J_{V}$ acts. A first step in this direction was taken in [18], where an extension of the Ashtekar-Streubel phase space was identified and $J_{V}$ obtained using covariant phase space techniques. The treatment in [18], however, had certain limitations that forbid an evaluation of PBs between charges. The objective of the present work is to show that such limitations can be overcome. We will obtain a phase space on which GBMS acts canonically with Poisson brackets reproducing the GBMS algebra.

We proceed as follows (see next subsection for further details). Our starting point is the observation that a Poisson bracket between $P_{f}$ and $J_{V}$ should satisfy

$$
\begin{equation*}
\left\{J_{V}, P_{f}\right\}=\delta_{f} J_{V}=-\delta_{V} P_{f} \tag{1.4}
\end{equation*}
$$

where $\delta_{f}$ and $\delta_{V}$ are the infinitesimal transformations associated to supertranslations and superrotations respectively.

It turns out that $\delta_{V} P_{f}$ can be evaluated from well-established expressions of supermomentum yielding

$$
\begin{equation*}
\delta_{V} P_{f}=-P_{V(f)} \tag{1.5}
\end{equation*}
$$

from which it follows $J_{V}$ should satisfy

$$
\begin{equation*}
\delta_{f} J_{V}=P_{V(f)} \tag{1.6}
\end{equation*}
$$

Unfortunately, the expression for $J_{V}$ given in [1] does not satisfy (1.6) (as discussed later, this is directly related to the non-closure of BT charges). Our main observation is that it is possible to correct $J_{V}$ so that (1.6) holds. The corrected $J_{V}$ is such that (i) it reduces to the angular momentum $J_{V}$ when $V^{a}$ is CKV and (ii) its Ward identity with the S matrix reproduces the subleading soft graviton theorem. ${ }^{2}$ It thus satisfies the same conditions as the charge proposed in [1], with the advantage of being compatible with (1.4).

We will then verify the corrected $J_{V}$ satisfies the remaining algebra relation,

$$
\begin{equation*}
\delta_{V} J_{V^{\prime}}=-J_{\left[V, V^{\prime}\right]} . \tag{1.7}
\end{equation*}
$$

Finally, we will see that conditions (1.5), (1.6) and (1.7) can be used to determine an extension of the Ashtekar-Streubel phase space on which GBMS acts canonically, with Poisson

[^2]brackets reproducing the GBMS algebra.
A key technical tool we will rely upon is a novel 'superrotation-covariant' derivative which greatly facilitates some of the computations.

We conclude the introduction by describing recent literature that relates to our work.
Compere, Fiorucci and Ruzziconi [26] improved the treatment in [18] in several directions, in particular by controlling radial divergences and defining renormalized surface charges. Although their resulting GBMS charges and algebra are different from what we find here (in particular they exhibit extension terms), their analysis was indispensable for the development of the present work.

Flanagan, Prabhu and Shehzad [27] present a no-go theorem for a symplectic structure supporting GBMS charges. Our symplectic structure violates at least one the assumptions in their theorem and so there is in principle no contradiction with their result, see section V for details. Further subtleties in the construction of a phase space at null infinity are discussed in [28, 29].

Adjei et.al. [30] provide an interpretation of BT superrotations that leads to GBMS. Potentially observable consequences of superrotation charges are described in [31 36]. GBMSlike symmetries on null surfaces other than null infinity are discussed in [37 40].

## A. Strategy and outline

Recall that gravitational radiation at null infinity is encoded in a 2 d tensor $C_{a b}$ that captures the subleading (in $r$ ) angular components of the spacetime metric. On the other hand, the leading angular components define a 2 d metric $q_{a b}$ that is usually kept fixed. From the perspective of GBMS, however, one needs to allow for variations of $q_{a b}$ and it is here where difficulties appear.

Let us for a moment forget about such difficulties and consider Eq. (1.6), which, as argued, is a necessary condition for the existence of PBs. Since Eq. (1.6) does not involve variations of $q_{a b}$ (supertranslations do not change the 2-metric) we may try to solve it for $J_{V}$ with $q_{a b}$ given. The equation simplifies considerably when $q_{a b}$ is the unit round sphere (referred to as 'Bondi frame' 41]), and so we first focus on this case in section [II.

To discuss Eqs. (1.5) and (1.7) we need expressions for $P_{f}$ and $J_{V}$ on a general $q_{a b}$. The former is well known in the literature, whereas the latter requires a generalization of the results in section 【II. To do this generalization we revisit in section 【II the description of nonBondi frames. Using the 'superboost' field introduced in [26], we define a 'Diff( $S^{2}$ )-covariant' derivative that is covariant under the action of $\operatorname{Diff}\left(S^{2}\right) \subset$ GBMS transformations. We will find that several non-Bondi frame formulas acquire a simple geometrical meaning when written in terms of this derivative. Some of the results in this section rely on appendices A and B.

Using the tools of section III, in section IV we obtain a general-frame formula for $J_{V}$ by 'covariantizing' the expression obtained in section III. We verify the resulting $J_{V}$ satisfies Eqs. (1.6) and (1.7). Some of the results in this section rely on appendices $C$ and D.

Finally, in section $V$ we determine the symplectic structure on the space of pairs ( $q_{a b}, C_{a b}$ ) by demanding compatibility with the GBMS charges.

In the remainder of the section we describe our conventions and specify the assumed $|u| \rightarrow \infty$ fall-offs at null infinity. We also present a brief review of GBMS transformations
and BMS charges.

## B. Conventions and spacetimes under consideration

We work in units such that $32 \pi G=1$. We consider asymptotically flat metrics at (future) null infinity in Bondi gauge (see e.g. [42]). The spacetime coordinates are given by a radial coordinate $r$, an advanced time $u$, and angular coordinates $x^{a}, a=1,2$. The angular part of the spacetime metric has a $r \rightarrow \infty$ expansion of the form ${ }^{3}$

$$
\begin{equation*}
g_{a b}(r, u, x) \stackrel{r \rightarrow \infty}{=} r^{2} q_{a b}(x)+r\left[C_{a b}(u, x)+u T_{a b}(x)\right]+\cdots \tag{1.8}
\end{equation*}
$$

The leading part of the angular metric, $q_{a b}$, is usually regarded as kinematical and fixed once and for all. The simplest choice, referred to as Bondi frame, is to take $q_{a b}$ the unit round sphere metric. $C_{a b}$ satisfies $q^{a b} C_{a b}=0$ and encodes outgoing gravitational waves at future null infinity $\mathcal{I}$. The tensor $T_{a b}(x)$ is constructed entirely from $q_{a b}$ and vanishes in Bondi frame (see section III and appendix A for further details).

We consider $u \rightarrow \pm \infty$ fall-offs in $C_{a b}(u, x)$ compatible with a $O(1)$ subleading soft theorem (see e.g. [43])

$$
\begin{equation*}
\partial_{u} C_{a b}(u, x) \stackrel{u \rightarrow \pm \infty}{=} O\left(1 /|u|^{2+\epsilon}\right), \quad \epsilon>0 \tag{1.9}
\end{equation*}
$$

These are compatible with tree-level scattering but are too restrictive for a generic gravitational scattering where fall-offs are given by (1.9) with $\epsilon=0$, corresponding to a logarithmic subleading soft theorem [43, 44]. ${ }^{4}$

We also require that $C_{a b}(u, x)$ is asymptotically flat as $u \rightarrow \pm \infty$. In Bondi frame, this corresponds to the vanishing of the magnetic part of $C_{a b}(u, x)$ at $u= \pm \infty$ (see e.g. [9]):

$$
\begin{equation*}
\lim _{u \rightarrow \pm \infty} D_{[a} D^{c} C_{b] c}(u, x)=0 \tag{1.10}
\end{equation*}
$$

The non-Bondi frame version of (1.10) is described in section IIID,
GBMS is generated by vector fields that preserve the Bondi form of the spacetime metric but are not necessarily Killing, thus allowing for changes in the leading order angular metric $q_{a b}$. The GBMS vector fields are parametrized by functions $f(x)$ (supertranslations) and arbitrary smooth vector fields $V^{a}(x)$ (which, borrowing the BT terminology, we will call superrotations). They act on $q_{a b}$ and $C_{a b}$ according to (see [18] and section [III for details)

$$
\begin{gather*}
\delta_{f} q_{a b}=0, \quad \delta_{V} q_{a b}=\mathcal{L}_{V} q_{a b}-2 \alpha q_{a b}  \tag{1.11}\\
\delta_{f} C_{a b}=f \partial_{u} C_{a b}-2 D_{a} D_{b} f^{\mathrm{TF}}+f T_{a b}, \quad \delta_{V} C_{a b}=\mathcal{L}_{V} C_{a b}+\alpha u \partial_{u} C_{a b}-\alpha C_{a b}, \tag{1.12}
\end{gather*}
$$

where $D_{a}$ is the covariant derivative of $q_{a b}$, TF stands for Trace-Free part, $\alpha=D_{a} V^{a} / 2$, and $\mathcal{L}_{V}$ is the Lie derivative on the sphere.

Note that the action of superrotations on the 2-metric $q_{a b}$ is such that it preserves the area element, $\delta_{V} \sqrt{q}=0$. We will work in the space of metrics that can be reached from Bondiframe metrics by finite GBMS transformations (see appendix A), so that the area element of

[^3]all $q_{a b}$ 's coincides with the unit round sphere area element. In particular, $\alpha \equiv \frac{1}{\sqrt{q}} \partial_{a}\left(\sqrt{q} V^{a}\right) / 2$ is independent of $q_{a b}$.

We conclude with a comment regarding the description of asymptotically flat spacetimes at null infinity. There are two main approaches: The original due to Bondi and Sachs that we follow here, and the Penrose approach [45] that uses a rescaled, compactified spacetime. We expect the results presented here admit a direct translation into the second description. See [1, 27] for a discussion of GBMS in the Penrose approach.

## C. Review of BMS charges in Bondi frame

In Bondi frame ( $q_{a b}=$ unit sphere metric, $T_{a b}=0$ ) the asymptotic BMS Killing symmetries act on $C_{a b}(u, x)$ according to

$$
\begin{align*}
\delta_{f} C_{a b} & =f \partial_{u} C_{a b}-2 D_{a} D_{b} f^{\mathrm{TF}},  \tag{1.13}\\
\delta_{V} C_{a b} & =\mathcal{L}_{V} C_{a b}+\alpha u \partial_{u} C_{a b}-\alpha C_{a b}, \tag{1.14}
\end{align*}
$$

where $V^{a}$ are global CKVs of $q_{a b}$ (i.e. they satisfy $\delta_{V} q_{a b}=0$ ).
As shown by Ashtekar and Streubel [5], the transformations (1.13) and (1.14) are canonical with respect to the symplectic structure of gravitational radiation at null infinity,

$$
\begin{equation*}
\Omega=\int_{\mathcal{I}} d u d^{2} x \sqrt{q}\left(\delta \partial_{u} C^{a b} \wedge \delta C_{a b}\right) \tag{1.15}
\end{equation*}
$$

This allows one to find canonical charges associated to BMS symmetries: the supermomenta,

$$
\begin{equation*}
P_{f}=\int_{\mathcal{I}} d u d^{2} x \sqrt{q} \partial_{u} C^{a b} \delta_{f} C_{a b} \tag{1.16}
\end{equation*}
$$

and angular momenta

$$
\begin{equation*}
\stackrel{\circ}{J}_{V}=\int_{\mathcal{I}} d u d^{2} x \sqrt{q} \partial_{u} C^{a b} \delta_{V} C_{a b} . \tag{1.17}
\end{equation*}
$$

As mentioned in the introduction, these charges close under PBs, reproducing the BMS algebra.

## II. SUPER ANGULAR MOMENTUM IN BONDI FRAME

The proposal [1, 18] for an asymptotic GBMS symmetry provided the following candidate for super angular momentum: ${ }^{5}$

$$
\begin{equation*}
J_{V}^{\prime}=\int_{\mathcal{I}} d u d^{2} x \sqrt{q} \partial_{u} C^{a b} \delta_{V} C_{a b}+\int_{\mathcal{I}} d u d^{2} x \sqrt{q} u \partial_{u} C^{a b}\left(-4 D_{a} D_{b} \alpha+D_{a} D^{c} \delta_{V} q_{b c}-\delta_{V} q_{a b}\right) \tag{2.1}
\end{equation*}
$$

${ }^{5}$ In 18], strong $u \rightarrow \pm \infty$ fall-offs where assumed such that the soft part of $J_{V}^{\prime}$ was written as in (2.1) with the replacement $u \partial_{u} C^{a b} \rightarrow-C^{a b}$. The soft charge as written here was introduced in [26] and is valid for the more general fall-offs (1.9).
with $\delta_{V} C_{a b}$ and $\delta_{V} q_{a b}$ given in Eqs. (1.11), (1.12).
Compared to the BMS angular momentum (1.17), $J_{V}^{\prime}$ has an extra 'soft' term that vanishes when $V^{a}$ is a global CKV. We recall from [18] that if $J_{V}^{\prime}$ is written in terms of holomorphic coordinates on the sphere, the resulting expression coincides with the BT charge used in [12] to compute BT superrotation Ward identities.

As motivated in the introduction, it is of interest to evaluate the action of a supertranslation on $J_{V}^{\prime}$. To organize the calculation, we introduce the following notation: ${ }^{6}$

$$
\begin{align*}
\stackrel{0}{S}_{a b}^{f} & :=-2 D_{a} D_{b} f^{\mathrm{TF}}  \tag{2.2}\\
\stackrel{1}{S}_{a b}^{V} & :=\left[-4 D_{a} D_{b} \alpha+D_{(a} D^{c} \delta_{V} q_{b) c}-\delta_{V} q_{a b}\right]^{\mathrm{TF}}  \tag{2.3}\\
\stackrel{0}{N}_{a b}(x) & :=\int_{-\infty}^{\infty} \partial_{u} C_{a b}(u, x) d u  \tag{2.4}\\
\stackrel{1}{N}_{a b}(x) & :=\int_{-\infty}^{\infty} u \partial_{u} C_{a b}(u, x) d u, \tag{2.5}
\end{align*}
$$

and write the supermomentum (1.16) and super angular momentum (2.1) as

$$
\begin{align*}
& P_{f}=P_{f}^{\text {hard }}+\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{0}{N}^{a b} \stackrel{0}{S}_{a b}^{f},  \tag{2.6}\\
& J_{V}^{\prime}=J_{V}^{\mathrm{hard}}+\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{1}{N}^{a b}{ }_{S}^{1}{ }_{a b}^{V}, \tag{2.7}
\end{align*}
$$

where the 'hard' piece is the part of the charge that involves an integral over $\mathcal{I}$ of terms quadratic in $C_{a b}$. To evaluate $\delta_{f} J_{V}^{\prime}$ we note the following identities

$$
\begin{align*}
\delta_{f} J_{V}^{\text {hard }} & =P_{V(f)}^{\mathrm{hard}}+\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{0}{N}^{a b}\left(\mathcal{L}_{V}-\alpha\right) \stackrel{0}{S}_{a b}^{f}  \tag{2.8}\\
\delta_{f} \stackrel{1}{N}_{a b} & =-f \stackrel{0}{N} a b \tag{2.9}
\end{align*}
$$

from which we arrive at

$$
\begin{equation*}
\delta_{f} J_{V}^{\prime}=P_{V(f)}+K(f, V) \tag{2.10}
\end{equation*}
$$

where

$$
\begin{equation*}
K(f, V)=\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{0}{N}^{a b}\left(\left(\mathcal{L}_{V}-\alpha\right) \stackrel{0}{S}_{a b}^{f}-\stackrel{0}{S}_{a b}^{V(f)}-f \stackrel{1}{S}_{a b}^{V}\right) . \tag{2.11}
\end{equation*}
$$

$K(f, V)$ may be thought of as a non-CKV generalization of the BT extension term [8]; see subsection IIB for further comparison with BT charges.

We now make use of a non-trivial identity, shown in appendix C which expresses $K(f, V)$

[^4]as a total $\delta_{f}$ term plus a 'magnetic' piece, ${ }^{7}$
\[

$$
\begin{equation*}
K(f, V)=-\delta_{f} J_{V}^{\partial \mathcal{I}}+\operatorname{mag}(f, V), \tag{2.12}
\end{equation*}
$$

\]

where

$$
\begin{align*}
J_{V}^{\partial \mathcal{I}} & =\int_{\partial \mathcal{I}} d^{2} x \sqrt{q}\left(V^{a} C^{b c} D_{c} C_{a b}+\frac{3}{2} \alpha C^{a b} C_{a b}\right)  \tag{2.13}\\
\operatorname{mag}(f, V) & =-4 \int_{S^{2}} d^{2} x \sqrt{q}{ }^{0}{ }^{a b} D_{a} D^{c}\left(D_{[b} f V_{c]}-\frac{1}{2} f D_{[b} V_{c]}\right)=0 . \tag{2.14}
\end{align*}
$$

In (2.13) $\int_{\partial \mathcal{I}} \equiv \int_{\partial \mathcal{I}_{+}}-\int_{\partial \mathcal{I}_{-}}$is a difference of integrals over the $u= \pm \infty$ boundaries of $\mathcal{I}$. The vanishing of (2.14) is due to condition (1.10) which implies

$$
\begin{equation*}
D^{[c} D_{a} \stackrel{0}{N}^{b] a}=0 \tag{2.15}
\end{equation*}
$$

Thus, if we redefine the super angular momentum as

$$
\begin{equation*}
J_{V}=J_{V}^{\prime}+J_{V}^{\partial I} \tag{2.16}
\end{equation*}
$$

it will satisfy

$$
\begin{equation*}
\delta_{f} J_{V}=P_{V(f)} \tag{2.17}
\end{equation*}
$$

Let us make a few comments about the proposed expression for super angular momentum:

1. $J_{V}^{\partial \mathcal{I}}$ vanishes when $V^{a}$ is a global CKV thanks to condition (2.15). (This vanishing is not obvious at first sight, see next subsection for an explicit demonstration). Thus, $J_{V}$ reduces to the standard BMS angular momentum $\grave{J}_{V}$ when $V^{a}$ is a global CKV.
2. One can formally write an operator expression for $J_{V}^{\partial \mathcal{I}}$ in terms of graviton Fock operators [25]. The resulting expression has a trivial action on finite energy states and thus has no effect on the usual computation of single-charge Ward identities [1, 12]. It can, however, affect the evaluation of double-charge Ward identities [20], or singlecharge Ward identity if one of the external states is in a shifted vacuum [19]. The consequences of such term on Ward identities will be discussed elsewhere [25].
3. Eq. (2.12) defines $J_{V}^{\partial I}$ modulo terms that are annihilated by $\delta_{f}$. Such terms can be constructed from the leading mode $\stackrel{0}{N}_{a b}$ and its powers. We discard such possible contributions since (i) linear terms in $\stackrel{0}{N}_{a b}$ would spoil the S matrix Ward identities and (ii) higher powers in $\stackrel{0}{N}_{a b}$ are non-local in $u$ (they cannot be written as an integral over $\mathcal{I}$ of a density local in $C_{a b}$ ).
[^5]In the remainder of the section we will present an alternative form of $J_{V}^{\partial \mathcal{I}}$ that will be of later use. We will also make contact with the BT treatment by describing how expressions simplify in the case of local CKVs.

## A. Alternative form of $J_{V}^{\partial \mathcal{I}}$

Let

$$
\begin{equation*}
C_{a b}^{ \pm}(x):=\lim _{u \rightarrow \pm \infty} C_{a b}(u, x) \tag{2.18}
\end{equation*}
$$

be the asymptotic values of $C_{a b}$ at $u= \pm \infty$. The vanishing of their magnetic part (1.10) implies they can be written as

$$
\begin{equation*}
C_{a b}^{ \pm}=-2\left(D_{a} D_{b} C^{ \pm}\right)^{\mathrm{TF}} \equiv \stackrel{0}{S}_{a b}^{C^{ \pm}} \tag{2.19}
\end{equation*}
$$

for some functions $C^{ \pm}(x)$. Below we use (2.19) to provide an alternative expression for $J_{V}^{\partial I}$.
We start by writing the soft mode $\stackrel{0}{N}_{a b}$ as

$$
\begin{equation*}
\stackrel{0}{N}_{a b}=C_{a b}^{+}-C_{a b}^{-} . \tag{2.20}
\end{equation*}
$$

Substituting (2.20) in the definitions of $K(f, V)$ and $J_{V}^{\partial \mathcal{I}}$ we find they can be written as

$$
\begin{gather*}
K(f, V)=K^{+}(f, V)-K^{-}(f, V)  \tag{2.21}\\
J_{V}^{\partial \mathcal{I}}=J_{V}^{\partial \mathcal{I}_{+}}-J_{V}^{\partial \mathcal{I}_{-}} \tag{2.22}
\end{gather*}
$$

where $\pm$ indicate the terms that depend on $C_{a b}^{ \pm}$.
We next observe that identity (2.12) holds for each piece separately:

$$
\begin{equation*}
K^{ \pm}(f, V)=-\delta_{f} J_{V}^{\partial \mathcal{I}_{ \pm}} \tag{2.23}
\end{equation*}
$$

$\left(\operatorname{mag}^{ \pm}(f, V)=0\right)$. Finally, since $J_{V}^{\partial \mathcal{I}_{ \pm}}$is quadratic in $C^{ \pm}$and

$$
\begin{equation*}
\delta_{f} C^{ \pm}=f \tag{2.24}
\end{equation*}
$$

we have

$$
\begin{equation*}
J_{V}^{\partial \mathcal{I}_{ \pm}}=\left.\frac{1}{2} \delta_{f} J_{V}^{\partial \mathcal{I}_{ \pm}}\right|_{f=C^{ \pm}}=-\frac{1}{2} K^{ \pm}\left(C^{ \pm}, V\right), \tag{2.25}
\end{equation*}
$$

where in the last equality we used Eqs. (2.23).
Collecting the above results, we conclude $J_{V}^{\partial \mathcal{I}}$ can be written as

$$
\begin{equation*}
J_{V}^{\partial \mathcal{I}}=-\frac{1}{2}\left(K^{+}\left(C^{+}, V\right)-K^{-}\left(C^{-}, V\right)\right) . \tag{2.26}
\end{equation*}
$$

This form makes it manifest that $J_{V}^{\partial \mathcal{I}}$ vanishes for global CKV (see also next subsection).

## B. Holomorphic coordinates and local CKVs.

In holomorphic coordinates $(z, \bar{z})$ such that $q_{z z}=0=q_{\bar{z} \bar{z}}$ one has

$$
\begin{align*}
\stackrel{1}{S}_{z z}^{V} & =-2 \partial_{z}^{3} V^{z}  \tag{2.27}\\
\left(\mathcal{L}_{V}-\alpha\right) \stackrel{0}{S}_{z z}^{f}-\stackrel{0}{S}_{S_{z z}^{V(f)}}^{V(f)} & =-2 \delta_{V}\left(D_{z}^{2}\right) f \tag{2.28}
\end{align*}
$$

and (2.26) becomes

$$
\begin{equation*}
J_{V}^{\partial \mathcal{I}}=\int_{\partial \mathcal{I}} d^{2} z \sqrt{q} C^{z z}\left(\delta_{V}\left(D_{z}^{2}\right) C-C \partial_{z}^{3} V^{z}\right)+c . c . \tag{2.29}
\end{equation*}
$$

If $V^{a}$ is a global CKV then $\delta_{V}\left(D_{z}^{2}\right)=0=\partial_{z}^{3} V^{z}$ and $J_{V}^{\partial \mathcal{I}}$ vanishes as stated before.
If on the other hand $V^{a}$ is a local CKV (i.e. $\partial_{\bar{z}} V^{z}=0$ ) one can show that

$$
\begin{equation*}
\delta_{V}\left(D_{z}^{2}\right) C=\frac{1}{2} \partial_{z}^{3} V^{z} C \tag{2.30}
\end{equation*}
$$

In this case $J_{V}^{\partial \mathcal{I}}$ and $K(f, V)$ reduce to

$$
\begin{align*}
J_{V}^{\partial \mathcal{I}} & \left.=-\frac{1}{2} \int_{\partial \mathcal{I}} d^{2} z \sqrt{q} C^{z z} C \partial_{z}^{3} V^{z}\right)+c . c .  \tag{2.31}\\
K(f, V) & =\int d^{2} z \sqrt{q}{ }^{0}{ }^{z z} f \partial_{z}^{3} V^{z}+c . c . \tag{2.32}
\end{align*}
$$

Expression (2.32) corresponds to the BT extension as written in 20]. We see that $J_{V}^{\partial \mathcal{I}}$ is non-trivial for BT superrotations, and may also be used to cancel the extension term as in the smooth vector field case.

## III. NON-BONDI FRAMES

In the case where $q_{a b}$ is not round sphere metric, one needs an additional $u$-independent tensor $T_{a b}$ to appropriately describe the gravitational field at null infinity [26, 48]. This tensor was introduced by Geroch [48] in order to have a conformally invariant notion of gravitational radiation in the Penrose description of asymptotically flat spacetimes. Here, following [26], we introduce $T_{a b}$ in the definition of $C_{a b}$ as given in Eq. (1.8),

$$
\begin{equation*}
g_{a b} \stackrel{r \rightarrow \infty}{=} r^{2} q_{a b}+r\left[C_{a b}+u T_{a b}\right]+\cdots \tag{3.1}
\end{equation*}
$$

so that $C_{a b}=0$ represents a flat metric in a non-Bondi frame, see appendix A for details.
With this definition, the action of supertranslations and superrotations on $C_{a b}$ is given by

$$
\begin{align*}
\delta_{f} C_{a b} & =f \partial_{u} C_{a b}-2 D_{a} D_{b} f^{\mathrm{TF}}+f T_{a b},  \tag{3.2}\\
\delta_{V} C_{a b} & =\mathcal{L}_{V} C_{a b}+\alpha u \partial_{u} C_{a b}-\alpha C_{a b} . \tag{3.3}
\end{align*}
$$

We see that supertranslations acquire an extra term with respect to the Bondi-frame ex-
pression (1.13). The new expression generalizes to non-Bondi frames the fact that the inhomogeneous piece of $\delta_{f} C_{a b}$ vanishes for spacetime translations [48]:

$$
\begin{equation*}
-2 D_{a} D_{b} f^{\mathrm{TF}}+f T_{a b}=0 \Longleftrightarrow f=\text { spacetime translation. } \tag{3.4}
\end{equation*}
$$

Regarding superrotations (3.3), we note they lack an inhomogeneous term that appears with the usual definition of $C_{a b}[18,42]$. From this perspective, the role of $T_{a b}$ in (3.1) is to eliminate such inhomogeneous term, see section 4 of [18].

In the original literature of BMS in the Penrose approach, special care is taken to ensure frame-independence, see e.g. [48, 49]. In particular, the Ashtekar-Streubel expression for supermomenta is valid in any frame. When written in terms of the physical spacetime metric (1.8), the AS supermomentum takes the form

$$
\begin{equation*}
P_{f}=\int_{\mathcal{I}} d u d^{2} x \sqrt{q} \partial_{u} C^{a b} \delta_{f} C_{a b} \tag{3.5}
\end{equation*}
$$

with $\delta_{f}$ given in (3.2). Given the transformation rules of $C_{a b}$ and $q_{a b}$ under superrotations, we can compute $\delta_{V} P_{f}$ resulting in (see appendix (D)

$$
\begin{equation*}
\delta_{V} P_{f}=-P_{V(f)} \tag{3.6}
\end{equation*}
$$

In the next section we will obtain a general-frame expression of super angular momentum compatible with (3.6) in the sense of Eq. (1.4). A main tool we will use to this end is a 'superrotation-covariant' derivative that can be defined with the help of a potential we now introduce.

## A. $\psi$-potential

In [26] a 'superboost' field $\psi$ was introduced that serves as a potential for $T_{a b}$ in the sense that ${ }^{8}$

$$
\begin{equation*}
T_{a b}=2\left(D_{a} \psi D_{b} \psi+D_{a} D_{b} \psi\right)^{\mathrm{TF}} \tag{3.7}
\end{equation*}
$$

Under superrotations $\psi$ transforms according to [26]

$$
\begin{equation*}
\delta_{V} \psi=\mathcal{L}_{V} \psi-\alpha, \tag{3.8}
\end{equation*}
$$

which can be verified to be compatible with the transformation of $T_{a b}$ induced by $\delta_{V} q_{a b}$ [18]

$$
\begin{equation*}
\delta_{V} T_{a b}=\mathcal{L}_{V} T_{a b}-2 D_{a} D_{b} \alpha^{\mathrm{TF}} \tag{3.9}
\end{equation*}
$$

One aspect of $\psi$ we would like to bring attention to is that, unlike $T_{a b}$, it is not invariant under CKVs. In other words, $q_{a b}$ fixes $\psi$ modulo an ambiguity parametrized by the conformal isometries of $q_{a b} .{ }^{9}$ We will later see that all quantities of interest such as charges

[^6]and symplectic structure depend on $\psi$ only through the combination (3.7). That is, our expressions will in fact be independent of the ambiguity in $\psi$. This property will not always be manifest and in some cases it will require some work to establish it. Further details on $\psi$ and its relation with the Geroch tensor are given in appendices A and B ,

We now use $\psi$ to construct a 'superrotation-covariant' derivative.

## B. $\operatorname{Diff}\left(S^{2}\right)$-covariant derivative

Let us first define the notion of covariance of a tensor with respect to superrotations. We say a ( $u$-independent) tensor $t_{a_{1} \ldots}^{b_{1} \ldots}$ on the celestial sphere is covariant under superrotations if it satisfies the transformation rule,

$$
\begin{equation*}
\delta_{V} t_{a_{1} \ldots}^{b_{1} \ldots}=\mathcal{L}_{V} t_{a_{1} \ldots}^{b_{1} \ldots}+k \alpha t_{a_{1} \ldots}^{b_{1} \ldots} \tag{3.10}
\end{equation*}
$$

for some constant $k$. For instance, the metric $q_{a b}$ is a covariant tensor with $k=-2$. Other examples are the leading and subleading soft modes of the news tensor introduced in Eqs. (2.4), (2.5), whose transformation properties obtained from (3.3) are:

$$
\begin{align*}
& \delta_{V} \stackrel{0}{N}_{a b}=\mathcal{L}_{V} \stackrel{0}{N}_{a b}-\alpha \stackrel{0}{N}_{a b}  \tag{3.11}\\
& \delta_{V} \stackrel{1}{N}_{a b}=\mathcal{L}_{V} \stackrel{1}{N}_{a b}-2 \alpha \stackrel{1}{N}_{a b} . \tag{3.12}
\end{align*}
$$

On the other hand, the potential $\psi$ and the tensor $T_{a b}$ are examples of non-covariant tensors.
Given a covariant tensor as defined above, its regular covariant derivative will not be covariant under superrotations. For example, consider a scalar $\varphi$ such that $\delta_{V} \varphi=\left(\mathcal{L}_{V}+\right.$ $k \alpha) \varphi$. Then,

$$
\begin{equation*}
\delta_{V}\left(D_{a} \varphi\right)=D_{a}\left(\mathcal{L}_{V} \varphi+k \alpha \varphi\right)=\left(\mathcal{L}_{V}+k \alpha\right) D_{a} \varphi+k D_{a} \alpha \varphi \neq\left(\mathcal{L}_{V}+k \alpha\right) D_{a} \varphi \tag{3.13}
\end{equation*}
$$

The 'extra' $k D_{a} \alpha \varphi$ term can be canceled if we instead consider

$$
\begin{equation*}
\bar{D}_{a} \varphi:=D_{a} \varphi+k D_{a} \psi \varphi \tag{3.14}
\end{equation*}
$$

which, thanks to, (3.8) satisfies

$$
\begin{equation*}
\delta_{V}\left(\bar{D}_{a} \varphi\right)=\left(\mathcal{L}_{V}+k \alpha\right) \bar{D}_{a} \varphi \tag{3.15}
\end{equation*}
$$

Expression (3.14) is the desired definition of $\operatorname{Diff}\left(S^{2}\right)$-covariant derivative for scalars. For arbitrary tensors we can generalize the above reasoning by taking into account the variation of the Christoffel symbols of $D_{a}$. For example, the Diff $\left(S^{2}\right)$-covariant derivative of a covector $\omega_{a}$ is found to be given by

$$
\begin{equation*}
\bar{D}_{a} \omega_{b}=D_{a} \omega_{b}-\stackrel{\psi}{\Gamma}_{a b}^{c} \omega_{c}+k D_{a} \psi \omega_{b} \tag{3.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\stackrel{\psi}{\Gamma}_{a b}^{c}:=-2 D_{(a} \psi \delta_{b)}^{c}+q_{a b} D^{c} \psi \tag{3.17}
\end{equation*}
$$

For general tensors, expression (3.16) generalizes with the appropriate inclusion of ${ }_{\Gamma}^{\psi}{ }_{a b}^{c}$ symbols for each tensor index, see Eq. (3.19) for another example.

To summarize, given a general tensor that is covariant with respect to superrotations as in (3.10), its $\operatorname{Diff}\left(S^{2}\right)$-covariant derivative also transforms covariantly:

$$
\begin{equation*}
\delta_{V} \bar{D}_{a} t_{a_{1} \ldots}^{b_{1} \ldots}=\left(\mathcal{L}_{V}+k \alpha\right) t_{a_{1} \ldots}^{b_{1} \ldots} \tag{3.18}
\end{equation*}
$$

With the above definitions one can verify $\bar{D}_{a}$ satisfies Leibiniz rule with the 'weight' $k$ of the product of tensors given by the sum of the weights of each tensor.

The $\bar{D}_{a}$ derivative has a number of useful properties we now describe.

1. Its action on $q_{a b}$ is zero:

$$
\begin{equation*}
\bar{D}_{c} q_{a b}=-\Gamma_{c a}^{d} q_{d b}-\stackrel{\psi}{\Gamma}{ }_{c b}^{d} q_{a d}-2 D_{c} \psi q_{a b}=0 \tag{3.19}
\end{equation*}
$$

2. The commutator of $\bar{D}_{a}$ derivatives satisfies the same formulas as for ordinary covariant derivatives but with a 'covariantized' curvature tensor. For instance:

$$
\begin{equation*}
\left[\bar{D}_{a}, \bar{D}_{b}\right] \omega_{c}=\bar{R}_{a b c}{ }^{d} \omega_{d} \tag{3.20}
\end{equation*}
$$

where ${ }^{10}$

$$
\begin{equation*}
\bar{R}_{a b c d}=\bar{R} q_{a[c} q_{d]}, \quad \text { with } \quad \bar{R}=R+2 D^{2} \psi \tag{3.21}
\end{equation*}
$$

$R$ being the scalar curvature of $q_{a b}$. Notice that $\bar{R}_{a b c}{ }^{d}$ is independent of the 'weight' $k$ of $\omega_{c}$.
3. Finally, one can show the remarkable property (see appendix (B):

$$
\begin{equation*}
\bar{D}_{a} \bar{R}=0 \tag{3.22}
\end{equation*}
$$

Eq. (3.22) can be thought of as a 'covariantized' version of the constant 2 d curvature in Bondi frame. It can also be understood as a rewriting of the Geroch identity [48] $D_{[a} \rho_{b] c}=0$ where $\rho_{a b}=\frac{R}{2} q_{a b}-T_{a b}$. See appendix B for further details.

## C. Supermomentum revisited

Let us briefly revisit the expression for supermomentum in light of the previous discussion.
We start by noting that a supertranslation function $f(x)$ should be treated as a covariant scalar with $k=-1$, since $V(f)=\mathcal{L}_{V} f-\alpha f$. We can then compute its Diff $\left(S^{2}\right)$-covariant derivative according to the rules of the previous section. Doing so one finds

$$
\begin{equation*}
-2 \bar{D}_{a} \bar{D}_{b} f^{\mathrm{TF}}=-2 D_{a} D_{b} f^{\mathrm{TF}}+f T_{a b} \tag{3.23}
\end{equation*}
$$

[^7]The right hand side of (3.23) matches the inhomogeneous term of a supertranslation (3.2), and so this identity allows us to reinterpret the expression of supermomentum (3.5) as a 'Diff( $\left(S^{2}\right)$-covariantization' of the Bondi-frame expression (1.13), (1.16).

In the next sections we use this covariantization idea to extend to non-Bondi frames (i) the zero magnetic condition (1.10) and (ii) the super angular momentum of section (II)

## D. Asymptotic magnetic condition

Give the superrotation transformation rule of $C_{a b}(u, x)$ (3.3) and the $u \pm \infty$ fall-offs (1.9), it is easy to see that $C_{a b}^{ \pm}(x)=\lim _{u \rightarrow \pm \infty} C_{a b}(u, x)$ is a $k=-1$ covariant tensor. Applying the rules of $\bar{D}_{a}$ differentiation one then finds ${ }^{11}$

$$
\begin{equation*}
\bar{D}_{[a} \bar{D}^{c} C_{b] c}^{ \pm}=D_{[a} D^{c} C_{b] c}^{ \pm}-\frac{1}{2} T_{[a}^{c} C_{b] c}^{ \pm} . \tag{3.24}
\end{equation*}
$$

The vanishing of (3.24),

$$
\begin{equation*}
\bar{D}_{[a} \bar{D}^{c} C_{b] c}^{ \pm}=0 \tag{3.25}
\end{equation*}
$$

is the non-Bondi frame generalization of condition (1.10) and imposes that $C_{a b}^{ \pm}$is a 'pure supertranslation':

$$
\begin{equation*}
C_{a b}^{ \pm}=-2\left(\bar{D}_{a} \bar{D}_{b} C^{ \pm}\right)^{\mathrm{TF}} \quad \text { for some function } \quad C^{ \pm} \tag{3.26}
\end{equation*}
$$

where $C^{ \pm}$is a $k=-1$ covariant scalar.

## Comment:

Equation (3.24) illustrates a kind of complementary for writing expressions, either in terms of $\bar{D}_{a}$ or in terms of $D_{a}$ and $T_{a b}$. Some properties are more transparent in the first version but obscure in the second version and vice versa. For instance, the fact that $C_{a b}^{ \pm}$in (3.26) satisfies (3.25) is easily seen in the first version, the proof being identical to the one for the round sphere case. On the other hand, to see that condition (3.25) is independent of the $\psi$-ambiguity, we use the second version.

## IV. SUPER ANGULAR MOMENTUM IN A GENERAL FRAME

We now construct the general-frame candidate for super angular momentum $J_{V}$ by 'covariantizing' the Bondi-frame expression of section III. The charge is a sum of three terms,

$$
\begin{equation*}
J_{V}=J_{V}^{\mathrm{hard}}+J_{V}^{\text {soft }}+J_{V}^{\partial \mathcal{I}} \tag{4.1}
\end{equation*}
$$

[^8]\[

$$
\begin{align*}
J_{V}^{\mathrm{hard}} & =\int_{\mathcal{I}} d u d^{2} x \sqrt{q} \partial_{u} C^{a b} \delta_{V} C_{a b}  \tag{4.2}\\
J_{V}^{\mathrm{soft}} & =\int_{S^{2}} d^{2} x \sqrt{q} N^{a b} S_{a b}^{V}  \tag{4.3}\\
J_{V}^{\partial \mathcal{I}} & =\int_{\partial \mathcal{I}} d^{2} x \sqrt{q}\left(V^{a} C^{b c} \bar{D}_{c} C_{a b}+\frac{3}{2} \bar{\alpha} C^{a b} C_{a b}\right) \tag{4.4}
\end{align*}
$$
\]

where

$$
\begin{equation*}
\stackrel{1}{S}_{a b}^{V}=\left[-4 \bar{D}_{a} \bar{D}_{b} \bar{\alpha}+\bar{D}_{(a} \bar{D}^{c} \delta_{V} q_{b) c}-\frac{\bar{R}}{2} \delta_{V} q_{a b}\right]^{\mathrm{TF}} \tag{4.5}
\end{equation*}
$$

with $\bar{\alpha}=\bar{D}_{a} V^{a} / 2\left(V^{a}\right.$ is treated as a $k=0$ vector) and $\bar{R}$ the 'covariantized' scalar curvature defined in (3.21). ${ }^{12}$ Note that, as written, it is not obvious that $J_{V}$ is free from the ambiguity in $\psi$ described in section IIIA, We will later give alternative expressions for $J_{V}^{\text {soft }}$ and $J_{V}^{\partial \mathcal{I}}$ in which this property is manifest. For now, let us focus on establishing the identity

$$
\begin{equation*}
\delta_{f} J_{V}-P_{V(f)}=\operatorname{mag}(f, V) \tag{4.6}
\end{equation*}
$$

where $\operatorname{mag}(f, V)$ is the covariant version of the Bondi-frame magnetic term (2.14). This will again vanish due to the zero magnetic condition $\bar{D}_{[a} \bar{D}^{c} C_{b] c}^{ \pm}=0$.

We start by evaluating the action of a supertranslation on each term of (4.1). The calculation is essentially the same as that of section (II ) and gives:

$$
\begin{align*}
\delta_{f} J_{V}^{\mathrm{hard}} & =P_{V(f)}^{\mathrm{hard}}+\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{0}{N}^{a b}\left(\mathcal{L}_{V}-\alpha\right) \stackrel{0}{S_{a b}^{f}}  \tag{4.7}\\
\delta_{f} J_{V}^{\text {soft }} & =-\int_{S^{2}} d^{2} x \sqrt{q} f \stackrel{0}{N}^{a b} \stackrel{1}{S}_{a b}^{V}  \tag{4.8}\\
\delta_{f} J_{V}^{\partial \mathcal{I}} & =\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{0}{N}^{a b}\left(-\bar{D}^{c}\left(V_{a} \stackrel{0}{S}_{b c}^{f}\right)+V^{c} \bar{D}_{a} \stackrel{0}{S}_{b c}^{f}+3 \bar{\alpha} \stackrel{0}{S}_{a b}^{f}\right), \tag{4.9}
\end{align*}
$$

where

$$
\begin{equation*}
\stackrel{0}{S}_{a b}^{f} \equiv-2 \bar{D}_{a} \bar{D}_{b} f^{\mathrm{TF}} \tag{4.10}
\end{equation*}
$$

Let us now look at the contribution in (4.6) coming from $\delta_{f} J_{V}^{\mathrm{hard}}-P_{V(f)}$. The $P_{V(f)}^{\mathrm{hard}}$ terms cancel while the soft terms can be combined as

$$
\begin{equation*}
\left(\mathcal{L}_{V}-\alpha\right) \stackrel{0}{S}_{a b}^{f}-\stackrel{0}{S} V a b=\delta_{V} \stackrel{0}{S}_{a b}^{f} \tag{4.11}
\end{equation*}
$$

We thus obtain

$$
\begin{equation*}
\delta_{f} J_{V}^{\mathrm{hard}}-P_{V(f)}=\int_{S^{2}} d^{2} x \sqrt{q} \stackrel{0}{N}^{a b} \delta_{V} \stackrel{0}{S}_{a b}^{f} \tag{4.12}
\end{equation*}
$$

Since the remaining terms in Eq. (4.6) are also proportional to $\stackrel{0}{N}^{a b}$, it follows that Eq. (4.6)

12 The scalar curvature in the last term term of (4.5) also appears in [18] and is needed in order for this term to have the same weight as the first two $(k=0)$, thus ensuring the correct superrotation transformation properties of $J_{V}^{\text {soft }}$, see appendix D .
will be satisfied if and only if

$$
\begin{equation*}
\left[\delta_{V} \stackrel{0}{S}_{a b}^{f}-f \stackrel{1}{S}_{a b}^{V}-\bar{D}^{c}\left(V_{(a} \stackrel{0}{S}_{b) c}^{f}\right)+V^{c} \bar{D}_{(a} \stackrel{0}{S}_{b) c}^{f}+3 \bar{\alpha} \stackrel{0}{S}_{a b}^{f}\right]^{\mathrm{TF}}=-4\left[\bar{D}_{(a} \bar{D}^{c}\left(\bar{D}_{[b)} f V_{c]}-\frac{1}{2} f \bar{D}_{[b)} V_{c]}\right)\right]^{\mathrm{TF}} \tag{4.13}
\end{equation*}
$$

This identity is equivalent to (the covariant version of) identity (2.12) and can be proven by direct evaluation of both sides, see appendix C.

## A. Independence of the ambiguity in $\psi$

We here verify that $J_{V}$ depends on $\psi$ only through $T_{a b}$.

For $J_{V}^{\text {soft }}(4.3)$ we use the identity:

$$
\begin{equation*}
\stackrel{1}{S}_{a b}^{V}=\left[2 \delta_{V} T_{a b}+D_{(a} D^{c} \delta_{V} q_{b) c}-\frac{R}{2} \delta_{V} q_{a b}\right]^{\mathrm{TF}} \tag{4.14}
\end{equation*}
$$

which can be established by direct evaluation on both sides, noting that $\bar{\alpha}=-\delta_{V} \psi$ and taking into account algebraic 2 d identities as in earlier calculations. Eq. (4.14) shows that all the $\psi$ dependence of $\left(\stackrel{1}{S}_{a b}^{V}\right)^{\mathrm{TF}}$, and hence of $J_{V}^{\text {soft }}$, is in the term $\delta_{V} T_{a b}$ in (4.14).

For $J_{V}^{\partial \mathcal{I}}$ we use the covariant version of Eq. (2.26) to rewrite it as

$$
\begin{equation*}
J_{V}^{\partial \mathcal{I}}=-\frac{1}{2} \int_{\partial \mathcal{I}} d^{2} x \sqrt{q} C^{a b}\left(\delta_{V} \stackrel{0}{S}_{a b}^{C}-\stackrel{0}{S}_{a b}^{\delta_{V} C}-C{ }^{1}{ }_{a b}^{V}\right), \tag{4.15}
\end{equation*}
$$

where $\left.C\right|_{\partial I_{ \pm}} \equiv C^{ \pm}$as defined in (3.26). Since the $\psi$ dependence of both $\stackrel{0}{S}_{a b}$ and $\stackrel{1}{S}_{a b}$ is through $T_{a b}$ (Eqs. (3.23) and (4.14)), this form makes it manifest that $J_{V}^{\partial \mathcal{I}}$ is independent on the $\psi$-ambiguity.

Expression (4.15) can also be used to show that $J_{V}^{\partial \mathcal{I}}$ vanishes for global CKVs, since all terms depend either on $\delta_{V} q_{a b}$ or on $\delta_{V} T_{a b}$. We can see this explicitly by expanding the first two terms in (4.15):

$$
\begin{equation*}
C^{a b}\left(\delta_{V} \stackrel{0}{S}_{a b}^{C}-\stackrel{0}{S}_{a b}^{\delta_{V} C}\right)=C^{a b}\left(2 D^{c} C D_{a} \delta_{V} q_{b c}-D^{c} C D_{c} \delta_{V} q_{a b}+D^{2} C \delta_{V} q_{a b}+C \delta_{V} T_{a b}\right) \tag{4.16}
\end{equation*}
$$

whereas the last term in (4.15) depends on $\delta_{V} q_{a b}$ and $\delta_{V} T_{a b}$ according to (4.14).

## Comment:

Identities (4.14), (4.15) and (4.16) show that $J_{V}^{\text {soft }}$ and $J_{V}^{\partial \mathcal{I}}$ depend on $V$ through $\delta_{V} q_{a b}$ and $\delta_{V} T_{a b}$. This property will be crucially used in the next section.

## V. EXTENSION OF THE GRAVITATIONAL PHASE SPACE AT NULL INFINITY

In the previous section we constructed a super angular momentum $J_{V}$ that is compatible with supertranslations in the sense that

$$
\begin{equation*}
\delta_{f} J_{V}=-\delta_{V} P_{f}=P_{V(f)} \tag{5.1}
\end{equation*}
$$

On the other hand, the compatibility of $J_{V}$ with superrotations

$$
\begin{equation*}
\delta_{V} J_{V^{\prime}}=-J_{\left[V, V^{\prime}\right]}, \tag{5.2}
\end{equation*}
$$

can be established from the 'superrotation covariance' of the expressions defining $J_{V}$, see appendix D. Finally, compatibility of supermomenta with supertranslations,

$$
\begin{equation*}
\delta_{f} P_{f^{\prime}}=0 \tag{5.3}
\end{equation*}
$$

is a well known result that can be easily checked from the expressions of supertranslations and supermomenta.

As discussed in the introduction, we can think of properties (5.1), (5.2) and (5.3) as reflecting an underlying phase space. In this section we show these properties can be used to determine an extension of the Ashtekar-Streubel phase space on which GBMS acts canonically. Let

$$
\begin{equation*}
\Gamma_{q_{a b}}:=\left\{C_{a b}(u, x): q^{a b} C_{a b}=0, \quad \partial_{u} C_{a b} \stackrel{u \rightarrow \pm \infty}{=} O\left(1 /|u|^{2+\epsilon}\right), \quad \bar{D}_{[a} \bar{D}^{c} C_{b] c}^{ \pm}=0\right\} \tag{5.4}
\end{equation*}
$$

be the space of allowed $C_{a b}$ 's for a given $q_{a b}$. Each $\Gamma_{q_{a b}}$ provides a realization of the AshtekarStreubel phase space, with symplectic structure given by

$$
\begin{equation*}
\Omega_{q_{a b}}=\int_{\mathcal{I}} \sqrt{q}\left(\delta \partial_{u} C^{a b} \wedge \delta C_{a b}\right), \quad \delta \in \Gamma_{q_{a b}} . \tag{5.5}
\end{equation*}
$$

In the traditional interpretation, different choices of $q_{a b}$ (or frames) are akin to gauge choices. However, to implement superrotations we need to consider the larger space [18]

$$
\begin{equation*}
\Gamma:=\bigcup_{q_{a b}: \sqrt{q}=\sqrt{q}} \Gamma_{q_{a b}}, \tag{5.6}
\end{equation*}
$$

where $\sqrt{\mathscr{q}}$ is the area element of the unit round sphere. Our aim is to find a symplectic structure $\Omega$ on $\Gamma$ such that:
(i) $P_{f}$ and $J_{V}$ are the canonical charges of supertranslations and superrotations,

$$
\begin{align*}
& \Omega\left(\delta, \delta_{f}\right)=\delta P_{f} \quad \forall \delta \in \Gamma  \tag{5.7}\\
& \Omega\left(\delta, \delta_{V}\right)=\delta J_{V} \quad \forall \delta \in \Gamma \tag{5.8}
\end{align*}
$$

and
(ii) $\Omega$ reduces to $\Omega_{q_{a b}}$ when restricted to $\Gamma_{q_{a b}}$,

$$
\begin{equation*}
\left.\Omega\right|_{\Gamma_{a b}}=\Omega_{q_{a b}} . \tag{5.9}
\end{equation*}
$$

Our starting point is to write $\Omega$ as

$$
\begin{equation*}
\Omega=\Omega^{\mathcal{I}}+\Omega^{S^{2}} \tag{5.10}
\end{equation*}
$$

with $\Omega^{\mathcal{I}}$ as in the AS expression (5.5) but allowing for arbitrary variations in $\Gamma$,

$$
\begin{equation*}
\Omega^{\mathcal{I}}=\int_{\mathcal{I}} \sqrt{q}\left(\delta \partial_{u} C^{a b} \wedge \delta C_{a b}\right), \quad \delta \in \Gamma \tag{5.11}
\end{equation*}
$$

and $\Omega^{S^{2}}$ a reminder to be determined.
Next, we evaluate $\Omega^{\mathcal{I}}$ on supertranslations and superrotations. A straightforward calculation gives

$$
\begin{align*}
& \Omega^{\mathcal{I}}\left(\delta, \delta_{f}\right)=\delta P_{f}^{\mathrm{hard}}+\int_{S^{2}} \sqrt{q} \delta N^{a b}{ }_{S}^{0}{ }_{a b}^{f}  \tag{5.12}\\
& \Omega^{\mathcal{I}}\left(\delta, \delta_{V}\right)=\delta J_{V}^{\mathrm{hard}} \tag{5.13}
\end{align*}
$$

Using these expressions, conditions (5.7) and (5.8) translate into the following conditions on $\Omega^{S^{2}}$ :

$$
\begin{align*}
& \Omega^{S^{2}}\left(\delta, \delta_{f}\right)=\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} \delta \stackrel{0}{S}_{a b}^{f}  \tag{5.14}\\
& \Omega^{S^{2}}\left(\delta, \delta_{V}\right)=\delta J_{V}^{\text {soft }}+\delta J_{V}^{\partial \mathcal{I}} \tag{5.15}
\end{align*}
$$

The strategy now will be to use Eq. (5.15) to determine $\Omega^{S^{2}}$ and then verify Eq. (5.14). We assume $\Omega^{S^{2}}$ is of the form

$$
\begin{equation*}
\Omega^{S^{2}}=\Omega^{\mathrm{soft}}+\Omega^{\partial \mathcal{I}} \tag{5.16}
\end{equation*}
$$

with

$$
\begin{equation*}
\Omega^{\cdots}\left(\delta, \delta_{V}\right)=\delta J_{V}^{\cdots} \tag{5.17}
\end{equation*}
$$

where "..." stands for either "soft" or " $\partial \mathcal{I}$ ". We think of each $\Omega$ " as defining a symplectic structure on its own such that $\delta_{V}$ acts canonically with charge $J_{V}$.

It will be convenient to express $\Omega^{\cdots}$ in terms of a symplectic potential $\theta^{\cdots}$,

$$
\begin{equation*}
\Omega^{\cdots}\left(\delta, \delta^{\prime}\right)=\delta \theta^{\cdots}\left(\delta^{\prime}\right)-\delta^{\prime} \theta^{\cdots}(\delta)-\theta^{\cdots}\left(\left[\delta, \delta^{\prime}\right]\right) \tag{5.18}
\end{equation*}
$$

and consider a $\theta \cdots$ compatible with $\delta_{V}$ (see e.g. [50])

$$
\begin{equation*}
\theta^{\cdots}\left(\delta_{V}\right)=J_{V}^{\cdots} \tag{5.19}
\end{equation*}
$$

so that Eq. (5.17) becomes ${ }^{13}$

$$
\begin{equation*}
\delta_{V} \theta \cdots(\delta)+\theta^{\cdots}\left(\left[\delta, \delta_{V}\right]\right)=0 \tag{5.20}
\end{equation*}
$$

If we now look at the expressions for $J_{V}^{\text {soft }}$ and $J_{V}^{\partial \mathcal{I}}$ as given by Eqs. (4.3), (4.14) and (4.15), we can easily find candidates for $\theta^{\cdots}(\delta)$ satisfying (5.19) by doing the replacement

[^9]$\delta_{V} \rightarrow \delta$ in such expressions. Defining
\[

$$
\begin{equation*}
\stackrel{1}{S}_{a b}(\delta):=\left[2 \delta T_{a b}+D_{(a} D^{c} \delta q_{b) c}-\frac{R}{2} \delta q_{a b}\right]^{\mathrm{TF}} \tag{5.21}
\end{equation*}
$$

\]

so that $\stackrel{1}{S}_{a b}\left(\delta_{V}\right)=\stackrel{1}{S}_{a b}^{V}$, we find the following candidates for symplectic potentials:

$$
\begin{align*}
\theta^{\text {soft }}(\delta) & =\int_{S^{2}} \sqrt{q} N^{a b}{ }_{S}^{1}(\delta)  \tag{5.22}\\
\theta^{\partial \mathcal{I}}(\delta) & =-\frac{1}{2} \int_{\partial \mathcal{I}} \sqrt{q} C^{a b}\left(\delta \stackrel{0}{S}_{a b}^{C}-\stackrel{0}{S}_{a b}^{\delta C}-C \stackrel{1}{S}_{a b}(\delta)\right) \tag{5.23}
\end{align*}
$$

By construction, (5.22) and (5.23) satisfy Eq. (5.19). Condition (5.20) can then be shown to be a a consequence of (i) the fact that $\theta^{\cdots}(\delta)$ is only sensitive to variations of $q_{a b}$, (ii) the fact that any variation $\delta q_{a b}$ can be written as $\delta q_{a b}=\delta_{W} q_{a b}$ for some vector field $W^{a}$ and (iii) the 'superrotation covariance' of $J_{V}$. Indeed, recall that $T_{a b}$ is fully determined by $q_{a b}$ and note that the term $\delta \stackrel{0}{S}_{a b}^{C}-\stackrel{0}{S}_{a b}^{\delta C}$ is independent of variations of $C$. Writing $\delta q_{a b}=\delta_{W} q_{a b}$ condition (5.20) becomes

$$
\begin{equation*}
\delta_{V} \theta^{\cdots}\left(\delta_{W}\right)+\theta^{\cdots}\left(\left[\delta_{W}, \delta_{V}\right]\right)=0 . \tag{5.24}
\end{equation*}
$$

Using $\left[\delta_{W}, \delta_{V}\right]=-\delta_{[W, V]}$ [18] and (5.19), Eq. (5.24) can be seen to be a direct consequence of the superrotation covariance of $J_{W}^{\cdots}$ (see appendix (D),

$$
\begin{equation*}
\delta_{V} J_{W}^{\cdots}=-J_{[V, W]} . \tag{5.25}
\end{equation*}
$$

Summing the resulting $\Omega^{\text {soft }}$ and $\Omega^{\partial \mathcal{I}}$, we obtain $\Omega^{S^{2}}$ satisfying (5.15). It now remains to verify that such $\Omega^{S^{2}}$ satisfies (5.14). This can be shown to be a consequence of the supertranslation transformation properties of $J_{V}^{\text {hard }}$ and $J_{V}$ as follows.

Written in terms of the symplectic potential $\theta^{S^{2}}=\theta^{\text {soft }}+\theta^{\partial \mathcal{I}}$, condition (5.14) takes the form,

$$
\begin{equation*}
\delta \theta^{S^{2}}\left(\delta_{f}\right)-\delta_{f} \theta^{S^{2}}(\delta)-\theta^{S^{2}}\left(\left[\delta, \delta_{f}\right]\right)=\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} \delta^{0} S_{a b}^{f} \tag{5.26}
\end{equation*}
$$

As before, we note that Eq. (5.26) is only sensitive to variations of $q_{a b}$. Writing $\delta q_{a b}=$ $\delta_{W} q_{a b}$ for some $W^{a}$ and using the fact that $\theta^{S^{2}}$ vanishes if evaluated on variations with $\delta q_{a b}=0$ (first and third term in (5.26)), the condition reduces to

$$
\begin{equation*}
-\delta_{f} \theta^{S^{2}}\left(\delta_{W}\right)=\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} \delta_{W} \stackrel{0}{S}_{a b}^{f} \tag{5.27}
\end{equation*}
$$

or, using (5.19), to

$$
\begin{equation*}
-\delta_{f}\left(J_{W}^{\text {soft }}+J_{W}^{\partial \mathcal{I}}\right)=\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} \delta_{W} \stackrel{0}{S}_{a b}^{f} . \tag{5.28}
\end{equation*}
$$

By writing $-\left(J_{W}^{\text {soft }}+J_{W}^{\partial \mathcal{I}}\right)=\left(J_{W}^{\text {hard }}-J_{W}\right)$, Eq. (5.28) can be seen to be a consequence of the supertranslation transformation formulas (4.12) and (5.1) for $J_{W}^{\text {hard }}$ and $J_{W}$ respectively.

## A. Summary

We found a symplectic structure $\Omega$ on the space $\Gamma$ (5.6) satisfying Eqs. (5.7) and (5.8). $\Omega$ is written as a sum of 'bulk' and 'boundary' pieces:

$$
\begin{equation*}
\Omega=\Omega^{\mathcal{I}}+\Omega^{S^{2}} \tag{5.29}
\end{equation*}
$$

where $\Omega^{\mathcal{I}}$ (5.11) is the extension to $\Gamma$ of the AS symplectic structure and $\Omega^{S^{2}}$ is given in terms of a symplectic potential

$$
\begin{equation*}
\theta^{S^{2}}=\theta^{\text {soft }}+\theta^{\partial \mathcal{I}} \tag{5.30}
\end{equation*}
$$

with $\theta^{\text {soft }}$ and $\theta^{\partial \mathcal{I}}$ defined in Eqs. (5.22) and (5.23) respectively.
By integration by parts on the sphere, one can bring $\theta^{S^{2}}$ and $\Omega^{S^{2}}$ into a form

$$
\begin{align*}
\theta^{S^{2}} & =\int_{S^{2}} \sqrt{q}\left(p^{a b} \delta q_{a b}+\Pi^{a b} \delta T_{a b}\right)  \tag{5.31}\\
\Omega^{S^{2}} & =\int_{S^{2}} \sqrt{q}\left(\delta p^{a b} \wedge \delta q_{a b}+\delta \Pi^{a b} \wedge \delta T_{a b}\right) \tag{5.32}
\end{align*}
$$

where

$$
\begin{align*}
p^{a b} & =D^{(a} D_{c} \stackrel{1}{N}^{b) c}-\frac{R}{2} \stackrel{1}{N}^{a b}+\left.(\text { quadratic in } C)^{a b}\right|_{\partial \mathcal{I}}  \tag{5.33}\\
\Pi^{a b} & =2 N^{a b}+\left.\frac{1}{2} C C^{a b}\right|_{\partial \mathcal{I}} . \tag{5.34}
\end{align*}
$$

The terms quadratic in $C^{ \pm}$in (5.33) can be obtained with the help of Eq. (4.16). Written in this form, it is clear that $\left.\Omega^{S^{2}}\right|_{\Gamma_{a b}}=0$ and so condition (5.9) is also satisfied.

The terms linear in $N^{a b}$ in (5.33) correspond to those found in [18, 26] by covariant phase space methods. An important question we leave open is whether the full $\Omega$ can be understood from a covariant phase space perspective. In this respect, we note the recent work [27] which shows there cannot be a symplectic structure - constructed from a local and covariant symplectic current- at null infinity that supports the action of GBMS. There is in principle no contradiction with our results, since we do not assume a symplectic current and furthermore our symplectic structure contains non-local terms due to appearance of $T_{a b}$ (which depends non-locally on $q_{a b}$ ) and of $C^{ \pm}$(which depends non-locally on $C_{a b}^{ \pm}$). It would be interesting to see if the analysis of [27] can be extended to include such non-local terms.

We conclude by noting that conditions (5.7), (5.8) together with (5.1), (5.2) and (5.3) imply the PBs

$$
\begin{equation*}
\left\{P_{f}, P_{f^{\prime}}\right\}=0, \quad\left\{J_{V}, P_{f}\right\}=P_{V(f)}, \quad\left\{J_{V}, J_{V^{\prime}}\right\}=J_{\left[V, V^{\prime}\right]} \tag{5.35}
\end{equation*}
$$

## VI. DISCUSSION

If the symmetries of gravity at null infinity are to include all diffeomorphisms on the celestial sphere [1], one should be able to associate canonical charges to this Diff $\left(S^{2}\right)$ group. A prerequisite for this is to have a phase space on which these symmetries act. It is clear
that such phase space should include, in addition to gravitational radiation, the 'kinematical' 2-metric at null infinity [18]. Doing so, however, introduces several divergences in the symplectic structure at null infinity that are notoriously difficult to control [18, 26, 27].

Here we have taken an alternative route. Rather than trying to obtain a finite symplectic structure from the beginning, we started by noting certain conditions the GBMS charges should satisfy if such finite symplectic structure exists. In particular, we noted that the Diff $\left(S^{2}\right)$ transformation properties of supermomenta imply there should not be an extension term in the Poisson bracket between supermomenta and super angular momenta. ${ }^{14}$ This led us to consider a correction term in the expression of super angular momentum that cancels (a non-CKV analogue of) the BT extension term [8]. The corrected super angular momentum may offer a better understanding of the charge algebra at the level of the gravitational $S$ matrix [25]: The BT extension does not commute with the $S$ matrix [20] and hence appears to contradict the idea of (extended/generalized) BMS as a symmetry of gravitational scattering [9, 10, 12].

We finally showed there exists a natural symplectic structure at null infinity that is compatible with the expressions of GBMS charges described above. A by product of our analysis was the introduction of a $\operatorname{Diff}\left(S^{2}\right)$-covariant derivative at null infinity that may be of interest beyond the scope of this paper.

There are many directions this work should be improved upon. Whereas the finding of a symplectic structure at null infinity supporting the $\operatorname{Diff}\left(S^{2}\right)$ action is a non-trivial fact -there was no guarantee of its existence- the question remains open as to whether this structure can be obtained from covariant phase space methods. A related question is whether there can be a $3+1$ realization of GBMS as was recently shown to exist for BMS 51].

We have worked under the assumption of 'tree-level' $u$ fall-offs in which the news tensor decays faster than $1 / u^{2}$. However, in generic gravitational scattering the news tensor has a leading component falling exactly as $1 / u^{2}$ [43, 44] that should be incorporated in the analysis.

The GBMS group has a direct analogue in higher dimensions and one may ask if the results presented here can be generalized to higher dimensions as in the BMS case [524 54]. Finally, one may wonder if there could exist an additional extension of the gravitational phase space that supports large diffeomorphisms [55, 56] associated to the sub-subleading soft graviton theorem [13, 57].

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## Appendix A: Finite $\operatorname{Diff}\left(S^{2}\right) \subset$ GBMS transformations

In this appendix, following [26, 58], we calculate the action of finite superrotations $\phi \in$ $\operatorname{Diff}\left(S^{2}\right)$ on the asymptotic spacetime metric. Restricting to the case where the 'initial' spacetime metric is in a Bondi-frame, we will obtain a parametrization of various non-Bondi frame quantities in terms of $\phi$.

The idea is to proceed in the same way as for infinitesimal superrotations [18, 42] but for finite diffeos. Namely, we look for spacetime diffeomorphisms that preserve the space-time metric in Bondi gauge,

$$
\begin{equation*}
g_{r r}=0=g_{r a}, \quad \sqrt{\operatorname{det} g_{a b}}=r^{2} \sqrt{q}, \tag{A1}
\end{equation*}
$$

with the standard $1 / r$ fall-offs [42] . We express the diffeo as ${ }^{15}$

$$
\begin{equation*}
\left(r, u, x^{a}\right) \rightarrow\left(R, U, X^{A}\right) \tag{A2}
\end{equation*}
$$

and assume an $1 / r$ expansion compatible with that of the spacetime metric:

$$
\begin{align*}
R(r, u, x) & =\stackrel{0}{R}(x) r+\stackrel{1}{R}(u, x)+O(1 / r)  \tag{A3}\\
U(r, u, x) & =\stackrel{0}{R^{-1}(x) u+O(1 / r)}  \tag{A4}\\
X^{A}(r, u, x) & =\phi^{A}(x)+\frac{1}{r} X^{A}(u, x)+O\left(r^{-2}\right) \tag{A5}
\end{align*}
$$

In the above expressions we have already fixed some of the $u$-dependence that is required for compatibility with the Bondi metric [18, 42]. We have also excluded supertranslations, which correspond to a $u$-independent term in the $O\left(r^{0}\right)$ part of $U$ (see [58] for expressions of finite supertranslations).

We proceed as follows. The 'initial' spacetime metric in $\left(R, U, X^{A}\right)$ coordinates is taken to be in Bondi frame so that its angular components take the form

$$
\begin{equation*}
g_{A B}(R, U, X)=R^{2} q_{A B}+R C_{A B}(U, X)+\cdots \tag{A6}
\end{equation*}
$$

with $q_{A B}$ the unit round sphere metric. Next, we compute the various components of the pullback metric in the $\left(r, u, x^{a}\right)$ coordinates under the spacetime diffeo (A3, A4, A5). By imposing the Bondi gauge conditions on the pullback metric we then determine the spacetime diffeo coefficients in terms of $\phi \in \operatorname{Diff}\left(S^{2}\right)$.

The angular part of the pullback metric is found to be given by

$$
\begin{equation*}
g_{a b}(r, u, x)=r^{2} q_{a b}^{\phi}(x)+r C_{a b}^{\phi}(u, x)+\cdots \tag{A7}
\end{equation*}
$$

[^11]where
\[

$$
\begin{equation*}
q_{a b}^{\phi}(x)=\stackrel{0}{R}^{2} \partial_{a} \phi^{A} \partial_{b} \phi^{B} q_{A B}(\phi), \tag{A8}
\end{equation*}
$$

\]

and

$$
\begin{align*}
C_{a b}^{\phi}(u, x)=\stackrel{0}{R} \partial_{a} \phi^{A} \partial_{b} \phi^{B}\left(C_{A B}\left(\stackrel{0}{R}^{-1} u, \phi\right)+\right. & \left.2 \stackrel{1}{R} q_{A B}(\phi)+\stackrel{0}{R}{ }_{X}^{1}{ }^{C} \partial_{C} q_{A B}(\phi)\right) \\
& +2 \stackrel{0}{R^{2}} \partial_{a} \phi^{A} \partial_{b} \stackrel{1}{X}^{B} q_{A B}(\phi)+2 \stackrel{0}{R}^{-2} \partial_{a} \stackrel{0}{R} \partial_{b} \stackrel{0}{R} \tag{A9}
\end{align*}
$$

In the RHS of the above equations it is understood that $\phi, \stackrel{0}{R}, \stackrel{1}{R}$, etc. are evaluated at $(u, x)$ as in Eqs. (A3, A4, A5).

To leading order, the determinant condition (A1) implies $\operatorname{det} q^{\phi}(x)=\operatorname{det} q(x)$, which fixes $\stackrel{0}{R}(x)$ to be

$$
\begin{equation*}
\stackrel{0}{R}(x)=\frac{\operatorname{det}^{1 / 4} q(x)}{\operatorname{det}^{1 / 4} q(\phi(x))} \frac{1}{\operatorname{det}^{1 / 2} \partial \phi(x)} \tag{A10}
\end{equation*}
$$

To subleading order, the determinant condition implies

$$
\begin{equation*}
q_{\phi}^{a b} C_{a b}^{\phi}=0 \tag{A11}
\end{equation*}
$$

which fixes $\stackrel{1}{R}$ in terms of the other quantities. The remaining diffeo component to be determined is $\stackrel{1}{X}^{A}$, which can be obtained from condition $g_{r a}=0$. The pullback for such metric components is found to be

$$
\begin{equation*}
g_{r a}(r, u, x)=u \partial_{a} \ln \stackrel{0}{R}-\stackrel{1}{X}^{A} \partial_{a} \phi^{B} \stackrel{0}{R}^{2} q_{A B}(\phi)+O\left(r^{-1}\right) \tag{A12}
\end{equation*}
$$

To solve $g_{r a}=0$ it is convenient to write $\stackrel{1}{X}^{A}$ as a pushforward of a vector field $Y^{a}$,

$$
\begin{equation*}
\stackrel{1}{X}^{A}=\partial_{a} \phi^{A} Y^{a} \tag{A13}
\end{equation*}
$$

Eq. (A12) can then be written as

$$
\begin{equation*}
g_{r a}=u \partial_{a} \ln \stackrel{0}{R}-q_{a b}^{\phi} Y^{b}+O\left(r^{-1}\right), \tag{A14}
\end{equation*}
$$

from which we obtain

$$
\begin{equation*}
Y^{a}=u q_{\phi}^{a b} \partial_{a} \ln \stackrel{0}{R} \tag{A15}
\end{equation*}
$$

We now have all elements to express $C_{a b}^{\phi}$ in (A9) in terms of $\phi$. The expression simplifies when written in terms of the covariant derivative $D_{a}^{\phi}$ compatible with the metric $q_{a b}^{\phi}$. After some work, it can be expressed as

$$
\begin{equation*}
C_{a b}^{\phi}(u, x)=\stackrel{0}{R} \partial_{a} \phi^{A} \partial_{b} \phi^{B} C_{A B}\left(\stackrel{0}{R}^{-1} u, \phi\right)+u N_{a b}^{\mathrm{vac}} \tag{A16}
\end{equation*}
$$

where

$$
\begin{equation*}
N_{a b}^{\mathrm{vac}}=2\left(D_{a}^{\phi} \ln \stackrel{0}{R} D_{b}^{\phi} \ln \stackrel{0}{R}+D_{a}^{\phi} D_{b}^{\phi} \ln \stackrel{0}{R}\right)^{\mathrm{TF}} \tag{A17}
\end{equation*}
$$

This result is essentially that of section 3 of [26], except that here our 'initial' 2 d metric is the round unit sphere, whereas in [26] it is the Euclidean plane.

The above expressions can be used to identify the potential $\psi$ of section III A in terms of $\phi$. From (A16) and (A17) we see that

$$
\begin{equation*}
T_{a b}=N_{a b}^{\mathrm{vac}}, \quad \psi=\ln \stackrel{0}{R} \tag{A18}
\end{equation*}
$$

where $\stackrel{0}{R}$ in terms of $\phi$ is given in Eq. (A10).
After this identification, we can write Eq. (A8) as

$$
\begin{equation*}
q_{a b}^{\phi}(x)=e^{2 \psi} \partial_{a} \phi^{A} \partial_{b} \phi^{B} q_{A B}(\phi) \tag{A19}
\end{equation*}
$$

From this perspective, $\psi$ appears as a conformal rescaling that makes $q_{a b}$ diffeomorphic to the unit sphere metric. We finally use Eq. (A19) to obtain a formula for the scalar curvature $\mathcal{R}$ of $q_{a b}^{\phi}$ in terms of $\psi$ :

$$
\begin{equation*}
\mathcal{R}=2\left(e^{-2 \psi}-\left(D^{\phi}\right)^{2} \psi\right) \tag{A20}
\end{equation*}
$$

This is the analogue of Eq. (3.11) in [26].

## Appendix B: Geroch tensor

In the Geroch approach [48], $T_{a b}$ is (minus) the trace-free part of a tensor

$$
\begin{equation*}
\rho_{a b}=\frac{R}{2} q_{a b}-T_{a b}, \tag{B1}
\end{equation*}
$$

that satisfies

$$
\begin{equation*}
D_{[a} \rho_{b] c}=0 . \tag{B2}
\end{equation*}
$$

Geroch shows there is a unique tensor $\rho_{a b}$ satisfying the above conditions. We here verify our $T_{a b}$ satisfies Eq. (B2).

Inserting B1 in B2, the Geroch condition becomes

$$
\begin{equation*}
D_{[a} R q_{b] c}=2 D_{[a} T_{b] c} \tag{B3}
\end{equation*}
$$

Using 2 d algebraic relations, this can be shown to be equivalent to ${ }^{16}$

$$
\begin{equation*}
D_{a} R=-2 D^{b} T_{a b} . \tag{B4}
\end{equation*}
$$

Eq. (B4) corresponds to Eq. (3.12) of [26] and can be shown to follow from the expressions of $R\left(\right.$ A20) and $T_{a b}$ (3.7) in terms of $\psi$ :

$$
\begin{align*}
R & =2\left(e^{-2 \psi}-D^{2} \psi\right)  \tag{B5}\\
T_{a b} & =2\left(D_{a} \psi D_{b} \psi+D_{a} D_{b} \psi\right)^{\mathrm{TF}} \tag{B6}
\end{align*}
$$

We conclude by discussing the implications of these identities on the 'covariantized' scalar

[^12]curvature $\bar{R}$. We first note that $\bar{R}$ given in Eq. (3.21) can be rewritten, using Eq. (B5), as:
\[

$$
\begin{equation*}
\bar{R}=2 e^{-2 \psi} \tag{B7}
\end{equation*}
$$

\]

In this form, the superrotation covariance of $\bar{R}$ appears as a direct consequence of the transformation property of $\psi$ (3.8), from which one obtains

$$
\begin{equation*}
\delta_{V} \bar{R}=\mathcal{L}_{V} \bar{R}+2 \alpha \bar{R} . \tag{B8}
\end{equation*}
$$

Finally, if we compute $\bar{D}_{a} \bar{R}$ with the rules of section IIIB one can verify

$$
\begin{equation*}
\bar{D}_{a} \bar{R}=D_{a} R+2 D^{b} T_{a b}, \tag{B9}
\end{equation*}
$$

which vanishes due to the Geroch identity (B4).

## Appendix C: Main identity

Let us first recall that $\delta_{f} J_{V}^{\partial \mathcal{I}}$ and $\operatorname{mag}(f, V)$ can be written in terms of $\stackrel{0}{N}^{a b}$ as

$$
\begin{align*}
\delta_{f} J_{V}^{\partial \mathcal{I}} & =\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} B_{a b}  \tag{C1}\\
\operatorname{mag}(f, V) & =\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} M_{a b} \tag{C2}
\end{align*}
$$

where

$$
\begin{align*}
B_{a b} & =\left[-\bar{D}^{c}\left(V_{a} \stackrel{0}{S}_{b c}^{f}\right)+V^{c} \bar{D}_{a} \stackrel{0}{S}_{b c}^{f}+3 \bar{\alpha} \stackrel{0}{S}_{a b}^{f}{ }^{\mathrm{STF}}\right.  \tag{C3}\\
M_{a b} & =\left[\bar{D}_{a} \bar{D}^{c}\left(-2 \bar{D}_{b} f V_{c}+2 \bar{D}_{c} f V_{b}+f \bar{D}_{b} V_{c}-f \bar{D}_{c} V_{b}\right)\right]^{\mathrm{STF}} \tag{C4}
\end{align*}
$$

where STF stands for symmetric, trace-free part in the indices $(a, b)$.
Equation (4.6)

$$
\begin{equation*}
\delta_{f} J_{V}-P_{V(f)}=\operatorname{mag}(f, V) \tag{C5}
\end{equation*}
$$

then becomes

$$
\begin{equation*}
\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b}\left(\delta_{V} \stackrel{0}{S}_{a b}^{f}-f \stackrel{1}{S}_{a b}^{V}+B_{a b}\right)=\int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} M_{a b} \tag{C6}
\end{equation*}
$$

which is satisfied if and only if

$$
\begin{equation*}
\left[\delta_{V} \stackrel{0}{S}_{a b}^{f}\right]^{\mathrm{TF}}-f \stackrel{1}{S}_{a b}^{V}+B_{a b}=M_{a b} \tag{C7}
\end{equation*}
$$

This is the identity we wish to prove, which corresponds to Eq. (4.13) of section [IV] ${ }^{17}$
We start by evaluating $B_{a b}$. Expanding the derivatives in (C3) and using the properties

[^13]listed at the end of section IIIB one finds
\[

$$
\begin{equation*}
B_{a b}=\left[2 \bar{D}^{c} V_{a} \bar{D}_{b} \bar{D}_{c} f+\bar{R} V_{a} D_{b} f+2 V_{a} \bar{D}_{b} \bar{D}^{2} f-\frac{1}{2} \gamma_{a b} \bar{D}^{2} f-2 V^{c} \bar{D}_{a} \bar{D}_{b} \bar{D}_{c} f-6 \bar{\alpha} \bar{D}_{a} \bar{D}_{b} f\right]^{\mathrm{STF}} \tag{C8}
\end{equation*}
$$

\]

where we have introduced the notation,

$$
\begin{equation*}
\gamma_{a b}=2\left[\bar{D}_{a} V_{b}\right]^{\mathrm{STF}}=\delta_{V} q_{a b} . \tag{C9}
\end{equation*}
$$

Notice that the second equality in (C9) involves a non-trivial identity $\left[\bar{D}_{a} V_{b}\right]^{\mathrm{STF}}=$ $\left[D_{a} V_{b}\right]^{\text {STF }}$. We will use the notation (C9) in all remaining expressions.

Expanding now (C4) and comparing with (C8) one finds

$$
\begin{align*}
& B_{a b}-M_{a b}=f\left[-2 \bar{D}_{a} \bar{D}_{b} \bar{\alpha}+\bar{D}_{a} \bar{D}^{c} \gamma_{b c}-\frac{\bar{R}}{2} \gamma_{a b}-\bar{D}_{a} \bar{R} V_{b}\right]^{\mathrm{STF}} \\
&+\left[\bar{D}_{a} f \bar{D}^{c} f \gamma_{b c}-\bar{D}^{c} f \bar{D}_{a} \gamma_{b c}+\bar{D}_{a} \bar{D}^{c} f \gamma_{b c}-\frac{3}{2} \gamma_{a b} \bar{D}^{2} f\right]^{\mathrm{STF}} \tag{C10}
\end{align*}
$$

where we note that the last term in the first line is actually zero since $\bar{D}_{a} \bar{R}=0$.
Finally, we evaluate the first term in (C7)

$$
\begin{equation*}
\left[\delta_{V} \stackrel{0}{S}_{a b}^{f}\right]^{\mathrm{TF}}=\left[-2 f \bar{D}_{a} \bar{D}_{b} \bar{\alpha}+2 \bar{D}^{c} f \bar{D}_{a} \gamma_{b c}-\bar{D}^{c} f \bar{D}_{c} \gamma_{a b}+\gamma_{a b} \bar{D}^{2} f\right]^{\mathrm{STF}} \tag{C11}
\end{equation*}
$$

and recall the definition of $\stackrel{1}{S}_{a b}^{V}$ (4.5)

$$
\begin{equation*}
\stackrel{1}{S}{ }_{a b}^{V}=\left[-4 \bar{D}_{a} \bar{D}_{b} \bar{\alpha}+\bar{D}_{a} \bar{D}^{c} \gamma_{b c}-\frac{\bar{R}}{2} \gamma_{a b}\right]^{\mathrm{STF}} \tag{C12}
\end{equation*}
$$

Collecting (C10), (C11) and (C12) we finally arrive at

$$
\begin{align*}
& {\left[\delta_{V} \stackrel{0}{S}_{a b}^{f}\right]^{\mathrm{STF}}-f \stackrel{1}{S}_{a b}^{V}+B_{a b}-M_{a b}=} \\
& \quad\left[\bar{D}^{c} f \bar{D}_{a} \gamma_{b c}-\bar{D}^{c} f \bar{D}_{c} \gamma_{a b}+\bar{D}_{a} f \bar{D}^{c} \gamma_{b c}+\bar{D}_{a} \bar{D}^{c} f \gamma_{b c}-\frac{1}{2} \gamma_{a b} \bar{D}^{2} f\right]^{\mathrm{STF}} \tag{C13}
\end{align*}
$$

One can now check that the right hand side of (C13) is trivially zero. As for similar 2d algebraic identities, this can be easily seen by writing the expression in holomorphic coordinates. This concludes the proof of identity (C7) and hence of (C5).

## Appendix D: Superrotation covariance of charges

In this appendix we show the relations

$$
\begin{align*}
\delta_{V} P_{f} & =-P_{V(f)}  \tag{D1}\\
\delta_{V} J_{V^{\prime}} & =-J_{\left[V, V^{\prime}\right]} \tag{D2}
\end{align*}
$$

which express the superrotation covariance of charges.

Let us first verify the 'hard' parts of (D1), (D2). For the supermomentum we have

$$
\begin{align*}
\delta_{V} P_{f}^{\mathrm{hard}} & =\int_{\mathcal{I}} \sqrt{q} f\left(\partial_{u} \delta_{V} C^{a b} \partial_{u} C_{a b}+\partial_{u} C^{a b} \partial_{u} \delta_{V} C_{a b}\right)  \tag{D3}\\
& =\int_{\mathcal{I}} \sqrt{q} f\left(\left(\mathcal{L}_{V} \dot{C}^{a b}+\alpha u \ddot{C}^{a b}+4 \alpha \dot{C}^{a b}\right) \dot{C}_{a b}+\dot{C}^{a b}\left(\mathcal{L}_{V} \dot{C}_{a b}+\alpha u \ddot{C}_{a b}\right)\right)  \tag{D4}\\
& =\int_{\mathcal{I}} \sqrt{q} f\left(\mathcal{L}_{V} \dot{C}^{2}+\alpha u \partial_{u} \dot{C}^{2}+4 \alpha \dot{C}^{2}\right)  \tag{D5}\\
& =\int_{\mathcal{I}} \sqrt{q}\left(-\mathcal{L}_{V} f \dot{C}^{2}+f \alpha \dot{C}^{2}\right)=-P_{V(f)}^{\mathrm{hard}}, \tag{D6}
\end{align*}
$$

where in the last step we integrated by parts and used that $\mathcal{L}_{V} \sqrt{q}=2 \alpha \sqrt{q}$. To simplify notation we denote $u$-derivatives with dots and $\dot{C}^{2} \equiv \dot{C}^{a b} \dot{C}_{a b}$.

Similarly, for the super angular momentum we have

$$
\begin{align*}
\delta_{V} J_{V^{\prime}}^{\mathrm{hard}} & =\int_{\mathcal{I}} \sqrt{q}\left(\partial_{u} \delta_{V} C^{a b} \delta_{V^{\prime}} C_{a b}+\dot{C}^{a b} \delta_{V} \delta_{V^{\prime}} C_{a b}\right.  \tag{D7}\\
& =\int_{\mathcal{I}} \sqrt{q}\left(\left(\mathcal{L}_{V} \dot{C}^{a b}+\alpha u \ddot{C}^{a b}+4 \alpha \dot{C}^{a b}\right) \delta_{V^{\prime}} C_{a b}+\dot{C}^{a b} \delta_{V} \delta_{V^{\prime}} C_{a b}\right)  \tag{D8}\\
& =\int_{\mathcal{I}} \sqrt{q}\left(\dot{C}^{a b}\left(-\mathcal{L}_{V} \delta_{V^{\prime}} C_{a b}-\alpha u \partial_{u} \delta_{V^{\prime}} C_{a b}+\alpha \delta_{V^{\prime}} C_{a b}\right)+\dot{C}^{a b} \delta_{V} \delta_{V^{\prime}} C_{a b}\right)  \tag{D9}\\
& =\int_{\mathcal{I}} \sqrt{q} \dot{C}^{a b}\left(-\delta_{V^{\prime}} \delta_{V} C_{a b}+\delta_{V} \delta_{V^{\prime}} C_{a b}\right)=-J_{\left[V, V^{\prime}\right]}^{\mathrm{hard}}, \tag{D10}
\end{align*}
$$

where in the third step we integrated by parts and in the fourth we recognized the combination $-\delta_{V^{\prime}} \delta_{V} C_{a b}$ in the first term of the third line. Finally we used $\left[\delta_{V}, \delta_{V^{\prime}}\right] C_{a b}=-\delta_{\left[V, V^{\prime}\right]} C_{a b}$ [18].

We now discuss the remaining, $u$-independent terms of the charges. We will use the notion of superrotation covariance of section IIIB to facilitate the calculation. Let us start by noting that a 'covariant' scalar $\rho(x)$ with $k=+2$ has a superrotation invariant integral over the sphere,

$$
\begin{align*}
\delta_{V} \int_{S^{2}} \sqrt{q} \rho & =\int_{S^{2}} d^{2} x \sqrt{q} \delta_{V} \rho  \tag{D11}\\
& =\int_{S^{2}} \sqrt{q}\left(\mathcal{L}_{V} \rho+2 \alpha \rho\right)=0 \tag{D12}
\end{align*}
$$

where in the last equality we integrated by parts and used $\mathcal{L}_{V} \sqrt{q}=2 \alpha \sqrt{q}$. Next, we note that eventhough $f$ and $V^{\prime a}$ are parameters and hence do not transform under the action of $\delta_{V}$, they can be thought of as 'covariant' tensors with $k=-1$ and $k=0$ respectively, due to the GBMS algebra relations (1.1). One can then check that all the integrands of $P_{f}^{\text {soft }}, J_{V^{\prime}}^{\text {soft }}$ and $J_{V^{\prime}}^{\partial I}$ have $k=2$. However because $f$ and $V^{\prime a}$ are parameters that do not change under
$\delta_{V}$ one gets

$$
\begin{align*}
\delta_{V} P_{f}^{\text {soft }} & =-P_{V(f)}^{\text {soft }}  \tag{D13}\\
\delta_{V} J_{V^{\prime}}^{\text {soft }} & =-J_{\left[V, V^{\prime}\right]}^{\text {soft }}  \tag{D14}\\
\delta_{V} J_{V^{\prime}}^{\partial \mathcal{I}} & =-J_{\left[V, V^{\prime}\right]}^{\partial I} \tag{D15}
\end{align*}
$$

rather than zero.
Let us do the calculation in detail for (D13), the others following along the same lines:

$$
\begin{align*}
\delta_{V} \int_{S^{2}} \sqrt{q} \stackrel{0}{N}^{a b} \stackrel{0}{S}_{a b}^{f} & =\int_{S^{2}} \sqrt{q}\left(\delta_{V} \stackrel{0}{N}^{a b} \stackrel{0}{S}_{a b}^{f}+\stackrel{0}{N} a b \delta_{V} \stackrel{0}{S}_{a b}\right)  \tag{D16}\\
& =\int_{S^{2}} \sqrt{q}\left(-\stackrel{0}{N^{a b}}\left(\mathcal{L}_{V}-\alpha\right) \stackrel{0}{S_{a b}^{f}}+\stackrel{0}{N^{a b}} \delta_{V} \stackrel{0}{S_{a b}^{f}}\right)  \tag{D17}\\
& =-\int_{S^{2}} \sqrt{q} \stackrel{0}{S}_{a b}^{V(f)}=-P_{V(f)}^{\mathrm{soft}}, \tag{D18}
\end{align*}
$$

where in the second line we integrated by parts on the sphere and in the last line we used Eq. (4.11).
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[^1]:    ${ }^{1}$ In this paper, we use the word 'charge' as a synonym of canonical generator on the gravitational phase space at null infinity. This differs from other common usage of 'charge' as a boundary term associated to a Cauchy slice that ends at null infinity, see e.g. 6].

[^2]:    ${ }^{2}$ The old and new versions of $J_{V}$ lead to the same Ward identities if evaluated between finite energy states. To see their difference, one needs to evaluate Ward identities on states with zero energy gravitons 19] or study the charge algebra on the $S$ matrix [20]. Both calculations require double soft graviton formulas [21 24] and it is here where one can distinguish the new and old versions of $J_{V}$ [25].

[^3]:    ${ }^{3}$ We follow the parametrization used in [26] with $\left(C_{a b}\right)_{\text {here }}=\left(\hat{C}_{A B}\right)_{\text {there }}$ and $\left(T_{a b}\right)_{\text {here }}=\left(N_{A B}^{\text {vac }}\right)_{\text {there }}$.
    ${ }^{4}$ We thank Biswajit Sahoo for correcting a wrong statement in the first version of the manuscript.

[^4]:    ${ }_{6}^{6}$ The notation is inspired from the role of charges on soft theorems. $\stackrel{0}{N}_{a b}(x)$ and $\stackrel{1}{N}_{a b}(x)$ are the leading and subleading soft modes 46] of the news tensor $N_{a b}(u, x)=\partial_{u} C_{a b}(u, x)$.

[^5]:    ${ }^{7}$ We become aware of identity (2.12) from the expressions of $\delta_{f} N_{a}$ in [42] ( $N_{a}=$ angular momentum aspect). The improved super angular momentum (2.16) corresponds to $J_{V}=-4 \int d u d^{2} x \sqrt{q} V^{a} \partial_{u} N_{a}^{H P S}$ where $N_{a}^{H P S}$ is, modulo the soft piece, the angular momentum aspect as defined in [47]. See [26] for a comparison of the different conventions for $N_{a}$.

[^6]:    ${ }^{8}$ We are deviating from the notation in [26]: $\psi_{\text {here }}=-\Phi_{\text {there }} / 2$.
    ${ }^{9}$ This ambiguity can be fixed by defining $\psi$ with respect to a reference 2 d metric from which all other $q_{a b}$ 's are obtained by finite superrotations. Here we do so by considering a reference unit sphere metric, see appendix A Another natural choice is to consider the Euclidean plane as a reference metric [26].

[^7]:    ${ }^{10}$ In establishing (3.21) one needs to use algebraic identities of 2 d tensors that may not be manifest in an abstract index notation. These identities are easily seen in holomorphic coordinates $(z, \bar{z})$ such that $q_{z z}=q_{\bar{z} \bar{z}}=0$.

[^8]:    ${ }^{11}$ The same comments as those in footnote 10 apply here. In the present case, the identity $D_{[a} \psi D^{c} C_{b] c}-$ $D^{c} \psi D_{[a} C_{b] c}=0$ was used to obtain (3.24).

[^9]:    ${ }^{13}$ Eq. (5.20) can be thought of as a $\delta_{V}$-invariance condition on $\theta^{\cdots} 50$.

[^10]:    ${ }^{14}$ We remind the reader that our statements refer to generators on the phase space associated to the entire null infinity. We are not making any claim about surface charges -as those studied in [8, 26]- associated to finite cuts of null infinity.

[^11]:    ${ }^{15}$ In this appendix $R$ denotes a radial coordinate and $\mathcal{R}$ the scalar curvature of $q_{a b}$. In the rest of paper $R$ is used for the scalar curvature of $q_{a b}$.

[^12]:    ${ }^{16}$ As for similar identities used in the paper, this equivalence is easily seen in holomorphic coordinates.

[^13]:    ${ }^{17}$ Alternatively, in the version of section (II) Eq. (2.12) corresponds to $\left[\delta_{V}{ }^{0} S_{a b}^{f}\right]^{\mathrm{TF}}-f{ }^{1} S_{a b}^{V}=-B_{a b}+M_{a b}$.

