Quantum fluctuating geometries and the information paradox II

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In a previous paper we discussed corrections to Hawking radiation from a collapsing shell due to quantum fluctuations of the shell and the resulting horizon. For the computation of the quantum corrections we used several approximations. In this paper we take into account effects that were neglected in the previous one. We find important corrections including non-thermal contributions to the radiation at high frequencies and a frequency dependent time scale at which the emission of thermal radiation of frequency ω cuts off. Such scale tends to infinity in the limit of a classical shell. The fact that one has almost from the outset non-thermal radiation has significant implications for the information paradox. In particular the amount of non-thermality is considerably larger than what we had estimated before. A naive estimate of the evaporation time leads to a much faster evaporation than in the usual Hawking analysis.

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I. INTRODUCTION

Hawking radiation has been studied for a collapsing shell going all the way back to Boulware in 1976 [1]. Most studies have treated the shell as a classical collapsing object. In a previous paper [2] we have studied the Hawking radiation produced by a collapsing quantum shell. We did it in the geometric optics approximation. In it, one considers ingoing and outgoing light rays and how they relate to each other via the parameters of the shell, namely its ADM mass and the position at past infinity from which the shell is launched. When the ADM mass and the position are turned into quantum operators acting on a Hilbert space for the geometry created by the shell, so do the relations between ingoing and outgoing rays. In the standard geometric optic treatment of Hawking radiation, the relations are used to construct the Bogoliubov coefficients. In our case the latter become quantum operators acting on the Hilbert space of the geometry. We found that the profile of the Hawking radiation as a function of time contains information about the initial state of the collapsing quantum shell. In particular, certain correlations of the Hawking radiation that vanish in the classical case, are non-vanishing in the quantum one. Since the Hawking radiation for a large black hole occurs entirely in a low-curvature region of space-time, our calculation shows that non-trivial quantum effects can occur in such types of regions. Our calculations involved several approximations, which we study in greater detail in the current paper. We find important corrections. The main message is that, even at rather short times, the radiation becomes non-thermal. The emission of the total mass of the black hole is faster than in the case of traditional Hawking radiation. In section 2 we summarize the previous results, showing that they correspond to a naive semi-classical limit that omits certain important quantum effects. Section 3 carries out the full quantum calculation which we characterize in terms of effective c-number Bogoliubov coefficients. In section 4 we compute the number of particles emitted, showing the non-thermal nature of their spectrum and bounds for the total the total thermal energy emitted. We end with a discussion.

II. GENERAL FRAMEWORK

A. Summary of previous results

The computation of Hawking radiation using the geometric optics approximation has a long history going back all the way to Hawking's original 1975 calculation [3]. Boulware was the first to consider the radiation of a collapsing null shell [1]. The metric of a collapsing shell is given by,

$$ds^2 = -\left(1 - \frac{2M\theta(v - v_s)}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2, \tag{1}$$

where v_s represents the position of the shell (in ingoing Eddington–Finkelstein coordinates) and M its mass. We are using units where $G = c = 1$ Its associated Penrose diagram is given in figure 1.

To use the geometric optics approximation one considers light rays that leave I^- with coordinate v less than $v_0 = v_s - 4M$ and escape to I^+ with the rest trapped in the black hole that forms. They reach I^+ with a coordinate,

$$u(v) = v - 4M \ln\left(\frac{v_0 - v}{4M_0}\right), \tag{2}$$

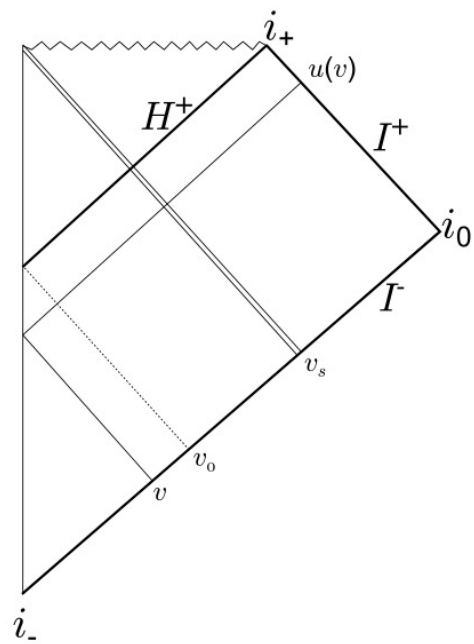


FIG. 1: The Penrose diagram of collapsing shell. v_s is the position at I^- from which the shell is sent in. Light rays sent in to the left of v_0 make it to I^+, and rays sent in to the right of v_0 get trapped in the black hole.

where M_0 is an arbitrary parameter that is usually chosen as $M_0 = M$. In our case, since we are considering a quantum black hole we will take M_0 to be the mean value of the mass. M_0 is related with the definition of the tortoise coordinate u, which involves a constant of integration. One uses this identity to relate the "in" modes of the scalar field at I^-,

$$\psi_{lm\omega'}(r, v, \theta, \phi) = \frac{e^{-i\omega' v}}{4\pi r\sqrt{\omega'}} Y_{lm}(\theta, \phi),$$

with the "out" modes at I^+,

$$\chi_{lm\omega}(r, u, \theta, \phi) = \frac{e^{-i\omega u}}{4\pi r\sqrt{\omega}} Y_{lm}(\theta, \phi),$$

to compute the Bogoliubov coefficients,

$$\alpha_{\omega\omega'} = \langle \chi_{lm\omega}, \psi_{lm\omega'} \rangle ,$$

$$\beta_{\omega\omega'} = - \langle \chi_{lm\omega}, \psi^*_{lm\omega'} \rangle .$$

That was the summary of the calculation of Hawking radiation on the background of a classical collapsing shell in the geometric optics approximation. To consider the case of quantum collapsing shells we recall that the ADM mass M and the position at I^- from which the shell is sent in are a complete set of Dirac observables and canonically conjugate to each other [2, 4]. One can promote them to quantum operators with commutators,

$$\left[\widehat{M}, \widehat{v}_s\right] = i\hbar\widehat{I}, \tag{3}$$

with $\widehat{I}$ the identity operator. It is actually more convenient to use $v_0 = v_s - 4M$ instead of v_s. In terms of the quantum operators one can write a quantum operatorial relationship between the operator associated with the ingoing position in I^- of a light ray and the outgoing position at I^+,

$$\hat{u}\left(v, \widehat{v}_0, \widehat{M}\right) = v\widehat{I} - 2\left[\widehat{M}\ln\left(\frac{\widehat{v}_0 - v\widehat{I}}{4M_0}\right) + \ln\left(\frac{\widehat{v}_0 - v\widehat{I}}{4M_0}\right)\widehat{M}\right]. \tag{4}$$

The operators act on states of the quantum geometry, which in the mass representation are given by $\varphi(M)$. With the above operators one can now promote the Bogoliubov coefficients to operators acting on the states of the quantum geometry,

$$\hat{\beta}_{\omega\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\delta\to 0}\int_{-\infty}^{+\infty} dv\theta\left(\widehat{v}_0 - v\widehat{I}\right)\exp\left(-i\omega\hat{u}_\delta(v) - i\omega' v\right)\theta\left(\widehat{v}_0 - v\widehat{I}\right). \tag{5}$$

The above computation requires the extension of the operator $\hat{u}$ to include all the range of rays that start at I^- including those that would fall into the black hole. Details can be seen in our previous paper [2]. The result is the operator $\hat{u}_\delta$. We need to solve its eigenvalue problem. It turns out that the spectrum of $\hat{u}_\delta$ is degenerate with degeneracy two. This leads us to choose two independent eigenstates of $\hat{u}$,

$$\psi_u^1(x) = \begin{cases} \frac{1}{\sqrt{8\pi\hbar|\ln(\delta)|}}\exp\left(\frac{iM_0}{\hbar}(u-v)\frac{x-\delta}{\ln(\delta)}\right), & x<\delta \\ \frac{1}{\sqrt{8\pi\hbar|\ln(x)|}}\exp\left(\frac{iM_0}{\hbar}(u-v)\left[\mathrm{li}\,(x)-\mathrm{li}\,(\delta)\right]\right), & \delta\le x<1 \\ 0, \quad x\ge 1 \end{cases} \tag{6}$$

$$\psi_u^2(x) = \begin{cases} 0, \quad x\le 1 \\ \frac{1}{\sqrt{8\pi\hbar|\ln(x)|}}\exp\left(\frac{iM_0}{\hbar}(u-v)\left[\mathrm{li}\,(x)-\mathrm{li}\,(\delta)\right]\right), & x>1 \end{cases} \tag{7}$$

where $x = (v_0 - v)/(4M_0)$ and we have chosen them as orthonormal. We adopt the notation $|u, J\rangle_\delta$ with $J = 1, 2$ for these states.

With this we can compute the expectation values of the Bogoliubov coefficients for different states of the quantum geometry,

$$\left\langle\hat{\beta}\right\rangle_{\omega\omega'} = -\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\delta\to 0}\langle\Phi|\int_{-\infty}^{+\infty} dv\int_{-\infty}^{+\infty} dv_0\,|v_0\rangle\langle v_0|\,\theta\left(\widehat{v}_0 - v\widehat{I}\right)e^{-i\omega\hat{u}_\delta(v)-i\omega' v}\times$$

$$\times\sum_{J=1,2}\int_{-\infty}^{+\infty} du\,|u,J\rangle_\delta\langle u,J|_\delta\int_{-\infty}^{+\infty} dv_0'\,|v_0'\rangle\langle v_0'|\,\theta\left(\widehat{v}_0 - v\widehat{I}\right)|\Phi\rangle.$$

We also found an expression for the expectation value of the density matrix of a scalar field on the background of a quantum shell,

$$\left\langle\rho^{QS}_{\omega_1\omega_2}\right\rangle \equiv \langle\Phi|\langle 0_{in}|\,\hat{a}^\dagger_{\omega_1}(\hat{M},\hat{v}_0)\hat{a}_{\omega_2}(\hat{M},\hat{v}_0)\,|0_{in}\rangle\,|\Phi\rangle = \int_0^\infty d\omega'\left\langle\hat{\beta}_{\omega_1\omega'}\hat{\beta}^*_{\omega_2\omega'}\right\rangle,$$

where $|\Phi\rangle$ is the quantum state of the shell centered in given values $\bar{M}, \bar{v_0}$ of the ADM mass and the position along I^- of the last light ray that escapes to I^+ (as we mentioned, it is equivalent to use this quantity in lieu of the position of the shell at I^-). In the above expression $\hat{a}, \hat{a}^\dagger$ are the annihilation and creation operators of the quantum field of the Hawking radiation (we take it to be a scalar field for simplicity) and $|0_{in}\rangle$ is the vacuum of the Hawking radiation.

We take a state for the shell that in the v_0 representation can be written as,

$$\psi\left(v_0\right) \equiv \langle v_0|\,\varphi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dM\varphi\left(M\right)\exp\left(-i\frac{M\left[v_0-\bar{v}_0\right]}{\hbar}\right) \tag{8}$$

with $\varphi(M)$ a complex function centered in $\bar{M}$ that satisfies $\int dM\left|\varphi\left(M\right)\right|^2 = 1$.

The result found in our previous paper [2] (equation above (39)) for the expectation value density matrix of the Hawking radiation for a quantum shell is,

$$\left\langle\rho^{QS}_{\omega_1\omega_2}\right\rangle = \frac{\left(2M_0\right)^2}{\pi^2}\int_0^\infty d\omega'\frac{\omega'}{\sqrt{\omega_1\omega_2}}\int_0^{+\infty} dxdx'\exp\left(-i4M_0\left[\omega'+\omega_2\right]x'\right)\exp\left(i4M_0\left[\omega'+\omega_1\right]x\right)\times$$

$$\times\exp\left(i4M_0\left[\omega'+\bar{\omega}\right]\Delta_{\omega_1\omega_2}(x,x')\right)\sqrt{\frac{\left|\ln(\bar{x}_{\omega_1}(x))\right|\left|\ln(\bar{x}_{\omega_2}(x'))\right|}{\left|\ln(x)\right|\left|\ln(x')\right|}}$$

$$\times\int_{-\infty}^{+\infty} ds e^{i\Delta\omega s}\psi^*(s-2M_0\Delta_{\omega_1\omega_2}(x,x'))\psi(s+2M_0\Delta_{\omega_1\omega_2}(x,x')),$$

where $\Delta_{\omega_1\omega_2}(x,x')=\Delta_{\omega_2}(x')-\Delta_{\omega_1}(x)$, $\Delta_\omega(x)=x-\bar{x}_\omega(x)=x-\mathrm{li}^{-1}\left(\mathrm{li}(x)-\delta_\omega\right)$, $\bar{x}_{\omega_i}(x)=\mathrm{li}^{-1}\left[\mathrm{li}(x)-\omega_i\hbar/M_0\right]$, li is the logarithmic integral, $\delta_\omega=\frac{\hbar\omega}{M_0}$, $\Delta\omega=\omega_2-\omega_1$ and $\bar{\omega}=\frac{\omega_1+\omega_2}{2}$.

This can be rewritten as,

$$\left\langle\rho^{QS}_{\omega_1\omega_2}\right\rangle=\int dM\varphi^*\left(M-\frac{\Delta\omega\hbar}{2}\right)\varphi\left(M+\frac{\Delta\omega\hbar}{2}\right)\int_0^\infty d\omega'\beta^{QS}_{\omega_1\omega'}(M,\bar{\omega})\left[\beta^{QS}_{\omega_2\omega'}(M,\bar{\omega})\right]^*, \tag{9}$$

in terms of the c-number quantity,

$$\begin{aligned}\beta^{QS}_{\omega\omega'}(M,\bar{\omega}) &= -\frac{2M_0e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\\ &\times\lim_{\epsilon\to0}\int_0^{+\infty}dx\exp\left(-\epsilon x\right)\exp\left(i4M_0\left[\omega'+\omega\right]x\right)\sqrt{\frac{|\ln(\bar{x}_\omega(x))|}{|\ln(x)|}}\exp\left(i4\omega m(\omega',\bar{\omega})\frac{\Delta_\omega(x)}{\delta_\omega}\right),\end{aligned} \tag{10}$$

that plays the role of an effective Bogoliubov coefficient, with $m(\omega',\bar{\omega})=M-(\omega'+\bar{\omega})\hbar$. The regulator ϵ is introduced in order to make the integral convergent since we are in a basis of plane waves.

B. Semiclassical approximation: naive version of corrections to the limit $\hbar\to0$

Expression (9) has the complete information of the geometric optics approximation for Hawking radiation on the background of a quantum shell and should therefore include the usual results for Hawking radiation from a classical collapsing shell when $\hbar\to0$ (the shell variables become classical but the radiation is kept quantum, otherwise it vanishes). To take such a limit is to set $\hbar\to0$ in the integrand of (10).One gets

$$\beta^{QS}_{\omega\omega'}(M,\bar{\omega})\to\beta^{CS}_{\omega\omega'}(M)=-\frac{2M_0e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to0}\int_0^{+\infty}dxe^{-\epsilon x}\exp\left(i4M_0\left(\omega'+\omega\right)x\right)\exp\left(i4M\omega\ln(x)\right)$$

that agrees with the standard result for Hawking radiation for classical shells (CS).

In our previous paper [2] we considered an approximation in which we kept the states of the quantum geometry but took the limit $\hbar\to0$ in the Bogoliubov coefficients in the manner discussed. We will call this approximation the "naive limit". We did this in the expectation that the corrections this approximation neglected were small. We will see in this paper that they are not.

To evaluate the expectation value of the density matrix in the naive limit, we start with the change of variable $y=\ln(x)$. The above expression becomes

$$\begin{aligned}\beta^{CS}_{\omega\omega'}(M) &= -\frac{2M_0e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to0}\int_{-\infty}^{+\infty}dy\exp\left(-\left[\epsilon-i4M_0\left(\omega'+\omega\right)\right]e^y\right)\exp\left((1+i4M\omega)y\right)\\ &= -\frac{2M_0e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to0}\left[\int_0^{+\infty}dy\exp\left(-\left[\epsilon-i4M_0\left(\omega'+\omega\right)\right]e^{-y}\right)\exp\left(-(1+i4M\omega)y\right)\right.\\ &\left.+\int_0^{+\infty}dy\exp\left(-\left[\epsilon-i4M_0\left(\omega'+\omega\right)\right]e^y\right)\exp\left((1+i4M\omega)y\right)\right].\end{aligned} \tag{11}$$

Using the identities for the upper and lower incomplete Gamma functions,

$$\Gamma(a,z)=z^a\int_0^{+\infty}\exp\left(at-ze^t\right)dt,\quad \mathrm{Re(z)}>0, \tag{12}$$

$$\gamma(a,z)=z^a\int_0^{+\infty}\exp\left(-at-ze^{-t}\right)dt,\quad \mathrm{Re(a)}>0, \tag{13}$$

$$\Gamma(a)=\gamma(a,z)+\Gamma(a,z).$$

we get

$$\begin{aligned}\beta^{CS}_{\omega\omega'}(M) &= -\frac{2M_0 e^{-i[\omega+\omega']\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to 0}\exp\left(-(1+i4M\omega)\ln\left[\epsilon - i4M_0\left(\omega'+\omega\right)\right]\right)\right) \\ &\quad\times\left[\gamma\left(1+i4M\omega,\epsilon-i4M_0\left[\omega'+\omega\right]\right)+\Gamma\left(1+i4M\omega,\epsilon-i4M_0\left[\omega'+\omega\right]\right)\right] \\ &= \frac{-i\exp\left(-i\left[\omega+\omega'\right]\bar{v}_0\right)e^{-2M\omega\pi}}{2\pi\sqrt{\omega}}\frac{\omega'}{\omega'+\omega}\frac{\exp\left(-i4M\omega\ln\left(4M_0\left[\omega'+\omega\right]\right)\right)}{\sqrt{\omega'}}\Gamma\left(1+i4M\omega\right). \end{aligned} \tag{14}$$

This is the expression of the Bogoliubov coefficient β that we obtained in the previous paper and that includes Hawking radiation in the long time limit but includes non-thermal corrections for early times, as one expects for the radiation of a collapsing shell. Substituting in (9) we get

$$\begin{aligned}\left\langle\rho^{QS}_{\omega_1\omega_2}\right\rangle_{\text{Naive limit}} &= \int dM\varphi^*\left(M-\frac{\Delta\omega\hbar}{2}\right)\varphi\left(M+\frac{\Delta\omega\hbar}{2}\right)\frac{8M^2\sqrt{\omega_1\omega_2}\exp\left(i\Delta\omega\bar{v}_0\right)}{\pi e^{4M\bar{\omega}\pi}}\Gamma\left(i4M\omega_1\right)\Gamma\left(-i4M\omega_2\right) \\ &\quad\times\frac{1}{2\pi}\int_0^{+\infty}\frac{d\omega'}{\omega'}\frac{(\omega')^2}{(\omega'+\omega_1)(\omega'+\omega_2)}\exp\left(i4M\left[\omega_2\ln\left(4M_0\left[\omega'+\omega_2\right]\right)-\omega_1\ln\left(4M_0\left[\omega'+\omega_1\right]\right)\right]\right) \end{aligned} \tag{15}$$

Notice that we have taken $\beta_{\omega\omega'}$ to be classical but kept the $\hbar$ dependence in the quantum states. This was the approximation we used in our previous paper and only partially captures the departures from thermality of the distribution of radiated energies. In this paper we will develop a better approximation and we will see significantly different behavior.

C. Computation of the radiated energy in the naive limit

In order to compare with the result we will obtain in this paper we need an expression for the amount of energy radiated as Hawking radiation in this naive limit. The radiated energy can be obtained from the diagonal terms of (15), that is, the number of particles per unit frequency. From there, the time of evaporation of the black hole, assuming the radiation maintains the same form (i.e. ignoring backreaction) can be estimated.

To compute the integral, we start by rewriting the Gamma function,

$$\Gamma(ix) = \sqrt{\frac{\pi}{x\sinh(\pi x)}}e^{i\phi(x)}, \tag{16}$$

with $\phi(x) = \text{Im}\left[\ln\left(\Gamma(ix)\right)\right]$ and carry out the change of variable,

$$t = \frac{4M}{\Delta\omega}\left[\omega_2\ln\left(4M_0\left[\omega'+\omega_2\right]\right)-\omega_1\ln\left(4M_0\left[\omega'+\omega_1\right]\right)\right], \tag{17}$$

and recalling that $x = 4M\omega_1$ or $x = -4M\omega_2$, this leads to,

$$\left\langle\rho^{QS}_{\omega_1\omega_2}\right\rangle_{\text{Naive limit}} = \int dM\varphi^*\left(M-\frac{\Delta\omega\hbar}{2}\right)\varphi\left(M+\frac{\Delta\omega\hbar}{2}\right)\frac{e^{i[\phi(4M\omega_1)-\phi(4M\omega_2)]}e^{i\Delta\omega\bar{v}_0}}{\sqrt{e^{8M\omega_1\pi}-1}\sqrt{e^{8M\omega_2\pi}-1}}\frac{1}{2\pi}\int_{t_0}^{+\infty}dte^{it\Delta\omega}, \tag{18}$$

with $t_0 = 4M\left[\frac{\bar{\omega}}{\Delta\omega}\ln\left(\frac{\omega_2}{\omega_1}\right)+\ln\left(4M_0\sqrt{\omega_1\omega_2}\right)\right]$. This density matrix is a distribution in $\omega_1-\omega_2$ whose diagonal yields a divergent term proportional to the number of particles. The divergence stems from assuming a basis of waves of definite frequency for the scalar field. As we will see later, the result can be made finite considering wavepackets with a finite spread in frequency and time. Computing the diagonal terms we have,

$$\left\langle N^{QS}_{\omega}\right\rangle_{\text{Naive limit}} = \left\langle\rho^{QS}_{\omega,\omega}\right\rangle_{\text{Naive limit}} = \int dM\left|\varphi\left(M\right)\right|^2\frac{1}{e^{8M\omega\pi}-1}\frac{1}{2\pi}\int_{\hat{t}_0}^{+\infty}dt, \tag{19}$$

with $\hat{t}_0 = 4M\left[1+\ln\left(4M_0\omega\right)\right]$. The divergence in $\omega_1 = \omega_2 = \omega$ appears because we have computed the Bogoliubov coefficients for a continuous basis of plane waves at I^+,

$$\left\{\phi_\omega(u) = \frac{1}{\sqrt{2\pi\omega}}, e^{-iu\omega}, \omega > 0\right\},$$

and therefore we have considered emission for all time. Formally, from here we can compute the total energy emitted as

$$E_{\text{Naive limit}} = \int_0^\infty d\omega\hbar\omega \left\langle N_\omega^{QS} \right\rangle_{\text{Naive limit}}, \tag{20}$$

and of course this will give an infinite result. We should compare this limit, where we consider fluctuations in the quantum states, with the ordinary Hawking radiation calculation, where only the energies of the particles emitted are quantized. Alternatively we can compute a bounded density matrix using a discrete basis of wavepackets,

$$\left\{ \Phi_{\omega_j}(u_n) = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} d\omega \phi_\omega(u) e^{iu_n\omega}, j \in \mathbb{N}, n \in \mathbb{Z} \right\},$$

centered around time ($u_n = n\frac{2\pi}{\varepsilon}$) and frequency ($\omega_j = [j + 1/2]\,\varepsilon$), with $\varepsilon << \omega_j$ a narrow frequency window. With this, the density matrix for the wavepackets becomes,

$$\left\langle \rho_{\omega_j,\omega_k}^{QS} \right\rangle_{\text{Naive limit}} (u_n) = \frac{1}{\varepsilon} \int_{j\varepsilon}^{(j+1)\varepsilon} \int_{k\varepsilon}^{(k+1)\varepsilon} d\omega_1 d\omega_2 e^{-iu_n\Delta\omega} \left\langle \rho_{\omega_1\omega_2}^{QS} \right\rangle_{\text{Naive limit}} \tag{21}$$

and the rate of emission of particles is,

$$\left\langle N_{\omega_j}^{QS} \right\rangle_{\text{Naive limit}} (u_n) = \left\langle \rho_{\omega_j,\omega_j}^{QS} \right\rangle_{\text{Naive limit}} (u_n).$$

Finally this leads to $E_{\text{Naive limit}}(u_n)$, the power emitted at time u_n, through integral (20). This way of computing the energy allows us to deal with finite quantities and also shows us the role of the frequency ω' which appears in the calculation of the Bogoliubov coefficients and determines their evolution in terms of the physical time (u_n) at I.

To do the explicit calculation we need an expression for (18) that we can handle when $\omega_1 \sim \omega_2$. Approximating the integrand to the lowest order in $\Delta\omega/\bar{\omega}$ both in amplitude and in phase in all the factors involved, we get [5],

$$\left\langle \rho_{\omega_1\omega_2}^{QS} \right\rangle_{\text{Naive limit}} \sim \int dM \left|\varphi(M)\right|^2 e^{i\sigma'(M)\Delta\omega\hbar} \frac{e^{-i4M\Delta\omega\Re\left[\psi^{(0)}(i4M\bar{\omega})\right]} e^{i\Delta\omega\bar{v}_0}}{e^{8M\bar{\omega}\pi} - 1} \frac{1}{2\pi} \int_{\tilde{t}_0}^{+\infty} dt e^{it\Delta\omega}, \tag{22}$$

where the change of variable variable (17) from ω' to t becomes

$$t = 4M\left[1 + \ln\left(4M_0\left[\omega' + \bar{\omega}\right]\right)\right] \tag{23}$$

and therefore $\tilde{t}_0 = 4M\left[1 + \ln\left(4M_0\bar{\omega}\right)\right]$.

In the previous expression we have expanded the phase of the function Γ of equation (16) as,

$$\phi(x) = \sum_{n=0}^{\infty} \phi^{(n)}(x_0) \frac{(x - x_0)^n}{n!},$$

where $\phi^{(n)} = \Im\left[i^n \psi^{(n-1)}(ix)\right]$ and $\psi^{(n)}$ is the polygamma function of order n. We have also assumed that the wavefunction in the mass representation takes the form,

$$\varphi(M) = \left|\varphi(M)\right| e^{i\sigma(M)},$$

with σ a smooth function and we have expanded φ assuming that $M >> \hbar\Delta\omega$. It is convenient to reorder the expression (22) in the following way,

$$\left\langle \rho_{\omega_1\omega_2}^{QS} \right\rangle_{\text{Naive limit}} \sim \int dM \frac{\left|\varphi(M)\right|^2}{e^{8M\bar{\omega}\pi} - 1} \frac{1}{2\pi} \int_{T_0(\bar{\omega})}^{+\infty} dt e^{i\Delta\omega t},$$

absorbing the phase in t such that,

$$t \to t + \sigma'(M)\hbar - 4M\Re\left[\psi^{(0)}(i4M\bar{\omega})\right] + \bar{v}_0, \tag{24}$$

and therefore,

$$T_0(\bar{\omega}) = \sigma'(M)\hbar - 4M\Re\left[\psi^{(0)}(i4M\bar{\omega})\right] + \bar{v}_0 + 4M + 4M\ln\left(4M_0\bar{\omega}\right). \tag{25}$$

We are now in position to incorporate the wavepackets by computing,

$$\langle N^{QS}_{\omega_j}\rangle_{\text{Naive limit}}(u_n) = \frac{1}{\epsilon}\int\int_{j\epsilon}^{(j+1)\epsilon} d\omega_1 d\omega_2 e^{-u_n\Delta\omega i}\left\langle \rho^{QS}_{\omega_1\omega_2}\right\rangle_{\text{Naive limit}} \sim$$

$$\sim \int dM \frac{|\varphi(M)|^2}{e^{8M\omega_j\pi}-1}\frac{1}{2\pi\epsilon}\int_{-\epsilon}^{\epsilon} d\left(\Delta\omega\right)\int_{\omega_j-\frac{\epsilon-|\Delta\omega|}{2}}^{\omega_j+\frac{\epsilon-|\Delta\omega|}{2}}\int_{T_0(\bar{\omega})}^{+\infty} dt e^{-(u_n-t)\Delta\omega i} d\bar{\omega} \sim$$

$$\sim \int dM \frac{|\varphi(M)|^2}{e^{8M\omega_j\pi}-1}\frac{1}{2\pi}\int_{T_0(\omega_j)}^{+\infty} dt \int_{-\epsilon}^{\epsilon} d\left(\Delta\omega\right)\frac{\epsilon-|\Delta\omega|}{\epsilon} e^{-(u_n-t)\Delta\omega i} =$$

$$= \int dM \frac{|\varphi(M)|^2}{e^{8M\omega_j\pi}-1}\frac{\epsilon}{2\pi}\int_{T_0(\omega_j)}^{+\infty} dt\, \text{sinc}^2\left[\frac{\epsilon}{2}(u_n-t)\right] \equiv \int dM \frac{|\varphi(M)|^2}{e^{8M\omega_j\pi}-1} S\left[u_n - T_0(\omega_j)\right], \tag{26}$$

where $\text{sinc}(x) = \sin(x)/x$ is the cardinal sine function and integrates to a function S which is a smooth version of the Heaviside function. This expression represents a superposition of thermal radiation that starts at time $u_n = T_0(\omega_j)$ and continues to be emitted for later times. The expression for the emitted power is

$$E_{\text{Naive limit}}(u_n) \sim \int dM \left|\varphi(M)\right|^2 \int_0^{\infty} d\omega \frac{\hbar\omega}{e^{8M\omega\pi}-1} S\left[u_n - T_0(\omega)\right]. \tag{27}$$

Although this is a computation for a quantum shell, by choosing a state with small uncertainty in the mass (and taking the naive limit $\hbar \to 0$ in the Bogoliubov coefficients), we are effectively obtaining the classical limit and therefore the final result coincides with the usual one quoted for ordinary Hawking radiation for a classical collapsing shell.

III. CORRECTIONS TO THE BOGOLIUBOV COEFFICIENTS

To develop a better approximation, we will evaluate $\beta^{QS}_{\omega\omega'}$ without taking the limit $\hbar \to 0$ in the integral (10). The latter expression depends on $\hbar$ through $\Delta_\omega(x)$, δ_ω and $m(\omega', \bar{\omega})$. We will show that the approximation described in the previous section fails to capture important properties of the radiation, in particular its non-thermal aspects. To see this, it is convenient to examine the region $x = 0$ of the integral making the change of variable $y = \ln(x)$. Given that the dependence in $\hbar$ is in the function $\bar{x}_\omega(x)$ we redefine,

$$\ln\left[\bar{x}_\omega(x)\right] = \ln\left(\text{li}^{-1}(\text{li}(x) - \delta_\omega)\right) = \text{Ei}^{-1}\left(\text{Ei}(y) - \delta_\omega\right) \equiv y_\omega(y), \tag{28}$$

with Ei the exponential integral. In figure 2 we show this function and the approximations to it we will later use. Notice that $y_\omega[y] \to y$ for large values of y. The function $y_\omega(y)$ involves the exponential integral and is not straightforward to integrate. This will require the use of approximations.

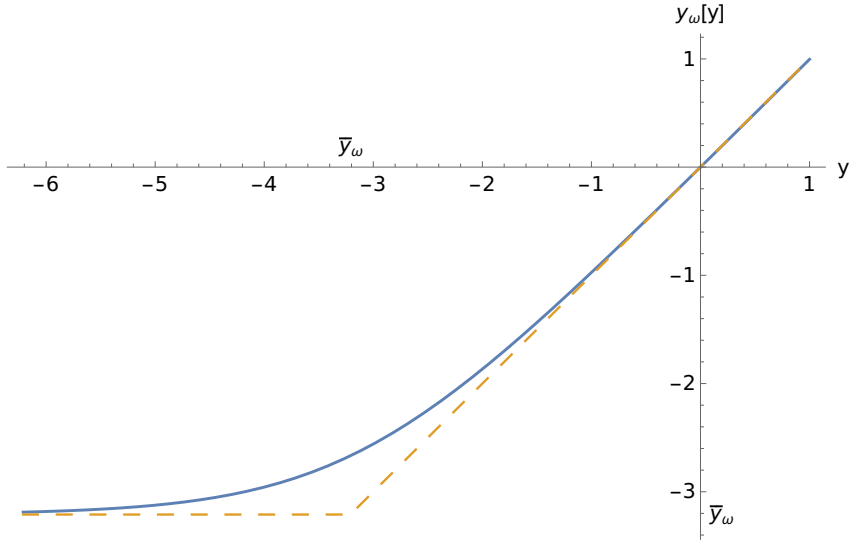
The effective Bogoliubov coefficients (10) of the quantum shell can be rewritten in terms of y as,

$$\begin{aligned}\beta^{QS}_{\omega\omega'}(M,\bar{\omega}) \;=\; & -\frac{2M_0 e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty} dy \exp\left(y - \epsilon e^y\right)\exp\left(i4M_0\left[\omega'+\omega\right]e^y\right) \\ & \times\sqrt{\frac{|y_\omega(y)|}{|y|}}\exp\left(i4\omega m(\omega',\bar{\omega})\frac{\exp(y)-\exp(y_\omega(y))}{\delta_\omega}\right).\end{aligned} \tag{29}$$

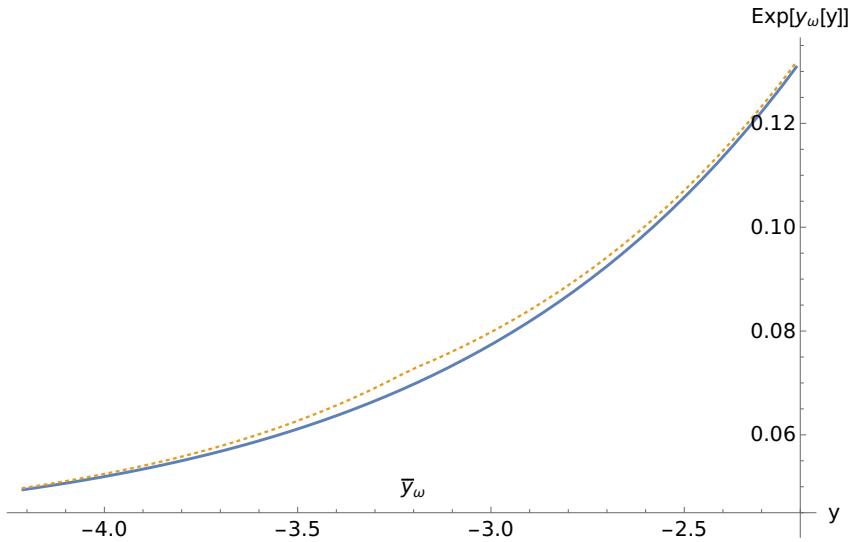
As we see in figure (2a) there exists a region of the integral where it is incorrect to take the limit $\delta_\omega \to 0$ (which is equivalent to $y_\omega = y$) in order to approximate $y_\omega(y)$ for small values of x, which corresponds to negative values of y. In particular for large negative values $y_\omega(y)$ becomes constant. As figure (2a) shows, such constant

$$\bar{y}_\omega \equiv y_\omega(-\infty) = \text{Ei}^{-1}(-\delta_\omega),$$

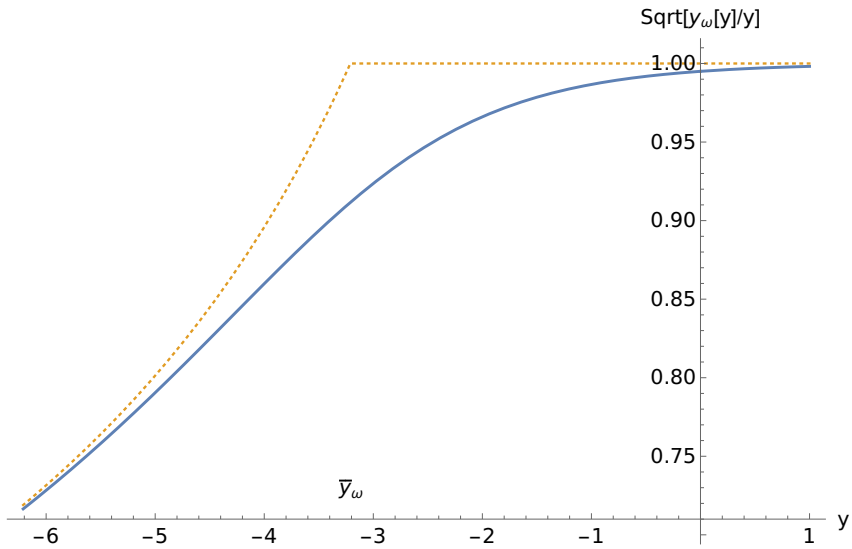
is a good indicator of the value of y where such departure takes place.



(a) Function $y_\omega(y)$ and its asymptotes.



(b) Approximation for $\exp(y_\omega(y))$ (dotted line) and the exact value.



(c) Approximation for $\sqrt{\left|\frac{y_\omega(y)}{y}\right|}$ (dotted line) and the exact value.

FIG. 2: Approximations to the function $y_\omega(y)$ and derived functions including the regions at both sides of $y = \bar{y}_\omega$. Plots correspond to $\delta_\omega = 10^{-2}$ in order to have visible departures from the exact functions.

A. Asymptotic Approximation

To take into account these two zones in the calculation of the integral we can approximate the function (28) using its asymptotic forms for $y \ll \bar{y}_\omega$ and $y \gg \bar{y}_\omega$. When $y > \bar{y}_\omega$ we have that $|\mathrm{Ei}(y)| > \delta_\omega$ so we consider the approximations,

$$\exp\left[y_\omega(y > \bar{y}_\omega)\right] \approx \exp[y] - y\delta_\omega, \tag{30}$$

$$\sqrt{\frac{|y_\omega(y > \bar{y}_\omega)|}{|y|}} \approx 1, \tag{31}$$

and we will explain why we keep an order less in the square root shortly. This can be obtained expanding (28) around $\mathrm{Ei}(y)$. Analogously, when $y < \bar{y}_\omega$, $|\mathrm{Ei}(y)| < \delta_\omega$ and we approximate,

$$\exp\left[y_\omega(y < \bar{y}_\omega)\right] \approx \exp[\bar{y}_\omega] + \bar{y}_\omega \mathrm{Ei}(y), \tag{32}$$

$$\sqrt{\frac{|y_\omega(y < \bar{y}_\omega)|}{|y|}} \approx \sqrt{\frac{|\bar{y}_\omega|}{|y|}}. \tag{33}$$

As we show in figure (2), the two approximations considered coincide when $y = \bar{y}_\omega$.

These approximations will be used to represent the phase and modulus in the integral (29). The integral is more sensitive to the phase, that is why we keep an additional order of approximation in the exponential with respect to the one taken in the square roots.

The point where both approximations agree is also where they give their worst result so is crucial to place a bound on the error introduced in that region and to find conditions such that the error is small. Studying the phase and modulus in (29) and comparing the approximations with the exact values, the conditions are,

$$\frac{4M_0}{\hbar}\left|m(\omega',\omega)\right|\left|\exp[\bar{y}_\omega] - \bar{y}_\omega\delta_\omega - \exp\left[\bar{y}_{2\omega}\right]\right| \ll 1, \tag{34}$$

$$\frac{\sqrt{|\bar{y}_{2\omega}|} - \sqrt{|\bar{y}_\omega|}}{\sqrt{|\bar{y}_{2\omega}|}} \ll 1, \tag{35}$$

where $\bar{y}_{2\omega} = y_\omega(y = \bar{y}_\omega) = \mathrm{Ei}\left(-2\delta_\omega\right)$. The first condition is imposed because the approximation appears in the phase of the integrand and the second because the approximation appears in its modulus. It is important to notice that the first condition imposes limits for the range of ω' and the other one in the range of ω for which the approximation is valid. We will now use these expressions to compute $\beta^{QS}_{\omega\omega'}$.

B. Approximate computation of the effective Bogoliubov coefficients

Considering approximations (30, 31, 32 and 33) we get the following expression for the effective Bogoliubov coefficients by breaking up the integral into the two regions involved,

$$\beta^{QS}_{\omega\omega'}(M,\bar{\omega}) \equiv \beta^{QS(+)}_{\omega\omega'}(M,\bar{\omega}) + \beta^{QS(-)}_{\omega\omega'}(M,\bar{\omega}),$$

with

$$\beta^{QS(+)}_{\omega\omega'}(M,\bar{\omega}) \sim -\frac{2M_0 e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to 0}\int_{\bar{y}_\omega}^{+\infty} dy \exp\left(-(\epsilon - i4M_0\left[\omega'+\omega\right])e^y\right)\exp\left(\left[1 + i4\omega m(\omega',\bar{\omega})\right]y\right), \tag{36}$$

for the region $y > \bar{y}_\omega$ and,

$$\begin{aligned}\beta^{QS(-)}_{\omega\omega'}(M,\bar{\omega}) \;\sim\; & -\frac{2M_0 e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{\bar{y}_\omega} dy \exp\left(y\right)\exp\left(i4M_0\left[\omega'+\omega\right]e^y\right) \\ & \times\sqrt{\frac{|\bar{y}_\omega|}{|y|}}\exp\left(i4\omega m(\omega',\bar{\omega})\frac{e^y - e^{\bar{y}_\omega} - \bar{y}_\omega \mathrm{Ei}(y)}{\delta_\omega}\right),\end{aligned} \tag{37}$$

for the region $y < \bar{y}_\omega$. We will now study these expressions assuming $\delta_\omega \ll 1$. The latter is essentially the energy of the emitted particle divided by the mass of the black hole, so it is very well satisfied. We also need to recall that these expressions are valid only when conditions (34) and (35) are met.

1. Study of $\beta^{QS(+)}_{\omega\omega'}$

We start by computing $\beta^{QS(+)}_{\omega\omega'}$. The integral in (36) can be computed with the change of variable $t = y - \bar{y}_\omega$. Thus,

$$\begin{aligned}\beta^{QS(+)}_{\omega\omega'}(M,\bar{\omega}) &= -\frac{2M_0 \exp\left(-i\left[\omega+\omega'\right]\bar{v}_0\right)}{\pi}\sqrt{\frac{\omega'}{\omega}}\lim_{\epsilon\to 0}\exp\left(\left[1+i4\omega m(\omega',\bar{\omega})\right]\bar{y}_\omega\right)\\ &\quad\times\int_0^{+\infty} dt \exp\left(-(\epsilon - i4M_0\left[\omega'+\omega\right])e^{\bar{y}_\omega}e^t\right)\exp\left(\left[1+i4\omega m(\omega',\bar{\omega})\right]t\right), \end{aligned} \tag{38}$$

and carrying out the integral in t,

$$\begin{aligned}\beta^{QS(+)}_{\omega\omega'}(M,\bar{\omega}) &= -\frac{i\exp\left(-i\left[\omega+\omega'\right]\bar{v}_0\right)}{2\pi}\frac{\exp\left(-2m(\omega',\bar{\omega})\omega\pi\right)}{\sqrt{\omega'\omega}}\frac{\omega'}{\omega'+\omega}\exp\left(-i4\omega m(\omega',\bar{\omega})\ln\left(4M_0\left[\omega'+\omega\right]\right)\right)\\ &\quad\times\Gamma\left[1+i4\omega m(\omega',\bar{\omega}), -i\xi(\omega,\omega',M_0)\right] \end{aligned} \tag{39}$$

where $\Gamma(a,z)$ is the incomplete Gamma function and where,

$$\xi(\omega,\omega',M_0) \equiv 4M_0\left[\omega'+\omega\right]\exp\left(\bar{y}_\omega\right).$$

Expression (39) reduces to the classical expression (14) when $m(\omega',\bar{\omega}) \to M$ and $\xi \to 0$ (equivalent to $\hbar \to 0$, which implies $\bar{y}_\omega \to -\infty$), but outside this regime, it has a very different behavior as can be seen in figure (3), particularly for large values of ω'/ω_0.

Notice that in these plots the pre-factor

$$-\frac{i\exp\left(-i\left[\omega+\omega'\right]\bar{v}_0\right)}{2\pi}\sqrt{\frac{\omega'}{\omega}}\frac{1}{\omega'+\omega} \tag{40}$$

is omitted and instead of ω' the plot is made against the dimensionless variable $\log\left(\frac{\omega'+\omega}{\omega_0}\right)$ with $\omega_0 \sim 2.82144/(8\pi M_{BH})$ the principal frequency, at which the peak of Hawking emission occurs. The reasons for the choice of the logarithm will be apparent in the next section.

2. Study of $\beta^{QS(-)}_{\omega\omega'}$

Let us now concentrate on $\beta^{QS(-)}_{\omega\omega'}$ given by (37). In this case we do not know how to compute the integral in closed form. However, unlike $\beta^{QS(+)}_{\omega\omega'}$ this contribution is an integral that converges very fast (due to the real exponentials in the integrand). The change of variable $t = \sqrt{|y|}$ makes it very explicit,

$$\begin{aligned}\beta^{QS(-)}_{\omega\omega'}(M,\bar{\omega}) &\sim -\frac{2M_0 e^{-i\left[\omega+\omega'\right]\bar{v}_0}}{\pi}\sqrt{\frac{\omega'}{\omega}}\int_{\sqrt{|\bar{y}_\omega|}}^{+\infty} 2dt\exp\left(-t^2\right)\exp\left(i4M_0\left[\omega'+\omega\right]e^{-t^2}\right)\\ &\quad\times\sqrt{|\bar{y}_\omega|}\exp\left(i4\omega m(\omega',\bar{\omega})\frac{e^{-t^2}-e^{\bar{y}_\omega}-\bar{y}_\omega \mathrm{Ei}(-t^2)}{\delta_\omega}\right). \end{aligned} \tag{41}$$

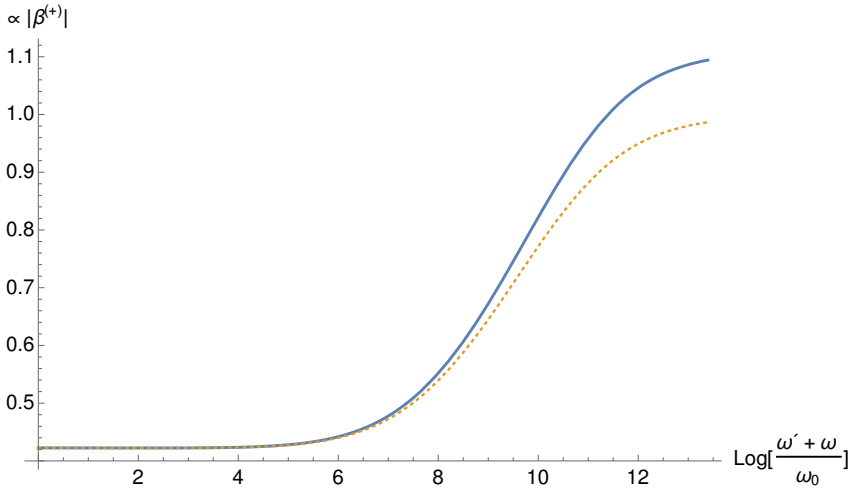
In fact this integral converges absolutely to,

$$\sqrt{\pi|\bar{y}_\omega|}\mathrm{Erfc}(\sqrt{|\bar{y}_\omega|}),$$

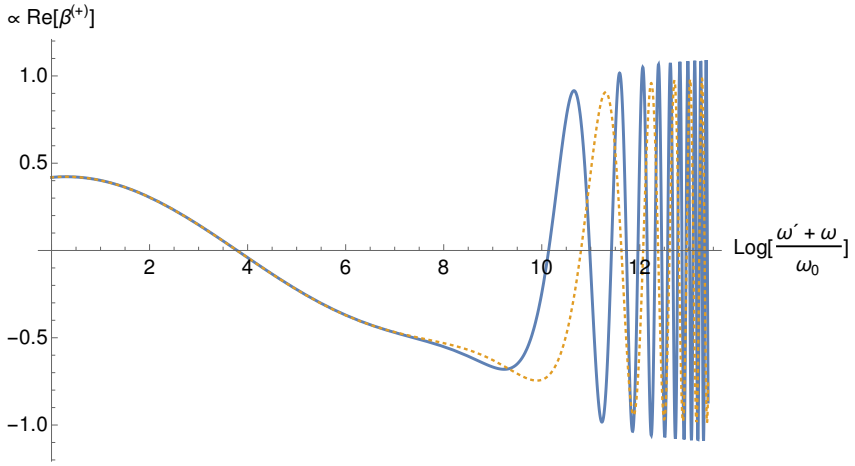
where Erfc is the complementary error function. In this case the limit $\hbar \to 0$ corresponds to $\bar{y}_\omega \to -\infty$ and therefore,

$$\beta^{QS(-)}_{\omega\omega'}(M,\bar{\omega}) \to 0, \tag{42}$$

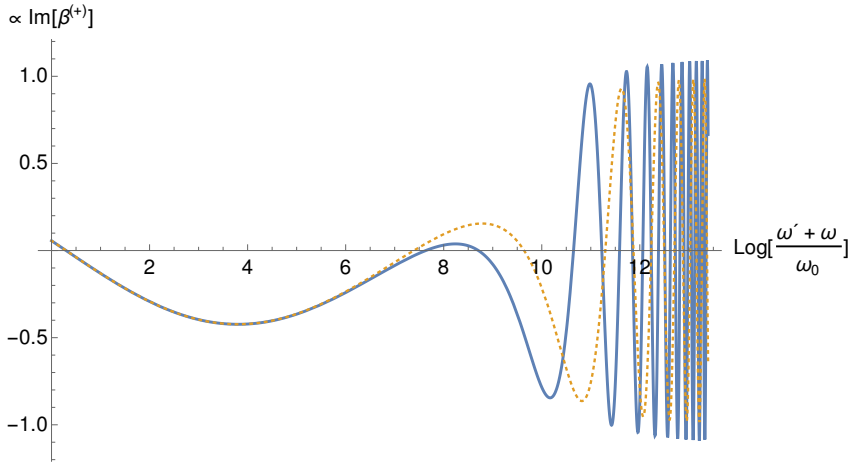
as is expected for the classical shell. In figure (4) we see the significant departure of this expression from the corresponding contribution in the case of the Hawking radiation for the classical shell, particularly for large values of ω'/ω_0.



(a) Modulus.

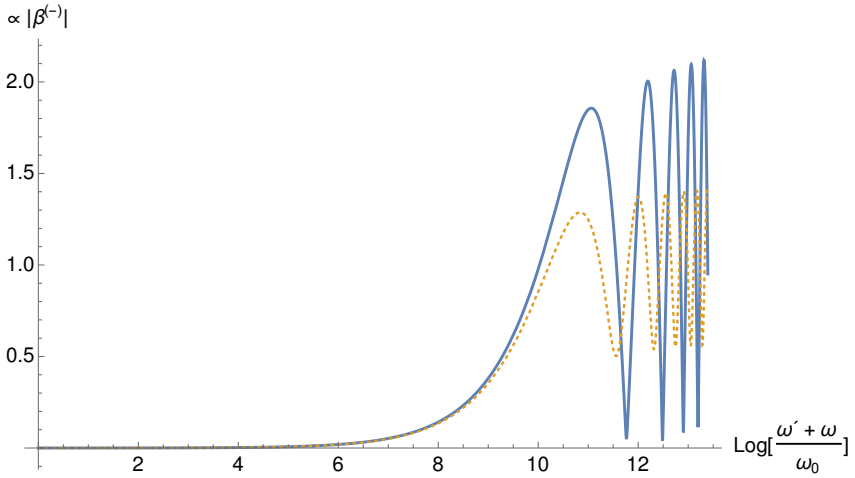


(b) Real part.

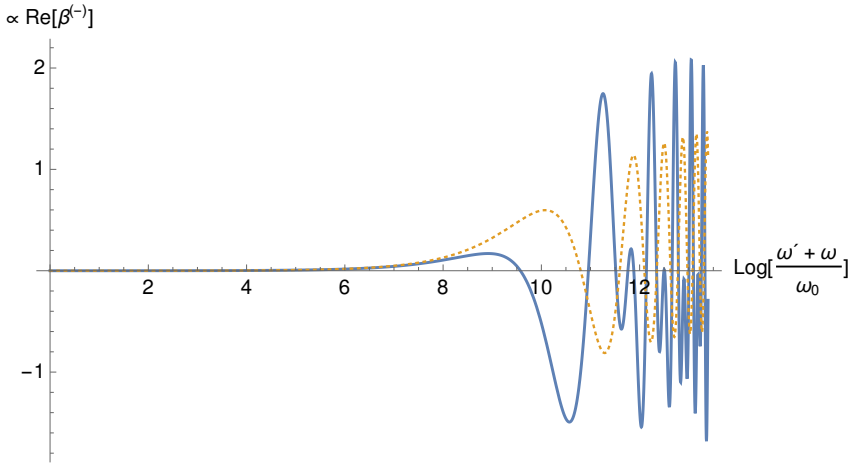


(c) Imaginary part.

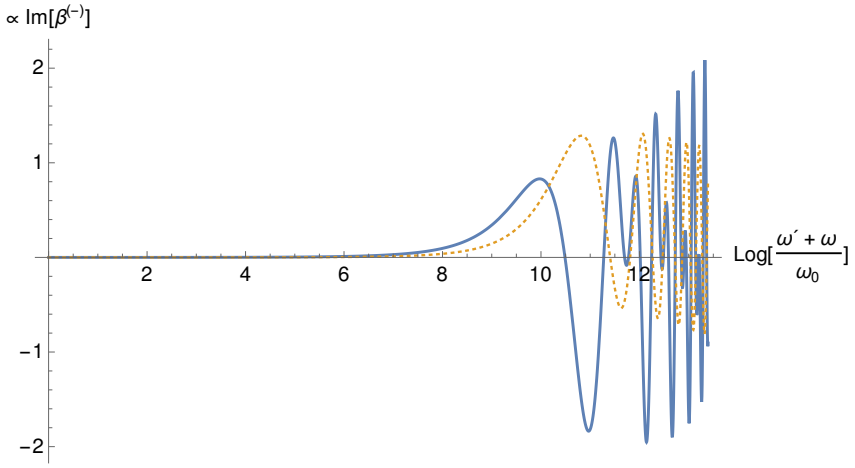
FIG. 3: Comparison between $\beta^{QS(+)}_{\omega\omega'}$ (solid line) and $\beta^{CS(+)}_{\omega\omega'}$ (dashed line). The latter is the integral of the ordinary Hawking calculation in the t variable starting in $\sqrt{|\bar{y}_\omega|}$. The plot considers $\omega = \omega_0$ (frequency of maximum emission for Hawking radiation) and $M_0 = M$.



(a) Modulus.



(b) Real part.



(c) Imaginary part.

FIG. 4: Comparison between $\beta^{QS(-)}_{\omega\omega'}$ (solid line) and $\beta^{CS(-)}_{\omega\omega'}$ (dashed line). The latter is the integral of the ordinary Hawking calculation in the t variable ending in $\sqrt{|\bar{y}_\omega|}$. The plot considers $\omega = \omega_0$ (frequency of maximum emission for Hawking radiation) and $M_0 = M$.

3. $\beta^{QS}_{\omega\omega'}$ vs $\beta^{CS}_{\omega\omega'}$

Adding the two contributions previously discussed, we get an expression for the effective Bogoliubov coefficients $\beta^{QS}_{\omega\omega'}$ that can be compared with the result (14) for the classical shell. In particular, the modulus $\left|\beta^{QS}_{\omega\omega'}\right|$, evaluated numerically, departs from that of the classical shell,

$$\left|\beta^{CS}_{\omega\omega'}\right| = \sqrt{\frac{4M}{2\pi}\frac{\omega'}{\omega'+\omega}\frac{1}{\exp(8M\omega\pi)-1}}. \tag{43}$$

Figure (5) depicts this departure which is most apparent in (5a). The oscillatory behavior at the end of the plot is not to be trusted. At this point the frequencies approach the regime where condition (34) is violated and the numerical result is no longer valid.

In the next section we will discuss the computation of the number of particles and energy emitted for which the Bogoliubov coefficients β are the basic ingredient. We will use this numerical result to comment on the departure from the result for the classical shell.

IV. NUMBER OF PARTICLES AND ENERGY EMISSION BASED ON THE PREVIOUS APPROXIMATIONS

Here we will use the expressions developed in the previous section for effective Bogoliubov coefficients to compute the number of radiated particles and the energy emitted. We will compare this result with the corresponding one that appears in the naive limit discussed in (II B). In the first place we are interested in the formal integral for continuous frequencies,

$$\left\langle N^{QS}_{\omega}\right\rangle = \left\langle \rho^{QS}_{\omega,\omega}\right\rangle = \int dM\,|\varphi(M)|^2 \int_0^\infty d\omega' \left|\beta^{QS}_{\omega\omega'}(M)\right|^2, \tag{44}$$

and finally in the energy

$$E = \int_0^{\omega_P} d\omega\hbar\omega \left\langle N^{QS}_{\omega}\right\rangle = \int dM\,|\varphi(M)|^2 \int_0^{\omega_P} d\omega\hbar\omega \int_0^\infty d\omega' \left|\beta^{QS}_{\omega\omega'}(M)\right|^2. \tag{45}$$

In this expression we have chosen to cutoff the frequency at the Planck frequency $\hbar\omega_P = M_P$ with M_P the Planck mass. Higher frequencies would lead to quantum gravity effects and our analysis would not be valid.

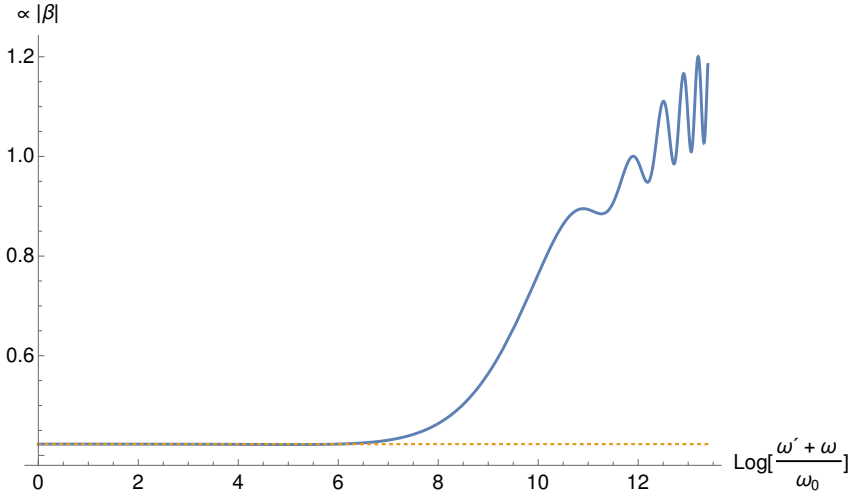
Since these are divergent integrals we can not compare them directly with (26) and (27). We could, for example, consider an alternative basis of modes (like the wave packets considered in section II B). However, since we lack an analytic expression for the effective Bogoliubov coefficients it would require a numerical evaluation that turns out to be very expensive from a computational point of view. Instead, we will study the integrand and compare it to the one already studied in the naive limit $\hbar \to 0$.

Assuming the wave function of the shell is highly peaked around the expectation value for the mass $\bar{M}$, we can ignore the integration in M and focus in the double integral in ω and ω'. As we did before, we choose to study these expressions as functions of $\log\left(\frac{\omega'+\omega}{\omega_0}\right)$ and $\frac{\omega}{\omega_0}$ where ω_0 is the principal frequency of emission. These are not only dimensionless but also better related to the physical variables of the problem (time and energy). Also, we need to fix the free parameter M_0. We chose to set it to $\bar{M}$ because is the usual choice for a classical shell and also because it makes the conditions $\hbar\omega \ll \bar{M}$ (negligible back-reaction) and $\delta_\omega = \frac{\hbar\omega}{M_0} \ll 1$ (semi-classical regime) coincide.

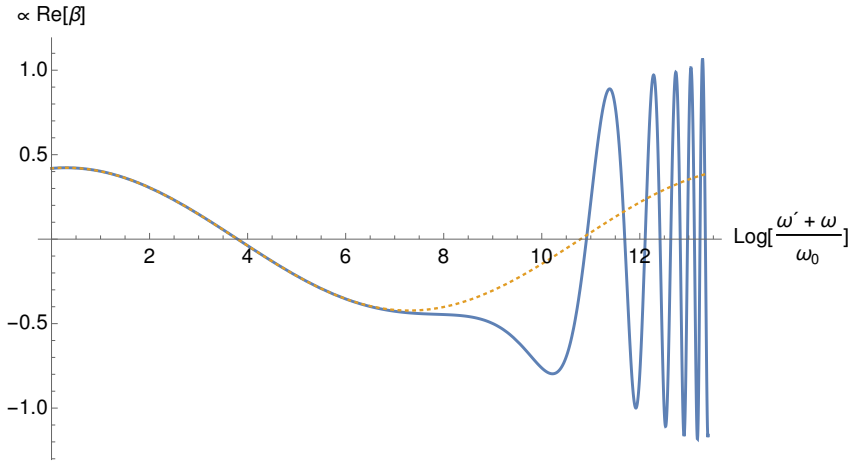
In figure (6) we plot the naive limit $\left|\beta^{QS}_{\omega\omega'}(M)_{\text{Naive limit}}\right|^2$, $\left|\beta^{QS}_{\omega\omega'}(M)\right|^2$ and its two contributions $\left|\beta^{QS(+)}_{\omega\omega'}(M)\right|^2$ and $\left|\beta^{QS(-)}_{\omega\omega'}(M)\right|^2$. As we did before we omit the prefactor,

$$\frac{1}{(2\pi)^2\omega}\frac{\omega'}{(\omega'+\omega)^2}. \tag{46}$$

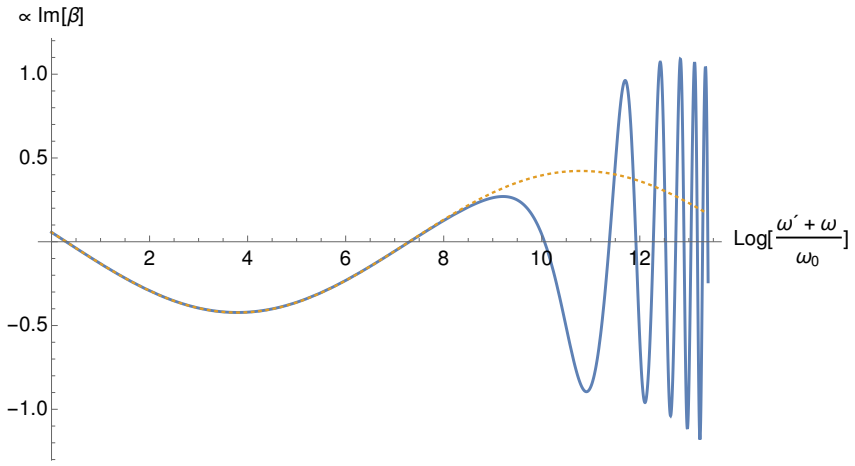
We see that $\left|\beta^{QS}_{\omega\omega'}(M)\right|^2$ approaches the naive limit when $\omega' \to 0$ and also that this behavior is controlled by the $\left|\beta^{QS(+)}_{\omega\omega'}(M)\right|^2$ contribution. This implies the existence of a regime in which the radiation is thermal and a strong



(a) Modulus.



(b) Real part.



(c) Imaginary part.

FIG. 5: Comparison between $\beta^{QS}_{\omega\omega'}$ (solid line) and $\beta^{CS}_{\omega\omega'}$ (dashed line). The plot considers $\omega = \omega_0$ (frequency of maximum emission for Hawking radiation) and $M_0 = M$.

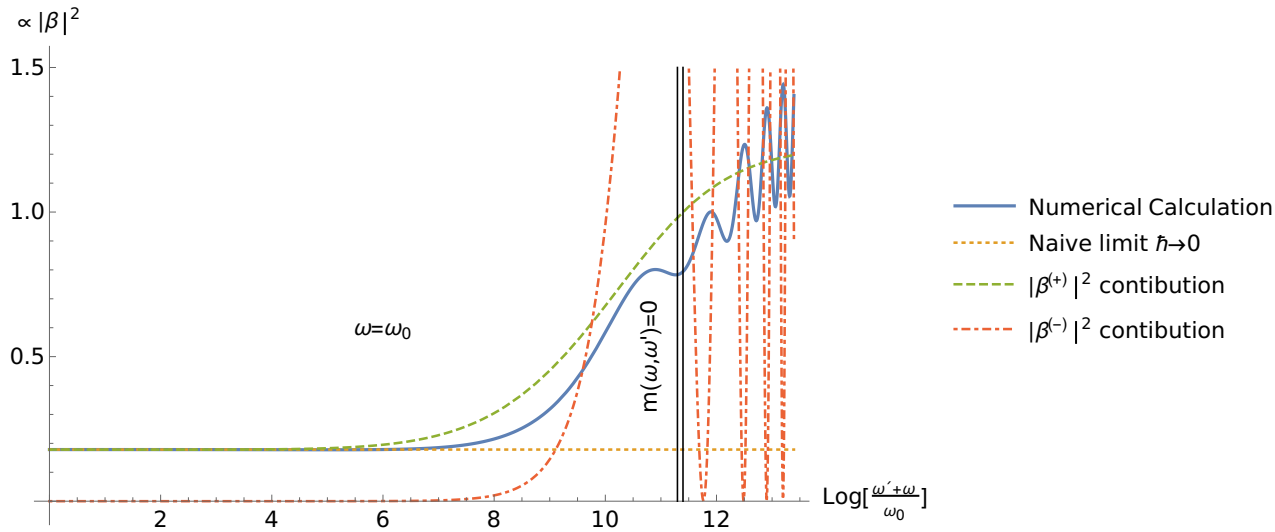


FIG. 6: Comparison between the numerical calculation for $\left|\beta^{QS}_{\omega\omega'}\right|^2$, its $(+)$ and $(-)$ contributions and the naive limit $\hbar \to 0$ (corresponding to a superposition of Hawking radiation). The plot considers $\omega = \omega_0$ and $M_0 = M$. The double line represents the bound $\log[\frac{M}{\hbar\omega_0}]$ for the independent variable. In addition to the plotted squares there are contributions of crossed terms in the evaluation of the density matrix, not shown.

departure when ω' grows characterized by an increased rate of emission. The condition $m(\omega', \omega) = 0$ sets an upper bound for region of thermal radiation. This is not the only (or best) bound we can find but is has the advantage of being frequency independent in the variable $\log\left(\frac{\omega'+\omega}{\omega_0}\right)$, corresponding to the constant value $\log\left(\frac{M}{\hbar\omega_0}\right)$. From these qualitative analyses we can estimate the amount of energy radiated as Hawking radiation and also the time when the departure starts.

Assuming the radiation is thermal until $m(\omega', \omega) = 0$ then the amount of thermal radiation is,

$$E_{\text{thermal}} = \int dM\, |\varphi\,(M)|^2 \int_0^{\omega_P} d\omega\hbar\omega \int_0^{m(\omega',\omega)=0} d\omega' \left|\beta^{QS}_{\omega\omega'}(M)_{\hbar=0}\right|^2 = \tag{47}$$

$$= \int dM\, |\varphi\,(M)|^2 \int_0^{\omega_P} d\omega\hbar\omega \frac{1}{\exp(8M\omega\pi) - 1} \frac{4M}{2\pi} \int_0^{m(\omega',\omega)=0} d\omega' \frac{\omega'}{(\omega' + \omega)^2} = \tag{48}$$

$$= \int dM\, |\varphi\,(M)|^2 \int_0^{\omega_P} d\omega\hbar\omega \frac{1}{\exp(8M\omega\pi) - 1} \frac{4M}{2\pi} \left[\log\left(\frac{M}{\hbar\omega}\right) + \frac{\hbar\omega}{M} - 1\right] = \tag{49}$$

$$= \int dM\, |\varphi\,(M)|^2\, M \times I\left(\frac{M}{M_P}\right), \tag{50}$$

where

$$I(x) = x^2 \int_0^{M_P/M} ds \frac{s}{\exp(8\pi x^2 s) - 1} \frac{s - 1 - \log(s)}{2\pi}. \tag{51}$$

This estimate represents less that 0.1% of the mass of any black hole with a mass larger than the Planck mass.

Finally we can make an estimation of the amount of time thermal radiation lasts. Introducing the same basis of wave packets of section (II B) and performing the same calculation, in particular the change of variable (23) and the subsequent phase absorption (24), we arrive to the analogous expression for the rate of emitted particles

$$\langle N^{QS}_{\omega_j}\rangle_{\text{thermal}}(u_n) = \int dM \frac{|\varphi(M)|^2}{e^{8M\omega_j\pi} - 1} \frac{\epsilon}{2\pi} \int_{T_0(\omega_j)}^{T_f(\omega_j)} dt\, \text{sinc}^2\left[\frac{\epsilon}{2}(u_n - t)\right], \tag{52}$$

where $T_f(\omega_j) = T_0(\omega_j) + 4M \log\left(\frac{M}{\hbar\omega_j}\right)$. This expression represents thermal radiation lasting

$$\Delta T = 4M \log\left(\frac{M}{\hbar\omega_j}\right), \tag{53}$$

which ranges from $\Delta T = 4M\log(M/M_P)$ for $\omega_j \to \omega_P$ to $\Delta T = +\infty$ for $\omega_j \to 0$.

V. CONCLUSIONS

By considering Hawking radiation on the background of a quantum collapsing null shell, we discovered significant deviations from the usual Hawking radiation on classical backgrounds. To begin with, we obtain thermal radiation that is emitted for a short time (of the order of the scrambling time, one millisecond for a solar mass black hole), insufficient to emit a substantial portion of the mass of the black hole. After that, a different type of radiation appears with a non-thermal profile and that can be emitted by a long enough time to evaporate the black hole. The details of this interval of time depend on trans-Planckian physics that our model does not capture. For low frequencies we get thermal radiation that cuts off after a time that depends on the frequency, giving way to non-thermal radiation of larger intensity. For lower frequencies the longer the time of emission of thermal radiation. It always holds that the limit time of emission of thermal radiation is infinite in the classical limit when $\hbar \to 0$ whereas the emission time of the non-thermal radiation tends to zero in that limit and the radiation is always the usual one. It should be noted that the approximations made are only valid for relatively short periods of emission. They do not allow to compute correctly the emitted energy for an arbitrary time of emission. A naive estimate of the evaporation time with the new type of radiation found, leads to black holes evaporating considerably faster that what traditional Hawking radiation predicts.

The fact that one has non thermal radiation may imply that there does not exist an information paradox, although a detailed analysis would be needed of how information could be retrieved, particularly for collapsing situations that are more realistic than a simple shell.

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