PEDECIBA Informática
Instituto de Computación – Facultad de Ingeniería
Universidad de la República
Montevideo, Uruguay

Tesis de Maestría
en Informática

Topological properties for a wide area network planning and a dynamic programming approach for designing the access network

Luis Stábile

2011

Topological properties for a wide area network planning and adynamic programming approach for designing the access network

Stábile, Luis

ISSN 0797-6410

Tesis de Maestría en Informática

Reporte Técnico RT 11-14

PEDECIBA
Instituto de Computación – Facultad de Ingeniería
Universidad de la República.

Montevideo, Uruguay, mayo de 2011

THESIS

presented

to Universidad de la República

to obtain

the title of: MAGISTER EN INFORMÁTICA - PEDECIBA

by

Ing. Luis STÁBILE

Facultad de Ingeniería
UNIVERSIDAD DE LA REPÚBLICA

Thesis title:

Topological Properties For A Wide Area Network Planning And A Dynamic Programming Approach For Designing The Access Network.

May 2^{nd}, 2011

M. :	Dr. Julio	OROZCO TORRENTERA	President
MM. :	Dra. Ing. Libertad	TANSINI	Examiners
	Dra. Ing. Adriana	MAROTTA	
MM. :	Dr. Ing. Franco	ROBLEDO	Thesis Director

Acknowledges

First of all I want to thank my advisor Dr. Ing. Franco Robledo Amoza for supervising my master thesis, providing so many interesting ideas and being a great help during the last months.

My special thanks also belong to my family, especially to my parents for always believing in me and for all their support during the last years.

I gratefully acknowledge ANTEL (Uruguay's state-owned telecommunications company), especially for Ing. Andrés Ferrari and the president Ing. Carolina Cosse.

Last but not least I want to thank my girlfriend Yohana for the wonderful time we had together, as well as for all her understanding and her outstanding support.

Resumen

Una red de área extendida (Wide Area Network - WAN) puede ser considerada como un conjunto de sitios interconectados por líneas de comunicación. Topológicamente una red WAN esta organizada en dos niveles: la Red Dorsal (*Backbone*) y la Red de Acceso (*Access Network*) compuesta por un cierto número de *Redes de Acceso Locales*. Cada red de acceso local usualmente tiene topología de árbol, teniendo como raíz un nodo de la Red Dorsal (sitio dorsal). Los sitios terminales (o clientes) se conectan directamente al sitio dorsal correspondiente a una red de acceso o bien a un sitio concentrador de la misma. La Red Dorsal tiene usualmente topología de malla y su propósito es permitir comunicación eficiente y confiable entre nodos de la Red Dorsal que actúan como puntos de entrada para las Redes de Acceso Locales.

En esta tesis atacamos el problema del diseño de una red WAN descomponiéndola en dos sub-problemas interrelacionados: el diseño de la Red de Acceso (the Access Network Design Problem - ANDP) y el diseño de la Red Dorsal (the Backbone Network Design Problem - BNDP). En ambos modelos consideramos solamente los costos de construcción, por ejemplo, los costos de dragados para el tendido de líneas y la puesta en servicio del cableado de la red.

Nuestro objetivo es estudiar los problemas ANDP y BNDP. Nos concentramos en ANDP con el objetivo de proponer un nuevo enfoque para resolverlo. Introducimos diferentes resultados en lo que concierne a las propiedades estructurales de las soluciones del ANDP. Presentamos el enfoque de clúster como una de las estrategias más usadas por las herramientas comerciales de diseño. Modelamos el ANDP como una variante del Problema de Steiner en Gráfos (the Steiner Problem in Graphs - SPG). Dada la complejidad del problema (es NP-Hard); es útil proveer técnicas para reducir la dimensión del problema original en uno equivalente de menor tamaño. Luego nos concentramos en algunas propiedades estructurales de las soluciones óptimas del BNDP.

Finalmente proponemos recurrencias para resolver los problemas ANDP y BNDP basadas en las metodologías de Programación Dinámica y Programación Dinámica con Relajación del Espacio de Estados.

Key words: Diseño topológico, Red de Acceso, Red Dorsal, Programación Dinámica con Relajación del Espacio de Estados.

Abstract

A wide area network (WAN) can be considered as a set of sites and a set of communication lines that interconect the sites. Topologically a WAN is organized in two levels: the *Backbone Network* and the *Access Network* composed of a certain number of *Local Access Network*. Each local access network usually has a tree-like structure, rooted at a single site of the backbone, and connected users (terminal sites) either directly to this backbone site or to a hierarchy of intermediate concentrator sites which are connected to the backbone site. The backbone network has usually a meshed topology, and this purpose is to allow efficient and reliable communication between the switch sites that act as connection points for the local access networks.

In this thesis we tackled the problem of designing a WAN by breaking it down into two inter-related sub-problems: the Access Network Design Problem (ANDP) and the Backbone Network Design Problem (BNDP). In both models we considered only the construction costs, e.g. the costs of digging trenches and placing a fiber cable into service.

Our aim in this thesis is the study of the ANDP and the BNDP problems. We concentrate on the ANDP with the objective of to propose a new approach for solving this problem. We study differents results related to the topological structure of the ANDP solutions. We present the clustering approach as one of the strategies more frequently used by the commercial tools of the design. We also formulate the ANDP as a Steiner Problem in Graphs (SPG). Given the complexity of the ANDP (the problem belongs to the NP-Hard class); it is very useful to provide techniques capable of reducing the dimension of the original problem to an equivalent smaller problem. After we concentrate on some structural property about the BNDP.

Finally we propose recurrences to solve the ANDP and BNDP which are based on Dynamic Programming and Dynamic Programming with State-Space Relaxation methodology.

Key words: Topological Design, Access Network, Backbone Network, Dynamic Programming with State-Space Relaxation.

Contents

Part I

INTRODUCTION

Chapter 1

Global presentation of the thesis

1.1 Motivation and General Context

Telecomunication networks have become strategic resources for private and state-owned institutions and its economic importance continuously increases. There are series of recent tendencies that have a considerable impact on the economy evolution such as growing integration of networks in the productive system, integration of different services in the same communication system, important modification in the telephone network structure (voice and data integration, mobility, telephony development on IP plataforms, etc). Such evolutions accompany a significant growth of the design complexity of these systems. The integration of different sorts of traffics and services, the necessity of a more accurate management of the service quality, in particular on IP plataforms (but on which the management evolution and the coexistence of technologies have not been anticipated), are factors that make this type of systems very hard to design, to dimension and therefore to optimize. This situation is aggravated with a very high competitiveness context, in an area of critical strategic importance.

In this work, we will focus on modern communication network planning. This field has considerably developed recently mostly owing to the introduction of optic fiber technologies which have very good performance. The planning and design of telecommunication networks is a very complex and generally expensive task. It integrates optimization process loops, analysis activities and quantitative evaluations. The planning team must consider the already existent or anticipated needs, the costs of the different elements that compose the systems, the restrictions on the performance, the reliability, the evolutiveness, the service quality, etc., besides specific restrictions on each particular system and, as a function of these, design a network as adapted as possible to the technical and the business plan. In the case of small size networks, the team may consist of a single person while in large-scale nerworks as a wide telecommunication networks (a WAN: Wide Area Network) the planning team may be constituted by several people working at different organization levels.

The conception of a WAN is a process in which dozens of sites with different characteristics require to be connected in order to satisfy certain reliability and performance restrictions with minimal costs. This design process involves the terminal sites location, the concentrators location, the backbone (central network or kernel) design, the routing procedures, as well as the lines and nodes dimensioning. A key aspect on WAN design is the high complexity of the problem, as much in its globality as in the principal sub-problems in which it is necessary to descompose it. Due to the high investment levels a cost decrease of very few percentage points while preserving the service quality results in high economic benefits.

Tipically, a WAN network global topology can be descomposed into two main components: the *access network* and the *backbone network*. These components have very different properties, and consequently they introduce specific design problems (although they are strongly interdependent). On one hand, this causes complicated problems (particulary algorithmic ones); on the other hand, it leads to stimulating and difficult research problems.

A WAN access network is composed of a certain number of access sub-networks, having tree-like topologies; and the flow concentration nodes allow to diminish the costs. These integrated flows reach the backbone which has a meshed topology, in order to satisfy security, reliability, vulnerability, survivability and performance criteria. Consequently, the backbone is usually formed by high capacity communication lines such as optic fiber links. In general, this WAN topological feature is valid in the case of a datagram based network (as in IP technology) and also in circuit switching (as in the current telephone network or other technologies like X25, Frame Relay or ATM).

Globaly, the designing team manages an important amount of data to propose a model that fulfills the preestablished requirements. For instance, it has information about the set of the terminal sites positions (the company customers, the service subscribers, etc.) and about the characteristics (most of the time estimated) of the inter-site flows (volume, temporary behaviour, etc.). Also there is information about the performance restrictions (for example delays), about the service quality (for example of video data quality), and on aspects such as reliability, vulnerability, connectivity, security and availability. On the network components aspect, the designer has a list of possible components according to the involved network nature, with its characteristics and costs. The technical nature of the considered network leads to specific routing procedures that should be taken into account of searching efficient solutions, or if possible optimal solutions. In general there are many other complementary data such as the sites in which it is suitable to install concentrator, which ones are no suitable for that, which backbone sites must have switch servers, special characteristics or security restrictions for some flows, etc.

Based on this data set, the designer must specify the access network and the backbone network topologies as well as the characteristics of the different sites and connections, the traffic routing, etc. The result of this process are specific optimization models for the design of both subnetworks and for the global WAN design problem. This global set of problems typically include the evaluation of performance, reliability among others.

1.2 A general WAN design process

Modelling a WAN design by means of the formulation of a single mathematical optimization problem is very intricate due to the interdependence of its large amount of parameters. Therefore the design of a WAN is usually divided into different sub-problems. A good example of a possible descomposition approach for the WAN design process is the following [53]:

I) Access and backbone network topologies design. Specific knowledge about the cost of laying lines between different network sites (terminals, concentrators and backbone) is assumed. Frequently, these costs are independent of the type of line that will effectively be installed since they model the fixed one-time costs (cost of digging trenches in the case of optic fiber, installing cost, placing a fiber cable into service). A high percentage from the total construction network budget is spent in this phase [57].

II) Dimensioning of the lines that will connect the different sites of the access and backbone networks, and the equipment to be settled in the mentioned sites.

III) Definition of the routing strategy of the flow on the backbone network.

These three sub-problems have different types of constraints:

- For the sub-problem (**I**):
  - The terminal sites (the clients) must be connected either directly to a backbone node, or through a hierarchy of intermediate concentrator sites which are connected to the backbone. Usually, there exist additional restrictions such as limiting the number of concentrators connected in cascade, so that in the case of a trunk line cut a significant number of terminal sites will not be affected.
  - The backbone must satisfy reliability restrictions that allow it to remain operational (connected) when failures occur in its servers or links. These reliability restrictions are often expressed in terms of the network connectivity. For instance, telecommunication network topologies which have proved being highly performant are 2-node-connected optic fiber networks. Its physical components have very low failure rates; and the network itself is resilient in the presence of a failure in a node or link. In the same direction, 3 node-connected topologies have been used in optic fiber networks connecting critical sites of a aircraft carrier [38, 57].
- Once the topological structure of the WAN is designed, its components are dimensioned in order to fulfill the performance requirements. A routing plan design and the projection of flow over the backbone must be done so that the performance restrictions imposed be respected. In this way, it can be noticed that (**II**) and (**III**) are not independent. Taking into account the technologies used, some of the usual performance restrictions are:
  - the traffic delay should not be greater than a certain prefixed limit. This restriction is imposed to the access network as well as to the backbone.
  - the blocking factor (the probability of a new connection to not succeed) must be lower than a certain prefixed value.
  - the packet loss rate must be relatively low. A level of 10^{-4} constitutes the agreed maximal level of the packet loss rate for a network normally working, for today standards.

We give below a generic WAN planning process as well as references related to other works in this area, including topics such as hierarchical network design, multitechnology network design, etc. Taken from [53]:

1. Backbone nodes localization. This implies producing a hypothesis *H* regarding the backbone sites localization or modifying the precedent hypothesis. These hypothesis must consider the switches installation in the core of the most dense zones.

2. Access network design:

   (a) Depending on the hypothesis *H*, an access topology is constructed by optimally placing the concentrator equipments. If this is not possible, at least a local optimum should be reached as the result of applying clustering strategies.

   (b) Determination of the needed capacities in the access network (links and nodes).

   (c) Determination of the access network performances by tuning them to the required level (specified in *2b*).

   (d) Determination of the access network reliability by tunning corresponding parameters to meet the required level (specified in *2b*).

   (e) Compute the access network cost.

   (f) Determination of the reduced matrix of point-to-point traffic between the backbone switch nodes which are entries of the access sub-networks induced by *2a*. These flows must be routed over the backbone topology once this last one has been designed.

3. Backbone network design:

   (a) Based on the hypothesis *H*, a backbone network topology is built adjusted to the reliability demands.

   (b) A routing strategy is defined. The point-to-point flows are projected into the network designed in *3a*. Thus, the paths of the backbone on which will circulate the effective traffic are obtained.

   (c) Determination of the needed capacities in the backbone network (links and nodes).

   (d) Determination of the backbone network performance in order to check if the required levels are fulfilled. If necessary the netwok of *3a* or the capacities of *3c* are redefined.

   (e) Determination of the backbone network reliability by adjusting it to the required level. If necessary the topology computed in *3a* is redefined.

   (f) Determination of the network fairness in order to achieve some required level. If necessary the topology computed in *3a* is redefined.

   (g) Computation of the backbone network cost.

4. Results consolidation and gloabal balance:

   (a) Determination of the global performances involving the access and the backbone networks simultaneously. If appropriate, return to 2 or 3 depending on where performance restrictions violations happen (i.e. in the access network, the backbone or both).

(b) Determination of the global reliability involving the access and the backbone networks. If appropriate, return to 2 or 3 depending on where reliability restrictions violations happen (i.e. in the access network, the backbone or both).

(c) Compute the overall cost (composed of the access network cost and the backbone network cost). If the WAN cost is approved, the obtained topology is returned as a solution. Otherwise, return to *1* in order to produce a new hypothesis H.

Based on performance evaluation procedures and dimensioning rules common to both network levels (access network and backbone), in [53] each sub-problem is studied and specific algorithms to solve them are proposed. The connections cost taking into account the geographic distances among the involved sites and the annual connection tariffs provided by the telecom operators are estimated. For the topological design of the access network the authors use clustering approaches around the backbone switch nodes whereas for the topological design of the backbone network they apply a Hierarchical Method. Even if the testing cases presented are relatively small, the suggested metodology can be useful as a reference about the way of descomposing the WAN topological planning process into several sub-problems.

Increasingly, survivability is becoming an important criterion in the design of telecommunication networks. Several recent developments have prompted this change.The first is technological: fiber-optic and opto-electronic cables are replacing traditional copper cables as a telecommunication medium. Because these newer technologies can carry substantially more traffic (both more channels and at a higher frequency) than traditional copper cables, telecommunication networks designed solely to minimize costs will tend to be sparse. In this case, the failure of a single edge can create significant system-wide disruptions disabling traffic between many customer locations if the network does not provide alternate paths for routing. Second, customers, individual as well as industrial are increasingly using telecommunication networks not only for transmitting voice, but also to transmit video and data. The Multi-level Network Design problem seeks a fixed cost minimizing design that spans all the nodes and connects the nodes at each level by facilities of the corresponding or higher grade. Balakrishn in [7, 8, 10, 11] develops a dual-based algorithm for the Multi-Layer Network Design problem. He has identified some structural properties of optimal solutions that enable preprocessing, developed a dual ascent method for generating lower and upper bounds, and performed extensive computational testing. For other related works concerning the optimal design of a multi-level hierarchical network the reader can consult the references [19, 20, 23, 40, 41, 58].

In this thesis, we will concentrate on phase (**I**) of the descomposition of a WAN design process. More precisely, we are interested in the topology planning process concerning the the access network and the backbone network. Our motivation comes from the necessity of devising efficient algorithms for these topological design highly-combinatorial problems. Due to the NP-hard nature of the problem and even though there exist some results, there is still room for improving industrial practices applied today. In this sense, we believe it is of strategic importance to design powerful quantitative analysis techniques, potentially easy to integrate into tools. We introduce combinatorial optimization models to formally define the topological design of the access and backbone networks. Moreover, we introduce differents results related to the topological structure and we study one of the strategies more frequently used by the commercial tools. Finally, we propose different algorithms to solve the topological design which are based on Dynamic Programming and Dynamic Programming with State Space Relaxation metodology.

1.3 Access and Backbone Network Design Problems

We will define these problems in terms of graph theory; for this purpose we introduce the following notation:

- S_T is the set of terminal sites (clients) to be connected to the backbone.
- S_C is the set of feasible concentrator sites of the access network. On each one of these sites, an intermediate server equipment might be placed. From this one, a trunk line is laid towards the backbone or other concentrator site.
- S_D is the set of feasible switch sites of the backbone network. On each one of these sites, a powerful server might be placed and from it, connection lines towards other backbone server equipments.
- $V = S_T \cup S_C \cup S_D$ are all the feasible sites of the WAN network.
- $A = \{a_{ij}\}_{i,j \in V}$ is a matrix which gives for any pair of sites $i, j \in V$, the cost $a_{ij} \geq 0$ of laying a line between them. When the direct connection between i and j is not possible, we define $a_{ij} = \infty$
- $U = \{(i, j) | i, j \in V, a_{ij} < \infty\}$ is the set of all the feasible connections between the different sites of the WAN network.
- $G = (V, U)$ is the simple graph which models every node and feasible connection of the WAN.

The General Access Network Design Problem (GANDP) consists of finding a minimum-cost subgraph $H \subset G$ such that all the sites of S_T are communicated with some node of the backbone. This connection can be direct or through intermediate concentrators. The use of terminal sites as intermediate nodes is not allowed; this implies that they must have degree one in the solution.
We simplified the GANDP problem by collapsing the backbone into a fictitious node. We call it the Access Network Design Problem (ANDP) and the equivalence between both problems, GANDP and ANDP, is proved in [4]. The ANDP belongs to NP-Hard class [4].
Given a subset of switch sites $S_D^{(I)} \subseteq S_D$ and a non-negative integer matrix $R = \{r_{ij}\}_{i,j \in S_D^{(I)}}$ the Backbone Network Design Problem (denoted by BNDP) consists of finding a minimum-cost subgraph $H \subseteq G(S_D)$ such that $S_D^{(I)} \subseteq H$ and $\forall i, j \in S_D^{(I)}$ there exist at least r_{ij} node-disjoint paths connecting i with j in H. This problem can be modelled as a Generalized Steiner Problem in Graphs with Node-Connectivity constraints (denoted by GSP-NC) which is NP-Hard in the general case [62].

For further details on the formulations of the Generalized Steiner Problem in its both versions, Edge-Connectivity (denoted by GSP-EC) and Node-Connectivity , the reader may consult [1, 25, 60–62].
A particular case of the BNDP is when $r_{ij} = 2, \forall i, j \in S_D^{(I)}$. This is known in the literature as the Steiner 2-node-survivable network problem (denoted by STNSNP) [4, 6, 26].
We call $S_D^{(I)}$ the set of fixed switches nodes. These will necessarily have to be integrated to the solution, either because they are access sub-network entry points to the backbone or due to specific conception requirements. The sites $S_D \backslash S_D^{(I)}$ are optional (commonly named Steiner nodes) and may be used to reduce the backbone cost.

This thesis work divides the WAN network topological design problem into two separared parts: backbone (BNDP) and access (ANDP) network. Our aim in this thesis is the study of the ANDP and the BNDP problems. We concentrate on the ANDP with the objective of proposing a new approach for solving this problem. We study differents results related to the topological structure of the ANDP solutions. In particular we present results that characterize the topologies of the feasible solutions of an ANDP instance. Moreover, for certain types of network classes we present results that characterize the structure of the global optimal solution. We present the clustering approach as one of the strategies more frequently used by the commercial tools of the design. We also formulate the ANDP as a Steiner Problem in Graphs (SPG). Given the complexity of the ANDP (the problem belongs to the NP-Hard class), it is very useful to provide techniques capable of reducing the dimension of the original problem to an equivalent smaller problem. After we concentrate on some structural property about the BNDP.

Finally we propose algoriths to solve the ANDP and BNDP which are based on Dynamic Programming and Dynamic Programming with State-Space Relaxation metodology.

1.4 Related Literature

As we already mentioned when we are talking of networks in this thesis, we are interested only in their topology, that is, we see a network as a set of sites and links that are placed between sites. Survivability in this context means that between any two sites there exists a pre-specified number of paths (consisting of nodes and links) that have no node or link in common. The problems ANDP, BNDP, and BNDP2NS correspond to this context. In practice, the topology of a network with low placement costs is created first, and in a second optimization stage, traffic and routing costs are considered [57].

We concentrate first in the literature related to ANDP. In [2, 5, 12, 27, 34, 36, 45, 47, 54, 55], the authors propose different approximate algorithms for the topological design of local and large-scale access networks. They are based on different approaches, and consider different parameters and restrictions, including aspects such as: the design of the access network is restricted to specific topologies; the number of concentrators to be placed is limited; network components dimensioning, etc. The resolution techniques used in these works include: Lagrangian Relaxation mixed with the Sub-gradient Method [47], Simulated Annealing [45], Linear Programming Relaxation [5], Lagrangian Heuristic, Greedy Heuristics [36], Branch-and-Bound mixed with Lagrangian Relaxation, Branch-and-Bound with Benders decomposition [54, 55], Neural Networks [2], Tabu Search [34], Genetic Algorithms, plus other specific methods.

Next we will focus on the GSP-NC and STNSNP (which are the reference models of base for our BNDP and BNDP2NS problems) and their related survivability models like those presented in [57, 61, 62].

Winter [60–62] demonstrated that the GSP-NC can be solved in linear time if the network is series-parallel, outerplanar or a Halin graph. Nextly, we will give a summary of the survivability problems related to the GSP-NC and STNSNP. Gröstchel, Monma and Stoer [37] consider a particular case of the GSP-NC working on a slightly different context. They called it the NCON problems. In [57], Stoer gives an extensive survey for the NCON and the ECON (the version with edge-connectivity constraints), and some particular cases. The NCON (resp. ECON) is formalized as follows. Given an undirected graph $G = (V, E)$ such that each edge $e \in E$ has a fixed weight c_e representing the cost of establishing the direct link connection. In particular, each node $i \in V$ has an associated nonnegative integer r_i, the type of i (the survivability requirement or “importance" of a node is modeled by node types). Let $H = (W, F)$ be a subgraph of G. We say that H satisfies the node-survivability conditions (also called node-connectivity constraints or requirements), if, for each pair $i, j \in V$ of distinct nodes, H contains at least $r_{ij} = \min\{r_i, r_j\}$ node-disjoint paths communicating i with j. Similarly, we say that H satisfies the edge-survivability conditions (also called edge-connectivity constraints or requirements), if, for each pair $i, j \in V$ of distinct nodes, H contains at least $r_{ij} = \min\{r_i, r_j\}$ edge-disjoint paths communicating i with j. These conditions ensure that some communication path between i and j will survive a prespecified level of node (or edge) failures.

Let us observe that the GSP-NC model generalizes the model given above since in the GSP-NC there exist general survivability requirements r_{ij} that are specified for each pair i, j of fixed nodes independently. Nevertheless, Grötschel, Monma and Stoer [38, 39] introduce the use of node types to define survivability requirements based on the premise that these adequately express the relative importance placed on maintaining connectivity between offices. They classify the different problem types according to the largest occurring node type and according to whether the node types represent node or edge connectivity requirements. In this way, given a graph $G = (V, E)$ and a vector $r = (r_i)_{i\in V}$, by assuming (without loss of generality) that there exist at least two node types of type k (which is defined as the largest node type), they speak of the kNCON problem (resp. kECON) when the objective is to find a minimum-cost network that satisfies the node survivability conditions (resp. the edge survivability conditions). If the highest value of k is not specified, these problems are called NCON and ECON respectively. In particular, if all node types have the same value k, the problem NCON (resp. ECON) is reduced to find k-node-survivable (resp. k-edge-survivable) networks having minimum cost.

Let us note that there exist many specializations of the survivability problems which can be formulated by varying its parameters as follows:

- As mentioned previously, the GSP-NC and GSP-EC are more general models of survivability than NCON and ECON, since the connectivity requirements are associated to pairs of nodes in independent form and not necessarily involving all the nodes of V.

- In the NCON and ECON, we have $r_{ij} = \min\{r_i, r_j\}$ for given nodes types r_i, r_j, which in turn may be:
  - general (kECON or kNCON problem),
  - uniform (k-edge or k-node connected graphs),
  - in $\{0, 1\}$ (Steiner trees)
- general or euclidean or uniform costs.

There exist polynomially solvable cases of the NCON and ECON problem. They result from relaxing the original problem with restrictions like uniform costs, 0/1 costs, restricted node types, and special underlying graphs such as outerplanar, series-parallel, and Halin graphs. All these particular cases are referenced and briefly exposed in [57]. On the other hand, lower bounds and heuristics with worst-case guarantees for kECON problems were found for restricted costs, e.g., uniform costs or costs satisfying the triangle inequality, as well as very important results on the structure of optimal survivable networks for this cost structure. Details of these works can be seen in [13, 35, 49] and in a summarized form in [57]. In [57], Stoer also summarizes heuristic procedures to solve general kNCON and kECON problems. Monma and Shallcross [50] give heuristics for the 2ECON and 2NCON problems. Frederickson and Jájá [28] propose a heuristic for the 2NCON problem with worst-case guarantee of $3/2$ under costs satisfying triangle inequality. Consider the NCON problem where instead a vector $r = (r_i)_{i \in X}$ we have a matrix $R = (r_{ij})_{ij \in V}$. They developed a simple heuristic for this problem which basically consists of a randomized starting routine and an optimizing routine where local transformations are applied to a feasible solution. Recently, Balakrishnan, Magnanti and Mirchandani [9] presented a family of new mixed-integer programming formulations for the GSP-EC, whose associated linear programming relaxations can be tighter than those of the usual cutset formulation. They provide several combinatorial heuristics for these formulations, which satisfy that the bounds on the heuristic costs relative to the optimal values of the integer program and the linear programming relaxation of the tighter formulation are stronger than some previously known performance bounds for combinatorial heuristics. For further details of these works (and their performance tests) the reader may consult the cited references.

We find in the literature other works related to our BNDP. In [14, 21, 33, 46] the authors provide different approaches for the topological design of a backbone network. Most of these works are not only focused in the topological design, but they also consider aspects such as network dimensioning, routing mechanisms, etc. They are based on different optimization models which include selection of network topology and other additional objectives. We can see these ones as network planning processes where the goal is to find backbone topologies with lowest possible overall network price, while keeping all requirements (such as availability, maximal number of, maximal blocking probability, etc.) satisfied. The resolution techniques used in these works include: Genetic Algorithms, Branch-and-Bound method mixed with the algorithm of Ford-Fulkerson, Tabu Search, Greedy Heuristic combined with Tabu Search heuristic as improver, Lagrangian Relaxation embedded in a sub-gradient optimization procedure, Dual-Based lower bounding procedure incorporated in a Branch-and-Bound algorithm, Dual-Based solution procedure, Hybrid approach of a genetic algorithm and local search algorithms as improver, Tabu-Search heuristic with a post-optimization algorithm, and other specific heuristics.

Other problem in this area can be found in [3, 15–18, 29–32, 44, 52, 59], where the authors propose several models for designing low-cost network topologies with additional constraints such as fault tolerence and performance restrictions, considering in addition in some of them network components dimensioning.

Part II

A MODEL FOR A WAN DESIGN

Chapter 2

A Model For A WAN Design

In this chapter, a model for the design of a WAN is introduced. The model tries to show the most essential aspects which are considered when designing access and backbone networks. In this model, some parameters are not considered: the operation probability of the lines and equipments, the number of equipment ports, and the memory capacity of the equipments. The objective is to design a WAN with the smallest possible installation cost, so that the constraints are satisfied.

In what follows, the data of the model are presented as well as its formalization as a combinatorial optimization problem on weighted graphs; in this way is tried to find the optimal topology that satisfies the imposed constraints to the access and backbone networks.

2.1 Data of the Model.

The information available for each type of equipment (switch & concentrator) and each type of connection line, as well as the line laying, is the following:

- E_a is the set of types of connection lines available. Furthermore $\forall e \in E_a$ the following data are given:
  - c_e is the cost by kilometer of the line type e. Here the laying cost is not included.
  - v_e is the speed in kbits/s of the line type e.
- K is the set of types of concentrator equipments available. Furthermore $\forall k \in K$ the following data are given:
  - c_k is the installation cost of the concentrator type k.
  - v_k is the speed in kbits/s of the concentrator type k.
- W is the set of types of switch equipments available. Furthermore $\forall w \in W$ the following data are given:
  - c_w is the installation cost of the switcher type w.

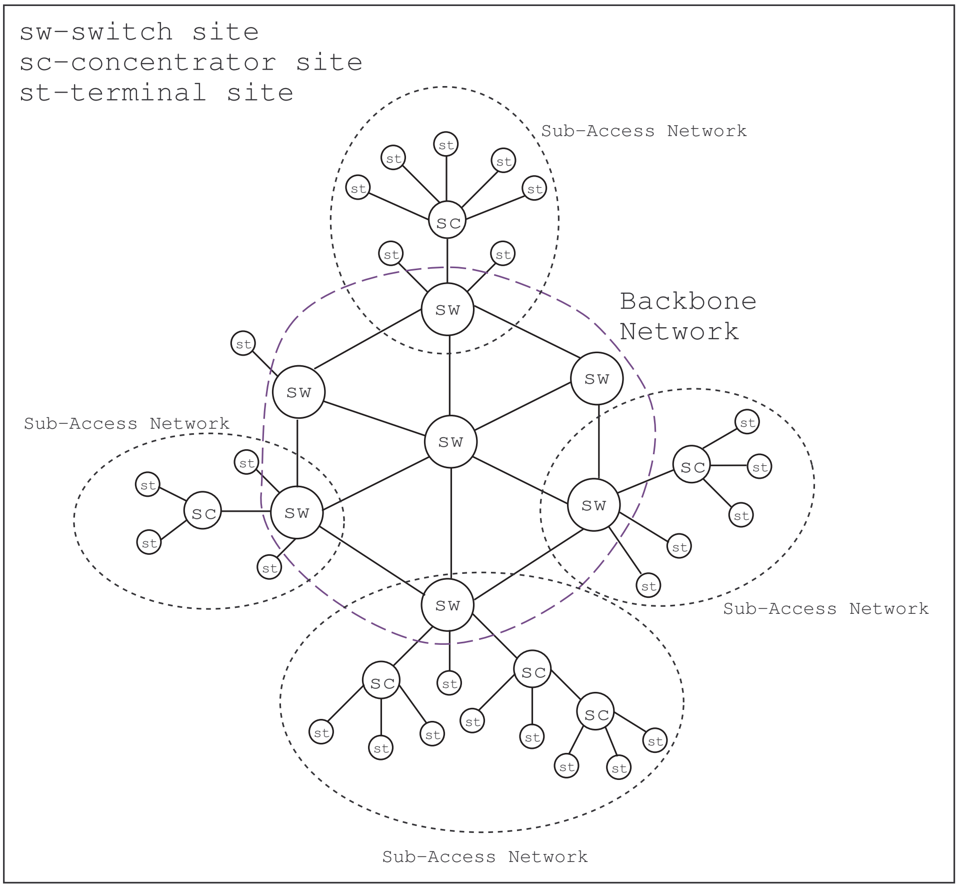

Figure 2.1: WAN example.

  - v_w is the speed in kbits/s of the switcher type w.

- $C = F_{cost}(L) = \{c_{ij} =$ *direct connection cost between the sites* $i, j; \forall i \in S, \forall j \in S_C \cup S_D\}$ this matrix gives us for a site of S and a site of $S_C \cup S_D$, the cost of laying a line among them. When the direct connection among both places is not possible, we assume that $c_{ij} = \infty$.

2.2 Graph Model for the Problem.

Here is presented, in terms of graph theory, a model for the design of a WAN, based on the problem. Previously to the problem formal definition, some notations are introduced.

- $E_1 = \{(i, j); \forall i \in S_T, \forall j \in S_C \cup S_D / d_{ij} < \infty\}$, this is the set of feasible connections between a terminal site and a concentrator or switch site.

- $E_2 = \{(i,j); \forall i \in S_C, \forall j \in S_C \cup S_D / d_{ij} < \infty\}$, this is the set of feasible connections between a concentrator site and a switcher or another concentrator site.
- $E_3 = \{(i,j); \forall i \in S_D, \forall j \in S_D / d_{ij} < \infty\}$, this is the set of feasible connections between two switch sites.
- $E = E_1 \cup E_2 \cup E_3$, this is the set of all feasible connections on the WAN.
- $D_{S_T} = \{D_{t_i}, t_i \in S_T\}$, where D_{t_i} is the set of terminal nodes which demand connections with $t_i \in S_T$.
- $V_{S_T} = \{v_{i,j}\}_{i,j \in S_T}$, is the traffic demand matrix.

Definition 2.2.1 (WANDP-Wide Area Network Design Problem)**.** *Let $G = (S, E)$ be the graph of feasible connections on the WAN. The Wide Area Network Design Problem $WANDP(S, E, K, W, E_a, C, D_{S_T}, V_{S_T})$ consists in finding a subnetwork of G of minimum cost which satisfies the following points:*

1. *The backbone network topology must be at least 2-node-connected.*
2. *The access & backbone networks must be able to support the demand of connection & traffic required by the terminal sites.*

2.3 Problem Decomposition

Given the complexity of the WANDP, to facilitate its solution, the topological design problem is divided into 3 subproblems:

1. The Access Network Design Problem (ANDP),
2. The Backbone Network Design Problem (BNDP),
3. The Routing (or Flow Assignment) and Capacity Assignment Problem (RCAP).

In this section formal definitions of these problems based on the original parameters of the WANDP are introduced. In addition some basic definitions and notations are presented.

Definition 2.3.1 (ANDP-Access Network Design Problem)**.** *Let $G_A = (S, E_1 \cup E_2)$ be the graph of feasible connections on the Access Network and C the matrix of connection costs defined previously. The Access Network Design Problem $ANDP(S, E_1 \cup E_2, C)$ consists in finding a subgraph of G_A of minimum cost such that $\forall i \in S_T$ there exists a path from i towards some site $j \in S_D$ of the backbone network.*

Notation 2.3.2. *We denote by Γ_{ANDP} the space of feasible solutions of $ANDP(S, E_1 \cup E_2, C)$ that do not have any cycle, and with an only output towards the backbone network $\forall t \in S_T$. These have forest topology as we ilustrate in figure 2.2*

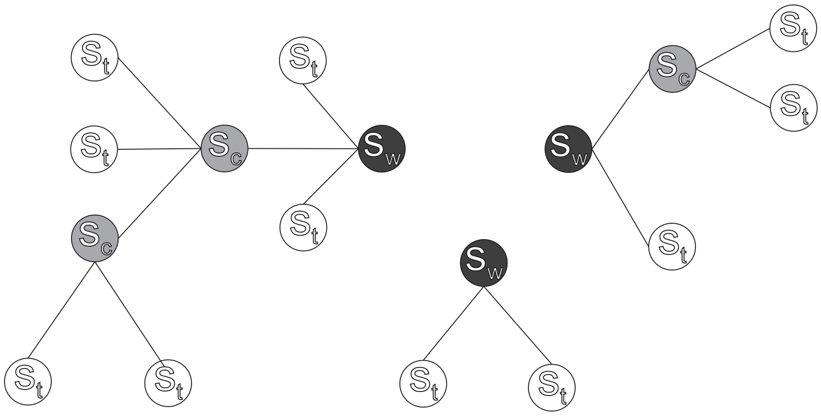

Figure 2.2: A feasible solution of ANDP.

Definition 2.3.3. *Given the $ANDP(S, E_1 \cup E_2, C)$ and $\mathcal{H}_A \in \Gamma_{ANDP}$ a feasible solution. We define the set of fixed sites of the Backbone $S_D^{(I)}$ as the subset of sites of S_D that belong to the topological structure of $\mathcal{H}_A$. In addition we define the set of Steiner nodes of the Backbone as $\bar{S}_D^{(I)} = S_D \setminus S_D^{(I)}$.*

Definition 2.3.4. *Given the $ANDP(S, E_1 \cup E_2, C)$ and $\mathcal{H}_A \in \Gamma_{ANDP}$ a feasible solution. We define the matrix of requirement of data flow between pairs of sites of $S_D^{(I)}$ as:*

$$F = \left\{ f_{ij} = \sum_{t_1 \in S_T, t_2 \in D_{t_1}} v_{t_1 t_2} \text{ such that } t_1 \in H_A^{(i)} \text{ and } t_2 \in H_A^{(j)} \right\}_{i,j \in S_D^{(I)}},$$

where $H_A^{(s_w)}$, with $s_w \in S_D^{(I)}$, is the sub-access-network of $\mathcal{H}_A$ that has to s_w as output towards to the Backbone Network.

Definition 2.3.5 (BNDP-Backbone Network Design Problem)**.** *Let $G_B = (S_D, E_3)$ be the graph of feasible connections on the Backbone Network and C the matrix of connection costs defined previously. Given the $ANDP(S, E_1 \cup E_2, C)$ and $\mathcal{H}_A \in \Gamma_{ANDP}$ a feasible solution. Let $S_D^{(I)}$ be the set of fixed sites of the Backbone and $R = \{r_{ij}\}_{i,j \in S_D^{(I)}}$ a matrix of requirement of connection between pairs of sites of $S_D^{(I)}$; the Backbone Network Design Problem $BNDP(S_D, E_3, C, R)$ consists in finding a subgraph $\mathcal{H}_B$ of G_B of minimum cost such that $\mathcal{H}_B$ satisfies the connection requirements presents in R.*

In the next chapter we present the Access Network Design Problem ANDP with different results related to the topological structure. We also introduce one of the strategies more frequently used in the commercial design tools.

Part III

THE ACCESS NETWORK DESIGN PROBLEM

Chapter 3

The Access Network Design Problem

In this chapter, the ANDP is analyzed with the objective of proposing a new approach for solving this problem. The organization of this chapter is the following. Section 3.1 presents different results related to the topological structure of the ANDP solutions. Section 3.2 presents the clustering approach as one of the strategies more frequently used by the commercial tools of design; we introduce results that show that this strategy does not give good results for a great variety of network topologies, and moreover we establish conditions so that the clustering strategy gives a global optimal solution for the ANDP. In section 3.3, given an instance of ANDP, we introduce methods to reduce the dimension of this, and therefore allows to find goods feasible solutions in a smaller time. Section 3.4 introduces the relation between the cost of the best solution given by the clustering strategy and the optimal cost. In addition another structural relations are presented. In Section 3.5 we formulate the ANDP as a Steiner Problem in Graphs (SPG).

3.1 Topological properties.

Here are presented results that characterize the topologies of the feasible solutions of an ANDP instance. Moreover, for certain types of network classes we present results that characterize the structure of the global optimal solution.
The following Proposition shows the topological form of the feasible solutions of Γ_{ANDP} for a given ANDP instance.

Proposition 3.1.1. *Given an* ANDP *with associated graph* $G_A = (S, E_1 \cup E_2)$ *and matrix of connection costs* C. *If the subnetwork* $\mathcal{H} = (S_T \cup \bar{S}, \bar{E})$ *(with* $\bar{S} \subseteq S_C \cup S_D$ *and* $\bar{E} \subseteq E_1 \cup E_2$*) is an optimal solution of* Γ_{ANDP}*, it is composed by a set of disjoint trees* $\mathcal{H} = \{H_1, \dots, H_m\}$ *that satisfy:*

1. $\forall H_l \in \mathcal{H}$, $\exists j \in S_D$ *unique* $/ j \in H_l$,
2. $\forall H_l \in \mathcal{H}$, $\exists$ *a subset* $S_T^{(l)} \subset S_T, S_T^{(l)} \neq \emptyset / S_T^{(l)} \subseteq NODES(H_l)$,
3. $\bigcup_{l=1}^{m} S_T^{(l)} = S_T$.

Proof. Trivial. □

The following Propositions present results that characterize the structure of the global optimal solution.

Proposition 3.1.2. *Let $ANDP(S, E_1 \cup E_2, C)$ be a problem where given $s_c \in S_C$, $\bar{s} \in S_C \cup S_D$, and $s \in S_T \cup S_C$ such that $\{(s, s_c), (s_c, \bar{s})\} \subset E_1 \cup E_2$ and $\exists s_w \in S_D / c_{(s,s_w)} < c_{(s,s_c)} + c_{(s_c,\bar{s})}$. Then, if $\mathcal{T}_A \in \Gamma_{ANDP}$ is a globally optimal solution, it is fulfilled that $g(s_c) \geq 3$ in $\mathcal{T}_A$, $\forall s_c \in \mathcal{T}_A, s_c \in S_C$.*

Proof. Let us suppose that there exists $\mathcal{T}_A \in \Gamma_{ANDP}$ global optimal solution such that $\exists s_c \in \mathcal{T}_A$ a concentrator site with $g(s_c) < 3$ in $\mathcal{T}_A$. If $g(s_c) = 1$ then s_c is a pendant in $\mathcal{T}_A$, therefore eliminating this, a feasible solution of smaller cost would be obtained. This is a contradiction, hence $g(s_c) \neq 1$. If $g(s_c) = 2$, let $\bar{s} \in S_C \cup S_D$ be the site adjacent to s_c in $\mathcal{T}_A$ which is its output site towards the backbone network. Let $s \in S_T \cup S_C$ be the other adjacent site in $\mathcal{T}_A$. Considering the network $\mathcal{H} = (\mathcal{T}_A \setminus \{s_c\}) \cup \{(s, s_w)\}$, where $s_w \in S_D$ satisfies $c_{(s,s_w)} < c_{(s,s_c)} + c_{(s_c,\bar{s})}$, it is fulfilled:

$$\text{COST}(\mathcal{H}) = \text{COST}(\mathcal{T}_A) - c_{(s,s_c)} - c_{(s_c,\bar{s})} + c_{(s,s_w)} < \text{COST}(\mathcal{T}_A).$$

Furthermore, it is easy to see that $\mathcal{H} \in \Gamma_{ANDP}$. Hence, this implies that $\mathcal{H}$ is a better feasible solution compared with $\mathcal{T}_A$. This is a contradiction, implicating $g(s_c) \geq 3$ in $\mathcal{T}_A$, as required and completing the proof. □

Proposition 3.1.3. *Given an $ANDP(S, E_1 \cup E_2, C)$ such that for any three sites (s_1, s_2, s_3), with $s_1 \in S_T \cup S_C$, $s_2 \in S_C$, and $s_3 \in S_C \cup S_D$, the strict triangular inequality is satisfied, i.e., $c_{(s_i,s_k)} < c_{(s_i,s_j)} + c_{(s_j,s_k)}$, $i, j, k \in (1 \dots 3)$. Then, if $\mathcal{T}_A \in \Gamma_{ANDP}$ is a globally optimal solution, it is fulfilled that $g(s_c) \geq 3$ in $\mathcal{T}_A$, $\forall s_c \in \mathcal{T}_A, s_c \in S_C$.*

Proof. As in the previous Proposition, let us suppose that there exists $\mathcal{T}_A \in \Gamma_{ANDP}$ global optimal solution such that $\exists s_c \in \mathcal{T}_A$ a concentrator site with $g(s_c) < 3$ in $\mathcal{T}_A$. Clearly $g(s_s)$ must be different to 1. Now, let us consider the case $g(s_c) = 2$ in $\mathcal{T}_A$. Let s_1, s_2 be the adjacent sites to s_c in $\mathcal{T}_A$. By hypothesis $c_{(s_1,s_2)} < c_{(s_1,s_c)} + c_{(s_c,s_2)}$. Considering the network $\bar{\mathcal{T}}_A = (\mathcal{T}_A \setminus \{s_c\}) \cup \{(s_1, s_2)\}$ we have a feasible solution and moreover:

$$\text{COST}(\bar{\mathcal{T}}_A) = \text{COST}(\mathcal{T}_A) - c_{(s_1,s_c)} - c_{(s_c,s_2)} + c_{(s_1,s_2)} < \text{COST}(\mathcal{T}_A),$$

this is a contradiction, therefore $g(s_c) \geq 3$ in $\mathcal{T}_A$, and hence completing the proof. □

3.2 The Clustering Approach applied to the ANDP.

The clustering strategies frequently are used for the solution of numerous problems of combinatorial optimization. Particularly this approach is applied in several telecommunication network design problems. In the ANDP case, there exist commercial tools of design that use the clustering strategy as base to find access network topologies. Here, we will see characteristics and properties of solutions given by this methodology, furthermore we show ANDP examples where this approach, in the best case, gives as result locally optimal solutions. Consequently, it is important to know under what conditions these locally optimal solutions become globally optimal solutions. We propose conditions under which it is guaranteed to produce the best solution for an ANDP instance applying the clustering approach.
Firstly, a series of auxiliary definitions and notations are introduced.

Definition 3.2.1 (SPG-Steiner Problem in Graphs)**.** *Let $G = (V, U)$ be a connected undirected graph, where V is the set of nodes and U denotes the set of edges. Given a non-negative cost function (also denominated weight function) $C : U \rightarrow \Re$ associated with its edges and a subset $X \subseteq V$ of terminal nodes, the Steiner problem $SPG(V, U, C, X)$ consists in finding a minimum weighted connected subgraph of G spanning all terminal nodes in X.*

Notation 3.2.2. *The solution of $SPG(V, U, C, X)$ is a Steiner minimum tree with terminal nodes in X. The non-terminal nodes of V are called Steiner nodes. We denote by $T_{V,U,C}(X)$ the Steiner minimum tree obtained solving the Steiner problem $SPG(V, U, C, X)$ formulated above.*

Definition 3.2.3. *Given a connected undirected graph $G = (V, E, C)$, where C is the matrix of connection costs, the minimum spanning tree problem $MSTP(V, E, C)$ consists in finding a minimum weighted subtree of G spanning all nodes in V.*

This problem can be seen as particular case of the $SPG(V, E, C, X)$, in which $X = V$. Accordingly, we will denote indifferently by $\mathrm{MST}(V, E, C)$ or $T_{V,E,C}(V)$ the minimum spanning tree solving $MSTP(V, E, C)$. It is easy to see that we can associate a feasible solution of the $SPG(V, E, C, X)$ with each subset $Y \subseteq (V \setminus X)$ of Steiner nodes such that the graph $G_{(X \cup Y)} = (X \cup Y, E(X \cup Y))$ is connected, given by a minimum spanning tree solving problem $MSTP(X \cup Y, E(X \cup Y), C)$. Let S^* be the set of Steiner nodes in the optimal solution of $SPG(V, E, C, X)$, the optimal solution $T_{V,E,C}(X)$ is a minimum spanning tree of the graph induced in G by the set of nodes $S^* \cup X$, i.e., the solution to the minimum spanning tree problem $MSTP(S^* \cup X, E(S^* \cup X), C)$.

The SPG definition will be used to model in alternative form the ANDP and also to model the problem of to find an optimal access sub-network. The following definitions introduce the concept of covering areas associated to concentrator & switch sites; these are used to establish the resulting components by the application of the clustering process.

Definition 3.2.4. *Given a switch site $s_w \in S_D$, the switch covering area associated to s_w is defined by:*

$$A_{s_w} = \{s_w\} \cup \{s \in S_T \cup S_C / cp_{min}(s, s_w) = \min\{cp_{min}(s, v); \forall v \in S_D\}\},$$

where $cp_{min}(u, v)$, with $u, v \in S$, is the cost of the shortest path from u to v in $G_A = (S, E_1 \cup E_2)$.

Notation 3.2.5. *We denote by S_T^w and S_C^w, respectively, the sets of sites of S_T and S_C present in A_{s_w}.*

Definition 3.2.6. *A site $s \in S_T \cup S_C$ is denominated a border site with respect to two switch sites, if it is to minimal distance to them in $G_A = (S, E_1 \cup E_2)$. The border sites of this type are assigned to a unique covering area associated to a switch site.*

Definition 3.2.7. *Given a concentrator site $s_c \in S_C$, the concentrator covering area associated to s_c is defined by:*

$$A_{s_c} = \{s_c\} \cup \left\{s \in S_T \cup S_C / c_{(s,s_c)} = \min\{c_{(s,s_v)}; \forall s_v \in S_C \cup S_D\}\right\}.$$

Notation 3.2.8. *We denote by S_T^c and S_C^c, respectively, the sets of sites of S_T and S_C present in A_{s_c}.*

Definition 3.2.9. *A site $s \in S_T \cup S_C$ is denominated a border site with respect to two concentrator sites or between concentrator-switch sites, if it is to minimal distance to them in $G_A = (S, E_1 \cup E_2)$. The border sites of this type are assigned to a unique covering area associated to a concentrator site. Moreover could happen that there exist $s_c \in S_C$ and $s_w \in S_D$ such that $cp_{min}(s, s_c) = cp_{min}(s, s_w)$ and $s \in A_{s_c}$, $s \in A_{s_w}$.*

Definition 3.2.10. *Given a site $s \in S_T \cup S_C$, we say that the site s is not in its covering area when there exists $s_c \in S_C, s_c \neq s$ such that $s \in A_{s_w}$, $s \in A_{s_c}$ and $s_c \notin A_{s_w}$, with $s_w \in S_D$.*

The following Proposition establishes an equivalence between models ANDP and SPG to design an optimal access subnetwork restricted to a certain switch covering area.

Proposition 3.2.11. *Given a switch site $s_w \in S_D$, the design of an optimal access subnetwork associated to A_{s_w}, that is to say $ANDP(A_{s_w}, (E_1 \cup E_2)|_{A_{s_w}}, C)$, is equivalent to finding an optimal solution of $SPG(A_{s_w}, E(A_{s_w}), C, S_T^w \cup \{s_w\})$.*

Proof. Trivial. □

Remark 3.2.12. *The feasible solution of $ANDP(S, E_1 \cup E_2, C)$ given by the union of optimal solutions associated with $SPG(A_{s_w}, E(A_{s_w}), C, S_T^w \cup \{s_w\})$, $\forall s_w \in S_D$, that is to say $\mathcal{T} = \bigcup_{\forall s_w \in S_D} T_{A_{s_w}, E(A_{s_w}), C}(S_T^w \cup \{s_w\})$, it is not guaranteed to be globally optimal.*

Now, it is presented an ANDP example that illustrates the difference between the globally optimal solution and the resulting optimal solution of to apply the SPG model to each one of the switch covering areas. Figure 3.1 shows the set of feasible connections of the ANDP instance chosen as example.

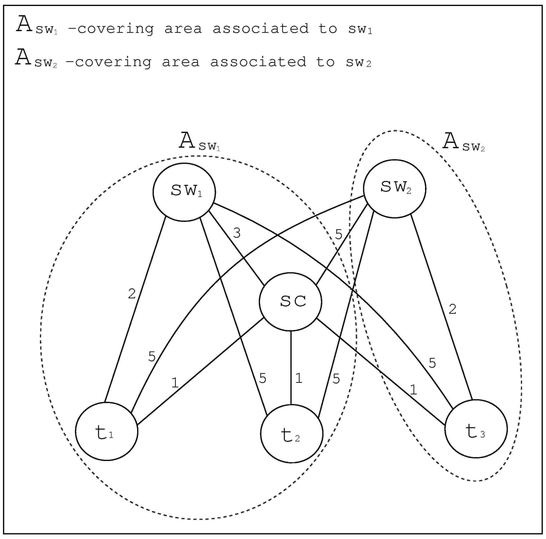

Figure 3.1: Feasible connections of an Access Network.

Figure 3.2 shows the switch covering areas associated to the sites s_{w_1} and s_{w_2}. The $SPG(A_{s_{w_1}}, E(A_{s_{w_1}}), C, S_T^{w1} \cup \{s_{w_1}\})$ and the $SPG(A_{s_{w_2}}, E(A_{s_{w_2}}), C, S_T^{w2} \cup \{s_{w_2}\})$ give the best solutions corresponding to the ANDP restricted to $A_{s_{w_1}}$ and $A_{s_{w_2}}$ respectively.

It is easy to see that the optimal solution of $SPG(A_{s_{w_2}}, E(A_{s_{w_2}}), C, S_T^{w2} \cup \{s_{w_2}\})$ is the same $A_{s_{w_2}}$. Figure 3.3 shows the optimal solution of $SPG(A_{s_{w_1}}, E(A_{s_{w_1}}), C, S_T^{w1} \cup \{s_{w_1}\})$ and the global optimal solution of the ANDP.

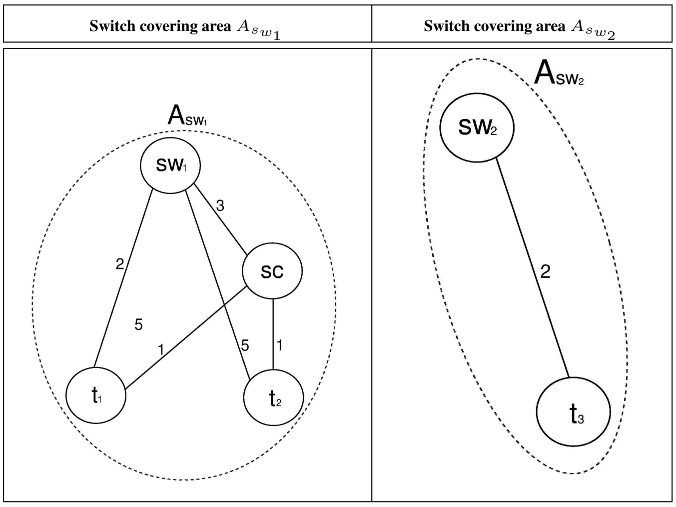

Figure 3.2: Switch covering areas of s_{w_1} and s_{w_2} respectively.

The cost of the feasible solution given by the union of the optimal access subnetworks has cost 7, and the optimum cost is 6.

The following Proposition describes the different situations that can happen between two concentrator covering areas of an ANDP.

Proposition 3.2.13. *Given an $ANDP(S, E_1 \cup E_2, C)$ and $s_{c_1}, s_{c_2} \in S_C$, then it is fulfilled the following:*

1. *if $s_{c_2} \in A_{s_{c_1}} \Longrightarrow \nexists s_{c_3} \in S_C, s_{c_3} \neq s_{c_1}, s_{c_2} / s_{c_2} \in A_{s_{c_3}}$,*
2. *$A_{s_{c_1}} \cap A_{s_{c_2}} \subseteq \{s_{c_1}, s_{c_2}\}$.*

Proof. Let us suppose that given $s_{c_1}, s_{c_2} \in S_C$ such that $s_{c_2} \in A_{s_{c_1}}$; there exists $s_{c_3} \in S_C, s_{c_3} \neq s_{c_1}, s_{c_2}$, where it satisfies $s_{c_2} \in A_{s_{c_3}}$. This implies that s_{c_2} is a border site that belongs to two concentrator covering areas associated to two different concentrator sites. This contradicts Definition 3.2.9, therefore (1) is demonstrated.
Now, by Definition 3.2.9 it is easy to see that $\nexists s_t \in S_T$ such that $s_t \in A_{s_{c_1}}$ and $s_t \in A_{s_{c_2}}$. Moreover could happen that $s_{c_1} \in A_{s_{c_2}}$ and $s_{c_2} \in A_{s_{c_1}}$. Since by (1) we known that $\nexists s \in S_C, s \neq s_{c_1}, s_{c_2}$, such that $s \in A_{s_{c_1}}$ and $s \in A_{s_{c_2}}$; hence we have that $A_{s_{c_1}} \cap A_{s_{c_2}} \subseteq \{s_{c_1}, s_{c_2}\}$, as required, completing the proof of (2). □

Figure 3.4 shows the different situations presented in Proposition 3.2.13.

In order to illustrate cases of topologies where the utilization of covering areas do not provide of “good" feasible solutions, the following results are introduced.

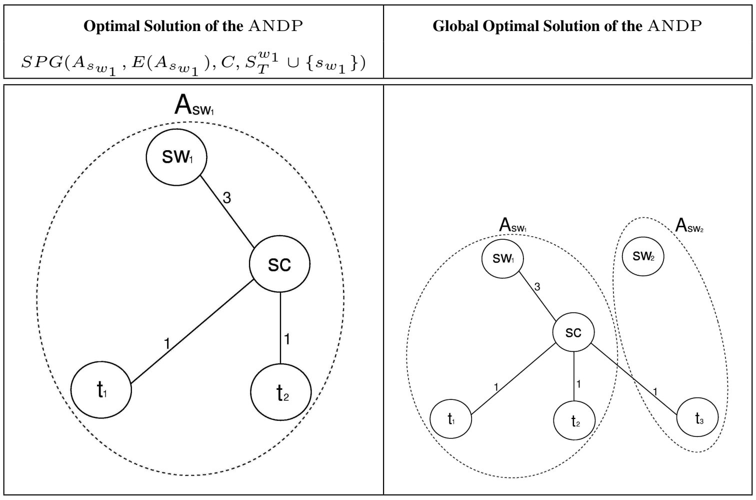

Figure 3.3:

Proposition 3.2.14. *In the worst case an $ANDP(S, E_1 \cup E_2, C)$ such that $\exists s_c \in A_{s_w} / A_{s_c} \not\subset A_{s_w}$, with $s_c \in S_C$ and $s_w \in S_D$, has at the most $n_C + n_T - 1$ sites in A_{s_c} that does not belong to A_{s_w}.*

Proof. Fixed the set of backbone sites S_D, we will demonstrate by induction in the amount of terminal and concentrator sites, that there exists an ANDP in the hypotheses of the Proposition 3.2.14 such that $n_C + n_T - 1$ is the number of sites of A_{s_c} that does not belong to A_{s_w}.
Case 1.$n_T + n_C = 1$. In this case $n_T = 0$ and therefore the Proposition 3.2.14 is satisfied in empty form.
Case 2.$n_T + n_C > 1$. As inductive hypothesis (denoted by I.H.) we have that $\forall k / 1 < k < h$ there exists an $ANDP(S, E_1 \cup E_2, C)$ in the conditions of the Proposition, with $k = n_T + n_C$, such that the number of sites of A_{s_c} that does not belong to A_{s_w} is $k - 1$. As inductive thesis (denoted by I.T.) the Proposition is satisfied when $k = h$.
By I.H. we known that there exists an ANDP with a set of terminal sites S_T and a set of concentrator sites S_C, with $|S_T \cup S_C| = n_T + n_C = h - 1$, such that the number of sites of A_{s_c} that does not belong to A_{s_w} is $h - 2$, for a certain concentrator site $s_c \in S_C$ and a backbone site $s_w \in S_D$ that fulfill $s_c \in A_{s_w}$ and $A_{s_c} \not\subset A_{s_w}$. We added a new site $\bar{s}$ (indifferently of the concentrator or terminal type) and new feasible connections that satisfy:

1. $c_{(\bar{s}, s_c)} < c_{(\bar{s}, s)}, \forall s \in S_C \cup S_D, s \neq s_c$,
2. At least one of the following conditions is satisfied:
   - $\exists s_{c_2} \in S_C, s_{c_2} \neq s_c$, $\exists s_{w_2} \in S_D, s_{w_2} \neq s_w$ such that:
   $$c_{(\bar{s}, s_{c_2})} + cp_{min}(s_{c_2}, s_{w_2}) < c_{(\bar{s}, s_c)} + c_{(s_c, s_w)},$$
   - $\exists s_{w_2} \in S_D, s_{w_2} \neq s_w$ such that $c_{(\bar{s}, s_{w_2})} < c_{(\bar{s}, s_c)} + c_{(s_c, s_w)}$,

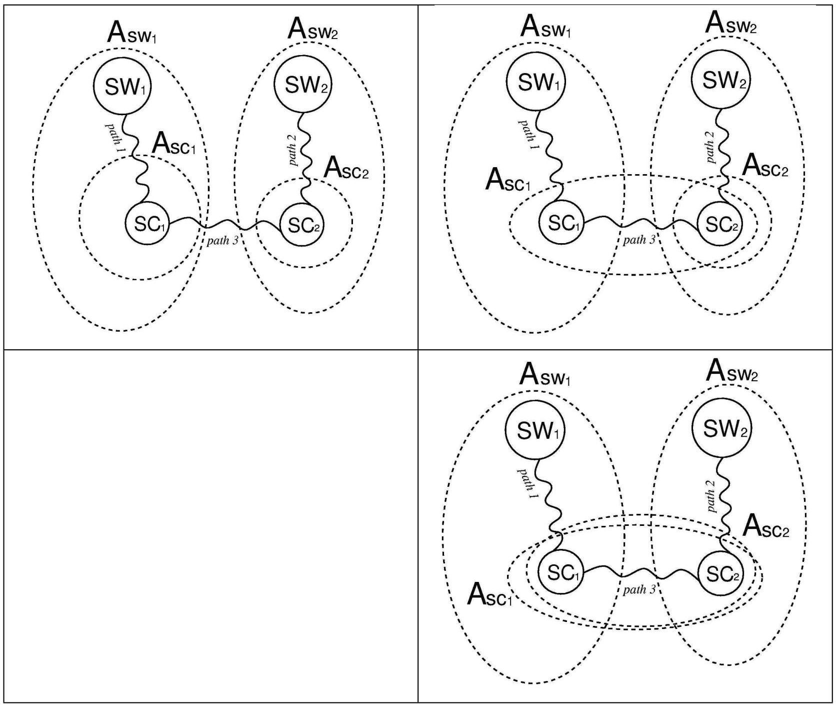

Figure 3.4: Relation between different concentrator covering areas.

3. If $\bar{s}$ is a concentrator site then $c_{(s_{c_2},\bar{s})} + c_{(\bar{s},s_c)} > c_{(s_{c_2},s_c)}$, $\forall s_{c_2} \in S_C, s_{c_2} \neq s_c$.

Other feasible connections can be considered depending of the type of site. Condition (1) implies that $\bar{s} \in A_{s_c}$, condition (2) implies that $\exists s_{w_2} \in S_D, s_{w_2} \neq s_w$, such that $\bar{s} \in A_{s_{w_2}}$, i.e. $\bar{s} \notin A_{s_w}$, and the condition (3) guarantees that the concentrator sites of $S_C \backslash \{s_c\}$ continue belonging to A_{s_c} considering the new connections towards $\bar{s}$, in the case that $\bar{s}$ is a concentrator site. Let $\mathcal{F}_{\bar{s}}$ be the set of the new feasible connections added from $\bar{s}$ to $G_A = (S, E_1 \cup E_2)$. We will denote as $\bar{C}$ the matrix of connection costs extended to $E_1 \cup E_2 \cup \mathcal{F}_{\bar{s}}$. Now, let us consider $ANDP(S \cup \{\bar{s}\}, E_1 \cup E_2 \cup \mathcal{F}_{\bar{s}}, \bar{C})$, it is easy to see that the number of sites in A_{s_c} that does not belong to A_{s_w} considering the network $\bar{G}_A = (S, E_1 \cup E_2 \cup \mathcal{F}_{\bar{s}})$ is $(h-2)+1 = h-1$. This completes the proof of Proposition 3.2.14. □

Figure 3.5 illustrates the situation presented in Proposition 3.2.14.

Corollary 3.2.15. *In the worst case an $ANDP(S, E_1 \cup E_2, C)$ with $n_C \geq 2$ concentrator sites, all the sites of $S_T \cup S_C$ are not in their covering areas.*

Proof. We will demonstrate by induction in the amount of concentrator sites, that there exists an ANDP in the conditions of the Corollary 3.2.15 where all the sites of $S_T \cup S_C$ are not in their covering areas.

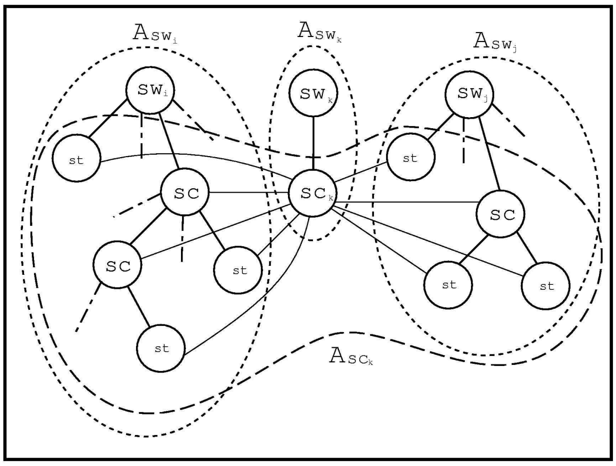

Figure 3.5: Concentrator covering area associated to s_{c_k}.

Case 1. $n_C = 2$. In this case we must find an instance of ANDP with exactly two concentrator sites such that all the sites of $S_T \cup S_C$ are not in their covering areas.
Let $ANDP(\{s_{w_1}, s_{w_2}\} \cup \{s_{c_1}, s_{c_2}\} \cup (S_T^{(1)} \cup S_T^{(2)}), E_1 \cup E_2, C)$ be an instance of the problem where $\{s_{w_1}, s_{w_2}\}$ are switch sites, $\{s_{c_1}, s_{c_2}\}$ are concentrator sites, $S_T^{(1)} \cup S_T^{(2)}$ are terminal sites, and their feasible connections verify:

1. $c_{(s_{c_1}, s_{w_1})} < Min\left\{c_{(s_{c_1}, s_{w_2})}, c_{(s_{c_1}, s_{c_2})} + c_{(s_{c_2}, s_{w_2})}\right\}$,

2. $c_{(s_{c_2}, s_{w_2})} < Min\left\{c_{(s_{c_2}, s_{w_1})}, c_{(s_{c_2}, s_{c_1})} + c_{(s_{c_1}, s_{w_1})}\right\}$,

3. $c_{(s_{c_1}, s_{c_2})} < Min\left\{c_{(s_{c_1}, s_{w_1})}, c_{(s_{c_1}, s_{w_2})}, c_{(s_{c_2}, s_{w_1})}, c_{(s_{c_2}, s_{w_2})}\right\}$,

4. $c_{(s_t, s_{c_2})} < Min\left\{c_{(s_t, s_{w_1})}, c_{(s_t, s_{w_2})}\right\}$, $\forall s_t \in S_T^{(1)}$,

5. $c_{(s_t, s_{c_1})} < Min\left\{c_{(s_t, s_{w_1})}, c_{(s_t, s_{w_2})}\right\}$, $\forall s_t \in S_T^{(2)}$,

6. $c_{(s_t, s_{w_1})} < Min\left\{c_{(s_t, s_{w_2})}, c_{(s_t, s_{c_2})} + c_{(s_{c_2}, s_{w_2})}\right\}$, $\forall s_t \in S_T^{(1)}$,

7. $c_{(s_t, s_{w_2})} < Min\left\{c_{(s_t, s_{w_1})}, c_{(s_t, s_{c_1})} + c_{(s_{c_1}, s_{w_1})}\right\}$, $\forall s_t \in S_T^{(2)}$.

We assume that these feasible connections exist only. Conditions (1) and (2) imply that $s_{c_1} \in A_{s_{w_1}}$ and $s_{c_2} \in A_{s_{w_2}}$ respectively. Condition (3) implies that $s_{c_1} \in A_{s_{c_2}}$ and $s_{c_2} \in A_{s_{c_1}}$. Conditions (4) and (5) imply that $s_t \in A_{s_{c_2}}$ $\forall s_t \in S_T^{(1)}$, and $s_t \in A_{s_{c_1}}$ $\forall s_t \in S_T^{(2)}$ respectively. Conditions (6) and (7) imply that $s_t \in A_{s_{w_1}}$ $\forall s_t \in S_T^{(1)}$, and $s_t \in A_{s_{w_2}}$ $\forall s_t \in S_T^{(2)}$. Therefore by Definition 3.2.10, the concentrator sites s_{c_1}, s_{c_2}, and the terminal sites of $S_T^{(1)} \cup S_T^{(2)}$ are not

in their covering areas.
Case 2. $n_C > 2$. As inductive hypothesis (denoted by I.H.) we assume that given $n_C/2 < |S_C| = n_C < h$ there exists an $ANDP(S, E_1 \cup E_2, C)$ in the conditions of the Corollary, such that all the sites of $S_T \cup S_C$ are not in their covering areas. As inductive thesis (denoted by I.T.) the Corollary is fulfilled when $n_C = h$.
By I.H. there exists an $ANDP(S, E_1 \cup E_2, C)/n_C = h-1$ and all the sites of $S_T \cup S_C$ are not in their covering areas. Let us consider a new concentrator site $\bar{s}_c$ and new feasible connections that satisfy:

1. We choose a concentrator site $s_{c_i} \in S_C$ and added a new feasible connection between s_{c_i} and $\bar{s}_c$.
2. Let $s_{c_j} \in S_C$ be the concentrator site that satisfies $s_{c_i} \in A_{s_{c_j}}$. We choose a switch site $s_{w_k} \in S_D$ and added a new connection between $\bar{s}_c$ and s_{w_k} so that:
   (a) $c_{(\bar{s}_c, s_{c_i})} > c_{(s_{c_i}, s_{c_j})}$,
   (b) $c_{(\bar{s}_c, s_{c_i})} < c_{(\bar{s}_c, s_{w_k})}$,
   (c) $c_{(\bar{s}_c, s_{w_k})} < c_{(\bar{s}_c, s_{c_i})} + cp_{min}(s_{c_i}, s_{w_i})$, where s_{w_i} is the switch site that satisfies $s_{c_i} \in A_{s_{w_i}}$.
3. $c_{(s_t, \bar{s}_c)} > c_{(s_t, s)}, \forall s \in S_C \cup S_D$.

We assume that only these feasible connections are added. Condition $(2.b)$ implies that the new concentrator site $\bar{s}_c$ belongs to the concentrator covering area $A_{s_{c_i}}$. Condition $(2.c)$ implies that $\bar{s}_c$ belongs to the switch covering area $A_{s_{w_k}}$. Condition $(2.a)$ implies that s_{c_i} continues belonging to the concentrator covering area $A_{s_{c_j}}$. Also, condition (3) implies that all the terminal sites of S_T continue belonging to their concentrator covering areas. Let $\mathcal{F}_{\bar{s}_c}$ be the set of the new feasible connections defined above. We will denote as $\bar{C}$ the matrix of connection costs extended to $E_1 \cup E_2 \cup \mathcal{F}_{\bar{s}_c}$. Considering the $ANDP(S \cup \{\bar{s}_c\}, E_1 \cup E_2 \cup \mathcal{F}_{\bar{s}_c}, \bar{C})$, it is easy to see that all the sites of $S_T \cup S_C \cup \{\bar{s}_c\}$ are not in their covering areas, and in addition the total number of concentrator sites is $n_C + 1 = (h-1)+1 = h$, as required, and completing the proof. $\square$

The following Lemma introduces conditions that must satisfy the solution constructed by the application of clustering approach to each one of the switch sites, so that it fulfills being a global optimal solution of the ANDP. This solution is the union of the optimal access sub-networks given by the global optimal solutions of the Steiner problems
$SPG(A_{s_w}, E(A_{s_w}), C, S_T^w \cup \{s_w\}), \forall s_w \in S_D$.

Lemma 3.2.16. *If $\mathcal{T} = \bigcup_{\forall s_w \in S_D} T_{A_{s_w}, E(A_{s_w}), C}(S_T^w \cup \{s_w\})$ is a global optimal solution of $ANDP(S, E_1 \cup E_2, C)$, then the following statements are fulfilled.*

1. *$\forall s_{c_1}, s_{c_2} \in \mathcal{T}$ such that $s_{c_1} \in A_{s_w}, s_{c_2} \notin A_{s_w}$ (with $s_w \in S_D$), the sets $A_{s_{c_1}}$ and $A_{s_{c_2}}$ satisfy $A_{s_{c_1}} \cap A_{s_{c_2}} = \emptyset$; unless s_{c_1} or s_{c_2} will be a border site.*

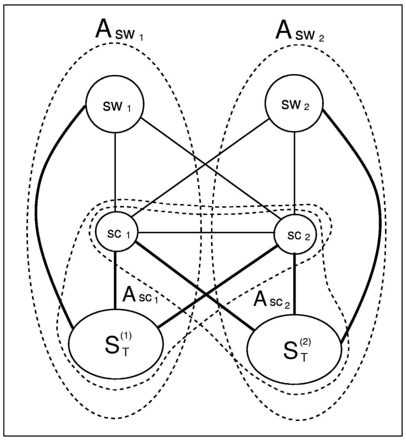

Figure 3.6: Example of Corollary 3.2.15 with $n_C = 2$.

2. *$\forall$ triangle $(s, s_{c_1}, s_{c_2}) \in G_A$ such that $s, s_{c_1} \in \mathcal{T}$, $s, s_{c_1} \in A_{s_w}$, $s_{c_2} \notin \mathcal{T}$, and $s_{c_2} \notin A_{s_w}$, where $s_w \in S_D$, the inequality $c_{(s,s_{c_1})} \leq c_{(s,s_{c_2})} + c_{(s_{c_1},s_{c_2})}$ is satisfied.*

3. *$\forall (s, s_c) \in \mathcal{T} / s \notin A_{s_c}$, if $\exists s_{c_2} \in S_C / s \in A_{s_{c_2}}$ then $s_{c_2} \notin \mathcal{T}$.*

Proof. We will demonstrate the three statements separately.
Statement 1. Let s_{c_1} and s_{c_2} be concentrator sites such that $s_{c_1} \in A_{s_w}$ and $s_{c_2} \in A_{s_w}$, with $s_w \in S_D$. Let us suppose that $A_{s_{c_1}} \cap A_{s_{c_2}} \neq \emptyset$. This implies that s_{c_1} or s_{c_2} or both are in the intersection. Suppose that $s_{c_1} \in A_{s_{c_2}}$. By definition of concentrator covering area $c_{(s_{c_1},s_{c_2})} = \min\{c_{(s_{c_1},s_v)}; (s_{c_1}, s_v) \in E_2\}$. Suppose, in addition, that s_{c_1} is not a border site, that is to say $c_{(s_{c_1},s_{c_2})} < c_{(s_{c_1},s_v)}, \forall s_v \in S_C \cup S_D, s_v \neq s_{c_2}$. Let $\bar{s} \in \mathcal{T}$ ($\bar{s} \in S_C \cup S_D$) be the output site towards the backbone network, associated to s_{c_1}. Let $\bar{\mathcal{H}}$ be a network defined by $\bar{\mathcal{H}} = (\mathcal{T} \setminus \{(s_{c_1}, \bar{s})\}) \cup \{(s_{c_1}, s_{c_2})\}$. It is easy to see that $\bar{\mathcal{H}}$ is a feasible solution of Γ_{ANDP}; we will analyze its cost.

$$\text{COST}(\bar{\mathcal{H}}) = \text{COST}(\mathcal{T}) - c_{(s_{c_1},\bar{s})} + c_{(s_{c_1},s_{c_2})} \overset{\overset{c_{(s_{c_1},s_{c_2})}<c_{(s_{c_1},\bar{s})}}{\uparrow}}{<} \text{COST}(\mathcal{T})$$

This is a contradiction, therefore $s_{c_1} \notin A_{s_{c_2}}$. By symmetry $s_{c_2} \notin A_{s_{c_1}}$, completing the proof of Statement 1.

Statement 2. Let $(s, s_{c_1}, s_{c_2}) \in G_A$ be a triangle such that $s, s_{c_1} \in \mathcal{T}$, $s, s_{c_1} \in A_{s_w}$, $s_{c_2} \notin \mathcal{T}$, and $s_{c_2} \notin A_{s_w}$. Let us suppose that $c_{(s,s_{c_1})} > c_{(s,s_{c_2})} + c_{(s_{c_1},s_{c_2})}$. Let $\bar{\mathcal{H}}$ be a network defined by $\bar{\mathcal{H}} = (\mathcal{T} \setminus \{(s, s_{c_1})\}) \cup \{(s, s_{c_2}), (s_{c_1}, s_{c_2})\}$. It is easy to see that $\bar{\mathcal{H}}$ is a feasible solution of Γ_{ANDP}; and moreover its cost is given by:

$$\text{COST}(\bar{\mathcal{H}}) = \text{COST}(\mathcal{T}) - c_{(s,s_{c_1})} + c_{(s,s_{c_2})} + c_{(s_{c_1},s_{c_2})} < \text{COST}(\mathcal{T}),$$

this is a contradiction, hence $c_{(s,s_{c_1})} \leq c_{(s,s_{c_2})} + c_{(s_{c_1},s_{c_2})}$ as required.

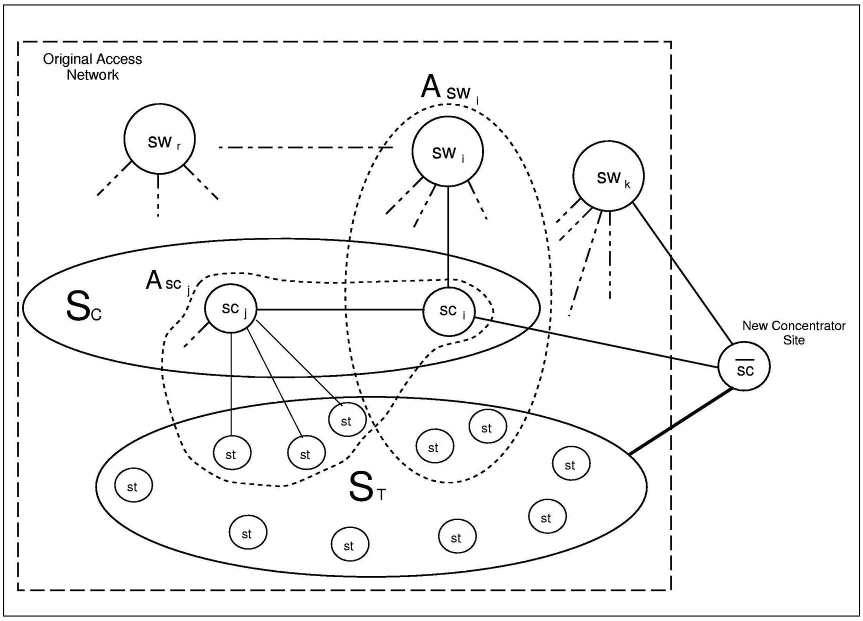

Figure 3.7: Example of Corollary 3.2.15 with $n_C > 2$.

Statement 3. Let us consider a connection $(s, s_c) \in \mathcal{T}/s \notin A_{s_c}$. Suppose, in addition, that $\exists s_{c_2} \in S_C/s \in A_{s_{c_2}}$ and $s_{c_2} \in \mathcal{T}$. Let $\bar{\mathcal{H}}$ be a network defined by $\bar{\mathcal{H}} = (\mathcal{T} \setminus \{(s, s_c)\}) \cup \{(s, s_{c_2})\}$. It is easy to see that $\bar{\mathcal{H}}$ is a feasible solution of Γ_{ANDP}; we will see its cost.

$$\text{COST}(\bar{\mathcal{H}}) = \text{COST}(\mathcal{T}) - c_{(s,s_c)} + c_{(s,s_{c_2})} \overset{\overset{c_{(s,s_{c_2})}-c_{(s,s_c)}<0}{\uparrow}}{<} \text{COST}(\mathcal{T}),$$

this is a contradiction, therefore $\nexists s_{c_2} \in S_C/s \in A_{s_{c_2}}$ and $s_{c_2} \in \mathcal{T}$. This completes the proof of Lemma 3.2.16 $\square$

Definition 3.2.17. *Given a switch site $s_w \in S_D$, the ratio of the covering area A_{s_w} is defined by:*

$$r(A_{s_w}) \overset{\text{def}}{=} \begin{cases} \max\{cp_{min}(s, s_w)/s \in A_{s_w}\} & \textit{if } |A_{s_w}| > 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 3.2.18. *Given two switch sites $s_{w_1}, s_{w_2} \in S_D$, the distance between their covering areas $A_{s_{w_1}}$ and $A_{s_{w_2}}$ is defined by:*

$$d_r(A_{s_{w_1}}, A_{s_{w_2}}) \overset{\text{def}}{=} \begin{cases} \min\{c_{(s_1,s_2)}/s_1 \in A_{s_{w_1}}, s_2 \in A_{s_{w_2}}\} & \textit{if } \exists \textit{ a connection between both,} \\ +\infty & \textit{otherwise.} \end{cases}$$

The following Lemma introduces a sufficient condition so that the feasible solution constructed by the application of clustering approach to each one of the switch sites is globally optimal. This condition establishes a relation between the ratio of covering area associated to a switch site and its distance with other switch covering areas.

Lemma 3.2.19. *Let $ANDP(S, E_1 \cup E_2, C)$ be such that $\forall s_{w_1}, s_{w_2} \in S_D, s_{w_1} \neq s_{w_2}$, it is fulfilled that $r(A_{s_{w_1}}) \leq d_r(A_{s_{w_1}}, A_{s_{w_2}})$, then the feasible solution given by $\mathcal{T} = \bigcup_{\forall s_w \in S_D} T_{A_{s_w}, E(A_{s_w}), C}(S_T^w \cup \{s_w\})$ is globally optimal.*

Proof. The Lemma will be demonstrated by induction in $n_D = |S_D|$.
Case 1.$n_D = 1$. In this case there exists an only switch site and furthermore we assume that $A_{s_w} = \{s_w\} \cup S_T \cup S_C$, ($S_T^w = S_T$ and $S_C^w = S_C$). The ANDP is formulated as $ANDP(A_{s_w}, (E_1 \cup E_2)|_{A_{s_w}}, C)$.
By Proposition 3.2.11, considering the $SPG(A_{s_w}, E(A_{s_w}), S_T \cup \{s_w\})$, both problems are equivalent. Therefore, if $\mathcal{T}_1 = T_{A_{s_w}, E(A_{s_w}), C}(S_T \cup \{s_w\})$ is an optimal solution of the SPG, also it is an optimal solution for the ANDP.
Case 2.$n_D > 1$. As inductive hypothesis (denoted by I.H.) the Lemma is fulfilled $\forall ANDP$ such that $1 < n_D = k < h$, and as inductive thesis (denoted by I.T.) the Lemma is fulfilled when $n_D = h$.
Let $ANDP(S, E_1 \cup E_2, C)$ be in the hypothesis of the Lemma, such that $n_D = |S_D| = h$. Let $s_{w_1} \in S_D$ be a switch site such that $|A_{s_{w_1}}| > 1$ (we assume that at least one in this condition there exists). Let us consider the sets $\bar{S}_D = S_D \setminus \{s_{w_1}\}$ and $\bar{S} = S \setminus A_{s_{w_1}}$, and the problems $ANDP(A_{s_{w_1}}, (E_1 \cup E_2)|_{A_{s_{w_1}}}, C)$ and $ANDP(\bar{S}, (E_1 \cup E_2)|_{\bar{S}}, C)$. It is easy to see that $\forall \bar{s}_{w_1}, \bar{s}_{w_2} \in \bar{S}_D$, it is fulfilled that $r(A_{\bar{s}_{w_1}}) \leq d_r(A_{\bar{s}_{w_1}}, A_{\bar{s}_{w_2}})$ considering $r(\cdot)$ and $d_r(\cdot,\cdot)$ restricted to $G_{\bar{S}} = (\bar{S}, (E_1 \cup E_2)|_{\bar{S}})$. Then by H.I., (we know that $|\bar{S}_D| = h-1$), $\bar{\mathcal{T}} = \bigcup_{\forall s_w \in \bar{S}_D} T_{A_{s_w}, E(A_{s_w}), C}(S_T^w \cup \{s_w\})$ is an optimal solution of the problem $ANDP(\bar{S}, (E_1 \cup E_2)|_{\bar{S}}, C)$. Moreover, by Proposition 3.2.11, $\mathcal{T}_1 = T_{A_{s_{w_1}}, E(A_{s_{w_1}}), C}(S_T^{w_1} \cup \{s_{w_1}\})$ is an optimal solution of the problem $ANDP(A_{s_{w_1}}, (E_1 \cup E_2)|_{A_{s_{w_1}}}, C)$.
We will prove now that an optimal solution of $ANDP(\bar{S}, (E_1 \cup E_2)|_{\bar{S}}, C)$, union an optimal solution of $ANDP(A_{s_{w_1}}, (E_1 \cup E_2)|_{A_{s_{w_1}}}, C)$, form an optimal solution of $ANDP(S, E_1 \cup E_2, C)$. Let $\mathcal{H}_{\bar{S}}$ and $\mathcal{H}_{A_{sw_1}}$ be optimal solutions of the problems $ANDP(\bar{S}, (E_1 \cup E_2)|_{\bar{S}}, C)$ and $ANDP(A_{s_{w_1}}, (E_1 \cup E_2)|_{A_{s_{w_1}}}, C)$ respectively. It is easy to see that $\mathcal{H}_{\bar{S}} \cup \mathcal{H}_{A_{sw_1}}$ is a feasible solution of the $ANDP(S, E_1 \cup E_2, C)$; we will prove that it is optimal. Let c_{opt} be the cost of an optimal solution of the $ANDP(S, E_1 \cup E_2, C)$. Let us suppose that $\textsc{cost}(\mathcal{H}_{\bar{S}}) + \textsc{cost}(\mathcal{H}_{A_{sw_1}}) > c_{opt}$. This is certain if and only if, all optimal solution of $ANDP(S, E_1 \cup E_2, C)$ has at least a connection in the set $B = (E_1 \cup E_2) \setminus ((E_1 \cup E_2)|_{\bar{S}} \bigcup (E_1 \cup E_2)|_{A_{sw_1}})$. The set B can also be seen as $B = \{(s_i, s_j) \in E_1 \cup E_2 / s_i \in A_{s_{w_1}}, s_j \in \bar{S}\}$. Let $\mathcal{H}_{opt}$ be an optimal solution of the $ANDP(S, E_1 \cup E_2, C)$. Let B_H be the set of connections defined by $B_H = \{(s_i, s_j) \in B / (s_i, s_j) \in \mathcal{H}_{opt}\}$. Let I_H be the set of sites defined by $I_H = \{s_i \in A_{s_{w_1}} / (s_i, s_j) \in B_H, \textit{with } s_j \in \bar{S}\}$. Now let $\bar{\mathcal{H}}$ be a network obtained from $\mathcal{H}$ such that $\bar{\mathcal{H}} = (\mathcal{H}_{opt} \setminus B_H) \cup \left(\bigcup_{\forall s_i \in I_H} p_{(s_i, s_{w_1})}\right)$, where $p_{(s_i, s_{w_1})}$ is the shortest path between s_i and s_{w_1} in $G_{A_{sw_1}} = (A_{s_{w_1}}, (E_1 \cup E_2)|_{A_{s_{w_1}}})$. Clearly $\bar{\mathcal{H}}$ is a feasible solution of the

$ANDP(S, E_1 \cup E_2, C)$, next we will analyze its cost.

$$\begin{aligned}
\textsc{cost}(\bar{\mathcal{H}}) &= \textsc{cost}(\mathcal{H}_{opt}) - \sum_{\forall (s_i,s_j)\in B_H} c_{(s_i,s_j)} + \sum_{\forall s_i \in I_H} \textsc{cost}(p_{(s_i,s_{w_1})}) \\
&\overset{\overset{\text{COST}(p_{(s_i,s_{w_1})})\leq r(A_{s_{w_1}})}{\uparrow}}{\leq} \textsc{cost}(\mathcal{H}_{opt}) - \sum_{\forall (s_i,s_j)\in B_H} c_{(s_i,s_j)} + |I_H| \cdot r(A_{s_{w_1}}) \\
&\overset{\overset{\min\{d_r(A_{s_{w_1}},A_{\bar{s}_w});\bar{s}_w\in \bar{S}_D\}\leq c_{(s_i,s_j)}}{\uparrow}}{\leq} \textsc{cost}(\mathcal{H}_{opt}) - |B_H| \cdot d_r^{min} + |I_H| \cdot r(A_{s_{w_1}}) \\
&\overset{\overset{|B_H|=|I_H|}{\uparrow}}{=} \textsc{cost}(\mathcal{H}_{opt}) + |I_H| \cdot (r(A_{s_{w_1}}) - d_r^{min}) \overset{\overset{d_r^{min}\geq r(A_{s_{w_1}})}{\uparrow}}{\leq} \textsc{cost}(\mathcal{H}_{opt})
\end{aligned}$$

Proving that $\textsc{cost}(\bar{\mathcal{H}}) \leq \textsc{cost}(\mathcal{H}_{opt})$. This implies that in the hypothesis of the Lemma, there exists an optimal solution of the $ANDP(S, E_1 \cup E_2, C)$ such that cannot have any connection in B. Then $\textsc{cost}(\mathcal{H}_{\bar{S}}) + \textsc{cost}(\mathcal{H}_{A_{s_{w_1}}}) = c_{opt}$ and therefore $\mathcal{H}_{\bar{S}} \cup \mathcal{H}_{A_{s_{w_1}}}$ is an optimal solution of the $ANDP(S, E_1 \cup E_2, C)$. In particular $\bar{\mathcal{T}} \cup \mathcal{T}_1$ is an optimal solution of the $ANDP(S, E_1 \cup E_2, C)$. Furthermore, we have that:

$$\begin{aligned}
\bar{\mathcal{T}} \cup \mathcal{T}_1 &= \left(\bigcup_{\forall s_w \in \bar{S}_D} T_{A_{s_w},E(A_{s_w}),C}(S_T^w \cup \{s_w\}) \right) \bigcup \left(T_{A_{s_{w_1}},E(A_{s_{w_1}}),C}(S_T^{w_1} \cup \{s_{w_1}\}) \right) \\
&= \bigcup_{\forall s_j \in S_D} T_{A_{s_j},E(A_{s_j}),C}(S_T^j \cup \{s_j\}) = \mathcal{T},
\end{aligned}$$

completing the induction and the proof of Lemma 3.2.19. □

3.3 Reducing the dimension of the problem.

Given the complexity of the ANDP and ANDP$^{(\leq 1)}$ (by [4] and Theorem 3.3.2 respectively, both of them belong to the NP-Hard class), it is very useful to provide techniques capable of reducing the dimension of the original problem to an equivalent smaller problem. We introduce a form of determining a subset $X_C \subseteq S_C$ of concentrator sites, which satisfies that $\forall s_c \in X_C$, s_c belongs to any global optimal solution of Γ_{ANDP}. Firstly, we present a general result where the proposed method has exponential complexity, and later a result that proposes a method of polynomial complexity in order to determine a subset with the previous characteristics and consequently transforming the problem to a reduced problem with smaller amount of non-fixed nodes (sites of S_C that do not necessarily belong to an optimal solution).

Definition 3.3.1 (ANDP$^{(\leq k)}$ where k is an integer$/0 \leq k \leq n_C$). *Let $G_A = (S, E_1 \cup E_2)$ be the graph of feasible connections on the Access Network and C the matrix of connection* *costs. The Access Network Design Problem of k^{th} level* $\mathrm{ANDP}^{(\leq k)}(S, E_1 \cup E_2, C)$ *consists in finding a subgraph of G_A of minimum cost such that $\forall i \in S_T$ there exists a path from i towards some site $j \in S_D$ of the backbone network and at the most have k concentrators connected in line.*

Theorem 3.3.2. *The* $\mathrm{ANDP}^{(\leq 1)}$ *belongs to the NP-Hard class.*

Proof. Let $S_T = \{t_1, ..., t_n\}$ and $S_C = \{s_1, ...s_m\}$ denote the set of terminal and concentrator sites respectively. Let Z a fictitious node representing the backbone. We define $\hat{S}^z = \{z_0, z_1, ..., z_m\}$ with $z_0 = Z$ and $z_i = s_i$ $i \in 1..m$. Let $C = \{c_{i,j}\}_{i \in S_T, j \in \hat{S}^z}$ denotes a matrix which gives us for a site of S_T and a site of $\hat{S}^z$, the cost of laying a line among them; and $\hat{C} = \{\hat{c}_i\}_{i \in 1..m}$ denotes the costs of laying a line among a site of S_C and the backbone.

The following Mathematical Programming Problem resolves the $\mathrm{ANDP}^{(\leq 1)}$ exactly.

$$(P_{\mathrm{ANDP}^{(\leq 1)}}) \begin{cases} \min_{i,j} \sum_{i=1}^{n} \sum_{j=0}^{m} c_{ij} \times x_{ij} + \sum_{j=1}^{m} \hat{c}_j \times y_j \\ s.t. : \\ \sum_{j=0}^{m} x_{ij} = 1, i \in 1..n \\ \sum_{i=1}^{n} x_{ij} \leq \hat{C} \times y_j, j \in 1..m \\ x_{ij} \in \{0,1\}, y_j \in \{0,1\} \end{cases}$$

where $\hat{C}$ is a constant such that $\hat{C} \gg n$.

It is easy to see that with $\hat{C} = 1$, $\hat{c}_j = 0$ and $y_j = 1$, $\forall j \in \{1..m\}$ we obtain the Asymmetric Assignment Problem which is NP-Hard [51]. Thus $\mathrm{ANDP}^{(\leq 1)}$ belongs to the NP-Hard class, completing the proof. □

Proposition 3.3.3. *Given an $ANDP(S, E_1 \cup E_2, C)$, the following sets are defined:*

$$\begin{cases} X_C^{(1)} \stackrel{\text{def}}{=} \{s_c \in S_C / \exists (s_c, s_w) \in E_2, \text{with } s_w \in S_D, \text{ such that } (\mathbf{i}) \text{ is satisfied}\}, \\ X_C^{(k)} \stackrel{\text{def}}{=} X_C^{(k-1)} \bigcup \{s_c \in (S_C \setminus (X_C^{(k-1)} \cup S_C^*)), \text{such that } (\mathbf{ii}) \text{ is satisfied}\}, \ \forall k > 1, \end{cases}$$

where

$$c_{(s_c, s_w)} < \sum_{\forall s_t \in S_T / s_t \in N_{(s_c)}} (\Phi(s_t, s_c) - c_{(s_t, s_c)}), \qquad (\mathbf{i})$$

$$\Phi_k(s_c) < \sum_{\forall s_t \in S_T / s_t \in N_{(s_c)}} (\Phi(s_t, s_c) - c_{(s_t, s_c)}), \qquad (\mathbf{ii})$$

$S_C^* \stackrel{\text{def}}{=} \{s_c \in S_C / \exists s_t \in S_T / N_{(s_t)} = \{s_c\}\}$, $\Phi(s_t, s_c) \stackrel{\text{def}}{=} \min\{c_{(s_t,s)} / (s_t, s) \in E_1, s \neq s_c\}$, *and* $\Phi_k(s_c) \stackrel{\text{def}}{=} \min\{c_{(s_c,s)} / (s_c, s) \in E_2, s \in S_D \cup X_C^{(k-1)}\}$. *Then the set $X_C^{(l_{max})}$ belongs to all optimal solution of Γ_{ANDP}, with l_{max} the maximum length of a chain of concentrators in $G_A = (S, E_1 \cup E_2)$.*

Proof. The Proposition will be demonstrated by induction in k.
Case 1.k $= 1$. Let us consider the set $X_C^{(1)}$ defined above. Let $s_c \in X_C^{(1)}$ be a concentrator site. By definition $\exists (s_c, s_w) \in E_2$, with $s_w \in S_D$, such that the inequality $(\mathbf{i})$ is satisfied. Let $\mathcal{H}_{opt} = (V, U)$ be an optimum solution of Γ_{ANDP}. Let us suppose that $s_c \notin \mathcal{H}_{opt}$. We define the sets $U_1 = \{(s_t, s) \in \mathcal{H}_{opt}/s_t \in S_T, s_t \in N_{(s_c)}\}$ and $U_2 = \{(s_t, s_c) \in E_1/s_t \in S_T, s_t \in N_{(s_c)}\}$. Let $\bar{\mathcal{H}} = (\bar{V}, \bar{U})$ be a network defined by $\bar{V} = V \cup \{s_c\}$ and $\bar{U} = (U \setminus U_1) \cup U_2 \cup \{(s_c, s_w)\}$. It is easy to see that $\bar{\mathcal{H}}$ is a feasible solution of Γ_{ANDP}. Let us analyze the cost of $\bar{\mathcal{H}}$ respect to $\mathcal{H}_{opt}$.

$$\textsc{cost}(\bar{\mathcal{H}}) = \textsc{cost}(\mathcal{H}_{opt}) - \textsc{cost}(U_1) + \textsc{cost}(U_2) + c_{(s_c, s_w)}$$

Since $\forall (s_t, s) \in \mathcal{H}_{opt}/s_t \in N_{(s_c)}, \Phi(s_t, s_c) \leq c_{(s_t, s)}$ then,

$$\textsc{cost}(\bar{\mathcal{H}}) \leq \textsc{cost}(\mathcal{H}_{opt}) - \sum_{\forall s_t \in S_T/s_t \in N_{(s_c)}} (\Phi(s_t, s_c) - c_{(s_t, s_c)}) + c_{(s_c, s_w)}.$$

Moreover, since $(\mathbf{i})$ is satisfied, we have that $\textsc{cost}(\bar{\mathcal{H}}) < \textsc{cost}(\mathcal{H}_{opt})$. This is a contradiction, therefore $s_c \in \mathcal{H}_{opt}$, completing the proof of Case 1.
Case 2.k > 1. As inductive hypothesis (denoted by I.H.) $X_C^{(j)}$ belongs to all optimal solution of Γ_{ANDP},$\forall j/1 \leq j < k$. As inductive thesis (denoted by I.T.) the set $X_C^{(k)}$ belongs to all optimal solution of Γ_{ANDP}.
By I.H. we know that the set $X_C^{(k-1)}$ belongs to all optimal solution of Γ_{ANDP}. We must also prove that $S_C^{(k)} = \{s_c \in (S_C \setminus (X_C^{(k-1)} \cup S_C^*))$, *such that* $(\mathbf{ii})$ *is satisfied*$\}$ belongs to all optimal solution of Γ_{ANDP}. Once more, let $\mathcal{H}_{opt} = (V, U)$ be an optimal solution of Γ_{ANDP} and $s_c \in S_C^{(k)}$ a concentrator site. Again we suppose that $s_c \notin \mathcal{H}_{opt}$. Let $\bar{\mathcal{H}} = (\bar{V}, \bar{U})$ be a network defined by $\bar{V} = V \cup \{s_c\}$ and $\bar{U} = (U \setminus U_1) \cup U_2 \cup \{(s_c, \bar{s})\}$, where $\bar{s} \in S_D \cup X_C^{(k-1)}/(s_c, \bar{s}) \in E_2$, and $c_{(s_c, \bar{s})} = \Phi_k(s_c)$ (we assumed that $S_C^{(k)} \neq \emptyset$). It is easy to see that $\bar{\mathcal{H}}$ is a feasible solution of Γ_{ANDP}. The cost relation between $\bar{\mathcal{H}}$ and $\mathcal{H}_{opt}$ is the following.

$$\textsc{cost}(\bar{\mathcal{H}}) = \textsc{cost}(\mathcal{H}_{opt}) - \textsc{cost}(U_1) + \textsc{cost}(U_2) + c_{(s_c, \bar{s})}$$

Since $\forall (s_t, s) \in \mathcal{H}_{opt}/s_t \in N_{(s_c)}, \Phi(s_t, s_c) \leq c_{(s_t, s)}$ then,

$$\textsc{cost}(\bar{\mathcal{H}}) \leq \textsc{cost}(\mathcal{H}_{opt}) - \sum_{\forall s_t \in S_T/s_t \in N_{(s_c)}} (\Phi(s_t, s_c) - c_{(s_t, s_c)}) + c_{(s_c, \bar{s})}.$$

We know that $c_{(s_c, \bar{s})} = \Phi_k(s_c)$, and moreover $(\mathbf{ii})$ is satisfied, obtaining so the inequality $\textsc{cost}(\bar{\mathcal{H}}) < \textsc{cost}(\mathcal{H}_{opt})$. This is a contradiction, hence $s_c \in \mathcal{H}_{opt}$, completing the proof of Case 2. Furthermore, since at the most there exists a chain of l_{max} concentrator sites connected in line, then $X_C^{(k)} = X_C^{(k+1)}\ \forall k \geq l_{max}$, completing the proof of Proposition 3.4.6. □

Proposition 3.3.4. *Given an* $ANDP(S, E_1 \cup E_2, C)$*, we redefine the sets* $\{X_C^{(k)}\}_{k \geq 1}$ *defined in Proposition 3.4.6, changing the condition* $(\mathbf{ii})$ *by:*

$$(\mathbf{ii})_b \left\{ \begin{array}{l} \exists \text{ a set } I_{(s_c)} \subseteq (N_k(s_c) \cup N_T(s_c)) \text{ and } \bar{s} \in (S_D \cup (X_C^{(k-1)} \setminus I_{(s_c)})) \\ \text{such that: } c_{(s_c, \bar{s})} < \sum_{\forall s \in I_{(s_c)}} (\Phi(s, s_c) - c_{(s, s_c)}), \end{array} \right.$$

where $N_k(s_c) \overset{\text{def}}{=} \{s \in X_C^{(k-1)}/s \in N_{(s_c)}\}$, $N_T(s_c) \overset{\text{def}}{=} \{s_t \in N_{(s_c)}/s_t \in S_T\}$ *and* $\Phi(s,s_c) \overset{\text{def}}{=} \min\{c_{(s,s_v)}/(s,s_v) \in E_1 \cup E_2, s_v \neq s_c\}$. *With this new definition,* $X_C^{(l_{max})}$ *also belongs to all optimal solution of* Γ_{ANDP}.

Proof. Again, we proceed by induction in k in order to demonstrate the Proposition.
Case 1.$k = 1$. It is the same that in the previous Proposition.
Case 2.$k > 1$. The inductive step is similar to the Proposition 3.4.6, with the difference that here we must demonstrate that the set of concentrator sites defined by $\bar{S}_C^{(k)} = \{s_c \in (S_C \setminus (X_C^{(k-1)} \cup S_C^*))$, *such that* $(\mathbf{ii})_b$ *is satisfied*$\}$ belongs to all optimal solution of Γ_{ANDP}. Once again we will consider $\mathcal{H}_{opt} = (V,U)$ as an optimal solution of Γ_{ANDP} and $s_c \in \bar{S}_C^{(k)}$ a concentrator site. By definition, there exists a set $I_{(s_c)} \subseteq (N_k(s_c) \cup N_T(s_c))$ and $\bar{s} \in (S_D \cup (X_C^{(k-1)} \setminus I_{(s_c)}))$ such that the inequality of $(\mathbf{ii})_b$ is fulfilled. Let us suppose that $s_c \notin \mathcal{H}_{opt}$. Let us consider the sets defined by $B_1 = \{(s,s_v) \in \mathcal{H}_{opt}/s \in I_{(s_c)}\}$ and $B_2 = \{(s,s_c) \in E_1 \cup E_2/s \in I_{s_c}\}$. Let $\bar{\mathcal{H}} = (\bar{V},\bar{U})$ be a network defined by $\bar{V} = V \cup \{s_c\}$ and $\bar{U} = (U \setminus B_1) \cup B_2 \cup \{(s_c,\bar{s})\}$ (we assume that $\bar{S}_C^{(k)} \neq \emptyset$). It is easy to see that $\bar{\mathcal{H}}$ is a feasible solution of Γ_{ANDP}. The cost of $\bar{\mathcal{H}}$ is given by:

$$\textsc{cost}(\bar{\mathcal{H}}) = \textsc{cost}(\mathcal{H}_{opt}) - \textsc{cost}(B_1) + \textsc{cost}(B_2) + c_{(s_c,\bar{s})}$$

Since $\forall (s,s_v) \in \mathcal{H}_{opt}/s \in I_{(s_c)}, \Phi(s,s_c) \leq c_{(s,s_v)}$ then,

$$\textsc{cost}(\bar{\mathcal{H}}) \leq \textsc{cost}(\mathcal{H}_{opt}) - \sum_{\forall s \in I_{(s_c)}} (\Phi(s,s_c) - c_{(s,s_c)}) + c_{(s_c,\bar{s})}.$$

Moreover, as the inequality in $(\mathbf{ii})_b$ is fulfilled, then $\textsc{cost}(\bar{\mathcal{H}}) < \textsc{cost}(\mathcal{H}_{opt})$. This is a contradiction, hence $s_c \in \mathcal{H}_{opt}$, completing the proof of Case 2. By the same argument of the previous Proposition, $X_C^{(l_{max})}$ belongs to all optimal solution of Γ_{ANDP}. □

3.4 Relation between feasible solutions.

In this section some relations between the costs of feasible solutions are introduced. To begin with, Theorem 3.4.1 introduces a relationship between the cost of the best solution given by the clustering approach and the optimal cost. In addition, Proposition 3.4.2 gives us the relationship between the cost of a feasible solution and the optimal cost.

Theorem 3.4.1. *Given an* $ANDP(S, E_1 \cup E_2, C)$ *and the feasible solution of* Γ_{ANDP} *formed by the optimal access subnetworks associated to the covering areas of the sites of* S_D*, that is to say* $\mathcal{T} = \bigcup_{\forall s_w \in S_D} T_{A_{s_w},E(A_{s_w}),C}(S_T^w \cup \{s_w\})$. *Let* $\mathcal{H} \in \Gamma_{ANDP}$ *be the feasible solution derived from* $\mathcal{T}$*, which is constructed, reconnecting with the smallest possible cost and maintaining the feasibility, the sites of* $\mathcal{T}$ *that are not in their covering areas to their respective areas. Then the following inequality is fulfilled:*

$$\frac{\textsc{cost}(\mathcal{T})}{\textsc{cost}(\mathcal{H})} \leq 1 + \frac{(c_{max} - c_B) \cdot \beta}{c_{min} \cdot n_T},$$

where $B = \{(s, s_c) \in E_1 \cup E_2 / (s, s_c) \notin \mathcal{T}; s, s_c \in \mathcal{T}; s \in S_T \cup S_C; s_c \in S_C; s \in A_{s_c}\}$, $c_{max} = \max\{c_{ij}; (i,j) \in E_1 \cup E_2\}$, $c_{min} = \min\{c_{ij}; (i,j) \in E_1 \cup E_2\}$, $c_B = \min\{c_{ij}; (i,j) \in B\}$, *and* $\beta = |B|$.

Proof. The set B defined above, it is the set of feasible connections that are not in $\mathcal{T}$, between a site $s_c \in \mathcal{T}$ ($s_c \in S_C$) and the sites of $S_T \cup S_C$ presents in $\mathcal{T}$ that belong to A_{s_c}.
If $B = \emptyset$ then $\mathcal{T} = \mathcal{H}$ and therefore $\frac{\text{COST}(\mathcal{T})}{\text{COST}(\mathcal{H})} = 1$. Furthermore in this case, $\mathcal{T}$ is an optimum solution of Γ_{ANDP}.
Let S_{B_1} denote the set induced by $B_1 \subseteq B$ of sites that are not in their covering areas, i.e., $S_{B_1} = \{s \in S_T \cup S_C / \exists s_c \in S_C, (s, s_c) \in B_1, s \in A_{s_c}\}$. In the case $B \neq \emptyset$, firstly we will demonstrate the following inequality.

$$\text{COST}(\mathcal{T}) - \text{COST}(\mathcal{H}_1) \leq (c_{max} - c_{B_1}) \cdot \beta_1, \ \forall B_1 \subseteq B, \tag{3.1}$$

where $c_{B_1} = \min\{c_{ij}; (i,j) \in B_1\}$, $\beta_1 = |B_1|$, and $\mathcal{H}_1 \in \Gamma_{ANDP}$ is the feasible solution derived from $\mathcal{T}$, which is constructed reconnecting with the smallest possible cost and maintaining the feasibility, the nodes of S_{B_1} to their respective covering areas. The inequality will be demonstrated by induction in β_1.
Case 1.$\beta_1 = 0$. In this case $B_1 = \emptyset$ and therefore $\mathcal{T} = \mathcal{H}_1$.
Case 2.$\beta_1 < h \leq \beta$. As inductive hypothesis (denoted by I.H.) the inequality 3.1 is satisfied $\forall \beta_1 < h \leq \beta$, and as inductive thesis (denoted by I.T.) the inequality 3.1 is satisfied when $\beta_1 = h$.
Let B_1 be a subset of B such that $\beta_1 = h \leq \beta$. Let U denote the subset of B_1 given by $U = \{(s_{c_1}, s_{c_2}) \in B_1 / s_{c_1} \in A_{s_{c_2}}, s_{c_2} \in A_{s_{c_1}}\}$.
Given $(s, s_c) \in B_1$ we denote by B_2 the subset $B_2 = B_1 \backslash \{(s, s_c)\}$. Let $\mathcal{H}_2 \in \Gamma_{ANDP}$ be the feasible solution derived from $\mathcal{T}$, which is constructed reconnecting with the smallest possible cost and maintaining the feasibility, the nodes of S_{B_2} to their respective covering areas. With respect to $\mathcal{H}_1$, if $(s, s_c) \in U$ the connection change is made only for one of the concentrator sites (s or s_c), because otherwise, if a direct connection between both is installed in $\mathcal{H}_1$ and their connections in $\mathcal{T}$ are eliminated, both would be without output towards the backbone network. If $(s, s_c) \notin U$ the connection of s in $\mathcal{T}$ is eliminated, adding the connection (s, s_c) to $\mathcal{H}_1$. In both cases, $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy the following:

$$\begin{aligned}
&\text{COST}(\mathcal{T}) - \text{COST}(\mathcal{H}_1) \leq \text{COST}(\mathcal{T}) - \text{COST}(\mathcal{H}_2) + (c_{max} - c_{(s,s_c)}) \overset{\overset{\textit{by I.H.}}{\uparrow}}{\leq} \\
&(c_{max} - c_{B_2}) \cdot \beta_2 + (c_{max} - c_{(s,s_c)}) = (c_{max} - c_{B_2}) \cdot (\beta_1 - 1) + (c_{max} - c_{(s,s_c)}) \\
&\leq \left(c_{max} - \min\{c_{B_2}, c_{(s,s_c)}\}\right) \cdot \beta_1 \overset{\overset{c_{B_1} = \min\{c_{B_2}, c_{(s,s_c)}\}}{\uparrow}}{=} (c_{max} - c_{B_1}) \cdot \beta_1,
\end{aligned}$$

completing the proof of the inequality 3.1.
If in the inequality 3.1 is chosen $B_1 = B$, then:

$$\text{COST}(\mathcal{T}) - \text{COST}(\mathcal{H}) \leq (c_{max} - c_B) \cdot \beta. \tag{3.2}$$

The inequality 3.2 is divided by $\text{COST}(\mathcal{H})$ obtaining the following:

$$\frac{\text{COST}(\mathcal{T})}{\text{COST}(\mathcal{H})} - 1 \leq \frac{(c_{max} - c_B) \cdot \beta}{\text{COST}(\mathcal{H})},$$

furthermore, we know that $\text{COST}(\mathcal{H}) \geq (c_{min} \cdot n_T)$ (the minimum cost in the best case). Therefore, it is fulfilled,

$$\frac{\text{COST}(\mathcal{T})}{\text{COST}(\mathcal{H})} - 1 \leq \frac{(c_{max} - c_B) \cdot \beta}{c_{min} \cdot n_T},$$

as required, and completing the proof of Theorem 3.4.1. □

Proposition 3.4.2. *Given an $ANDP(S, E_1 \cup E_2, C)$ with feasible solutions space Γ_{ANDP}; let $G_1, G_{opt} \in \Gamma_{ANDP}$ be feasible solutions where G_{opt} is globally optimal. Then the following inequality is fulfilled:*

$$\frac{\text{COST}(G_1)}{\text{COST}(G_{opt})} \leq \frac{c_{max}}{c_{min}} \cdot \left(1 + \frac{n_C}{n_T}\right),$$

where $c_{max} = \max\{c_{ij}; (i,j) \in E_1 \cup E_2\}$, $c_{min} = \min\{c_{ij}; (i,j) \in E_1 \cup E_2\}$ *and* $\text{COST}(G) = \sum_{\forall (i,j) \in G} c_{ij}$.

Proof. In the worst case a solution of Γ_{ANDP} has all the sites of S_C and all their connection lines have maximum cost. Then the maximum cost in the worst case is $(n_T + n_C) \cdot c_{max}$. In the best case a solution of Γ_{ANDP} has all the sites of S_T connected directly to the backbone network with minimum cost. Then the minimum cost in the best case is $(n_T \cdot c_{min})$. Therefore the costs of G_1 and G_{opt} satisfy: $\text{COST}(G_1) \leq (n_T + n_C) \cdot c_{max}$ and $\text{COST}(G_{opt}) \geq (n_T \cdot c_{min})$. Now, let us consider the quotient between the costs of the solutions G_1 and G_{opt} respectively; then:

$$\frac{\text{COST}(G_1)}{\text{COST}(G_{opt})} \leq \frac{(n_T + n_C) \cdot c_{max}}{(n_T \cdot c_{min})} = \frac{c_{max}}{c_{min}} \cdot \left(1 + \frac{n_C}{n_T}\right),$$

as required, completing the proof. □

In some cases, we would like to know how much money we can save with a new level of concentrators (a higher k according to the next notation). Theorem 3.4.4 helps us to answer that question. Besides, in Proposition 3.4.6, we will prove that the bound given is tight respect to k.After, Corollary 3.4.5 gives us an idea of how much we can save with regard to the optimal solution.

Notation 3.4.3. *Given an $ANDP(S, E_1 \cup E_2, C)$ and an integer $k/0 \leq k \leq n_C$, the set of feasible solutions of Γ_{ANDP} that at the most have k concentrators connected in line is denoted by $\Gamma_{ANDP}^{(\leq k)}$. Furthermore, it is easy to see that $\Gamma_{ANDP}^{(\leq k-1)} \subseteq \Gamma_{ANDP}^{(\leq k)}$, $\forall k/0 < k \leq n_C$.*

Theorem 3.4.4. *Given an $ANDP(S, E_1 \cup E_2, C)$ and an integer k, $0 < k \leq n_C$, such that:*

1. $\forall s_c \in S_C, \exists s_w \in S_D / c_{(s_c, s_w)} < +\infty$,
2. $\emptyset \neq \Gamma_{ANDP}^{(\leq k-1)} \subset \Gamma_{ANDP}^{(\leq k)}$.

If c_{opt}^{k-1} and c_{opt}^{k} are the costs of the best solutions of $\Gamma_{ANDP}^{(\leq k-1)}$ and $\Gamma_{ANDP}^{(\leq k)}$ respectively, then the following inequality is fulfilled:

$$\frac{c_{opt}^{k-1}}{c_{opt}^{k}} \leq 1 + \left\lfloor \frac{n_C}{k} \right\rfloor \cdot \left(\frac{1}{k + n_T} \right) \cdot \left(\frac{c_{max}}{c_{min}} - 1 \right).$$

Proof. By hypothesis, we know that $\Gamma_{ANDP}^{(\leq k-1)} \subset \Gamma_{ANDP}^{(\leq k)}$, this implies $c_{opt}^{k} \leq c_{opt}^{k-1}$.

Case 1. If $c_{opt}^{k} = c_{opt}^{k-1}$, then $\frac{c_{opt}^{k-1}}{c_{opt}^{k}} = 1$ satisfies the inequality.

Case 2. If $c_{opt}^{k} < c_{opt}^{k-1}$, firstly, the following inequality is demonstrated:

$$c_{opt}^{k-1} - c_{opt}^{k} \leq \max\{\textsc{cost}(G_{k-1}) - c_{opt}^{k}\}, \tag{3.3}$$

where the solution $G_{k-1} \in \Gamma_{ANDP}^{(\leq k-1)}$ is constructed from a solution of $\Gamma_{ANDP}^{(\leq k)}$ with cost c_{opt}^{k}, by means of the minimum amount of connection reassignments of its concentrator sites.

Let $\mathcal{H}_{k-1} \in \Gamma_{ANDP}^{(\leq k-1)}$ be an solution so that satisfies 3.3 strictly by means of equality $c_H - c_{opt}^{k} = \max\{\textsc{cost}(G_{k-1}) - c_{opt}^{k}\}$, where $c_H = \textsc{cost}(\mathcal{H}_{k-1})$. Let us suppose that the inequality 3.3 is not fulfilled, then we would have the following strict inequality: $c_{opt}^{k-1} - c_{opt}^{k} > c_H - c_{opt}^{k}$ which implies that $c_{opt}^{k-1} > c_H$, this is a contradiction. Therefore the inequality 3.3 is fulfilled.

An optimum solution of $\Gamma_{ANDP}^{(\leq k)}$, at the most has $\left\lfloor \frac{n_C}{k} \right\rfloor$ chains of concentrators connected in line, each one of length k. Since $\forall s_c \in S_C$ there exists at least a site $s_w \in S_W$ so that $c_{(s_c,s_w)} < +\infty$, then at the most with $\left\lfloor \frac{n_C}{k} \right\rfloor$ changes of connection on the concentrator sites, a solution of $\Gamma_{ANDP}^{(\leq k-1)}$ is obtained. Moreover for each connection change, we have a cost increase equal to $(c_{max} - c_{min})$ in the worst case.

Therefore, from the exposed previously, the following inequality is obtained:

$$c_{opt}^{k-1} - c_{opt}^{k} \leq \left\lfloor \frac{n_C}{k} \right\rfloor \cdot (c_{max} - c_{min}). \tag{3.4}$$

The right side of the inequality 3.4 is the cost difference when doing $\left\lfloor \frac{n_C}{k} \right\rfloor$ connection changes in the concentrator sites, in the worst case. The inequality 3.4 is divided by c_{opt}^{k} obtaining the following inequality:

$$\frac{c_{opt}^{k-1}}{c_{opt}^{k}} - 1 \leq \left\lfloor \frac{n_C}{k} \right\rfloor \cdot \frac{(c_{max} - c_{min})}{c_{opt}^{k}}.$$

Moreover c_{opt}^{k} satisfies the inequality $c_{opt}^{k} \geq (k + n_T) \cdot c_{min}$; since in the best case a solution of $\Gamma_{ANDP}^{(\leq k)} \backslash \Gamma_{ANDP}^{(\leq k-1)}$ has an only chain of concentrators connected in line, of length k and cost $k \cdot c_{min}$, and all terminal sites are connected directly to the chain with cost $(n_T \cdot c_{min})$. This implies that:

$$\frac{c_{opt}^{k-1}}{c_{opt}^{k}} \leq 1 + \left\lfloor \frac{n_C}{k} \right\rfloor \cdot \frac{(c_{max} - c_{min})}{c_{opt}^{k}} \leq 1 + \left\lfloor \frac{n_C}{k} \right\rfloor \cdot \frac{(c_{max} - c_{min})}{(k + n_T) \cdot c_{min}}.$$

As a result is deduced then:

$$\frac{c_{opt}^{k-1}}{c_{opt}^{k}} \leq 1 + \left\lfloor \frac{n_C}{k} \right\rfloor \cdot \left(\frac{1}{k + n_T} \right) \cdot \left(\frac{c_{max}}{c_{min}} - 1 \right),$$

as required, and completing the proof.

□

Corollary 3.4.5. *Given an $ANDP(S, E_1 \cup E_2, C)$ in the hypothesis of the previous theorem and an integer k, $0 \le k \le n_C$, such that $c_{opt} < c^k_{opt} < +\infty$, where c^k_{opt} is the cost of the best feasible solution in $\Gamma^{(\le k)}_{ANDP}$ and c_{opt} is the globally optimal cost. Then the following inequality is fulfilled:*

$$\frac{c^k_{opt}}{c_{opt}} \le \left(1 + \left\lfloor \frac{n_C}{k+1} \right\rfloor \cdot \left(\frac{1}{n_T + k + 1}\right) \cdot \left(\frac{c_{max}}{c_{min}} - 1\right)\right)^{n_{\Psi(k)}},$$

where Ψ and $n_{\Psi(k)}$ are defined by:
$\Psi(i) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } c^{i+1}_{opt} < c^i_{opt} \\ 0 & \text{otherwise} \end{cases}$, $n_{\Psi(k)} \stackrel{\text{def}}{=} \sum_{i=k}^{n_C} \Psi(i)$.

Proof. Firstly, the following inequality will be demonstrated by induction:

$$\frac{c^j_{opt}}{c_{opt}} \le B^{n_{\Psi(j)}}_{(j+1)},\ \forall j / k \le j < n_C, \tag{3.5}$$

where $B_{(j)} = \left(1 + \left\lfloor \frac{n_C}{j} \right\rfloor \cdot \left(\frac{1}{j+n_T}\right) \cdot \left(\frac{c_{max}}{c_{min}} - 1\right)\right)$.

Case 1.$j = n_C - 1$. Since $n_C - 1 \ge k$ then $\Gamma^{(\le k)}_{ANDP} \subseteq \Gamma^{(\le n_C - 1)}_{ANDP}$. Moreover, as we know that $c^{n_C-1}_{opt} \le c^k_{opt} < +\infty$ then $\Gamma^{(\le n_C-1)}_{ANDP} \ne \emptyset$.

If $c^{n_C-1}_{opt} = c_{opt}$ then $\frac{c^{n_C-1}_{opt}}{c_{opt}} = 1$. Furthermore, as $B_{(n_C-1)} > 0$ and $n_{\Psi(n_C-1)} = \Psi(n_C - 1) = 0$, $(c^{n_C-1}_{opt} = c_{opt} = c^{n_C}_{opt})$, then $B^{n_{\Psi(n_C-1)}}_{(n_C-1)} = 1$.

If $c^{n_C-1}_{opt} > c_{opt}$ then $\Gamma^{(\le n_C-1)}_{ANDP} \subset \Gamma^{(\le n_C)}_{ANDP} = \Gamma_{ANDP}$. By Theorem 3.4.4, this implies that $\frac{c^{n_C-1}_{opt}}{c^{n_C}_{opt}} \le B_{(n_C)}$; moreover as $n_{\Psi(n_C-1)} = \Psi(n_C - 1) = 1$, this implies that $\frac{c^{n_C-1}_{opt}}{c^{n_C}_{opt}} \le B^{n_{\Psi(n_C)}}_{(n_C)}$, completing the proof of Case 1.

Case 2.$j > h \ge k$. The inductive step is presented of the following way.

As inductive hypothesis (denoted by I.H.) the inequality 3.5 is satisfied $\forall j > h \ge k$, and as inductive thesis (denoted by I.T.) the inequality 3.5 is satisfied when $j = h$.

If $c^h_{opt} = c^{h+1}_{opt}$ then $n_{\Psi(h)} = n_{\Psi(h+1)}$. Therefore the following inequality is satisfied:

$$\frac{c^h_{opt}}{c_{opt}} = \frac{c^{h+1}_{opt}}{c_{opt}} \overset{\text{by I.H.}}{\le} B^{n_{\Psi(h+1)}}_{(h+2)} \overset{n_{\Psi(h)} = n_{\Psi(h+1)}}{=} B^{n_{\Psi(h)}}_{(h+2)} \overset{B_{(h+1)} \ge B_{(h+2)}}{\le} B^{n_{\Psi(h)}}_{(h+1)},$$

proving that the I.T. is fulfilled when $c^h_{opt} = c^{h+1}_{opt}$.

Let us consider now the case $c^h_{opt} > c^{h+1}_{opt}$. Since $c^h_{opt} \le c^k_{opt} < +\infty$ then $\Gamma^{(\le h)}_{ANDP} \ne \emptyset$.

The quotient $\frac{c^h_{opt}}{c_{opt}}$ can be seen as: $\frac{c^h_{opt}}{c_{opt}} = \frac{c^h_{opt}}{c^{h+1}_{opt}} \cdot \frac{c^{h+1}_{opt}}{c_{opt}}$. By I.H. it is fulfilled the inequality $\frac{c^{h+1}_{opt}}{c_{opt}} \le B^{n_{\Psi(h+1)}}_{(h+2)}$. Moreover by theorem 3.4.4 $\frac{c^h_{opt}}{c^{h+1}_{opt}} \le B_{(h+1)}$. Therefore the following inequality is satisfied:

$$\frac{c^h_{opt}}{c_{opt}} \le B_{(h+1)} \cdot B^{n_{\Psi(h+1)}}_{(h+2)} \overset{B_{(h+1)} \ge B_{(h+2)}}{\le} B^{1+n_{\Psi(h+1)}}_{(h+1)}.$$

Furthermore $n_{\Psi(h)} = \Psi(h) + n_{\Psi(h+1)} = 1 + n_{\Psi(h+1)}$, this implies that:

$$\frac{c^h_{opt}}{c_{opt}} \leq B^{1+n_{\Psi(h+1)}}_{(h+1)} = B^{n_{\Psi(h)}}_{(h+1)},$$

as required, and completing the inductive step.
To complete the proof, in the inequality 3.5 is chosen $j = k$. This completes the proof of the Corollary. □

Proposition 3.4.6. *The bound given by the theorem 3.4.4 is tight respect to k for certain topologies. That is to say, fixed k there exists at least an* ANDP *in the hypotheses of the theorem, such that the inequality is satisfied by means of strict equality.*

Proof. Firstly, the case $k = 1$ will be presented. Let $ANDP(S, E_1 \cup E_2, C)$ be a problem in the following conditions:

- $S_D = S^1_D \cup \{s_{w_1}\}, S_C = \{s_{c_1}\}, S_T = S^1_T \cup \{s_{t_1}, s_{t_2}\}; S^1_D, S^1_T \neq \emptyset$,
- the matrix C satisfies:
  - only two connection costs exist, these are c_{min} and $c_{max} = 2c_{min}$,
  - $c_{(s_t, s_{w_1})} = c_{min},\ \forall s_t \in S^1_T$,
  - $c_{(s_{t_1}, s_{w_1})} = c_{(s_{t_2}, s_{w_1})} = c_{max}$,
  - $c_{(s_{t_1}, s_{c_1})} = c_{(s_{t_2}, s_{c_1})} = c_{(s_{c_1}, s_{w_1})} = c_{min}$,
  - $c_{(s_t, s_w)} = c_{max},\ \forall s_w \in S^1_D,\ \forall s_t \in S_T$,
  - other feasible connections do not exist.

It is easy to see that the optimal costs of $\Gamma^{(\leq 0)}_{ANDP}$ and Γ^1_{ANDP} respectively, satisfy:

$$c^0_{opt} = (n_T - 2) \cdot c_{min} + 2c_{max} = c_{min} \cdot (n_T + 2),$$
$$c^1_{opt} = (n_T - 2) \cdot c_{min} + 3c_{min} = c_{min} \cdot (n_T + 1).$$

Therefore, the quotient between both has the value:

$$\frac{c^0_{opt}}{c^1_{opt}} = \frac{c_{min} \cdot (n_T + 2)}{c_{min} \cdot (n_T + 1)} = \frac{(n_T + 2)}{(n_T + 1)}.$$

Moreover, since $\left\lfloor \frac{n_C}{k} \right\rfloor = 1$ and $\left(\frac{c_{max}}{c_{min}} - 1\right) = 1$, the right side of the inequality has the value:

$$1 + \left(\frac{1}{k + n_T}\right) = 1 + \left(\frac{1}{1 + n_T}\right) = \frac{(n_T + 2)}{(n_T + 1)}.$$

Now, we will present the case $k > 1$. Let $ANDP(S, E_1 \cup E_2, C)$ be a problem in the following conditions:

- $S_D = \{s_{w_1}\} \cup S_D^1 / |S_D^1| > 1$,
- $S_C = \{s_{c_1}, \dots, s_{c_k}\}$,
- $S_T = \bigcup_{i=1}^k S_T^i / |S_T^i| > 1, S_T^i \cap S_T^j = \emptyset,\ \forall i, j \in 1 \dots k, i \neq j$,
- the matrix C satisfies:
  - only two connection costs exist, these are c_{min} and $c_{max} = 2c_{min}$,
  - $c_{(s_{w_1}, s_{c_1})} = c_{(s_{c_i}, s_{c_{i+1}})} = c_{min},\ \forall i \in 1 \dots k-1$,
  - $c_{(s_t, s_{c_i})} = c_{min},\ \forall s_t \in S_T^i,\ \forall i \in 1 \dots k$,
  - $\forall s_{c_i} \in S_C, \exists s_w \in S_D^1 / c_{(s_{c_i}, s_w)} = c_{max}$,
  - other feasible connections do not exist.

As in the previous example, it is easy to see that the optimal costs of $\Gamma_{ANDP}^{(\leq k-1)}$ and $\Gamma_{ANDP}^{(\leq k)}$ respectively, satisfy:

$$\begin{aligned} c_{opt}^{k-1} &= ((k-1) + n_T - n_T^k) \cdot c_{min} + n_T^k \cdot c_{min} + c_{max} = c_{min} \cdot (n_T + k + 1), \\ c_{opt}^k &= c_{min} \cdot (n_T + k), \end{aligned}$$

where $n_T^k = |S_T^k|$. The quotient between both has the value:

$$\frac{c_{opt}^{k-1}}{c_{opt}^k} = \frac{c_{min} \cdot (n_T + k + 1)}{c_{min} \cdot (n_T + k)} = \frac{(n_T + k + 1)}{(n_T + k)}.$$

Since $\left\lfloor \frac{n_C}{k} \right\rfloor = \left\lfloor \frac{k}{k} \right\rfloor = 1$ and $\left(\frac{c_{max}}{c_{min}} - 1 \right) = 1$, the right side of the inequality has the value:

$$1 + \left(\frac{1}{k + n_T} \right) = 1 + \left(\frac{1}{k + n_T} \right) = \frac{(n_T + k + 1)}{(n_T + k)}.$$

$\square$

3.5 The ANDP transformed to an instance of the SPG.

In this section we formulate the ANDP as a Steiner Problem in Graphs (SPG). Before starting, some definitions are introduced. Lemma 3.5.3 and Lemma 3.5.4 shows the relationship between both problems and both solutions respectively. After that, other properties are introduced.

Definition 3.5.1. *Given $\bar{E}_3 = \{(s_{w_1}, s_{w_2}) / s_{w_1}, s_{w_2} \in S_D\}$ the set of all the connections (feasible and non-feasible) between sites of S_D, it is defined the set $\bar{E} = E_1 \cup E_2 \cup \bar{E}_3$.*

Definition 3.5.2. *The matrix of line laying costs restricted to $E_1 \cup E_2$, is defined by:*

$$A = \{a_{ij}\}/a_{ij} \stackrel{\text{def}}{=} \begin{cases} c_{ij} & \text{if } i \in S_T \cup S_C,\ j \in S_C \cup S_D, \\ 0 & \text{if } i, j \in S_D. \end{cases}$$

Lemma 3.5.3 (ANDP-SPG Relation)**.** *Let $\mathcal{T}_A = T_{S,\bar{E},A}(S_T \cup S_D)$ be an optimal solution of $SPG(S, \bar{E}, A, S_T \cup S_D)$, where A is the matrix defined above. Then the sub-network $\mathcal{T}_A \backslash \bar{E}_3$ is a global optimal solution of $ANDP(S, E_1 \cup E_2, C)$.*

Proof. Let us suppose that $\mathcal{H}_A = \mathcal{T}_A \backslash \bar{E}_3$ is not a globally optimal solution in Γ_{ANDP}. This implies that there exists a feasible solution $\mathcal{G} \in \Gamma_{ANDP}$ so that $\text{COST}(\mathcal{G}) < \text{COST}(\mathcal{H}_A)$. Let $\mathcal{B}_\mathcal{G}$ be the set the switch sites of $\mathcal{G}$ which have associate one access sub-network. Let us consider the network $\mathcal{N} = (\mathcal{B}_\mathcal{G}, \bar{E}_3(\mathcal{B}_\mathcal{G}))$. Let $\mathcal{T}_\mathcal{N}$ be a spanning tree solving problem $MSTP(\mathcal{B}_\mathcal{G}, \bar{E}_3(\mathcal{B}_\mathcal{G}), A_{|\mathcal{B}_\mathcal{G}})$. Clearly the network $\mathcal{T}_\mathcal{N} \cup \mathcal{G}$ is a feasible solution of $SPG(S, \bar{E}, A, S_T \cup S_D)$. Let us analyze its cost:

$$\text{COST}(\mathcal{T}_\mathcal{N} \cup \mathcal{G}) = \text{COST}(\mathcal{G}) < \text{COST}(\mathcal{H}_A) = \text{COST}(\mathcal{T}_A),$$

this contradicts the optimality of $\mathcal{T}_A$. Then, necessarily $\mathcal{H}_A$ is a global optimal solution in Γ_{ANDP}. □

Lemma 3.5.4. *Let $\mathcal{T}_{opt}$ be an optimal solution of $ANDP(S, E_1 \cup E_2, C)$. Considering the set of switch sites $B_\mathcal{T} \stackrel{\text{def}}{=} \{s_w \in S_D / s_w \in \mathcal{T}_{opt}\}$, the network $\mathcal{T}_{opt} \cup \mathcal{H}_s$ is an optimal solution of $SPG(S, \bar{E}, A, S_T \cup S_D)$, where $\mathcal{H}_s$ is a spanning tree of $\mathcal{H} = (B_\mathcal{T}, \bar{E}_3(B_\mathcal{T}))$.*

Proof. Again, by contradiction, let us suppose that $\mathcal{T}_{opt} \cup \mathcal{H}_s$ is not a global optimal solution for the problem $SPG(S, \bar{E}, A, S_T \cup S_D)$. Therefore there exists a feasible solution $\mathcal{G}$ of $SPG(S, \bar{E}, A, S_T \cup S_D)$, such that $\text{COST}(\mathcal{G}) < \text{COST}(\mathcal{T}_{opt} \cup \mathcal{H}_s)$. Considering the network $\bar{\mathcal{G}} = \mathcal{G} \setminus \bar{E}_3$, it is easy to see that $\bar{\mathcal{G}} \in \Gamma_{ANDP}$, and in addition:

$$\text{COST}(\bar{\mathcal{G}}) = \text{COST}(\mathcal{G}) < \text{COST}(\mathcal{T}_{opt} \cup \mathcal{H}_s) = \text{COST}(\mathcal{T}_{opt}),$$

which would imply that $\bar{\mathcal{G}}$ is a better feasible solution than $\mathcal{T}_{opt}$ in the space Γ_{ANDP}. This is a contradiction, since by hypothesis $\mathcal{T}_{opt}$ is a global optimum in Γ_{ANDP}; completing therefore the proof. □

Definition 3.5.5. *Given a real $c_\varepsilon / 0 < c_\varepsilon < c_{min}$, we define the matrix A_ε of the following way:*

$$A_\varepsilon = \{a_{ij}\}/a_{ij} \stackrel{\text{def}}{=} \begin{cases} c_{ij} & \text{if } i \in S_T \cup S_C,\ j \in S_C \cup S_D, \\ c_\varepsilon & \text{if } i, j \in S_D. \end{cases}$$

Theorem 3.5.6 (ANDP-SPG Generalized Relation)**.** *Let $\mathcal{T}_A = T_{S,\bar{E},A_\varepsilon}(S_T \cup S_D)$ be an optimal solution of $SPG(S, \bar{E}, A_\varepsilon, S_T \cup S_D)$, where A_ε is the matrix defined above. Then the sub-network $\mathcal{T}_A \backslash \bar{E}_3$ is a global optimum of $ANDP(S, E_1 \cup E_2, C)$.*

Proof. Let us suppose that $\mathcal{T}_A \backslash \bar{E}_3$ is not globally optimal in Γ_{ANDP}. Then, there exists a feasible solution $\mathcal{G} \in \Gamma_{ANDP}$ such that $\textsc{cost}(\mathcal{G}) < \textsc{cost}(\mathcal{T}_A \backslash \bar{E}_3)$. Let $\mathcal{T}_B$ be a spanning tree solving problem $MSTP(S_D, \bar{E}_3, A_\varepsilon)$. It is easy to see that the network $\mathcal{T}_B \cup \mathcal{G}$ is a feasible solution for the $SPG(S, \bar{E}, A_\varepsilon, S_T \cup S_D)$, and furthermore its cost is $\textsc{cost}(\mathcal{G}) + (n_D - 1)c_\varepsilon$. On the other hand, considering the induced network $\mathcal{T}_{A(S_D)}$, and since in $\bar{G}_B = (S_D, \bar{E}_3)$ all the connections have cost $c_\varepsilon / 0 < c_\varepsilon < c_{min}$, easily we can infer that $\textsc{cost}(\mathcal{T}_{A(S_D)}) = (n_D - 1)c_\varepsilon$ (because, if S_C^* is an optimal set of concentrator sites, when computing the network $\mathrm{MST}(S_D \cup S_C^* \cup S_T, \bar{E}_{|(S_D \cup S_C^* \cup S_T)}, A)$ applying Kruskal's algorithm or Prim's algorithm, the $(n_D - 1)$ first selected connections necessarily belong to $\bar{G}_B$). We can immediately deduce the following relation;

$$\textsc{cost}(\mathcal{T}_A \backslash \bar{E}_3) \quad \overset{\overset{\textit{Kruskal's theorem}}{\uparrow}}{=} \quad \textsc{cost}(\mathcal{T}_A) - (n_D - 1)c_\varepsilon > \textsc{cost}(\mathcal{G}),$$

implying the equation:

$$\textsc{cost}(\mathcal{T}_A) > \textsc{cost}(\mathcal{G}) + (n_D - 1)c_\varepsilon = \textsc{cost}(\mathcal{T}_B \cup \mathcal{G}),$$

contradicting therefore that $\mathcal{T}_A$ is globally optimal for the $SPG(S, \bar{E}, A_\varepsilon, S_T \cup S_D)$. Hence, the network $\mathcal{T}_A \backslash \bar{E}_3$ must be a global optimal solution in Γ_{ANDP}, as required, and completing the proof. □

Definition 3.5.7. *Given $\mathcal{C} = (S_D, U_\mathcal{C})$ a cycle formed by all the switch sites of S_D, we define the matrix B_ϵ as follows:*

$$B_\varepsilon = \{b_{ij}\}/b_{ij} \overset{\text{def}}{=} \begin{cases} c_{ij} & \textit{if } i \in S_T \cup S_C,\ j \in S_C \cup S_D, \\ c_\varepsilon & \textit{if } (i,j) \in U_\mathcal{C}, \\ \infty & \textit{if } i,j \in S_D, (i,j) \notin U_\mathcal{C}, \end{cases}$$

where c_ε satisfies $0 < c_\varepsilon < c_{min}$.

Corollary 3.5.8. *Let $\mathcal{C} = (S_D, U_\mathcal{C})$ be a cycle formed by all the switch sites of S_D and B_ϵ the matrix defined above. Let $\mathcal{T}_A = T_{S,(U_\mathcal{C} \cup E_1 \cup E_2),B_\varepsilon}(S_T \cup S_D)$ be an optimal solution of $SPG(S, U_\mathcal{C} \cup E_1 \cup E_2, B_\varepsilon, S_T \cup S_D)$. Then the sub-network $\mathcal{T}_A \setminus U_\mathcal{C}$ is a global optimal solution of $ANDP(S, E_1 \cup E_2, C)$.*

Proof. Let us suppose that $\exists \mathcal{G} \in \Gamma_{ANDP}$ so that $\textsc{cost}(\mathcal{G}) < \textsc{cost}(\mathcal{T}_A \backslash U_\mathcal{C})$. We define the network $\mathcal{G}_B = \mathcal{G} \cup \mathcal{T}_{A(S_D)}$. Easily we could verify that $\mathcal{G}_B$ is a feasible solution for the $SPG(S, U_\mathcal{C} \cup E_1 \cup E_2, B_\varepsilon, S_T \cup S_D)$. By the exposed in the previous theorem, we know that;

$$\textsc{cost}(\mathcal{T}_A \backslash U_\mathcal{C}) \quad \overset{\overset{\textit{Kruskal's theorem}}{\uparrow}}{=} \quad \textsc{cost}(\mathcal{T}_A) - (n_D - 1)c_\varepsilon > \textsc{cost}(\mathcal{G}),$$

being established the following relation:

$$\text{COST}(\mathcal{T}_A) > \text{COST}(\mathcal{G}) + (n_D - 1)c_\varepsilon = \text{COST}(\mathcal{G}_B).$$

But this contradicts the optimality of $\mathcal{T}_A$, therefore $\mathcal{T}_A \setminus U_c$ is a global optimum in the space Γ_{ANDP}. $\square$

Proposition 3.5.9. *Given an $ANDP(S, E_1 \cup E_2, C)$ and a sub-set of concentrator sites $\bar{S}_C \subseteq S_C$. The network $\text{MST}(S_D \cup \bar{S}_C \cup S_T, \bar{E}_{|(S_D \cup \bar{S}_C \cup S_T)}, A) \setminus \bar{E}_3$ is the best feasible solution of Γ_{ANDP} containing exactly the set of concentrators $\bar{S}_C$.*

Proof. The problem is reduced to solve the $MSTP(S_D \cup \bar{S}_C \cup S_T, \bar{E}_{|(S_D \cup \bar{S}_C \cup S_T)}, A)$. $\square$

Corollary 3.5.10. *Given an $ANDP(S, E_1 \cup E_2, C)$, let us suppose that the set $S_C^* \subseteq S_C$ is a maximal subset of concentrator sites present in some optimal solution of Γ_{ANDP}. Then, the topology of the network $\text{MST}(S_D \cup S_C^* \cup S_T, \bar{E}_{|(S_D \cup S_C^* \cup S_T)}, A) \setminus \bar{E}_3$ is globally optimal.*

Proof. Let us suppose that $\mathcal{T}_{\text{MST}} = \text{MST}(S_D \cup S_C^* \cup S_T, \bar{E}_{|(S_D \cup S_C^* \cup S_T)}, A) \setminus \bar{E}_3$ is not globally optimal for the ANDP. Then, $\exists \mathcal{T} \in \Gamma_{ANDP} / \text{COST}(\mathcal{T}) < \text{COST}(\mathcal{T}_{\text{MST}})$. Let us consider the network $\bar{G}_B = (S_D, \bar{E}_3)$. Let $\mathcal{H}$ be a spanning tree on this network. Clearly, the network $\mathcal{H} \cup \mathcal{T}$ is a feasible solution for the $SPG(S, \bar{E}, A, S_T \cup S_D)$, and furthermore it is satisfied $\text{COST}(\mathcal{T}) = \text{COST}(\mathcal{H} \cup \mathcal{T})$. Now, since $\mathcal{T}_{\text{MST}}$ is a global optimum for the SPG, then the following relation is fulfilled:

$$\text{COST}(\mathcal{T}) = \text{COST}(\mathcal{H} \cup \mathcal{T}) \geq \text{COST}(\mathcal{T}_{\text{MST}}),$$

completing therefore the proof. $\square$

Definition 3.5.11. *We define the subset associated to $S_T \cup S_C$ of connections of minimum cost towards the backbone network as:*

$$E_m = \left\{(s_i, s_w) \in E_1 \cup E_2 / s_w \in S_D,\ \textit{and}\ \forall \in \bar{s}_w \in (S_D \setminus \{s_w\}),\ c_{(s_i, s_w)} \leq c_{(s_i, \bar{s}_w)}\right\}.$$

Definition 3.5.12. *Given an $ANDP(S, E_1 \cup E_2, C)$, we define the shrunk Steiner problem associated to the* ANDP *as $SPG(S, \mathcal{D}, C_{|\mathcal{D}}, S_T \cup \{\text{z}_\text{w}\})$, where:*

- z_w *is a fictitious node representing all the backbone network,*
- $\mathcal{D} = \{(s_i, s_c) \in E_1 \cup E_2 / s_i \in S_T \cup S_C, s_c \in S_C\} \cup \{(s_i, \text{z}_\text{w}) / \exists (s_i, s_w) \in E_m\}$.

Proposition 3.5.13. *Let $SPG(S, \mathcal{D}, C_{|\mathcal{D}}, S_T \cup \{\text{z}_\text{w}\})$ be the shrunk Steiner problem associated to the $ANDP(S, E_1 \cup E_2, C)$. Let $\mathcal{T}^{(\text{opt})} \in \Gamma_{ANDP}$ and $\mathcal{G}^{(\text{opt})}$ be optimal solutions for the instances* ANDP *and* SPG *respectively. Then, their costs satisfy:* $\text{COST}(\mathcal{T}^{(\text{opt})}) = \text{COST}(\mathcal{G}^{(\text{opt})})$.

Proof. According to the definitions of E_m and $\mathcal{D}$, it is evident that the inequality $\text{COST}(\mathcal{T}^{(\text{opt})}) \leq \text{COST}(\mathcal{G}^{(\text{opt})})$ is fulfilled. Now, let us analyze the case of strict inequality $\text{COST}(\mathcal{T}^{(\text{opt})}) < \text{COST}(\mathcal{G}^{(\text{opt})})$. It is easy to see that $\forall (s_i, s_w) \in \mathcal{T}^{(\text{opt})}$, with $s_i \in S_T \cup S_C$, $s_w \in S_D$, necessarily $(s_i, s_w) \in E_m$. Let $\hat{\mathcal{G}}$ be the network builded by means of the connection replacements: $\hat{\mathcal{G}} = \Upsilon_{(\mathcal{T}^{(\text{opt})})}$, where $\Upsilon_{(\cdot)}$ is the connection substitution operator which is defined by:

$$\Upsilon_{(s_i,s)} \overset{\text{def}}{=} \begin{cases} (s_i, s) & \text{if } s_i \in S_T \cup S_C,\ s \in S_C, \\ (s_i, \mathrm{z_w}) & \text{if } s \in S_D, \end{cases}$$

and given $(s_i, s) \in \mathcal{T}$, $\Upsilon_{(\mathcal{T})} = \Upsilon_{(s_i,s)} \cup \Upsilon_{(\mathcal{T} \setminus \{(s_i,s)\})}$.
Easily, we can see that the network $\hat{\mathcal{G}}$ is a feasible solution for the shrunk SPG, and in addition: $\text{COST}(\hat{\mathcal{G}}) = \text{COST}(\mathcal{T}^{(\text{opt})})$. This implies $\text{COST}(\hat{\mathcal{G}}) < \text{COST}(\mathcal{G}^{(\text{opt})})$, contradicting $\mathcal{G}^{(\text{opt})}$ being globally optimal, and implying therefore the equality of costs: $\text{COST}(\mathcal{T}^{(\text{opt})}) = \text{COST}(\mathcal{G}^{(\text{opt})})$, as required. □

Definition 3.5.14. *Given an $ANDP(S, E_1 \cup E_2, C)$, we define the set of concentrator sites of first level as:* $\mathrm{L}_C^{(1)} = \{s_c \in S_C / \nexists \bar{s}_c \in (S_C \setminus \{s_c\}),$ *so that* $s_c \in A_{\bar{s}_c}\}$.

Notation 3.5.15. *Given an $ANDP(S, E_1 \cup E_2, C)$, we introduce the following notation:*

- $\bar{E}_2 = \{(i,j) \in E_2 / i \in S_C, j \in S_D\}$*, and* $\bar{U} = E_1 \cup \bar{E}_2$.
- $\bar{C} = C_{|\bar{U}} = \{\bar{c}_{ij}\} / \bar{c}_{ij} = \begin{cases} c_{ij} & \textit{if } (i,j) \in \bar{U}, \\ \infty & \textit{otherwise.} \end{cases}$
- $\mathrm{F}_T^{(\text{ANDP})} = \{s_t \in S_T / \nexists s_c \in S_C / s_t \in A_{s_c}\}$.
- *Given $\mathcal{T} \in \Gamma_{ANDP}$ and $s_w \in S_D$, $\mathcal{T}_{(s_w)}$ denotes the tree access sub-network which has the switch site s_w as root.*

Lemma 3.5.16. *Given an $ANDP(S, E_1 \cup E_2, C)$, let us suppose that in all the sites of S_C concentrator machines were installed. Then,*

$$\text{COST}(\text{MST}(S, \bar{E}, A)) = \sum_{\forall s_w \in S_D} \text{COST}(\text{MST}(A_{s_w}, E(A_{s_w}), C_{|A_{s_w}})),$$

if and only if $\forall s_w \in S_D / |A_{s_w}| > 1$, it is fulfilled that $A_{s_c} \subset A_{s_w}$, $\forall s_c \in A_{s_w}$.

Proof. Without loss of generality, we will assume that do not exist border sites with respect to sites that do not belong to its switch covering area.
($\Rightarrow$). By contradiction, let us suppose that there exist $s_w \in S_D$, $s_c \in S_C$, $s_c \in A_{s_w}$, so that

$A_{s_c} \not\subset A_{s_w}$. This implies that $\exists s \in (S_T \cup S_C), \bar{s}_w \in S_D / s \in A_{\bar{s}_w}$, and $s \in A_{s_c}$. We will denote:

$$\mathcal{T}_{\mathrm{A}} = \bigcup_{\forall s_w \in S_D} \left\{ \mathrm{MST}(A_{s_w}, E(A_{s_w}), C_{|A_{s_w}}) \right\}.$$

Let $(s, \bar{s}) \in \mathcal{T}_{\mathrm{A}}$ be the connection such that $\bar{s} \in (S_C \cup S_D)$ is the adjacent site to s in the path towards the backbone network. We build the following network:

$$\mathcal{N}_{\mathrm{A}} = (\mathcal{T}_{\mathrm{A}} \setminus \{(s, \bar{s})\}) \cup \{(s, s_c)\}.$$

Clearly $\mathcal{N}_{\mathrm{A}}$ is a tree spanning all the sites of S, and in addition:

$$\textsc{cost}(\mathcal{N}_{\mathrm{A}}) = \textsc{cost}(\mathcal{T}_{\mathrm{A}}) - c_{(s,\bar{s})} + c_{(s,s_c)} \overset{\overset{s \in A_{s_C}}{\uparrow}}{<} \textsc{cost}(\mathcal{T}_{\mathrm{A}}),$$

but this would imply that $\textsc{cost}(\mathcal{N}_{\mathrm{A}}) < \textsc{cost}(\mathrm{MST}(S, \bar{E}, A))$, contradicting the optimality of $\mathrm{MST}(S, \bar{E}, A)$.
($\Leftarrow$). Again, we proceed by contradiction. Let us suppose that:

$$\textsc{cost}(\mathrm{MST}(S, \bar{E}, A)) < \sum_{\forall s_w \in S_D} \textsc{cost}(\mathrm{MST}(A_{s_w}, E(A_{s_w}), C_{|A_{s_w}})).$$

It is easy to see that necessarily $\exists (s, \bar{s}_c) \in \mathrm{MST}(S, \bar{E}, A) / s \in A_{s_w}$, and $\bar{s}_c \in A_{\bar{s}_w}$, where $s \in (S_T \cup S_C)$, $\bar{s}_c \in S_C$, $s_w, \bar{s}_w \in S_D$, $s_w \neq \bar{s}_w$. We will analyze the following cases.
Case A: $s \in (\mathrm{F}_T^{(\mathrm{ANDP})} \cup \mathrm{L}_C^{(1)})$. Considering the network:

$$\mathcal{G}_{\mathrm{A}} = \big(\mathrm{MST}(S, \bar{E}, A) \setminus \{(s, \bar{s}_c)\}\big) \cup \{(s, s_w)\},$$

we have that $\mathcal{G}_{\mathrm{A}}$ is a tree spanning the sites of S, and besides:

$$\textsc{cost}(\mathcal{G}_{\mathrm{A}}) = \textsc{cost}(\mathrm{MST}(S, \bar{E}, A)) - c_{(s,\bar{s}_c)} + c_{(s,s_w)} \overset{\overset{\nexists s_C / s \in A_{s_C}}{\uparrow}}{<} \textsc{cost}(\mathrm{MST}(S, \bar{E}, A)),$$

consequently $\mathrm{MST}(S, \bar{E}, A)$ would not be globally optimal. This is a contradiction.
Case B: $\exists s_c \in S_C / s \in A_{s_c}$. By hypothesis we know that $A_{s_c} \subset A_{s_w}$. Let us build the network:

$$\mathcal{G}_{\mathrm{A}} = \big(\mathrm{MST}(S, \bar{E}, A) \setminus \{(s, \bar{s}_c)\}\big) \cup \{(s, s_c)\}.$$

This network is a tree topology spanning S, and moreover:

$$\textsc{cost}(\mathcal{G}_{\mathrm{A}}) = \textsc{cost}(\mathrm{MST}(S, \bar{E}, A)) - c_{(s,\bar{s}_c)} + c_{(s,s_c)} \overset{\overset{s \in A_{s_C}}{\uparrow}}{<} \textsc{cost}(\mathrm{MST}(S, \bar{E}, A)),$$

implying that $\mathrm{MST}(S, \bar{E}, A)$ would not be globally optimal. But this is contradictory, completing therefore the proof.

$\square$

Proposition 3.5.17. *Considering the restricted problem $ANDP(S, E_1 \cup \bar{E}_2, \bar{C})$, let us suppose that $\forall s_t \in (S_T \setminus \mathrm{F}_T^{(\mathrm{ANDP})})$, with $s_t \in A_{s_w}$, it is fulfilled the following inequality:*

$$\min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in (S_C \setminus A_{s_w})\right\} > \max\left\{c_{(s_t,s)}; \forall s \in A_{s_w}\right\}.$$

Then the feasible solution $\bar{\mathcal{T}} = \bigcup_{\forall s_w \in S_D} T_{A_{s_w},\bar{U}(A_{s_w}),\bar{C}}(S_T^w \cup \{s_w\})$ belonging to the space Γ_{ANDP} and corresponding to the union of optimal solutions associated to the instances $SPG(A_{s_w}, \bar{U}(A_{s_w}), \bar{C}, S_T^w \cup \{s_w\})$, $\forall s_w \in S_D$, is the best feasible solution of $\Gamma_{ANDP}^{(\leq 1)}$.

Proof. Firstly, it is easy to demonstrate that for all best feasible solution of $\Gamma_{ANDP}^{(\leq 1)}$, all terminal site of $\mathrm{F}_T^{(\mathrm{ANDP})}$ is directly connected to the switch site $s_w \in S_D$ to which it belongs to its covering area A_{s_w}. Moreover, if $\mathcal{T}^{(\mathrm{opt})}$ is a feasible solution of minimum cost in the subspace $\Gamma_{ANDP}^{(\leq 1)}$, it is easy to see that for all concentrator site $s_c \in S_C / s_c \in \mathcal{T}^{(\mathrm{opt})}$ then $s_c \in \mathcal{T}_{(s_w)}^{(\mathrm{opt})}$, where $s_c \in A_{s_w}$, $s_w \in S_D$.
Let us suppose that $\bar{\mathcal{T}}$ is not the best feasible solution in $\Gamma_{ANDP}^{(\leq 1)}$. Then, for all $\mathcal{T}^{(\mathrm{opt})} \in \Gamma_{ANDP}^{(\leq 1)}$ ($\mathcal{T}^{(\mathrm{opt})}$ best solution of $\Gamma_{ANDP}^{(\leq 1)}$) $\exists s_t \in S_T / s_t \in \mathcal{T}_{(\bar{s}_w)}^{(\mathrm{opt})}$, with $\bar{s}_w \neq s_w$, $s_t \in A_{s_w}$, $s_t \notin A_{\bar{s}_w}$. Let $\bar{\mathcal{T}}^{(\mathrm{opt})} \in \Gamma_{ANDP}^{(\leq 1)}$ be a solution of minimum cost. Let us denote: $\mathrm{Y}_T = \{s_t \in S_T / s_t \in A_{s_w}, s_t \in \bar{\mathcal{T}}_{(\bar{s}_w)}^{(\mathrm{opt})}, s_t \notin A_{\bar{s}_w}\}$. Let us consider a terminal site $s_t \in \mathrm{Y}_T$, we know that $\exists s_w, \bar{s}_w \in S_D$ such that $s_t \in \bar{\mathcal{T}}_{(\bar{s}_w)}^{(\mathrm{opt})}$, $s_t \notin A_{\bar{s}_w}$, and $s_t \in A_{s_w}$. Let $p_{(s_t,s_w)}$ be the shortest path between s_t and s_w in $\bar{G}_A = (S, \bar{U})$. If $p_{(s_t,s_w)}$ has the form (s_t, s_c, s_w), with $s_c \in S_C$, necessarily $s_c \in A_{s_w}$, otherwise we would have that $s_t \notin A_{s_w}$. Therefore, it is fulfilled that $p_{(s_t,s_w)} \subset \bar{U}(A_{s_w})$. Let us analyze the following cases.
Case 1: $(s_t, \bar{s}_w) \in \bar{\mathcal{T}}^{(\mathrm{opt})}$. Since $s_t \in A_{s_w}$ then $cp_{min}(s_t, s_w) \leq c_{(s_t,\bar{s}_w)}$. Now, we define the solution:

$$\mathcal{H} = \left(\bar{\mathcal{T}}^{(\mathrm{opt})} \setminus \{(s_t, \bar{s}_w)\}\right) \cup \{p_{(s_t,s_w)}\}.$$

The solution $\mathcal{H}$ satisfies $\mathcal{H} \in \Gamma_{ANDP}^{(\leq 1)}$, $s_t \in \mathcal{H}_{(s_w)}$, and besides:

$$\mathrm{COST}(\mathcal{H}) = \mathrm{COST}(\bar{\mathcal{T}}^{(\mathrm{opt})}) - c_{(s_t,\bar{s}_w)} + cp_{min}(s_t, s_w) \leq \mathrm{COST}(\bar{\mathcal{T}}^{(\mathrm{opt})}).$$

Case 2: $(s_t, \bar{s}_v, \bar{s}_w) \in \bar{\mathcal{T}}^{(\mathrm{opt})}$. Where $\bar{s}_v \in A_{\bar{s}_w}$, $\bar{s}_v \in S_C$. Let $s_v \in A_{s_w}$ be the site that fulfills $c_{(s_t,s_v)} = \min\left\{(s_t, s); \forall s \in (\bar{\mathcal{T}}_{(s_w)}^{(\mathrm{opt})} \cap A_{s_w})\right\}$, (we assume that $c(s_t, s_w) < \infty$, if $s_t \in A_{s_w}$). The following inequalities are satisfied:

$$c_{(s_t,s_v)} \leq \max\left\{c_{(s_t,s)}; \forall s \in A_{s_w}\right\} \overset{\substack{\textit{by hyp.}\\ \uparrow}}{<} \min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in (S_C \setminus A_{s_w})\right\} \leq c_{(s_t,\bar{s}_v)}.$$

Again, we define a new solution given by:

$$\mathcal{H} = \left(\bar{\mathcal{T}}^{(\mathrm{opt})} \setminus \{(s_t, \bar{s}_v)\}\right) \cup \{(s_t, s_v)\}.$$

As in the previous case, this solution satisfies $\mathcal{H} \in \Gamma_{ANDP}^{(\leq 1)}$, $s_t \in \mathcal{H}_{(s_w)}$, and in addition the inequality:

$$\mathrm{COST}(\mathcal{H}) = \mathrm{COST}(\bar{\mathcal{T}}^{(\mathrm{opt})}) - c_{(s_t,\bar{s}_v)} + c_{(s_t,s_v)} < \mathrm{COST}(\bar{\mathcal{T}}^{(\mathrm{opt})}).$$

Repeating the reasoning seen in the above cases, for all terminal site $s_t \in \Upsilon_T$ we obtain a feasible solution $\bar{\mathcal{H}} \in \Gamma^{(\leq 1)}_{ANDP}$ which fulfills that $s_t \in \bar{\mathcal{H}}_{(s_w)}$, $\forall s_t \in \Upsilon_T$, and furthermore $\textsc{cost}(\bar{\mathcal{H}}) \leq \textsc{cost}(\bar{\mathcal{T}}^{(\mathrm{opt})})$. This implies the existence of an optimal solution of $\Gamma^{(\leq 1)}_{ANDP}$ where all the sites of $S_T \cup S_C$ belonging to the solution are in their respective switch covering areas. Hence $\bar{\mathcal{T}}$ defined above, is a feasible solution of minimum cost in $\Gamma^{(\leq 1)}_{ANDP}$, as required, and completing the proof. □

Proposition 3.5.18. *Given an $ANDP(S, E_1 \cup E_2, C)$ such that in $G_A = (S, E_1 \cup E_2)$ for any three sites (s_1, s_2, s_3) with $s_1 \in S_T \cup S_C$, $s_2 \in S_C$, and $s_3 \in S_C \cup S_D$, the triangular inequality is fulfilled, i.e., $c_{(s_1,s_3)} \leq c_{(s_1,s_2)} + c_{(s_2,s_3)}$. Then the following points are satisfy:*

1. $c_{(s,s_w)} = cp_{min}(s, s_w)$, $\forall s \in S_T \cup S_C$,
2. $c_{(s,s_w)} \leq c_{(s,\bar{s}_w)}$, $\forall s \in A_{s_w}$, $\bar{s}_w \neq s_w$, $s_w, \bar{s}_w \in S_D$.

Proof. Let us consider two sites of G_A, $s \in S_T \cup S_C$ and $s_w \in S_D$. Let $p_{min(s,s_w)}$ be the shortest path between s and s_w in G_A. We denote by $\ell_{(p)}$ the length of the path p. We will demonstrate by induction in $\ell_{(p)}$ the statement (1).
Case 1: $\ell_{(p_{min})} \leq 2$. If $\ell_{(p_{min})} = 1$ then $p_{min(s,s_w)} = (s, s_w)$. If $\ell_{(p_{min})} = 2$ then there exits $s_c \in S_C$ such that $p_{min(s,s_w)} = (s, s_c, s_w)$. Considering the triangle formed by the sites $\{s, s_c, s_w\}$, the following inequality is satisfies:

$$c_{(s,s_w)} \overset{\overset{\text{by } \triangle \text{ ineq.}}{\uparrow}}{\leq} c_{(s,s_c)} + c_{(s_c,s_w)} \overset{\overset{\text{by hyp.}}{\uparrow}}{=} cp_{min}(s, s_w),$$

this implies that $c_{(s,s_w)} = cp_{min}(s, s_w)$.
Case 2: $2 < \ell_{(p_{min})} = k \leq h$. The inductive step is formulated of the following way.
As inductive hypothesis (denoted by I.H.) the statement (1) is satisfied when $\ell_{(p_{min})} = k < h$, and as inductive thesis (denoted by I.T.) the statement (1) is satisfied when $\ell_{(p_{min})} = h$.
Let us suppose that the length of the shortest path between s and s_w has the value $2 < \ell_{(p_{min})} = h$. Then, there exits a concentrator site s_c such that $s_c \in p_{min(s,s_w)}$. Let $\bar{p}_{min(s_c,s_w)}$ be the shortest path between s_c and s_w in G_A, which is contained in $p_{min(s,s_w)}$, i.e., $p_{min(s,s_w)} = \big((s, s_c)\textsc{conc}(\bar{p}_{min(s_c,s_w)})\big)$, where CONC is the concatenation operator. Since $\ell_{(\bar{p}_{min})} = h-1$ then by I.H. it is fulfilled that $c_{(s_c,s_w)} = cp_{min}(s_c, s_w)$, implying $cp_{min}(s, s_w) = c_{(s,s_c)} + c_{(s_c,s_w)}$. Moreover, by triangular inequality we know that $c_{(s,s_w)} \leq c_{(s,s_c)} + c_{(s_c,s_w)}$, implying $c_{(s,s_w)} \leq cp_{min}(s, s_w)$. Discarding the case $c_{(s,s_w)} < cp_{min}(s, s_w)$, we have $c_{(s,s_w)} = cp_{min}(s, s_w)$, as required, and completing the induction on the statement (1).
In order to prove the statement (2), let us consider $s \in S_T \cup S_C$, $s_w \in S_D / s \in A_{s_w}$, and $\bar{s}_w \in S_D$, $\bar{s}_w \neq s_w$. By definition of switch covering area $cp_{min}(s, s_w) \leq c_{(s,\bar{s}_w)}$, and statement (1) implies that $c_{(s,s_w)} \leq c_{(s,\bar{s}_w)}$; concluding therefore the proof. □

Lemma 3.5.19. *Let $ANDP(S, E_1 \cup E_2, C)$ be an instance where in $G_A = (S, E_1 \cup E_2)$ for any three sites (s_1, s_2, s_3) with $s_1 \in S_T \cup S_C$, $s_2 \in S_C$, and $s_3 \in S_C \cup S_D$, the triangular inequality is fulfilled, i.e., $c_{(s_1,s_3)} \leq c_{(s_1,s_2)} + c_{(s_2,s_3)}$, and besides the following inequality is satisfied:*

$$\min\left\{c_{(s_l,s_q)}; \forall s_q \in (S_C \setminus A_{s_w})\right\} > \max\left\{c_{(s_l,s)}; \forall s \in (A_{s_w} \setminus S_T)\right\},$$

for all $s_l \in (A_{s_w} \setminus \{s_w\})$, and $s_w \in S_D$. Then, the feasible solution of Γ_{ANDP} given by $\mathcal{T} = \bigcup_{\forall s_w \in S_D} T_{A_{s_w},E(A_{s_w}),C}(S_T^w \cup \{s_w\})$ is globally optimal to less of reassignments of border sites.

Proof. Firstly, we know that in any global optimal solution of Γ_{ANDP}, all terminal site $s_t \in \mathrm{F}_T^{(\mathrm{ANDP})}$ is connected directly to the switch site $s_w / s_t \in A_{s_w}$. Now, let us see what happens with the sites of $S_C \cup (S_T \setminus \mathrm{F}_T^{(\mathrm{ANDP})})$. We will prove that $\forall s_c \in S_C$, if $\mathcal{T}^{(\mathrm{opt})}$ (where $\mathcal{T}^{(\mathrm{opt})}$ is globally optimal in Γ_{ANDP}) then $s_c \in \mathcal{T}^{(\mathrm{opt})}_{(s_w)}$, with $s_c \in A_{s_w}$, to less of a connection reassignment when s_c is a border site.
Let $\mathcal{T}^{(\mathrm{opt})}$ be a global optimal solution of Γ_{ANDP}. Let us suppose that $\exists s_c \in \mathcal{T}^{(\mathrm{opt})}$ such that $s_c \in \mathcal{T}^{(\mathrm{opt})}_{(\bar{s}_w)}$, with $s_c \in A_{s_w}$, $\bar{s}_w \neq s_w$, $\bar{s}_w \in S_D$. Let us analyze the following cases.
Case A: $(s_c, \bar{s}_w) \in \mathcal{T}^{(\mathrm{opt})}$. Since $s_c \in A_{s_w}$, by Proposition 3.5.18 we know that $c_{(s_c,s_w)} \leq c_{(s_c,\bar{s}_w)}$. Let $\mathcal{G}$ be a solution defined by:

$$\mathcal{G} = \left(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_c, \bar{s}_w)\}\right) \cup \{(s_c, s_w)\},$$

this network satisfies $\mathcal{G} \in \Gamma_{ANDP}$, $s_c \in \mathcal{G}_{(s_w)}$, and in addition the inequality:

$$\mathrm{COST}(\mathcal{G}) = \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s_c,\bar{s}_w)} + c_{(s_c,s_w)} \leq \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}).$$

Case B: $\exists \bar{s}_c \in \mathcal{T}^{(\mathrm{opt})}_{(\bar{s}_w)} / (s_c, \bar{s}_c) \in \mathcal{T}^{(\mathrm{opt})}$. Let $p_{(s_c,\bar{s}_w)}$ be the path that communicates the site s_c with $\bar{s}_w$ in $\mathcal{T}^{(\mathrm{opt})}$. We will distinguish two different subcases.
Subcase B.1: If $\forall s_v \in S_C / s_v \in p_{(s_c,\bar{s}_w)}$ it is satisfied that $s_v \in A_{s_w}$, let $\bar{s}_v \in p_{(s_c,\bar{s}_w)}$ be the adjacent site to $\bar{s}_w$ in $\mathcal{T}^{(\mathrm{opt})}$. Let us consider the network defined by:

$$\mathcal{G} = \left(\mathcal{T}^{(\mathrm{opt})} \setminus \{(\bar{s}_v, \bar{s}_w)\}\right) \cup \{(\bar{s}_v, s_w)\}.$$

The network thus defined fulfills that $\mathcal{G} \in \Gamma_{ANDP}$, and $s_c \in \mathcal{G}_{(s_w)}$ (particularly $s_v \in \mathcal{T}^{(\mathrm{opt})}_{(\bar{s}_w)}$, $\forall s_v \in p_{(s_c,\bar{s}_w)}$). Moreover, since $\bar{s}_v \in A_{s_w}$ by Proposition 3.5.18 we know that $c_{(\bar{s}_v,s_w)} \leq c_{(\bar{s}_v,\bar{s}_w)}$. Therefore, the following inequality is fulfilled:

$$\mathrm{COST}(\mathcal{G}) = \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}) - c_{(\bar{s}_v,\bar{s}_w)} + c_{(\bar{s}_v,s_w)} \leq \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}).$$

Subcase B.2: If $\exists s_u \in p_{(s_c,\bar{s}_w)} / s_u \notin A_{s_w}$; let $\bar{s}_u \in p_{(s_c,\bar{s}_w)}$ be the concentrator site closest to s_c in $p_{(s_c,\bar{s}_w)}$ so that $\bar{s}_u \notin A_{s_w}$. Furthermore let us consider $\bar{s}_r \in A_{s_w}$ the concentrator site adjacent to $\bar{s}_u$ in $p_{(s_c,\bar{s}_w)}$ closest to s_c (eventually $\bar{s}_r = s_c$, and $\bar{s}_u = \bar{s}_c$). Again, we define a new network given by:

$$\mathcal{G} = \left(\mathcal{T}^{(\mathrm{opt})} \setminus \{(\bar{s}_r, \bar{s}_u)\}\right) \cup \{(\bar{s}_r, s_w)\}.$$

This network satisfies $\mathcal{G} \in \Gamma_{ANDP}$ and $s_c \in \mathcal{G}_{(s_w)}$. Its cost fulfills:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}^{(\text{opt})}) - c_{(\bar{s}_r,\bar{s}_u)} + c_{(\bar{s}_r,s_w)}.$$

By hypothesis, since $\bar{s}_r \in A_{s_w}$, we know:

$$c_{(\bar{s}_r,\bar{s}_u)} \geq \min\left\{c_{(\bar{s}_r,s_q)}; \forall s_q \in (S_C \setminus A_{s_w})\right\} > \max\left\{c_{(\bar{s}_r,s)}; \forall s \in (A_{s_w} \setminus S_T)\right\}.$$

In addition $\max\left\{c_{(\bar{s}_r,s)}; \forall s \in (A_{s_w} \setminus S_T)\right\} \geq c_{(\bar{s}_r,s_w)}$. Therefore $c_{(\bar{s}_r,\bar{s}_u)} > c_{(\bar{s}_r,s_w)}$, and implying that $\text{COST}(\mathcal{G}) < \text{COST}(\mathcal{T}^{(\text{opt})})$.
Once analyzed the previous points, we infer that if $s_c \in \mathcal{T}^{(\text{opt})}$ then $s_c \in \mathcal{T}^{(\text{opt})}_{(s_w)}$ (with $s_c \in A_{s_w}$) unless s_c is a border site, where in this case it is easy to see that there exits $\bar{\mathcal{T}}^{(\text{opt})} \in \Gamma_{ANDP}$ such that it is globally optimal and obtained from $\mathcal{T}^{(\text{opt})}$, reconnecting the border concentrator sites to their respective switch sites (from the switch covering area to which belongs).
Now, let us consider the terminal sites of $(S_T \setminus \text{F}_T^{(\text{ANDP})})$. We will demonstrate that $\forall s_t \in (S_T \setminus \text{F}_T^{(\text{ANDP})})/s_t \in A_{s_w}$, it is fulfilled that $s_t \in \mathcal{T}^{(\text{opt})}_{(s_w)}$, $\forall \mathcal{T}^{(\text{opt})} \in \Gamma_{ANDP}$ global optimal solution.
Again, considering $\mathcal{T}^{(\text{opt})}$ as globally optimal, let us suppose that $\exists s_t \in (S_T \setminus \text{F}_T^{(\text{ANDP})})$ such that $s_t \in \mathcal{T}^{(\text{opt})}_{(s_w)}$, $s_t \in A_{s_w}$, $\bar{s}_w \in S_D$, and $\bar{s}_w \neq s_w$. We will differentiate the following cases.
Case C: $(s_t, \bar{s}_w) \in \mathcal{T}^{(\text{opt})}$. Since $s_t \in A_{s_w}$, by Proposition 3.5.18 we know that $c_{(s_t,s_w)} \leq c_{(s_t,\bar{s}_w)}$. Let $\mathcal{G}$ be a solution defined by:

$$\mathcal{G} = \left(\mathcal{T}^{(\text{opt})} \setminus \{(s_t, \bar{s}_w)\}\right) \cup \{(s_t, s_w)\},$$

this network satisfies $\mathcal{G} \in \Gamma_{ANDP}$, $s_t \in \mathcal{G}_{(s_w)}$, and furthermore:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}^{(\text{opt})}) - c_{(s_t,\bar{s}_w)} + c_{(s_t,s_w)} \leq \text{COST}(\mathcal{T}^{(\text{opt})}).$$

Case D: $\exists \bar{s}_c \notin A_{s_w}, \bar{s}_c \in S_C/(s_t, \bar{s}_c) \in \mathcal{T}^{(\text{opt})}$. Once again, let us define the following network:

$$\mathcal{G} = \left(\mathcal{T}^{(\text{opt})} \setminus \{(s_t, \bar{s}_c)\}\right) \cup \{(s_t, s_w)\}.$$

Clearly $\mathcal{G} \in \Gamma_{ANDP}$ and $s_t \in \mathcal{G}_{(s_w)}$. In addition, knowing that $s_t \in A_{s_w}$ and using the hypotheses is deduced the inequality:

$$c_{(s_t,\bar{s}_c)} \geq \min\left\{c_{(\bar{s}_t,s_q)}; \forall s_q \in (S_C \setminus A_{s_w})\right\} > \max\left\{c_{(\bar{s}_t,s)}; \forall s \in (A_{s_w} \setminus S_T)\right\} \geq c_{(s_t,s_w)}.$$

Hence, we have the following relation:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}^{(\text{opt})}) - c_{(s_t,\bar{s}_c)} + c_{(s_t,s_w)} < \text{COST}(\mathcal{T}^{(\text{opt})}).$$

Now, considering the cases C and D, we also prove that if $s_t \in \mathcal{T}^{(\text{opt})}$ then $s_t \in \mathcal{T}^{(\text{opt})}_{(s_w)}$ (with $s_t \in A_{s_w}$) unless s_t is a border site, where as we saw above, in this case there exits $\bar{\mathcal{T}}^{(\text{opt})} \in \Gamma_{ANDP}$ globally optimal and builded from $\mathcal{T}^{(\text{opt})}$, reconnecting the border terminal sites to their respective switch sites (from the switch covering area to which belongs).
Finally, according to the cases previously exposed, we conclude that under the hypotheses of the Lemma there exits a global optimal solution where all the sites of $S_T \cup S_D$ belong to access sub-networks formed by trees that have as root to the switch site of the covering area to which these belong. Therefore, $\mathcal{T}$ is globally optimal to less of reassignments of border sites. $\square$

Notation 3.5.20. *Let* $ANDP(S, E_1 \cup E_2, C)$ *be an* ANDP *instance. We denote by* $\Gamma^{(\text{SPG})}_{ANDP}/\Gamma^{(\text{SPG})}_{ANDP} \subseteq \Gamma_{ANDP}$ *to the set of feasible solutions in* Γ_{ANDP} *given by the union of the optimal solutions of*
$SPG(A_{s_w}, E(A_{s_w}), C, S^w_T \cup \{s_w\})$*,* $\forall s_w \in S_D$*.*

Proposition 3.5.21. *Let* $ANDP(S, E_1 \cup E_2, C)$ *be an instance where for any three sites* (s_1, s_2, s_3) *with* $s_1 \in S_T \cup S_C$*,* $s_2 \in S_C$*, and* $s_3 \in S_C \cup S_D$*, the triangular inequality is satisfied. If all global optimal solution of* Γ_{ANDP} *belongs to* $\Gamma^{(\text{SPG})}_{ANDP}$*, then given* $\mathcal{T}^{(\text{opt})} \in \Gamma^{(\text{SPG})}_{ANDP}$*, it is fulfilled the following points.*

1. $\forall (s_u, s_v) \in \mathcal{T}^{(\text{opt})}$ *(with* $s_u, s_v \in A_{s_w}$*,* $s_w \in S_D$*):*
   (i) $\forall \bar{s}_c \in (S_C \setminus A_{s_w})$*,* $c_{(s_u,s_v)} < c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,s_v)}$*.*
   (ii) $c_{(s_u,s_v)} < \min\left\{c_{(s_u,\bar{s})}; \forall \bar{s} \in (\mathcal{T}^{(\text{opt})} \setminus A_{s_w}), \bar{s} \in S_C \cup S_D)\right\}$*, (considering the connection* (s_u, s_v)*, we assume that* s_v *is the site closest to* s_w *in* $\mathcal{T}_{opt}$*).*
2. $\forall \bar{s}_c \in (S_C \setminus A_{s_w})$ *(with* $\bar{s}_c \in A_{\bar{s}_w}, s_w, \bar{s}_w \in S_D$*), and* $\forall X_{\mathcal{T}} \subseteq (\mathcal{T}^{(\text{opt})} \setminus S_D)$*;*

$$\sum_{(s_u,s) \in \mathcal{T}^{(\text{opt})}/s_u \in X_{\mathcal{T}}} c_{(s_u,s)} < \sum_{s_u \in X_{\mathcal{T}}} c_{(s_u,\bar{s}_c)} + \min\left\{c_{(\bar{s}_c,\bar{s})}; \forall \bar{s} \in (\mathcal{T}^{(\text{opt})}_{(\bar{s}_w)} \setminus S^{\bar{w}}_T)\right\}.$$

Proof. Firstly, we will demonstrate the statement $(1.i)$. Let $\mathcal{T}^{(\text{opt})} \in \Gamma^{(\text{SPG})}_{ANDP}$ be a global optimal solution. Let us suppose that $\exists (s_u, s_v) \in \mathcal{T}^{(\text{opt})}$ and $\bar{s}_c \in (S_C \setminus A_{s_w})$ such that: $c_{(s_u,s_v)} \geq c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,s_v)}$. Since in G_A the triangular inequality is satisfied, then $c_{(s_u,s_v)} = c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,s_v)}$. Let us define the following access network:

$$\mathcal{G} = \left(\mathcal{T}^{(\text{opt})} \setminus \{(s_u, s_v)\}\right) \cup \{(s_u, \bar{s}_c), (\bar{s}_c, s_v)\}.$$

This solution satisfies $\mathcal{G} \notin \Gamma^{(\text{SPG})}_{ANDP}$ (because $\bar{s}_c \notin A_{s_w}$), $\mathcal{G} \in (\Gamma_{ANDP} \setminus \Gamma^{(\text{SPG})}_{ANDP})$, and moreover:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}^{(\text{opt})}) - c_{(s_u,s_v)} + c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,s_v)} = \text{COST}(\mathcal{T}^{(\text{opt})}),$$

implying that $\mathcal{G}$ also would be a global optimal solution. This contradicts the hypothesis. Now, let us see the statement $(1.ii)$. Again, we suppose that $\exists (s_u, s_v) \in \mathcal{T}^{(\text{opt})}$ so that: $c_{(s_u,s_v)} \geq \min\left\{c_{(s_u,\bar{s})}; \forall \bar{s} \in (\mathcal{T}^{(\text{opt})} \setminus A_{s_w}), \bar{s} \in S_C \cup S_D)\right\} = c_{(s_u,\bar{s}_r)}$, with $\bar{s}_r \in (\mathcal{T}^{(\text{opt})} \setminus A_{s_w})$. Considering the network:

$$\mathcal{G} = \left(\mathcal{T}^{(\text{opt})} \setminus \{(s_u, s_v)\}\right) \cup \{(s_u, \bar{s}_r)\},$$

it satisfies $\mathcal{G} \notin \Gamma^{(\text{SPG})}_{ANDP}$ (because $\bar{s}_r \notin A_{s_w}$), $\mathcal{G} \in (\Gamma_{ANDP} \setminus \Gamma^{(\text{SPG})}_{ANDP})$, and in addition:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}^{(\text{opt})}) - c_{(s_u,s_v)} + c_{(s_u,\bar{s}_r)} \geq \text{COST}(\mathcal{T}^{(\text{opt})}),$$

therefore $\mathcal{G}$ also would be globally optimal. This is a contradiction. Hence we have proven the statement (1).

Now, we will prove the statement (2). Let $\mathcal{T}^{(\mathrm{opt})} \in \Gamma^{(\mathrm{SPG})}_{ANDP}$ be an optimal solution, let us consider $\bar{s}_c \in (S_C \setminus A_{s_w})$, $\bar{s}_c \in A_{\bar{s}_w}/\bar{s}_w \in S_D$, and $X_{\mathcal{T}} \subseteq (\mathcal{T}^{(\mathrm{opt})} \setminus S_D)$.

Case A: $X_{\mathcal{T}} = \{s_u\}$. Let $(s_u, s) \in \mathcal{T}^{(\mathrm{opt})}_{(s_w)}$ be the line connection present in $\mathcal{T}^{(\mathrm{opt})}$. Then, the following inequality is fulfilled:

$$c_{(s_u,s)} \overset{\substack{by\ (1.i) \\ \uparrow}}{<} c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,s)} \overset{\substack{\bar{s}_c \in A_{\bar{s}_w} \\ \uparrow}}{\leq} c_{(s_u,\bar{s}_c)} + \min\left\{c_{(\bar{s}_c,\bar{s})}; \forall \bar{s} \in (\mathcal{T}^{(\mathrm{opt})}_{(\bar{s}_w)} \setminus S^{\bar{w}}_T)\right\}.$$

Case B: $|Y| > 1$. Let $\bar{s}_m \in (\mathcal{T}^{(\mathrm{opt})}_{(\bar{s}_w)} \setminus S^{\bar{w}}_T)$ be the site that satisfies:

$$c_{(\bar{s}_c,\bar{s}_m)} = \min\left\{c_{(\bar{s}_c,\bar{s})}; \forall \bar{s} \in (\mathcal{T}^{(\mathrm{opt})}_{(\bar{s}_w)} \setminus S^{\bar{w}}_T)\right\}.$$

Let us suppose that:

$$\sum_{(s_u,s)\in\mathcal{T}^{(\mathrm{opt})}/s_u\in X_{\mathcal{T}}} c_{(s_u,s)} \geq \sum_{s_u\in X_{\mathcal{T}}} c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,\bar{s}_m)}.$$

Given the network:

$$\mathcal{H} = \left(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_u,s) \in \mathcal{T}^{(\mathrm{opt})}/s_u \in X_{\mathcal{T}}\}\right) \cup \{(s_u,\bar{s}_c)/s_u \in X_{\mathcal{T}}\} \cup \{(\bar{s}_c,\bar{s}_m)\},$$

it is easy to see that this network is a feasible solution of the ANDP and furthermore satisfies $\mathcal{H} \in \left(\Gamma_{ANDP} \setminus \Gamma^{(\mathrm{SPG})}_{ANDP}\right)$. Besides, analyzing its cost, we have:

$$\begin{aligned}\mathrm{COST}(\mathcal{H}) &= \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}) - \sum_{(s_u,s)\in\mathcal{T}^{(\mathrm{opt})}/s_u\in X_{\mathcal{T}}} c_{(s_u,s)} + \sum_{s_u\in X_{\mathcal{T}}} c_{(s_u,\bar{s}_c)} + c_{(\bar{s}_c,\bar{s}_m)} \\ &\leq \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}),\end{aligned}$$

but this implies that $\mathcal{H}$ is also a global optimal solution. This is a contradiction, therefore the statement (2) is fulfilled. □

Theorem 3.5.22. *Given an $ANDP(S, E_1 \cup E_2, C)$ instance and a fixed integer $k > 1$. Let us suppose that there exists a subset of terminal sites $X_T = \{s_{t_1}, \ldots, s_{t_r}\} \subseteq S_T$, so that $\mathcal{P} = \{p^{(1)}, \ldots, p^{(r)}\}$ is a set of shortest paths from the sites of X_T towards the backbone network in G_A, which satisfy $\ell_{(p^{(i)})} \leq k$ (where $\ell_{(\cdot)}$ is the length operator), $\forall p^{(i)} \in \mathcal{P}$. Let $X_C \subseteq S_C$ be the set of concentrator sites present in $\mathcal{P}$. If the following conditions are satisfied:*

1. $\forall p^{(i)} \in \mathcal{P}/p^{(i)} \cap (\mathcal{P} \setminus p^{(i)}) = \emptyset$, $\mathrm{COST}(p^{(i)}) < cp_{min}(s_{t_i}, \bar{s}_c), \forall \bar{s}_c \in (S_C \setminus p^{(i)})$.
2. $\forall s_c \in X_C/\exists p^{(i)}, p^{(j)} \in \mathcal{P}$, *with* $s_c \in (p^{(i)} \cap p^{(j)})$, *it is fulfilled:*
   (a) $\mathrm{COST}(p^{(i)}_{(s_c,s_w)}) < c_{(s_c,\bar{s}_c)}, \forall \bar{s}_c \in (S_C \setminus (p^{(i)} \cup p^{(j)}))$,

(b) $\textsc{cost}(p^{(i)}_{(s_{t_i},s_c)}) < c_{(s_{t_i},\bar{s}_c)}$, $\forall \bar{s}_c \in (S_C \setminus p^{(i)})$ *symmetrically for* $p^{(j)}$).

3. $S_T = X_T \cup \{s_t \in (S_T \setminus X_T)/\exists s_c \in X_C, \textit{with } s_t \in A_{s_c}\} \cup \mathrm{F}_T^{(\mathrm{ANDP})}$.

Then $\exists \mathcal{T}^{(\mathrm{k})} \in \Gamma^{(\leq k)}_{ANDP}$ *global optimal solution of* Γ_{ANDP}.

Proof. Firstly, it is easy to see that all terminal site of $\mathrm{F}_T^{(\mathrm{ANDP})}$ is directly connected to some switch site for any global optimal solution. We denote the set of direct connections from the sites of $\mathrm{F}_T^{(\mathrm{ANDP})}$ towards the backbone network by:

$$\mathcal{F} = \left\{(s_t, s_w) \in E_1/s_t \in \mathrm{F}_T^{(\mathrm{ANDP})}, s_t \in A_{s_w}, \textit{with } s_w \in S_D\right\}.$$

We denote $\bar{X}_T = S_T \setminus (X_T \cup \mathrm{F}_T^{(\mathrm{ANDP})})$. Let $\mathcal{L}$ be the set of connections given by:

$$\mathcal{L} = \{(s_t, s_c) \in E_1; \textit{where } s_t \in \bar{X}_T,\ s_c \in X_C, \textit{ and } s_t \in A_{s_c}\}.$$

By (3) all the nodes of $\bar{X}_T$ are considered in $\mathcal{L}$. Moreover, by definition of concentrator covering area, $\textsc{cost}(\mathcal{L})$ is equal to the minimum cost of connecting all the terminal sites of $\bar{X}_T$ to any site of $S_C \cup S_D$, and therefore it is a lower bound for the sum of the direct connections of the sites of $\bar{X}_T$ in all global optimal solution of Γ_{ANDP}.
We divide the sets X_T and $\mathcal{P}$ in the following subsets:

- $X_T^{(1)} = \{s_{t_i} \in X_T/p^{(i)} \cap (\mathcal{P} \setminus p^{(i)}) = \emptyset\}$,
- $X_T^{(2)} = X_T \setminus X_T^{(1)}$,
- $\mathcal{P}^{(1)} = \{p^{(i)} \in \mathcal{P}/s_{t_i} \in X_T^{(1)}\}$,
- $\mathcal{P}^{(2)} = \mathcal{P} \setminus \mathcal{P}^{(1)}$.

Now, we will see that $\mathcal{P}^{(1)} \subseteq \mathcal{T}^{(\mathrm{opt})}$ where $\mathcal{T}^{(\mathrm{opt})}$ is a global optimal solution of Γ_{ANDP}. Let us suppose that $\mathcal{P}^{(1)} \not\subseteq \mathcal{T}^{(\mathrm{opt})}$, this implies that $\exists s_{t_v} \in X_T$ such that if $\bar{p}^{(v)}$ is the path from s_{t_v} towards the backbone network in $\mathcal{T}^{(\mathrm{opt})}$ then $\bar{p}^{(v)} \neq p^{(v)}$. Let us analyze the following cases:
Case A: $p^{(v)} = (s_{t_v}, s_w)/s_w \in S_D$. If $\bar{p}^{(v)} = (s_{t_v}, \bar{s}_w)$, necessarily $c_{(s_{t_v},s_w)} = c_{(s_{t_v},\bar{s}_w)}$ (s_{t_v} would be a border site). Considering $\bar{\mathcal{T}}^{(\mathrm{opt})} = (\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_{t_v}, \bar{s}_w)\}) \cup \{(s_{t_v}, s_w)\}$, this is also globally optimal. If $\exists \bar{s}_c \in S_C/\bar{s}_c \in \bar{p}^{(v)}$, let $\bar{s} \in S_C$ be the concentrator site closest to s_{t_v} in $\bar{p}^{(v)}$. Considering $\bar{\mathcal{T}}^{(\mathrm{opt})} = (\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_{t_v}, \bar{s})\}) \cup \{(s_{t_v}, s_w)\}$, clearly this is a feasible solution of Γ_{ANDP}, and moreover applying (1) we have:

$$\textsc{cost}(\bar{\mathcal{T}}^{(\mathrm{opt})}) = \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s_{t_v},\bar{s})} + c_{(s_{t_v},s_w)} < \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}),$$

contradicting that $\mathcal{T}^{(\mathrm{opt})}$ is globally optimal.
Case B: $\exists s_c \in S_\mathcal{C}/s_c \in p^{(v)}$. If $\bar{p}^{(v)} = (s_{t_v}, \bar{s}_w)$, necessarily $\textsc{cost}(p^{(v)}) = c_{(s_{t_v},\bar{s}_w)}$ (s_{t_v} would be a border site). Again, $\bar{\mathcal{T}}^{(\mathrm{opt})} = (\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_{t_v}, \bar{s}_w)\}) \cup \{p^{(v)}\}$ is a feasible solution of Γ_{ANDP} and furthermore:

$$\textsc{cost}(\bar{\mathcal{T}}^{(\mathrm{opt})}) = \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s_{t_v},\bar{s})} + \textsc{cost}(p^{(v)}) \overset{\overset{s_{t_v} \textit{ is border site}}{\uparrow}}{=} \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}),$$

implying the optimality of $\bar{\mathcal{T}}^{(\mathrm{opt})}$. Let us see the case when both paths are only different by their last connection lines; $(\bar{s}, s_w) \in p^{(v)}$ and $(\bar{s}, \bar{s}_w) \in \bar{p}^{(v)}$. Considering $\bar{\mathcal{T}}^{(\mathrm{opt})} = (\mathcal{T}^{(\mathrm{opt})} \setminus \{(\bar{s}, \bar{s}_w)\}) \cup \{(\bar{s}, s_w)\}$, we have that $\bar{\mathcal{T}}^{(\mathrm{opt})} \in \Gamma_{ANDP}$, and in addition:

$$\textsc{cost}(\bar{\mathcal{T}}^{(\mathrm{opt})}) = \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}) - c_{(\bar{s},\bar{s}_w)} + c_{(\bar{s},s_w)} \overset{\overset{p^{(v)} \textit{ is shortest}}{\uparrow}}{\leq} \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}).$$

If the equality is fulfilled, then s_{t_v} would be a border site and $\bar{\mathcal{T}}^{(\mathrm{opt})}$ would be also globally optimal. Otherwise, the strict inequality contradicts the optimality of $\mathcal{T}^{(\mathrm{opt})}$.
Finally, let us suppose that $p^{(v)}$ and $\bar{p}^{(v)}$ are different at least in one concentrator site. Let $\bar{s}_q \in \bar{p}^{(v)}$ be the concentrator site closest to s_{t_v} in $\bar{p}^{(v)}$ such that $\bar{s}_q \notin p^{(v)}$, and $s \in p^{(v)}$ its preceding site. Let us consider the network:

$$\bar{\mathcal{T}}^{(\mathrm{opt})} = \left(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s, \bar{s}_q)\}\right) \cup \{p^{(v)}\}.$$

It is easy to see that $\bar{\mathcal{T}}^{(\mathrm{opt})} \in \Gamma_{ANDP}$, and besides the following relation is satisfied:

$$\textsc{cost}(\bar{\mathcal{T}}^{(\mathrm{opt})}) = \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s,\bar{s}_q)} + \textsc{cost}(p^{(v)}) \overset{\overset{\textit{by } (1)}{\uparrow}}{<} \textsc{cost}(\mathcal{T}^{(\mathrm{opt})}).$$

This is a contradiction since $\mathcal{T}^{(\mathrm{opt})}$ is globally optimal. Hence, of the cases previously analyzed we infer that $\mathcal{P}^{(1)} \subseteq \mathcal{T}^{(\mathrm{opt})}$ to less of reassignment of border sites.
Now, let us analyze the sites of $X_T^{(2)}$. We will prove that $\textsc{cost}(\mathcal{P}^{(2)})$ is the minimum cost of connecting the sites of $X_T^{(2)}$ to sites of $S_C \cup S_D$ acceding to the backbone network. Given a maximal subset $Y_T \subseteq X_T^{(2)} / \forall s_{t_i}, s_{t_j} \in Y_T, p^{(i)} \cap p^{(j)} \neq \emptyset$. By induction in $|Y_T|$, we will demonstrate that $\textsc{cost}(\mathcal{P}^{(2)}_{|Y_T|})$ is the minimum cost of connecting the terminal sites of Y_T to sites of $S_C \cup S_D$ acceding to the backbone network.
Basic Step: $|Y_T| = 2$, $Y_T = \{s_{t_i}, s_{t_j}\}$. Let $s_c \in S_C$ be the joint site in $p^{(i)} \cap p^{(j)}$ (we assume that from s_c, $p^{(i)}$ and $p^{(j)}$ are the same path). The following notation is introduced:

$$\begin{aligned}
\phi_{(s_{t_i},s_{t_j})} &= \min\left\{c_{(s_{t_i},\bar{s}_c)} / \bar{s}_c \in (S_C \setminus p^{(i)})\right\} + \min\left\{c_{(s_{t_j},\bar{s}_c)} / \bar{s}_c \in (S_C \setminus p^{(j)})\right\}, \\
\phi_{(s_c)} &= \min\left\{c_{(s_c,\bar{s}_c)} / \bar{s}_c \in (S_C \setminus (p^{(i)} \cup p^{(j)}))\right\}.
\end{aligned}$$

The minimum cost of connecting the terminal sites $\{s_{t_i}, s_{t_j}\}$ to sites of $S_C \cup S_D$ acceding to the backbone network is at the most:

$$\begin{aligned}
&\min\left\{\phi_{(s_{t_i},s_{t_j})}, \textsc{cost}(p^{(i)} \cup p^{(j)}), \textsc{cost}(p^{(i)}_{(s_{t_i},s_c)}) + \textsc{cost}(p^{(j)}_{(s_{t_j},s_c)}) + \phi_{(s_c)}\right\} \\
&\overset{\overset{\textit{by } (2.b)}{\uparrow}}{>} \min\left\{\textsc{cost}(p^{(i)} \cup p^{(j)}), \textsc{cost}(p^{(i)}_{(s_{t_i},s_c)}) + \textsc{cost}(p^{(j)}_{(s_{t_j},s_c)}) + \phi_{(s_c)}\right\} \\
&\overset{\overset{\textit{by } (2.a)}{\uparrow}}{>} \textsc{cost}(p^{(i)} \cup p^{(j)}),
\end{aligned}$$

completing therefore the demonstration of basic step.
Induction Step: $|Y_T| > 2$. Firstly, we denote by $\mathrm{C}_{min(Y_T)}$ to the minimum cost of connecting

the terminal sites of Y_T to sites of $S_C \cup S_D$ acceding to the backbone network. Let $\mathcal{G}_Y = \mathcal{P}^{(2)}_{|Y_T}$ be the tree induced by Y_T and their shortest paths. Let $s_c \in S_C$ be the joint site with greater depth in $\mathcal{G}_Y$. Let us choose a terminal site $s_{t_i} \in Y_T / p^{(i)}_{(s_{t_i}, s_c)} \subset \mathcal{G}_Y$. Since by $(2.b)$ we know that:

$$\textsc{cost}(p^{(i)}_{(s_{t_i}, s_c)}) < c_{(s_{t_i}, \bar{s}_c)}, \forall \bar{s}_c \in (S_C \setminus p^{(i)}),$$

considering $\bar{Y}_T = Y_T \setminus \{s_{t_i}\}$, we have the following relation:

$$\mathrm{C}_{min(Y_T)} = \mathrm{C}_{min(\bar{Y}_T \cup \{s_{t_i}\})} \overset{\overset{\text{by } (2.a),\,(2.b)}{\uparrow}}{=} \mathrm{C}_{min(\bar{Y}_T)} + \textsc{cost}(p^{(i)}_{(s_{t_i}, s_c)}) \overset{\overset{\text{by } I.H.}{\uparrow}}{=}$$
$$= \textsc{cost}(\mathcal{P}^{(2)}_{|\bar{Y}_T}) + \textsc{cost}(p^{(i)}_{(s_{t_i}, s_c)}) \overset{\overset{\text{by def.}}{\uparrow}}{=} \textsc{cost}(\mathcal{P}^{(2)}_{|Y_T}),$$

as required, and completing the induction step.
Hence, dividing to the set X_T in different disjoint components satisfying the above property, then we have that $\mathrm{C}_{min(X_T)} = \textsc{cost}(\mathcal{P}^{(2)})$. Now, let us build the following network:

$$\mathcal{T}^{(\mathrm{k})} = (\bigcup_{\forall p^{(i)} \in \mathcal{P}} \{p^{(i)}\}) \cup \mathcal{L} \cup \mathcal{F},$$

clearly $\mathcal{T}^{(\mathrm{k})} \in \Gamma^{(\leq k)}_{ANDP}$, its cost is $\textsc{cost}(\mathcal{F} \cup \mathcal{L}) + \textsc{cost}(\mathcal{P}^{(1)}) + \textsc{cost}(\mathcal{P}^{(2)})$, and by the previously exposed this cost is minimal, therefore $\mathcal{T}^{(\mathrm{k})}$ is globally optimal. □

Notation 3.5.23. *Let $ANDP(S, E_1 \cup E_2, C)$ be an* ANDP *instance. Given a feasible solution $\mathcal{T} \in \Gamma_{ANDP}$ and a terminal site $s_t \in S_T$. We introduce the following notation:*

- $p_{\mathcal{T}(s_t)}$ *is the path from s_t towards the backbone network on $\mathcal{T}$.*
- $p_{min(s_t)}$ *is the shortest path from s_t towards the backbone network on G_A.*
- $\mathcal{T}_{(s_c)}$ *denotes the sub-tree of $\mathcal{T}$ which has the concentrator site s_c as root.*
- $p_{\mathcal{T}(s_i, s_j)}$ *is the path from s_i to s_j on $\mathcal{T}$.*
- $\bar{\mathrm{d}}_{\mathcal{T}(s_i, s_j)}$ *is the cost of the shortest path between the site s_i and s_j restricted to the network $(G_A \setminus \textsc{edges}_{(\mathcal{T})})$, and we denote as $\bar{p}_{\mathcal{T}(s_i, s_j)}$ to this path.*

Proposition 3.5.24. *Given an $ANDP(S, E_1 \cup E_2, C)$ instance. Let us suppose that there exists $\mathcal{T} \in \Gamma^{(\leq k)}_{ANDP}$ global optimal solution of Γ_{ANDP}. Then the following points are satisfied:*

1. *If $\textsc{cost}(p_{\mathcal{T}(s_t)}) > \textsc{cost}(p_{min(s_t)})$ or $\ell_{(p_{min(s_t)})} > k+1$, $\exists X_T \subseteq S_T$, with $s_t \in X_T$, and at least one access sub-network $\mathcal{T}_X \subseteq G_A$ spanning X_T so that $\textsc{depth}_{(\mathcal{T}_X)} \leq k+1$ and besides:*

$$\textsc{cost}(\mathcal{T}_X) \leq \textsc{cost}(\bigcup_{s_t \in X_T} \{p_{min(s_t)}\}).$$

2. $\forall (s_c, s) \in p_{\mathcal{T}(s_t)}/\ell_{(p_{\mathcal{T}(s_t)})} \leq k+1$, *with* $s_t \in S_T$, *and* $s \in S_C \cup S_D$, *it is fulfilled:*

$$c_{(s_c,s)} \leq \min \left\{ \bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s}_c)}; \forall \bar{s}_c \in (X_C \setminus \mathcal{T}_{(s)}) \right\},$$

where X_C *is the set of concentrator sites present in* $\mathcal{T}$.

Proof. Firstly, we will demonstrate (1). Let us suppose that $\exists s_t \in S_T$ such that $\textsc{cost}(p_{\mathcal{T}(s_t)}) > \textsc{cost}(p_{min(s_t)})$ or $\ell_{(p_{min(s_t)})} > k+1$. Let $\mathcal{T}_X \subseteq \mathcal{T}$ be the access sub-network with tree topology spanning the terminal sites of $X_T/s_t \in X_T$. Clearly $\textsc{depth}_{(\mathcal{T}_X)} \leq k+1$. By contradiction, we suppose that the following inequality is fulfilled:

$$\textsc{cost}(\mathcal{T}_X) > \textsc{cost}(\bigcup_{s_t \in X_T} \{p_{min(s_t)}\}).$$

Let us consider the network: $\bar{\mathcal{T}} = \left(\mathcal{T} \setminus \textsc{edges}_{(\mathcal{T}_X)}\right) \cup \left(\bigcup_{s_t \in X_T} \{p_{min(s_t)}\}\right)$. It is easy to see that $\mathcal{T} \in \Gamma_{ANDP}$, let us see its cost:

$$\textsc{cost}(\bar{\mathcal{T}}) = \textsc{cost}(\mathcal{T}) - \textsc{cost}(\mathcal{T}_X) + \textsc{cost}(\bigcup_{s_t \in X_T} \{p_{min(s_t)}\}) < \textsc{cost}(\mathcal{T}),$$

this contradicts the optimality of $\mathcal{T}$, completing the proof of (1).
To demonstrate (2), let us suppose that $\exists s_t \in S_T/\ell_{(p_{\mathcal{T}(s_t)})} \leq k+1$ and $\exists (s_c, s) \in p_{\mathcal{T}(s_t)}$, with $s \in S_C \cup S_D$, such that: $c_{(s_c,s)} > \min \left\{ \bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s}_c)}; \forall \bar{s}_c \in (X_C \setminus \mathcal{T}_{(s)}) \right\}$. Let $\bar{s} \in S_C$ be the concentrator site that minimize this distance. Let us build the network: $\bar{\mathcal{T}} = (\mathcal{T} \setminus \{(s_c, s)\}) \cup \{\bar{p}_{\mathcal{T}(s_c,\bar{s})}\}$. Clearly $\bar{\mathcal{T}} \in \Gamma_{ANDP}$, and moreover:

$$\textsc{cost}(\bar{\mathcal{T}}) = \textsc{cost}(\mathcal{T}) - c_{(s_c,s)} + \bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s})} < \textsc{cost}(\mathcal{T}).$$

This is a contradiction since $\mathcal{T}$ is globally optimal, proving thus the point (2), and completing the proof.

□

Proposition 3.5.25. *In the hypothesis of the previous Proposition, if in addition does not exist a global optimal solution in* $\Gamma^{(\leq k-1)}_{ANDP}$, *then the following points are satisfied:*

1. $\exists p_{\mathcal{T}(s_t,s_w)}$, *with* $\textsc{depth}_{(p_{\mathcal{T}(s_t,s_w)})} = k+1/\forall (s_c, s) \in p_{\mathcal{T}(s_t,s_w)}$ *it is fulfilled:*

$$c_{(s_c,s)} \geq \min \left\{ \bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s}_c)}; \forall \bar{s}_c \in (X_C \setminus \mathcal{T}_{(s)})/\ell_{(p_{\mathcal{T}(s_t,s_c)}, \bar{p}_{\mathcal{T}(s_c,\bar{s}_c)}, p_{\mathcal{T}(\bar{s}_c,\bar{s}_w)})} \leq k \right\}.$$

2. *Let* $K_T \subseteq S_T$ *be the maximal subset so that* $\forall s_t \in K_T$, $\ell_{(p_{\mathcal{T}(s_t)})} = k+1$, *and* $\mathcal{P} = \{p_{\mathcal{T}(s_t)}/s_t \in K_T\}$. *Considering* $\mathcal{H}_A = (G_A \setminus \textsc{edges}_{(\mathcal{T})}) \cup \mathcal{P}$, *for all access sub-network* $\mathcal{G}_K \subseteq \mathcal{H}_A$ *spanning* K_T *and* $\textsc{depth}_{(\mathcal{G}_K)} \leq k$ *it is fulfilled:*

$$\textsc{cost}(\mathcal{G}_K \setminus \mathcal{P}) > \textsc{cost}(\mathcal{I}_K),$$

where $\mathcal{I}_K = \left\{ (s_i, s_j) \in p_{\mathcal{T}(s_t)}, \textit{with } s_t \in K_T/s_i \in \mathcal{G}_K \textit{ and } s_j \notin \mathcal{G}_K \right\}$.

Proof. In order to prove (1), let us suppose that $\forall p_{\mathcal{T}(s_t)}/\textsc{depth}_{(p_{\mathcal{T}(s_t)})} = k+1$, there exists a connection $(s_c, s) \in p_{\mathcal{T}(s_t)}$ such that:

$$c_{(s_c,s)} \geq \min\left\{\bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s}_c)}; \forall \bar{s}_c \in (X_C \setminus \mathcal{T}_{(s)})/\ell_{(p_{\mathcal{T}(s_t,s_c)},\bar{p}_{\mathcal{T}(s_c,\bar{s}_c)},p_{\mathcal{T}(\bar{s}_c,\bar{s}_w)})} > k\right\}.$$

Let us denote by $\mathcal{B}$ the set of connections that fulfill the previous equation. If in $\mathcal{B}$ there exists a connection fulfilling strictly the inequality, then we could build a feasible solution whose cost is smaller to $\textsc{cost}(\mathcal{T})$, contradicting therefore the optimality of $\mathcal{T}$. Otherwise, if in $\mathcal{B}$ all the connections fulfill the equality, then considering the set of paths that induce to the equality in the previous equation; $\bar{\mathcal{P}}_{\mathcal{T}} = \left\{\bar{p}_{\mathcal{T}(s_c,\bar{s}_c)}/c_{(s_c,s)} = \bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s}_c)}, \textit{with } (s_c,s) \in \mathcal{B}\right\}$, we build the following network: $\mathcal{G}_{\mathcal{T}} = (\mathcal{T} \setminus \mathcal{B}) \cup \bar{\mathcal{P}}_{\mathcal{T}}$. It is easy to see that $\mathcal{G}_{\mathcal{T}} \in \Gamma^{(\leq k-1)}_{ANDP}$. Now, let us compare its cost:

$$\textsc{cost}(\mathcal{G}_{\mathcal{T}}) = \textsc{cost}(\mathcal{T}) - \textsc{cost}(\mathcal{B}) + \textsc{cost}(\bar{\mathcal{P}}_{\mathcal{T}}) \quad \overset{\overset{c_{(s_c,s)}=\bar{\mathrm{d}}_{\mathcal{T}(s_c,\bar{s}_c)}}{\uparrow}}{=} \quad \textsc{cost}(\mathcal{T}),$$

but this implies that $\mathcal{G}_{\mathcal{T}}$ is also globally optimal, contradicting the hypothesis that in $\Gamma^{(\leq k-1)}_{ANDP}$ does not exist a global optimal solution. Hence, we have completed the proof of (1).
Now, we will prove the statement (2). In the hypotheses of (2), let us suppose that there exists an access sub-network $\mathcal{G}_K \subseteq \mathcal{H}_A$ spanning K_T, $\textsc{depth}_{(\mathcal{G}_K)} \leq k$ and furthermore: $\textsc{cost}(\mathcal{G}_K \setminus \mathcal{P}) \leq \textsc{cost}(\mathcal{I}_K)$. We build the following network:

$$\bar{\mathcal{T}} = (\mathcal{T} \setminus \mathcal{I}_K) \cup \left(\mathcal{G}_K \setminus \textsc{edges}_{(\mathcal{P})}\right).$$

By construction, the network $\bar{\mathcal{T}}$ is a feasible solution of Γ_{ANDP}. Moreover, analyzing its depth, we have: $\textsc{depth}_{(\bar{\mathcal{T}})} = \max\left\{\textsc{depth}_{(\mathcal{G}_K)}, \textsc{depth}_{(\mathcal{T}\setminus\mathcal{P})}\right\} \leq k$, implying that particularly $\bar{\mathcal{T}} \in \Gamma^{(\leq k-1)}_{ANDP}$. Next, we will analyze its cost:

$$\textsc{cost}(\bar{\mathcal{T}}) = \textsc{cost}(\mathcal{T}) - \textsc{cost}(\mathcal{I}_K) + \textsc{cost}(\mathcal{G}_K) - \textsc{cost}(\mathcal{P}) \leq \textsc{cost}(\mathcal{T}).$$

If the inequality is fulfilled strictly, we would have one better feasible solution that $\bar{\mathcal{T}}$, and this would imply the not-optimality of $\mathcal{T}$. Therefore it would be a contradiction. If the equality is fulfilled, then $\bar{\mathcal{T}}$ is also globally optimal and besides by the exposed above $\bar{\mathcal{T}}$ belongs to the subspace $\Gamma^{(\leq k-1)}_{ANDP}$. But by hypothesis we know that in $\Gamma^{(\leq k-1)}_{ANDP}$ does not exist optimal solutions. Hence (2) is established, as required, and completing the proof. □

Proposition 3.5.26. *Given an $ANDP(S, E_1 \cup E_2, C)$ such that the following points are fulfilled:*

1. $\forall s_t \in (S_T \setminus \mathrm{F}^{(\mathrm{ANDP})}_T)$, $\exists s_c \in \mathrm{L}^{(1)}_C / s_t \in A_{s_c}$.
2. $\forall s_t \in A_{s_c}$, $c_{(s_t,s_w)} < \min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in \left(S_C \setminus \mathrm{L}^{(1)}_C\right)\right\}$, *where* $s_c \in A_{s_w}$, *and* $s_w \in S_D$.

Then all the global optimal solutions of Γ_{ANDP} *belong to* $\Gamma^{(\leq 1)}_{ANDP}$.

Proof. We will assume that the sites of $\mathrm{L}_C^{(1)}$ are not border sites with respect to other concentrator sites. Let us suppose that $\exists \mathcal{T} \in \left(\Gamma_{ANDP} \setminus \Gamma_{ANDP}^{(\leq 1)}\right)$ globally optimal. Let $p_{\mathcal{M}(s_t,s_w)} \subset \mathcal{T}$ be the longest path from a terminal site to a switch site. Under this supposition necessarily $\ell_{(p_{\mathcal{M}(s_t,s_w)})} \geq 3$. Let $s_c \in S_C$ be the concentrator site adjacent to s_t in $p_{\mathcal{M}}$. We will analyze the following cases.

Case A: $s_c \notin \mathrm{L}_C^{(1)}$. Let $\mathcal{B}_T$ be the set of connections from terminal sites to s_c present in $\mathcal{T}$. Let us consider the set of connections given by:

$$\mathcal{B}_D = \left\{(\bar{s}_t, \bar{s}_w)/\bar{s}_w \in S_D, \exists \bar{s}_t \in \mathcal{B}_T, \textit{with } \bar{s}_t \in A_{s_{c_1}}, \textit{ and } s_{c_1} \in A_{\bar{s}_w}\right\}.$$

Let us build the network: $\bar{\mathcal{T}} = (\mathcal{T} \setminus \mathcal{B}_T) \cup \mathcal{B}_D$. It is easy to see that $\bar{\mathcal{T}} \in \Gamma_{ANDP}$, we will compare its cost:

$$\textsc{cost}(\bar{\mathcal{T}}) = \textsc{cost}(\mathcal{T}) - \textsc{cost}(\mathcal{B}_T) + \textsc{cost}(\mathcal{B}_D) \overset{\overset{\textit{by }(2)}{\uparrow}}{<} \textsc{cost}(\mathcal{T}).$$

This contradicts the optimality of $\mathcal{T}$.

Case B: $s_c \in \mathrm{L}_C^{(1)}$. Let s_{c_2} be the concentrator site adjacent to s_c in $p_{\mathcal{M}}$. Since $s_c \in \mathrm{L}_C^{(1)}$, by definition we know that $c_{(s_c,s_{c_2})} > c_{(s_c,\bar{s}_w)}$, where $s_c \in A_{\bar{s}_w}$. As above, we construct a new network given by: $\bar{\mathcal{T}} = (\mathcal{T} \setminus \{(s_c, s_{c_2})\}) \cup \{(s_c, \bar{s}_w)\}$. Clearly $\bar{\mathcal{T}} \in \Gamma_{ANDP}$, and moreover:

$$\textsc{cost}(\bar{\mathcal{T}}) = \textsc{cost}(\mathcal{T}) - c_{(s_c,s_{c_2})} + c_{(s_c,\bar{s}_w)} \overset{\overset{\textit{by }(2)}{\uparrow}}{<} \textsc{cost}(\mathcal{T}),$$

implying that $\mathcal{T}$ is not a global optimal solution. Again, this is a contradiction. Therefore the path $p_{\mathcal{M}}$ is due to fulfill $\ell_{(p_{\mathcal{M}(s_t,s_w)})} < 3$, as required, and finalizing the proof. $\square$

Definition 3.5.27. *Given an $ANDP(S, E_1 \cup E_2, C)$, we define the set of concentrator sites of second level as:* $\mathrm{L}_C^{(2)} = \left\{s_c \in S_C/\exists s_{c_1} \in \mathrm{L}_C^{(1)}, \textit{so that } s_c \in A_{s_{c_1}}\right\}$.

Proposition 3.5.28. *Given an $ANDP(S, E_1 \cup E_2, C)$ such that the following points are fulfilled:*

1. $S_C = \mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)}$.
2. $\forall s_c \in \mathrm{L}_C^{(2)}$*, such that* $s_c \in A_{s_{c_1}}$*, and* $s_{c_1} \in A_{s_w}$*, it is fulfilled:*

$$c_{(s_c,s_w)} \leq \min\left\{c_{(s_c,s_{c_2})}; \forall s_{c_2} \in \mathrm{L}_C^{(2)}\right\}.$$

Then, there exists a global optimal solution in $\Gamma_{ANDP}^{(\leq 2)}$.

Proof. Let us suppose that all the optimal solutions are in $\left(\Gamma_{ANDP} \setminus \Gamma_{ANDP}^{(\leq 2)}\right)$. Let $\mathcal{T}$ be globally optimal. We define the maximal set of paths of $\mathcal{T}$ (from a terminal site towards a switch site) that have length greater to 3:

$$\mathcal{B}_{p(\mathcal{T})} = \left\{ p \subset \mathcal{T} / \ell_{(p)} \geq 4, \forall p_i, p_j \in \mathcal{B}_{p(\mathcal{T})}, \text{NODES}_{(p_i)} \cap \text{NODES}_{(p_j)} \subset (S_C \cup S_D) \right\}.$$

We will demonstrate that under these hypotheses we can build a solution derived from $\mathcal{T}$ whose cost is smaller or equal to $\text{COST}(\mathcal{T})$ and its depth is smaller to 4. Specifically by induction in $|\mathcal{B}_{p(\mathcal{T})}|$, we will prove that by means of connection reassignments, we can obtain a feasible solution $\bar{\mathcal{T}} / \text{COST}(\bar{\mathcal{T}}) \leq \text{COST}(\mathcal{T})$ and $\text{DEPTH}_{(\bar{\mathcal{T}})} \leq 3$.
Basic Step: $|\mathcal{B}_{p(\mathcal{T})}| = 1$. Let $p_{\mathcal{T}(s_t, s_w)}$ be its unique path. Let us define the set of concentrator sites given by: $Y_C = \left\{ s_c \in (S_C \cap p_{\mathcal{T}(s_t, s_w)}) / \ell_{(p_{\mathcal{T}(s_c, s_w)})} \geq 3 \right\}$. Let us consider the connection between the concentrator sites that are to greater depth in $p_{\mathcal{T}}$, and let us denote it by (s_{c_i}, s_{c_j}). It is fulfilled:

$$p_{\mathcal{T}(s_t, s_w)} = (s_t, s_{c_i}) \text{CONC}(s_{c_i}, s_{c_j}) \text{CONC}(p_{\mathcal{T}(s_{c_j}, s_w)}).$$

We will analyze the following cases. If $s_{c_i} \in \mathrm{L}_C^{(1)}$, (with $s_{c_i} \in A_{\bar{s}_w}$, $\bar{s}_w \in S_C$), then building the network:

$$\mathcal{G} = \big(\mathcal{T} \setminus \{(s_{c_i}, s_{c_j})\}\big) \cup \{(s_{c_i}, \bar{s}_w)\},$$

we obtain a new feasible solution that in addition fulfills:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}) - c_{(s_{c_i}, s_{c_j})} + c_{(s_{c_i}, \bar{s}_w)} \overset{\mathrm{L}_C^{(1)}\ \textit{def.}}{\overset{\uparrow}{<}} \text{COST}(\mathcal{T}).$$

If $s_{c_i}, s_{c_j} \in \mathrm{L}_C^{(2)}$, by definition of $\mathrm{L}_C^{(2)}$ there exists a concentrator site $\bar{s}_c \in \mathrm{L}_C^{(1)}$ so that $s_{c_i} \in A_{\bar{s}_c}$, and $\bar{s}_c \in A_{s_r}$, with $s_r \in S_D$. Considering the network:

$$\mathcal{G} = \big(\mathcal{T} \setminus \{(s_{c_i}, s_{c_j})\}\big) \cup \{(s_{c_i}, s_r)\},$$

it is easy to see that $\mathcal{G} \in \Gamma_{ANDP}$, and besides:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}) - c_{(s_{c_i}, s_{c_j})} + c_{(s_{c_i}, s_r)} \overset{\textit{by}\ (2)}{\overset{\uparrow}{\leq}} \text{COST}(\mathcal{T}).$$

If $s_{c_i} \in \mathrm{L}_C^{(2)}$ and $s_{c_j} \in \mathrm{L}_C^{(1)}$, by definition of $\mathrm{L}_C^{(1)}$ we know that there exists a switch site $\bar{s} \in S_D / s_{c_j} \in A_{\bar{s}}$. Let $s_{c_k} \in Y_C$ be the site adjacent to s_{c_j} in the path $p_{\mathcal{T}(s_{c_j}, s_w)}$. Let us consider the following network:

$$\mathcal{G} = \big(\mathcal{T} \setminus \{(s_{c_j}, s_{c_k})\}\big) \cup \{(s_{c_j}, \bar{s})\},$$

clearly $\mathcal{G} \in \Gamma_{ANDP}$, and moreover verifies:

$$\text{COST}(\mathcal{G}) = \text{COST}(\mathcal{T}) - c_{(s_{c_j}, s_{c_k})} + c_{(s_{c_j}, \bar{s})} \overset{\mathrm{L}_C^{(1)}\ \textit{def.}}{\overset{\uparrow}{<}} \text{COST}(\mathcal{T}).$$

Analyzing the cases exposed above, in any case, we can obtain a new feasible solution that satisfies: $\text{COST}(\mathcal{G}) \leq \text{COST}(\mathcal{T})$ and $\text{DEPTH}_{(\mathcal{G})} \leq \text{DEPTH}_{(\mathcal{T})} - 1$ (eventually $\text{DEPTH}_{(\mathcal{G})} = \text{DEPTH}_{(\mathcal{T})} - 2$). Applying the reasoning exposed previously to the sites of $(Y_C \setminus \{s_c\})$, we will obtain a feasible solution $\bar{\mathcal{T}} \in \Gamma_{ANDP}$ which satisfies: $\text{COST}(\bar{\mathcal{T}}) \leq \text{COST}(\mathcal{T})$ and $\text{DEPTH}_{(\bar{\mathcal{T}})} \leq \text{DEPTH}_{(\mathcal{T})} - |Y_T| \leq 2$. Hence $\bar{\mathcal{T}} \in \Gamma_{ANDP}^{(\leq 2)}$, and furthermore it is globally optimal.
Induction Step: $|\mathcal{B}_{p(\mathcal{T})}| > 1$. Let us choose a path $\bar{p}_{\mathcal{T}(s_t,s_w)} \in \mathcal{B}_{p(\mathcal{T})}$, and let us consider the set $\bar{\mathcal{B}}_p = \left(\mathcal{B}_{p(\mathcal{T})} \setminus \{\bar{p}_{\mathcal{T}(s_t,s_w)}\}\right)$. Then, realizing the transformations exposed above, we can build a feasible solution $\mathcal{T}_1 \in \Gamma_{ANDP}$ such that only a subset of paths of $\bar{\mathcal{B}}_p$ have length greater to 3 in $\mathcal{T}_1$, and besides: $\text{COST}(\mathcal{T}_1) \leq \text{COST}(\mathcal{T})$ and $\text{DEPTH}_{(\mathcal{T}_1)} = \max\left\{\ell_{(p)}/p \in \mathcal{B}_{p(\mathcal{T}_1)}\right\}$. Now, considering the network $\mathcal{T}_1$ and using the inductive hypothesis we know that there exists a feasible solution $\bar{\mathcal{T}} \in \Gamma_{ANDP}$ obtained of $\mathcal{T}_1$ by means of changes in its connections and satisfying:

$$\text{COST}(\bar{\mathcal{T}}) \overset{\overset{\textit{by I.H.}}{\uparrow}}{\leq} \text{COST}(\mathcal{T}_1) \overset{\overset{\textit{by transformations}}{\uparrow}}{\leq} \text{COST}(\mathcal{T}),$$

and in addition fulfilling $\text{DEPTH}_{(\bar{\mathcal{T}})} \leq 3$, finalizing therefore the inductive step and culminating the proof. □

Lemma 3.5.29. *Let $ANDP(S, E_1 \cup E_2, C)$ be an* ANDP *instance such that for any three sites $\{s_{c_i}, s_{c_j}, s_{c_k}\}/s_{c_i} \in S_C, s_{c_j}, s_{c_k} \in (S_C \setminus \mathrm{L}_C^{(1)})$, and given a switch site $s_w \in S_D$ it is fulfilled:*

$$c_{(s_{c_i},s_{c_j})} + c_{(s_{c_i},s_{c_k})} \leq c_{(s_{c_j},s_{c_k})} + \min\left\{c_{(s_{c_i},s_{c_j})}, c_{(s_{c_i},s_{c_k})}\right\},$$

assuming that among these, s_{c_i} is the site that has minimum cost of direct connection towards s_w. Then, $\exists \mathcal{T} \in \Gamma_{ANDP}^{(\leq 2)}$ globally optimal.

Proof. In order to proof the Lemma, we will suppose that any global optimal solution belongs to $\left(\Gamma_{ANDP} \setminus \Gamma_{ANDP}^{(\leq 2)}\right)$. Let $\mathcal{T}$ be a global optimal solution. Let us denote by $p_{\mathcal{M}(s_c,s_w)}$ the longest path from a switch site towards a concentrator site in $\mathcal{T}$. Considering the set: $Y_C = \left\{\bar{s}_c \in (S_C \cap p_{\mathcal{M}(s_c,s_w)})/\ell_{(p_{\mathcal{M}(\bar{s}_c,s_w)})} \geq 3\right\}$, we will demonstrate by induction in $\ell_{(p_{\mathcal{M}(s_c,s_w)})}$, that we can find a feasible solution $\bar{\mathcal{G}}$ derived from $\mathcal{T}$ where all the concentrator sites of $\{s_{c_i} \in p_{\mathcal{M}(s_c,s_w)}\}$ belong to paths of depth smaller to 4 in $\bar{\mathcal{G}}$.
Basic Step: $\ell_{(p_{\mathcal{M}(s_c,s_w)})} = 3$. Let us see the situations that could happen. Let $\bar{s}_c$ be the concentrator site adjacent to s_c in $p_{\mathcal{M}}$. If s_c or $\bar{s}_c$ belongs to $\mathrm{L}_C^{(1)}$ clearly connecting them to the switch sites corresponding to its covering areas, we obtain a feasible solution whose cost is smaller or equal to $\text{COST}(\mathcal{T})$. Otherwise, if we have $s_c, \bar{s}_c \in (S_C \setminus \mathrm{L}_C^{(1)})$, let us consider s_{c_1} the concentrator site adjacent to $\bar{s}_c$ in $p_{\mathcal{M}}$. Without loss of generality, we will assume that among these three concentrator sites, s_{c_1} is the site that has minimum cost of direct connection towards s_w. By hypothesis, we know that the triangle formed by the concentrators $\{s_c, \bar{s}_c, s_{c_1}\}$ satisfies: $c_{(s_{c_1},s_c)} + c_{(s_{c_1},\bar{s}_c)} \leq c_{(s_c,\bar{s}_c)} + \min\left\{c_{(s_{c_1},s_c)}, c_{(s_{c_1},\bar{s}_c)}\right\}$. Then, building the following network: $\mathcal{G}_1 = (\mathcal{T} \setminus \{(s_c, \bar{s}_c)\}) \cup \{(s_c, s_{c_1})\}$, it is easy to see that $\mathcal{G}_1 \in \Gamma_{ANDP}$, $\text{COST}(\mathcal{G}_1) \leq \text{COST}(\mathcal{T})$, and furthermore the sites s_c, $\bar{s}_c$, and s_{c_1} belong to paths of depth smaller to 4 in $\mathcal{G}_1$.

Induction Step: $\ell_{(p_{\mathcal{M}(s_c,s_w)})} = h > 3$. We will differentiate the following situations.
Case A: $s_c \in \mathrm{L}_C^{(1)}$. The network $\mathcal{G}_1 = (\mathcal{T} \setminus \{(s_c, \bar{s}_c)\}) \cup \{(s_c, \bar{s}_w)\}$, (with $s_c \in A_{\bar{s}_w}$, $\bar{s}_w \in S_D$) is a feasible solution of Γ_{ANDP} satisfying $\textsc{cost}(\mathcal{G}_1) \leq \textsc{cost}(\mathcal{T})$. Moreover as $\ell_{(p_{\mathcal{M}(\bar{s}_c,s_w)})} = h - 1 \geq 3$ in $\mathcal{G}_1$, by inductive hypothesis there exists a feasible solution $\bar{\mathcal{G}}$ derived from $\mathcal{G}_1$ where all the concentrator sites of $\{s_{c_i} \in p_{\mathcal{M}(\bar{s}_c,s_w)}\}$ belong to paths of depth smaller to 4 in $\bar{\mathcal{G}}$.
Case B: $s_c \in (S_C \setminus \mathrm{L}_C^{(1)})$, $\bar{s}_c \in \mathrm{L}_C^{(1)}$. The network $\mathcal{G}_1 = (\mathcal{T} \setminus \{(\bar{s}_c, s_{c_1})\}) \cup \{(\bar{s}_c, \bar{s}_r)\}$, (with $\bar{s}_c \in A_{\bar{s}_r}$, $\bar{s}_r \in S_D$) is a feasible solution fulfilling $\textsc{cost}(\mathcal{G}_1) \leq \textsc{cost}(\mathcal{T})$. Now, if $\ell_{(p_{\mathcal{M}(s_{c_1},s_w)})} = h - 2 \geq 3$ in $\mathcal{G}_1$, by inductive hypothesis there exists a feasible solution $\bar{\mathcal{G}}$ derived from $\mathcal{G}_1$ where all the concentrator sites of $\{s_{c_i} \in p_{\mathcal{M}(s_{c_1},s_w)}\}$ belong to paths of depth smaller to 4 in $\bar{\mathcal{G}}$.
Case C: $s_c \in (S_C \setminus \mathrm{L}_C^{(1)})$, $\bar{s}_c \in (S_C \setminus \mathrm{L}_C^{(1)})$. As above $\mathcal{G}_1 = (\mathcal{T} \setminus \{(s_c, \bar{s}_c)\}) \cup \{(s_c, s_{c_1})\}$ is a feasible solution and furthermore, since $\ell_{(p_{\mathcal{M}(\bar{s}_c,s_w)})} = h - 1 \geq 3$ in $\mathcal{G}_1$, by inductive hypothesis there exits a feasible solution $\bar{\mathcal{G}}$ constructed from $\mathcal{G}_1$ with all the concentrator sites of $\{s_{c_i} \in p_{\mathcal{M}(\bar{s}_c,s_w)}\}$ belonging to paths of depth smaller to 4 in $\bar{\mathcal{G}}$. Hence, by the exposed previously, we have proven the inductive step.
Now, let us consider the maximal set of paths present in $\bar{\mathcal{G}}$ which go from a concentrator site (located in last level on its access sub-network) towards the backbone network and have length greater to 3:

$$\mathcal{P}_{\bar{\mathcal{G}}} = \left\{ p_{(s_c,s_w)} \subset \bar{\mathcal{G}} / \ell_{(p_{(s_c,s_w)})} \geq 3, \textit{and } \nexists \bar{p} \in \mathcal{P}_{\bar{\mathcal{G}}} / p_{(s_c,s_w)} \subset \bar{p} \right\}.$$

Then, repeating recurrently on $\mathcal{P}_{\bar{\mathcal{G}}}$ the process described previously, we will be able to find a feasible solution $\bar{\mathcal{T}} \in \Gamma_{ANDP}$ satisfying: $\textsc{cost}(\bar{\mathcal{T}}) \leq \textsc{cost}(\mathcal{T})$, and $\textsc{depth}_{(\bar{\mathcal{T}})} \leq 3$, completing therefore the proof. □

Theorem 3.5.30. *Given an $ANDP(S, E_1 \cup E_2, C)$ so that the following points are fulfilled:*

1. $c_{(s_t,s_w)} < \min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in (S_C \setminus A_{s_w})\right\}$, $\forall s_t \in A_{s_w}$, $s_w \in S_D$.
2. $\mathrm{d}_{min(s_c)} < c_{(s_c,\bar{s}_c)}$, $\forall s_c \in (A_{s_w} \setminus \mathrm{L}_C^{(1)})$, $\forall \bar{s}_c \in A_{\bar{s}_w}$, $\bar{s}_w \in S_D$, *where* $\mathrm{d}_{min(s_c)}$ *is the cost of the shortest path from* s_c *to* s_w *in* $\mathcal{G}_{(s_w)} = (A_{s_w}, E(A_{s_w}))$.
3. $c_{(s_{c_i},s_{c_j})} \geq \min\left\{c_{(s_{c_i},s_w)}, c_{(s_{c_j},s_w)}\right\}$, $\forall s_{c_i}, s_{c_j} \in \left(A_{s_w} \cap (S_C \setminus \mathrm{L}_C^{(1)})\right)$, *with* $s_w \in S_D$.

Then, there exists a global optimal solution in the subspace $\left(\Gamma_{ANDP}^{(\leq 2)} \cap \Gamma_{ANDP}^{(\mathrm{SPG})}\right)$.

Proof. Firstly, we will prove the following property.
Property I: Under these hypothesis, all the optimal solutions belong to the space $\Gamma_{ANDP}^{(\mathrm{SPG})}$. In order to prove this, let us consider $\mathcal{T} \in \Gamma_{ANDP}$ globally optimal. Then, supposing that $\mathcal{T} \notin \Gamma_{ANDP}^{(\mathrm{SPG})}$, necessarily there exists at least one connection $(s, s_c) \in \mathcal{T}$ such that $s \in S_T \cup S_C$, $s_c \in S_C$, $s_c \in A_{s_w}$, $s \in A_{s_c}$, and $s \in A_{\bar{s}_w}$, where $s_w, \bar{s}_w \in S_D$. We will analyze the following cases.
Case A: $s \in S_T$. Clearly, by (1) the network $\bar{\mathcal{T}} = (\mathcal{T} \setminus \{s, s_c)\}) \cup \{(s, \bar{s}_w)\}$, it would be one

better feasible solution than $\mathcal{T}$.
Case B: $s \in \mathrm{L}_C^{(1)}$. Assuming that s_c is the site closest to the backbone network in $\mathcal{T}$, and building the network like above, by definition of $\mathrm{L}_C^{(1)}$, $\bar{\mathcal{T}}$ is a better feasible solution than $\mathcal{T}$.
Case C: $s \in \left(S_C \setminus \mathrm{L}_C^{(1)}\right)$. Again, we assume that s_c is the site closest to the backbone network in $\mathcal{T}$. Let p_s be the shortest path from s to $\bar{s}_w$ in $\mathcal{G}_{(\bar{s}_w)} = (A_{\bar{s}_w}, E(A_{\bar{s}_w}))$. By (2) we know that its cost fulfills $\mathrm{d}_{min(s)} < c_{(s,s_c)}$. This would imply that the network $\bar{\mathcal{T}} = (\mathcal{T} \setminus \{s, s_c)\}) \cup \{p_s\}$ is a better feasible solution compared with $\mathcal{T}$.
Hence, since anyone of the previous cases contradicts the optimality of $\mathcal{T}$, we infer that any global optimal solution of Γ_{ANDP} belongs to $\Gamma_{ANDP}^{(\mathrm{SPG})}$.
Now, we must prove in addition that there exists $\bar{\mathcal{G}} \in \Gamma_{ANDP}^{(\leq 2)}$ derived from $\mathcal{T}$ which is globally optimal, and besides that it belongs to the subspace $\Gamma_{ANDP}^{(\mathrm{SPG})}$ like $\mathcal{T}$.
For each switch covering area $A_{s_w}/|A_{s_w}| > 1$, we define:

$$\mathcal{B}_{s_w} = \left\{ p_{(s_c,s_w)} \subset \mathcal{T}_{(s_w)}/\ell_{(p_{(s_c,s_w)})} \geq 3, s_c \in S_C, \textit{and } \nexists \bar{p} \in \mathcal{B}_{s_w}/p_{(s_c,s_w)} \subset \bar{p} \right\}.$$

It is the set of paths of $\mathcal{T}_{(s_w)}$ that have maximum length greater to 2 and go from a concentrator site $s_c \in A_{s_w}$ towards s_w.
Property II: Fixed $s_w \in S_D/|A_{s_w}| > 1$, and given $p_{(s_c,s_w)} \in \mathcal{B}_{s_w}$, we will use induction in $\ell_{(p_{(s_c,s_w)})}$ to prove that by means of reassignments of connections on $\mathcal{T}_{(s_w)}$ we can obtain a new global feasible solution $\mathcal{G}_1 \in \Gamma_{ANDP}^{(\mathrm{SPG})}$, where all the concentrator sites present in $p_{(s_c,s_w)}$ belong to paths of length smaller to 4 in $\mathcal{G}_{1(s_w)}$.
Basic Step: $\ell_{(p_{(s_c,s_w)})} = 3$. Let $\bar{s}_c$ be the concentrator adjacent to s_c in p. If s_c or $\bar{s}_c$ belongs to $\mathrm{L}_C^{(1)}$ clearly reconnecting them to s_w we maintain the feasibility not being increased the solution cost, and moreover the concentrator sites of p are on paths of length smaller to 4 in the new solution. Otherwise, if s_c and $\bar{s}_c$ belong to $(S_C \setminus \mathrm{L}_C^{(1)})$, by (3) we know that $c_{(s_c,\bar{s}_c)} \geq \min\left\{c_{(s_c,s_w)}, c_{(\bar{s}_c,s_w)}\right\}$. Independently of which of both connections it has the smaller cost, we can build a new optimal solution reconnecting the concentrator sites s_c or $\bar{s}_c$ (according to the connection of minimum cost) to s_w, maintaining the feasibility without increasing the cost. Furthermore, by construction in the new solution the concentrators sites of p belong to paths of length smaller to 4.
Inductive Step: $\ell_{(p_{(s_c,s_w)})} = k > 3$. We will analyze the following cases.
Case 1*:* $s_c \in \mathrm{L}_C^{(1)}$. Let us consider the network $\mathcal{H} = (\mathcal{T} \setminus \{(s_c, \bar{s}_c)\}) \cup \{(s_c, s_w)\}$. This is also globally optimal and in addition the concentrator site s_c is to depth 1 in $\mathcal{H}$. Since $p_{(\bar{s}_c,s_w)} \subset \mathcal{H}$ and $\ell_{(p_{(\bar{s}_c,s_w)})} = k - 1 \geq 3$, by inductive hypothesis we know that there exists another solution $\mathcal{G}_1 \in \Gamma_{ANDP}^{(\mathrm{SPG})}$ globally optimal and derived from $\mathcal{H}$ so that all the concentrator sites pertaining to $p_{(\bar{s}_c,s_w)}$ are on paths of length smaller to 4 in $\mathcal{G}_{1(s_w)}$.
Case 2*:* $s_c \in (S_C \setminus \mathrm{L}_C^{(1)})$, $\bar{s}_c \in \mathrm{L}_C^{(1)}$. Let $\bar{s}$ be the next site adjacent to $\bar{s}_c$ in $p_{(\bar{s}_c,s_w)}$. We build the network $\mathcal{H} = (\mathcal{T} \setminus \{(\bar{s}_c, \bar{s})\}) \cup \{(\bar{s}_c, s_w)\}$. If $\ell_{(p_{(\bar{s},s_w)})} = k - 2 \geq 3$, knowing that $p_{(\bar{s},s_w)} \subset \mathcal{H}$ and applying the inductive hypothesis, we obtain another global optimal solution $\mathcal{G}_1 \in \Gamma_{ANDP}^{(\mathrm{SPG})}$ transformed from $\mathcal{H}$ where all the concentrator sites belonging to $p_{(\bar{s},s_w)}$ are in depth smaller to 3 in $\mathcal{G}_{1(s_w)}$. If $\ell_{(p_{(\bar{s},s_w)})} < 3$ the network $\mathcal{H}$ in itself verifies the property.
Case 3*:* $s_c \in (S_C \setminus \mathrm{L}_C^{(1)})$, $\bar{s}_c \in (S_C \setminus \mathrm{L}_C^{(1)})$. By hypothesis we know that these concentrators fulfill: $c_{(s_c,\bar{s}_c)} \geq \min\left\{c_{(s_c,s_w)}, c_{(\bar{s}_c,s_w)}\right\}$. We will suppose, without loss of generality, that

$c_{(s_c,\bar{s}_c)} = c_{(\bar{s}_c,s_w)}$ (the analysis of the other possible situations would be similar). Again, let $\bar{s}$ be the next site adjacent to $\bar{s}_c$ in $p_{(\bar{s}_c,s_w)} \subset \mathcal{T}$. We will see the following subcases.
Subcase 3.1*:* $\bar{s} \in \mathrm{L}_C^{(1)}$. Let us denote by $\hat{s}$ the site adjacent to $\bar{s}$ in $p_{(\bar{s},s_w)} \subset \mathcal{H}$. We replace the connection $(\bar{s}, \hat{s})$ by the connection $(\bar{s}, s_w)$ obtaining a solution $\mathcal{H}$ globally optimal. If $\mathcal{H}$ fulfills: $\ell_{(p_{(\hat{s},s_w)})} = k - 3 \geq 3$, by inductive hypothesis there exists a global optimal solution $\mathcal{G}_1 \in \Gamma_{ANDP}^{(\mathrm{SPG})}$ such that all the concentrator sites from $p_{(\hat{s},s_w)}$ are in depth smaller to 3 in $\mathcal{G}_{1(s_w)}$. If $\ell_{(p_{(\hat{s},s_w)})} < 3$ the network $\mathcal{H}$ in itself fulfill the property.
Subcase 3.2*:* $\bar{s} \in (S_C \setminus \mathrm{L}_C^{(1)})$. By hypothesis $c_{(\bar{s}_c,\bar{s})} \geq \min\left\{c_{(\bar{s}_c,s_w)}, c_{(\bar{s},s_w)}\right\}$. Therefore replacing the connection $(s_c, \bar{s})$ or $(\bar{s}, \hat{s})$ (according to the case) by the connection line which produces the minimum value we obtain another optimal solution $\mathcal{H}$. In any case, the inductive hypothesis implies the existence of a global optimal solution $\mathcal{G}_1 \in \Gamma_{ANDP}^{(\mathrm{SPG})}$ so that all the concentrators present in $p_{(s_c,s_w)}$ are located with depth smaller to 3 in $\mathcal{G}_{1(s_w)}$. This completes the proof of the Property (II) enunciated above.
To complete the proof, we apply recursively the Property (II) to all the paths of $\mathcal{B}_{s_w}$, and later repeating the process for all the sites of $(S_D \setminus \{s_w\})$, we obtain a global optimal solution $\bar{\mathcal{G}} \in \left(\Gamma_{ANDP}^{(\leq 2)} \cap \Gamma_{ANDP}^{(\mathrm{SPG})}\right)$ derived of $\mathcal{T}$ according to different changes realized in the connection lines. $\square$

Definition 3.5.31. *Given an $ANDP(S, E_1 \cup E_2, C)$, we define the sets of terminal sites of first level and second level respectively as:*

- $S_T^{(1)} = \left\{s_t \in S_T / \exists s_c \in \mathrm{L}_C^{(1)}, \textit{so that } s_t \in A_{s_c}\right\}$,
- $S_T^{(2)} = \left\{s_t \in S_T / \exists s_c \in \mathrm{L}_C^{(2)}, \textit{so that } s_t \in A_{s_c}\right\}$.

Proposition 3.5.32. *Let $ANDP(S, E_1 \cup E_2, C)$ be an* ANDP *instance such that $S_T = S_T^{(1)} \cup S_T^{(2)}$ and in addition the following points are fulfilled.*

1. *Given $s_c, \bar{s}_c \in \mathrm{L}_C^{(2)}$, these satisfy $\ell_{(p_{min(s_c)})} \leq 2$ and $\mathrm{COST}(p_{min(s_c)}) \leq c_{(s_c,\bar{s}_c)}$.*
2. $\forall s_t \in S_T^{(1)}$*:* $\max\left\{\Psi_{(s_t)}^{(1)}, \Psi_{(s_t)}^{(2)}\right\} \leq \min\left\{c_{(s_t,s_c)}; \forall s_c \in (S_C \setminus \mathrm{L}_C^{(1)})\right\}$,
3. $\forall s_t \in S_T^{(2)}$*:* $\max\left\{\Psi_{(s_t)}^{(2)}, \Psi_{(s_t)}^{(3)}\right\} \leq \min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in (S_C \setminus (\mathrm{L}_C^{(1)} \cup \{s_{c_2}\}))\right\}$, *with* $s_t \in A_{s_{c_1}}$, $s_{c_1} \in A_{s_{c_2}}$*, and where:*
   - $\Psi_{(s_t)}^{(1)} = \min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in (\mathrm{L}_C^{(1)} \setminus \{\bar{s}\})\right\}$, *with* $s_t \in A_{\bar{s}}$, $\bar{s} \in \mathrm{L}_C^{(1)}$,
   - $\Psi_{(s_t)}^{(2)} = \min\left\{c_{(s_t,s_w)}; \forall s_w \in S_D\right\}$,
   - $\Psi_{(s_t)}^{(3)} \min\left\{c_{(s_t,\bar{s}_c)}; \forall \bar{s}_c \in \mathrm{L}_C^{(1)}\right\}$.

Then, there exists a global optimal solution $\mathcal{T} \in \Gamma_{ANDP}^{(\leq 2)}$ so that their concentrator sites are including in $(\mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)})$.

Proof. Firstly, we will prove that given a solution globally optimal $\mathcal{T}^{(\mathrm{opt})} \in \Gamma_{ANDP}$ and a terminal site $s_t \in S_T$; s_t is directly connected to some site of $S_D \cup \mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)}$, to less of reassignments of connections.
Case 1: $s_t \in S_T^{(1)}$. Let us suppose that $(s_t, \hat{s}) \in \mathcal{T}^{(\mathrm{opt})} / \hat{s} \in (S_C \setminus (\mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)}))$. Let $s_{c_1} \in \mathrm{L}_C^{(1)}$ and $\bar{s} \in (\mathrm{L}_C^{(1)} \setminus \{s_{c_1}\})$ be the concentrator sites that fulfill $s_t \in A_{s_{c_1}}$, and $\Psi_{(s_t)}^{(1)} = c_{(s_t,\bar{s})}$ respectively. Let us consider in addition the switch site $\bar{s}_w$ so that $\Psi_{(s_t)}^{(2)} = c_{(s_t,\bar{s}_w)}$. If $\bar{s} \in \mathcal{T}^{(\mathrm{opt})}$ and $\Psi_{(s_t)}^{(1)} \leq \Psi_{(s_t)}^{(2)}$ then building the network:

$$\mathcal{G} = \big(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_t, \hat{s})\}\big) \cup \{(s_t, \bar{s})\},$$

we obtain a new feasible solution which satisfies:

$$\mathrm{COST}(\mathcal{G}) = \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s_t,\hat{s})} + c_{(s_t,\bar{s})} \overset{\overset{\text{by (2)}}{\uparrow}}{\leq} \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}),$$

and therefore this would imply that $\mathcal{G}$ is globally optimal (eventually it would be one better solution which contradicts the optimality of $\mathcal{T}^{(\mathrm{opt})}$).
Now, if $\Psi_{(s_t)}^{(1)} > \Psi_{(s_t)}^{(2)}$ then the following network is also globally optimal:

$$\mathcal{H} = \big(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_t, \hat{s})\}\big) \cup \{(s_t, \bar{s}_w)\}.$$

The optimality of $\mathcal{H}$ is guaranteed by the cost relation:

$$\mathrm{COST}(\mathcal{H}) = \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s_t,\hat{s})} + c_{(s_t,\bar{s}_w)} \overset{\overset{\text{by (2)}}{\uparrow}}{\leq} \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}).$$

Case 2: $s_t \in S_T^{(2)}$. Again, we will suppose that $(s_t, \hat{s}) \in \mathcal{T}^{(\mathrm{opt})} / \hat{s} \in (S_C \setminus (\mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)}))$. Let $\bar{s}_{c_1} \in \mathrm{L}_C^{(1)}$ be the concentrator site that fulfills $\Psi_{(s_t)}^{(3)} = c_{(s_t,\bar{s}_{c_1})}$. If $\bar{s}_{c_1} \in \mathcal{T}^{(\mathrm{opt})}$ and $\Psi_{(s_t)}^{(3)} \leq \Psi_{(s_t)}^{(2)}$ then considering the network:

$$\mathcal{G} = \big(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_t, \hat{s})\}\big) \cup \{(s_t, \bar{s}_{c_1})\},$$

we have a new feasible solution, and moreover:

$$\mathrm{COST}(\mathcal{G}) = \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}) - c_{(s_t,\hat{s})} + c_{(s_t,\bar{s}_{c_1})} \overset{\overset{\text{by (3)}}{\uparrow}}{\leq} \mathrm{COST}(\mathcal{T}^{(\mathrm{opt})}),$$

implying the optimality of $\mathcal{G}$.
If $\Psi_{(s_t)}^{(3)} > \Psi_{(s_t)}^{(2)}$ then the network $\mathcal{H}$ builded as in the previous case, it would be furthermore a global optimal solution.
We will demonstrate now that we can obtain a new solution $\bar{\mathcal{T}}^{(\mathrm{opt})} \in \Gamma_{ANDP}^{(\leq 2)}$ globally optimal and obtained from $\mathcal{T}^{(\mathrm{opt})}$ applying appropriate changes of connection that maintain the feasibility and optimality. Assuming that in $\mathcal{T}^{(\mathrm{opt})}$ are only concentrator sites belonging to $(\mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)})$, let us suppose that there exists a least one path of length greater to 2 from a switch site towards a concentrator site. Let $p_{(s_c,s_w)} \subset \mathcal{T}^{(\mathrm{opt})}$ be a path such that $\ell_{(p_{(s_c,s_w)})} \geq 3$. Let

$(s_c, \bar{s}_c) \in p_{(s_c,s_w)}$ be the last connection on this path. Let us suppose that $s_c, \bar{s}_c \in \mathrm{L}_C^{(2)}$ (for another type of connection, we could clearly obtain a solution with smaller depth with respect to the concentrator sites present in $p_{(s_c,s_w)}$). Building the network:

$$\bar{\mathcal{T}} = \big(\mathcal{T}^{(\mathrm{opt})} \setminus \{(s_c, \bar{s}_c)\}\big) \cup \{p_{min(s_c)}\},$$

and applying (1), we maintain the feasibility and moreover $\textsc{cost}(\bar{\mathcal{T}}) \leq \textsc{cost}(\mathcal{T}^{(\mathrm{opt})})$. Hence, $\bar{\mathcal{T}}$ would be also globally optimal. Then, repeating recurrently this process for all connection between sites of $\mathrm{L}_C^{(2)}$ present in the builded solution, we will be able to build a global optimal solution $\bar{\mathcal{T}}^{(\mathrm{opt})}$ belonging to the subspace $\Gamma_{ANDP}^{(\leq 2)}$ and whose concentrator sites are including in $(\mathrm{L}_C^{(1)} \cup \mathrm{L}_C^{(2)})$, as required, and completing the proof. $\square$

In next chapter we present algorithms applied to the $\mathrm{ANDP}^{(\leq k)}$ with $k \in \{1, 2\}$. We obtain a way of computing the global optimal solution cost of it using the Dynamic Programming approach. Considering that the $\mathrm{ANDP}^{(\leq 1)}$ is a NP-hard problem, we obtain lower bounds to the global optimal solution cost by Dynamic Programming with State-Space Relaxation in polynomial time.

Chapter 4

Algorithms applied to the ANDP

This chapter presents the Dynamic Programming approach as alternative methodology to find a global optimal solution cost for the $\text{ANDP}^{(\leq 1)}$ and $\text{ANDP}^{(\leq 2)}$. After we introduce the Dynamic Programming with State-Space Relaxation as a method to obtain lower bounds for the original problem.

4.1 Dynamic Programming

Proposition 4.1.1. *Given an $ANDP(S, E_1 \cup E_2, A)$, the cost of a global optimal solution of $\Gamma^{(\leq 1)}_{ANDP}$ is given by $f_{(S_T,Z,A^Q)}$, with $f_{(\cdot,\cdot,\cdot)}$ defined by the following expression of Dynamic Programming,*

$$f_{S_C}(S_T, Z, A^Q) = \begin{cases} \min\limits_{s_t \in S_T} \left\{ \begin{array}{l} \text{COST}(s_t, Z) + f_{S_C}(S_T \setminus \{s_t\}, Z, A^Q), \\ \min\limits_{s_c \in S_C} \left\{ \begin{array}{l} \text{COST}(s_t, s_c) + \text{COST}(s_c, Z) + \\ f_{S_C}(S_T \setminus \{s_t\}, Z, A^{Q \cup \{(s_c, Z)\}}) \end{array} \right\} \end{array} \right\} & \textit{if } S_T \neq \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

where: $\text{COST}(s, Z) = \min\limits_{z \in S_D} \{\text{COST}(s, z)\}$, $(s, Z) = \operatorname*{argmin}\limits_{z \in S_D} \{\text{COST}(s, z)\}$ *and the matrix of connection costs* $A^Q = \{a_{i,j}\}_{(i,j) \in E_1 \cup E_2}$ *is defined by*

$$a_{i,j} = \begin{cases} \text{COST}(i, j) & if (i, j) \notin Q \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 4.1.2. *Given an $ANDP(S, E_1 \cup E_2, A)$, the cost of a global optimal solution of $\Gamma^{(\leq 2)}_{ANDP}$ is given by $f_{(S_T,Z,A^Q)}$, with $f_{(\cdot,\cdot,\cdot)}$ defined by the following expression of Dynamic*

Programming,

$$f_{S_C}(S_T, Z, A^Q) = \begin{cases} \min\limits_{s_t \in S_T} \left\{ \begin{array}{l} \textsc{cost}(s_t, Z) + f_{S_C}(S_T \setminus \{s_t\}, Z, A^Q), \\ \min\limits_{s_c \in S_C} \left\{ \begin{array}{l} \textsc{cost}(s_t, s_c) + \textsc{cost}(s_c, Z) + \\ f_{S_C}(S_T \setminus \{s_t\}, Z, A^{Q \cup \{(s_c, Z)\}}) \end{array} \right\}, \\ \min\limits_{(s_c^u, s_c^v) \in E_2} \left\{ \begin{array}{l} \textsc{cost}(s_t, s_c^u) + \\ \textsc{cost}(s_c^u, s_c^v) + \textsc{cost}(s_c^v, Z) + \\ f_{S_C}(S_T \setminus \{s_t\}, Z, A^{Q \cup \{(s_c^u, s_c^v), (s_c^v, Z)\}}) \end{array} \right\} \end{array} \right\} & \textit{if } S_T \neq \emptyset \\ 0 & \textit{otherwise.} \end{cases}$$

where: $\textsc{cost}(s, Z) = \min\limits_{z \in S_D} \{\textsc{cost}(s, z)\}$, $(s, Z) = \operatorname*{argmin}\limits_{z \in S_D} \{\textsc{cost}(s, z)\}$ *and the matrix of connection costs* $A^Q = \{a_{i,j}\}_{(i,j) \in E_1 \cup E_2}$ *is defined by*

$$a_{i,j} = \begin{cases} \textsc{cost}(i, j) & if (i, j) \notin Q \\ 0 & \textit{otherwise.} \end{cases}$$

4.2 Dynamic Programming with State-Space Relaxation

In order to find a lower bound of $f_{S_C}(S_T, Z, A^Q)$, we use the Dynamic Programming with State-Space Relaxation. It is a general relaxation procedure proposed by Christofides, Mingozzi and Toth for a number of routing problems [22, 48]. The motivation for this methodology stems from the fact that very few combinatorial optimization problems can be solved by Dynamic Programming alone due to the dimensionality of their state-space. To overcome this difficulty, the number of states is reduced by mapping the state-space associated with a given Dynamic Programming recursion to a smaller cardinality space. This mapping, denoted by g, must associate to every transition from a state S_1 to a state S_2 in the original state-space, a transition $g(S_1)$ to $g(S_2)$ in the new state-space. To be effective, the function g must give rise to a transformed recursion over the relaxed state-space which can be computed in polynomial time. Furthermore, this relaxation must generate a good lower bound for the original problem.

With the aim of illustrating this methodology, we present this approach in the context of the minimization of the total schedule time for TSPTW (Travelling Salesman Problem with Time Window), after we apply it to the Dynamic Programming recursion presented in Proposition 4.1.2.

The objective of the TSPTW is to find an optimal tour where a single vehicle is required to visit each of a given set of locations (customers) exactly once and then return to its starting location. The vehicle must visit each location within a specified time window, defined by an earliest service start time and latest service start time. If the vehicle arrives at a service location before the earliest service start time, it is permitted to wait until the earliest service start time is reached. The vehicle conducts its service for a known period of time and immediately departs for the location of the next scheduled customer. Assume that the time constrained path starts at fixed time value a_o. Define $F(S, i)$ as the shortest time it takes for a feasible path starting at node o, passing through every node of $S \subseteq N$ exactly once, to end at node $i \in S$. Note that optimization of the total arc cost would involve an additional dimension to account for the arrival time at a node. The function $F(S,i)$ can be computed by solving the following recurrence equations:

$$F(S,j) = \min_{(i,j)\in E}\{F(S-\{j\},i) + t_{ij} | i \in S-\{j\}\} \forall S \subseteq N, j \in S. \tag{4.1}$$

The recursion formula initialized by

$$F(\{j\},j) = \begin{cases} \max\{a_j, a_o + t_{oj}\} & \text{if } (o,j) \in E \\ +\infty & \text{Otherwise.} \end{cases}$$

The optimal solution to the TSPTW is given by:

$$\min_{j\in N}\{F(N,j) + t_{jd}\}. \tag{4.2}$$

Note that equation 4.1 is valid if $a_j \le F(S,j) \le b_j$. If however $F(S,j) < a_j$, then $F(S,j) = a_j$; if $F(S,j) > b_j$, $F(S,j) = \infty$. Formulas 4.1- 4.2 define a shortest path algorithm on a state graph whose nodes are the states (S,i) and whose arcs represent transitions from one state to another. This algorithm is a forward dynamic programming algorithm where at step s, $s = 1, \ldots, n+1$, a path of length s is generated. The state (S,i) of cost $F(S,i)$ are defined as follows: S is an unordered set of visited nodes and i is the last visited node, $i \in S$.

Christofides, Mingozzi and Toth [22, 48] suggest several alternatives for the mapping g. Here, we present the shortest $r - path$ relaxation, i.e., $g(S) = r = \sum_{i\in S} r_i$, where $r_i \ge 1$ is an integer associated with node $i \in N$; then $g(S \setminus \{i\}) = g(S) - r_i$. Define $R = \sum_{i\in S} r_i$. Hence the transofrmed recursion equations are:

$$F(r,j) = \min_{(i,j)\in E}\{F(r-r_j,i) + t_{ij} | r - r_j \ge r_i\},\ r \in \{1,\ldots,R\}, j \in N. \tag{4.3}$$

Recursion 4.3 hold if $a_j \le F(r,j) \le b_j$. Otherwise, if $F(r,j) < a_j$, then $F(r,j) = a_j$; if $F(r,j) > b_j$, $F(r,j) = \infty$. The recursion formula is initialized by

$$F(\{j\},j) = \begin{cases} \max\{a_j, a_o + t_{oj}\} & \text{if } (o,j) \in E \text{ and } q = q_j \\ +\infty & \text{Otherwise, for } q \in \{1,\ldots,Q\}, j \in N. \end{cases}$$

The lower bound is given by:

$$\min_{j\in N}\{F(R,j) + t_{jd}\}. \tag{4.4}$$

The complexity of the bounding procedure is $O(n^2 \times Q)$ for a n-node problem [22, 48]. Now, we present this approach in the context of find a “good" lower bound for the solution of $\text{ANDP}^{(\le 2)}$. The following Proposition gives a lower bound for the $f_{S_C}(S_T, Z, A^Q)$ presented in Proposition 4.1.2 (the optimum value of the $\text{ANDP}^{(\le 2)}$).

Proposition 4.2.1. *Given an $ANDP(S, E_1 \cup E_2, C)$, a lower bound for the value of $f_{S_C}(S_T, Z, A^Q)$ is derived from the following expression of Dynamic Programming with State-Space Relaxation,*

$$g_{S_C}(r, Z, A^Q) = \begin{cases} \min\limits_{s_t^i \in S_T} \left\{ \begin{array}{l} \textsc{cost}(s_t^i, Z) + g_{S_C}(r - r_i, Z, A^Q), \\ \min\limits_{s_c^j \in S_C} \left\{ \begin{array}{l} \textsc{cost}(s_t^i, s_c^j) + \textsc{cost}(s_c^j, Z) + \\ g_{S_C}(r - r_i, Z, A^{Q \cup \{(s_c^j, Z)\}}) | r - \hat{R} - r_i \geq r_j \end{array} \right\}, \\ \min\limits_{(s_c^j, s_c^k) \in E_2} \left\{ \begin{array}{l} \textsc{cost}(s_t^i, s_c^j) + \textsc{cost}(s_c^j, s_c^k) + \textsc{cost}(s_c^k, Z) + \\ g_{S_C}(r - r_i, Z, A^{Q \cup \{(s_c^j, s_c^k), (s_c^k, Z)\}}) | r - \hat{R} - r_i \geq r_j + r_k \end{array} \right\} \end{array} \right\} & \textit{if } r > \hat{R} \\ 0 & \textit{otherwise.} \end{cases}$$

where $1 \leq r_i \leq R$ is an integer associated with the site $i \in S_T \cup S_C$; $R = \sum_{i \in S_T \cup S_C} r_i$; $\hat{R} = \sum_{j \in S_C} r_j$ and the matrix of connection costs $A^Q = \{a_{i,j}\}_{(i,j) \in E_1 \cup E_2}$ is defined by

$$a_{i,j} = \begin{cases} \textsc{cost}(i, j) & if(i, j) \notin Q \\ 0 & \textit{otherwise.} \end{cases}$$

The lower bound is given by $g_{(R,Z,A^{\emptyset})}$.

4.3 Computational Results

We present here the experimental results obtained with the recursions of above. The algorithms were implemented in ANSI C. The experiments were made on a *Intel(R) Core(TM)2 CPU T5200 @ 1.60GHz*, with *1GB RAM* running under *Linux Ubuntu 9.10*. The recursions presented in Propositions 4.1.1 and 4.1.2 were applied to the $\text{ANDP}^{(\leq 1)}$ and the $\text{ANDP}^{(\leq 2)}$ respectively, whereas the recursion presented in Proposition 4.2.1 was applied to $\text{ANDP}^{(\leq 2)}$. They were tested using a large test set, by modifying the Steiner Problem in Graphs (SPG) instances from SteinLib [43]. This library contains many problem classes of widely different graph topologies. We extracted most of the problems in the classes: C, MC, X, PUC, I080, I160, P6E, P6Z and WRP3. We customized the SPG problems, transforming them into ANDP instances by means of the following changes. For each considered problem:

i) we selected the terminal node with greatest degree as the z node (modelling the backbone),

ii) the Steiner nodes model the concentrator sites, and the terminal nodes model the terminal sites,

iii) all the edges between terminal sites were deleted (as they are not allowed in feasible ANDP solutions).

Moreover, if the resulting topology was unconnected, the problem instance was discarted. Let us notice that since in the ANDP the terminals can not be used as intermediate nodes (which implies also that edges between pairs of terminals are not allowed), the cost of a SPG optimum is a lower bound for the optimum of the corresponding ANDP. Therefore are for $ANDP^{(\leq k)}$ with $k \in 1..2$.

Set	Name	$\lvert V\rvert$	$\lvert E\rvert$	$\lvert T\rvert$	LB_{SPG}	c^1_{opt}	c^2_{opt}	$LB\_GAP^{(1)}_{SPG}$	$LB\_GAP^{(2)}_{SPG}$
I080	i080-001	80	120	6	1787	∞	2187		22.38%
I080	i080-011	80	350	6	1479	∞	1499		1.35%
I080	i080-012	80	350	6	1484	∞	1497		0.88%
I080	i080-013	80	350	6	1381	∞	1383		0.14%
I080	i080-014	80	350	6	1397	∞	1505		7.73%
I080	i080-111	80	350	8	2051	∞	2159		5.27%
I080	i080-112	80	350	8	1885	2201	1887	16.76%	0.11%
I080	i080-113	80	350	8	1884	∞	1884		0%
I080	i080-114	80	350	8	1895	∞	2099		10.77%
I080	i080-115	80	350	8	1868	2174	1969	16.38%	5.41%
I080	i080-233	80	160	16	4354	∞	4564		4.82%
I160	i160-011	160	812	7	1677	∞	1875		11.81%
I160	i160-012	160	812	7	1750	∞	1891		8.06%
I160	i160-013	160	812	7	1661	∞	1862		12.1%
I160	i160-014	160	812	7	1778	∞	1991		11.98%
I160	i160-015	160	812	7	1768	2281	1864	29.02%	5.43%
PUC	cc3-4p	64	288	8	2338	∞	2553		9.2%
PUC	cc3-4u	64	288	8	23	∞	25		8.7%
Average								20.72%	7.01%

Table 4.1: Results obtained by applying Dinamic Programming to c^1_{opt} and c^2_{opt}.

Table 4.1 shows the results obtained by applying the recurrences presented in Propositions 4.1.1 and 4.1.2. In each one of them the first column contains the names of the original SteinLib classes with the name of the customized instance. The entries from left to right are:

- the size of the selected instance in terms of number of nodes, edges and terminal sites respectively,
- a lower bound for the optimal cost; the SPG optimum cost (LB_{SPG}),
- c^1_{opt} and c^2_{opt} where c^k_{opt} is the cost of the best feasible solution found in $\Gamma^{(\leq k)}_{ANDP}$,
- the gap of the cost for the best feasible solution of $\Gamma^{(\leq k)}_{ANDP}$ (c^k_{opt}) with respect to the lower bound $LB^{(k)}_{SPG}$ with $k \in \{1, 2\}$ ($LB\_GAP^{(k)}_{SPG}$).

The $LB\_GAP^{(k)}_{SPG}$ is computed as:

$$LB\_GAP^{(k)}_{SPG} = 100 \times \frac{c^k_{opt} - LB_{SPG}}{LB_{SPG}}$$

We obtained feasible solutions only for *i080-112*, *i080-115* and *i160-015* with $k = 1$ because, as we can see, the cost is finite. The optimal values of the SPG instances (LB_{SPG}) provided lower bounds for the optimal values of the ANDP (therefore to ANDP$^{(\leq k)}$ with $k \geq 0$),

Name	n_T	n_C	c_{min}	c_{max}	$\frac{c^1_{opt}}{c^2_{opt}}$	$1 + \lfloor \frac{n_C}{2} \rfloor \cdot \left(\frac{1}{2+n_T}\right) \cdot \left(\frac{c_{max}}{c_{min}} - 1\right)$
i080-112	7	72	85	209	1.166401	5.997385619
i080-115	7	72	86	302	1.1004114	10.325581395
i160-015	6	153	86	300	1.223712	23.639534884

Table 4.2: Relation between optimal solutions of $\text{ANDP}^{(\leq 1)}$ and $\text{ANDP}^{(\leq 2)}$.

considering that in the ANDP generation process, all the connections between terminal nodes were deleted, and further that ANDP's feasible solutions space is more restrictive than of SPG. The experimental results obtained for c^1_{opt} have an average gaps with respect to the lower bound of 20.72 percient. Increasing k to 2 (appliying the recursion presented in Proposition 4.1.2) we obtained feasibles solution for all the testing networks; and the experimental results obtained have an average gaps with respect to the lower bound of 7.01 percent.

Increasing k, by Theorem 3.4.4 we know that the following inequality is fullfilled:

$$\frac{c^{k-1}_{opt}}{c^k_{opt}} \leq 1 + \left\lfloor \frac{n_C}{k} \right\rfloor \cdot \left(\frac{1}{k + n_T}\right) \cdot \left(\frac{c_{max}}{c_{min}} - 1\right).$$

Table 4.2 shows the results obtained applying the Theorem 3.4.4. Despite the bound was not good in these cases (due the heterogeneity of costs of the lines), it can help us in some cases to answer the following question: how much can we save with a higher k?.

Table 4.3 shows the results obtained by applying the recursion presented in Proposition 4.2.1. As before the first column contains the names of the original SteinLib classes with the name of the customized instance. The entries from left to right are:

- the size of the selected instance in terms of number of nodes, edges and terminal sites respectively,
- the cost of a global optimal solution of $\Gamma^{(\leq 2)}_{ANDP}$ (c^2_{opt}),
- the execution time, in seconds, for c^2_{opt} ($t_{c^2_{opt}}$)
- a lower bound for the cost of a global optimal solution of $\Gamma^{(\leq 2)}_{ANDP}$ obtained by applying Dynamic Programming with State-Space Relaxation (presented in Proposition 4.2.1) ($LB^{(2)}_{SSR}$),
- the execution time, in seconds, for $LB^{(2)}_{SSR}$ ($t_{LB^{(2)}_{SSR}}$)
- the gap of the cost for a global optimal solution of $\Gamma^{(\leq 2)}_{ANDP}$ (c^2_{opt}) with respect to the lower bound $LB^{(2)}_{SSR}$; $LB\_GAP^{(2)}_{SSR}$.

The $LB\_GAP^{(2)}_{SSR}$ is computed as:

Set	Name	$\|V\|$	$\|E\|$	$\|T\|$	c^2_{opt}	$t_{c^2_{opt}}$	$LB^{(2)}_{SSR}$	$t_{LB^{(2)}_{SSR}}$	$LB\_GAP^{(2)}_{SSR}$
I080	i080-001	80	120	6	2187	0	1698	0	28.8%
I080	i080-011	80	350	6	1499	6.04	1307	0.27	14.69%
I080	i080-012	80	350	6	1497	5.33	1486	0.16	0.74%
I080	i080-013	80	350	6	1383	8.20	1000	0.92	38.3%
I080	i080-014	80	350	6	1505	4.89	1211	0.25	24.28%
I080	i080-111	80	350	8	2159	3.09	1982	0.45	8.93%
I080	i080-112	80	350	8	1887	1812	1501	7.52	25.72%
I080	i080-113	80	350	8	1884	1809	1591	393.8	18.42%
I080	i080-114	80	350	8	2099	44.81	1988	6.65	5.58%
I080	i080-115	80	350	8	1969	479.8	1496	15.41	31.62%
I080	i080-233	80	160	16	4564	361.1	3997	6.75	14.19%
I160	i160-011	160	812	7	1875	45.67	1399	2.17	34.02%
I160	i160-012	160	812	7	1891	8.83	1502	1.13	25.9%
I160	i160-013	160	812	7	1862	6.58	1381	1.81	34..83%
I160	i160-014	160	812	7	1991	6.06	1783	0.86	11.67%
I160	i160-015	160	812	7	1864	70.28	1793	6.21	3.96%
PUC	cc3-4p	64	288	8	2553	79.37	2177	2.54	17.27%
PUC	cc3-4u	64	288	8	25	80.04	21	5.18	19.05%
Average									19.89%

Table 4.3: Lower bounds obtained to $\mathrm{ANDP}^{(\leq 2)}$ by applying Dinamic Programming with State-Space Relaxation.

$$LB\_GAP^{(2)}_{SSR} = 100 \times \frac{c^2_{opt} - LB^{(2)}_{SSR}}{LB^{(2)}_{SSR}}$$

In general, the gaps related to the lower bounds were low. The r_i to each terminal site and concentrator site were distinct integers chosen from $\{1, \ldots |S_T \cup S_C|\}$. This lower bound can be increased by modifying the state-space through the application of subgradient optimization to r_i. As future work, it is possible to incorporate is method for a better choice of r_i. For more information refer to [22, 48].

We noticed that the execution times of computing global optimal solution costs were much longer than using Dynamic Programming with State-Space Relaxation.

4.4 Conclusions

By modelling the access network design problem as a variant of the Steiner problem in graphs, we were able to develop two algorithms. The implementation of our algorithms was tested on a number of different problems with heterogeneous characteristics. In particular, we built a set of ANDP instances transforming 18 SPG instances (extracted from SteinLib [43]) to our problem. The optimal values for the selected SPG instances are lower bound for the corresponding ANDP. When computing the weighted average over all the classes, the average gaps of the solutions obtained related to this bounds were lower than 21% a 7% to $\text{ANDP}^{(\leq 1)}$ and $\text{ANDP}^{(\leq 2)}$ respectively. It is reasonable supposing that the gaps related to the global optimum of the ANDP instances be even lower since the feasible solutions of the ANDP that are also feasible solutions of the original SPG, but not reciprocally. In this sense, remember that in any ANDP instance generated, all the edges between pairs of terminal nodes were deleted (because in our ANDP such connections are not allowed) having the additional constraint that the terminal nodes must have degree one in the solution.

Besides we were able to develop a Dynamic Programming with State-Space Relaxation algorithm which can give a lower bound in polynomial time. The average gaps with respect to the global optimal solution costs were lower than 20%.

We noticed that, as expected, the execution times of the proposed algorithms are strongly dependant on the number of sites, edges and the terminal sites.

To sum up, as far as we are concerned, the results obtained with the recurrences above are very good as we consider that computing the global optimal solution of an $\text{ANDP}^{(\leq 2)}$ is a NP-hard problem.

In next chapter we present the Backbone Network Design Problem BNDP. We introduce some topological results about the BNDP. We propose a recurrence to provide a lower bound for the BNDP which is based on Dynamic Programming with State-Space Relaxation methodology.

Part IV

THE BACKBONE NETWORK DESIGN PROBLEM

Chapter 5

The Backbone Network Design Problem

In general, a typical WAN backbone network has a meshed topology, and its purpose is to allow efficient and reliable communication between the switch sites of the network that act as connection points for the local access network. The Topological design of a backbone network basically consists of finding a minimum cost topology which satisfies some additional requirements, generally chosen to improve the survivability of the network (that is its capacity to resists the failures of some of its components). One way to do this is to specify a connectivity level, and to search for topologies which have at least this number of disjoint paths (either edge disjoint or node disjoint) between pairs of switch sites. In the most general case, the connectivity level can be fixed independently for each pair of switch sites (heterogeneous connectivity requirements). This problem can be modelled as a *Generalized Steiner Problem with Node-Connectivity* (denoted by GSP-NC) and it is a NP-Hard problem [42, 56, 61, 62]. Winter [60–62] demostrated that the GSP-NC can be solved in linear time if the network is series-parallel, outplanar or a Halin graph (see 5.1.1). Topologies verifying edge-disjoint path connectivity constraints assure that the network can survive to failures in the connection lines. whereas node-disjoint path constraints assure that the network can survive to failures both in switch sites as well as in the connection lines.

Theorem 5.1.2 presents an structural results about the WANDP assuming that the *Backbone Network* must be *two-node-connected*. In Proposition 5.1.4, we propose a recurrence to provide a lower bound for the BNDP which is based on Dynamic Programming with State-Space Relaxation methodology.

5.1 Structural Theorems

Before starting, we introduce the *Halin* graph definition.

Definition 5.1.1. *Halin graph is a planar graph constructed from a plane embedding of a tree with at least four vertices and with no vertices of degree 2, by connecting all the leaves of the tree (the vertices of degree 1) with a cycle that passes around the tree in the natural cyclic order defined by the embedding of the tree.*

Theorem 5.1.2. *Given a* WANDP *where the connection matrix satisfies the following:*

1. *In $G_B = (S_D, E_3)$ it is fulfilled the triangular inequality for any three switch sites of S_D.*

2. *In $G_A = (S, E_1 \cup E_2)$ it is fulfilled the triangular inequality for any three sites $s_1, s_2, s_3 \in S$, where $s_1 \in S_T \cup S_C$, $s_2 \in S_C$, and $s_3 \in S_C \cup S_D$.*

3. *$\Delta c_\beta \leq \frac{c^\beta_{min}}{n_D}$, with $c^\beta_{min} = \min\{c_{(i,j)}\}_{i,j \in S_D}$, $c^\beta_{max} = \max\{c_{(i,j)}\}_{i,j \in S_D}$, and $\Delta c_\beta = c^\beta_{max} - c^\beta_{min}$.*

Assuming that the Backbone Network must be two-node-connected and connect all the sites of S_D; then there exits a Halin topology $\mathcal{H} = (S^*, \mathcal{C}_B \cup \mathcal{T}_A)$ global optimal solution of the WAN *topological design problem.*

Proof. Let Γ_B be the space of two-node-connected solutions that cover the set S_D. Let us denote $\Gamma^{(3)}_B$ the *two-node-connected* feasible solutions of the Backbone Network that cover the set S_D and where sites of degree 3 exist. Let us denote $\Gamma^{(2)}_B$ the feasible solutions of the Backbone Network conformed by a cycle and that cover the set S_D.
By Theorem A.0.8 and Corollary A.0.9, in the best case a feasible solution of $\Gamma^{(3)}_B$ has exactly two nodes of degree 3 as minimum amount of nodes of degree 3, and in addition its topology has the structure illustrated in Figure 5.1. It is easy to see that in this case the best possible cost of the solution is $c^{(3)}_{best} = (n_D{+}1)\cdot c^\beta_{min}$, therefore $\text{COST}(\mathcal{N}_B) \geq (n_D{+}1)\cdot c^\beta_{min}$, $\forall \mathcal{N}_B \in \Gamma^{(3)}_B$. Let us consider $\Gamma^{(2)}_B$; in the worst case a feasible solution in this subspace has cost $c^{(2)}_{worst} = n_D \cdot c^\beta_{max}$, and therefore $\text{COST}(\mathcal{C}_B) \leq n_D \cdot c^\beta_{max}$, $\forall \mathcal{C}_B \in \Gamma^{(2)}_B$.
Let us suppose that does not exist a global optimal solution of Γ_B with cycle topology. Now, let $\bar{\mathcal{C}}_B$ and $\bar{\mathcal{H}}_B$ be the best feasible solutions of $\Gamma^{(2)}_B$ and $\Gamma^{(3)}_B$ respectively. Considering the difference $(\text{COST}(\mathcal{C}_B) - \text{COST}(\mathcal{H}_B))$, we have the inequality:

$$0 < \text{COST}(\bar{\mathcal{C}}_B) - \text{COST}(\bar{\mathcal{H}}_B) \leq n_D \cdot c^\beta_{max} - (n_D + 1) \cdot c^\beta_{min} =$$
$$n_D \cdot (c^\beta_{max} - c^\beta_{min}) - c^\beta_{min} = n_D \cdot \Delta c_\beta - c^\beta_{min}.$$

Implying that $\Delta c_\beta > \frac{c^\beta_{min}}{n_D}$, this is a contradiction. Hence, since by hypothesis $\Delta c_\beta \leq \frac{c^\beta_{min}}{n_D}$, then there exists a global optimal solution of Γ_B with cycle topology, i.e., $\bar{\mathcal{C}}_B$ is globally optimal.
Now, by statement (2) of the hypothesis and Proposition 3.1.3 we have that there exists a global optimal solution $\bar{\mathcal{T}}_A \in \Gamma_{ANDP}$ which satisfies that for all concentrator site $s_c \in \bar{\mathcal{T}}_A$ $g(s_c) \geq 3$ in $\bar{\mathcal{T}}_A$. Let S^*_C be the subset of sites of S_C present in $\bar{\mathcal{T}}_A$.
Let us consider the network $\mathcal{H}_{\text{wan}} = (S_D \cup S^*_C \cup S_T, \bar{\mathcal{C}}_B \cup \bar{\mathcal{T}}_A)$, it fulfills to be a global optimal solution for the WAN topological design problem, and clearly it has Halin topology, as required, completing the proof. □

Corollary 5.1.3. *Given a* WANDP *which satisfies the following points:*

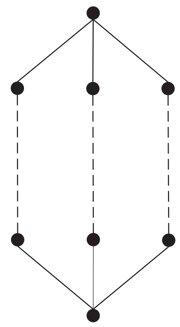

Figure 5.1: A feasible solution of $\Gamma_B^{(3)}$

1. *There exists a solution $\mathcal{T}_A \in \Gamma_{ANDP}$ globally optimal of the* ANDP *such that $g(s_c) \geq 3$, $\forall s_c \in S_C / s_c \in \mathcal{T}_A$.*
2. *In $G_B = (S_D, E_3)$ it is fulfilled the triangular inequality for any three switch sites of S_D.*
3. $\Delta c_\beta \leq \frac{c^\beta_{min}}{\bar{n}_D}$,

with $\bar{n}_D = |S_D^{(I)}|$, $S_D^{(I)} \subseteq S_D$ is the set of fixed sites of the Backbone Network induced by $\mathcal{T}_A$. Assuming that the Backbone Network must be two-node-connected with respect to $S_D^{(I)}$; then exists a Halin topology $\mathcal{G} = (S^*, \mathcal{C}_B \cup \mathcal{T}_A)$ globally optimal solution of the WAN *topological design problem.*

Proof. Let Γ_{BS} be the set of feasible solutions conformed by the subnetworks of G_B that satisfy: $\mathcal{N}_B \in \Gamma_{BS}$ if for every pair of sites $s_{w_1}, s_{w_2} \in S_D^{(I)}$, $s_{w_1} \neq s_{w_2}$, there is at least two node-disjoint paths which connect them in $\mathcal{N}_B$. Let us denote $\Gamma_{BS}^{(3)}$ the feasible solutions of Γ_{BS} where sites of degree 3 exist, and $\Gamma_{BS}^{(2)}$ the feasible solutions of Γ_{BS} conformed by a cycle.
Again, as the previous theorem, by Theorem A.0.8 and Corollary A.0.9, in the best case a feasible solution of $\Gamma_{BS}^{(3)}$ has exactly two nodes of degree 3 and its topology is illustrated by Figure 5.1. Moreover, in this case the minimum possible cost is $c_{best}^{(3)} = (\bar{n}_D + 1) \cdot c^\beta_{min}$, therefore $\textsc{cost}(\mathcal{G}_B) \geq (\bar{n}_D + 1) \cdot c^\beta_{min}$, $\forall \mathcal{G}_B \in \Gamma_{BS}^{(3)}$. It is easy to see that the best feasible solution of $\Gamma_{BS}^{(2)}$, in the worst case has cost $c^{(2)}_{opt\_worst} = \bar{n}_D \cdot c^\beta_{max}$ (it is a cycle only conformed by the sites of $S_D^{(I)}$). Let us suppose that does not exist a solution $\mathcal{C}_B \in \Gamma_{BS}^{(2)}$ globally optimal. Let $\bar{\mathcal{C}}_B$ and $\bar{\mathcal{G}}_B$ be the best feasible solutions of $\Gamma_{BS}^{(2)}$ and $\Gamma_{BS}^{(3)}$ respectively. The following inequality is fulfilled:

$$0 < \textsc{cost}(\bar{\mathcal{C}}_B) - \textsc{cost}(\bar{\mathcal{G}}_B) \leq \bar{n}_D \cdot c^\beta_{max} - (\bar{n}_D + 1) \cdot c^\beta_{min} = \bar{n}_D \cdot \Delta c_\beta - c^\beta_{min}.$$

Hence, we have that $\Delta c_\beta > \frac{c^\beta_{min}}{\bar{n}_D}$. This is a contradiction, therefore $\bar{\mathcal{C}}_B$ is globally optimal. Now, considering $\mathcal{G}_{\text{wan}} = (S_D^{(I)} \cup S_C^* \cup S_T, \bar{\mathcal{C}}_B \cup \mathcal{T}_A)$, with S_C^* the concentrator sites present in $\mathcal{T}_A$, it satisfies to be globally optimal for the WAN topological design problem, and besides it has Halin topology. $\square$

Proposition 5.1.4. *Let us suppose that in $G_B = (S_D, E_3)$ the triangular inequality is fulfilled and in addition $\Delta c_\beta \le \frac{c^\beta_{min}}{n_D}$. Then, we can compute in polynomial time a lower bound for the optimal cost of a backbone network with two-node-connected topology.*

Proof. Assuming that the backbone network topology must be at least *two-node-connected*, by Theorem A.0.8, we know that there exists at least one optimal *two-node-connected* solution for the BNDP with cycle topology. Let us choose a switch site $s^0_w \in S_D$. Let $\hat{z}^0_w$ be a new fictitious site which has the same feasible connections than $s^0_w \in S_D$. We define E_z as the extended connections set, and Z as the matrix of connections cost extended to $S_D \cup \{\hat{z}^0_w\}$. Now, we will introduce a recurrence of dynamic programming to calculate the optimal cost of a backbone network with cycle topology, i.e., the *two-node-connected* topology of minimum cost. We define $g_{(\bar{S}_D, s^i_w)}$ as the cost of the shortest path with origin $s^0_w \in \bar{S}_D \subseteq S_D$, passing through all the sites of $\bar{S}_D$ exactly once, and finalizing at site $s^i_w \in \bar{S}_D$. This value can be computed by the expression of Dynamic Programming.

$$g_{(\bar{S}_D, s^i_w)} = \min_{(s^i_w, s^j_w) \in E_z} \left\{ g_{(\bar{S}_D, s^j_w)} + Z_{(s^j_w, s^i_w)}; s^j_w \in (\bar{S}_D \setminus \{s^i_w\}) \right\},$$
$$g_{(\{s^i_w\}, s^i_w)} = Z_{(s^0_w, s^i_w)}, \ \forall \bar{S}_D \subseteq S_D / s^0_w, s^i_w \in \bar{S}_D.$$

The optimum value is: $\text{OPT\_VALUE} = \min_{s^i_w \in S_D} \left\{ g_{(S_D, s^i_w)} + Z_{(s^i_w, \hat{z}^0_w)} \right\}$. Clearly the time-complexity to compute this expression is exponential. Now, applying the State-Space Relaxation, specifically the shortest r-path relaxation, we obtain the following transformed recursion.

$$g_{(r, s^i_w)} = \min_{(s^i_w, s^j_w) \in E_z} \left\{ g_{(r - r_i, s^j_w)} + Z_{(s^j_w, s^i_w)}; r_j \le r - r_i \right\},$$
$$g_{(r, s^i_w)} = \begin{cases} Z_{(s^0_w, s^i_w)} & \text{if } (s^0_w, s^i_w) \in E_z \text{ and } r = r_j, \\ \infty & \text{otherwise.} \end{cases}$$

where $r \in \{1, \ldots, \mathrm{R}\}$, and $s^i_w \in S_D$. In this relaxation, each switch site $s^i_w \in S_D$ has associate a different integer $r_i \ge 1$, and moreover $\mathrm{R} = \sum_{s^i_w \in S_D} r_i$. Therefore, the state-space is mapping to a new space state which can be polynomially computed. Hence, the inequality:

$$\min_{s^i_w \in S_D} \left\{ g_{(\mathrm{R}, s^i_w)} + Z_{(s^i_w, \hat{z}^0_w)} \right\} \le \text{OPT\_VALUE},$$

is established, and furthermore the bounding associated algorithm can be computed in $\mathrm{O}(\mathrm{R} n^2_D)$, completing the proof. □

Lemma 5.1.5. *Given a $BNDP(S, E_3, C, R)$ such that $R_{ij} = 2, \forall s^i_w, s^j_w \in S_D$. There exists a lower bound for the optimal cost of the backbone network, given by a Dynamic Programming recursion with State-Space relaxation.*

Proof. Firstly, by Proposition A.0.7, we know that any *two-node-connected* topology is built from a cycle adding successively a new path having both extremities in the current network (usually denominated decomposition in H-paths, being H the network to which new paths are added to him recurrently). Let $\Gamma^{(\text{opt})}_{BNDP} \subseteq \Gamma_{BNDP}$ be the sub-space of optimal solutions of the BNDP. Particulary, it is easy to see that the decomposition in H-paths corresponding to an optimal solution belonging to $\Gamma^{(\text{opt})}_{BNDP}$ does not have any path only formed by one connection. That is to say, given $\mathcal{G} \in \Gamma^{(\text{opt})}_{BNDP}$ and $\mathcal{H}_{(\mathcal{G})} = \{H_0, H_{p1}, \ldots, H_{pk}\}$ is its H-paths decomposition ($H_{(j)} = H_{(j-1)} \cup H_{pj}$, $\forall j \in 0 \ldots k$), then, each H-path H_{pj} has at least some new site not including in $H_{(j)}$. Otherwise, if a simple connection were added, $\mathcal{G}$ would not be globally optimal. Now, in order to compute the optimum value corresponding to an optimal *two-node-connected* topology, we will introduce a recurrence of dynamic programming.
Let $\bar{S}_D \subseteq S_D$ be a sub-set of switch sites. We define $\mathrm{f}_{(\bar{S}_D)}$ as the minimum cost of a *two-node-connected* sub-network spanning only the set $\bar{S}_D$. The expression that computes $\mathrm{f}_{(\bar{S}_D)}$ is given by:

$$\mathrm{f}_{(\bar{S}_D)} = \min\left\{\mathrm{cy}_{(\bar{S}_D)}, \mathrm{f}_{(\bar{S}_D \setminus \hat{S}_D)} + \mathrm{hp}_{(\hat{S}_D, s^i_w, s^j_w)}; \forall \hat{S}_D \subset \bar{S}_D,\ s^i_w, s^j_w \in (\bar{S}_D \setminus \hat{S}_D)\right\},$$
$$\mathrm{f}_{(\emptyset)} = \infty,\ \mathrm{f}_{(\{s^i_w\})} = \infty,$$

where the expressions $\mathrm{cy}_{(\bar{S}_D)}$ and $\mathrm{hp}_{(\hat{S}_D, s^i_w, s^j_w)}$ are defined as:

- $\mathrm{cy}_{(\bar{S}_D)}$ is the optimal cost of a cycle (with ring topology) only conformed by sites of $\bar{S}_D$.
- $\mathrm{hp}_{(\hat{S}_D, s^i_w, s^j_w)}$ is the minimum cost of a path with origin in s^i_w, passing through every site of $\hat{S}_D$ exactly once, to end at site s^j_w.

Let $\bar{s}_w \in \bar{S}_D$ be a switch site. Let us consider a new fictitious site $\bar{z}_w$ which has the same feasible connections than $\bar{s}_w$. It is easy to see that $\mathrm{cy}_{(\bar{S}_D)}$ is equivalent to compute the expression $g_{(\bar{S}_D, \bar{z}_w)}$ whose definition was introduced in Proposition 5.1.4. Furthermore, $\mathrm{hp}_{(\hat{S}_D, s^i_w, s^j_w)}$ is equivalent to compute the expression $g_{(\hat{S}_D \cup \{s^j_w\}, s^j_w)}$ fixing as departure site the switch site s^i_w. Now, applying the r-path relaxation, we associate to each site of S_D a different integer r_i and besides we define the value $\mathrm{R} = \sum_{s_w \in S_D} r_i$. Let us notice that any subset $\bar{S}_D \subseteq S_D$ is mapping to an integer r so that $r = \sum_{s^i_w \in \bar{S}_D} r_i$. Hence, the recursion of dynamic programming with state-space relaxation is given by the equations:

$$\mathrm{f}_{(r)} = \min\left\{\mathrm{cy}_{(r)}, \mathrm{f}_{(r-\bar{r})} + \mathrm{hp}_{(\bar{r}, s^i_w, s^j_w)}; \forall \bar{r} < r,\ r_i, r_j \leq (r - \bar{r})\right\},$$
$$\forall r \in \{1, \ldots, \mathrm{R}\},\ \mathrm{f}_{(0)} = \infty,\ \mathrm{f}_{(r)} = \infty \textit{ if } \exists s^i_w / r = r_i.$$

The lower bound is: $\mathrm{f}_{(\mathrm{R})} \leq \mathrm{f}_{(S_D)}$, and in addition we can compute it in polynomial time. In fact, it is polynomially computed with time-complexity $\mathrm{O}(2\mathrm{R}^2 n_D^2)$. □

In next chapter we conclude the thesis and introduce some future works.

Part V

GENERAL CONCLUSIONS AND PERSPECTIVES

Chapter 6

Conclusions

In this thesis we have studied the topological design of a WAN (Wide Area Network) considering only the construction costs, for instance, the costs of digging trenches and putting a fiber cable into service [57]. The reason for this following approach is that construction costs have the largest share in the overall cost of a WAN planning and design stage. Let us point out that even a very small reduction in this cost may represent many million dollars of savings for, say, telephone companies.

We tackled the problem of designing a WAN by breaking it down into two inter-related sub-problems: the Access Network Design Problem (ANDP) and the Backbone Network Design Problem (BNDP). We modeled the ANDP as a variant of the *Steiner Problem in Graphs* (SPG), and the BNDP on the basis of the *Generalized Steiner Problem with Node-Connectivity Constraints* (GSP-NP) [57]. We studied differents results related to the topological structure of the ANDP solutions. In particular we presented results that characterize the topologies of the feasible solutions of ANDP and BNDP instances. Moreover, for certain types of network classes we present results that characterize the structure of the global optimal solution. We presented the clustering approach as one of the strategies more frequently used by the commercial design tools. We also formulated the ANDP as a Steiner Problem in Graphs (SPG). Given the complexity of the ANDP we provided techniques capable of reducing the dimension of the original problem to an equivalent smaller problem. We also presented some structural properties about the BNDP.

We now provide a summary of the experimental results obtained for each one of the problems referred to above. For ANDP we designed an algorithm for the optimal solution to $ANDP^{(\leq k)}$ with $k \in 1..2$ using Dynamic Programming and another one that gives a lower bound for the global optimal solution cost of $ANDP^{(\leq 2)}$ using Dinamic Programming with State-Space Relaxation. The numerical experiments were done on a testing set containing 18 SPG instances extracted from the SteinLib repository and customized for ANDP. The optimal values of the SPG instances provided lower bounds for the optimal values of $ANDP$ (Considering that in the ANDP generation process all the connections with terminal nodes were deleted, and further that ANDP's feasible solutions space is more restrictive than of SPG). Therefore are for $ANDP^{(\leq k)}$ with $k \in 1..2$. Other lower bounds were obtained by applying Dynamic Programming with State-Space Relaxation approach. The experimental results obtained were successful. The average gaps of the global optimal solution costs obtained with respect to SGP bounds were lower than 21% a 7% to $ANDP^{(\leq 1)}$ and $ANDP^{(\leq 2)}$ respectively whereas the gap of the optimum with respect to the lower bound obtained by applying Dynamic Programming with State-Space Relaxation was lower than 20%.

To sum up, as far as we are concerned, the results obtained with the algorithms proposed are very good as we consider that the computing the global optimal solution of the ANDP is a NP-hard problem.

Furthermore, we establish conditions to the WANDP that ensure that there exists Halin topology global optimal solution of the WAN topological design problem.

Future research, based on this line of work, could add new topological restrictions like maximum number of incident links at concentrator nodes. It is also worth minimizing the impact of failures on the access network, as there is no redundancy in its topology. It might be relevant balancing the number of subscribers per switch.

Part VI

APPENDICES

Appendix A

Additional Theorems

Definition A.0.6. *(**H-Path**) Given a graph H, we call a path p an H - path if p is non-trivial and meets H exactly in its ends. In particular, the edge of any H - path of length 1 is never an edge of H.*

Proposition A.0.7. *A graph is two-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed. A proof can be found in [24].*

Theorem A.0.8. *For any set of nodes V with distance function $d(\cdot)$ on $V \times V$, there exists a minimum weight two-node-connected graph $G = (V, E)$ satisfying the following conditions.*

(a) Every node in G has degree 2 or 3.

(b) Deleting any edge or pair of edges in G leaves a bridge in one of the resulting connected components of G [49].

Corollary A.0.9. *Any two-node-connected graph $G = (V, E)$ satisfying conditions (a) and (b) of Theorem A.0.8, and which is not a cycle, contains the graph shown in Figure 5.1 as a node-induced subgraph [49].*

Bibliography

[1] A. Agrawal, P. Klein, and R. Ravi. When trees collide: an approximation algorithm for the generalized Steiner problem on networks. *SIAM Journal on Computing*, 24(3):440–456, 1995.

[2] K. Altinkemer and A. Chaturvedi. Neural networks for topological design of local access tree networks. In *Proceeding of Telecommunication Systems, Modelling and Analysis Conference*, pages 256–263, 1993.

[3] K. Altinkemer and Z. Yu. Topological design of wide area communication networks. *Annals of Operations Research*, 36:365–382, 1992.

[4] Franco Robledo Amoza. *GRASP heuristics for a Wide Area Network design*. PhD thesis, Universidad de la República Oriental del Uruguay and Université de Rennes I, 2005.

[5] M. Andrews and L. Zhang. The Access Network Design Problem. In *39th Annual Symposium on Foundations of Computer Science*, pages 40–49, 1998.

[6] Mourad Baïou. *Le problème du suos-graphe Steiner 2-arête connexe: approache polyédrale*. PhD thesis, Université de Rennes I, 1996.

[7] A. Balakrishnan, T. Magnanti, and P. Mirchandi. A dual-based algorithm for multi-level network design. *Management Science*, 40:567–581, 1994.

[8] A. Balakrishnan, T. Magnanti, and P. Mirchandi. Intuitive solution-doubling techniques for worst-case analysis of some survivable network design problems. *Operations Research Letters*, 29:99–106, 2001.

[9] A. Balakrishnan, T. Magnanti, and P. Mirchandi. Connectivity-splitting models for survivable network design. *Networks*, 43(1):10–27, 2004.

[10] A. Balakrishnan, T.L. Magnanti, and P. Mirchandani. Modeling and heuristic worst-case performance analysis of the two-level network design problem. *Management Science*, 40(7):846–867, 1994.

[11] A. Balakrishnan, T.L. Magnanti, and P. Mirchandani. Designing hierarchical survivable networks. *Operations Research*, 46(1):116–136, 1998.

[12] R.T. Berger and S. Raghavan. Long-Distance Access Network Design. *Management Science*, 50:309–325, 2004.

[13] D. Bienstock, E.F. Brickell, and C.L. Monma. On the structure of minimum weight k-connected spanning networks. *SIAM Journal on Discrete Mathematics*, 3(3):320–329, 1990.

[14] Z.R. Bogdanowicz. A new optimal algorithm for backbone topology design in communications networks. *Mathematical and Computer Modelling*, 17(8):49–61, 1993.

[15] R.R. Boorstlyn and H. Frank. Large-scale network topological optimization. *IEEE Transactions on Communications*, 25(1):29–47, 1977.

[16] H. Cancela, F. Robledo, and G. Rubino. A GRASP algorithm for designing a Wide Area Network backbone. In *Proceedings of the International Network Optimization Conference (INOC'03)*, pages 138–143, Evry/Paris, France, October 2003.

[17] H. Cancela, F. Robledo, and G. Rubino. A GRASP algorithm with RNN based local search for designing a WAN access network. In *Electronic Notes in Discrete Mathematics - special issue including the Proceedings of the Latin-American Conference on Combinatorics, Graphs and Applications (LACGA'04)*, volume 18C, pages 53–58, 2004.

[18] H. Cancela, F. Robledo, and G. Rubino. Designing low-cost access network topologies. In *Proceeding of the International Network Optimization Conference (INOC'05)*, pages 825–832, University of Lisbon, Portugal, October 2005.

[19] S. Chamberland and B. Sansò. On the design problem of multitechnology networks. *INFORMS Journal on Computing*, 13(3):245–256, 2001.

[20] S. Chamberland, B. Sansò, and O. Marcotte. Topological design of two-level telecommunication networks with modular switches. *Operations Research*, 48(5):745–760, 2000.

[21] Sheng-Tzong Cheng. Topological optimization of a reliable communication network. *IEEE Transactions on Reliability*, 47(3):225–233, 1998.

[22] Nicos Christofides, A. Mingozzi, and P. Toth. State-space relaxation procedures for the computation of bounds to routing problems. *Networks*, 11(2):145–164, 1981.

[23] F.R.B. Cruz, J. MacGregor Smith, and G.R. Mateus. Algorithms for a multi-level network optimization problem. *European Journal of Operational Research*, 118:164–180, 1999.

[24] R. Diestel. *Graph theory*. Springer-Verlag, 2 edition, 2000.

[25] Eduardo Canale Franco Robledo. Designing backbone networks using the generalized steiner problem. *Proceedings of 7th International Workshop on the Design of Reliable Communication Networks*, pages 327–334, 2009.

[26] Gerardo Rubino Franco Robledo-Amoza, Hector Cancela. Solving the steiner two-node-survivable network problem. *International Journal of Logistics Systems and Management*, 6(2):218–234, 2010.

[27] L.F. Frantzeskakis and H. Luss. The network redesign problem for access telecommunications networks. *Naval Research Logistics*, 46:487–506, 1999.

[28] Greg N. Frederickson and Joseph Ja'ja'. On the relationship between the biconnectivity augmentation and travelling salesman problems. *Theoretical Computer Science*, 19(2):189 – 201, 1982.

[29] B. Gavish. A general model for the topological design of computer networks. In *GLOBECOM'86 - IEEE International Global Telecommunications Conference*, pages 1584–1588, 1986.

[30] B. Gavish. Configuring wide area computer networks-problems and models. *OR Spektrum*, 14(3):115–128, 1992.

[31] B. Gavish. Topological design of computer communication networks-the overall design problem. *European Journal of Operations Research*, 58(2):149–172, 1992.

[32] B. Gavish. Topological design of computer communication networks. In *Proceedings of the 2nd International Workshop on All-Optical Networks - WAON 2001 (Zagreb, Croatia)*, volume 3, pages 103–107, 2001.

[33] L. Ghosh, A. Mukherjee, and D. Saha. Optimal design of backbone topology for a communication network cost constraint. In *Proceedings of ICCC 2002 - 15th International Conference on Computer Communication*, volume 2, pages 471–485, 2002.

[34] A. Girard, B. Sansò, and L. Dadjo. A tabu search algorithm for access network design. *Annals of Operations Research*, 106(1-4):229–262, 2001.

[35] M.X. Goemans and D.J. Bertsimas. Survivable networks, linear programming relaxations and the parsimonious property. *Mathematical Programming*, 60:143–166, 1993.

[36] L. Gouveia and M.J. Lopes. Using generalized capacitated trees for designing the topology of local access networks. *Telecommunications Systems*, 7(4):315–337, 1997.

[37] M. Grötschel and C.L. Monma. Integer polyhedra associated with certain network design problems with connectivity constraints. *SIAM Journal on Discrete Mathematics*, 3:502–523, 1990.

[38] M. Grötschel, C.L. Monma, and M. Stoer. Computational results with a cutting plane algorithm for designing communication networks with low-connectivity constraints. *Operations Research*, 40(2):309–330, 1992.

[39] M. Grötschel, C.L. Monma, and M. Stoer. Polyhedral and computational investigations for designing communications networks with high survivability requirements. *Operations Research*, 43(6):1012–1024, 1995.

[40] Jae gyun Kim and Dong wan Tcha. Optimal design of a two-level hierarchical network with tree-star configuration. *Computers & Industrial Engineering*, 22(3):273–281, 1992.

[41] Sung hark Chung, Young soo Myung, and Dong wan Tcha. Optimal design of a distributed network with two-level hierarchical structure. *European Journal of Operational Research*, 62(1):105–115, 1992.

[42] A. Ridha Mahjoub Hervé Kerivin. Design of survivable networks: A survey. *Networks*, 26(1):1–21, 2005.

[43] T. Koch, A. Martin, and S. Voß. SteinLib: An updated library on Steiner tree problems in graphs. Technical Report ZIB-Report 00-37, Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustr. 7, Berlin, 2000.

[44] M. Kos, M. Mikac, and D. Mikac. Topological planning of communication networks. *Journal of Information and Organizational Sciences*, 26(1-2):57–68, 2002.

[45] B. Lukic. An approach of designing local access network using Simulated Annealing method. In *ConTEL'99 - 5th International Conference on Telecommunications*, pages 241–247, 1999.

[46] S. Mandal, D. Saha, R. Mukherjee, and A. Roy. An efficient algorithm for designing optimal backbone topology for a communication networks. In *Proceedings of ICCT 2003 - International Conference on Communication Technology (Beijing, China)*, volume 1, pages 103–106, 2003.

[47] G.R. Mateus and R.V.L. Franqueira. Model and heuristics for a generalized access network design problem. *Telecommunication Systems - Modelling, Analysis, Design and Management*, 15(3-4):257–271, 2000.

[48] Aristide Mingozzi. State space relaxation and search strategies in dynamic programming. In *Proceedings of the 5th International Symposium on Abstraction, Reformulation and Approximation*, pages 51–, London, UK, 2002. Springer-Verlag.

[49] C.L. Monma, B.S. Munson, and W.R. Pulleyblank. Minimum-weight two connected spanning networks. *Mathematical Programming*, 46:153–171, 1990.

[50] Clyde L. Monma and David F. Shallcross. Methods for designing communications networks with certain two-connected survivability constraints. *Operations Research*, 37(4):pp. 531–541, 1989.

[51] Zuordnungsproblem Mit Nebenbedingungen, Ulrich Bauer, Aufgabenstellerin Prof, Dr. Angelika Steger, and Betreuer Andreas Weißl. Technische Universität München Fakulät für informatik assignment problem with constraints. http://num.math.uni-goettingen.de/ bauer/pub/ConstrainedAssignment.pdf, 2005.

[52] S. Pierre and A. Elgibaoui. A tabu-search approach for designing computer-network topologies with unreliable components. *IEEE Transactions on Reliability*, 46(3):350–359, 1997.

[53] M. Priem and F. Priem. *Ingénierie des WAN (text in French)*. Dunod InterEditions, 1999.

[54] C.D. Randazzo and H.P.L. Luna. A comparison of optimal methods for local access uncapacited network design. *Annals of Operations Research*, 106:263–286, 2001.

[55] C.D. Randazzo, H.P.L. Luna, and P. Mahey. Benders decomposition for local access network design with two technologies. *Discete Mathematics & Theoretical Computer Science*, 4:235–246, 2001.

[56] K. Steiglitz, P. Weiner, and D.J. Kleitman. The design of minimum-cost survivable networks. *IEEE Transactions on Circuit Theory*, 16:455–460, 1969.

[57] M. Stoer. *Design of Survivable Networks*, volume 1532. Springer-Verlag, 1992.

[58] T. Thomadsen and J. Clausen. Hierarchical network design using simulated annealing. Technical report, Informatics and Mathematical Modelling, Technical University of Denmark, DTU, Richard Petersens Plads, Building 305, DK-2800 Kgs. Lyngby, sep 2002.

[59] L. Tran and P.A. Beling. A heuristic for the topological design of two-tiered networks. In *SMC'98 - IEEE International Conference on Systems, Man, and Cybernetics*, volume 3, pages 2962–2967, 1998.

[60] P. Winter. Generalized Steiner problem in outerplanar graphs. *BIT*, 25(3):485–496, 1985.

[61] P. Winter. Generalized Steiner problem in series-parallel networks. *Journal of Algorithms*, 7:549–566, 1986.

[62] P. Winter. Steiner problem in networks: A survey. *Networks*, 17(2):129–167, 1987.