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Abstract

Objects of different quality are to be assigned to agents. Agents can be assigned at most one object and there are not enough high-quality objects for every agent. The social planner is unable to use transfers to give incentives for agents to convey their private information; instead, she is able to imperfectly verify their reports. We characterize a mechanism that maximizes welfare, where agents face different lotteries over the various objects, depending on their report. We then apply our main result to the case of college admissions. We find that optimal mechanisms are, in general, ex-post inefficient and do strictly better than the standard mechanisms that are typically studied in the matching literature.

JEL classification: C7, D8.

Keywords: imperfect verification, evidence, mechanism design, matching.

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Resumen

Objetos de distinta calidad deben ser asignados a agentes. Cada agente puede recibir a lo sumo un objeto, y no hay suficientes objetos de alta calidad para todos los agentes. El planificador central no puede usar transferencias monetarias para dar incentivos a los agentes de forma que reporten su información privada. En lugar de transferencias puede verificar los reportes de los agentes aunque imperfectamente. Caracterizamos un mecanismo que maximiza el bienestar, en el cual distintos agentes enfrentan distintas loterías sobre los objetos dependiendo de su reporte. Aplicamos luego nuestro resultado principal al caso de las admisiones a las universidades. Encontramos que los mecanismos óptimos en este caso son, en general, ex-post ineficientes y tienen una performance estrictamente mejor que los mecanismos estudiados en esta literatura.

Palabras clave: Verificación imperfecta, evidencia, diseño de mecanismos, matching.

1 Introduction

We consider an object assignment problem, where objects of high and low quality are assigned to agents. Each agent can be assigned at most one object and there are less high-quality objects than agents. The value to a social planner from giving certain objects to any given agent depends on that agent's private information (his type). We consider a setting without transfers and assume that the social planner is able to *imperfectly* verify the agents' private information. We characterize allocation rules that maximize the social planner's expected payoff, who is assumed to prefer to assign the high-quality objects to the high-type agents, so that, in the social planner's preferred allocation, there would be assortative matching.

There are several important applications that fit this description. Throughout the paper, we make references to the college admissions' problem (Balinski and Sonmez, 1999), where seats at various universities are assigned to students. Students have a common ranking of the universities (universities have different quality) and differ in "talent". The social planner prefers to assign the "more talented" students to the higher quality universities, but does not observe talent.[1] Instead, she observes a signal of talent, which may include the grade of some exam, letters of recommendation, etc. Other examples include the housing assignment problem, where the social planner assigns houses to those who cannot afford one; the school choice problem, where seats at public schools are assigned to students, etc.

In the optimal mechanism, agents are initially asked to choose one of many "tracks". After that, signals, which are correlated with the agents' types, are realized. The object the agent is awarded, if any, depends on the track he chose and on the signal that is realized. Specifically, each track is characterized by two thresholds for the signal: an upper threshold and a lower threshold. If the agent's signal exceeds the upper threshold, the agent is assigned a high-quality object; if his signal is in-between the two thresholds, he is assigned a low-quality object; finally, if his signal is below the lower threshold, he is not assigned any object. Different tracks have different pairs of thresholds; for some tracks, the two thresholds are very close, while for some others, they are very far apart. Figure 1 illustrates.[2]

In the framework of the college admissions' problem, tracks can be thought of as different thresholds for each university, and signals as the realized scores in some cen-

[1]The word "talent" does not have a latent meaning; it is simply an indicator of how the social planner ranks students.

[2]Not being assigned any object can alternatively be interpreted as receiving an object with even lesser quality, provided there is enough supply of that object.

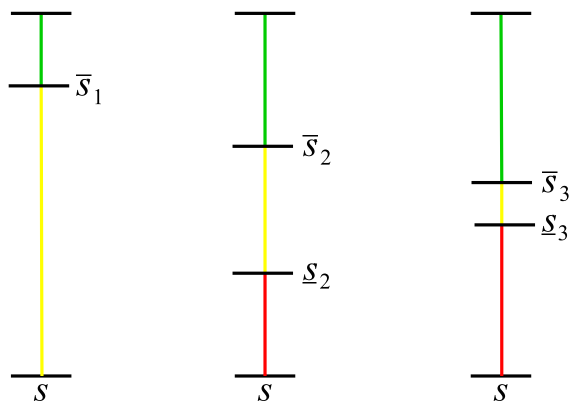

Figure 1: The optimal mechanism when there are only three types. In equilibrium, if the agent's type is the highest one, he chooses the track on the right; if it is second highest one, he chooses the middle track; if it is smallest one, he chooses the track on the left. Once a track has been chosen, the agent is assigned the high-quality object if his signal s lands in the green area, a low-quality object if it lands in the yellow area and no object if it lands in the red area.

tralized exam. If the score is above the upper threshold of the chosen track, the student is assigned a high-quality university; if it is in-between the two, he is assigned a low-quality university; if it is below both thresholds he is not assigned any university. The optimal mechanism induces the more talented students to choose tracks which involve lower upper thresholds and higher lower thresholds. Thus, it is easier for them to get assigned to a high-quality university, but also to end up unassigned. More talented students are more willing to choose these tracks because they are more confident that their scores will be high. As a result, this self-selection leads to larger types effectively facing lower upper thresholds than lower types. Therefore, in the optimal mechanism, some of the lower types who would have been assigned the high-quality objects if there was only a single track (because they would have been lucky enough to have had a high score) are being replaced by some of the larger types who would have been unlucky to have had a lower score. Given that the goal of the social planner is to match the quality of the universities with the students' talent, this ends up being beneficial.

In addition to its theoretical interest, the optimal mechanism we propose rationalizes some real-life assignment rules. For example, in Hungary, the centralized matching scheme that assigns students to public universities and colleges has some of the same features as our optimal mechanism. Before being asked to take a final exam, which, in conjunction with the student's secondary school's grades, determines his ranking, each student is asked to choose between a normal and a high-level track. In the high-level track, the exam is more difficult to pass but it may receive an extra score (10% of the maximum score of the normal track). As in our optimal mechanism, students in the high-level track have a higher probability of getting one of their most preferred programs, but also a higher probability of being unassigned.[3]

Object assignment problems have been studied by a literature on mechanism design and a literature on matching.

Related literature on mechanism design. In mechanism design, the closest references to our work are Ben-Porath, Dekel and Lipman (2014), Mylovanov and Zapechelnyuk (2017), Li (2019b) and Chua, Hu and Liu (2019). These papers consider the problem of assigning homogeneous objects to agents in settings without transfers. The social planner, who prefers to assign the objects to the agents with the largest types, is able to verify at least some of them. All of these papers either assume that verifying the agents' types is costly or that the social planner is unable to destroy (some of) the objects. If, instead, one assumes that verification is costless and that the objects can be fully destroyed, the optimal mechanism in all of these papers would be the following: ask every agent to report their type and assign the objects to the largest reports after verifying they are truthful; if some report is false, destroy all objects. Notice that every agent has an incentive to report truthfully because any false report has no chance of being awarded any object. Moreover, not only is this mechanism optimal; it is a first best mechanism.

In this paper, we depart from this literature in two fundamental ways. **First**, we assume that the type verification is costless and that objects can be completely destroyed. We do this not only because we want to focus on different aspects of the problem but also because there are several applications where the trade-offs explored in this literature do not seem to be of first order importance. For example, if we again consider the college admissions' problem, the verification costs do not seem to be a factor when determining which mechanism to use, as students are asked to do a variety of tests in virtually every college assignment mechanism that has been used worldwide.

The **second** and main difference is that we assume that the type verification is imperfect, i.e., the social planner is able to obtain signals that are only imperfectly correlated with the agents' types, like the exam grades or the letters of recommendation

[3] See Biro (2011) for more details about the Hungarian system.

in the college admissions' problem.[4] This assumption is crucial because the previous first best mechanism is no longer incentive compatible. In particular, an agent with the lowest of types, who, when reporting truthfully, would be given the lowest quality object assigned, if any, would prefer to claim to having the largest type, because there would be a perhaps small but positive chance that his report would be considered truthful. As a result, there is no first best mechanism that is incentive compatible if the social planner only has imperfect evidence about the agents' private information. A plausible alternative to assigning the objects to the agents with the largest types would be to simply assign the objects to the agents with the largest signals, i.e., a one-track mechanism. In a model with a continuum of agents, we find that such a mechanism is only optimal when all objects being assigned have the same quality. If objects can be of high or low quality, the optimal mechanism asks agents to self-select into different tracks as described above.[5]

Related literature on matching. In the matching literature that studies object assignment problems (for example, Abdulkadiroglu and Sonmez, 2003, or Balinski and Sonmez, 1999), the focus is on characterizing mechanisms that have certain desirable properties like strategy-proofness (incentive compatibility), efficiency, and the elimination of "justified envy". This last concept is closely related to stability, and, in the college admissions' problem, it implies that a student with a larger score does not prefer the assignment of a student with a lower score. The key difference between our approach and the one followed by the matching literature is that the latter (implicitly) assumes that scores are perfectly correlated with talent.[6] Under this assumption, one of the most famous mechanisms that is widely used in practice, the Deferred Acceptance (DA) mechanism by Gale and Shapley (1962), would be optimal in our model.[7] This mechanism is effectively a one-track mechanism, because it considers only the scores; it assigns students with the largest scores to the best universities. The DA mechanism is strategy-proof, (ex-post) efficient, and eliminates justified envy. However, we

[4]Here we also depart from Li (2019a) and Etritopou and Vohra (2019), who consider perfect verification models (the former paper considers bidimensional types, while the latter assumes that agents arrive sequentially).

[5]There is also a literature on the design of allocation rules without transfers that differs from our approach in that it allows agents to send costly messages (McAfee and McMillan 1992, Hartline and Roughgarden 2008, Yoon 2011, Condorelli 2012 and Chakravarty and Kaplan 2013).

[6]The only exception is Lien, Zheng and Zhong (2017). However, they do not study optimal mechanisms.

[7]In our model, where all universities can be interpreted as having the same ranking of students, the DA mechanism is equivalent to the Top Trading Cycles of Gale and Scarf (1974).

show that, when scores and talent are only imperfectly correlated, not only is the DA mechanism no longer optimal, but also that, in general, the optimal mechanism is neither (ex-post) efficient nor does it eliminate justified envy. The optimal mechanism is not efficient, because there might be students who are not assigned to any university despite there being vacancies at low-quality universities. It also does not eliminate justified envy because, as can be seen in Figure 1, a student with a low type may have a higher score than a student with a higher type and, nevertheless, be assigned to a lower quality university. These findings suggest that the focus on mechanisms that are efficient and eliminate justified envy might be detrimental for the social planner when signals are imperfect.

Related literature on imperfect evidence: Finally, the paper is also related to the recent literature on mechanism design with imperfect evidence, which generally focuses on single agent problems (Silva, 2019a, and Siegel and Strulovici, 2019). Silva (2019b) does consider multiple agents but each agent's problem ends up being independent from one another, unlike what happens in this paper, because the measure of high-quality objects is smaller than the measure of agents.

In the next section, we discuss a very simple example based on the college admissions' application discussed above. In section 3, we present the general model. In section 4, we characterize the optimal mechanism. In section 5, we discuss the case of college admissions, while in section 6 we conclude. In the appendix, we briefly discuss the model with a finite number of agents. In particular, we show that we can construct a mechanism that converges to the optimal mechanism characterized in the main text as the number of agents grows. All the proofs are in the online appendix.

2 An illustrative example

The purpose of this example is to illustrate some of the intuitions behind our results in the context of the college admissions' application. Assume that there is a continuum of students of measure 1 and two types of universities; those with high quality (denoted as h) and those with low quality (denoted as l). There is only enough space for 50% of the students at the high-quality universities, but unlimited space at the low-quality universities (this is generalized in the text).

Each student privately observes his "talent" level θ. For the purposes of this example, let us say that each student's talent is either "high" ($\theta = \theta_H$) or "low" ($\theta = \theta_L$) with equal probability. Each student's payoff depends on his talent and on the university he is assigned to in the following way:

$$u(\theta_H, h) = 4, \quad u(\theta_H, l) = 2, \quad u(\theta_L, h) = 2, \quad u(\theta_L, l) = 1,$$

where $u(\theta_i, j)$ is the payoff when his type is $\theta_i \in \{H, L\}$ and he is assigned to university $j \in \{h, l\}$. If the agent is not assigned to any university, his payoff is normalized to 0. Notice that the marginal gain from increasing the university's quality is increasing with the student's talent.[8] Therefore, if one conceives of a social planner who wants to maximize the ex-ante expected utility of an arbitrary student, she should prefer an assortative matching; i.e., to assign the 50% more talented students to high-quality universities and the 50% less talented students to low-quality universities. In that case, each agents' ex-ante expected utility would be of 2.5.

The problem the social planner faces, however, is that she cannot observe talent. Instead, the social planner observes the score $s \in [0, 1]$ of an exam that students take (like the SAT or the GRE), which is imperfectly correlated with the students' talent. In particular, assume that the probability density function of s is:

$$p(s|\theta) = \begin{cases} 2s & \text{if } \theta = \theta_H \\ 2(1-s) & \text{if } \theta = \theta_L \end{cases}.$$

This means that, while more talented students are more likely to obtain better scores than less talented students ($p(\cdot|\theta_H)$ first order stochastically dominates $p(\cdot|\theta_L)$), there is no perfect correlation between talent and score.

Most mechanisms that have been used to assign students to universities ignore this imperfect correlation. Let us take as an example the deferred acceptance (DA) mechanism. In most countries where the DA mechanism is used it works as follows: after students have reported on their preferences over universities, an algorithm determines which university they are assigned to. The algorithm works in stages. In the first stage, it only considers for each university those students who have ranked it as their first option. It then tentatively assigns its seats to the students one at a time following the

[8] Indeed, going from being unassigned to being assigned to a low-quality university increases the more talented students' payoff by 2 but only increases the less talented students' payoff by 1; the same happens when going from being assigned to a low-quality university to being assigned to a high-quality university.

score of the exam. Any remaining students are rejected. In the second stage, for each university the algorithm compares the students who were conditionally accepted in the first stage with those who have ranked it second and have not been conditionally assigned to any other university. The process continues in this way until every candidate has either been conditionally accepted to a university or rejected by all universities in his list, at which point all conditional acceptances become real acceptances.

In our setting, where all students have the same preferences over universities, this algorithm would be completed in two stages: in the first stage, the students with the 50% highest scores would be accepted by the high-quality universities, while, in the second stage, everyone else would be accepted by the low-quality universities. Clearly, if talent and score were perfect correlated, this system would be optimal as it would implement the social planner's preferred outcome. However, because they are not, regardless of their talent level, students are assigned to a high-quality university only if their score exceeds threshold $\bar{s}_{DA} = 0.5$ and are assigned to a low-quality university otherwise. This means that the more talented students only have a 75% chance of being assigned a high-quality university, while less talented students have a 25% chance. Therefore, the ex-ante expected payoff of any given student is

$$\frac{1}{2} * \left(\frac{3}{4} * 4 + \frac{1}{4} * 2\right) + \frac{1}{2} * \left(\frac{1}{4} * 2 + \frac{3}{4} * 1\right) = \frac{19}{8}.$$

As we show in the text, the **DA mechanism is not optimal** (Proposition 1).[9] Moreover, in the main result we **characterize the optimal mechanism** (Theorem 1), which we have described in the introduction. In this example, it has the following form.

Optimal mechanism. Before completing their exams, students are asked to choose between tracks A and B. If they choose **track A**, they are assigned to a high-quality university if their score s is above $\bar{s}^A \simeq 0.36$, to a low-quality university if s is between $\bar{s}^A$ and $\underline{s}^A \simeq 0.15$, but will not be assigned any university if their score s is below $\underline{s}^A$. If instead, they choose **track B** they are guaranteed to be assigned at least to a low-quality university, and are assigned to a high-quality university if their score is above $\bar{s}^B \simeq 0.64$.

In the next sections, we discuss in detail why this is an optimal mechanism. However, this example is sufficient to understand why it is preferred to the DA mechanism.

[9]There is another popular mechanism called the Boston mechanism (also known as the Immediate Acceptance mechanism), which induces the same allocation as the one induced by the DA mechanism and is also not optimal.

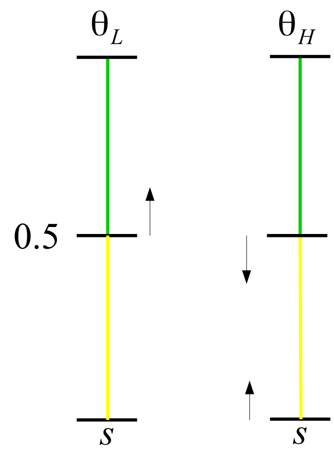

Figure 2: In the DA mechanism, students are assigned a high-quality university (the green area) if their score is above the threshold at 0.5, and are assigned to a low-quality university (the yellow area) otherwise.

The new mechanism we present induces students to self-select; those who are more talented prefer track A, while those who are less talented prefer track B. As a result, whether a student is assigned to a university depends not only on his score but also on his actual talent level.

Figure 2 presents how students are assigned in the DA mechanism, and the arrows represent what changes in the new mechanism. Two things happen. On the one hand, the threshold to access high-quality universities goes up for the students who are less talented (from 0.5 to 0.64) and down for those who are more talented (from 0.5 to 0.36). The social planner is made better off by this change because she prefers to assign those who are more talented to the universities with more quality. On the other hand, the threshold to access low-quality universities goes up (from 0 to 0.15) for the more talented students, which means that more talented students are no longer guaranteed to be assigned a university. While this second effect is necessary to prevent the less talented students from mimicking the more talented students and choosing track A, it makes the social planner worse off. In the text, we show that under certain (quite general) conditions that are satisfied in this example, the first effect dominates. Indeed, a more talented student has an 87% chance of being assigned to a high-quality university, an 11% chance of being assigned to a low-quality university and only a 2% chance of being unassigned, while those numbers for a less talented student are 13%, 87% and 0%, respectively. Therefore, the ex-ante expected payoff of a student is

$2.415 > \frac{19}{8}$.

Finally, this example also illustrates that, in contrast to the majority of mechanisms studied in the literature or used in practice, **the optimal mechanism has justified envy** (proposition 3) - for example, a more talented student who ends up with a score of 0.5 will be assigned to a high-quality university while a less talented student with a score of 0.6 will be assigned to a low-quality university - and, in general, **is ex-post inefficient** (proposition 4) - the more talented students who have a score lower than 0.15 will not be assigned even though there is enough space for them in the low-quality universities.

3 Model

3.1 Fundamentals

There is a continuum of agents of mass 1 and a continuum of objects to be assigned to the agents. Each object can be of high (h) or low (l) quality. There is a measure $\alpha_h \in (0,1)$ of high-quality objects and a measure $\alpha_l \in (0,1)$ of low-quality objects. Each agent has a private type $\theta \in \Theta$, where $\Theta = \{\theta_1, .., \theta_J\} \subset \mathbb{R}$. Each θ is independent and identically distributed across agents and the prior probability of each type $\theta \in \Theta$ is denoted by $q(\theta) \in (0,1)$.[10] Without loss of generality, we assume that $\theta_{j+1} > \theta_j$ for all $j = 1, ..., J-1$. Each agent generates a public signal $s \in [0,1]$, which is only correlated with that agent's type θ. Denote the conditional density of s given θ by $p(s|\theta)$ and assume that it is continuous. We also assume that $\frac{p(s'|\theta)}{p(s|\theta)}$ is strictly increasing with θ for all $s' > s$, i.e., densities $\{p(\cdot, \theta) : \theta \in \Theta\}$ have the strict monotone likelihood ratio property. This guarantees that larger types are the ones that are more likely to generate larger signals.

Each agent's payoff depends on his type and on the quality of the object he is assigned. When an agent of type θ is assigned the high-quality object, his payoff is denoted by $u(\theta, h)$; if he is assigned the low-quality object it is $u(\theta, l)$. If the agent is not assigned any object, his payoff is normalized to 0. We assume that all agents have the same ordinal preferences over objects - $u(\theta, h) > u(\theta, l) > 0$ for all $\theta \in \Theta$ - and

[10] We also interpret $q(\theta)$ as the fraction of agents of type θ. We rely on the argument of Judd (1985) in order to identify probabilities with fractions when the population is a continuum.

that type and quality are complements.[11] Formally, we assume that i) the marginal benefit of quality is strictly increasing with θ, i.e., $u(\theta, l)$ and $(u(\theta, h) - u(\theta, l))$ are both strictly increasing in θ, and ii) $\frac{u(\theta,h)}{u(\theta,l)}$ is weakly increasing with θ. To understand the meaning of condition ii), notice that the expected payoff of an agent is given by

$$u(\theta, h) \Pr\{\text{receiving the } h \text{ object}\} + u(\theta, l) \Pr\{\text{receiving the } l \text{ object}\},$$

which is proportional to

$$\frac{u(\theta, h)}{u(\theta, l)} \Pr\{\text{receiving the } h \text{ object}\} + \Pr\{\text{receiving the } l \text{ object}\}.$$

Therefore, condition ii) implies that larger types value receiving the high-quality object relative to receiving the low-quality object weakly more than lower types.[12] A simple example that the reader might want to keep in mind is the following: $u(\theta, h) = \theta h$ and $u(\theta, l) = \theta l$, with $\theta > 0$ for all $\theta \in \Theta$ and $h > l > 0$.[13]

To summarize, our setting is basically an auction setting with products of different quality but where there are no transfers; instead incentives are given through the agents' signals.

3.2 Definitions

Our goal is to find the optimal mechanism for the social planner. By the revelation principle, it is enough to consider only revelation mechanisms, i.e., allocations that are incentive compatible. We focus on symmetric allocations. A symmetric allocation (henceforth, simply allocation) is a mapping $x = (x_h, x_l) : \Theta \times [0, 1] \to [0, 1]^2$ such that

$$x_h(\theta, s) + x_l(\theta, s) \leq 1$$

[11]In the school choice context, there is evidence that students' preferences are highly correlated (Abdulkadiroglu, Che, Yasuda, 2011) and that parents value similar things (Bosetti, 2004). A similar assumption was made in Akin (2019) and in Lien, Zheng and Zhong (2017) for example.

[12]Notice that if $u(\theta, l)$ is strictly increasing and $\frac{u(\theta,h)}{u(\theta,l)}$ is weakly increasing, $(u(\theta, h) - u(\theta, l))$ is strictly increasing, so we only really need to assume the first two.

[13]For example, in a college admissions' setting, one can think of $\theta \in (0, 1)$ as the probability that the student completes his studies and h and l as the discounted sum of future earnings after completing a degree in high and low-quality universities respectively.

for all $\theta \in \Theta$ and $s \in [0,1]$, where $x_h(\theta, s)$ and $x_l(\theta, s)$ represent the probability that an agent with type θ and signal s is assigned object h and l respectively.[14]

An allocation x is *feasible* if the measure of assigned objects does not exceed the measure of available objects, i.e., if

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_h(\theta, s)\, ds \leq \alpha_h$$

and

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_l(\theta, s)\, ds \leq \alpha_l.$$

An allocation x is *incentive compatible* (IC) if each agent prefers to report truthfully, i.e., for all $\theta \in \Theta$,

$$\theta \in \arg\max_{\theta' \in \Theta} U(\theta, x(\theta')),^{15}$$

where

$$U(\theta, z) \equiv \int_0^1 p(s|\theta)\,(z_h(s)\, u(\theta, h) + z_l(s)\, u(\theta, l))\, ds$$

for any $z = (z_h, z_l) : [0,1] \to [0,1]^2$. Notice that each agent reports his type *before* observing his signal. That is how incentives are given to the agents; different types have different beliefs about signal s.

An allocation x is *ordered* if, for all $\theta \in \Theta$, there is $\underline{s}_\theta, \overline{s}_\theta$ such that $0 \leq \underline{s}_\theta \leq \overline{s}_\theta \leq 1$ and

$$x_h(\theta, s) = \begin{cases} 1 \text{ if } s \geq \overline{s}_\theta \\ 0 \text{ if } s < \overline{s}_\theta \end{cases} \text{ and } x_l(\theta, s) = \begin{cases} 1 \text{ if } s \in [\underline{s}_\theta, \overline{s}_\theta] \\ 0 \text{ if } s \notin [\underline{s}_\theta, \overline{s}_\theta] \end{cases}.$$

In an ordered allocation, the only randomness an agent of some type θ faces comes from the signal s, i.e., conditional on his type and on the signal, there is no random-

[14]The assumption that there is a continuum of agents implies that the probability of being assigned either object only depends on the agent's type and signal and not on everyone else's type and signal. By assuming there is a continuum of agents, we are assuming that each agent is only uncertain about whether their signal will reflect their type; they are not uncertain about how large their type is relative to others. It is a relatively standard assumption that is also made in Li (2019a), Avery and Levin (2010) and Chade, Lewis and Smith (2014) for example. Nevertheless, in the appendix, we discuss the case with finite agents and construct an allocation that approximates the optimal allocation that we characterize in the main text when the number of agents is sufficiently large.

[15]In order to make the notation lighter we write $x(\theta) \equiv x(\theta, \cdot) : [0,1] \to [0,1]^2$. That is, for a fixed type θ, $x(\theta)$ gives the probability that an agent of type θ is assigned object h and l as a function of the signal.

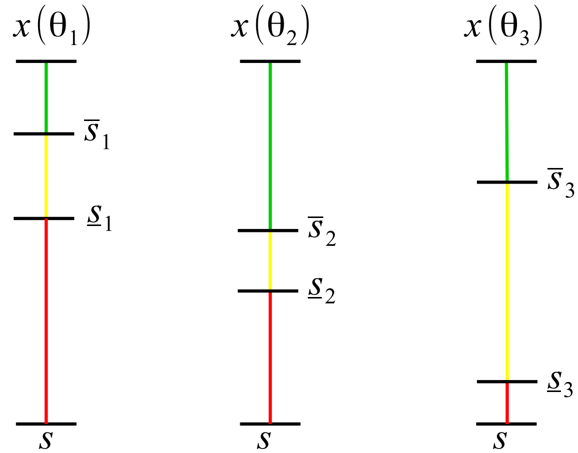

Figure 3: An example of an ordered allocation. This particular ordered allocation is not IC because type θ_1 would prefer to report to being type θ_2.

ization. Furthermore, the agent always prefers to have a larger signal than a lower signal; the rewards are at the top. Figure 3 presents an example of an ordered allocation. Notice that an ordered allocation is completely characterized by its thresholds $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta\in\Theta}$.

Finally, we assume that the social planner wants to maximize the ex-ante expected utility of each agent. Let $W(x)$ denote the welfare of allocation x and define it as

$$W(x) \equiv \sum_{\theta\in\Theta} q(\theta)\, U(\theta, x(\theta)).$$

We say that an allocation x is optimal if it maximizes W.

4 Optimal allocations

In this section, we characterize an optimal feasible IC allocation.

Theorem. *There is an ordered allocation $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta\in\Theta}$ that is an optimal feasible IC allocation. It has the following properties: i) $\overline{s}_\theta$ is weakly decreasing; ii) $\underline{s}_\theta$ is weakly increasing; iii) type θ_j is indifferent to reporting to being type θ_{j+1} for all $j < J$.*

Figure 1 of the introduction displays the optimal allocation when there are three types. Below, we guide the reader through the argument. All the more technical details are left for the online appendix.

4.1 The single-crossing problem

At first glance, the problem of finding optimal feasible IC allocations might appear relatively standard. Recall that the agent's expected utility is given by

$$u(\theta, h) \Pr\{\text{receiving the } h \text{ object}\} + u(\theta, l) \Pr\{\text{receiving the } l \text{ object}\},$$

where the two goods - the probability of being assigned the high-quality object and the probability of being assigned the low-quality object - enter linearly. Furthermore, the condition that $\frac{u(\theta,h)}{u(\theta,l)}$ is increasing (we only assume it is weakly increasing, but, for the sake of argument, say it is strictly increasing) looks a lot like the typical single crossing condition that is standard in mechanism design. So, the problem appears to be a variation of Myerson and Satterthwaite (1983).[16] However, unlike what is standard in mechanism design, the fact that there is "evidence" in the problem makes it so that the probability of receiving each good depends not only on the agent's report but also on his true type (the probability that an agent receives the high-quality object for example might depend on the signal that is realized, whose distribution depends on the agent's true type). As it turns out, this complicates matters considerably, because incentive compatibility no longer implies that types that are closer together receive distributions of goods that are also closer. For example, in Myerson and Satterthwaite (1983), who study bilateral trade, and in much of the literature that followed, larger types are more likely to receive the object that is being traded than lower types in *every* incentive compatible allocation. By contrast, as we illustrate below, in our setting, there are incentive compatible allocations where the probability of receiving the high quality object is not monotone with the agent's type.

Consider the following example. Say that θ belongs to $\{\theta_1, \theta_2, \theta_3\}$ with $\theta_i = i$ for $i = 1, 2, 3$, $u(\theta, h) = 2\theta$ and $u(\theta, l) = \theta$, and:

$$p(s|\theta) = \begin{cases} 2s & \text{if } \theta = \theta_3 \\ 1 & \text{if } \theta = \theta_2 \\ 2(1-s) & \text{if } \theta = \theta_1 \end{cases}$$

Consider the following allocation, illustrated in Figure 4:

[16]Indeed, a similar version of this model without evidence is studied by Hafalir and Miralles (2015).

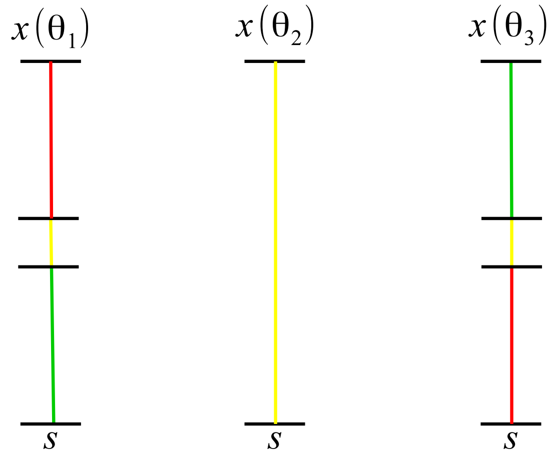

Figure 4: Example of an incentive compatible allocation where types θ_1 and θ_3 have the same probability of being assigned the high-quality object and the same probability of being assigned the low-quality object, while type θ_2 is assigned the low-quality object with certainty.

$$x_h(\theta_1, s) = \begin{cases} 1 \text{ if } s \leq 0.4 \\ 0 \text{ if } s > 0.4 \end{cases} \text{ and } x_l(\theta_1, s) = \begin{cases} 1 \text{ if } 0.4 \leq s \leq 0.6 \\ 0 \text{ if } s > 0.6 \end{cases}.$$

$$x_l(\theta_2, s) = 1 \text{ for all } s \in [0, 1].$$

$$x_h(\theta_3, s) = \begin{cases} 1 \text{ if } s \geq 0.6 \\ 0 \text{ if } s < 0.6 \end{cases} \text{ and } x_l(\theta_3, s) = \begin{cases} 1 \text{ if } 0.4 \leq s \leq 0.6 \\ 0 \text{ if } s < 0.4 \end{cases}.$$

It is straightforward to check that, not only is this allocation incentive compatible, but also that the distribution of objects that are assigned to type θ_1 is exactly the same as type θ_3 (both types have a 64% probability of being assigned the high-quality object and a 20% probability of being assigned the low-quality object) but very different than type θ_2 (type θ_2 is assigned the low-quality object with certainty).

In order to overcome this difficulty, we first show that we would not have this problem if we were to restrict attention to the class of ordered allocations and then show that there are ordered allocations which are optimal.

4.2 Single crossing and ordered allocations

In ordered allocations, we can reinterpret the problem by thinking of the two goods as being the two thresholds $\overline{s}$ and $\underline{s}$ (instead of the probability of receiving the high quality object and the low quality object, respectively). The advantage of framing the problem in this manner is that the thresholds $\overline{s}$ and $\underline{s}$ the agent is assigned depend only on his report and not on his true type. Specifically, in any ordered allocation, we can write the expected utility of any given agent of type θ when reporting θ' as

$$\widehat{U}\left(\theta, \overline{s}_{\theta'}, \underline{s}_{\theta'}\right) \equiv u\left(\theta, h\right) \int_{\overline{s}_{\theta'}}^{1} p\left(s|\theta\right) d\theta + u\left(\theta, l\right) \int_{\underline{s}_{\theta'}}^{\overline{s}_{\theta'}} p\left(s|\theta\right) d\theta.$$

Notice that $\widehat{U}\left(\theta, \overline{s}, \underline{s}\right)$ is decreasing with both $\overline{s}$ and $\underline{s}$, so if we were to draw indifference curves of the different types on the space $\left(\underline{s}, \overline{s}\right)$ they would be downward sloping. Using the property that $\frac{p(s'|\theta)}{p(s|\theta)}$ is strictly increasing with θ for all $s' > s$, we are able to show that those indifference curves cross at most once as figure 5 illustrates. The intuition is that larger types are more confident that their signals will be above the upper thresholds. So, if a lower type is indifferent between any two tracks, a larger type will prefer the track with the lower upper threshold. In the case of figure 1, this implies for example that if some type prefers the third (second) track over the second (third) track, all larger (lower) types will also prefer the third (second) track over the second (third) track.

Formally, we have the following lemma:

Lemma 1. *Take any ordered allocation x and any two types $\theta' \in \Theta$ and $\theta'' \in \Theta$ such that such that $\overline{s}_{\theta'} > \overline{s}_{\theta''} \geq \underline{s}_{\theta''} > \underline{s}_{\theta'}$. It follows that for all $\theta \in \Theta$,*

$$U\left(\theta, x\left(\theta'\right)\right) \geq U\left(\theta, x\left(\theta''\right)\right) \Rightarrow U\left(\widehat{\theta}, x\left(\theta'\right)\right) > U\left(\widehat{\theta}, x\left(\theta''\right)\right)$$

for all $\widehat{\theta} < \theta$ and

$$U\left(\theta, x\left(\theta''\right)\right) \geq U\left(\theta, x\left(\theta'\right)\right) \Rightarrow U\left(\widehat{\theta}, x\left(\theta''\right)\right) > U\left(\widehat{\theta}, x\left(\theta'\right)\right)$$

for all $\widehat{\theta} > \theta$.

The previous lemma makes it much easier to deal with ordered allocations (as compared to general allocations) because it implies that one only needs to be concerned

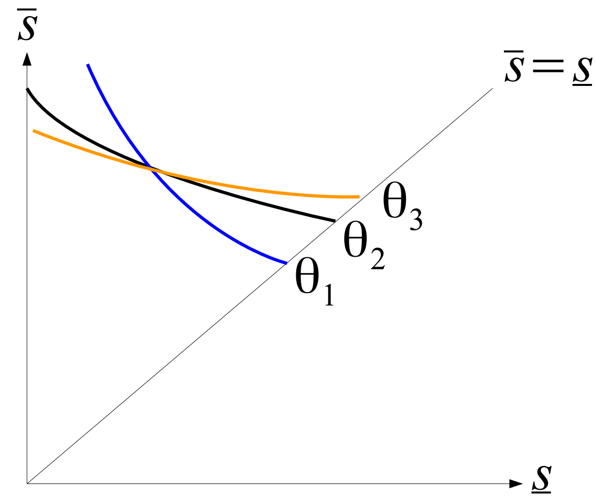

Figure 5: Indifference curves for three types which only cross once. Note that pairs nearer the origin are preferred to pairs away from the origin.

with "local" incentive constraints. Indeed, the way to find the optimal ordered allocation is precisely through a series of arguments of a local nature. Of course, one still has to show that ordered allocations are optimal, which we do in the following sections.

4.3 The relaxed problem

The optimal allocation maximizes $W(x)$ subject to the i) feasibility conditions, ii) upper incentive constraints, i.e., for all θ,

$$U(\theta, x(\theta)) \geq U(\theta, x(\theta')) \text{ for all } \theta' > \theta,$$

and iii) lower incentive constraints, i.e., for all θ,

$$U(\theta, x(\theta)) \geq U(\theta, x(\theta')) \text{ for all } \theta' < \theta.$$

We define the relaxed problem as maximizing $W(x)$ subject only to i) and ii). We start by showing that there is an ordered allocation that solves the relaxed problem.

Lemma 2. *Let x be any allocation that satisfies all the incentive constraints of the* *relaxed problem. Let $\widehat{x}$ be an ordered allocation with thresholds $\{(\underline{s}_\theta, \overline{s}_\theta)\}_{\theta\in\Theta}$ such that*

$$\int_{\overline{s}_\theta}^{1} p(s|\theta)\, ds = \int_{0}^{1} x_h(\theta, s)\, p(s|\theta)\, ds$$

and

$$\int_{\underline{s}_\theta}^{\overline{s}_\theta} p(s|\theta)\, ds = \int_{0}^{1} x_l(\theta, s)\, p(s|\theta)\, ds$$

for all $\theta \in \Theta$. It follows that $\widehat{x}$ satisfies all the incentive constraints of the relaxed problem.

Allocation $\widehat{x}$ in lemma 2 is such that the probability that each type is assigned each object is the same as in allocation x. The only difference is that the rewards are all "brought up to the top". Therefore, by definition, $W(x) = W(\widehat{x})$. To see why allocation $\widehat{x}$ satisfies all upper incentive constraints, it might be convenient to go through the finite steps of transforming allocation x into allocation $\widehat{x}$. Take allocation x and reorder only type θ_1's track as described in the lemma; call that allocation x^1. It follows that allocation x^1 satisfies all incentive constraints because the only incentive constraints considered that involve type θ_1 are the ones that prevent him from mimicking higher types. Seeing as his expected utility is the same under allocations x^1 and x, those incentive constraints are satisfied.

Now, do the same reordering with type θ_2 and call the corresponding allocation x^2. Once again, by the same reason, type θ_2 does not want to mimic any larger type under allocation x^2, so we only need to show that type θ_1 does not want to mimic type θ_2. That is the case because lower types are less likely to draw large signals; therefore, if type θ_2 is made indifferent by bringing all his rewards up, lower types would be made worse off as a result. By continuing with this logic for all the J types, we get to allocation $\widehat{x}$.

To find the optimal allocation, we solve for the optimal ordered allocation of the relaxed problem. We are able to completely characterize it because of the single crossing property that ordered allocations have; in particular, our arguments are all of a "local" nature as we describe next. The final part of the argument is to show that the optimal ordered allocation of the relaxed problem satisfies the lower incentive constraints.

4.4 The optimal ordered allocation of the relaxed problem

Lemma 3. *Any ordered allocation x that solves the relaxed problem is such that i) $\overline{s}_\theta$ is weakly decreasing; ii) $\underline{s}_\theta$ is weakly increasing; iii) type θ_j is indifferent to reporting to being type θ_{j+1} for all $j < J$.*

To prove the statement, we use an argument by induction: take any two consecutive types θ_j and θ_{j+1} and, for ease of exposition, assume that $\left(\overline{s}_{j+1}, \underline{s}_{j+1}\right) \neq \left(\overline{s}_{j+2}, \underline{s}_{j+2}\right)$. By induction, assume that $\overline{s}_{j+k}$ is weakly decreasing with k for $k \geq 1$, that $\underline{s}_{j+k}$ is weakly increasing with k for $k \geq 1$ and that $U\left(\theta_{j+k}, x\left(\theta_{j+k}\right)\right) = U\left(\theta_{j+k}, x\left(\theta_{j+k+1}\right)\right)$ for all $k \geq 1$. We show that $\overline{s}_j \geq \overline{s}_{j+1}$, $\underline{s}_j \leq \underline{s}_{j+1}$ and $U\left(\theta_j, x\left(\theta_j\right)\right) = U\left(\theta_j, x\left(\theta_{j+1}\right)\right)$. We do so in two steps (the details are in the online appendix).

The first step is to show that if $\underline{s}_j \leq \underline{s}_{j+1}$, then $U\left(\theta_j, x\left(\theta_j\right)\right) = U\left(\theta_j, x\left(\theta_{j+1}\right)\right)$. The argument is as follows: Suppose that $\underline{s}_j \leq \underline{s}_{j+1}$ and that $U\left(\theta_j, x\left(\theta_j\right)\right) > U\left(\theta_j, x\left(\theta_{j+1}\right)\right)$. By lemma 1, it follows that $U\left(\theta, x\left(\theta_j\right)\right) > U\left(\theta, x\left(\theta_{j+1}\right)\right)$ for all $\theta < \theta_j$, so that no type mimics type θ_{j+1}. In that case, it is always possible to transfer some objects from type θ_j to type θ_{j+1} in a way that satisfies the considered incentive constraints, which, by the way our welfare function is constructed, increases welfare. Specifically, if $\underline{s}_{j+1} > 0$, one could raise $\underline{s}_j$ by some small $\varepsilon > 0$ and lower $\underline{s}_{j+1}$ by some $\delta\left(\varepsilon\right) > 0$, where $\delta\left(\varepsilon\right)$ is such that the measure of low-quality objects used remains the same. If, instead, $\underline{s}_{j+1} = 0$, one could do a similar ε-transfer but with the high-quality objects.

The second step is to show that it must be that $\underline{s}_j \leq \underline{s}_{j+1}$. Suppose not, so that $\underline{s}_j > \underline{s}_{j+1}$ and, as a consequence, $\overline{s}_j < \overline{s}_{j+1}$. By a similar argument from before, we can show that some type $\widehat{\theta} \leq \theta_j$ must be indifferent to mimicking type θ_{j+1}; otherwise, we could just do the ε-transfers of objects from lower types to type θ_{j+1} of the previous paragraph. The contradiction is found by perturbing allocation x as follows: raise $\overline{s}_j$ by a small $\varepsilon > 0$, lower $\overline{s}_{j+1}$ by $\delta\left(\varepsilon\right)$, raise $\underline{s}_{j+1}$ by $\gamma\left(\varepsilon\right)$ and lower $\underline{s}_j$ by $\beta\left(\varepsilon\right)$. Choose $\delta\left(\varepsilon\right)$, $\gamma\left(\varepsilon\right)$ and $\beta\left(\varepsilon\right)$ such that the measure of objects being assigned remains constant and type $\widehat{\theta}$ is made indifferent between reporting θ_j and θ_{j+1} if $\widehat{\theta} = \theta_j$; if not, make type $\widehat{\theta}$ be indifferent to reporting θ_{j+1} before and after the perturbation. Notice that this perturbation leaves us with a feasible allocation and improves welfare, because, essentially, one is just shifting the objects of better quality to the larger types at the expense of the lower types. The argument is completed in the online appendix, where we show that the perturbed allocation satisfies all considered incentive constraints because of lemma 1.

The final step of the proof is to show that the ordered allocation that solves the relaxed problem satisfies the relaxed incentive constraints.

Lemma 4. *Let x be an ordered allocation that solves the relaxed problem. Then x is also an optimal feasible IC allocation.*

Lemma 4 directly follows from lemmas 1 and 3 as can be seen by considering figure 1. If type θ_1 is indifferent to mimicking type θ_2, it follows by lemma 1 that type θ_2, who is more confident that he can land in the green zone, strictly prefers to report θ_2 over mimicking type θ_1. By the same reasons, type θ_3 strictly prefers to report θ_3 over reporting θ_2 and strictly prefers that over reporting θ_1.

5 The case of college admissions

The college admissions' problem has been studied for decades by the matching literature. The basic problem is how to assign a set of students to a set of universities such that each student is assigned at most one university and each university does not exceed its maximum capacity. The goal is to design a mechanism to be run by a central clearinghouse that assigns students based on their reported preferences and on each university's priorities. Priorities are assumed to be based almost exclusively on the students' scores on centralized exams. A student with a higher score is said to have priority over another student who is ranked lower.

The approach that the matching literature has followed has been to propose mechanisms that have certain desirable properties like stability, efficiency or incentive compatibility (usually referred to as strategy proofness). In the college admissions' literature, the most used stability concept is that of the elimination of justified envy; a mechanism eliminates justified envy if there is no student with a higher priority who prefers the assignment of another student with a lower priority. Some of the most well-known mechanisms that have been proposed are: the deferred acceptance (DA) mechanism, the immediate acceptance or Boston mechanism, and the top trading cycles mechanism.[17] Our approach is different: we specify an objective function and look

[17]The DA mechanims was first introduced by Gale and Shapley (1962) and then adapted to the college admissions' problem by Balinski and Sonmez (1999), and to the school choice problem by Abdulkadiroglu and Sonmez (2003). The immediate acceptance mechanism was used for some years in the city of Boston, and is described in Abdulkadiroglu and Sonmez (2003). Finally, the top trading cycles was introduced by Shapley and Scarf (1974) and is also discussed by Abdulkadiroglu and Sonmez (2003).

for the optimal incentive compatible allocation, which we have characterized previously. In this section, we discuss it in the context of the matching literature on college admissions.

5.1 The suboptimality of the DA mechanism

Our first result is that the optimal mechanism is not the DA mechanism or any of the other mechanisms discussed before in the matching literature. At first glance, this finding might unsettle the reader; one could argue that while the matching literature does not attempt to maximize the same objective function we do per se, classical mechanisms like the DA mechanism should perform fairly well under any reasonable criterion. Indeed, as discussed in the example of section 2, what makes the DA mechanism and others sub-optimal in our setting is that we explicitly model the imperfect correlation between the students' talent and the publicly available signals of talent.

Since we assume that all students have the same preferences over schools, the DA mechanism can simply be defined as follows:[18]

Definition 1. *The DA mechanism is such that there is a single track with thresholds $\bar{s}_{DA}$ and $\underline{s}_{DA}$, where $\bar{s}_{DA}$ is such that*

$$\sum_{\theta \in \Theta} q(\theta) \int_{\bar{s}_{DA}}^{1} p(s|\theta) ds = \alpha_h,$$

and $\underline{s}_{DA}$ is such that

$$\sum_{\theta \in \Theta} q(\theta) \int_{\underline{s}_{DA}}^{\bar{s}_{DA}} p(s|\theta) ds = min\{\alpha_l, 1 - \alpha_h\}.$$

The DA mechanism induces the DA allocation described below.[19]

Definition 2. *The DA allocation is an ordered allocation $\{(\underline{s}_\theta, \bar{s}_\theta)\}_{\theta \in \Theta}$ such that $\underline{s}_\theta = \underline{s}_{DA}$, and $\bar{s}_\theta = \bar{s}_{DA}$, for all $\theta \in \Theta$.*

In the following proposition, we show that the DA mechanism is not optimal because it does not induce students to self-select.

Proposition 1. *(DA is not optimal) The DA allocation is not an optimal feasible IC allocation if*

[18] See the example of section 2 for a more thorough description of the DA mechanism.

[19] The same allocation is also induced by the top trading cycles' mechanism and the Boston mechanism.

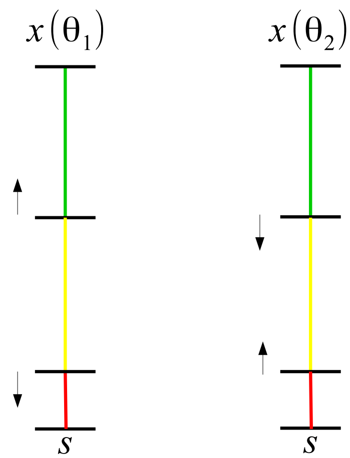

Figure 6: Comparison between the DA allocation and the optimal allocation

i) $\alpha_l + \alpha_h < 1$, *or*
ii) $\alpha_l + \alpha_h \geq 1$, *and* $p(0|\theta_1) > p(0|\theta_j) = 0$ *for all* $j > 1$.

To see why that the DA allocation is not optimal, let us assume that $\Theta = \{\theta_1, \theta_2\}$. When the DA mechanism is applied, all types face the same track with the same two thresholds as displayed in figure 6. The optimal allocation characterized in the previous section transfers some of the seats at the high-quality university from the less talented students (type θ_1) to the more talented students (type θ_2). This is done by raising the upper threshold of the track of the less talented students and lowering the upper threshold of the more talented students. That is, some of the less talented students who would have been lucky enough to have had a score above the upper threshold in the DA mechanism get replaced by some of the more talented students who would have been unlucky to have had a score below that threshold. Naturally, by changing the allocation in this manner, one induces less talented students to mimic the more talented students. To prevent that, one must lower the lower threshold of the less talented students and raise the lower threshold of the more talented students just enough to make the less talented students indifferent. As a result, we end up with an allocation where the same measure of objects is being assigned as in the DA allocation; the difference is that more seats at the high-quality university are assigned to the more talented students, which improves welfare.

When $\alpha_l + \alpha_h \geq 1$, the DA allocation is such that every agent is assigned an object just like in the example (see figure 2). In that case, the same exact argument does not follow because one cannot lower the lower threshold of the less talented students; the increase of the lower threshold of the more talented students must be enough to prevent the less talented students from mimicking. Therefore, altering the allocation in this manner generates an inefficiency: some agents will not be assigned any object. The condition of part ii) of proposition 1 (which holds for the case of the example) ensures that this inefficiency is small enough to make the DA allocation suboptimal; effectively, it guarantees that raising the lower threshold of the more talented students has a very large dissuading effect on the incentives of the less talented students to mimic and a very small impact on the more talented student's expected payoff (we revisit the issue of ex-post inefficiency below).

5.2 The optimality of the DA mechanism when the objects have the same quality

While in general the DA mechanism is not optimal, we show in this section that there are circumstances in which it is. In particular, we show that when all universities have the same quality the DA mechanism is optimal. To that end, we define what we call *full* allocations as allocations where every agent is assigned a university (which naturally requires that $\alpha_l + \alpha_h \geq 1$).

Definition 3. *An allocation x is called full if it is feasible and every student is assigned to a university, i.e., $x_h(\theta, s) + x_l(\theta, s) = 1$ for all $(\theta, s) \in \Theta \times [0,1]$.*

Notice that if one considers only the set of full allocations, one essentially considers a problem with homogeneous objects: each agent is either assigned a high-quality object or he is not (and is assigned a low-quality object). In the college admissions' problem, we can reinterpret the model by saying that being assigned the high-quality object is equivalent to being assigned to a university, while not being assigned the high-quality object is equivalent to not being assigned to any university. This problem is then the natural extension to imperfect evidence of the mechanism design literature described in the introduction, which focuses on homogeneous objects (for example, Ben-Porath, Dekel and Lipman, 2014).

Proposition 2. *The DA allocation is optimal among all full IC allocations.*

Recall from proposition 1 that the DA allocation is not optimal even when there is enough capacity to assign every student ($\alpha_l + \alpha_h \geq 1$); in the proof of proposition 1 we build another allocation that does better. However, that alternative allocation is not

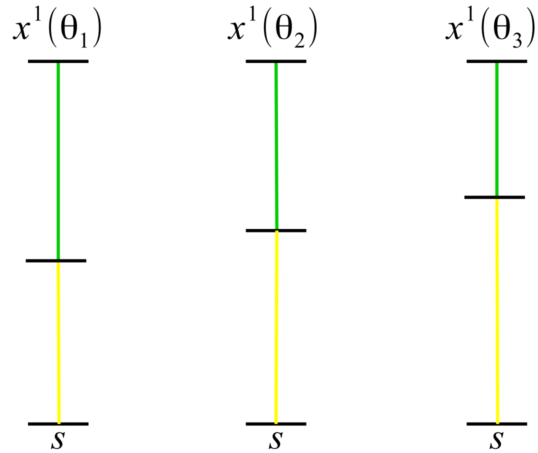

Figure 7: Example of an ordered allocation that does not solve the considered relaxed problem

full; there are some students who are left unassigned. Indeed, as proposition 2 shows, the DA allocation is optimal if one is obliged to assign every student.

In order to prove the result, we consider a relaxed problem where the only incentive constraints that are considered are the upper incentive constraints. Just like in the proof of the theorem, one can reorder every type's track by pushing all the rewards to the top and still satisfy all the incentive constraints considered. Call the optimal ordered allocation x^1 and notice that, because those incentive constraints are still satisfied, it must be that the corresponding thresholds $\overline{s}_\theta$ are weakly increasing with θ. The argument is completed by showing that $\overline{s}_\theta$ is in fact constant with θ, because in that case, allocation x^1 is just the DA allocation.

Suppose $\overline{s}_\theta$ was not constant with θ, so that it was something like what is displayed in figure 7. Because type θ_1 is not indifferent to choosing type θ_2's track, the social planner could raise $\overline{s}_1$ by $\varepsilon > 0$ and lower $\overline{s}_2$ by $\delta(\varepsilon)$, where $\delta(\varepsilon)$ is such that the measure of high-quality objects that is assigned remains the same. Provided ε is small, this change would increase welfare and satisfy all considered incentive constraints, a contradiction to the optimality of x^1.

5.3 Elimination of justified envy

There are two properties of mechanisms that are often considered desirable by the matching literature: the elimination of justified envy and (ex-post) efficiency. As a consequence, not much attention has been devoted to mechanisms that do not fulfil at least one of these two properties. Moreover, in our context, all the standard mechanisms (like the DA mechanism, the Boston mechanism or the top trading cycles mechanism) eliminate justified envy and are ex-post efficient. In light of this, we find it interesting that, in general, the optimal mechanism has justified envy and might be ex-post inefficient.

In the case of ordered allocations, the elimination of justified envy would imply that all thresholds be constant.

Definition 4. *An ordered allocation* $\{(\underline{s}_\theta, \bar{s}_\theta)\}_{\theta \in \Theta}$ *eliminates justified envy if, and only if,* $\underline{s}_\theta$ *and* $\bar{s}_\theta$ *are constant with* θ.

Proposition 3. *The optimal feasible IC ordered allocation does not eliminate justified envy if* $\alpha_l + \alpha_h < 1$*, or if* $\alpha_l + \alpha_h \geq 1$ *and* $p(0|\theta_1) > p(0|\theta_j) = 0$ *for all* $j > 1$.

It is straightforward to see why the optimal mechanism does not eliminate justified envy: students of different talent self-select into tracks of different thresholds; as a result, it is possible that a less talented student with a larger score is assigned to the low-quality university, while a more talented student with a lower score is assigned to a high-quality university. The condition of the proposition simply ensures that the DA allocation, which does eliminate justified envy, is not optimal.

5.4 The trade-off between ex-post efficiency and optimality

When agents have the same preferences over objects (as in our case), ex-post efficiency is equivalent to non-wastefulness.

Definition 5. *An allocation* x *is called non-wasteful if*

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_h(\theta, s)\, ds = \alpha_h$$

and

$$\sum_{\theta \in \Theta} q(\theta) \int_0^1 p(s|\theta)\, x_l(\theta, s)\, ds = min\{\alpha_l, 1 - \alpha_h\}.$$

In words, an allocation is non-wasteful if the high-quality university is at capacity and either the low-quality university is also at capacity or no student is left unassigned. If the measure of low-quality objects is sufficiently large, we find that any optimal allocation is wasteful. In particular, we find that even though the high-quality university is always at capacity, the low-quality university will not be completely filled up despite there being students who are left unassigned.

Proposition 4. *Assume $p(0|\theta_1) > p(0|\theta_j) = 0$ for all $j > 1$. Then, for every $\alpha_h \in (0,1)$, there is some threshold $\overline{\alpha}_l \in (0, 1-\alpha_h)$ such that,*

i) for all $\alpha_l \leq \overline{\alpha}_l$, the optimal feasible IC allocation is non-wasteful,

ii) for all $\alpha_l > \overline{\alpha}_l$, the optimal feasible IC allocation is wasteful, because, even though all high-quality university are at capacity, there are both unassigned students and free space at the low quality universities.

The argument is as follows. Consider the problem when $\alpha_h + \alpha_l \geq 1$, i.e., the problem when there are enough vacancies for every student. In that case, we have seen that the optimal full allocation is the DA allocation (proposition 2) which, however, is not optimal (proposition 1). Seeing as non-wasteful allocations must be full whenever $\alpha_h + \alpha_l \geq 1$, it follows that the optimal allocation is wasteful. In particular, one can show that, while the high-quality university is always at capacity, there are some students who are not assigned to any university despite there being vacancies at the low-quality university. As a result, the statement follows by letting $\overline{\alpha}_l$ denote the measure of students assigned to the low-quality university when $\alpha_h + \alpha_l \geq 1$ under the optimal allocation (characterized in the theorem).

The fact that all optimal mechanisms are ex-post inefficient if the measure of low-quality universities is large suggests that the focus on efficient mechanisms might harm welfare. In particular, requiring ex-post efficiency limits the mechanism when it comes to providing incentives for students to self-select, a key feature of any optimal mechanism when there is imperfect correlation between types and signals.

5.5 The binary mechanism

One last concern we want to address is a practical one; the reader might worry that the optimal mechanism is too hard to implement, particularly in the college admissions' application. Recall that, in the optimal mechanism, before doing their exams, students are asked to choose one of many tracks, each with different standards of admission to the various universities. In principle, one could have as many tracks as there are types, so that number could be enormous.

In this section, we introduce a simpler class of mechanisms where agents can also self-select but only between two tracks. We call them *binary mechanisms*. Allocations that are implemented by binary mechanisms are called binary allocations.

Definition 6. *A binary allocation is an ordered allocation* $\{(\underline{s}_\theta, \bar{s}_\theta)\}_{\theta\in\Theta}$ *such that there exist* $(\underline{s'}_\theta, \bar{s'}_\theta)$ *and* $(\underline{s''}_\theta, \bar{s''}_\theta)$ *such that, for all* θ,

$$(\underline{s}_\theta, \bar{s}_\theta) = (\underline{s'}_\theta, \bar{s'}_\theta) \text{ or } (\underline{s}_\theta, \bar{s}_\theta) = (\underline{s''}_\theta, \bar{s''}_\theta).$$

In any binary mechanism, students self-select between one track with lower standards of admission to the better university and another with lower standards of admission to the lower quality university. Binary mechanisms seem eminently feasible - as we mentioned in the introduction, they are very similar to the college assignment mechanism that is used in Hungary - and, as stated below, generate allocations with a larger welfare than that generated by the DA mechanism, where students face a single track.

Proposition 5. *(Binary mechanisms dominate DA) There is a feasible and IC binary allocation that generates a larger welfare than the DA allocation if* $\alpha_l + \alpha_h < 1$, *or if* $\alpha_l + \alpha_h \geq 1$ *and* $p(0|\theta_1) > p(0|\theta_j) = 0$ *for all* $j > 1$.

6 Conclusion

In this paper, we have considered a basic auction setting with heterogeneous objects, but where transfers are replaced by evidence. We have shown how (imperfect) evidence can be used to elicit the private information held by the agents: agents have different incentives because they have different beliefs about what the evidence will be. In the college admissions' setting, the optimal mechanism we characterize generates a larger welfare than the standard mechanisms discussed in the matching literature (like the deferred acceptance mechanism), which depend only on the agents' signals (mostly exam scores and recommendation letters). Moreover, we show that optimal mechanisms do not eliminate justified envy and might not be (ex-post) efficient, which suggests that the focus on efficient mechanisms that eliminate justified envy might be misguided.

The fact that optimal mechanisms need not be efficient is also interesting in and of itself. While one could argue that mechanisms that are efficient could be implemented through decentralized systems, where each university decides independently what students to accept, it is much harder to see how an inefficient mechanism could be implemented in such a manner. In general (if the measure of low-quality universities is sufficiently large in the model), there will be students who are unassigned and universities with room in any optimal mechanism. It is hard to see how this could be the outcome of a decentralized system; surely, the university with unassigned vacancies would contact the unassigned students to have them attend the university. In that sense, our results contribute to the debate over the decentralization of colleges' admissions markets by demonstrating that there is a cost to decentralization.[20]

Finally, the reader might be concerned that the optimal mechanism gives an unfair advantage to risk loving agents. Indeed, all else the same, a more risk loving agent is more likely to be assigned the high-quality object (and also more likely of being unassigned). However, we believe that there is no problem with risk loving agents having an advantage in getting the high-quality object per se; the problem is that risk aversion might be correlated with other agents' characteristics. For example, there is suggestive evidence that low income students tend to be more risk averse (see, for example, Calsamiglia and Guell, 2018; and also Calsamiglia, Martinez-Mora and Miralles, 2020, for a theoretical analysis), so a reasonable concern is that the optimal mechanism further increases income inequality. One option to mitigate that negative effect (that is not considered by the social planner of our model) would be to add discriminatory clauses to the mechanism, so that the set of tracks an agent has available depends on his socioeconomic status. In the simpler case where all agents are either high or low income, and all high income (low income) agents have the same risk aversion level, that would actually be optimal, as it is easy to see that one could find the optimal mechanism by treating each set of agents independently.

7 Appendix

7.1 The case with finite agents

Let us rewrite the model for the case when there are N agents. Each agent $i = 1, ..., N$ has a private type $\theta_i \in \Theta$ and generates a signal $s_i \in [0, 1]$ with the same distributions of the text. There is a total number of τ_H and τ_L high and low-quality objects respectively. An allocation is $x = \left(x^1, ..., x^N\right)$, where

$$x^i = \left(x^i_h, x^i_l\right) : \Theta^N \times [0, 1]^N \to [0, 1] \times [0, 1]$$

[20]Decentralized school choice systems have been studied by Avery and Levin (2010) and Chade, Lewis and Smith (2014). In the latter paper, the low-quality school might end up with vacancies in equilibrium, but that is a product of assuming that schools are able to commit to an acceptance threshold before students choose whether to accept their offers. If schools were unable to commit, there would be no student left unassigned.

such that

$$x_h^i(\theta, s) + x_l^i(\theta, s) \leq 1$$

for all vectors $\theta \in \Theta^N$ and $s \in [0,1]^N$ and for all $i = 1, ..., N$. An allocation x is feasible if

$$\sum_{i=1}^{N} x_h^i(\theta, s) \leq \tau_H$$

and

$$\sum_{i=1}^{N} x_l^i(\theta, s) \leq \tau_L$$

for all $\theta \in \Theta^N$ and $s \in [0,1]^N$. An allocation x is incentive compatible if

$$\begin{aligned} & \mathbb{E}_{\theta_{-i},s} \left(u(\theta_i, h)\, x_h^i((\theta_i, \theta_{-i}), s) + u(\theta_i, l)\, x_l^i((\theta_i, \theta_{-i}), s)\right) \\ \geq\ & \mathbb{E}_{\theta_{-i},s} \left(u(\theta_i, h)\, x_h^i((\theta_i', \theta_{-i}), s) + u(\theta_i, l)\, x_l^i((\theta_i', \theta_{-i}), s)\right) \end{aligned}$$

for all $\theta_i' \in \Theta$, $\theta_i \in \Theta$ and $i = 1, ..., N$. Finally, the value for the social planner of allocation x is given by

$$W(x) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left(u(\theta_i, h)\, x_h^i(\theta, s) + u(\theta_i, l)\, x_l^i(\theta, s)\right).$$

Let us momentarily go back to the main model with a continuum of agents. Let us fix any parameters $\alpha = (\alpha_H, \alpha_L) \in (0,1)^2$ and define the value for the social planner of implementing the optimal feasible IC allocation to be W^*. We show that it is possible to construct feasible IC allocations in the finite version of the model whose value for the social planner converges to W^* as N goes to infinity when the availability of objects is the same in both versions of the model. In order to do that, consider the floor function $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{Z}$, such that $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. Assume that $\tau_H = \lfloor N\alpha_H \rfloor$ and $\tau_L = \lfloor N\alpha_L \rfloor$.

Proposition 6. *For every $\delta > 0$, there is an allocation $x = \left(x^1, ..., x^N\right)$ such that*

$$\lim_{J \to \infty} W(x) \geq W^* - \delta.$$

Proof. In order to construct allocation x, let us first go back to the model with a continuum of agents and consider the optimal ordered feasible IC allocation when the parameters of the model are $\alpha = (\alpha_H - \varepsilon, \alpha_L - \varepsilon)$ for some $\varepsilon \in (0, \min\{\alpha_L, \alpha_H\})$.

Let the thresholds of that allocation be denoted by $\{\underline{s}_\theta(\varepsilon), \overline{s}_\theta(\varepsilon)\}_{\theta\in\Theta}$ and denote the value for the social planner of implementing it by W_ε. Notice that by continuity of the objective function of the social planner and of all the incentive constraints considered, it follows that $\lim_{\varepsilon\to 0} W_\varepsilon = W^*$. As a result, in order to prove the proposition, it is enough to construct an allocation x such that $\lim_{N\to\infty} W(x) = W_\varepsilon$ and then, for each $\delta > 0$, select a small enough ε.

Construct each $x^i : \Theta^N \times [0,1]^N \to [0,1]\times[0,1]$ of allocation $x = (x^1, ..., x^N)$ as follows: i) for all $\theta \in \Theta^N$ and $s \in [0,1]^N$ such that either

$$\sum_{j\neq i} \mathbf{1}\left\{(\theta_j, s_j) : s_j \geq \overline{s}_{\theta_j}(\varepsilon)\right\} \geq \lfloor N\alpha_H \rfloor$$

or

$$\sum_{j\neq i} \mathbf{1}\left\{(\theta_j, s_j) : s_j \in \left(\underline{s}_{\theta_j}(\varepsilon), \overline{s}_{\theta_j}(\varepsilon)\right)\right\} \geq \lfloor N\alpha_L \rfloor$$

then $x_h^i(\theta, s) = x_l^i(\theta, s) = 0$; if not, then

$$x_h^i(\theta, s) = \begin{cases} 1 \text{ if } s_i \geq \overline{s}_{\theta_i}(\varepsilon) \\ 0 \text{ if not} \end{cases}$$

and

$$x_l^i(\theta, \mathbf{s}) = \begin{cases} 1 \text{ if } s_i \in \left(\underline{s}_{\theta_i}(\varepsilon), \overline{s}_{\theta_i}(\varepsilon)\right) \\ 0 \text{ if not} \end{cases}.$$

In words, agent i receives the exact same lottery of rewards as if we were in the model with a continuum of agents provided all other agents do not already exhaust the availability of each object.

By construction, it follows that allocation x is feasible and incentive compatible for any $N \geq 1$. In addition, the law of large numbers implies that the probability that condition i) is realized converges to 0 as $N \to \infty$ because $\varepsilon > 0$. Therefore, it follows that each agent is assigned the same lottery of rewards as in the case of the ordered allocation $\{\underline{s}_\theta(\varepsilon), \overline{s}_\theta(\varepsilon)\}_{\theta\in\Theta}$ of the model with a continuum of agents with probability 1, which implies that $\lim_{N\to\infty} W(x) = W_\varepsilon$. □

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Online Appendix

1 Proof of the theorem

The theorem follows by combining lemmas 1-4. Below, we show lemmas 1-3. Lemma 4 directly follows from lemmas 1 and 3 as described in the text.

1.1 Proof of Lemma 1

Proof. Take any $\theta \in \Theta$ and notice that

$$U\left(\theta, x\left(\theta'\right)\right) \geq U\left(\theta, x\left(\theta''\right)\right) \Leftrightarrow \frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p\left(s|\theta\right) ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p\left(s|\theta\right) ds} \geq \frac{u\left(\theta, h\right)}{u\left(\theta, l\right)} - 1.$$

The statement of the lemma follows because $\frac{u(\theta,h)}{u(\theta,l)}$ is (weakly) increasing with θ and, as we prove in the following paragraph, the left hand side of the final inequality is strictly decreasing with θ.

Consider any two types θ and $\hat{\theta}$, with $\theta > \hat{\theta}$. We will show that:

$$\frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p\left(s|\theta\right) ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p\left(s|\theta\right) ds} < \frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\hat{\theta}) ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p(s|\hat{\theta}) ds}.$$

We know that densities $\{p(\,\cdot\,|\theta) : \theta \in \Theta\}$ have the MLRP. Then, by Proposition 4 in Milgrom (1981), it follows that signal $\{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}$ is "more favorable" than signal $\{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}$. By definition, this implies that for every nondegenerate prior distribution G for θ, the posterior distribution $G(\,\cdot\,|\{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})$ first order stochastic dominates $G(\,\cdot\,|\{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})$.

Consider G such that it assigns positive and equal probability only to θ and $\hat{\theta}$. First order stochastic dominance implies:[21]

$$P(\theta|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}) > P(\theta|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}), \text{ and}$$

$$P(\hat{\theta}|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}) < P(\hat{\theta}|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}).$$

Then,

$$\frac{P(\theta|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})}{P(\theta|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})} > \frac{P(\hat{\theta}|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})}{P(\hat{\theta}|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})}$$

or equivalently,

$$\frac{P(\theta|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})}{P(\hat{\theta}|s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})} > \frac{P(\theta|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})}{P(\hat{\theta}|s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})}$$

By Bayes' theorem:

$$\frac{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\theta)}{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\hat{\theta})} > \frac{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\theta)}{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\hat{\theta})}.$$

Then we have:

$$\frac{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\hat{\theta})}{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\hat{\theta})} > \frac{P(s \in \{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\}|\theta)}{P(s \in \{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\}|\theta)}.$$

Finally, the last relation implies:

$$\frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\hat{\theta})ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p(s|\hat{\theta})ds} > \frac{\int_{\underline{s}_{\theta'}}^{\underline{s}_{\theta''}} p(s|\theta)ds}{\int_{\bar{s}_{\theta''}}^{\bar{s}_{\theta'}} p(s|\theta)ds}.$$

□

1.2 Proof of Lemma 2

Proof. Take any $\theta, \theta' \in \Theta$ such that $\theta' > \theta$. We want to show that:

[21] Given G, there are only two types which have positive probability; as a result, first order stochastic dominance implies that $G(\,\cdot\,|\{s \in [\bar{s}_{\theta''}, \bar{s}_{\theta'}]\})$ should put more probability than $G(\,\cdot\,|\{s \in [\underline{s}_{\theta'}, \underline{s}_{\theta''}]\})$ on the high type, and vice-versa with respect to the low type.

$$U\left(\theta,\widehat{x}\left(\theta'\right)\right)\leq U\left(\theta,\widehat{x}\left(\theta\right)\right).$$

Allocation x satisfies the upper incentives constraints, so we know that $U\left(\theta,x\left(\theta'\right)\right)\leq U\left(\theta,x\left(\theta\right)\right)$. Also, by the definition of $\hat{x}$ we have that $U\left(\theta,\widehat{x}\left(\theta\right)\right)=U\left(\theta,x\left(\theta\right)\right)$. So it is enough to prove that:

$$U\left(\theta,\widehat{x}\left(\theta'\right)\right)\leq U\left(\theta,x\left(\theta'\right)\right).$$

Therefore, it is sufficient to show that $\widehat{x}\left(\theta'\right)$ solves the following program:

$$\min_{(f_h,f_l):[0,1]\rightarrow[0,1]^2} U\left(\theta,f\right)$$

s.t.

$$\int_0^1 f_h\left(s\right)p\left(s|\theta'\right)ds=\int_{\overline{s}_{\theta'}}^1 p\left(s|\theta'\right)ds, \tag{1}$$

$$\int_0^1 f_l\left(s\right)p\left(s|\theta'\right)ds=\int_{\underline{s}_{\theta'}}^{\overline{s}_{\theta'}} p\left(s|\theta'\right)ds, \tag{2}$$

$$0\leq f_h\left(s\right)\leq 1 \text{ for all } s\in\left[0,1\right], \tag{3}$$

and

$$0\leq f_l\left(s\right)\leq 1-f_h\left(s\right) \text{ for all } s\in\left[0,1\right]. \tag{4}$$

In words, the mapping (f_h,f_l) that solves this problem minimizes the deviation payoff of type θ, while keeping the payoff of type θ' constant. Notice that the statement follows trivially if $\underline{s}_{\theta'}=1$; if $\underline{s}_{\theta'}=0$ and $\overline{s}_{\theta'}=1$; and if $\overline{s}_{\theta'}=0$. We focus on the remaining cases.

We do pointwise optimization. Let the solution be denoted by f^* and consider the following Lagrangian function:

$$\mathcal{L}=-\int_0^1[u(\theta,h)f_h(s)p(s|\theta)+u(\theta,l)f_l(s)p(s|\theta)]ds+\lambda_h\left(\int_0^1 f_h(s)p\left(s|\theta'\right)ds-\int_{\overline{s}_{\theta'}}^1 p(s|\theta')ds\right)+$$

$$\lambda_l\left(\int_0^1 f_l(s)p\left(s|\theta'\right)ds-\int_{\underline{s}_{\theta'}}^{\overline{s}_{\theta'}} p(s|\theta')ds\right)+$$

$$\overline{\mu}_h(s) f_h(s) + \underline{\mu}_h(s)(1 - f_h(s)) + \overline{\mu}_l(s) f_l(s) + \underline{\mu}_l(s)(1 - f_h(s) - f_l(s)),$$

where $\lambda_h \in \mathbb{R}$, $\lambda_l \in \mathbb{R}$, and for all $s \in [0,1]$, $\overline{\mu}_h(s) \geq 0$, $\underline{\mu}_h(s) \geq 0$, $\overline{\mu}_l(s) \geq 0$ and $\underline{\mu}_l(s) \geq 0$.

The first order conditions are given by

$$\begin{cases} -u(\theta, h) p(s|\theta) + \lambda_h p(s|\theta') - \underline{\mu}_l(s) + \overline{\mu}_h(s) - \underline{\mu}_h(s) = 0 \\ -u(\theta, l) p(s|\theta) + \lambda_l p(s|\theta') + \overline{\mu}_l(s) - \underline{\mu}_l(s) = 0 \end{cases}.$$

Case 1: For all $s \in [0,1]$,

$$-u(\theta, l) p(s|\theta) + \lambda_l p(s|\theta') \leq 0.$$

In that case, it follows that $f_l^*(s) =^{a.e.} 0$, which is only possible if $\overline{s}_{\theta'} = \underline{s}_{\theta'} \in (0,1)$. Using the first order conditions, it follows that

$$f_h^*(s) =^{a.e.} \mathbf{1}\left\{\lambda_h \frac{p(s|\theta')}{p(s|\theta)} > u(\theta, h)\right\}.$$

Because $\overline{s}_{\theta'} < 1$, it must be that $\lambda_h > 0$ so that the statement holds because $\frac{p(s|\theta')}{p(s|\theta)}$ is strictly increasing with s.

Case 2: For all $s \in [0,1]$,

$$-u(\theta, l) p(s|\theta) + \lambda_l p(s|\theta') \geq 0.$$

In that case, it follows that $f_l^*(s) =^{a.e.} 1 - f_h^*(s)$, which is only possible if $\underline{s}_{\theta'} = 0$. Using the first order conditions, it follows that

$$f_h^*(s) = \mathbf{1}\left\{(\lambda_h - \lambda_l) \frac{p(s|\theta')}{p(s|\theta)} \geq u(\theta, h) - u(\theta, l)\right\}.$$

Because $\overline{s}_{\theta'} < 1$, it must be that $\lambda_h > \lambda_l$, so that the statement holds because $\frac{p(s|\theta')}{p(s|\theta)}$ is strictly increasing with s.

Case 3: There is some $\underline{s} \in (0,1)$ such that

$$\lambda_l \frac{p(\underline{s}|\theta')}{p(\underline{s}|\theta)} = u(\theta, l).$$

In that case, it follows that

$$f_l^*(s) =^{a.e.} \begin{cases} 1 - f_h^*(s) \text{ if } s > \underline{s} \\ 0 \text{ if } s < \underline{s} \end{cases},$$

because $\frac{p(s|\theta')}{p(s|\theta)}$ is strictly increasing with s and $\lambda_l > 0$. Using the other first order condition, we get that:

$$f_h^*(s) = \begin{cases} 1 \text{ if } s < \underline{s} \text{ and } \lambda_h \frac{p(s|\theta')}{p(s|\theta)} > u(\theta, h) \\ 1 \text{ if } s > \underline{s} \text{ and } (\lambda_h - \lambda_l) \frac{p(s|\theta')}{p(s|\theta)} \geq u(\theta, h) - u(\theta, l) \\ 0 \text{ otherwise} \end{cases}.$$

We consider three cases.

Case 3.1: Suppose there is some $\bar{s} \in (0, \underline{s}]$ such that

$$\lambda_h \frac{p(\bar{s}|\theta')}{p(\bar{s}|\theta)} = u(\theta, h).$$

Then, for all $s < \bar{s}$, $f_h^*(s) = f_l^*(s) = 0$; for all $\bar{s} < s < \underline{s}$, $f_l^*(s) = 0$ and $f_h^*(s) = 1$. For $s > \underline{s}$, we have that $\lambda_h \frac{p(s|\theta')}{p(s|\theta)} > u(\theta, h)$, and $\lambda_l \frac{p(s|\theta')}{p(s|\theta)} > u(\theta, l)$. These two conditions imply that $(\lambda_h - \lambda_l) \frac{p(s|\theta')}{p(s|\theta)} > u(\theta, h) - u(\theta, l)$, so $f_h^*(s) = 1$ for all $s \geq \underline{s}$. In sum, $f_l^*(s) = 0$ for all s, $f_h^*(s) = 0$ if $s \leq \bar{s}$, and $f_h^*(s) = 1$ if $s > \bar{s}$. Then, it must be that $\underline{s}_{\theta'} = \bar{s}_{\theta'} \in (0, 1)$, in which case the statement follows with $\underline{s}_{\theta'} = \bar{s}_{\theta'} = \bar{s}$.

Case 3.2 Suppose not. And assume there is some signal $\bar{s} \in [\underline{s}, 1]$ such that

$$(\lambda_h - \lambda_l) \frac{p(\bar{s}|\theta')}{p(\bar{s}|\theta)} = u(\theta, h) - u(\theta, l).$$

In this case, $f_l^*(s) = 0$, and $f_h^*(s) = 1$ for $s \geq \bar{s}$; $f_l^*(s) = 1$, and $f_h^*(s) = 0$ for $\bar{s} < s < \underline{s}$. For $s \leq \underline{s}$ we have that:

$$(\lambda_h - \lambda_l) \frac{p(s|\theta')}{p(s|\theta)} < u(\theta, h) - u(\theta, l) \iff \lambda_h \frac{p(s|\theta')}{p(s|\theta)} - u(\theta, h) < \lambda_l \frac{p(s|\theta')}{p(s|\theta)} - u(\theta, l).$$

Given that $\lambda_l \frac{p(s|\theta')}{p(s|\theta)} - u(\theta, l) < 0$ for all $s < \underline{s}$, $\lambda_h \frac{p(s|\theta')}{p(s|\theta)} - u(\theta, h) < 0$. Thus, $f_h^*(s) = f_l^*(s) = 0$ for all $s < \underline{s}$, and the statement follows with $\underline{s}_{\theta'} = \underline{s}$ and $\bar{s}_{\theta'} = \bar{s}$.

Case 3.3: Finally, assume that for all $s > \underline{s}$,

$$(\lambda_h - \lambda_l) \frac{p(s|\theta')}{p(s|\theta)} < u(\theta, h) - u(\theta, l),$$

.

Then, we have that $f_h^*(s) = 0$, and $f_l^*(s) = 1$ for all $s > \underline{s}$. Moreover, as the same inequality holds for $s \leq \underline{s}$, we have as before that $\lambda_h \frac{p(s|\theta')}{p(s|\theta)} - u(\theta, h) < 0$. And then, $f_h^*(s) = f_l^*(s) = 0$ for all $s < \underline{s}$. Then, we must have $\overline{s}_{\theta'} = 1$ and the statement follows with $\underline{s}_{\theta'} = \underline{s}$. □

1.3 Proof of Lemma 3

Consider any ordered allocation $\widehat{x}$ that solves the relaxed problem. Let the associated thresholds be denoted by $\left(\overline{s}_j, \underline{s}_j\right)_{j=1}^J$. Consider any type $\theta_j \in \Theta$. We proceed by induction. Assume that, for all $k > 0$,

$$\overline{s}_{j+k} \geq \overline{s}_{j+k+1} \geq \underline{s}_{j+k+1} \geq \underline{s}_{j+k}$$

and

$$U\left(\theta_{j+k}, \widehat{x}\left(\theta_{j+k}\right)\right) = U\left(\theta_{j+k}, \widehat{x}\left(\theta_{j+k+1}\right)\right).$$

We complete the proof by showing that[22]

$$\overline{s}_j \geq \overline{s}_{j+1} \geq \underline{s}_{j+1} \geq \underline{s}_j$$

and

$$U\left(\theta_j, \widehat{x}\left(\theta_j\right)\right) = U\left(\theta_j, \widehat{x}\left(\theta_{j+1}\right)\right).$$

Let $j' \geq j+1$ be such that $\theta_{j'} \in \Theta$ is the largest type such that $\overline{s}_{j+1} = \overline{s}_{j'}$. Then, between θ_{j+1} and $\theta_{j'}$, all upper thresholds are equal. This, in turn, implies that the lower thresholds are also equal (otherwise the corresponding upper incentive constraints do not hold).

Let

$$\widehat{q}\left(\theta_{j+1}\right) \equiv \sum_{i=j+1}^{j'} q\left(\theta_i\right)$$

and

$$\widehat{p}\left(s|\theta_{j+1}\right) \equiv \sum_{i=j+1}^{j'} \frac{q\left(\theta_i\right)}{\widehat{q}\left(\theta_{j+1}\right)} p\left(s|\theta_i\right).$$

Notice that for any $s' > s$,

[22] Notice that the proof also applies to the case where $j = J - 1$.

$$\frac{p(s'|\theta')}{p(s|\theta')} > \frac{p(s'|\theta_{j'})}{p(s|\theta_{j'})} > \frac{\widehat{p}(s'|\theta_{j+1})}{\widehat{p}(s|\theta_{j+1})} > \frac{p(s'|\theta_{j+1})}{p(s|\theta_{j+1})} > \frac{p(s'|\theta'')}{p(s|\theta'')} \tag{5}$$

for any $\theta' > \theta_{j'} \geq \theta_{j+1} > \theta''$.

Indeed,

$$\frac{\widehat{p}(s'|\theta_{j+1})}{\widehat{p}(s|\theta_{j+1})} = \frac{\sum_{i=j+1}^{j'} q(\theta_i)\, p(s'|\theta_i)}{\sum_{i=j+1}^{j'} q(\theta_i)\, p(s|\theta_i)}.$$

We know that $\frac{p(s'|\theta_i)}{p(s|\theta_i)} < \frac{p\left(s'|\theta_{j'}\right)}{p\left(s|\theta_{j'}\right)}$ for every $i = j+1, \ldots, j'$. Then:

$$\frac{\widehat{p}(s'|\theta_{j+1})}{\widehat{p}(s|\theta_{j+1})} < \frac{\sum_{i=j+1}^{j'} q(\theta_i)\, \frac{p\left(s'|\theta_{j'}\right)}{p\left(s|\theta_{j'}\right)} p(s|\theta_i)}{\sum_{i=j+1}^{j'} q(\theta_i)\, p(s|\theta_i)} = \frac{p(s'|\theta_{j'})}{p(s|\theta_{j'})}.$$

By the same reasoning we show $\frac{\widehat{p}(s'|\theta_{j+1})}{\widehat{p}(s|\theta_{j+1})} > \frac{p(s'|\theta_{j+1})}{p(s|\theta_{j+1})}$.

We divide the argument into four claims.

Claim 1: If $\underline{s}_j \leq \underline{s}_{j+1}$, then i) $\overline{s}_j \geq \overline{s}_{j+1}$ and ii) $U(\theta_j, \widehat{x}(\theta_j)) = U(\theta_j, \widehat{x}(\theta_{j+1}))$.

Proof. i) follows because of ii), so it is enough to show ii).

Case 1: $\underline{s}_{j+1} > 0$ and $\underline{s}_j < \overline{s}_j$.

Suppose not, so that $U(\theta_j, \widehat{x}(\theta_j)) > U(\theta_j, \widehat{x}(\theta_{j+1}))$. By lemma 1, this implies that $U(\theta, \widehat{x}(\theta_j)) > U(\theta, \widehat{x}(\theta_{j+1}))$ for all $\theta < \theta_j$. Consider a new ordered allocation x', where $x' = \hat{x}$ except that $\underline{s}'_j = \underline{s}_j + \varepsilon$, while $\underline{s}'_{\widehat{j}} = \underline{s}_{\widehat{j}} - \gamma(\varepsilon)$, where

$$q(\theta_j) \int_{\underline{s}_j}^{\underline{s}_j+\varepsilon} p(s|\theta_j)\, ds = \widehat{q}(\theta_{j+1}) \int_{\underline{s}_{j+1}-\gamma(\varepsilon)}^{\underline{s}_{j+1}} \widehat{p}(s|\theta_{j+1})\, ds$$

and $\varepsilon > 0$, for all $\widehat{j}$ such that $j+1 \leq \widehat{j} \leq j'$. Notice that, provided ε is sufficiently small, allocation x' is feasible and satisfies all the incentive constraints of the relaxed problem, because any type $\theta_{\widehat{j}}$ with $j+1 \leq \widehat{j} \leq j'$ is made better off. It is also the case that $W(x') > W(\widehat{x})$, because $u(\theta, l)$ is increasing with θ (allocation x' just shifts low-quality objects from type θ_j to larger types), which is a contradiction.

Case 2: $\underline{s}_{j+1} = 0$ or $\underline{s}_j = \overline{s}_j$

Suppose not, so that $U(\theta_j, \widehat{x}(\theta_j)) > U(\theta_j, \widehat{x}(\theta_{j+1}))$. Consider a new ordered allocation x', where $x' = x$ except that $\overline{s}'_j = \overline{s}_j + \varepsilon$, while $\overline{s}'_{\widehat{j}} = \overline{s}_{\widehat{j}} - \gamma(\varepsilon)$, where

$$q(\theta_j) \int_{\overline{s}_j}^{\overline{s}_j + \varepsilon} p(s|\theta_j)\, ds = \widehat{q}(\theta_{j+1}) \int_{\overline{s}_{j+1} - \gamma(\varepsilon)}^{\overline{s}_{j+1}} \widehat{p}(s|\theta_{j+1})\, ds$$

and $\varepsilon > 0$, for all $\widehat{j}$ such that $j + 1 \leq \widehat{j} \leq j'$. By the same argument as in case i), if ε is sufficiently small, allocation x' is feasible, satisfies all the incentive constraints of the relaxed problem and is such that $W(x') > W(\widehat{x})$, because $(u(\theta, h) - u(\theta, l))$ is increasing with θ (allocation x' is such that type θ_j trades low-quality objects for high-quality objects with the larger types), which is a contradiction. □

Claim 2: If $\underline{s}_j > \underline{s}_{j+1}$, *then there is some type* $\theta_{\widetilde{j}} \leq \theta_j$ *such that* $U\left(\theta_{\widetilde{j}}, \widehat{x}\left(\theta_{\widetilde{j}}\right)\right) = U\left(\theta_{\widetilde{j}}, \widehat{x}(\theta_{j+1})\right)$.

Proof. Suppose not, so that $U(\theta_{j''}, \widehat{x}(\theta_{j''})) > U(\theta_{j''}, \widehat{x}(\theta_{j+1}))$ for all $j'' \leq j$. Then, we can proceed in the same manner of before. In particular, by considering the allocation of case 2 of the proof of the previous claim, an allocation that is feasible, satisfies the considered incentive constraints and attains a larger welfare, we find a contradiction. □

Claim 3: *If* $\underline{s}_j > \underline{s}_{j+1}$, *then* $U(\theta_j, \widehat{x}(\theta_j)) > U(\theta_j, \widehat{x}(\theta_{j+1}))$.

Proof. Suppose not, so that $U(\theta_j, \widehat{x}(\theta_j)) = U(\theta_j, \widehat{x}(\theta_{j+1}))$. Consider the following ordered allocation x', where $x' = \widehat{x}$ except that $\overline{s}'_j = \overline{s}_j + \varepsilon$, $\overline{s}'_{\widehat{j}} = \overline{s}_{j+1} - \delta(\varepsilon)$, $\underline{s}'_j = \underline{s}_j - \beta(\varepsilon)$ and $\underline{s}'_{\widehat{j}} = \underline{s}_{j+1} + \gamma(\varepsilon)$, where

$$\widehat{q}(\theta_{j+1}) \int_{\overline{s}_{j+1} - \delta(\varepsilon)}^{\overline{s}_{j+1}} \widehat{p}(s|\theta_{j+1}) = q(\theta_j) \int_{\overline{s}_j}^{\overline{s}_j + \varepsilon} p(s|\theta_j)\, ds,$$

$$\widehat{q}(\theta_{j+1}) \int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} \widehat{p}(s|\theta_{j+1})\, ds = q(\theta_j) \int_{\underline{s}_j-\beta(\varepsilon)}^{\underline{s}_j} p(s|\theta_j)\, ds$$

and

$$\begin{aligned} & u(\theta_j, l) \int_{\underline{s}_j-\beta(\varepsilon)}^{\underline{s}_j} p(s|\theta_j)\, ds - (u(\theta_j, h) - u(\theta_j, l)) \int_{\overline{s}_j}^{\overline{s}_j+\varepsilon} p(s|\theta_j)\, ds \\ = \; & (u(\theta_j, h) - u(\theta_j, l)) \int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} p(s|\theta_j)\, ds - u(\theta_j, l) \int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} p(s|\theta_j)\, ds \end{aligned}$$

In words, we are perturbing allocation $\widehat{x}$ by increasing the measure of h objects and reducing the measure of l objects assigned to types $\theta_{\widehat{j}}$ with $j+1 \leq \widehat{j} \leq j'$, while keeping the total measures constant and type θ_j indifferent between reporting to being θ_j and θ_{j+1}. Once again, if ε is sufficiently small, allocation x' is feasible, functions δ, β and γ are all differentiable and all converge to 0 when $\varepsilon = 0$. After some algebra, we have that

$$\delta'(0) = \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\overline{s}_j|\theta_j)}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}, \tag{6}$$

$$\gamma'(0) = \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\underline{s}_j|\theta_j)}{\widehat{p}(\underline{s}_{j+1}|\theta_{j+1})} \beta'(0) \tag{7}$$

and

$$\begin{aligned} & u(\theta_j, l)\, p(\underline{s}_j|\theta_j)\, \beta'(0) - (u(\theta_j, h) - u(\theta_j, l))\, p(\overline{s}_j|\theta_j) \\ = \; & (u(\theta_j, h) - u(\theta_j, l))\, p(\overline{s}_{j+1}|\theta_j)\, \delta'(0) - u(\theta_j, l)\, p(\underline{s}_{j+1}|\theta_j)\, \gamma'(0). \end{aligned} \tag{8}$$

We start by showing if ε is sufficiently small, $W(x') > W(\widehat{x})$. Let $V(\varepsilon)$ denote the increase in welfare from allocation $\widehat{x}$ to allocation x' as a function of ε, i.e.,

$$V(\varepsilon) = \left\{ \begin{array}{c} \sum_{\widehat{j}=j+1}^{j'} q(\theta_{\widehat{j}}) \left(\begin{array}{c} \left(u\left(\theta_{\widehat{j}}, h\right) - u\left(\theta_{\widehat{j}}, l\right)\right) \int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} p\left(s|\theta_{\widehat{j}}\right) ds \\ -u\left(\theta_{\widehat{j}}, l\right) \int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} p\left(s|\theta_{\widehat{j}}\right) ds \end{array} \right) + \\ q(\theta_j) \left(u(\theta_j, l) \int_{\underline{s}_j-\beta(\varepsilon)}^{\underline{s}_j} p(s|\theta_j)\, ds - (u(\theta_j, h) - u(\theta_j, l)) \int_{\overline{s}_j}^{\overline{s}_j+\varepsilon} p(s|\theta_j)\, ds \right) \end{array} \right\}.$$

Notice that $V(\varepsilon) \geq \widehat{V}(\varepsilon)$ for all $\varepsilon > 0$, where

$$\widehat{V}(\varepsilon) = \left\{ \begin{array}{c} \widehat{q}(\theta_{j+1}) \left(\begin{array}{c} (u(\theta_{j+1}, h) - u(\theta_{j+1}, l)) \int\limits_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} \widehat{p}(s|\theta_{j+1})\, ds \\ -u(\theta_{j+1}, l) \int\limits_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} \widehat{p}(s|\theta_{j+1})\, ds \end{array} \right) + \\ q(\theta_j) \left(u(\theta_j, l) \int\limits_{\underline{s}_j-\beta(\varepsilon)}^{\underline{s}_j} p(s|\theta_j)\, ds - (u(\theta_j, h) - u(\theta_j, l)) \int\limits_{\overline{s}_j}^{\overline{s}_j+\varepsilon} p(s|\theta_j)\, ds \right) \end{array} \right\},$$

because $u(\theta, l)$ and $\frac{u(\theta,h)}{u(\theta,l)}$ are increasing with θ. After replacing (6), (7) and (8), we that $\widehat{V}'(0) > 0$ if and only if

$$\begin{aligned} & u(\theta_{j+1}, h) - u(\theta_{j+1}, l) - u(\theta_j, h) + u(\theta_j, l) \\ > \; & \frac{(u(\theta_j, h) - u(\theta_j, l))}{u(\theta_j, l)} (u(\theta_{j+1}, l) - u(\theta_j, l)) \frac{\left(1 + \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\overline{s}_{j+1}|\theta_j)}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}\right)}{\left(1 + \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\underline{s}_{j+1}|\theta_j)}{\widehat{p}(\underline{s}_{j+1}|\theta_{j+1})}\right)}. \end{aligned}$$

Notice that

$$\frac{\left(1 + \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\overline{s}_{j+1}|\theta_j)}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}\right)}{\left(1 + \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\underline{s}_{j+1}|\theta_j)}{\widehat{p}(\underline{s}_{j+1}|\theta_{j+1})}\right)} < 1$$

because, because by equation (5),

$$\frac{p(\overline{s}_{j+1}|\theta_j)}{p(\underline{s}_{j+1}|\theta_j)} < \frac{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}{\widehat{p}(\underline{s}_{j+1}|\theta_{j+1})}.$$

Therefore, it is sufficient to show that

$$u(\theta_{j+1}, h) - u(\theta_{j+1}, l) - u(\theta_j, h) + u(\theta_j, l) \geq \frac{(u(\theta_j, h) - u(\theta_j, l))}{u(\theta_j, l)} (u(\theta_{j+1}, l) - u(\theta_j, l)),$$

which is equivalent to

$$\frac{u(\theta_{j+1}, h)}{u(\theta_{j+1}, l)} \geq \frac{u(\theta_j, h)}{u(\theta_j, l)},$$

which is true. Therefore, if ε is sufficiently small, $W(x') > W(\widehat{x})$.

We find a contradiction by showing that all incentive constraints of the relaxed problem are satisfied if ε is sufficiently small. First, type θ_j, by definition, does not want to deviate to mimicking any type $\theta_{\widehat{j}}$ for $\widehat{j} \leq j'$. Furthermore, by lemma 1, it also follows that $U(\theta_j, x'(\theta_j)) > U(\theta_j, \widehat{x}(\theta_{j''}))$ for any $j'' > j'$, so that he does not deviate under x' provided ε is sufficiently small.

As for any type $\theta_{j''} < \theta_j$, by lemma 1, it follows that $U(\theta_{j''}, \widehat{x}(\theta_{j+1})) > U(\theta_{j''}, \widehat{x}(\theta_j))$, so that one only has to verify that type $\theta_{j''}$ does not mimic type θ_{j+1}, provided ε is sufficiently small. We start by showing that $U(\theta_j, x'(\theta_j)) < U(\theta_j, \widehat{x}(\theta_j))$.

Let $B(\varepsilon)$ denote the payoff change in the expected utility of type θ_j, i.e.,

$$B(\varepsilon) = u(\theta_j, l) \int_{\underline{s}_j - \beta(\varepsilon)}^{\underline{s}_j} p(s|\theta_j)\, ds - (u(\theta_j, h) - u(\theta_j, l)) \int_{\overline{s}_j}^{\overline{s}_j + \varepsilon} p(s|\theta_j)\, ds.$$

Notice that

$$B'(0) = u(\theta_j, l)\, p\left(\underline{s}_j|\theta_j\right) \beta'(0) - (u(\theta_j, h) - u(\theta_j, l))\, p(\overline{s}_j|\theta_j).$$

After replacing $\beta'(0)$, we get that $B'(0) < 0$ if and only if

$$\frac{\left(1 + \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p(\overline{s}_{j+1}|\theta_j)}{\widehat{p}(\overline{s}_{j+1}|\theta_{j+1})}\right)}{\left(1 + \frac{q(\theta_j)}{\widehat{q}(\theta_{j+1})} \frac{p\left(\underline{s}_{j+1}|\theta_j\right)}{\widehat{p}\left(\underline{s}_{j+1}|\theta_{j+1}\right)}\right)} < 1,$$

which is true as established above. Therefore, if ε is sufficiently small, we have that

$$U(\theta_j, x'(\theta_{j+1})) = U(\theta_j, x'(\theta_j)) < U(\theta_j, \widehat{x}(\theta_j)) = U(\theta_j, \widehat{x}(\theta_{j+1})).$$

By lemma 1, we have that $U(\theta_{j''}, x'(\theta_{j+1})) < U(\theta_{j''}, \widehat{x}(\theta_{j+1}))$ for all $\theta_{j''} < \theta_j$, so type $\theta_{j''}$ does not deviate under x'.

Finally, notice that $U(\theta_{j'}, x'(\theta_{j'})) > U(\theta_{j'}, \widehat{x}(\theta_{j'}))$ because $W(x') > W(\hat{x})$. Therefore, type $\theta_{j'}$ does not want to deviate, which, by lemma 1, implies that types $\theta_{j''}$ such that $j+1 \leq j'' \leq j'$ do not want to deviate either. □

Claim 4: If $\underline{s}_j > \underline{s}_{j+1}$, then $U(\theta_{j''}, \widehat{x}(\theta_{j''})) > U(\theta_{j''}, \widehat{x}(\theta_{j+1}))$ for all $j'' < j$.

Proof. Suppose not and let $\theta_{\tilde{j}} < \theta_j$ denote the largest type such that $U\left(\theta_{\tilde{j}}, \widehat{x}\left(\theta_{\tilde{j}}\right)\right) = U\left(\theta_{\tilde{j}}, \hat{x}(\theta_{j+1})\right)$. Consider ordered allocation x', where $x' = \widehat{x}$ except that $\overline{s}'_j = \overline{s}_j + \varepsilon$, $\overline{s}'_{j+1} = \overline{s}_{j+1} - \delta(\varepsilon)$, $\underline{s}'_j = \underline{s}_j - \beta(\varepsilon)$ and $\underline{s}'_{j+1} = \underline{s}_{j+1} + \gamma(\varepsilon)$, where

$$q(\theta_{j+1}) \int_{\overline{s}_{j+1} - \delta(\varepsilon)}^{\overline{s}_{j+1}} p(s|\theta_{j+1})\, ds = q(\theta_j) \int_{\overline{s}_j}^{\overline{s}_j + \varepsilon} p(s|\theta_j)\, ds,$$

$$q\left(\theta_{j+1}\right)\int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} p\left(s|\theta_{j+1}\right)ds = q\left(\theta_{j}\right)\int_{\underline{s}_{j}-\beta(\varepsilon)}^{\underline{s}_{j}} p\left(s|\theta_{j}\right)ds$$

and

$$\left(u\left(\theta_{\tilde{j}},h\right)-u\left(\theta_{\tilde{j}},l\right)\right)\int_{\overline{s}_{j+1}-\delta(\varepsilon)}^{\overline{s}_{j+1}} p\left(s|\theta_{\tilde{j}}\right)ds = u\left(\theta_{\tilde{j}},l\right)\int_{\underline{s}_{j+1}}^{\underline{s}_{j+1}+\gamma(\varepsilon)} p\left(s|\theta_{\tilde{j}}\right)ds.$$

In words, we are perturbing allocation $\widehat{x}$ by increasing the measure of h objects and reducing the measure of l objects assigned to type θ_{j+1}, while keeping the total measures constant and type $\theta_{\tilde{j}}$ indifferent to mimicking type θ_{j+1}.

Notice that

$$\delta'\left(0\right)=\frac{q\left(\theta_{j}\right)}{q\left(\theta_{j+1}\right)}\frac{p\left(\overline{s}_{j}|\theta_{j}\right)}{p\left(\overline{s}_{j+1}|\theta_{j+1}\right)},$$

$$\beta'\left(0\right)=\frac{\left(u\left(\theta_{\tilde{j}},h\right)-u\left(\theta_{\tilde{j}},l\right)\right)}{u\left(\theta_{\tilde{j}},l\right)}\frac{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)}{p\left(\overline{s}_{j+1}|\theta_{j+1}\right)}\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}\frac{p\left(\overline{s}_{j}|\theta_{j}\right)}{p\left(\underline{s}_{j}|\theta_{j}\right)}$$

and

$$\gamma'\left(0\right)=\frac{q\left(\theta_{j}\right)}{q\left(\theta_{j+1}\right)}\frac{\left(u\left(\theta_{\tilde{j}},h\right)-u\left(\theta_{\tilde{j}},l\right)\right)}{u\left(\theta_{\tilde{j}},l\right)}\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}\frac{p\left(\overline{s}_{j}|\theta_{j}\right)}{p\left(\overline{s}_{j+1}|\theta_{j+1}\right)}.$$

We start by showing that if ε is sufficiently small, $W\left(x'\right)>W\left(\widehat{x}\right)$ by showing that $V'\left(0\right)>0$, where function V is as in the previous proof. Notice that $V'\left(0\right)>0$ if and only if

$$\begin{aligned}
& u\left(\theta_{j+1},h\right)-u\left(\theta_{j+1},l\right)-u\left(\theta_{j},h\right)+u\left(\theta_{j},l\right) \\
> \quad & \left(u\left(\theta_{j+1},l\right)-u\left(\theta_{j},l\right)\right)\frac{\left(u\left(\theta_{\tilde{j}},h\right)-u\left(\theta_{\tilde{j}},l\right)\right)}{u\left(\theta_{\tilde{j}},l\right)}\frac{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)}{p\left(\overline{s}_{j+1}|\theta_{j+1}\right)}\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}.
\end{aligned}$$

From equation (5), notice that

$$\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}<\frac{p\left(\overline{s}_{j+1}|\theta_{j+1}\right)}{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)},$$

so it is enough to show that

$$u(\theta_{j+1},h) - u(\theta_{j+1},l) - u(\theta_j,h) + u(\theta_j,l) \geq (u(\theta_{j+1},l) - u(\theta_j,l)) \frac{\left(u\left(\theta_{\tilde{j}},h\right) - u\left(\theta_{\tilde{j}},l\right)\right)}{u\left(\theta_{\tilde{j}},l\right)}$$

in order to show that $V'(0) > 0$, which can be written as

$$\frac{u(\theta_{j+1},l)}{u(\theta_j,l)} \left(\frac{u(\theta_{j+1},h)}{u(\theta_{j+1},l)} - \frac{u\left(\theta_{\tilde{j}},h\right)}{u\left(\theta_{\tilde{j}},l\right)} \right) \geq \frac{u(\theta_j,h)}{u(\theta_j,l)} - \frac{u\left(\theta_{\tilde{j}},h\right)}{u\left(\theta_{\tilde{j}},l\right)},$$

which is true, because $\frac{u(\theta_{j+1},l)}{u(\theta_j,l)} > 1$ and $\frac{u(\theta,h)}{u(\theta,l)}$is increasing with θ.

By definition, allocation x' is feasible if ε is small. Let us now turn to the incentive constraints. Consider type $\theta_{\tilde{j}}$, who,, by definition, does not want to mimic type θ_{j+1}. Let $C(\varepsilon)$ denote the increase in the expected utility of type $\theta_{\tilde{j}}$ when mimicking type θ_j as a function of ε, i.e.,

$$C(\varepsilon) = u\left(\theta_{\tilde{j}},l\right) \int_{\underline{s}_j - \beta(\varepsilon)}^{\underline{s}_j} p\left(s|\theta_{\tilde{j}}\right) ds - \left(u\left(\theta_{\tilde{j}},h\right) - u\left(\theta_{\tilde{j}},l\right)\right) \int_{\overline{s}_j}^{\overline{s}_j+\varepsilon} p\left(s|\theta_{\tilde{j}}\right) ds.$$

Notice that $C'(0) < 0$ if and only if

$$\frac{\frac{p(\overline{s}_j|\theta_j)}{p\left(\underline{s}_j|\theta_j\right)}}{\frac{p\left(\overline{s}_j|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\tilde{j}}\right)}} < \frac{\frac{p(\overline{s}_{j+1}|\theta_{j+1})}{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}},$$

which is true, because

$$\begin{aligned} \frac{\frac{p(\overline{s}_{j+1}|\theta_{j+1})}{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}} &= \frac{\frac{p(\overline{s}_j|\theta_{j+1})}{p\left(\underline{s}_j|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_j|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\tilde{j}}\right)}} \frac{\frac{p(\overline{s}_{j+1}|\theta_{j+1})}{p(\overline{s}_j|\theta_{j+1})}}{\frac{p\left(\overline{s}_{j+1}|\theta_{\tilde{j}}\right)}{p\left(\overline{s}_j|\theta_{\tilde{j}}\right)}} \frac{\frac{p\left(\underline{s}_j|\theta_{j+1}\right)}{p\left(\underline{s}_{j+1}|\theta_{j+1}\right)}}{\frac{p\left(\underline{s}_j|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_{j+1}|\theta_{\tilde{j}}\right)}} \\ &> \frac{\frac{p(\overline{s}_j|\theta_{j+1})}{p\left(\underline{s}_j|\theta_{j+1}\right)}}{\frac{p\left(\overline{s}_j|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\tilde{j}}\right)}} > \frac{\frac{p(\overline{s}_j|\theta_j)}{p\left(\underline{s}_j|\theta_j\right)}}{\frac{p\left(\overline{s}_j|\theta_{\tilde{j}}\right)}{p\left(\underline{s}_j|\theta_{\tilde{j}}\right)}}. \end{aligned}$$

Therefore, we have that

$$U\left(\theta_{\tilde{j}}, x'(\theta_j)\right) < U\left(\theta_{\tilde{j}}, \widehat{x}(\theta_j)\right),$$

so that type $\theta_{\widetilde{j}}$ does not deviate.

Now, consider any type $\theta_{j''}$ with $\widetilde{j} < j'' \leq j$. Type θ_j was not indifferent to mimicking type θ_{j+1} under allocation $\widehat{x}$, so that still carries over to allocation x' provided ε is sufficiently small. Moreover, notice that

$$U\left(\theta_{\widetilde{j}}, x'(\theta_j)\right) < U\left(\theta_{\widetilde{j}}, \widehat{x}(\theta_j)\right) \Rightarrow U\left(\theta_{j''}, x'(\theta_j)\right) < U\left(\theta_{j''}, \widehat{x}(\theta_j)\right)$$

by lemma 1, which implies that type $\theta_{j''}$ does not prefer to mimic type θ_j under allocation x'. Any other deviation by type θ_j is ruled out if ε is sufficiently small.

Now, consider any type $\theta_{j''}$ with $j'' < \widetilde{j}$. Recall that

$$U\left(\theta_{\widetilde{j}}, \widehat{x}\left(\theta_{\widetilde{j}}\right)\right) = U\left(\theta_{\widetilde{j}}, \widehat{x}(\theta_{j+1})\right) \geq U\left(\theta_{\widetilde{j}}, \widehat{x}(\theta_j)\right),$$

which, by lemma 1, implies that

$$U\left(\theta_{j''}, \widehat{x}(\theta_{j''})\right) \geq U\left(\theta_{j''}, \widehat{x}(\theta_{j+1})\right) > U\left(\theta_{j''}, \widehat{x}(\theta_j)\right)$$

for all $j'' < \widetilde{j}$. Therefore, if ε is small enough, for any $j'' < \widetilde{j}$, type $\theta_{j''}$ does not want to mimic type θ_j under allocation x'. Notice also that

$$U\left(\theta_{\widetilde{j}}, \widehat{x}(\theta_{j+1})\right) = U\left(\theta_{\widetilde{j}}, x'(\theta_{j+1})\right) \Rightarrow U\left(\theta_{j''}, \widehat{x}(\theta_{j+1})\right) \geq U\left(\theta_{j''}, x'(\theta_{j+1})\right)$$

for all $j'' < \widetilde{j}$ by lemma 1. Therefore, we can conclude that, for any $j'' < \widetilde{j}$, type $\theta_{j''}$ does not want to deviate.

Finally, type θ_{j+1} does not want to deviate because $U\left(\theta_{j+1}, x'(\theta_{j+1})\right) > U\left(\theta_{j+1}, \widehat{x}(\theta_{j+1})\right)$ because welfare went up with allocation x'. As a result, all considered incentive constraints are satisfied, the new allocation is feasible and it increases welfare, which is a contradiction. □

Claims 2, 3 and 4 are inconsistent, so it must be that $\underline{s}_j \leq \underline{s}_{j+1}$, which implies the statement of lemma 3 by claim 1.

1.4 Proof of Proposition 1

Proof of i). Let x^* denote the DA allocation and notice that it is an ordered allocation where, for all $\theta_j \in \Theta$, $\overline{s}_j = \overline{s}^*$ and $\underline{s}_j = \underline{s}^*$, where $0 < \underline{s}^* < \overline{s}^* < 1$. Let

$$\widehat{q}(\theta_2) = \sum_{j=2}^{J} q(\theta_j)$$

and

$$\widehat{p}(s|\theta_2) = \sum_{j=2}^{J} \frac{q(\theta_j)\, p(s|\theta_j)}{\widehat{q}(\theta_2)}.$$

Notice that for any $s' > s$,

$$\frac{p(s'|\theta_J)}{p(s|\theta_J)} > \frac{\widehat{p}(s'|\theta_2)}{\widehat{p}(s|\theta_2)} \geq \frac{p(s'|\theta_2)}{p(s|\theta_2)} > \frac{p(s'|\theta_1)}{p(s|\theta_1)}.$$

Consider the following alternative ordered allocation x', where $\overline{s}_1 = \overline{s}^* + \varepsilon$, $\overline{s}'_j = \overline{s}^* - \delta(\varepsilon)$ for all $j > 1$, $\underline{s}'_1 = \underline{s}^* - \beta(\varepsilon)$ and $\underline{s}'_j = \underline{s}^* + \gamma(\varepsilon)$ for all $j > 1$, where

$$q(\theta_1) \int_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds = \widehat{q}(\theta_2) \int_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} \widehat{p}(s|\theta_2)\, ds,$$

$$q(\theta_1) \int_{\underline{s}^*-\beta(\varepsilon)}^{\underline{s}^*} p(s|\theta_1)\, ds = \widehat{q}(\theta_2) \int_{\underline{s}^*}^{\underline{s}^*+\gamma(\varepsilon)} \widehat{p}(s|\theta_2)\, ds$$

and

$$\begin{aligned} & u(\theta_1, l) \int_{\underline{s}^*-\beta(\varepsilon)}^{\underline{s}^*} p(s|\theta_1)\, ds - (u(\theta_1, h) - u(\theta_1, l)) \int_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds \\ = \; & (u(\theta_1, h) - u(\theta_1, l)) \int_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} p(s|\theta_1)\, ds - u(\theta_1, l) \int_{\underline{s}^*}^{\underline{s}^*+\gamma(\varepsilon)} p(s|\theta_1)\, ds. \end{aligned}$$

In words, we are perturbing allocation x^* by shifting some of the h objects from type θ_1 to larger types, while keeping type θ_1 indifferent and assigning the same measure of h and l objects.

Notice that

$$\delta'(0) = \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)},$$

$$\gamma'(0) = \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\underline{s}^*|\theta_1)}{\widehat{p}(\underline{s}^*|\theta_2)} \frac{(u(\theta_1,h) - u(\theta_1,l))}{u(\theta_1,l)} \frac{p(\overline{s}^*|\theta_1)}{p(\underline{s}^*|\theta_1)} \frac{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)}\right)}{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\underline{s}^*|\theta_1)}{\widehat{p}(\underline{s}^*|\theta_2)}\right)}$$

and

$$\beta'(0) = \frac{(u(\theta_1,h) - u(\theta_1,l))}{u(\theta_1,l)} \frac{p(\overline{s}^*|\theta_1)}{p(\underline{s}^*|\theta_1)} \frac{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)}\right)}{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\underline{s}^*|\theta_1)}{\widehat{p}(\underline{s}^*|\theta_2)}\right)}.$$

Let $V(\varepsilon)$ denote the increase in welfare as a function of ε, i.e.,

$$V(\varepsilon) = \left\{ \begin{array}{c} q(\theta_1) \left(u(\theta_1,l) \int\limits_{\underline{s}^*-\beta(\varepsilon)}^{\underline{s}^*} p(s|\theta_1)\, ds - (u(\theta_1,h) - u(\theta_1,l)) \int\limits_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds \right) + \\ \sum\limits_{j=2}^{J} q(\theta_2) \left((u(\theta_j,h) - u(\theta_j,l)) \int\limits_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} p(s|\theta_j)\, ds - u(\theta_j,l) \int\limits_{\underline{s}^*}^{\underline{s}^*+\gamma(\varepsilon)} p(s|\theta_j)\, ds \right) \end{array} \right\}$$

and notice that $V(\varepsilon) \geq \widehat{V}(\varepsilon)$, where

$$\widehat{V}(\varepsilon) = \left\{ \begin{array}{c} q(\theta_1) \left(u(\theta_1,l) \int\limits_{\underline{s}^*-\beta(\varepsilon)}^{\underline{s}^*} p(s|\theta_1)\, ds - (u(\theta_1,h) - u(\theta_1,l)) \int\limits_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds \right) + \\ \widehat{q}(\theta_2) \left((u(\theta_2,h) - u(\theta_2,l)) \int\limits_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} \widehat{p}(s|\theta_2)\, ds - u(\theta_2,l) \int\limits_{\underline{s}^*}^{\underline{s}^*+\gamma(\varepsilon)} \widehat{p}(s|\theta_2)\, ds \right) \end{array} \right\}.$$

Notice also that $\widehat{V}'(0) > 0$ if and only if

$$\begin{array}{ll} & u(\theta_2,h) - u(\theta_2,l) - u(\theta_1,h) + u(\theta_1,l) \\ > & \frac{(u(\theta_1,h) - u(\theta_1,l))}{u(\theta_1,l)} (u(\theta_2,l) - u(\theta_1,l)) \frac{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)}\right)}{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\underline{s}^*|\theta_1)}{\widehat{p}(\underline{s}^*|\theta_2)}\right)} \end{array}$$

Given that

$$\frac{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)}\right)}{\left(1 + \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\underline{s}^*|\theta_1)}{\widehat{p}(\underline{s}^*|\theta_2)}\right)} < 1$$

because

$$\frac{p(\overline{s}^*|\theta_1)}{p(\underline{s}^*|\theta_1)} < \frac{\widehat{p}(\overline{s}^*|\theta_2)}{\widehat{p}(\underline{s}^*|\theta_2)},$$

it follows that $\widehat{V}'(0) > 0$ if

$$u(\theta_2, h) - u(\theta_2, l) - u(\theta_1, h) + u(\theta_1, l) > \frac{(u(\theta_1, h) - u(\theta_1, l))}{u(\theta_1, l)} (u(\theta_2, l) - u(\theta_1, l)),$$

which is equivalent to

$$\frac{u(\theta_2, h)}{u(\theta_2, l)} > \frac{u(\theta_1, h)}{u(\theta_1, l)},$$

which is true. Therefore, it follows that $W(x') > W(x^*)$ if ε is sufficiently small. Furthermore, notice that type θ_1 is indifferent as to what to report, which, by lemma 1, implies that for every $j > 1$, type θ_j does not want to misreport. As a result, allocation x' is not only feasible but also incentive compatible. Therefore, the DA mechanism is not optimal. □

Proof of ii). If $\alpha_l + \alpha_h \geq 1$, then the DA allocation x^*, which is an ordered allocation, is such that for all $\theta_j \in \Theta$, $\overline{s}_j = \overline{s}^*$ and $\underline{s}_j = 0$, where $0 < \overline{s}^* < 1$. Consider the following alternative ordered allocation x', where $\overline{s}_1 = \overline{s}^* + \varepsilon$, $\overline{s}'_j = \overline{s}^* - \delta(\varepsilon)$ for all $j > 1$ and $\underline{s}'_j = \gamma(\varepsilon)$ for all $j > 1$, where

$$q(\theta_1) \int_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds = \widehat{q}(\theta_2) \int_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} \widehat{p}(s|\theta_2)\, ds,$$

and

$$\begin{aligned} & -(u(\theta_1, h) - u(\theta_1, l)) \int_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\, ds \\ = \;& (u(\theta_1, h) - u(\theta_1, l)) \int_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} p(s|\theta_1)\, ds - u(\theta_1, l) \int_{0}^{\gamma(\varepsilon)} p(s|\theta_1)\, ds. \end{aligned}$$

In words, we are perturbing allocation x^* by shifting some of the high-quality objects from type θ_1 to the higher types. Unlike the previous proof, we are only keeping constant the measure of high-quality objects being assigned; there will less low-quality objects assigned in order to create enough incentives for type θ_1 not to deviate.

Notice that

$$\delta'(0) = \frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)}$$

and

$$\gamma'(0) = \frac{(u(\theta_1,h) - u(\theta_1,l))}{u(\theta_1,l)} \frac{p(\overline{s}^*|\theta_1)}{p(0|\theta_1)} \left(\frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)} + 1 \right).$$

Once again, let $V(\varepsilon)$ denote the increase in welfare as a function of ε, i.e.,

$$V(\varepsilon) = \left\{ \begin{array}{c} -q(\theta_1)(u(\theta_1,h) - u(\theta_1,l)) \int\limits_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\,ds + \\ \sum\limits_{j=2}^{J} q(\theta_j) \left((u(\theta_j,h) - u(\theta_j,l)) \int\limits_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} p(s|\theta_j)\,ds - u(\theta_j,l) \int\limits_{0}^{\gamma(\varepsilon)} p(s|\theta_j)\,ds \right) \end{array} \right\}$$

and notice that $V(\varepsilon) \geq \widehat{V}(\varepsilon)$ for all $\varepsilon > 0$, where

$$\widehat{V}(\varepsilon) = \left\{ \begin{array}{c} -q(\theta_1)(u(\theta_1,h) - u(\theta_1,l)) \int\limits_{\overline{s}^*}^{\overline{s}^*+\varepsilon} p(s|\theta_1)\,ds + \\ \widehat{q}(\theta_2) \left((u(\theta_2,h) - u(\theta_2,l)) \int\limits_{\overline{s}^*-\delta(\varepsilon)}^{\overline{s}^*} \widehat{p}(s|\theta_2)\,ds - u(\theta_2,l) \int\limits_{0}^{\gamma(\varepsilon)} \widehat{p}(s|\theta_2)\,ds \right) \end{array} \right\}.$$

Notice that $\widehat{V}'(0) > 0$ if and only if

$$\begin{aligned} & u(\theta_2,h) - u(\theta_2,l) - u(\theta_1,h) + u(\theta_1,l) \\ > \; & \frac{\widehat{q}(\theta_2)}{q(\theta_1)} u(\theta_2,l) \frac{(u(\theta_1,h) - u(\theta_1,l))}{u(\theta_1,l)} \frac{\widehat{p}(0|\theta_2)}{p(0|\theta_1)} \left(\frac{q(\theta_1)}{\widehat{q}(\theta_2)} \frac{p(\overline{s}^*|\theta_1)}{\widehat{p}(\overline{s}^*|\theta_2)} + 1 \right), \end{aligned}$$

which holds whenever $\frac{\widehat{p}(0|\theta_2)}{p(0|\theta_1)} = 0$.

Given that type θ_1 is indifferent, by lemma 1, it follows that if $\varepsilon > 0$ is sufficiently small, allocation x' generates a larger welfare than the DA allocation, is incentive compatible and is feasible. □

1.5 Proof of Proposition 2

Proof. Consider the same relaxed problem as in the proof of the Theorem, where the only incentive constraints considered are the upward ones, and add to it an additional constraint that states that every agent must be assigned an object. By lemma 2, it follows that there is an ordered allocation x^1 that solves the relaxed problem with thresholds $\{\underline{s}_\theta, \overline{s}_\theta\}$. It also follows that $\underline{s}_\theta = 0$, because all considered allocations are full, and that $\overline{s}_\theta$ is weakly increasing with θ, in order for the incentive constraints that are considered to hold. The proof is completed by showing that $\overline{s}_\theta$ must be constant with θ.

Suppose not, so that there is some j such that $\overline{s}_{\theta_j} < \overline{s}_{\theta_{j+1}}$. Consider alternative allocation x' that is equal to allocation x^1 except that

$$\overline{s}'_{\theta_j} = \overline{s}_{\theta_j} + \varepsilon \text{ and } \overline{s}'_{\theta_{j+1}} = \overline{s}_{\theta_j} - \delta(\varepsilon)$$

for some small enough $\varepsilon > 0$, where $\delta(\varepsilon)$ is such that the total proportion of students attending the h school is the same as under allocation x^1. It follows that allocation x' would generate a strictly larger value welfare because

$$u(\theta_{j+1}, h) - u(\theta_{j+1}, l) > u(\theta_j, h) - u(\theta_j, l).$$

Furthermore, provided $\varepsilon > 0$ is small enough, no type $\widehat{j} < j+1$ would like to mimic type $j+1$, because

$$U\left(\theta_{\widehat{j}}, x^1(\theta_j)\right) > U\left(\theta_{\widehat{j}}, x^1(\theta_{j+1})\right)$$

for all $\widehat{j} < j+1$, which is a contradiction. □

1.6 Proof of Proposition 4

Consider the optimal ordered allocation when $\alpha_h + \alpha_l \geq 1$ and let $\overline{\alpha}_l$ denote the measure of low-quality objects assigned in that optimal allocation. By combining propositions 1 and 2, we get that $\overline{\alpha}_l < \alpha_l$, which implies that the l-feasibility constraint does not bind. Because the optimization problem is linear, it follows that the allocation that is the solution of a relaxed problem where the l-feasibility constraint is ignored is an optimal allocation whenever $\alpha_l \geq \overline{\alpha}_l$. That proves ii) ($h$ quality objects are all assigned because, if not, the h-feasibility constraint would not bind either, which cannot be). If $\alpha_l < \overline{\alpha}_l$, then the l-feasibility constraints binds, which implies i).

1.7 Proof of Proposition 5

When proving that the DA allocation is not optimal (proof of proposition 1), we introduce a binary allocation and show that it generates a larger welfare than the DA allocation.